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The algebraic characterization of the exteriors of certain 2-knots

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Summary. We study 2-knots with virtually solvable group by applying recent work of Freedman to the 4-manifolds obtained by surgery on such knots. In particular we show that "Gluck reconstruction" is the only ambiguity in recovering the Cappell-Shaneson knots (as TOP locally flat knots) from their groups alone.

In [18] Gluck showed that if K is a 2-knot then there is at most one other knot with homeomorphic exterior. The optimistic expectation that the exterior might determine the knot was confounded by Cappell and Shaneson [6] and Gordon [19]. Here we shall show that nevertheless the 2-knots of Cappell and Shaneson are determined as TOP locally flat knots (up to Gluck reconstruction and change of orientation) by information which is apparently much weaker, namely by their groups alone. Similarly any 2-knot whose group is torsion free and virtually poly-$\mathbb{Z}$ is determined thus by its group together with the conjugacy class of a normal generator (information which is redundant when the group is metabelian). This would also be so for Gordon's examples if certain standard conjectures hold for their groups.

Our argument begins with the observation that the closed 4-manifold $M(K)$ obtained by surgery on such a knot is aspherical (unless the group is $\mathbb{Z}$, in which case the knot must be trivial). The result then follows easily from the work of Freedman on 4-dimensional surgery and s-cobordism theorems over virtually solvable fundamental groups [16] and that of Farrell and Hsiang on the topological rigidity of almost flat manifolds [13].

In [22] we showed that if a 2-knot group π is virtually solvable and has a nontrivial torsion free abelian normal subgroup then it is either torsion free and virtually poly-$\mathbb{Z}$ or has finite commutator subgroup π' or is the group Φ with presentation $\langle a, t | ta = a^2 t \rangle$. Freedman's work applies to the latter cases also. If π' is finite then the homotopy type of $M(K)$ is determined by π and a k-invariant in $\mathbb{Z}/|\pi'|\mathbb{Z}$; the difficulties in classifying such knots lie in determining Whitehead torsions, the Wall group $L^s_5(\pi)$ and its action on the structure set of $M(K)$. When π' is odd cyclic some computations are feasible, and we

can show that there are infinitely many 2-knots K such that $M(K)$ is simple homotopy equivalent to $M(T)$, where T is the 2-twist spun trefoil. If the Poincaré conjecture is true, then among these knots only T has a minimal Seifert manifold, and the manifolds $M(K)$ for the other knots are counter examples to the 4-dimensional analogue of the Farrell fibration theorem.

In the case of the group Φ the Whitehead group is 0 and the Wall group acts trivially, so the only difficulty is in constructing a map to a standard model. We shall show that the group alone now determines the algebraic 2-type of $M(K)$, and that the homotopy type of $M(K)$ determines the exterior of the knot. The group determines the equivariant intersection pairing on $\pi_2(M(K))$ up to sign, but we have been unable to show that this specifies the full homotopy type.

Gluck reconstruction

A 2-*knot* is an embedding $K\colon S^2 \to S^4$ which extends to an embedding $\phi\colon S^2 \times D^2 \to S^4$ such that $\phi | S^2 \times (0) = K$, with image a regular neighbourhood N of $\mathrm{im}(K)$. We shall assume that the spheres have been oriented. The embedding is then unique up to isotopy rel $S^2 \times (0)$. The *exterior* of K is $X(K) = S^4 \setminus \mathrm{int}\, N$, and the *group* of K is $\pi(K) = \pi_1(X(K))$. The orientations of the spheres determine a choice of *meridians* (the oriented boundaries of the transverse discs) which represent a single conjugacy class in $\pi(K)$ whose normal closure is the whole group. As any self homeomorphism of $\partial X(K) = S^2 \times S^1$ extends across $D^3 \times S^1$, the closed oriented 4-manifold $M(K) = X(K) \cup D^3 \times S^1$ obtained by surgery on K is well defined, and clearly $\pi_1(M(K)) = \pi(K)$ also. We shall let $\tilde{M}(K)$ and $M'(K)$ denote the universal and infinite cyclic covering spaces, respectively, of $M(K)$. (In working with the homology of such spaces, we shall often use implicitly the fact that the compact manifold $M(K)$ is homotopy equivalent to a finite cell complex [37]). We shall abbreviate $X(K), \pi(K), M(K)$ etc. as X, π, M etc. where this creates no ambiguity.

