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Some observations on motivic cohomology of arithmetic schemes

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Contents

Introduction

Let X be a regular Noetherian scheme. A recent prevailing trend is to seek the *motivic cohomology theory* on X, that is a reasonable cohomology theory on X which should play a universal role in the geometric and arithmetic aspects on X. In [Be] Beilinson conjectured the existence of certain complexes of Zariski sheaves on X, whose cohomology groups are related to the algebraic K-theory of X in the same way as classical cohomology groups are to the topological K-theory. On the other hand Lichtenbaum [L-1] predicted the existence of certain complexes of etale sheaves on X. When X is an arithmetic scheme, namely a scheme of finite type over Spec($\mathbf{Z}$), he expected that the cohomology groups arising from these complexes should play a central role to describe special values of the zeta function of X. Together with some complements given later by Milne [M-2], precise conjectures are formulated when X is a proper smooth scheme over a finite field. In honor of this fact, he called this hoped-for cohomology theory the *arithmetic cohomology theory* and conjectured that it coincides with the cohomology theory arising from the Beilinson's conjectured complexes. He also expected that it satisfies an appropriate duality theorem. In the case that X is faithfully flat over Spec($\mathbf{Z}$), it should compensate the lacking of a satisfactory duality of Poincare type for l-adic etale cohomology on X by the fact that any prime number can have ramification on X. In this respect some results are obtained in [Sa-4] in the case that $\dim(X) = 2$ by using the wight-two arithmetic cohomology constructed in [L-2]. Candidates for the hoped-for cohomology theory are given in [B1] and [B-M-S] though it is not verified that they satisfy all of desired properties. In [B-M-S] it is named the *motivic cohomology theory* following Grothendieck who looked for a cohomology theory with certain universal properties.

In this paper we will observe how the notion of motivic cohomology gives a wide overview of classical arithmetic theories for a number field or a function field of one variable over a finite field, such as the class field theory, the theory of Brauer groups and the reciprocity uniqueness theorem, and their generalizations to higher dimensional global fields. We establish a formalism which exercise general and systematic control over all these theories in terms of a duality between motivic cohomology groups of an arithmetic scheme and its idele class groups constructed by using the algebraic K-theory of X. Also we will investigate its relation with certain arithmetic theories such as the Brauer-Grothendieck group of an arithmetic surface (cf. [T-1]), a Tate's conjecture on algebraic cycles on a variety over a finite field (cf. [T-2]) and a Hasse principle for a variety over a global field.

To be more precise, fix an integral scheme X of dimension d which is proper over Spec($\mathbf{Z}$) and let K be its function field. Note that in the case $d = 1$, K is a number field or a function field in one variable over a finite field which is called a classical global field. We assume that X has no $\mathbf{R}$-valued point. In general case all statements hold true modulo two torsions. Now our fundamental implement is the following general reciprocity homomorphism

$$\Phi_K^i : H^{i+2}(K, \mathbf{Z}(i)) \to \varinjlim_{I \subset \mathcal{O}_X} \mathrm{Hom}(C_I^{d-i}(X), \mathbf{Q}/\mathbf{Z}) .$$

First, for an integer $i \geqq -1$, $H^{i+2}(K, \mathbf{Z}(i))$ is a motivic cohomology group of K whose precise definition will be given in §1. By definition

$$H^1(K, \mathbf{Z}(-1)) = \mathrm{Hom}(\mu(K), \mathbf{Q}/\mathbf{Z}) ,$$
$$H^2(K, \mathbf{Z}(0)) = H^1(K, \mathbf{Q}/\mathbf{Z}) \simeq \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K^{ab}/K), \mathbf{Q}/\mathbf{Z}) ,$$
$$H^3(K, \mathbf{Z}(1)) = H^2(K, \mathbf{G}_m) \simeq \mathrm{Br}(K) ,$$

where $\mu(K)$ is the group of all roots of unity in K, K^{ab} is the maximal abelian extension of K and $\mathrm{Br}(K)$ is the Brauer group of K. Secondly, for an integer $j \geqq 0$ and for an ideal $I \subset \mathcal{O}_X$, $C_I^j(X)$ is the *j-th idele class group of X with the modulus I* which is defined by

$$C_I^j(X) = H^d(X_{\mathrm{Nis}}, K_j^M(\mathcal{O}_X, I)) ,$$

where X_{Nis} is a certain Grothendieck topology on X introduced by Nisnevich [N] and $K_j^M(\mathcal{O}_X, I)$ is a relative version of the sheaf of the Milnor K-group on X_{Nis}. The precise definitions are given in §2. We will see that $C_I^j(X)$ has an explicit presentation by symbols in the j-th Milnor K-groups of various henselizations of K. In particular, in the case that $\dim(X) = 1$ and X is regular, we get

$$C_I^j(X) \simeq \mathrm{Coker}\left(K_j^M(K) \to \bigoplus_{v \in X_0} (K_j^M(K_v)/U^{I_v}K_j^M(\mathcal{O}_v)\right).$$

Here X_0 denotes the set of all closed points of X. For $v \in X_0$, K_v denotes the henselization of K at v and $\mathcal{O}_v$ is its ring of integers. Finally $K_j^M(*)$ denotes the Milnor K-group and $U^{I_v}K_j^M(\mathcal{O}_v)$ denotes the I_v-th unit group, where $I_v = I\mathcal{O}_v$. (For precise definitions, see Notations below.) Thus the main results of the classical arithmetic theories such as the theory of $\mathrm{Br}(K)$, the class field theory of K and the reciprocity uniqueness theorem, are rephrased that Φ_K^i is an isomorphism for $i = 1$, 0 and -1 respectively. (cf. §4)

The map Φ_K^i is constructed in the same way as Φ_K^0, which is given in [K-S, §3]. We will give its brief review in §6. In §4 and §5 the lower dimensional cases ($\dim(X) = 1$ and 2) are treated, which may help to understand the general higher dimensional case which is treated in §6. In §3 we also explain a philosophical idea for the existence of Φ_K^i from motivic cohomological point of view.

Now the main results known so far are the followings (see §7 for more details).

Theorem(0-1). (1) *The map* Φ_K^0 *is an isomorphism.*

(2) *The map* Φ_K^{-1} *is an isomorphism.*

(3) *Assume that* X *is regular of dimension two. The map* Φ_K^2 *is an isomorphism.*

(4) *Assume that* X *is regular of dimension two. The kernel of the map* Φ_K^1 *is equal to the Brauer-Grothendieck group* $\mathrm{Br}(X)$ *of* X (cf. [G, I]). *For any prime number* l, $\mathrm{Coker}(\Phi_K^1)(l) = 0$ *if and only if* $\mathrm{Br}(X)(l)$ *is finite.*

(5) *Assume that* X *is a proper smooth scheme of dimension* d *over a finite field of characteristic* p. *Fix a prime number* $l \neq p$ *and for an integer* $i \geqq 0$, *let*

$$\rho_l^i : CH^i(X)\hat{\otimes}\mathbf{Z}_l \to H^{2i}(X, \mathbf{Z}_l(i))$$

be the cycle map where $CH^i(X)$ *denotes the Chow group of cycles of codimension* i *on* X (cf. SGA4½ *Cycles*). *Then, if* ρ_l^{d-1} *is surjective,* $\mathrm{Coker}(\Phi_K^1)(l) = 0$.

(0–1)(1) is the higher dimensional global class field theory established in [K-S] and (0-1)(2) is the higher dimensional version of the reciprocity uniqueness theorem which is also proved in [K-S]. (0-1)(3) is a rephrasing of a certain Hasse principle for the motivic cohomology $H^4(K,\mathbf{Z}(2))$ of a two-dimensional global field K which is due to K. Kato [K-5]. (0-1)(4) is proved in [Sa-4] and the proof of (0-1)(5) will be given in §7 of this paper. Note that (0-1)(4) and (5) are related to a work of Tate [T-1], where he proved the equivalence of the finiteness of $\mathrm{Br}(X)(l)$ and the surjectivity of ρ_l^1 in case that X is a proper smooth surface over a finite field. It should be noted also that a well-known conjecture by Tate [T-2] implies that ρ_l^i is surjective modulo torsion.

In the last section we consider the following problem. Let k be a classical global field which is assumed to have no real place. Let P be the set of all places of k. Let X be a proper smooth geometrically connected scheme over k. For $v \in P$, let k_v be the henselization of k at v and put $X_v = X \times_k k_v$. Consider the following statement.

(H^*) Assume that there exists a 0-cycle of degree 1 on X_v for every $v \in P$. Then there exists a 0-cycle of degree 1 on X.

Here, for a scheme Z, a zero cycle on Z is a formal finite sum $c = \Sigma n_z(z)$ with $n_z \in \mathbf{Z}$, where z ranges over all closed points of Z and if Z is geometrically connected over a field k, its degree is $\Sigma n_z[\kappa(z):k]$. There have been found some cases where (H^*) is true and fails to be true (cf. [Sal], [San] and [C-S-S]). In fact, following Manin [Ma], for each collection $\mathbf{c} = (c_v)_{v \in P}$ of 0-cycles c_v of degree 1 on X_v and for each $\omega \in \mathrm{Br}(X)$, we introduce a certain element $\omega(\mathbf{c}) \in \mathbf{Q}/\mathbf{Z}$ which satisfies the following conditions.

(a) If there exists a 0-cycle on X which gives rise to c_v on each X_v, then $\omega(\mathbf{c}) = 0$ for any $\omega \in \mathrm{Br}(X)$. In other words, if for any given $\mathbf{c}$ as above we can find $\omega \in \mathrm{Br}(X)$ such that $\omega(\mathbf{c}) \neq 0$, then (H^*) fails to be true. In this sense $\omega(\mathbf{c})$ can be viewed as an obstruction for (H^*).

(b) If ω lies in the image of $\iota: \mathrm{Br}(k) \to \mathrm{Br}(X)$, then $\omega(\mathbf{c}) = 0$. In other words $\omega(\mathbf{c})$ depends only on the class of ω in $\mathrm{Coker}(\iota)$.

Now our question is whether $\omega(\mathbf{c})$ is the only obstruction for (H^*). Namely, is the following true?

(M^*) If there exists $\mathbf{c}$ as above such that $\omega(\mathbf{c}) = 0$ for any $\omega \in \mathrm{Br}(X)$, then there exists a 0-cycle on X of degree 1.

Concerning this, we give the following results (0-2) and (0-3). Put $S = \mathrm{Spec}(\mathcal{O}_k)$ if k is a number field and $\mathcal{O}_k$ is its ring of integers and let S be the proper smooth model if k is a function field. Fix a normal model $\mathcal{X}$ over S of X/k, namely $\mathcal{X}$ is a connected normal scheme which is proper flat over S and such that $\mathcal{X} \times_S \mathrm{Spec}(k) \simeq X$. Assume that $\dim(X) = d$ so that $\dim(\mathcal{X}) = d + 1$. Let K be the function field of X. For a prime number l, we consider the following condition.

(P_l) The natural map

$$\mathrm{Br}(X)(l) \to \mathrm{Ker}\left(\mathrm{Br}(K)(l) \to \bigoplus_{y \in X^1} \mathrm{Br}(K_y)/\mathrm{Br}(\mathcal{O}_y)\right)$$

is surjective. Here X^1 denotes the set of all points of X of codimension one and $\mathcal{O}_y$ is the henselization of the local ring of X at y and K_y is its quotient field.

(P_l) is conjectured to hold in general and known to be true in the case that $\dim(X) \leqq 2$ (cf. [G-II, §2] and [Sa-4, (7-3)]).

Theorem(0-2). *Assume that there exists* $\mathbf{c}$ *as above such that* $\omega(\mathbf{c}) = 0$ *for any* $\omega \in \mathrm{Br}(X)$. *Let* l *be a prime number and assume* (P_l). *If the image of the map*

$$\Phi_K^1: \mathrm{Br}(K) \to \varinjlim_{I \subset \mathcal{O}_X} \mathrm{Hom}(C_I^d(\mathcal{X}), \mathbf{Q}/\mathbf{Z})$$

contains all homomorphism of order l, *then there exists a* 0*-cyle on* X *whose degree is prime to* l.

As a corollary of (0-1)(4), (0-1)(5) and (0-2), we get the following

Theorem(0-3). *Let* X *and* $\mathcal{X}$ *be as before. Assume that there exists* $\mathbf{c}$ *as above such that* $\omega(\mathbf{c}) = 0$ *for any* $\omega \in \mathrm{Br}(X)$.

(1) *Assume that* $d = 1$ *and* $\mathscr{X}$ *is regular. Then, if* $\mathrm{Br}(\mathscr{X})(l)$ *is finite, there exist a* 0*-cycle on* X *whose degree is prime to* l*. In particular, if* $\mathrm{Br}(\mathscr{X})$ *is finite,* (M^*) *holds true.*

(2) *Assume that* k *is a function field and that* $\mathscr{X}$ *is a proper smooth scheme over a field* $\mathbf{F}_q$*. Let* l *be a prime number different from* $\mathrm{ch}(\mathbf{F}_q)$ *and assume* (P_l)*. Then, if the cycle map*

$$CH^d(\mathscr{X})\hat{\otimes}\mathbf{Z}_l \to H^{2d}(\mathscr{X}, \mathbf{Z}_l(d))$$

is surjective, there exist a 0*-cycle on* X *whose degree is prime to* l*.*

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Notations

For an abelian group M, we put $M^\vee = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$. For an integer $n \geqq 0$, M_n and M/n denote the kernel and the cokernel of $M \xrightarrow{n} M$ respectively. For a prime number l, we put

$$M(l) = \bigcup_{\nu \geqq 0} M_{l^\nu}, \quad T_l M = \mathrm{Hom}(\mathbf{Q}_l/\mathbf{Z}_l, M) \quad \text{and} \quad M \hat{\otimes} \mathbf{Z}_l = \varprojlim_{\nu} M/l^\nu .$$

For a scheme X, $\mathrm{Br}(X)$ denotes the Brauer-Grothendieck group of X (cf. [G-I]). If $\dim(X) \leqq 1$ or X is regular of dimension $\leqq 2$, we have $\mathrm{Br}(X) \simeq H^2(X, \mathbf{G}_m)$ (cf. [G-II Cor. (2-2)]). For $x \in X$, $\mathcal{O}_{X,x}$ denotes the local ring of X at x. For an integer $i \geqq 0$, X^i (resp. X_i) denotes the set of all points $x \in X$ such that $\dim(\mathcal{O}_{X,x}) = i$, (resp. $\dim(\overline{\{x\}}) = i$, where $\overline{\{x\}}$ is the closure of x in X.)

