Werk

Titel: I. Complex immersions and resolutions.

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for other currents which were considered in Sect. 2. The results of Sect. 4 will be of essential importance in [BGS4, 5] to construct Bott-Chern singular currents.

In Sect. 5, we restrict the complex (ξ, v) considered in the previous sections to a submanifold M_1, which may have non transversal intersection with the manifold M'. In this more general case, under adequate assumptions on the various metrics, the explicit computation of the limit current uses a fundamental property of the Berezinian in supergeometry [M, p. 166].

Part of this paper is quite technical since we need precise estimates to control speeds of convergence, wave front sets, or more simply because we have to use dominated convergence. Also rather complicate algebraic identities are proved in the course of the proofs, which account for the simplicity of the final answers.

The results contained in this paper were anounced in [B2].

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I. Complex immersions and resolutions

In this Section, we introduce our basic datas which are:

- An immersion of complex manifolds $M' \to M$.
- A holomorphic vector bundle η on M'.
- A projective resolution of the sheaf of holomorphic sections of η by a chain complex of holomorphic vector bundles (ξ, v) on M.
- A submersion of complex manifolds $\pi: M \to B$ which restricts to a submersion $\pi: M' \to B$.

This last data is useful since we ultimately deal with integrals along the fibers of π. However the reader way well assume that B is reduced to a point.

Also in Sect. 1c, we introduce assumption (A), which is a compatibility assumption for metrics on the vector bundles ξ, η and on the normal bundle N to M' in M.

This Section is organized as follows. In (a) we introduce our main assumptions and notations. In (b), we relate the homology of $(\xi, v)_{|M'}$ to the Koszul complex of N. In (c), we introduce Hermitian metrics on our vector bundles. Finally in (d), we describe the holomorphic Hermitian connection on the homology of the complex $(\xi, v)_{|M'}$.

a. Assumptions and notations

Let $l, l', l_1, \ldots, l_n$ be nonnegative integers such that $l_j \leqq l$, $j=1, \ldots, n$.

Let M be a compact connected complex manifold of complex dimension $l+l'$. Let $M' = \bigcup_1^n M'_j$ be a finite union of compact connected complex submanifolds of M such that if $j \neq j'$, $M'_j \cap M'_{j'} = \emptyset$. For $1 \leqq j \leqq n$, we assume that M'_j has complex dimension $l_j + l'$. Let i be the embedding $M' \to M$.

Let B be a compact connected complex manifold of complex dimension l'. Let $\pi: M \to B$ be a holomorphic submersion from M on B, whose fibers Z are compact connected complex submanifolds of M. The fibers Z have complex dimension l.

We assume that for $1 \leqq j \leqq n$, the restriction of π to M'_j is a holomorphic submersion from M'_j on B, whose fibers Y_j are compact connected complex submanifolds of M'_j. For $1 \leqq j \leqq n$, the fibers Y_j have complex dimension l_j.

Set $Y = \bigcup_1^n Y_j$. Clearly

$$Y_j = Z \cap M'_j; \qquad Y = Z \cap M'.$$

We still denote by i the embedding of fibers $Y \to Z$.

In the sequel, $T_R M$ denotes the real tangent bundle to M, and TM the $(1,0)$ part of the bundle $T_C M = T_R M \otimes_R \mathbb{C}$. When necessary, we will also use the notation $T^{(1,0)} M$ instead of TM. We define $T_R M'$, TM', $T_R B$, TB in the same way.

$T_R Z$ denotes the real tangent bundle to the fiber Z, TZ the $(1,0)$ part of $TZ = T_R Z \otimes_R \mathbb{C}$. When necessary, we still also write $T^{(1,0)} Z$ instead of TZ. We also define $T_R Y$, TY, $T^{(1,0)} Y$ in the same way. Of course we will use the notation $T_R Y_j$, TY_j, $T^{(1,0)} Y_j$ when necessary.

For $1 \leqq j \leqq n$, we denote by $N_{R,j}$ the real normal bundle to M'_j in M, and by N_j the $(1,0)$ part of $N_{R,j} \otimes_R \mathbb{C}$. Then N_j has complex dimension $e_j = l - l_j$. We will often write N_R, N, e instead of $N_{R,j}$, N_j, e_j. Note that the restrictions of N_R, N to one given fiber Y are exactly the real and complex normal bundles to Y in Z.