Let τ be the self homeomorphism of $S^2 \times S^1$ defined by $\tau(x, \theta) = (\rho(\theta)(x), \theta)$, where $\rho(\theta)$ is rotation of S^3 about its polar S^1 through the angle θ. Then the manifold $S^* = X(K) \cup_{\rho\tau} S^2 \times D^2$ is homeomorphic to S^4 (by [15] and [18]) and the inclusion of $S^2 \times (0)$ determines another 2-knot K^*, the *Gluck reconstruction* of K. Clearly $X(K^*) = X(K)$ and $M(K^*) = M(K)$. Gluck [18] showed that any 2-knot with exterior homeomorphic to $X(K)$ was equivalent to either K or K^* and his argument was extended to all higher dimensions by Browder [3] and Kato [23]. (An analogue for the classical case has recently been given by Culler, Gordon, Luecke and Shalen [9]). Litherland [26] has shown that if K is a 2-twist spun classical knot then $K = K^*$, whereas Plotnick [30] has extended work of Gordon [19] to show that if K is fibred with monodromy of odd finite order then $K \neq K^*$. Ribbon 2-knots, including spun classical knots, are also determined by their exterior [35; Theorem 9.11].

The conjugacy class of the meridians in $\pi = \pi_1(M)$ is represented by an isotopy class of embeddings of S^1 in M with a product regular neighbourhood. The two ways of identifying this neighbourhood with $D^3 \times S^1$ lead back to

the two knots K and K^* after surgery. More generally, call an element of π a *weight element* if its normal closure is the whole group, and call its conjugacy class a *weight class*. Then surgery on a weight class in M gives back a simply connected 4-manifold with Euler characteristic 2, hence S^4, and which contains a 2-knot K which may be neither K nor K^*, if the weight class is not that of a meridian or its inverse. This cannot happen if the group is metabelian.

Lemma 1. *If π is a metabelian knot group then it has an unique weight class, up to inversion.*

Proof. Let t be a meridian. Then any other weight element has the form $(gt)^{\pm 1}$ for some g in π'. Moreover π'/π'' is a module over $\mathbb{Z}[t, t^{-1}]$, under the action induced by conjugation, and $t-1$ acts invertibly on this module. Therefore if π' is abelian we must have $g=(1-t)h=hth^{-1}t^{-1}$ for some h in π', and so $gt=hth^{-1}$. □

More generally, if t is a weight element in a knot group then any other must be conjugate to $(kt)^e$ for some k in G'' and $e=1$ or -1.

Lemma 2. *Let K be a 2-knot. Then K has a Seifert hypersurface which contains no fake 3-cells.*

Proof. By the standard obstruction theoretical argument and TOP transversality every 2-knot has a Seifert hypersurface. Thus K bounds a locally flat 3-submanifold V which has trivial normal bundle in S^4. If Δ is a homotopy 3-cell in V then $\Delta \times R = D^3 \times R$, by simply connected surgery, and the submanifold $\partial \Delta$ of $\partial(\Delta \times R)=\partial(D^3 \times R)$ is isotopic there to the boundary of a standard 3-cell in $D^3 \times R$ which we may use instead of Δ. □

We shall say that a Seifert hypersurface for a 2-knot K is *minimal* if the "pushoff" maps on either side into its complement in $X(K)$ each induce injections of the fundamental group.

The aspherical case

Cappell and Shaneson [6] showed that examples of n-knots which are not determined by their exterior could be constructed from the mapping tori of suitable self homeomorphisms of the $(n+1)$-torus $(S^1)^{n+1}$. (The existence of such self homeomorphisms reduces to a question of algebraic number theory; at present examples are only known for $n=2$, 3 and 4). The result of surgery on such a knot is this mapping torus, and so is aspherical. (Dyer and Vasquez showed that if $n>1$ the exterior of an n-knot is aspherical only if it is a homotopy circle, in which case the knot must be trivial [11]. When $n=2$ the last point follows from Freedman [16]).