$D^b(X_{et})$ denotes the derived category of complexes of etale sheaves of X_{et} with bounded cohomology sheaves (cf. [SGA$4\frac{1}{2}$, C.D.]).

Unless indicated otherwise, all cohomology groups are taken over etale topology.

For a field F, $\bar{F}$ (resp. F^{ab}) denotes the separable closure (resp. the maximal abelian extension) of F. For an integer $i \geqq 0$, $K_i^M(F)$ denotes the Milnor K-group of F.

Let R be a discrete valuation ring with the quotient field F. For $i > 0$, $K_i^M(R)$ denotes the subgroup of $K_i^M(F)$ generated by all symbols $\{x_1, \ldots, x_i\}$ with $x_\mu \in R^*$ for $1 \leqq \mu \leqq i$. For an ideal $I \subset R (I \neq R)$, $U^I K_i^M(R)$ denotes the subgroup of $K_i^M(R)$ generated by all symbols $\{x_1, \ldots, x_i\}$ with $x_\mu \in R^*$ for $1 \leqq \mu \leqq i$ and $x_1 \in 1 + I$. For $I = R$ we put $U^I K_i^M(R) = K_i^M(R)$ by convention. We also define

$$U^I K_0^M(R) = \begin{cases} \mathbf{Z} & \text{if } I = R, \\ 0 & \text{otherwise.} \end{cases}$$

For a positive integer n invertible on X, μ_n denotes the sheaf of n-th roots of unity and for an integer i, we put $\mathbf{Z}/n\mathbf{Z}(i) = \mu_n^{\otimes i}$. If X is a scheme over $\mathbf{F}_p$, $\nu_n(i)$ denotes the additive subsheaf of the de Rham-Witt complex $W_n\Omega^i_{X/\mathbf{F}_p}$ generated by all logarithmic differentials (cf. [I]) and $\mathbf{Z}/p^n\mathbf{Z}(i) = \nu_n(i)[-i]$. If $\mathbf{Z}/l^n\mathbf{Z}(i)$ is defined on X for a prime number l, we put

$$H^j(X, \mathbf{Q}_l/\mathbf{Z}_l(i)) = \varinjlim_n H^j(X, \mathbf{Z}/l^n\mathbf{Z}(i)) \quad \text{and} \quad H^j(X, \mathbf{Z}_l(i)) = \varprojlim_n H^j(X, \mathbf{Z}/l^n\mathbf{Z}(i)) .$$

§1. Motivic cohomology

In this section, we introduce a formalism of motivic cohomology theory following Lichtenbaum [L-1]. Let X be a Noetherian regular scheme. For each non-negative integer i, he postulated the existence of an object $\mathbf{Z}(i)$ of $D^b(X_{et})$ which satisfies the following Axioms.

(A-0) $\mathbf{Z}(0) = \mathbf{Z}$ and $\mathbf{Z}(1) = \mathbf{G}_m[-1]$.

(A-1) For $i \neq 0$, $\mathbf{Z}(i)$ is acyclic outside $[1, i]$.

(A-2) If $X = \mathrm{Spec}(F)$, where F is a field, then $H^{i+1}(F, \mathbf{Z}(i)) = 0$.

(A-3) (1) For any positive integer n which is invertible on X, there exists a triangle

$$\mathbf{Z}(i) \xrightarrow{n} \mathbf{Z}(i) \to \mathbf{Z}/n\mathbf{Z}(i) \to \mathbf{Z}(i)[1] . \qquad \text{(cf. Notations)}$$

(2) (This is given later in [M-2, §2].) Assume that X is an essentially smooth scheme over a perfect field of characteristic $p \neq 0$. Then for any positive integer n, there exists a triangle

$$\mathbf{Z}(i) \xrightarrow{p^n} \mathbf{Z}(i) \to \mathbf{Z}/p^n\mathbf{Z}(i) \to \mathbf{Z}(i)[1] . \qquad \text{(cf. Notations)}$$

(A-4) There is a product map

$$\mathbf{Z}(i) \overset{L}{\otimes} \mathbf{Z}(j) \to \mathbf{Z}(i+j) .$$

(A-5) If F is a field, we have a canonical isomorphism

$$H^i(F, \mathbf{Z}(i)) \simeq K_i^M(F) .$$

The following is easy to see.

Lemma (1-1). *Let K be a field. Assume that there is $\mathbf{Z}(i)$ in $D^b(\mathrm{Spec}(K)_{et})$ satisfying* (A-1) *and* (A-3).

(1) *$H^j(K, \mathbf{Z}(i))$ is a torsion group for $j \geqq i+1$.*

(2) *Let i and j be integers such that $j \geqq i+2$ and let l be any prime number. Then we have*

$$H^j(K, \mathbf{Z}(i))(l) \simeq H^{j-1}(K, \mathbf{Q}_l/\mathbf{Z}_l(i)) . \qquad \text{(cf. Notations)}$$

(3) *Let l be a prime number. If $cd_l(K) = r < \infty$, then $H^j(K, \mathbf{Z}(i))(l) = 0$ for $j \geqq \max\{r+2, i+2\}$.*

By (1-1)(2) we are naturally led to propose the following

Definition(1-2). Let K be a field and let i, j be integers such that $i \geqq -1$ and $j \geqq i+2$. Then we define $H^j(K, \mathbf{Z}(i))$ to be the torsion group whose l-primary part is given as in (1-1)(2). By definition we have

$$H^1(K, \mathbf{Z}(-1)) = \mathrm{Hom}(\mu(K), \mathbf{Q}/\mathbf{Z}), \tag{1-2-1}$$

$$H^2(K, \mathbf{Z}(0)) = H^2(K, \mathbf{Z}) \simeq \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K^{ab}/K), \mathbf{Q}/\mathbf{Z}), \tag{1-2-2}$$

$$H^3(K, \mathbf{Z}(1)) = H^2(K, \mathbf{G}_m) \simeq \mathrm{Br}(K), \tag{1-2-3}$$

where $\mu(K)$ denotes the group of all roots of unity in K.

Lemma(1-3). *Let i, j and h be integers such that $i \geqq -1$, $j \geqq i+2$ and $h \geqq 0$. Then there exists a canonical pairing*

$$H^j(K, \mathbf{Z}(i)) \times K_h^M(K) \to H^{j+h}(K, \mathbf{Z}(i+h)).$$

Proof. Let l be a prime number. If $l \neq \mathrm{ch}(K)$, we have Galois symbols by Tate [T-3]

$$K_h^M(K) \to H^h(K, \mathbf{Z}/l^n\mathbf{Z}(h)).$$

If $l = \mathrm{ch}(K)$, we have differential symbols (cf. [I] and [K-1, II])

$$K_h^M(K) \to H^0(K, \nu_n(h)) = H^h(K, \mathbf{Z}/p^n\mathbf{Z}(h)).$$

Now the above pairing is obtained by the cup product on the Galois cohomology together with the product map for $\mathbf{Z}/l^n\mathbf{Z}(i)$ and $\mathbf{Z}/p^n\mathbf{Z}(i)$. Note that if we admit (A-5), the pairing should come from the product structure (A-4).

Now we give an important result on the arithmetic cohomology of a discrete valuation field.

Theorem(1-4). ([K-1, II] *and* [K-2]) *Assume that K is an excellent henselian discrete valuation field with residue field F. Let l be a prime number.*

(1) *Let i and j be integers such that $j \geqq i+2$. If $l \neq \mathrm{ch}(F)$, there exists a canonical exact sequence*

$$0 \to H^j(F, \mathbf{Z}(i))(l) \to H^j(K, \mathbf{Z}(i))(l) \to H^{j-1}(F, \mathbf{Z}(i-1))(l) \to 0.$$

(2) *Assume that $l = p := \mathrm{ch}(F)$ and $[F:F^p] = p^r < \infty$. Then there exists a canonical isomorphism*

$$H^{r+2}(K, \mathbf{Z}(r))(p) \simeq H^{r+1}(F, \mathbf{Z}(r-1))(p).$$

Corollary(1-5). *Assume that K is a henselian higher local field of dimension d, that is, there exists a sequence of fields;*

$$k_0, k_1, \ldots, k_d = K$$

such that k_0 is a finite field $\mathbf{F}_q$ and that k_{i+1} is an excellent henselian discrete valuation field with residue field k_i for $0 \leqq i \leqq d-1$. Then there exists a canonical isomorphism

$$H^{d+2}(K, \mathbf{Z}(d)) \cong \mathbf{Q}/\mathbf{Z}.$$

Proof. By (1-4) and (1-1)(3) we get isomorphisms

$$H^{d+2}(K, \mathbf{Z}(d)) \simeq H^{d+1}(k_{d-1}, \mathbf{Z}(d-1)) \simeq \ldots \simeq H^2(k_0, \mathbf{Z}(0)) .$$

Hence the desired isomorphism is obtained by the natural isomorphism (cf. (1-2-2))

$$H^2(k_0, \mathbf{Z}(0)) \simeq \mathrm{Hom}(\mathrm{Gal}(\mathbf{F}_q^{ab}/\mathbf{F}_q), \mathbf{Q}/\mathbf{Z}) \simeq \mathbf{Q}/\mathbf{Z}; \chi \to \chi(f) ,$$

where f is the Frobenius element over $\mathbf{F}_q$.

Let K be a henselian higher local field of dimension d. By (1-5) and (1-3) we get a canonical pairing

$$\langle\, , \rangle_\lambda : H^{i+2}(K, \mathbf{Z}(i)) \times K_{d-i}^M(K) \to H^{d+2}(K, \mathbf{Z}(d)) \cong \mathbf{Q}/\mathbf{Z} . \qquad (1\text{-}6)$$

Theorem(1-7). ([K-1, II] and [K-3]) (1) *Let $c \in H^{i+2}(K, \mathbf{Z}(i))$ and let L/K be a finite separable extension such that the image of c in $H^{i+2}(L, \mathbf{Z}(i))$ is trivial. Let*

$$N_{L/K} : K_{d-i}^M(L) \to K_{d-i}^M(K)$$

be the norm map for Milnor K-groups for fields (cf. [B-T] *and* [K-1, II]). *Then we have*

$$\langle c, N_{L/K}(a) \rangle_\lambda = 0 \quad \text{for any } a \in K_{d-i}^M(L) .$$

(2) *The pairing $\langle\, , \rangle_\lambda$ induces a canonical homomorphism*

$$\Phi_\lambda^i : H^{i+2}(K, \mathbf{Z}(i)) \to \varinjlim_{I \subset R} \mathrm{Hom}(K_{d-i}^M(K)/U^I K_{d-i}^M(R), \mathbf{Q}/\mathbf{Z}) ,$$

where R is the ring of integers of K and I ranges over all ideals of R. (By definition K is a discrete valuation field.)

(3) *The map Φ_λ^i is injective.*

Remark (1-8). (1-7)(2) follows from (1-7)(1) together with the fact that if we take a sufficiently small ideal $I \subset R$, then $1 + I \subset N_{L/K}(L^*)$.

§2. Idele class groups

Let X be a Noetherian scheme. In this section we introduce the idele class groups of X and review its fundamental properties. All details and proofs of results in this section are in [K-S]. First we introduce a certain Grothendieck topology X_{Nis} on X introduced by Nisnevich [N]. As a site it is defined as follows; as a category, X_{Nis} is the same as the small etale site over X. A family of morphisms $(f_\mu : U_\mu \to U)_\mu$ in X_{Nis} is a covering if and only if for any $x \in U$, there exists an index μ and a point $y \in U_\mu$ such that

$$f_\mu(y) = x \quad \text{and} \quad \kappa(x) \simeq \kappa(y) .$$

By definition, X_{Nis} is an intermediate topology between X_{Zar} and X_{et}. For $x \in X$ the localization of X at x in this topology is nothing other than the henselization of X at x. First we recall some basic properties of X_{Xis}.

Fact (2-1). Let $f\colon Y \to X$ be a finite morphism between Noetherian schemes. Then the direct image functor

$$f_*: Y_{\mathrm{Nis}} \to X_{\mathrm{Nis}}$$

is exact

Fact (2-2). Let $\mathscr{F}$ be an abelian sheaf on X_{Nis}. The theory of supports exists for X_{Nis} and we have a spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^p} H_x^{p+q}(X_{\mathrm{Nis}}, \mathscr{F}) \Rightarrow H^*(X_{\mathrm{Nis}}, \mathscr{F}) . \tag{2-2-1}$$

For each point $x \in X$, let X_x be the henselization of X at x. We have an isomorphism

$$H^{i-1}((X_x - x)_{\mathrm{Nis}}, \mathscr{F}) \simeq H_x^i(X_{\mathrm{Nis}}, \mathscr{F}) \quad \text{for } i \geqq 2 , \tag{2-2-2}$$

and an exact sequence

$$H^0((X_x)_{\mathrm{Nis}}, \mathscr{F}) \to H^0((X_x - x)_{\mathrm{Nis}}, \mathscr{F}) \to H_x^1(X_{\mathrm{Nis}}, \mathscr{F}) \to 0 . \tag{2-2-3}$$

By the induction on $\dim(X)$ we can see

$$E_1^{p,q} = 0 \quad \text{for } q > 0 \text{ or } p > \dim(X) , \tag{2-2-4}$$

$$H^i(X_{\mathrm{Nis}}, \mathscr{F}) = 0 \quad \text{for } i > \dim(X) . \tag{2-2-5}$$

Fact (2-3). Assume that X is integral of dimension d. Let $\mathscr{F}, \mathscr{G}$ be abelian sheaves on X_{Nis}. Then we have an isomorphism

$$H^d(X_{\mathrm{Nis}}, \mathscr{F}) \simeq H^d(X_{\mathrm{Nis}}, \mathscr{G})$$

under the following conditions. For a point $x \in X$, let $\mathscr{F}_x$, $\mathscr{G}_x$ denote their stalks at x.

(1) There exists an isomorphism $\varphi\colon \mathscr{F}_\Lambda \simeq \mathscr{G}_\Lambda$ where Λ is the generic point of X.

(2) For each point $\eta \in X^1$ put

$$\mathscr{F}_\Lambda(\eta) = \mathrm{Im}(\mathscr{F}_\eta \to \mathscr{F}_\Lambda) \quad \text{and} \quad \mathscr{G}_\Lambda(\eta) = \mathrm{Im}(\mathscr{G}_\eta \to \mathscr{G}_\Lambda) .$$

Then φ induces an isomorphism $\mathscr{F}_\Lambda(\eta) \simeq \mathscr{G}_\Lambda(\eta)$ for each $\eta \in X^1$.

(2-3) follows from (2-2) and will be used in the next section.

Now we can introduce our idele class groups of X.