We have the exact sequences of holomorphic vector bundles

$$\begin{aligned} 0 &\to TZ \to TM \to \pi^* TB \to 0 \\ 0 &\to TY \to TM' \to \pi^* TB \to 0 \\ 0 &\to TM' \to TM_{|M'} \to N \to 0 \\ 0 &\to TY \to TZ_{|M'} \to N \to 0. \end{aligned} \tag{1.1}$$

Let η be a holomorphic vector bundle on the manifold M', let η_j be the restriction of η to M'_j. Of course the dimension of η_j depends in general on j. Let

$$(\xi, v): 0 \to \xi_m \xrightarrow[v]{} \xi_{m-1} \xrightarrow[v]{} \dots \xrightarrow[v]{} \xi_0 \to 0 \tag{1.2}$$

be a holomorphic chain complex of holomorphic vector bundles on M. Let r be a holomorphic restriction map $\xi_{0|M'} \to \eta$.

For $0 \leqq k \leqq m$, let $\mathcal{O}_M(\xi_k)$ be the sheaf of holomorphic sections of ξ_k on M. Similarly let $\mathcal{O}_{M'}(\eta)$ be the sheaf of holomorphic sections of η on M'. In the whole paper, we assume that the sequence of sheaves

$$0 \to \mathcal{O}_M(\xi_m) \xrightarrow[v]{} \mathcal{O}_M(\xi_{m-1}) \to \dots \xrightarrow[v]{} \mathcal{O}_M(\xi_0) \xrightarrow[r]{} i_* \mathcal{O}_{M'}(\eta) \to 0 \tag{1.3}$$

is exact. In particular the complex (ξ, v) is acyclic on $M \setminus M'$.

b. The Koszul complex of N and the homology of the complex $(\xi, v)_{|M'}$

For $1 \leqq j \leqq n$, let N_j^* be the vector bundle on M'_j dual to N_j. We will often write N^* instead of N_j^*. Let $\Lambda(N_j^*) = \bigoplus_1^{e_j} \Lambda^q(N_j^*)$ be the exterior algebra of N_j^*. If $y \in N_j$, let i_y be the interior multiplication operator acting on $\Lambda(N_j^*)$.

Let ρ_j be the projection $N_j \to M'_j$. If E is a vector bundle on M'_j, we will still denote by E the vector bundle $\rho_j^*(E)$ on the total space of N_j.

M'_j can be embedded in N_j as the zero section of N_j. Then the Koszul complex on N_j $(\Lambda(N_j^*) \otimes \eta_j, i_y)$ provides a resolution of η_j. Equivalently, if k_j is the embedding $M'_j \to N_j$ then we have the exact sequence of sheaves on N_j

$$0 \to \mathcal{O}_{N_j}(\Lambda^{e_j}(N_j^*) \otimes \eta_j) \xrightarrow[i_y]{} \dots \xrightarrow[i_y]{} \mathcal{O}_{N_j}(\eta_j) \to k_{j*}\mathcal{O}_{M'_j}(\eta_j) \to 0. \tag{1.4}$$

Take $x \in M'_j$. There exists holomorphic coordinates $(z^1, \dots, z^l)$ on an open neighborhood U of x in M such that $M'_j \cap U$ is represented by $(z^1 = 0, \dots, z^{e_j} = 0)$. On $M'_j \cap U$, N_j^* is spanned by the forms $dz^1, \dots, dz^{e_j}$, which extend to the whole open set U. Therefore the vector bundle N_j^* on $M'_j \cap U$ extends into a holomorphic vector bundle on U, which we note $\tilde{N}_j^*$. Then $y = \sum_1^{e_j} z^k \frac{\partial}{\partial z^k}$ is a holomorphic section of $\tilde{N}_j$ (which is now the dual of $\tilde{N}_j^*$ on U) which exactly vanishes on $M'_j \cap U$.