Theorem 1. *Let K and K_1 be 2-knots with groups isomorphic to π, a torsion free virtually solvable group other than $\mathbb{Z}$ or Φ. Then $M(K)$ and $M(K_1)$ are homeomorphic. Hence K is determined up to Gluck reconstruction and change of orientation by π together with a weight class. In particular K is thus determined by π alone if $\pi'=\mathbb{Z}^3$.*

Proof. It is shown in [22] that $M(K)$ and $M(K_1)$ are aspherical, so there is an homotopy equivalence $f\colon M(K_1)\to M(K)$. Moreover f is simple as π is torsion free and virtually poly-$\mathbb{Z}$ [2; Theorem 9.23], so $Wh(\pi)=0$ by [12]. Freedman [16] has established the exactness of the Sullivan-Wall surgery sequence and the s-cobordism theorem for such groups and 4-manifolds. Let $\theta_i\colon [S^{i-4}(M^+), G/\mathrm{TOP}]\to L_i(\pi)$ be the surgery obstruction map. Now π' is virtually nilpotent, by [22; Theorem 13]. Therefore the surgery obstruction map from $[S^{i-3}(K(\pi',1)^+), G/\mathrm{TOP}]$ to $L_i(\pi')$ is an isomorphism for $i\geqq 6$, by [13; Theorem 5.1). Applying the argument of the inductive step $((a)\Rightarrow(b))$ of [38; Theorem 15.B.1] and the periodicity of G/TOP and $L_*(\pi')$ we find that θ_4 and θ_5 are isomorphisms. The first assertion now follows from the exactness of the surgery sequence, and the others then follow from the discussion in the previous section. □

This result applies to the examples of Cappell and Shaneson, and also to the other groups of Theorem 4 and parts (i) to (iii) of Theorem 14 of [22]. (We take this opportunity to correct a minor error in [22]. The centres of $G(+)$ and $G(-)$ are infinite cyclic. In fact $G(+)$ is the group of the 3-twist spin of the figure eight knot). Except for the Cappell-Shaneson groups, each such group has infinitely many weight classes. An argument similar to that of Theorem 1 above shows that when they exist, Cappell-Shaneson n-knots are determined (up to Gluck reconstruction and change of orientation) by their groups together with the condition that $\pi_i(X)=0$ for $2\leqq i\leqq(n+1)/2$.

The class of 2-knots such that M is aspherical (i.e. such that $\tilde{M}$ is contractible) is considerably larger than that covered by Theorem 1. For instance, if K is a twist spun prime knot then M fibres over S^1 with fibre an irreducible 3-manifold, and so if moreover G' is infinite then M is aspherical. The universal cover $\tilde{M}$ is then in fact homeomorphic to R^4. We shall prove this under an assumption which covers all cases presently known for which M is aspherical. (Note however that Davis has given examples of aspherical closed 4-manifolds whose universal covers are not Euclidean space [10]).

Theorem 2. *Let K be a 2-knot whose group G has a nontrivial torsion free abelian normal subgroup A, and suppose that if the rank of A is 1 then G/A has one end. Then $\tilde{M}(K)$ is homeomorphic to R^4.*

Proof. The assumptions imply that M is aspherical, and hence that G is an orientable PD_4-group [22]. If A has rank at least 3 then G is virtually poly-$\mathbb{Z}$ [22; Theorem 3] and the result follows from Theorem 1. Therefore we may assume that the rank of A is at most 2. Then the virtual homological dimension of G/A is at least $4-h.d.A\geqq 2$, and so G/A cannot have 0 or 2 ends. If it had infinitely many ends then it would have a generalized free product structure, with amalgamation over a finite subgroup, and so G would have a corresponding structure with amalgamation over a finite extension of A. The building blocks of such a structure must have infinite index in G and so have homological dimension less than 4, by [2; Proposition 9.22], and the amalgamating subgroup would have homological dimension at most 2. A Mayer-Vietoris calculation would then give $H_4(G;\mathbb{Z})=0$, contrary to G being a PD_4-group. Thus G/A has 1 end, and so G is 1-connected at ∞, by [27; Theorem 1]. Therefore $\tilde{M}$

is a contractible open 4-manifold which is 1-connected at ∞ and so must be homeomorphic to R^4, by [15; Corollary 1.1]. □