Definition (2-4). Let $d = \dim(X)$. For an integer $i \geqq 0$ and for an ideal $I \subset \mathcal{O}_X$, the i-th idele class group of X with modulus I is

$$C_I^i(X) = H^d(X_{\mathrm{Nis}}, K_i^M(\mathcal{O}_X, I)) .$$

Here, for any scheme Z, we put

$$K_0^M(\mathcal{O}_Z) = \mathbf{Z} \quad \text{and} \quad K_i^M(\mathcal{O}_Z) = \overbrace{\mathcal{O}_Z^* \otimes \ldots \otimes \mathcal{O}_Z^*}^{i \text{ times}}/J \quad \text{for } i > 0 ,$$

where J is the subgroup generated by all local sections of the forms $x_1 \otimes \ldots \otimes x_i$

such that $x_\mu + x_\nu = 1$ for some $1 \leqq \mu \neq \nu \leqq i$. For $i > 0$ we put

$$K_i^M(\mathcal{O}_X, I) = \mathrm{Ker}(K_i^M(\mathcal{O}_X) \to K_i^M(\mathcal{O}_X/I)) .$$

For $i = 0$ we put

$$K_0^M(\mathcal{O}_X, I) = j_!\mathbf{Z} ,$$

where $j: U \subsetneq X$ is the complement of $\mathrm{Supp}(\mathcal{O}_X/I)$ in X and $j_!$ denotes the extension by zero outside U.

Remark (2-5). The choice of the above notations is rather confusing. For the stalks of the sheaves $K_i^M(\mathcal{O}_X)$ and $K_i^M(\mathcal{O}_X, I)$ do not necessarily coincide with the usual notations for the Milnor K-groups of rings and its relative version. This inconsistency is remedied by (2-3) and (2-7) below.

In what follows, we give some basic results known for $C_I^i(X)$.

Fact (2-6). (*Transition maps*) For $J \subset I \subset \mathcal{O}_X$, the natural map

$$K_i^M(\mathcal{O}_X, J) \to K_i^M(\mathcal{O}_X, I)$$

induces surjective homomorphisms

$$C_J^i(X) \to C_I^i(X) .$$

In fact this follows at once from (2-2-5).

Fact (2-7). Assume that X is integral of dimension d and let K be its function field. For an ideal $I \subset \mathcal{O}_X$, let $\bar{K}_i^M(\mathcal{O}_X, I)$ be the image of $K_i^M(\mathcal{O}_X, I)$ in the constant sheaf $K_i^M(K)$. Then we have an isomorphism

$$C_I^i(X) \simeq H^d(X_{\mathrm{Nis}}, \bar{K}_i^M(\mathcal{O}_X, I)).$$

This follows at once from (2-3).

Fact (2-8). (Natural maps) Let $f: Y \to X$ be a finite morphism of integral schemes. Let K (resp. L) be the function field of X (resp. Y). Let $I \subset \mathcal{O}_X$ and put $J = I\mathcal{O}_Y$. Then the natural map $K_i^M(K) \to K_i^M(L)$ induces a natural map

$$\bar{K}_i^M(\mathcal{O}_X, I) \to \bar{K}_i^M(\mathcal{O}_Y, J)$$

and we get a canonical map

$$R: C_I^i(X) \to C_J^i(Y) .$$

Fact (2-9). (*Norm maps*) Let the assumptions be as in (2-8). Let

$$N: K_i^M(L) \to K_i^M(K)$$

be the norm map for Milnor K-groups of fields (cf. [B-T] and [K-1, II]). Then, for any $I \subset \mathcal{O}_X$, if we take a sufficiently small $J \subset \mathcal{O}_Y$, N induces

$$f_* \bar{K}_i^M(\mathcal{O}_Y, J) \to \bar{K}_i^M(\mathcal{O}_X, I) .$$

Thus, by (2-1) we get a norm map

$$N: C_J^i(Y) \to C_I^i(X) .$$

We can see that the composite map

$$\varprojlim_{I\subset\mathcal{O}_X} C_I^i(X) \xrightarrow{R} \varprojlim_{J\subset\mathcal{O}_Y} C_J^i(X) \xrightarrow{N} \varprojlim_{I\subset\mathcal{O}_X} C_I^i(X)$$

coincides with the multiplication by $[L:K]$.

Fact (2-10). The idele class group $C_I^i(X)$ has an explicit presentation by symbols in the i-th Milnor K-groups of various henselizations of K. The general description will be given in §6.

§3. Reciprocity maps and motivic interpretation

Let X be an integral scheme which is proper over Spec($\mathbf{Z}$). Let K be its function field. We assume that X has no $\mathbf{R}$-valued point. The following theorem plays a fundamental role in this paper.

Theorem (3-1). *Let $d = \dim(X)$. For an integer $-1 \leqq i \leqq d$, there exists a canonical homomorphism*

$$\Phi_K^i : H^{i+2}(K, \mathbf{Z}(i)) \to \varinjlim_{I\subset\mathcal{O}_X} \mathrm{Hom}(C_I^{d-i}(X), \mathbf{Q}/\mathbf{Z}).$$

Remark (3-2). (1) By [Se-1, Ch. II Pr. 11 and Pr. 14], we know that $cd(K) = d+1$. Hence, by (1-1-3) we have

$$H^{i+2}(K, \mathbf{Z}(i)) = 0 \quad \text{for } i \geqq d+1.$$

(2) Let $f: Y \to X$ be a finite morphism and let L be the function field of Y. Then we have commutative diagrams

$$\begin{array}{ccc} H^{i+2}(K,\mathbf{Z}(i)) & \xrightarrow{\Phi_K^i} & \varinjlim_{I\subset\mathcal{O}_X} \mathrm{Hom}(C_I^{d-i}(X), \mathbf{Q}/\mathbf{Z}) \\ \downarrow \mathrm{Res}_{K/L} & & \downarrow N^\vee \\ H^{i+2}(L,\mathbf{Z}(i)) & \xrightarrow{\Phi_L^i} & \varinjlim_{J\subset\mathcal{O}_Y} \mathrm{Hom}(C_J^{d-i}(Y), \mathbf{Q}/\mathbf{Z}), \end{array}$$

$$\begin{array}{ccc} H^{i+2}(L,\mathbf{Z}(i)) & \xrightarrow{\Phi_L^i} & \varinjlim_{J\subset\mathcal{O}_Y} \mathrm{Hom}(C_J^{d-i}(Y), \mathbf{Q}/\mathbf{Z}) \\ \downarrow \mathrm{Cor}_{L/K} & & \downarrow R^\vee \\ H^{i+2}(K,\mathbf{Z}(i)) & \xrightarrow{\Phi_K^i} & \varinjlim_{I\subset\mathcal{O}_X} \mathrm{Hom}(C_I^{d-i}(X), \mathbf{Q}/\mathbf{Z}). \end{array}$$

The construction of Φ_K^i is essentially the same as that of Φ_K^0 given in [K-S]. Its outline will be given in §6. In this section we will explain its philosophical idea from motivic cohomological point of view, which is not really necessary for understanding the rest of the paper.

First assume the following

Assumption (3-3). For a sufficiently small dense regular open subscheme $U \subsetneq X$ and for each integer $i \geqq 0$, there exists $\mathbf{Z}(i)$ in $D^b(U_{et})$ which satisfies (A-0) and (A-1) and satisfies (A-3) and (A-5) at its generic point.

We should have

$$H^{i+2}(K, \mathbf{Z}(i)) \simeq \varinjlim_{U \subset X} H^{i+2}(U, \mathbf{Z}(i)),$$

where U ranges over all dense regular open subschemes of X. Hence we are reduced to construct a homomorphism

$$H^{i+2}(U, \mathbf{Z}(i)) \to \varinjlim_{I \in \mathscr{I}_U} \mathrm{Hom}(C_I^{d-i}(X), \mathbf{Q}/\mathbf{Z}), \tag{3-4}$$

where $\mathscr{I}_U$ denotes the set of all ideals $I \subset \mathcal{O}_X$ such that $I\mathcal{O}_U = \mathcal{O}_U$.

Theorem (3-5). *Let $j: U \subsetneq X$ be a dense open regular subscheme and assume* (3-3). *Then there exists a canonical homomorphism*

$$\theta: H^{2d+2}(X, j!\mathbf{Z}(d)) \to \mathbf{Q}/\mathbf{Z}.$$

In [Sa-4] the canonical map θ is constructed and proved to be an isomorphism in the case that $d = 2$ by using $\mathbf{Z}(2)$ defined by [L-2]. The construction of θ in general case is essentially similar to this case but much more complicate. When X is a proper integral scheme over a finite field $\mathbf{F}_q$, θ can be constructed also by using (A-3) and the trace map for etale cohomology, at least neglecting the p-part (cf. [M-2]).

Now (3-5) gives a canonical pairing

$$H^{i+2}(U, \mathbf{Z}(i)) \times H^{2d-i}(X, j_!\mathbf{Z}(d-i)) \to H^{2d+2}(X, j_!\mathbf{Z}(d)) \xrightarrow{\theta} \mathbf{Q}/\mathbf{Z}$$

which induces a canonical homomorphism

$$H^{i+2}(U, \mathbf{Z}(i)) \to \mathrm{Hom}(H^{2d-i}(X, j_!\mathbf{Z}(d-i)), \mathbf{Q}/\mathbf{Z}). \tag{3-6}$$

Next we assume the following

Assumption (3-7). Let R be an excellent discrete valuation ring with the quotient field L. Then, for each ideal $I \subset R$ $(I \neq R)$ and for each non-negative integer i, there exists an object $\mathbf{Z}(i)_I$ in $D^b(\mathrm{Spec}(R/I)_{et})$ which satisfies the following

(0) For $I = (0)$, $\mathbf{Z}(i)_I = \mathbf{Z}(i)$ conjectured by [L-1] (cf. §1).

(1) $\mathbf{Z}(0)_I = \mathbf{Z}$ on $\mathrm{Spec}(R/I)_{et}$ and $\mathbf{Z}(1)_I = \mathbf{G}_{m,R/I}[-1]$.

(2) There exists a canonical isomorphism

$$H^i(R, \mathbf{Z}(i)_I) \simeq K_i^M(R)/U^I K_i^M(R).$$

(3) There are natural maps

$$\iota_I: \mathbf{Z}(i) \to \mathbf{Z}(i)_I \quad \text{and} \quad \iota_{J,I}: \mathbf{Z}(i)_J \to \mathbf{Z}(i)_I,$$

where $J \subset I \subset R$. These maps should satisfy the obvious compatibility. Also they are compatible with the isomorphism in (2) in an evident sense.

Fix a non-zero ideal $I \subset \mathcal{O}_X$. For $\eta \in X^1$ let R_η be the local ring of X at η and put $I_\eta = I\mathcal{O}_X$. Let X_I^1 be the subset of X^1 consisting of all $\eta \in \mathrm{Supp}(\mathcal{O}_X/I)$. By (2-9) we may replace X by its normalization for the construction of Φ_K^i. Thus we may assume that R_η is a discrete valuation ring for all $\eta \in X^1$. Then (3-7) gives an object

$$\widetilde{\mathbf{Z}(i)}_I := \bigoplus_{\eta \in X_I^1} \mathbf{Z}(i)_{I_\eta} \quad \text{in} \quad D^b\bigg(\bigg(\coprod_{\eta \in X_I^1} \mathrm{Spec}(R_\eta/I_\eta)\bigg)_{et}\bigg).$$

Let $\mathbf{Z}(i)_I = \tau_* \widetilde{\mathbf{Z}(i)}_I$ be its extension by the natural morphism $\tau : \coprod_{\eta \in X_I^1} \mathrm{Spec}(R_\eta/I_\eta) \to X$. Finally we put

$$j_I \mathbf{Z}(i) = \mathrm{Ker}(\mathbf{Z}(i) \to \mathbf{Z}(i)_I).$$

For $J \subset I \subset \mathcal{O}_X$, we have natural maps

$$j_! \mathbf{Z}(i) \to j_J \mathbf{Z}(i) \to j_I \mathbf{Z}(i)$$

which induce the natural maps

$$H^{2d-i}(X, j_! \mathbf{Z}(i)) \to H^{2d-i}(X, j_J \mathbf{Z}(i)) \to H^{2d-i}(X, j_I \mathbf{Z}(i)).$$

Assumption (*3-8*). The map (3-6) factors through a canonical homomorphism

$$H^{i+2}(U, \mathbf{Z}(i)) \to \varinjlim_{I \in \mathcal{I}_U} \mathrm{Hom}(H^{2d-i}(X, j_I \mathbf{Z}(d-i)), \mathbf{Q}/\mathbf{Z}).$$

Now (3-8) reduces the construction of the map (3-6) to that of a canonical map

$$C_I^j(X) \to H^{d+j}(X_{et}, j_I \mathbf{Z}(j)) \quad \text{for } j \geqq 0. \tag{3-9}$$

Let $\pi : X_{et} \to X_{\mathrm{Nis}}$ be the natural morphism of sites and put $\mathcal{F} = R^j \pi_* j_I \mathbf{Z}(j)$ and $\mathcal{G} = K_j^M(\mathcal{O}_X, I)$. Let Λ be the generic point of X and let $\eta \in X^1$. Then, by (3-3) and (3-7)(2), we have (cf. (2-3))

$$\mathcal{F}_\Lambda \simeq K_j^M(K) \simeq \mathcal{G}_\Lambda \quad \text{and} \quad \mathcal{F}_\Lambda(\eta) \simeq U^{I_\eta} K_j^M(R_\eta) \simeq \mathcal{G}_\Lambda(\eta).$$

Hence (2-3) gives an isomorphism

$$C_I^j(X) = H^d(X_{\mathrm{Nis}}, K_j^M(\mathcal{O}_X, I)) \simeq H^d(X_{\mathrm{Nis}}, R^j \pi_* j_I \mathbf{Z}(j)). \tag{3-10}$$

Now the desired homomorphism (3-9) is obtained as the composite of (3-10) and the edge homomorphism

$$H^d(X_{\mathrm{Nis}}, R^j \pi_* j_I \mathbf{Z}(j)) \to H^{d+j}(X_{et}, j_I \mathbf{Z}(j))$$

in view of (2-2-5).

§4. One dimensional case

In this section we assume that X is a one-dimensional regular scheme which is proper over Spec($\mathbf{Z}$). Let K be its function field. We assume that X has no embedding into **R**. For $v \in X_0$ let $\mathcal{O}_v$ be the henselization of the local ring of X at v and let K_v be its quotient field. We have the following explicit description of our idele class group.

Lemma (4-1). *For any integer $i \geqq 0$, there exists a canonical isomorphism*

$$C_I^i(X) \simeq \operatorname{Coker}\left(K_i^M(K) \to \bigoplus_{v \in X_0} K_i^M(K_v)/U^{I_v} K_i^M(\mathcal{O}_v) \right),$$

where $I_v = I\mathcal{O}_v$.