Also if U is small enough, the holomorphic vector bundle $\eta_{j|M'_j \cap U}$ extends into a holomorphic vector bundle on U, which we note $\tilde{\eta}_j$.

Let $(\Lambda(\tilde{N}_j^*) \otimes \tilde{\eta}_j, i_y)$ denote the corresponding $\mathbb{Z}$ graded Koszul complex on U. By the local uniqueness of resolutions [S, Chap. IV, Appendix 1], [E, Theorem 8], we know that if U is small enough, there exists a holomorphic acyclic $\mathbb{Z}$ graded chain complex (A, a) on U such that on U, we have an isomorphism of holomorphic $\mathbb{Z}$ graded chain complexes

$$(\xi, v) \simeq (\Lambda(\tilde{N}_j^*) \otimes \tilde{\eta}_j, i_y) \oplus (A, a). \tag{1.5}$$

In particular in the local description (1.5) of the complex (ξ, v), the restriction map $r\colon \xi_{0|M'_j} \to \eta_j$ is given by

$$r\colon h \oplus a_0 \in \eta_j \oplus A_0 \to h \in \eta_j. \tag{1.6}$$

As before, we will often omit the subscripts j.

Definition 1.1. For $x \in M'$, let $F_{0,x}, F_{1,x}, \dots, F_{m,x}$ denote the homology groups of the $\mathbb{Z}$ graded chain complex $(\xi, v)_x$. Set

$$F_x = \bigoplus_0^m F_{k,x}. \tag{1.7}$$

Take now $x \in M'$ and choose a holomorphic trivialization of the chain complex (ξ, v) on an open neighborhood U of x in M. If $X \in (T_R M)_x$, we can define

the derivative $\partial_X v(x)$ of the map $x \to v(x)$ in the direction X. Since $v^2=0$, we find that

$$\partial_X v v + v \partial_X v = 0. \tag{1.8}$$

From (1.8), we deduce that $\partial_X v$ acts naturally on the vector space F_x. If $\partial'_X v(x)$ is the derivative of v in the direction X with respect to another holomorphic trivialization of ξ, there exists $A_x(X)$ acting linearly on $\xi_{0,x}, \ldots, \xi_{m,x}$ such that

$$\partial'_X v(x) = \partial_X v(x) + [A_x(X), v(x)]. \tag{1.9}$$

From (1.9) we deduce that $\partial_X v(x)$ and $\partial'_X v(x)$ act in the same way on F_x.

We now denote by $\partial_X v(x)$ the action of any of the $\partial'_X v(x)$ on F_x, which is unambiguously defined by (1.9). In particular $\partial_X v(x)$ decreases by one the grading in F_x. Also since v is a holomorphic section of $\mathrm{End}(\xi)$, if $X \in T_x^{(0,1)} M$, then $\partial_X v(x) = 0$.

Theorem 1.2. *The vector spaces $F_{0,x}, \ldots, F_{m,x}$ are the fibers of smooth vector bundles $F_0, \ldots, F_m$, which inherit a canonical holomorphic structure from the holomorphic vector bundles $\xi_0, \ldots, \xi_m$.*

For any $x \in M'_j$, $X \in (T_R M'_j)_x$, $\partial_X v(x) = 0$, so that the linear map $y \in N_{j,x} \to \partial_y v(x) \in \mathrm{End}\, F_x$ is well defined, and depends smoothly on x, y. Also for any $x \in M'_j$, $y \in N_{j,x}$

$$(\partial_y v(x))^2 = 0. \tag{1.10}$$

On the normal bundle N_j, the complex

$$(F, \partial_y v)\colon 0 \to F_m \xrightarrow[\partial_y v]{} F_{m-1} \cdots \xrightarrow[\partial_y v]{} F_0 \to 0$$

is a holomorphic $\mathbb{Z}$ graded chain complex, which is canonically isomorphic to the holomorphic $\mathbb{Z}$ graded Koszul complex $(\Lambda(N_j^*) \otimes \eta_j, i_y)$.