If 4-dimensional surgery and s-cobordism, $Wh=0$ and a strong form of the Novikov conjecture can be established for all orientable Poincaré duality groups then the exterior of any 2-knot with such a group would also be determined by its group togheter with a weight class. Note also that if π is an orientable PD_4-group then generators for $H_4(\pi;\mathbb{Z})$ correspond to orientations for M and hence for S^4; a weight class then determines a transverse orientation and hence an orientation for K. (Moreover if a 2-knot group π with $H_1(\pi';\mathbb{Z}/2\mathbb{Z})\neq 0$ is a PD_4-group then it must be orientable [21]).

When π' is the group of an aspherical closed Seifert fibred 3-manifold the algebraic part of this programme can be carried out. For then π' has a subgroup σ of finite index which is a central extension by $\mathbb{Z}$ of the group of a closed surface, and so is the amalgamation of $\mathbb{Z}\times F(m)$ with $\mathbb{Z}\times F(n)$ over $\mathbb{Z}^2$ for suitable free groups $F(m)$ and $F(n)$. Therefore $\mathbb{Z}[\sigma]$ and hence $\mathbb{Z}[\pi']$ are regular coherent and so $Wh(\pi)=0$ by [36]. Moreover Stark [34] has shown that in most cases the surgery maps θ_j over π' are isomorphisms for j large, and by using the argument of the inductive step of Theorem 15.B.1 of [38] it can be shown that this is also true over π. These observations apply in particular to most of the examples of Gordon [19], which are obtained by k-twist spinning torus knots, with k odd, and for which M is fibred, with fibre a Brieskorn Seifert fibred 3-manifold.

In Theorem 1 we did not need the full strength of the information drawn from the work of Farrell and Hsiang. Much (but not all) of what we needed is contained in the next theorem, which we shall use in the following sections. Weinberger [39; Proposition 1] shows that for any group G there is a natural homomorphism $I_G\colon G\to L_5(G)$ defined by $I_G(g)=g_*(1)$ where $g_*\colon \mathbb{Z}=L_5(\mathbb{Z})\to L_5(G)$ is induced by the map sending 1 in $\mathbb{Z}$ (canonically identified with $L_5(\mathbb{Z})$) to g in G. When G is a knot group this map splits the projection onto $L_5(\mathbb{Z})$ induced by abelianization in the sense that if $\alpha\colon G\to\mathbb{Z}$ is either of the possible surjections then $\alpha=L_5(\alpha)\circ I_G$. Let $\hat{I}_G=I_G\circ\alpha^{-1}\circ I_{\mathbb{Z}}^{-1}$.

Theorem 3. *Let K be a 2-knot with group π. Then (i) the surgery obstruction map from $[M(K), G/\mathrm{TOP}]$ to $L_4(\pi)$ is injective; (ii) the image of $L_5(\mathbb{Z})$ under $\hat{I}_\pi$ acts trivially on $S_{\mathrm{TOP}}(M)$. (iii) if $f\colon N\to M(K)$ is a 2-connected degree 1 map and $\chi(N)=0$ then f is a homotopy equivalence.*

Proof. Since $[M, G/\mathrm{TOP}]\simeq\mathbb{Z}$ and the composition with the projection onto $L_4(1)$ is given by the signature difference map (cf. [38; page 237]), (i) is clear.

Let P be the E_8 manifold [15] and delete the interior of a submanifold homeomorphic to $D^3\times I$ to obtain P_0. There is a normal map $p\colon P_0\to D^3\times I$ which we may assume is the identity on the boundary (and in particular respects the "corners" of the boundary). Let U be an open regular neighbourhood of a meridian for M and form the union of $(M\setminus U)\times I$ with $P_0\times S^1$, identifying $\partial U\times I=S^2\times S^1\times I$ with $S^2\times I\times S^1$ in $\partial P_0\times S^1$. We may also match together $\mathrm{id}_{(M\setminus U)\times I}$ and $p\times\mathrm{id}_{S^1}$ to obtain a normal cobordism Q from id_M to itself. The surgery obstruction of this normal cobordism generates the image of $\hat{I}_\pi$, which proves (ii).