Proof. By (2-2) we have an exact sequence

$$K_i^M(K) = H^0(\operatorname{Spec}(K)_{\mathrm{Nis}}, K_i^M(\mathcal{O}_X, I)) \to \bigoplus_{v \in X_0} H_v^1(X_{\mathrm{Nis}}, K_i^M(\mathcal{O}_X, I)) \to C_I^i(X) \to 0$$

and for $v \in X_0$,

$$\begin{aligned} H_v^1(X_{\mathrm{Nis}}, K_i^M(\mathcal{O}_X, I) &\simeq \operatorname{Coker}(H^0(\operatorname{Spec}(\mathcal{O}_v)_{\mathrm{Nis}}, K_i^M(\mathcal{O}_X, I)) \\ &\to H^0(\operatorname{Spec}(K_v)_{\mathrm{Nis}}, K_i^M(\mathcal{O}_X, I))) \\ &\simeq K_i^M(K_v)/U^{I_v} K_i^M(\mathcal{O}_v) . \end{aligned}$$

This proves our assertion. Note that in the case $i = 0$ we get

$$C_I^0(X) \simeq \operatorname{Coker}\left(\mathbf{Z} \to \bigoplus_{v \in \Sigma_I} \mathbf{Z} \right),$$

where $\sum_I$ denotes the subset of X_0 consisting of all $v \in \operatorname{Supp}(\mathcal{O}_X/I)$.

By (1-5) we have a canonical isomorphism for each $v \in X_0$

$$\theta_v : H^3(K_v, \mathbf{Z}(1)) \simeq \mathbf{Q}/\mathbf{Z} .$$

Noting (1-2-3), this is nothing but the classical isomorphism for the Brauer group of a local field (cf. [Se-2, XII]). Furthermore, by (1-6) and (1-7), we have a canonical pairing for $-1 \leqq i \leqq 1$,

$$H^{i+2}(K_v, \mathbf{Z}(i)) \times K_{1-i}^M(K_v) \to \mathbf{Q}/\mathbf{Z}$$

which induces a canonical homomorphism

$$\Phi_v^i : H^{i+2}(K_v, \mathbf{Z}(i)) \to \varinjlim_{I_v \subset \mathcal{O}_v} \operatorname{Hom}(K_{1-i}^M(K_v)/U^{I_v} K_{1-i}^M(\mathcal{O}_v), \mathbf{Q}/\mathbf{Z}).$$

From classical arithmetic theories, we get the following results.

Fact (4-2). In view of (1-2-3) the map Φ_v^1 gives an homomorphism

$$H^3(K_v, \mathbf{Z}(1)) \to \mathbf{Q}/\mathbf{Z} .$$

It is nothing but the map θ_v and it is an isomorphism.

Fact (4-3). In view of (1-2-2) the map Φ_v^0 gives

$$H^1(K_v, \mathbf{Q}/\mathbf{Z}) \to \varinjlim_{I_v \subset \mathcal{O}_v} \operatorname{Hom}(K_v^*/U_v^{I_v}, \mathbf{Q}/\mathbf{Z}) ,$$

where $U_v^{I_v}$ is the I_v-th unit subgroup in $\mathcal{O}_v^*$. It is nothing other than the dual of the classical reciprocity map for K_v (cf. [Se-2]) and it is an isomorphism.

Fact (4-4). In view of (1-2-1) the dual of Φ_v^{-1} gives

$$\delta_v : K_2^M(K_v) \to \mu(K_v)\,.$$

It is nothing other than the classical Hilbert symbol and it is an isomorphism. In fact let $\hat{K}_v$ be the completion of K_v and let

$$\hat{\delta}_v : K_2^M(\hat{K}_v) \to \mu(\hat{K}_v) = \mu(K_v)$$

be the Hilbert symbol for $\hat{K}_v$. Moore proved that $\hat{\delta}_v$ is surjective and $\operatorname{Ker}(\hat{\delta}_v)$ is divisible (cf. [Mi, Appendix]). Tate [T-4] and Merkurjev [Me] proved that $\operatorname{Ker}(\hat{\delta}_v)$ has no torsion so that it is uniquely divisible. Finally, by [Su, Cor. 3-12],

$$K_2^M(K_v) \simeq K_2^M(\hat{K}_v)_{\text{tor}}\,.$$

This proves our assertion. In particular, we can see that $U^{I_v}K_2^M(K_v) = 0$ for a sufficiently small $I_v \subset \mathcal{O}_v$.

Finally our fundamental homomorphism

$$\Phi_K^i : H^{i+2}(K, \mathbf{Z}(i)) \to \varinjlim_{I \subset \mathcal{O}_X} \operatorname{Hom}(C_I^{1-i}(X), \mathbf{Q}/\mathbf{Z})$$

is obtained from all maps Φ_v^i for $v \in X_0$ together with the following classical result.

Theorem (4-5). (*Reciprocity law*). *For $c \in H^3(K, \mathbf{Z}(1)) \simeq \operatorname{Br}(K)$, its image c_v in $H^3(K_v, \mathbf{Z}(1)) \simeq \operatorname{Br}(K_v)$ is trivial for almost all $v \in X_0$. Moreover we have*

$$\sum_{v \in X_0} \theta_v(c_v) = 0\,.$$

By the classical arithmetic theories, we have the following results.

Fact (4-6). The map Φ_K^2 is an isomorphism, which is equivalent to say that the following sequence is exact.

$$0 \to \operatorname{Br}(K) \xrightarrow{\iota} \bigoplus_{v \in X_0} \mathbf{Q}/\mathbf{Z} \xrightarrow{\sigma} \mathbf{Q}/\mathbf{Z} \to 0\,,$$

where σ is the addition map and

$$\iota(\omega) = \{\theta_v(\omega_v)\}_{v \in X_0}\,.$$

Here for $\omega \in \operatorname{Br}(K)$, ω_v denotes its image in $\operatorname{Br}(K_v)$.

Fact (4-7). The map Φ_K^0 gives

$$H^1(K, \mathbf{Q}/\mathbf{Z}) \to \varinjlim_{I \subset \mathcal{O}_X} \operatorname{Hom}(C_I^1(X), \mathbf{Q}/\mathbf{Z})\,.$$

By (4-1) $C_I^1(X)$ is the finite part of the classical idele class group with the modulus I and the map is nothing but the dual of the classical reciprocity map of the global class field theory. Thus it is an isomorphism.

Fact (4-8). The map Φ_K^{-1} is an isomorphism, or more precisely we have a canonical isomorphism

$$C_I^2(X) \simeq \mu(K)$$

for a sufficiently small $I \subset \mathcal{O}_X$. In view of (4-4) this is equivalent to say that the following sequence is exact, which is known as the reciprocity uniqueness theorem due to C. Moore (cf. [Mi, §16]).

$$K_2(K) \xrightarrow{\iota} \bigoplus_{v \in X_0} \mu(K_v) \xrightarrow{\rho} \mu(K) \to 0 .$$

Here the map ι is given by

$$\iota(a) = \{\delta_v(a_v)\}_{v \in X_0} ,$$

where a_v is the image of $a \in K_2(K)$ in $K_2(K_v)$. The map ρ is given by

$$\rho(\{\zeta_v\}_{v \in X_0}) = \prod_{v \in X_0} \zeta_v^{d_v} ,$$

where $d_v = [\mu(K_v) : \mu(K)]$.

§5. Two dimensional case

In this section we assume that X is a two-dimensional normal scheme which is proper over Spec(**Z**) with the function field K and give the construction of our fundamental homomorphism Φ_K^i. For this we introduce some conventions. For $x \in X_0$ (resp. $\eta \in X_1$), $\mathcal{O}_x$ (resp. $\mathcal{O}_\eta$) denotes the henselization of the local ring of X at x (resp. η), K_x (resp. K_η) denotes its field of fractions, and $\kappa(x)$ (resp. $\kappa(\eta)$) denotes its residue field. For a fixed $x \in X_0$, put $P_x = \mathrm{Spec}(\mathcal{O}_x)_1$. For $\eta \in X_1$, η_0 denotes the set of all $x \in X_0$ which lie on the closure of η in X. Put

$$P(X) = \{\delta = (x, \eta) | x \in X_0, \eta \in X_1, x \in \eta_0\} .$$

For $\delta = (x, \eta) \in P(X)$, let P_δ denote the subset of P_x consisting of all $\lambda \in P_x$ which lie over η. For a fixed $\eta \in X_1$, put $P_\eta = \coprod_{x \in \eta_0} P_{(x,\eta)}$. It is in one-to-one correspondence with the set of all finite places of $k(\eta)$. For $\lambda \in P_x$ let $\mathcal{O}_\lambda$ denotes the henselization of $\mathcal{O}_x$ at λ, $\kappa(\lambda)$ the residue field of λ and K_λ denotes the field of fractions of $\mathcal{O}_\lambda$. By definition, there are natural maps

$$\iota_{x,\lambda} : K_x \to K_\lambda \quad \text{for } x \in X_0 \text{ and } \lambda \in P_x,$$

$$\iota_{\eta,\lambda} : K_\eta \to K_\lambda \quad \text{for } \delta = (x, \eta) \in P(X) \text{ and } \lambda \in P_\delta .$$

Lemma (5-1). *For an integer $i \geqq 0$ and an ideal $I \subset \mathcal{O}_X$, we have a canonical isomorphism*

$$C_I^i(X) \simeq \left(\bigoplus_{\delta \in P(X)} \bigoplus_{\lambda \in P_\delta} K_i^M(K_\lambda)/U^{I_\lambda} K_i^M(\mathcal{O}_\lambda) \right) / \Delta ,$$

where $I_\lambda = I\mathcal{O}_\lambda$ and Δ is the subgroup generated by all elements $(a_\lambda)_\lambda$ of either of the following types.

(1) *There exists* $x \in X_0$ *and* $b_x \in K_i^M(K_x)$ *such that*

$$a_\lambda \equiv \begin{cases} \iota_{x,\lambda}(b_x) & \textit{if } \lambda \in P_x, \\ 0 & \textit{otherwise}. \end{cases}$$

(2) *There exist* $\eta \in X_1$ *and* $b_\eta \in K_i^M(K_\eta)$ *such that*

$$a_\lambda \equiv \begin{cases} \iota_{\eta,\lambda}(b_\eta) & \textit{if } \lambda \in P_\eta, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Put $\mathscr{F} = K_i^M(\mathcal{O}_X, I)$. By (2-2) we get an exact sequence

$$\bigoplus_{\eta \in X_1} H_\eta^1(X_{\mathrm{Nis}}, \mathscr{F}) \to \bigoplus_{x \in X_0} H_x^2(X_{\mathrm{Nis}}, \mathscr{F}) \to C_I^i(X) \to 0\,, \tag{5-1-1}$$

and a surjective homomorphism

$$K_i^M(K_\eta) = H^0(\mathrm{Spec}(K_\eta)_{\mathrm{Nis}}, \mathscr{F}) \to H_\eta^1(X_{\mathrm{Nis}}, \mathscr{F})\,, \tag{5-1-2}$$

and an isomorphism

$$H_x^2(X_{\mathrm{Nis}}, \mathscr{F}) \simeq H^1((X_x - x)_{\mathrm{Nis}}, \mathscr{F})\,.$$

Applying (2-2) to the one-dimensional scheme $X_x - x$, we get an exact sequence

$$K_i^M(K_x) = H^0(\mathrm{Spec}(K_x)_{\mathrm{Nis}}, \mathscr{F}) \to \bigoplus_{\lambda \in P_x} H_\lambda^1((X_x - x)_{\mathrm{Nis}}, \mathscr{F})$$
$$\to H^1((X_x - x)_{\mathrm{Nis}}, \mathscr{F}) \to 0$$

and an isomorphism

$$H_\lambda^1((X_x - x)_{\mathrm{Nis}}, \mathscr{F}) \simeq \mathrm{Coker}(H^0(\mathrm{Spec}(\mathcal{O}_\lambda)_{\mathrm{Nis}}, \mathscr{F}) \to H^0(\mathrm{Spec}(K_\lambda)_{\mathrm{Nis}}, \mathscr{F}))$$
$$\simeq K_i^M(K_\lambda)/U^{I_\lambda} K_i^M(\mathcal{O}_\lambda)\,.$$

Hence we get an isomorphism

$$H_x^2(X_{\mathrm{Nis}}, \mathscr{F}) \simeq \mathrm{Coker}\left(K_i^M(K_x) \to \bigoplus_{\lambda \in P_x} K_i^M(K_\lambda)/U^{I_\lambda} K_i^M(\mathcal{O}_\lambda) \right).$$

This together with (5-1-1) and (5-1-2) proves our assertion.

Now we give the construction of our fundamental homomorphism

$$\Phi_K^i : H^{i+2}(K, \mathbf{Z}(i)) \to \varinjlim_{I \subset \mathcal{O}_X} \mathrm{Hom}(C_I^{2-i}(X), \mathbf{Q}/\mathbf{Z})\,.$$

Let $c \in H^{i+2}(K, \mathbf{Z}(i))$ and let L/K be a separable extension such that the image of c in $H^{i+2}(L, \mathbf{Z}(i))$ is trivial. By definition, for $\delta \in P(X)$ and $\lambda \in P_\delta$, K_λ is a henselian local field of dimension two in the sense of (1-5). Hence (1-5) gives a canonical isomorphism

$$\theta_\lambda : H^4(K_\lambda, \mathbf{Z}(2)) \simeq \mathbf{Q}/\mathbf{Z}\,,$$

and (1-6) and (1-7) give a homomorphism

$$\Phi_\lambda^i(c_\lambda) : K_{2-i}^M(K_\lambda) \to \mathbf{Q}/\mathbf{Z}\,,$$

where c_λ is the image of c in $H^{i+2}(K_\lambda, \mathbf{Z}(i))$. Moreover, if we take a sufficiently small $I \subset \mathcal{O}_X$, then it satisfies the following condition for any λ: If the extension LK_λ/K_λ is unramified with respect to the discrete valuation of $\mathcal{O}_\lambda$, then $I\mathcal{O}_\lambda = \mathcal{O}_\lambda$. If it ramifies, then the subgroup $1 + I\mathcal{O}_\lambda$ of $\mathcal{O}_\lambda^*$ is contained in the image of the norm map $(LK_\lambda)^* \to K_\lambda^*$. Then, by (1-7)(1) the collection of the homomorphisms $\Phi_\lambda^i(c_\lambda)$ for all λ gives a canonical homomorphism

$$\tilde{\Phi}_K^i(c): \bigoplus_{\delta \in P(X)} \bigoplus_{\lambda \in P_\delta} K_{2-i}^M(K_\lambda)/U^{I_\lambda}K_{2-i}^M(\mathcal{O}_\lambda) \to \mathbf{Q}/\mathbf{Z},$$

where $I_\lambda = I\mathcal{O}_\lambda$. Finally, in view of (5-1), $\tilde{\Phi}_K^i(c)$ induces the desired homomorphism

$$\Phi_K^i(c): C_I^{2-i}(X) \to \mathbf{Q}/\mathbf{Z}$$

by the following two reciprocity laws in the two-dimensional context.