Proof. Take $x \in M'_j$, and let U be an open neighborhood of x in M taken as in (1.5). We also use the notations of (1.5). Since the complex (A, a) is acyclic, for any $x' \in M'_j \cap U$, we have the non canonical isomorphism

$$F_{k,x'} \simeq (\Lambda^k N_j^* \otimes \eta_j)_{x'}; \qquad 0 \leqq k \leqq m. \tag{1.11}$$

From (1.11), we deduce in particular that the vector spaces $\{F_{k,x'}\}_{0 \leqq k \leqq m}$ have constant dimension on M'_j. Since the F_k's are the homology groups of the holomorphic chain complex $(\xi, v)_{|M'_j}$ and have constant dimension, they are the fibers of smooth holomorphic vector bundles on M'_j. Tautologically, on $M'_j \cap U$, the identification (1.11) is an identification of smooth holomorphic vector bundles.

Using the coordinate system $(z^1, \ldots, z^l)$ on the open set U, we find that on $M'_j \cap U$, TM splits holomorphically into $TM = TM'_j \oplus N_j$. If $X \in TM$, and if X' is the component of X in N_j, we deduce from (1.5) that under the identification (1.11), $\partial_X v$ corresponds to $i_{X'}$. Therefore if $X \in TM'_j$, then $\partial_X v = 0$. So the map $y \in N_j \to \partial_y v \in \mathrm{End}\, F$ is well defined. Moreover we have the identity $(\partial_y v)^2 = 0$. Also $(\partial_y v)(x)$ depends holomorphically on x, y.

We then find that on $M'_j \cap U$, we have an identification of holomorphic $\mathbb{Z}$ graded chain complexes

$$(F, \partial_y v) \simeq (\Lambda N_j^* \otimes \eta_j, i_y); \qquad y \in N_j. \tag{1.12}$$

Since the description of (ξ, v) given in (1.5) is only local, the identification (1.12) is a priori noncanonical. We now prove it is indeed canonical.

Clearly the holomorphic map $r: \xi_{0|M'_j} \to \eta_j$ induces a holomorphic map $F_0 \to \eta_j$, which we still note r. From (1.6), it follows that $r: F_0 \to \eta_j$ is an isomorphism of holomorphic vector bundles on M'_j. We see from (1.5), (1.6) that in (1.12), the isomorphism $F_0 \simeq \eta_j$ on $M'_j \cap U$ is obtained by the map $r: F_0 \to \eta_j$, i.e. is canonical.

Using (1.12) we find that if $y_1, \dots y_{e_j}$ is a base of N_j, if $f \in \eta_j \simeq F_0$, there is a unique $\omega \in F_{e_j}$ such that

$$(\partial_{y_{e_j}} v \partial_{y_{e_j - 1}} v \dots \partial_{y_1} v)\, \omega = f. \tag{1.13}$$

If $y^1, \dots, y^{e_j}$ is the base of N_j^* dual to $y_1, \dots, y_{e_j}$, we identify ω with $(y^1 \wedge y^2 \wedge \dots \wedge y^{e_j}) \otimes f$. So F_{e_j} is now canonically identified with $\Lambda^{e_j}(N_j^*) \otimes \eta_j$ and this identification is exactly the one in (1.12). Finally if $y'_1, \dots, y'_k \in N_j$, $k \leqq e_j$, we identify $(\partial_{y'_1} v \dots \partial_{y'_k} v)\, \omega$ with $(i_{y'_1} \dots i_{y'_k})\, (y^1 \wedge \dots \wedge y^{e_j}) \otimes f \in \Lambda^{e_j - k}(N_j^*) \otimes \eta_j$. Still this identification is compatible with (1.12). We then find that the isomorphism in (1.12) is canonical. For different open sets U, the corresponding isomorphisms (1.12) patch together on M'_j.

Our Theorem is proved. □

Remark 1.3. Observe that for every fiber Z the sequence of sheaves

$$0 \to \mathcal{O}_Z(\xi_m) \to \mathcal{O}_Z(\xi_{m-1}) \to \dots \to \mathcal{O}_Z(\xi_0) \to i^* \mathcal{O}_Y(\eta) \to 0 \tag{1.14}$$

is still exact.