If $f: N \to M$ is a 2-connected degree 1 map then its failure to be an homotopy equivalence is measured by the homology kernel $K_2(f)$, which is a stably free module over the group ring $\mathbb{Z}[\pi]$, by Proposition 2.3 of [38]. By counting bases for the cellular chain complexes of the universal covering spaces of N and M we see that this module has rank $\chi(N)-\chi(M)$. Now Kaplansky has shown that a stably free $\mathbb{Z}[\pi]$-module of rank 0 must be 0 (cf. [32]). This proves (iii). □

Corollary. *If the abelianization homomorphism induces an isomorphism from $L_5(\pi)$ to $L_5(\mathbb{Z})$ then simple homotopy equivalent closed orientable 4-manifolds with fundamental group π and $H_2=0$ are s-cobordant.*

Proof. Let $f: N \to M$ be such a homotopy equivalence. By part (i) of the theorem f is normally cobordant to id_M, and by part (ii) and our assumption on $L_5(\pi)$ we may find a normal cobordism with obstruction 0. □

At present there is not yet a "plumbing theorem" over arbitrary groups: we do not know whether the whole of $L_5(\pi)$ acts on $S_{\mathrm{TOP}}(M)$, except when π is virtually solvable, or when the above corollary applies.

The spherical spaceform case

Suppose now that π is a 2-knot group such that the commutator subgroup π' is finite. Then it is shown in [20] that π' has cohomological period dividing 4. (We reprove this in the course of Theorem 4). It is also shown there to be of the form $A \times (\mathbb{Z}/n\mathbb{Z})$, where A is trivial, the quaternion group Q of order 8, the extended binary tetrahedral group $T(k)$ of order $3^k 8$, or the binary icosahedral group I^* of order 120, and $(n, 2|A|)=1$. The structure of the group π is determined by the class in the outer automorphism group of π' represented by conjugation by a meridian. This outer automorphism class is in fact unique (up to inversion) [20, 31]. Weight classes for π correspond to certain classes of automorphisms of π' modulo conjugation by inner automorphisms; up to inversion, the weight class is unique if $A=1$ or Q, there are two classes if $A=T(k)$, and four classes if $A=I^*$ [31]. Many of these groups are realized by 2-twist spun classical knots [41] and such 2-knots are determined by their exteriors [26].

Theorem 4. *Let K be a 2-knot with group π such that π' is finite. Then $\tilde{M}$ is homotopy equivalent to S^3, and so π' has cohomological period dividing 4. The homotopy type of M is determined by the group π and the class of the first nontrivial k-invariant, which is a generator k_4 of $H^4(\pi;\mathbb{Z}) \simeq \mathbb{Z}/|\pi'|\,\mathbb{Z}$, modulo the action of* $\mathrm{Aut}(\pi)$ *and* $\mathrm{Aut}(\mathbb{Z})$.

Proof. Let C be a maximal infinite cyclic central subgroup of π, and let M_C be the covering space of M with group C. Then M_C is a closed orientable 4-manifold with fundamental group Z and Euler characteristic $\chi(M_C) = [\pi: C]\,\chi(M(K))=0$. The homology groups of $\tilde{M}$ may be regarded as modules over the ring $\Gamma=\mathbb{Z}[C] \simeq \mathbb{Z}[c, c^{-1}]$. By the Wang sequence for the projection of $\tilde{M}$ onto M_C, multiplication by $c-1$ maps $H_2(\tilde{M};\mathbb{Z})$ onto itself. But by equi-

variant Poincaré duality and the Universal Coefficient spectral sequence we have $H_2(\tilde{M};\mathbb{Z}) \simeq \overline{\mathrm{Hom}}_\Gamma(H_2\tilde{M};\mathbb{Z}),\Gamma)$, where the overbar denotes the conjugate Γ-action. Hence $H_2(\tilde{M};\mathbb{Z})=0$. Since $\pi_1(M_C)$ has two ends $H_3(\tilde{M};\mathbb{Z}) \simeq \mathbb{Z}$ and since $\tilde{M}$ is an open 4-manifold $H_4(\tilde{M};\mathbb{Z})=0$. Thus $\tilde{M}$ is homotopy equivalent to S^3 and the first assertion follows.