Theorem (5-2). (cf. [Sa-1, Ch. I]) *For $c \in H^4(K_x, \mathbf{Z}(2))$, its image c_λ in $H^4(K_\lambda, \mathbf{Z}(2))$ is trivial for almost all $\lambda \in P_x$. Moreover we have*

$$\sum_{\lambda \in P_x} \theta_\lambda(c_\lambda) = 0 .$$

Theorem (5-3). *Assume that $\kappa(\eta)$ has no embedding into $\mathbf{R}$. For $c \in H^4(K_\eta, \mathbf{Z}(2))$, its image c_λ in $H^4(K_\lambda, \mathbf{Z}(2))$ is trivial for almost all $\lambda \in P_\eta$. Moreover we have*

$$\sum_{\lambda \in P_\eta} \theta_\lambda(c_\lambda) = 0 .$$

In view of the isomorphism (cf. (1-3-2) and (1-2-3))

$$H^4(K_\eta, \mathbf{Z}(2)) \simeq H^3(\kappa(\eta), \mathbf{Z}(1)) \simeq \mathrm{Br}(\kappa(\eta)),$$

(5-3) follows at once from the classical reciprocity law for $\kappa(\eta)$.

§6. Higher dimensional case

In this section we treat the general dimensional case. First we must recall the following (cf. [K-S, §1])

Definition (6-1). Let X be a Noetherian integral scheme of dimension d.
(1) A chain on X is a sequence $\delta = (p_0, \ldots, p_r)$ of points p_i of X such that

$$\overline{\{p_0\}} \subset \overline{\{p_1\}} \subset \ldots \subset \overline{\{p_r\}} .$$

(2) Let $P(X)$ denote the set of all chains $\delta = (p_0, \ldots, p_d)$ on X such that $p_i \in X_i$.
(3) For $0 \leqq s \leqq d$, let $Q_s(X)$ denote the set of all chains $\vartheta = (q_0, \ldots, q_{d-1})$ on X such that $q_i \in X_i$ for $0 \leqq i \leqq s-1$ and $q_i \in X_{i+1}$ for $s \leqq i \leqq d-1$.
(4) For $\vartheta \in Q_s(X)$, let $S(\vartheta)$ denote the set of all $x \in X_s$ such that

$$\vartheta(x) := (q_0, \ldots, q_{s-1}, x, q_s, \ldots, q_{d-1}) \in P(X)$$

Definition (6-2). Let X be as in (6-1) and let $\delta = (p_0, \ldots, P_r)$ be a chain on X.

(1) We define the ring $\mathcal{O}_\delta$ as follows. If $r = 0$, let $\mathcal{O}_\delta$ be the henselization of the local ring of X at p_0. Assume that $r \geqq 1$ and put $\delta' = (p_0, \ldots, p_{r-1})$. By induction we have defined the ring $R = \mathcal{O}_{\delta'}$. Let T be the set of all prime ideals of R which lie over p_r. For $\eta \in T$ let R_η be the henselization of R at η. We define

$$\mathcal{O}_\delta = \prod_{\eta \in T} R_\eta .$$

(2) For $\delta \in P(X)$, let P_δ denote the set of all minimal points of $\mathrm{Spec}(\mathcal{O}_\delta)$. Then we have

$$\mathcal{O}_\delta = \prod_{\lambda \in P_\delta} K_\lambda$$

where K_λ is the localization of $\mathcal{O}_\delta$ at λ and it is a field.
(3) For $\vartheta \in Q_s(X)$ with $0 \leqq s \leqq d - 1$, let P_ϑ denote the set of all minimal points of $\mathrm{Spec}(\mathcal{O}_\vartheta)$. We have

$$\mathcal{O}_\vartheta = \prod_{v \in P} K_v$$

where K_v is the localization of $\mathcal{O}_\vartheta$ at v and it is a field.
(4) If $\vartheta \in Q_d(X)$, $S(\vartheta)$ consists of the unique generic point Λ of X and if X is normal we have

$$\mathcal{O}_\vartheta = \prod_{\lambda \in P_\delta} \mathcal{O}_\lambda .$$

Here $\delta = \vartheta(\Lambda)$ and for $\lambda \in P_\delta$ $\mathcal{O}_\lambda$ is the localization of $\mathcal{O}_\vartheta$ at the unique minimal point of $\mathrm{Spec}(\mathcal{O}_\vartheta)$ which lies under $\lambda \in P_\delta$. If $\vartheta = (p_0, \ldots, p_{d-1})$, $\mathcal{O}_\lambda$ is a henselian discrete valuation ring which dominates the discrete valuation ring $\mathcal{O}_{X, p_{d-1}}$ and K_λ is the quotient field of $\mathcal{O}_\lambda$.

Now we have the following explicit presentation of idele class groups. Here, for a product of fields, its Milnor K-group denotes the product of the Milnor K-group of each field.

Theorem (6-3). ([K-S, §1]). *Let X be an integral normal Noetherian scheme of dimension d and let K be its function field. Let $I \subset \mathcal{O}_X$ be an ideal. For any integer $i \geqq 0$, there exists a canonical isomorphism*

$$C_I^i(X) \simeq \left(\bigoplus_{\delta \in P(X)} \bigoplus_{\lambda \in P_\delta} K_i^M(K_\lambda)/U^{I_\lambda} K_i^M(\mathcal{O}_\lambda) \right) \Big/ \Delta ,$$

where $I_\lambda = I\mathcal{O}_\lambda$ and Δ is the subgroup generated by all elements $(a_\lambda)_\lambda$ of the following type: There exists an integer $0 \leqq s \leqq d - 1$, $\vartheta \in Q_s(X)$ and an element $b_\vartheta \in K_i^M(\mathcal{O}_\vartheta)$ such that

$$a_\delta := (a_\lambda)_{\lambda \in P_\delta} \equiv \begin{cases} \iota_{\vartheta, \vartheta(x)}(b_\vartheta) & \text{if } \delta = \vartheta(x) \text{ for some } x \in S(\vartheta), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\iota_{\vartheta,\vartheta(x)}: K_i^M(\mathcal{O}_\vartheta) \to K_i^M(\mathcal{O}_{\vartheta(x)})$$

is the natural map.

Now we return to the case where X is an arithmetic scheme, namely an integral scheme which is proper over Spec(**Z**). We assume that X has no **R**-valued point. For the definition of our fundamental homomorphism

$$\Phi_K^i: H^{i+2}(K, \mathbf{Z}(i)) \to \varinjlim_{I \subset \mathcal{O}_X} \operatorname{Hom}(C_I^{d-i}(X), \mathbf{Q}/\mathbf{Z}),$$

we may assume that X is normal by (2-9). By [K-S, §3], for $\delta \in P(X)$ and $\lambda \in P_\delta$, K_λ is naturally endowed with a structure of a henselian local field of dimension d in the sense of (1-5). Hence (1-5) gives a canonical isomorphism

$$\theta_\lambda: H^{d+2}(K_\lambda, \mathbf{Z}(d)) \simeq \mathbf{Q}/\mathbf{Z}.$$

Now the rest of the argument is exactly the same as that of the case of dimension two except that we need the following reciprocity law in this higher dimensional context instead of (5-2) and (5-3).

Theorem(6-4). ([K-S, §3]). *Let $\vartheta \in Q_s(X)$ with $0 \leqq s \leqq d-1$. For any element*

$$c \in H^{d+2}(\mathcal{O}_\vartheta, \mathbf{Z}(d)) := \prod_{v \in P_\vartheta} H^{d+2}(K_v, \mathbf{Z}(d))$$

its image $c_{\vartheta(x)}$ in

$$H^{d+2}(\mathcal{O}_{\vartheta(x)}, \mathbf{Z}(d)) := \prod_{\lambda \in P_{\vartheta(x)}} H^{d+2}(K_\lambda, \mathbf{Z}(d))$$

is trivial for almost all $x \in S(\vartheta)$. Moreover we have

$$\sum_{x \in S(\vartheta)} \theta_{\vartheta(x)}(c_{\vartheta(x)}) = 0.$$

Here $\theta_{\vartheta(x)}$ is the composite map

$$H^{d+2}(\mathcal{O}_{\vartheta(x)}, \mathbf{Z}(d)) \xrightarrow{\alpha} \prod_{\lambda \in P_{\vartheta(x)}} \mathbf{Q}/\mathbf{Z} \xrightarrow{\sigma} \mathbf{Q}/\mathbf{Z},$$

where $\alpha = \prod_{\lambda \in P_{\vartheta(x)}} \theta_\lambda$ and σ is the addition map.

Definition (6-5). Let X and K be as before. Then we put

$$D^i(X) = \bigcap_{\delta \in P(X)} \operatorname{Ker}\left(H^{i+2}(K, \mathbf{Z}(i)) \to \prod_{\lambda \in P_\delta} H^{i+2}(K_\lambda, \mathbf{Z}(i)) \right).$$

By the definition of Φ_K^i and the injectivity of Φ_λ^i (cf. (1-7)(3)), we get the following

Theorem (6-6). $\operatorname{Ker}(\Phi_K^i) = D^i(X)$.

In other words $c \in H^{i+2}(K, \mathbf{Z}(i))$ lies in $\operatorname{Ker}(\Phi_K^i)$ if and only if c trivialized under all henselizations of K attached to $\delta \in P(X)$ in (6-2).

Finally we consider the following problem.

(*) Let U be a nonempty open subscheme of X. Then, can one tell if an element $\omega \in H^{i+2}(K, Z(i))$ belongs to the image of $H^{i+2}(U, \mathbf{Z}(i))$ (though it is not defined in general) in terms of $\Phi_K^i(\omega)$.

Concerning this, we have the following results.

Proposition (6-7). *Assume that U is regular. Then $\omega \in H^2(K, \mathbf{Z}(0)) \simeq H^1(K, \mathbf{Q}/\mathbf{Z})$ belongs to the subgroup $H^1(U, \mathbf{Q}/\mathbf{Z})$ if and only if there exists a coherent ideal $J \subset \mathcal{O}_X$ such that $J\mathcal{O}_U = \mathcal{O}_U$ and that*

$$\Phi_K^0(\omega) \in \operatorname{Hom}(C_J^d(X), \mathbf{Q}/\mathbf{Z}) .$$

Proposition (6-8). *Assume that U is regular and put*

$$\widetilde{\operatorname{Br}}(U) = \operatorname{Ker}\left(\operatorname{Br}(K) \to \bigoplus_{y \in U^1} \operatorname{Br}(K_y)/\operatorname{Br}(\mathcal{O}_y)\right) .$$

Here, for $y \in X^1$, $\mathcal{O}_y$ is the henselization of the local ring of X at y and K_y is its quotient field. Then $\omega \in H^3(K, \mathbf{Z}(1)) \simeq \operatorname{Br}(K)$ belongs to the subgroup $\widetilde{\operatorname{Br}}(U)$ if and only if there exists a coherent ideal $J \subset \mathcal{O}_X$ such that $J\mathcal{O}_U = \mathcal{O}_U$ and that

$$\Phi_K^1(\omega) \in \operatorname{Hom}(C_J^{d-1}(X), \mathbf{Q}/\mathbf{Z}) .$$

Remark (6-9). Clearly we have $\operatorname{Br}(U) \subsetneq \widetilde{\operatorname{Br}}(U)$. It is conjectured that it is an isomorphism and it is true if $\dim(X) \leqq 2$.

Note that by [SGA1, *X* (3.1)] we have

$$H^1(U, \mathbf{Q}/\mathbf{Z}) \simeq \operatorname{Ker}\left(H^1(K, \mathbf{Q}/\mathbf{Z}) \to \bigoplus_{y \in U^1} H^1(K_y, \mathbf{Q}/\mathbf{Z})/H^1(\mathcal{O}_y, \mathbf{Q}/\mathbf{Z})\right).$$

Thus (6-7) and (6-8) follow from the following local theory in codimension one: Fix $y \in X^1$. Let ν denote 0 or 1. Then there exists a projective system of abelian groups

$$\{C_{i,J}^{d-\nu}(y)\}_{i,J} ,$$

where i ranges over all integers $\geqq 1$ and J ranges over all ideals of $\mathcal{O}_X$ such that $J_y = \mathcal{O}_{X,y}$. We put

$$C^{d-\nu}(y) = \varprojlim_{i,J} C_{i,J}^{d-\nu}(y) .$$

We have the following facts.

Fact(*a*). There exists a canonical homomorphism

$$\varphi_y^\nu : C^{d-\nu}(y) \to \varprojlim_{I \subset \mathcal{O}_X} C_I^{d-\nu}(X) ,$$

where I ranges over all non-zero coherent ideal of $\mathcal{O}_X$. For a fixed $I \subset \mathcal{O}_X$, the composite map $C^{d-\nu}(y) \to C_I^{d-\nu}(X)$ factors through $C_{i,J}^{d-\nu}(y)$ for some i and J.

Fact(*b*). Let T be a finite subset of X^1. Let I and J be non-zero ideals of $\mathcal{O}_X$ such that $I \subset J$, $I_y = J_y$ for any $y \in X^1 - T$ and that $J_y = \mathcal{O}_{X,y}$ for any $y \in T$. Then φ_y^ν for $y \in T$ yield an exact sequence

$$\bigoplus_{y \in T} C^{d-\nu}(y) \to C_I^{d-\nu}(X) \to C_J^{d-\nu}(X) \to 0 .$$

Fact(c). There exists canonical homomorphism

$$\Phi_y^0 : H^1(K_y, \mathbf{Q}/\mathbf{Z})/H^1(\mathcal{O}_y, \mathbf{Q}/\mathbf{Z}) \to \varinjlim_{i,J} \operatorname{Hom}(C^d_{i,J}(y), \mathbf{Q}/\mathbf{Z}) ,$$

$$\Phi_y^1 : \operatorname{Br}(K_y)/\operatorname{Br}(\mathcal{O}_y) \to \varinjlim_{i,J} \operatorname{Hom}(C^{d-1}_{i,J}(y), \mathbf{Q}/\mathbf{Z}) .$$

These maps are isomorphisms and are compatible with Φ_K^0 and Φ_K^1 with respect to the maps φ_y^ν and the natural maps

$$H^1(K, \mathbf{Q}/\mathbf{Z}) \to H^1(K_y, \mathbf{Q}/\mathbf{Z})/H^1(\mathcal{O}_y, \mathbf{Q}/\mathbf{Z}) ,$$

$$\operatorname{Br}(K) \to \operatorname{Br}(K_y)/\operatorname{Br}(\mathcal{O}_y) .$$

In [K-S, §8] it is proved that the above theory exists for the case $v = 0$. The other case is obtained in the similar way. The two-dimensional case is given in [Sa-4].