Remark 1.4. In previous work by Bismut-Gillet-Soulé [BGS1], the chain map v was supposed to increase the degree by 1, while here it decreases the degree in ξ by 1. We choose this convention to make the comparison with the Koszul complex easier. However this new convention leads to unescapable conflicts of signs with [BGS1].

c. A family of Hermitian metrics on (ξ, v) and the Koszul complex of N

Let $h^{\xi_0}, \dots, h^{\xi_m}$ be smooth Hermitian metrics on the vector bundles $\xi_0, \dots, \xi_m$. We equip the vector bundle $\xi = \bigoplus_{k=0}^{m} \xi_k$ with the metric $h^\xi = \bigoplus_{k=0}^{m} h^{\xi_k}$ which is the orthogonal sum of the metrics $h^{\xi_0}, \dots, h^{\xi_m}$.

Clearly v is a smooth section of $\operatorname{End} \xi$. Let v^* be the adjoint of v with respect to the metric h. For $x \in M'$, we may develop the "Hodge theory" of

the finite dimensional Hermitian complex $(\xi, v)_x$. Namely, for every $x \in M'$, $0 \leqq k \leqq m$, set

$$F'_{k,x} = \{f \in \xi_{k,x};\ v(x) f = 0;\ v^*(x) f = 0\}. \tag{1.15}$$

We then have a canonical identification of vector spaces

$$F_{k,x} \simeq F'_{k,x}; \qquad x \in M';\ 0 \leqq k \leqq m. \tag{1.16}$$

Now since the $F_{k,x}$ have constant dimension on each M'_j $(1 \leqq j \leqq n)$, the $F'_{k,x}$ also have constant dimension on each M'_j $(1 \leqq j \leqq n)$. Therefore $F'_{0,x}, \ldots, F'_{m,x}$ are the fibers of smooth vector bundles $F'_0, \ldots, F'_m$ on M'. One then easily verifies that the canonical identification (1.16) induces an identification of smooth vector bundles on M'.

As smooth vector subbundles of the vector bundles $\xi_{0|M'}, \ldots, \xi_{|M'}$, the vector bundles $F'_0, \ldots, F'_m$ on M' inherit smooth Hermitian metrics. Using Theorem 1.2 and (1.16), we then find that $F_0, \ldots, F_m$ are now holomorphic Hermitian vector bundles on M'. We denote by $h^{F_0}, \ldots, h^{F_m}$ the metrics on $F_0, \ldots, F_m$.

At this stage, we will often use the notation $F_0, \ldots, F_m$ instead of $F'_0, \ldots, F'_m$, and so we will consider $F_0, \ldots, F_m$ as vector subbundles of $\xi_0, \ldots, \xi_m$. We equip $F = \bigoplus_{k=0}^{k} F_k$ with the metric h^F which is the orthogonal sum of the metrics of the metrics $h^{F_0}, \ldots, h^{F_m}$.

Let $\nabla^\xi = \bigoplus_0^m \nabla^{\xi_k}$ denote the holomorphic Hermitian connection on $\xi = \bigoplus_0^m \xi_k$. Let $\nabla^{\xi'}, \nabla^{\xi''}$ be the holomorphic and antiholomorphic parts of ∇^ξ.

By proceeding as in Sect. 1b, we find that if $x \in M'$, $X \in (T_R M')_x$, and if $\partial_X v^*$ is the derivative of v^* in any given holomorphic trivialization of ξ near x, then $\partial_X v^*(x)$ acts naturally or F_x. Of course $\partial_X v^*(x)$ now depends on the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. Since $\nabla^{\xi''} v = 0$, then $\nabla^{\xi'} v^* = 0$. We then find that if $x \in M'$, $X \in T_x^{(1,0)} M$, then $\partial_X v^* = 0$.

Let f, g be smooth sections of ξ near $x \in M'$ which are such that $f_x, g_x \in F_x$. Since

$$\langle vf, g \rangle = \langle f, v^* g \rangle \tag{1.17}$$

and since at x, $vf = 0$, $vg = 0$, $v^* f = 0$, $v^* g = 0$, we deduce that if $x \in M'$, $X \in (T_R M)_x$

$$\langle (\partial_X v) f, g \rangle = \langle f, (\partial_X v^*) g \rangle. \tag{1.18}$$

So if $X \in (T_R M)_x$, $\partial_X v^*$ is the adjoint of $\partial_X v$. In particular if $X \in T_R M'$, we deduce from (1.18) that $\partial_X v^* = 0$.