In particular $\pi_2(M)=0$ and $\pi_3(M) \simeq \mathbb{Z}$, so the first nontrivial k-invariant is k_4 in $H^4(\pi;\mathbb{Z})$. By examining the effect on the Sylow subgroups of π', as in [29], we see that π acts trivially on $H^4(\pi';\mathbb{Z})$. By the Wang sequence for the extension $1 \to \pi' \to \pi \to \mathbb{Z} \to 1$ and the universal coefficient theorem, we then have $H^4(\pi;\mathbb{Z}) \simeq H^4(\pi';\mathbb{Z}) \simeq H_3(\pi';\mathbb{Z}) \simeq \mathbb{Z}/|\pi'|\mathbb{Z}$. Moreover k_4 clearly also represents the first k-invariant of the 3-dimensional Poincaré duality space M', and so must be a generator by [37; Theorem 4.3].

Now let N be the third stage of the Postnikov tower for $M(K)$ and let $j\colon M(K) \to N$ be the natural map. Then j is 4-connected and we may construct N by adjoining cells of dimension greater than 4 to $M(K)$. If K_1 is another 2-knot and $f\colon \pi(K_1) \to \pi(K)$ is an isomorphism which identifies the k-invariants up to sign, then there is a 4-connected map $j_1\colon M(K_1) \to N$ inducing f, which is homotopic to a map with image in the 4-skeleton of N, and so there is a map $h\colon M(K_1) \to M(K)$ such that j_1 is homotopic to jh. The map h induces isomorphisms on π_i for $i \leqq 3$, since j and j_1 are 4-connected, and so the lift $h\colon \tilde{M}_1 \simeq S^3 \to \tilde{M} \simeq S^3$ is a homotopy equivalence, by the theorems of Hurewicz and Whitehead. It follows that h is itself a homotopy equivalence. □

Note that in fact M_C and $S^3 \times S^1$ are homeomorphic, by surgery over $\mathbb{Z}$ [16]. In particular the universal cover $\tilde{M}$ is homeomorphic to $S^3 \times R$, and if $\pi = \mathbb{Z}$ the knot is trivial.

Determination of the range of simple homotopy types of such M seems difficult, in particular because the Nil groups occuring in Waldhausen's sequence [36] relating $Wh(\pi)$ to the K-theory of π' seem so intractable. Beyond this there lies the problem of finding the Wall group and its action on the structure set.

However when π' is odd cyclic we can largely sidestep such problems. In this case $L^x_n(\pi')=0$ for n odd and $x=s$ or h, and $L^s_4(\pi')=\mathbb{Z}^r$ where $r=(|\pi'|+1)/2$ and the natural map to $L^h_4(\pi')$ is an injection with cokernel a quotient of $H^2(\mathbb{Z}/2\mathbb{Z};\tilde{K}_0(\pi'))$, as Bak [1] has shown. Moreover the involution of π' induces the identity on $L^s_4(\pi')$, as the elements of this group are detected by multisignatures. Therefore Cappell's Mayer-Vietoris sequences [4; page 502] for $L^x_n(\pi)$ collapse to give isomorphisms $L^s_5(\pi)=L^h_4(\pi')$ and $L^s_4(\pi')=L^s_4(\pi)$. If furthermore the involution of π' fixes no nonzero element of $\tilde{K}_0(\pi')$ then $L^s_4(\pi')=L^h_4(\pi')$ and so $L^s_5(\pi)=L^h_5(\pi)$ and $L^s_4(\pi)=L^h_4(\pi)$. (Note that $\tilde{K}_0(\pi')=0$ if $|\pi'| \leqq 13$. More generally, the condition on $\tilde{K}_0(\pi')$ holds if $|\pi'|$ is a prime p such that $h^+_p=1$ and h_p is odd). It is then clear from the exact sequence of surgery that any homotopy equivalence f from N to M is h-cobordant to a simple homotopy equivalence.