§7. Results and conjectures

Let X and K be as the beginning of §3. In this section we state main results known concerning our fundamental map (3-1) and give some conjectures.

Theorem (7-1). ([K-S]) (*Higher dimensional class field theory*) *Assume that* $\dim(X) = d$. *Then*

$$\Phi_K^0 : H^1(K, \mathbf{Q}/\mathbf{Z}) \to \varinjlim_{I \subset \mathcal{O}_X} \operatorname{Hom}(C^d_I(X), \mathbf{Q}/\mathbf{Z})$$

is an isomorphism.

Corollary (7-2). *Let X be as (7-1) and let U be a nonempty regular open subscheme.* (1) *The map Φ_K^0 induces an isomorphism*

$$\Phi_U^0 : H^1(U, \mathbf{Q}/\mathbf{Z}) \simeq \varinjlim_{I \in \mathcal{I}_U} \operatorname{Hom}(C^d_I(X), \mathbf{Q}/\mathbf{Z}) .$$

Here $\mathcal{I}_U$ denotes the set of all non-zero coherent ideals $I \subset \mathcal{O}_X$ such that $I\mathcal{O}_U = \mathcal{O}_U$. (2) *Assume that X is proper and smooth over a finite field of characteristic p. Fix an integer $n > 0$ which is prime to p. Then, for any $I \in \mathcal{I}_U$ such that* $\operatorname{Supp}(\mathcal{O}_X/I) = X - U$, *$\Phi_U^0$ induces an isomorphism*

$$H^1(U, \mathbf{Z}/n\mathbf{Z}) \simeq \operatorname{Hom}(C^d_I(X), \mathbf{Z}/n\mathbf{Z}) .$$

Proof. (7-2)(1) follows from (7-1) and (6-7). (7-2)(2) follows from (7-2)(1) together with the fact that if $I \subset J \subset \mathcal{O}_X$ such that $\operatorname{Supp}(\mathcal{O}_X/I) = \operatorname{Supp}(\mathcal{O}_X/J)$, then

$$\operatorname{Ker}(C^d_I(X) \to C^d_J(X))$$

is killed by some power of p (cf. Fact(b) in §6 and [K-S, §8]).

Theorem (7-3). ([K-S, §7]). *(Higher dimensional version of Moore's reciprocity uniqueness theorem) Assume that* $\dim(X) = d$. *Then* Φ_K^{-1} *is an isomorphism. More strongly, for a sufficiently small* $I \subset \mathcal{O}_X$, *we have an isomorphism*

$$C_I^{d+1}(X) \simeq \mu(K) .$$

Theorem (7-4). ([K-5]) *Assume that X is regular of dimension two. The map* Φ_K^2 *is an isomorphism. In other words the following sequence is exact* (cf. §5)

$$O \to H^4(K, \mathbf{Z}(2)) \xrightarrow{\iota} \bigoplus_{\delta \in P(X)} \bigoplus_{\lambda \in P_\delta} \mathbf{Q}/\mathbf{Z} \xrightarrow{\sigma} \left(\bigoplus_{\eta \in X_1} \mathbf{Q}/\mathbf{Z} \right) \oplus \left(\bigoplus_{x \in X_0} \mathbf{Q}/\mathbf{Z} \right).$$

Here ι *is obtained from the composite maps*

$$H^4(K, \mathbf{Z}(2)) \to H^4(K_\lambda, \mathbf{Z}(2)) \xrightarrow{\theta_\lambda} \mathbf{Q}/\mathbf{Z}$$

and

$$\sigma(\{a_\lambda\}_{\lambda \in P_\delta, \delta \in P(X)}) = \{\{b_\eta\}_{\eta \in X_1}, \{b_x\}_{x \in X_0}\}$$

where

$$b_\eta = \sum_{\lambda \in P_\eta} a_\lambda \qquad \text{and} \qquad b_x = \sum_{\lambda \in P_x} a_\lambda .$$

Theorem (7-5). ([Sa-4]) *Assume that X is regular of dimension two. The kernel of the map* Φ_K^1 *is equal to the Brauer-Grothendieck group* $\mathrm{Br}(X)$ *of X. For any prime number l, we have a canonical isomorphism*

$$\mathrm{Coker}(\Phi_K^1)(l) \simeq \mathrm{Hom}(T_l \mathrm{Br}(X), \mathbf{Q}/\mathbf{Z}) .$$

In particular, $\mathrm{Coker}(\Phi_K^1)(l) = 0$ *if and only if* $\mathrm{Br}(X)(l)$ *is finite.*

Finally we give the following result. Let X be a proper smooth scheme over a finite field of characteristic p. Fix a prime number $l \neq p$. For an integer $i \geqq 0$, let

$$\rho_l^i : CH^i(X) \hat{\otimes} \mathbf{Z}_l \to H^{2i}(X, \mathbf{Z}_l(i))$$

be the cycle map, where $CH^i(X)$ denotes the Chow group of cycles of codimension i on X. Put

$$\mathrm{Trans}_l^i(X) = \mathrm{Coker}(\rho_l^i) \quad \textit{(the group of transcendental cycles)} .$$

Theorem (7-6). *Assume that* $\dim(X) = d + 1$. *Then there exists a canonical surjective homomorphism*

$$\mathrm{Trans}_l^d(X) \to \mathrm{Coker}(\Phi_K^1)(l)^\vee ,$$

where the right hand side denotes the dual of the l-part of the cokernel of

$$\Phi_K^1 : H^3(K, \mathbf{Z}(1)) \simeq \mathrm{Br}(K) \to \varinjlim_{I \subset \mathcal{O}_X} \mathrm{Hom}(C_I^d(X), \mathbf{Q}/\mathbf{Z}) .$$

In particular, if ρ_l^d *is surjective, then* $\mathrm{Coker}(\Phi_K^1)(l) = 0$.

The proof of (7-6) will be given later. In view of the above results we propose the following conjectures. Assume that $\dim(X) = d$ and let

$$\Phi_K^i : H^{i+2}(K, \mathbf{Z}(i)) \to \varinjlim_{I \subset \mathcal{O}_X} \mathrm{Hom}(C_I^{d-i}(X), \mathbf{Q}/\mathbf{Z})$$

be our fundamental homomorphism.

Conjecture (7-7). (cf. (7-4)) Assume that X is regular. Then Φ_K^d is an isomorphism. In other words the following sequence is exact.

$$0 \to H^{d+2}(K, \mathbf{Z}(d)) \xrightarrow{\iota} \bigoplus_{\delta \in P(X)} \bigoplus_{\lambda \in P_\delta} \mathbf{Q}/\mathbf{Z} \xrightarrow{\sigma} \bigoplus_{0 \leqq s \leqq d-1} \bigoplus_{\vartheta \in Q_s(X)} \mathbf{Q}/\mathbf{Z}\,,$$

where ι is obtained from the composite maps

$$H^{d+2}(K, \mathbf{Z}(d)) \to H^{d+2}(K_\lambda, \mathbf{Z}(d)) \xrightarrow{\theta_\lambda} \mathbf{Q}/\mathbf{Z}$$

and

$$\sigma(\{a_\lambda\}_{\lambda \in P_\delta, \delta \in P(X)}) = \{\{b_\vartheta\}_{\vartheta \in Q_s(X)}\}_{0 \leqq s \leqq d-1}$$

where

$$b_\vartheta = \sum_{x \in S(\vartheta)} \sum_{\lambda \in P_{\vartheta(x)}} a_\lambda\,.$$

The conjecture (7-7) is essentially equivalent to the conjecture (0.3) in Kato [K-5] which involves a certain complex of Bloch-Ogus type on X.

Conjecture (7-8). (cf. (7-5)) Assume that X is regular. Then
(1) $D^1(X) \simeq \mathrm{Br}(X)$.
(2) The map Φ_K^1 is surjective.

By (6-8), (7-8)(1) is equivalent to the fact that $\mathrm{Br}(X) \simeq \widetilde{\mathrm{Br}}(X)$.

Conjecture(7-9). (cf. (6-6)) $\mathrm{Ker}(\Phi_K^i) = D^i(X)$ is finite.

The conjecture (7-9) can be viewed as a generalization of a conjecture on the finiteness of the Brauer-Grothendieck group of an arithmetic surface by Artin and Tate.

The rest of this section is devoted to the proof of (7-6). First we need the following facts.

Proposition(7-10). *Let X be a connected proper smooth scheme of dimension $d+1$ over a finite field of characteristic p. Let n be an integer prime to p. Let $I \subset \mathcal{O}_X$ be a non-trivial ideal and put $Y = \mathrm{Supp}(\mathcal{O}_X/I)$ with reduced scheme structure. Then there exists canonical isomorphisms*

$$\varphi : H^d(Y_{\mathrm{Nis}}, K_d^M(\mathcal{O}_X/I))/n \simeq H^{2d}(Y_{et}, \mathbf{Z}/n\mathbf{Z}(d))\,,$$
$$\psi : H^d(Y_{\mathrm{Nis}}, K_{d+1}^M(\mathcal{O}_X/I))/n \simeq H^{2d+1}(Y_{et}, \mathbf{Z}/n\mathbf{Z}(d+1))\,.$$

Proposition (7-11). *Let X be a smooth scheme over a field. Then, for any integer $0 \leqq r \leqq \dim(X)$, there exists a canonical homomorphism*

$$\tau_r : H^r(X_{\mathrm{Nis}}, K_r^M(\mathcal{O}_X)) \longrightarrow CH^r(X)\,.$$

Moreover τ_r *is an isomorphism for* $r = \dim(X)$ *and it is surjective for* $r = \dim(X) - 1$.

First, assuming (7-10) and (7-11) we finish the proof of (7-6). Let X be as in (7-6). For each closed subscheme $Y \subset X$ with reduced scheme structure, let $I_Y \subset \mathcal{O}_X$ be the ideal of definition. Note that for any $I \subset \mathcal{O}_X$ such that $\mathrm{Supp}(\mathcal{O}_X/I) = Y$ and for any integer $n > 0$ prime to p, we have $C_I^d(X)/n \simeq C_{I_Y}^d(X)/n$ (cf. the proof of (7-2)(2)). Thus we have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Br}(K)(l) \to & & \varinjlim_{Y\subset X} H_Y^3(X, \mathbf{Q}_l/\mathbf{Z}_l(1)) \to & & H^3(X, \mathbf{Q}_l/\mathbf{Z}_l(1)) \\ \downarrow \Phi_K^d & & \downarrow f_1 & & \downarrow f_2 \\ 0 \to \varinjlim_{Y\subset X} C_{I_Y}^d(X)^\vee(l) \to & & \varinjlim_{Y\subset X} H^d(Y_{\mathrm{Nis}}, K_d^M(\mathcal{O}_X/I_Y))^\vee(l) \to & & H^d(X_{\mathrm{Nis}}, K_d^M(\mathcal{O}_X))^\vee(l)\,, \end{array}$$

where Y ranges over all closed subschemes of X with reduced scheme structure. The upper horizontal sequence comes from the localization theory on X and the lower horizontal sequence comes from the exact sequence of sheaves

$$0 \to K_d^M(\mathcal{O}_X, I_Y) \to K_d^M(\mathcal{O}_X) \to K_d^M(\mathcal{O}_X/I_Y) \to 0\,.$$

By the Poincare duality for etale cohomology we have isomorphisms (cf. [M-1])

$$g_1 : H_Y^3(X, \mathbf{Q}_l/\mathbf{Z}_l(1)) \simeq H^{2d}(Y, \mathbf{Z}_l(d))^\vee\,,$$
$$g_2 : H^3(X, \mathbf{Q}_l/\mathbf{Z}_l(1)) \simeq H^{2d}(X, \mathbf{Z}_l(d))^\vee\,.$$

Then we have put

$$f_1 = \varphi^\vee \circ g_1 \quad \text{and} \quad f_2 = \tau_d^\vee \circ (\rho_l^d)^\vee \circ g_2\,.$$

Here φ (resp. τ_d) is the map in (7-10) (resp. (7-11)) and ρ_l^d is the cycle map. The map f_1 is an isomorphism by (7-10) and we have

$$\mathrm{Ker}(f_2) = (\mathrm{Trans}_l^d(X))^\vee$$

since τ_d is surjective by (7-11). Now (7-6) follows by an easy diagram chasing.

Proof of (7-10). First, to define the map φ and ψ, let $\pi\colon Y_{et} \to Y_{\mathrm{Nis}}$ be the natural morphism of sites. Then, for any integer $i \geqq 0$, we have a natural map of sheaves

$$K_i^M(\mathcal{O}_X/I) \to R^i\pi_* \mathbf{Z}/n\mathbf{Z}(i)\,,$$

by taking the cup product of the natural map

$$(\mathcal{O}_X/I)^* \to R^1\pi_* \mathbf{Z}/n\mathbf{Z}(1)$$

which is obtained from the Kummer theory on Y. Then it induces the desired maps

$$H^d(Y_{\mathrm{Nis}}, K_d^M(\mathcal{O}_X/I)) \to H^d(Y_{\mathrm{Nis}}, R^d\pi_*\mathbf{Z}/n\mathbf{Z}(d)) \to H^{2d}(Y_{et}, \mathbf{Z}/n\mathbf{Z}(d))\,,$$
$$H^d(Y_{\mathrm{Nis}}, K_{d+1}^M(\mathcal{O}_X/I)) \to H^d(Y_{\mathrm{Nis}}, R^{d+1}\pi_*\mathbf{Z}/n\mathbf{Z}(d+1))$$
$$\to H^{2d+1}(Y_{et}, \mathbf{Z}/n\mathbf{Z}(d+1))\,,$$

where the last maps are the edge homomorphisms in view of (2-2-5).