Let $\bar{N}$ be the vector bundle conjugate to N. If $y \in N$, let $\bar{y}$ denote the conjugate element in $\bar{N}$. From the previous considerations and especially from (1.18), we deduce that if $y \in N$, $\partial_{\bar{y}} v^*(x)$ acts naturally on F_x as the adjoint of $\partial_y v(x)$.

Let now g^N and g^η be smooth Hermitian metrics on the vector bundles N and η. Then N^*, and more generally the exterior algebra $\Lambda(N^*)=\bigoplus_0^e \Lambda^p(N^*)$ are naturally equipped with smooth Hermitian metrics.

We equip $\Lambda(N^*)\otimes\eta$ with the product of the metric on $\Lambda(N^*)$ and of the metric g^η.

Definition 1.5. Given metrics g^N and g^η on N and η, we will say that the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ verify assumption (A) with respect to g^N and g^η if the identification of holomorphic chain complexes on N

$$(F, \partial_y v)\simeq(\Lambda(N^*)\otimes\eta, i_y) \tag{1.19}$$

also identifies the metrics.

Proposition 1.6. *Given smooth Hermitian metrics g^N and g^η on N and η, there exist smooth Hermitian metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ which verify assumption (A) with respect to g^N and g^η.*

Proof. We first choose arbitrary smooth Hermitian metrics $h_{0|M'}, \ldots, h_{m|M'}$ on $\xi_{0|M'}, \ldots, \xi_{m|M'}$. We equip $\xi_{|M'}=\bigoplus_0^m \xi_{k|M'}$ with the orthogonal sum of the metrics $h_{k|M'}$. Let v^* denote the adjoint of v on M'.

For $0\leqq k\leqq m$, we define the vector bundle F'_k on M' by formula (1.15). For $0\leqq k\leqq m$, $x\in M'$ we then have the Hodge decomposition

$$\xi_{k,x}=v(\xi_{k+1})_x\oplus v^*(\xi_{k-1})_x\oplus F'_{k,x} \tag{1.20}$$

and the splitting in (1.20) is orthogonal with respect to the metric $h_{k|M'}$. Since $F'_0, \ldots, F'_m$ are smooth vector bundles on M', one deduces easily that for $0\leqq k\leqq m$, $v(\xi_{k+1})_{|M'}$ and $v^*(\xi_{k-1})_{|M'}$ are smooth vector subbundles of $\xi_{k|M'}$.

We now modify the metric $h_{k|M'}$ on $\xi_{k|M'}$. Namely let $h'_{k|M'}$ be the metric on $\xi_{k|M'}$ which is defined by the following three properties:

– The splitting (1.20) is still orthogonal for the metric $h'_{k|M'}$.
– The restrictions of $h_{k|M'}$ and $h'_{k|M'}$ to $v(\xi_{k+1})\oplus v^*(\xi_{k-1})$ coincide.
– When F'_k is equipped with the metric induced by $h'_{k|M'}$, the canonical identification $F'_k\simeq\Lambda^k(N^*)\otimes\eta$ also identifies the metrics.

If we equip $\xi_{|M'}$ with the orthogonal sum of the metric $h'_{k|M'}$, we find that on M', v^* is still the adjoint of v. In particular (1.20) is now the Hodge decomposition of ξ_k with respect to the new metrics $h'_{0|M'}, \ldots, h'_{m|M'}$.

By partition of unity, we extend the metrics $h'_{0|M'}, \ldots, h'_{m|M'}$ into smooth Hermitian metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$. Tautologically, the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ verify assumption (A) with respect to g^N and g^η. □

Remark 1.7. If N is equipped with a metric g^N, the metric g^N induces an isomorphism of smooth vector bundles $N^*\simeq\bar{N}$. If $y\in N$, the adjoint of the linear map i_y acting on $\Lambda(N^*)$ is given by

$$\omega\in\Lambda(N^*)\to\bar{y}\wedge\omega\in\Lambda(N^*). \tag{1.21}$$