In this case the map $\hat{I}_\pi$ of Theorem 3 may be identified with the composition $L_5(\mathbb{Z}) \simeq L_4(1) \to L_4(\pi') \simeq L_5(\pi)$. Moreover the whole of $L_5(\pi)$ acts on the structure set $S_{\mathrm{TOP}}(M)$, and by Theorem 3(i) the action is transitive, while by Theorem 3(ii) the image of $\hat{I}_\pi$ acts trivially. Since $[SM,G/\mathrm{TOP}] \simeq \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ it follows that the s-cobordism classes of simple homotopy equivalences are parametrized by

$L_4(\pi') \simeq \mathbb{Z}^{r-1}$, which is infinite if $\pi' \neq 1$. The next result then implies that there are infinitely many distinct topological 4-manifolds simple homotopy equivalent to M.

Theorem 5. *Let K be a 2-knot with group π such that π' is finite. Then the group of self homotopy equivalences of $M(K)$ is finite.*

Proof. If f is a self homotopy equivalence of $M(K)$ then we may assume that it fixes a base point. Since $\mathrm{Aut}(\pi)$ is finite, we may then assume also that f is orientation preserving and induces the identity on π. Since f lifts to orientation preserving self homotopy equivalences f_C and f of the covering spaces M_C and $\tilde{M}$, it follows that f induces the identity on $H_3(\tilde{M}; \mathbb{Z})$. Therefore it is homotopic to $\mathrm{id}_{\tilde{M}}$, since $\tilde{M}$ is homotopy equivalent to S^3, and so induces the identity on all homotopy groups. We may assume that M has a finite cell structure with a single 4-cell. There are no obstructions to constructing a homotopy from f to id_M on the 3-skeleton M_0, and since $\pi_4(M) = \pi_4(S^3) = \mathbb{Z}/2\mathbb{Z}$ there are just two possibilities for f. □

In the simplest nontrivial case, $\pi' = \mathbb{Z}/3\mathbb{Z}$ and π is then the group of the 2-twist spun trefoil knot T.

Theorem 6. *Let K be a 2-knot such that $\pi(K)' = \mathbb{Z}/3\mathbb{Z}$, and which has a minimal Seifert hypersurface. Then K is fibred.*

Proof. Let V be a minimal Seifert hypersurface for K. The modification of Lemma 2 preserves minimality and so we may assume that V is irreducible. The closed 3-manifold $\hat{V} = V \cup D^3$ is naturally embeddable in $M(K)$ with a product neighbourhood $N = \hat{V} \times [-1, 1]$ and the two inclusions $j_\pm$ of $\hat{V}$ as the boundary components of $W = M(K) \setminus \mathrm{int}\, N$ induce injections of the fundamental group. By Van Kampen's theorem π is an HNN extension with base $\pi_1(W)$ and associated subgroups the images of $\pi_1(\hat{V})$ under the homomorphisms induced by $j_\pm$. Since π' is finitely generated it follows that these monomorphisms are isomorphisms, and that $\pi_1(\hat{V}) \simeq \pi_1(W) \simeq \mathbb{Z}/3\mathbb{Z}$. Moreover $\chi(W) = 0$ and so, as in Theorem 2, we see that the universal covering space of W has the homotopy type of S^3. It follows that W is an h-cobordism from $\hat{V}$ to itself, and therefore $W = \hat{V} \times I$, for $S_{\mathrm{TOP}}(\hat{V} \times I \text{ rel } \partial)$ has just one element, by surgery over $\mathbb{Z}/3\mathbb{Z}$. (Note that $Wh(\mathbb{Z}/3\mathbb{Z}) = L_5(\mathbb{Z}/3\mathbb{Z}) = 0$). Therefore $M(K)$ fibres over the circle, and so K is fibred also. □