To prove that φ and ψ are isomorphisms, we may assume that Y is reduced and may replace I by the ideal of definition of Y in which case $\mathcal{O}_X/I = \mathcal{O}_Y$. Indeed the etale cohomology groups are not affected by such replacement. Let $W = \mathrm{Supp}(Y)$ with reduced subscheme structure. Let $J \subset \mathcal{O}_X$ be the ideal of the definition of W in X. By definition we have an exact sequence

$$H^d(Y_{\mathrm{Nis}}, \mathscr{K}_i) \to H^d(Y_{\mathrm{Nis}}, K_i^M(\mathcal{O}_X/I)) \to H^d(W_{\mathrm{Nis}}, K_i^M(\mathcal{O}_X/J)) \to 0 ,$$

where $\mathscr{K}_i = \mathrm{Ker}(K_i^M(\mathcal{O}_X/I) \to K_i^M(\mathcal{O}_X/J))$. It is easy to see that $H^d(Y_{\mathrm{Nis}}, \mathscr{K}_i)$ is killed by some power of p for $i \geqq 1$ and this proves our claim. Let $J \subset \mathcal{O}_Y$ be a sufficiently small ideal such that $Z = \mathrm{Supp}(\mathcal{O}_Y/J)$ contains the set of all singular points on Y. Let $U = Y \backslash Z$ and let $\{U_\lambda | \lambda \in \Lambda\}$ be the set of all irreducible components of dimension d of U. Let Y_λ be the closure of U_λ in Y for each $\lambda \in \Lambda$ and put $J_\lambda = J\mathcal{O}_{Y_\lambda}$. Then we have the following commutative diagrams

$$\begin{array}{ccccccc} H^{d-1}(Z_{\mathrm{Nis}}, K_d^M(\mathcal{O}_Y/J))/n & \to & H^d(Y_{\mathrm{Nis}}, K_d^M(\mathcal{O}_Y, J))/n & \xrightarrow{\alpha} & H^d(Y_{\mathrm{Nis}}, K_d^M(\mathcal{O}_Y))/n & \to & 0 \\ \downarrow \xi & & \downarrow \nu & & \downarrow \varphi & & \\ H^{2d-1}(Z, \mathbf{Z}/n\mathbf{Z}(d)) & \to & H_c^{2d}(U, \mathbf{Z}/n\mathbf{Z}(d)) & \xrightarrow{\beta} & H^{2d}(Y, \mathbf{Z}/n\mathbf{Z}(d)) & \to & 0 , \end{array} \tag{7-10-1}$$

$$\begin{array}{ccccc} H^d(Y_{\mathrm{Nis}}, K_{d+1}^M(\mathcal{O}_Y, J))/n & \xrightarrow{\gamma} & H^d(Y_{\mathrm{Nis}}, K_{d+1}^M(\mathcal{O}_Y))/n & \to & 0 \\ \downarrow \mu & & \downarrow \psi & & \\ H_c^{2d+1}(U, \mathbf{Z}/n\mathbf{Z}(d+1)) & \simeq & H^{2d+1}(Y, \mathbf{Z}/n\mathbf{Z}(d+1)) . & & \end{array} \tag{7-10-2}$$

Here ξ is defined for Z by the same way as ψ. The surjectivity of α and γ follows from (2-2-5). The surjectivity of β and the isomorphism in (7-10-2) follows from [A, Cor. 4.3.]. Moreover we have isomorphisms

$$H^d(Y_{\mathrm{Nis}}, K_d^M(\mathcal{O}_Y, J)) \simeq \bigoplus_{\lambda \in \Lambda} H^d((Y_\lambda)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{Y_\lambda}, J_\lambda)) ,$$

$$H^d(Y_{\mathrm{Nis}}, K_{d+1}^M(\mathcal{O}_Y, J)) \simeq \bigoplus_{\lambda \in \Lambda} H^d((Y_\lambda)_{\mathrm{Nis}}, K_{d+1}^M(\mathcal{O}_{Y_\lambda}, J_\lambda)) .$$

Also we have isomorphisms

$$H_c^{2d}(U, \mathbf{Z}/n\mathbf{Z}(d)) \simeq \bigoplus_{\lambda \in \Lambda} H_c^{2d}(U_\lambda, \mathbf{Z}/n\mathbf{Z}(d)) .$$

$$H_c^{2d+1}(U, \mathbf{Z}/n\mathbf{Z}(d+1)) \simeq \bigoplus_{\lambda \in \Lambda} H_c^{2d+1}(U_\lambda, \mathbf{Z}/n\mathbf{Z}(d+1)) .$$

Here we used again (2-2-5) and [A, Cor. 4.3.]. Moreover, by the Poincare duality for etale cohomology we have isomorphisms (cf. [M-1])

$$H_c^{2d}(U_\lambda, \mathbf{Z}/n\mathbf{Z}(d)) \simeq H^1(U_\lambda, \mathbf{Z}/n\mathbf{Z})^\vee ,$$

$$H_c^{2d+1}(U_\lambda, \mathbf{Z}/n\mathbf{Z}(d+1)) \simeq \mu_n(U_\lambda) .$$

Thus the maps ν and μ come from (7-2)(2) and (7-3) respectively and they are isomorphisms. Hence (7-10-2) shows that ψ is an isomorphism. Applying this to the map ξ for Z, (7-10-1) implies that φ is an isomorphism. This completes the proof of (7-10).

Proof of (7-11). The proof is due to K. Kato. By [K-4], for any Noetherian scheme X and for any integer $n \geqq 0$, we have a complex of sheaves on X_{Nis}

$$K_n^M(\mathcal{O}_X) \to \bigoplus_{x \in X^0} K_n^M(\kappa(x)) \to \bigoplus_{x \in X^1} K_{n-1}^M(\kappa(x)) \to \dots \to \bigoplus_{x \in X^i} K_{n-i}^M(\kappa(x)) \to \dots ,$$

where if $j < 0$ $K_j^M(\kappa(x)) = 0$ by convention. The boundary maps are given essentially by the tame symbols for discrete valuation fields. For each $x \in X^i$ this gives a canonical homomorphism

$$H_x^i(X_{\mathrm{Nis}}, K_n^M(\mathcal{O}_X)) \to K_{n-i}^M(\kappa(x)) . \tag{7-11-1}$$

Theorem (7-11-2). *Assume that X is smooth over a field or a Dedekind domain.*

(1) *If $n \geqq i$, (7-11-1) is an isomorphism.*

(2) *If $n < i$, $H_x^i(X_{\mathrm{Nis}}, K_n^M(\mathcal{O}_X)) = 0$.*

We prove (7-11-2) by the induction on the codimension i of x in X (X and x may vary). Let $\mathcal{O}_x$ be the henselization of the local ring of X at x. By the assumption it is a regular local ring of dimension i. If $i = 0$ (7-11-2) is trivial. If $i = 1$ it follows from (2-2-3) and the well-known exact sequence

$$K_n^M(\mathcal{O}_x) \to K_n^M(K_x) \to K_{n-1}^M(\kappa(x)) \to 0 ,$$

where K_x is the quotient field of $\mathcal{O}_x$. Assume that $i > 1$. By (2-2-2), for any sheaf $\mathcal{F}$ on X_{Nis} we have

$$H_x^i(X_{\mathrm{Nis}}, \mathcal{F}) \simeq H^{i-1}(Y_{\mathrm{Nis}}, \mathcal{F}) ,$$

where $Y = \mathrm{Spec}(\mathcal{O}_x) - x$. Nothing that $\dim(Y) = i - 1$, we apply (2-2-1), (2-2-4) and (2-2-5) for the computation of $H^{i-1}(Y_{\mathrm{Nis}}, \mathcal{F})$. We get an exact sequence

$$\bigoplus_{y \in Y^{i-2}} H_y^{i-2}(Y_{\mathrm{Nis}}, \mathcal{F}) \to \bigoplus_{y \in Y^{i-1}} H_y^{i-1}(Y_{\mathrm{Nis}}, \mathcal{F}) \to H^{i-1}(Y_{\mathrm{Nis}}, \mathcal{F}) \to 0 .$$

By the induction hypothesis we have canonical isomorphisms

$$H_y^{i-2}(Y_{\mathrm{Nis}}, K_n^M(\mathcal{O}_X)) \simeq K_{n-i+2}^M(\kappa(y)) \qquad \text{for } y \in Y^{i-2} ,$$
$$H_y^{i-1}(Y_{\mathrm{Nis}}, K_n^M(\mathcal{O}_X)) \simeq K_{n-i+1}^M(\kappa(y)) \qquad \text{for } y \in Y^{i-1} .$$

Thus (7-11-2) follows from the following

Theorem (7-11-3). (cf. [K-4]) *Let A be a local domain which is essentially etale over a ring smooth over a field or a Dedekind domain. Let k be the residue field of A and put $Y = \mathrm{Spec}(A) - x$ where $x = \mathrm{Spec}(k)$ is the closed point. Assume that $\dim(A) \geqq 2$.*

(1) *For any integer $m \geqq 0$, we have an exact sequence*

$$\bigoplus_{z \in Y_1} K_{m+2}^M(\kappa(y)) \to \bigoplus_{y \in Y_0} K_{m+1}^M(\kappa(y)) \to K_m^M(k) \to 0 .$$

(2) *The boundary map*

$$\bigoplus_{y \in Y_1} \kappa(y)^* \to \bigoplus_{y \in Y_0} \mathbf{Z}$$

is surjective.

Now we return to the proof of (7-11). Consider the spectral sequence (cf. (2-2-1))

$$E_1^{p,q} = \bigoplus_{x \in X^p} H_x^{p+q}(X_{\mathrm{Nis}}, K_r^M(\mathcal{O}_X)) \Rightarrow H^*(X_{\mathrm{Nis}}, K_r^M(\mathcal{O}_X)) .$$

By (2-2-4) and (7-11-2), we have

$$E_1^{p,q} = 0 \quad \text{for } q > 0 \quad \text{and} \quad E_1^{p,0} = 0 \quad \text{for } p > r$$

and we have an isomorphism

$$E_1^{p,0} \simeq \bigoplus_{x \in X^p} K_{r-p}^M(\kappa(x)) \qquad \text{for } 0 \leqq p \leqq r .$$

From this we get an edge homomorphism

$$H^r(X_{\mathrm{Nis}}, K_r^M(\mathcal{O}_X)) \to \mathrm{Coker}\left(\bigoplus_{x \in X^{r-1}} \kappa(x)^* \to \bigoplus_{x \in X^r} \mathbf{Z} \right) \simeq CH^r(X)$$

which is an isomorphism if $r = \dim(X)$ and surjective if $r = \dim(X) - 1$. This completes the proof of (7-11).

§8. Relation with Hasse principle

Let k be a classical global field, namely it is either a number field or a function field in one variable over a finite field. Put $S = \mathrm{Spec}(\mathcal{O}_k)$ in the former case and let S be the proper smooth model in the latter case. We assume that k has no real place. For $v \in S_0$ let k_v be the henselization of k at v. Let X be a proper smooth geometrically connected scheme over k. For each $v \in S_0$, we put $X_v = X \times_k k_v$.

The classical Hasse principle says

(H) Assume that X has a k_v-rational point for all $v \in S_0$. Then X has a k-rational point.

It is known that (H) is not true in general. On the other hand Manin ([Ma]) introduced the following obstruction for (H). For a collection $\mathbf{x} = (x_v)_{v \in S_0}$ of k_v-rational points of X and for $\omega \in \mathrm{Br}(X)$, put

$$\omega(\mathbf{x}) = \sum_{v \in S_0} \theta_v(\omega \otimes_X \kappa(x_v)) ,$$

where $\theta_v : \mathrm{Br}(k_v) \simeq \mathbf{Q}/\mathbf{Z}$ is the classical isomorphism (cf. §4 and [Se-2, XII]). The well-definedness, namely the fact that the sum is a finite sum, follows from (8-2)(1) below. If $\mathbf{x}$ comes from a single k-rational point x of X, we must have $\omega(\mathbf{x}) = 0$ by the classical reciprocity law for $\kappa(x)$. In this sense $\omega(\mathbf{x})$ can be viewed as an obstruction for (H). Now the question is whether it is the only obstruction or not, namely is the following true?

(M) Assume that $\omega(\mathbf{x}) = 0$ for any $\omega \in \mathrm{Br}(X)$. Then there exists a k-rational point of X.

(M) is verified in certain special cases (cf. [San] and [C-S-S]). In this section we consider the following modified version of (H).

(H^*) Assume that there exists $c_v \in Z_0(X_v)$ of degree 1 for each $v \in S_0$. Then there exists $c \in Z_0(X)$ of degree 1.

Here, for a smooth geometrically connected scheme Y over a field k, $Z_0(Y)$ denotes the group of all 0-cycles $c = \sum_{y \in Y_0} n_y(y)$ on Y, where $n_y = 0$ for almost all $y \in Y_0$. Let $CH_0(Y)$ be the Chow group of 0-cycles on Y and we put

$$\deg : CH_0(Y) \to \mathbf{Z} : c = \sum_{y \in Y_0} n_y(y) \to \sum_{y \in Y_0} n_y[\kappa(y):k]$$

the degree map. Certain cases were found where (H^*) is true and fails to be true (cf. [Sal], [San] and [C-S-S]). Following Manin we introduce the following obstructions for (H^*) as (8-2) below. For each $v \in S_0$ let

$$\langle\, ,\rangle_v : \mathrm{Br}(X_v) \times Z_0(X_v) \to \mathbf{Q}/\mathbf{Z}$$

be the pairing which is defined for $c = \sum_{x \in (X_v)_0} n_x(x) \in Z_0(X_v)$ and $\omega \in \mathrm{Br}(X_v)$, by

$$\langle \omega, c\rangle_v = \sum_{x \in (X_v)_0} n_x \,.\, \theta_v(N_{\kappa(x)/k_v}(\omega \otimes_X \kappa(x))) ,$$

where $N_{\kappa(x)/k_v} : \mathrm{Br}(\kappa(x)) \to \mathrm{Br}(k_v)$ is the norm map.

Lemma (8-1). *The above pairing induces*

$$\langle\, ,\rangle_v : \mathrm{Br}(X_v) \times CH_0(X_v) \to \mathbf{Q}/\mathbf{Z} .$$

Proof. We have to prove that if $c \in Z_0(X_v)$ is rationally equivalent to 0, then $\langle \omega, c\rangle_v = 0$ for any $\omega \in \mathrm{Br}(X_v)$. We may assume that there exists a proper smooth curve C over k_v, a non-trivial map $f: C \to X_v$ and $a \in Z_0(C)$ which is rationally equivalent to 0 and such that $f_*(a) = c$. Then $\langle \omega, c\rangle_v = \langle f^*(\omega), a\rangle_C$, where $\langle , \rangle_C$ is the pairing defined for C by the same way as before. This reduces the proof to the case where $\dim(X_v) = 1$, in which case (8-1) is proved in [L-3] and [Sa-2].

Lemma (8-2). (1) *We have a well-defined pairing*

$$\mathrm{Br}(X) \times \prod_{v \in S_0} CH_0(X_v) \to \mathbf{Q}/\mathbf{Z}; (\omega, \mathbf{c} = (c_v)_{v \in S_0}) \to \omega(\mathbf{c}) := \sum_{v \in S_0} \langle \omega_v, c_v\rangle_v ,$$

where for $\omega \in \mathrm{Br}(X)$, ω_v *is the image of* ω *in* $\mathrm{Br}(X_v)$.

(2) *If* $\mathbf{c}$ *lies in the diagonal image of* $CH_0(X)$, *then* $\omega(\mathbf{c}) = 0$.

(3) *Let* $\iota : \mathrm{Br}(k) \to \mathrm{Br}(X)$ *be the natural map. Then, for* $\omega \in \mathrm{Br}(k)$ *and for* $\mathbf{c} = (c_v)_{v \in S_0}$,

$$\iota(\omega)(\mathbf{c}) = \sum_{v \in S_0} \deg(c_v)\theta_v(\omega_v) ,$$

where ω_v *is the image of* ω *in* $\mathrm{Br}(k_v)$. *In particular, if* $\deg(c_v) = 1$ *for all* $v \in S_0$, $\iota(\omega)(\mathbf{c}) = 0$.

Proof. (8-2)(2) and (3) follow at once from the classical reciprocity law (4-5). The proof of (8-2)(1) will be given later.

Now we consider the following statement. Let l be a prime number.

(M_l^*) Suppose that there exists $\mathbf{c} = (c_v)_{v \in S_0}$ such that $\deg(c_v) = 1$ for all $v \in S_0$ and such that $\omega(\mathbf{c}) = 0$ for any $\omega \in \mathrm{Br}(X)$. Then $d := gcd\{[\kappa(x):k] | x \in X_0\}$ is prime to l.

Remark(8-3). (1) (M_l^*) for every prime number l implies that $\omega(\mathbf{c})$ for $\omega \in \mathrm{Br}(X)$ is the only obstruction for (H^*).

(2) Note that if the natural map $\iota: \mathrm{Br}(k) \to \mathrm{Br}(X)$ is surjective, the second condition for $\mathbf{c}$ in (M_l^*) is automatically satisfied by (8-2)(3). Thus, in this case (M_l^*) for every prime number l implies (H^*).

Let $d = \dim(X)$ and let K be the function field of X. Let $f: \mathscr{X} \to S$ be a normal model of X/k, namely $\mathscr{X}$ is a $(d+1)$-dimensional connected normal scheme and f is a proper flat morphism such that $\mathscr{X} \times_S \mathrm{Spec}(k) \simeq X$.

Theorem (8-4). *Let l be a prime number and let*

$$\Phi_K^1: \mathrm{Br}(K) \to \varinjlim_{I \subset \mathcal{O}_X} \mathrm{Hom}(C_I^d(\mathscr{X}), \mathbf{Q}/\mathbf{Z})$$

be the homomorphism (3-1). *Assume that there exists $w \in S_0$ where f is smooth and such that the image of Φ_K^1 contains all homomorphisms*

$$\chi: C_J^d(\mathscr{X}) \to \mathbf{Z}/l\mathbf{Z}\,,$$

where $J \subset \mathcal{O}_{\mathscr{X}}$ is the ideal of definition of $f^{-1}(w)$ in $\mathscr{X}$. Assume further (P_l) (cf. (0-2)). *Then (M_l^*) holds true.*

Corollary (8-5). *Suppose that* $\dim(X) = 1$. *If* $\mathrm{Br}(\mathscr{X})(l)$ *is finite, then (M_l^*) is true.*

Corollary (8-6). *Assume that S is a proper smooth curve over a finite field $\mathbf{F}_q$ and that $\mathscr{X}$ is a smooth over $\mathbf{F}_q$ of dimension $d+1$. Then, if the cycle map*

$$\rho_l^d: CH^d(\mathscr{X}) \hat{\otimes} \mathbf{Z}_l \to H^{2d}(\mathscr{X}, \mathbf{Z}_l(d))$$

is surjective for a prime number $l \neq \mathrm{ch}(\mathbf{F}_q)$ and if (P_l) is true, then (M_l^*) is true.

(8-5) (resp. (8-6)) follows from (8-4) together with (7-5) (resp. (7-6)).

For $v \in S_0$ let S_v be the henselization of S at v. Put $\mathscr{X}_v = \mathscr{X} \times_S S_v$ and $Y_v = \mathscr{X} \times_S \mathrm{Spec}(\kappa(v))$.

First we prove (8-2)(1). We must show that the sum is a finite sum. Note that for $\omega \in \mathrm{Br}(X)$ its image ω_v in $\mathrm{Br}(X_v)$ lies in the subgroup $\mathrm{Br}(\mathscr{X}_v)$ for almost all $v \in S_0$. Thus it suffices to show $\langle \omega_v, c_v \rangle_v = 0$ if $\omega_v \in \mathrm{Br}(\mathscr{X}_v)$. In fact it follows from the fact that $\omega_v \otimes_{X_v} \kappa(x) = 0$ for $x \in (X_v)_0$. For by the assumption, $\omega_v \otimes_{X_v} \kappa(x)$ lies in the subgroup $\mathrm{Br}(\mathcal{O}_{\kappa(x)})$ which is trivial, where $\mathcal{O}_{\kappa(x)}$ is the ring of integers of $\kappa(x)$.

Next we prove (8-4). Let $I_v \subset \mathcal{O}_{\mathscr{X}_v}$ be an ideal such that $\mathrm{Supp}(\mathcal{O}_{\mathscr{X}_v}/I_v) \subset Y_v$. By the localization theory on $\mathscr{X}_v$, we have an exact sequence

$$H^d((X_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{X_v})) \xrightarrow{\alpha} H_{Y_v}^{d+1}((\mathscr{X}_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{\mathscr{X}_v}, I_v))$$

$$\to H^{d+1}((\mathscr{X}_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{\mathscr{X}_v}, I_v))\,.$$

Lemma (8-7). (1) $H^{d+1}((\mathscr{X}_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{\mathscr{X}_v}, I_v)) = 0$. *In particular we get a canonical surjection*

$$\pi: CH_0(X_v) \xrightarrow{\tau^{-1}} H^d((X_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{\mathscr{X}_v})) \xrightarrow{\alpha} H^{d+1}_{Y_v}((\mathscr{X}_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{\mathscr{X}_v}, I_v)) ,$$

where τ is the isomorphism in (7-11).
(2) *Assume that $\mathscr{X}_v$ is smooth over S_v and let $I_v \subset \mathcal{O}_{\mathscr{X}_v}$ be the ideal of definition of Y_v. Then the map*

$$\iota: H^d((Y_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{Y_v})) = H^d_{Y_v}((\mathscr{X}_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{Y_v})) \to H^{d+1}_{Y_v}((\mathscr{X}_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{\mathscr{X}_v}, I_v))$$

is an isomorphism, where ι comes from the exact sequence of sheaves

$$0 \to K_d^M(\mathcal{O}_{\mathscr{X}_v}, I_v) \to K_d^M(\mathcal{O}_{\mathscr{X}_v}) \to K_d^M(\mathcal{O}_{Y_v}) \to 0 .$$

Moreover the composite map

$$CH_0(X_v) \xrightarrow{\pi} H^{d+1}_{Y_v}((\mathscr{X}_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{\mathscr{X}_v}, I_v)) \xrightarrow{\iota^{-1}} H^d((Y_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{Y_v})) \xrightarrow{\tau} CH_0(Y_v)$$

is given by the specialization map defined below.

Definition (8-8). (cf. [Sa-3, (3-9)]) Assume that $\mathscr{X}_v$ is smooth over S_v. We define

$$\mathrm{sp}: CH_0(X_v) \to CH_0(Y_0)$$

by the formula

$$\mathrm{sp}\left(\sum_{x \in (X_v)_0} n_x(x)\right) = \sum_{x \in (X_v)_0} n_x m_x(y_x) ,$$

where y_x is the unique closed point of Y_v which lies on the closure C_x of x in $\mathscr{X}_v$ and m_x is the intersection multiplicity of C_x and Y_v. We have the commutative diagram

$$\begin{array}{ccc} CH_0(X_v) & \xrightarrow{sp} & CH_0(Y_v) \\ \downarrow \deg & & \downarrow \deg \\ \mathbf{Z} & = & \mathbf{Z} \end{array}$$

We will give the proof of (8-7) later and return to the proof of (8-4).
Put

$$U^{I_v}CH_0(X_v) = \mathrm{Ker}(CH_0(X_v) \xrightarrow{\pi} H^{d+1}_{Y_v}((\mathscr{X}_v)_{\mathrm{Nis}}, K_d^M(\mathcal{O}_{\mathscr{X}_v}, I_v))) .$$

It can be viewed as a *congruence subgroup of $CH_0(X_v)$ with respect to the modulus I_v*. Let $\mathscr{I}_X$ be the set of all coherent ideal of $\mathcal{O}_{\mathscr{X}}$ such that $I\mathcal{O}_X = \mathcal{O}_X$. For $I \in \mathscr{I}_X$, put

$$CH_I(\mathscr{X}) = \mathrm{Coker}\left(CH_0(X) \to \bigoplus_{v \in S_0} CH_0(X_v)/U^{I_v}CH_0(X_v)\right) ,$$

where $I_v = I\mathcal{O}_{\mathscr{X}_v}$. Note that for a fixed $I \in \mathscr{I}_X$, $CH_0(X_v) = U^{I_v}CH_0(X_v)$ for almost all v.

Theorem (8-9). *The pairing* (8-2)(1) *induces a canonical homomorphism*

$$\Phi_X : \mathrm{Br}(X) \to \varinjlim_{I \in \mathscr{I}_X} \mathrm{Hom}(CH_I(\mathscr{X}), \mathbf{Q}/\mathbf{Z}) .$$

Proof. By its construction and (6-8), the homomorphism Φ_K^1 induces a canonical homomorphism

$$\mathrm{Br}(X) \to \varinjlim_{I \in \mathscr{I}_X} \mathrm{Hom}(C_I^d(\mathscr{X}), \mathbf{Q}/\mathbf{Z}) .$$

Thus (8-9) follows from the following

Lemma (8-10). *For $I \in \mathscr{I}_X$, there exists a canonical isomorphism*

$$C_I^d(\mathscr{X}) \simeq CH_I(\mathscr{X}) .$$

Proof. Let $I \in \mathscr{I}_X$. By the localization theory on $\mathscr{X}$, we have an exact sequence

$$H^d(X_{\mathrm{Nis}}, K_d^M(\mathscr{O}_X)) \to \bigoplus_{v \in S_0} H_{Y_v}^{d+1}((\mathscr{X}_v)_{\mathrm{Nis}}, K_d^M(\mathscr{O}_{\mathscr{X}_v}, I_v))$$
$$\to C_I^d(\mathscr{X}) \to H^{d+1}(X_{\mathrm{Nis}}, K_d^M(\mathscr{O}_X)) ,$$

where we put $I_v = I\mathscr{O}_{\mathscr{X}_v}$. By (2-2-5) $H^{d+1}(X_{\mathrm{Nis}}, K_d^M(\mathscr{O}_X)) = 0$, and by (7-11)

$$H^d(X_{\mathrm{Nis}}, K_d^M(\mathscr{O}_X)) \simeq CH_0(X) .$$

This finishes the proof of (8-10).

Now we can finish the proof of (8-4). Let $\mathbf{c} = (c_v)_{v \in S_0}$ be as in (M_I^*). Let $w \in S_0$ and $J \subset \mathscr{O}_{\mathscr{X}}$ be as the assumption of (8-4). In view of (8-10), the assumptions of (8-4) and (M_I^*) together with (6-8) imply that the image of $\mathbf{c}$ in $CH_J(X)/l$ is trivial. Thus we get an element $c \in CH_0(X)$ whose image in $\oplus_{v \in S_0} (CH_0(X_v)/U^{J_v}CH_0(X_v))/l$ coincides with the image of $\mathbf{c}$. By (8-7)(2) this implies that the image of c in $CH_0(Y_w)/l$ coincides with the image $\overline{c_w}$ of c_w under the specialization map $CH_0(X_w) \to CH_0(Y_w)$. Moreover, by the commutative diagram in (8-8), we have

$$\deg(c) \equiv \deg(\overline{c_w}) \equiv \deg(c_w) \equiv 1 \text{ modulo } l .$$

This completes the proof of (8-4).

Proof of (8-7). First (8-7)(1) follows from (6-3) and the following fact which is easy to see.

Lemma (8-11). *Let $\delta = (p_0, \ldots, p_{d+1}) \in P(\mathscr{X}_v)$ and assume that $p_{s-1} \in Y_v$ and $p_s \notin Y_v$. Define $\vartheta = (q_0, \ldots, q_d) \in Q_s(\mathscr{X}_v)$ as $q_i = p_i$ for $0 \leqq i \leqq s-1$ and $q_i = p_{i+1}$ for $s \leqq i \leqq d$. Then the set $S(\vartheta)$ is finite. For any non-zero ideal $I \in \mathscr{I}_X$ and any integer $n > 0$, the natural map*

$$K_n^M(K_\vartheta) \to \prod_{x \in S(\vartheta)} \prod_{\lambda \in P_{\vartheta(x)}} K_n^M(K_\lambda)/U^{I_\lambda}K_n^M(\mathscr{O}_\lambda)$$

is surjective.

Secondly (8-7)(2) follows from the following commutative diagram

$$\begin{array}{ccccc}
H^d_{Y_v}((\mathscr{X}_v)_{\mathrm{Nis}}, K^M_d(\mathcal{O}_{\mathscr{X}_v})) & \to & H^d_{Y_v}((\mathscr{X}_v)_{\mathrm{Nis}}, K^M_d(\mathcal{O}_{Y_v})) & \xrightarrow{\iota} & H^{d+1}_{Y_v}((\mathscr{X}_v)_{\mathrm{Nis}}, K^M_d(\mathcal{O}_{\mathscr{X}_v}, I_v)) \\
\downarrow & & \| & & \\
H^d((\mathscr{X}_v)_{\mathrm{Nis}}, K^M_d(\mathcal{O}_{\mathscr{X}_v})) & & H^d((Y_v)_{\mathrm{Nis}}, K^M_d(\mathcal{O}_{Y_v})) & & \uparrow \pi \\
\downarrow & & \downarrow \tau & & \\
H^d((X_v)_{\mathrm{Nis}}, K^M_d(\mathcal{O}_{X_v})) \xrightarrow{\tau} CH_0(X_v) & \xrightarrow{sp} & CH_0(Y_v) & \xleftarrow{sp} & CH_0(X_v)\,,
\end{array}$$

where the left vertical sequence comes from the localization theory on $\mathscr{X}_v$ and the upper horizontal sequence comes from the exact sequence of sheaves in (8-7)(2).

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