By Theorem 4, if K is any such knot then $M(K)$ is homotopy equivalent to $M(T)$ (and the homotopy equivalence is simple if $\mathrm{Nil}(\mathbb{Z}[\mathbb{Z}/3\mathbb{Z}], -) = 0$). However by the subsequent discussion there are infinitely many 2-knots such that the corresponding 4-manifold is simple homotopy equivalent to $M(T)$ but not homeomorphic to it. If the 3-dimensional Poincaré conjecture were true then a closed 3-manifold with fundamental group $\mathbb{Z}/3\mathbb{Z}$ would be the lens space $L(3, 1)$, by a recent result of Rubinstein [33]. It would then follow that none of the exotic versions of $M(T)$ could fibre over the circle, and so we would have counter examples to a 4-dimensional analogue of Farrell's fibration theorem. (Cf. also [40]).

We do not know whether any of these knots (other than T) is smoothable in some smooth structure on S^4. (An equivalent question is whether the corresponding manifold $M(K)$ is smoothable).

It may be possible to construct other exotic 2-knots with π' finite by wrapping together the ends of nontrivial s-cobordisms, such as those recently found by Cappell and Shaneson [7].

Groups of geometric dimension 2

We shall consider the case $\pi = \Phi$ in the broader context of knot groups which have finite 2-dimensional Eilenberg-Mac Lane spaces and one end, although thus far surgery and s-cobordism theorems have been established only for Φ among these groups. This class includes the classical knot groups (other than $\mathbb{Z}$), as is well known [28]. All of its members are in fact 2-knot groups, as we show next.

Lemma 3. *Let π be a high dimensional knot group which has a finite 2-dimensional $K(\pi, 1)$ complex. Then π has deficiency 1 and so is a 2-knot group.*

Proof. We may assume that $K(\pi, 1)$ has a single 0-cell, and that there are g 1-cells and r 2-cells. Then $1-g+r=\chi(K(\pi, 1))=0$, since $H_1(\pi; \mathbb{Z})=\mathbb{Z}$ and $H_2(\pi; \mathbb{Z})=0$. Thus we obtain a presentation of deficiency 1 from the complex. Kervaire showed that every such high dimensional knot group is the group of a smooth knot in some homotopy 4-sphere [24]; the 4-sphere is topologically standard by [15]. □

Whether all such groups have one end is related to the question of Kervaire as to whether a nontrivial free product $G * \mathbb{Z}$ can have weight 1 [25; Problem 5.7]. Another subclass of interest, including the group Φ as well as the torus knot groups, is described in the next result.

Theorem 7. *Let π be a group of cohomological dimension 2 with a non-trivial abelian normal subgroup A and such that $\pi/\pi' \simeq \mathbb{Z}$. Then either $\pi \simeq \Phi$ (and $A \simeq \Phi' \simeq \mathbb{Z}[\frac{1}{2}]$) or A is central and infinite cyclic, $A \cap \pi' = 1$, and π' is free of even rank. Hence π has one end. Moreover π has a finite presentation of deficiency 1, and the finite 2-dimensional cell complex determined by any such presentation is aspherical.*

Proof. It is shown in [22] that either π is solvable, in which case $\pi = \Phi$ by the classification of Gildenhuys [17], or $A \simeq \mathbb{Z}$. In the latter case let γ be the centralizer of A in π. Then $[\pi : \gamma] \leqq 2$ and so c.d. $\gamma = 2$. By [3; Theorem 8.8], γ' is free and so $A \cap \gamma' = 1$ (since π is not solvable). Since $\pi/\pi' \simeq \mathbb{Z}$, the subgroup γ' is characteristic in π and A maps injectively to $\rho = \pi/\gamma'$. Since ρ has an index 2 abelian normal subgroup and $\rho/\rho' \simeq \mathbb{Z}$, we must in fact have $\rho \simeq \mathbb{Z}$. It now follows easily that A is central in π, i.e. that $\gamma = \pi$, so that π' is free, hence of rank equal to the degree of the "Alexander polynomial" of π. Since the centre of π maps nontrivially to the abelianization, this polynomial must be cyclotomic, and so of even degree. Since π is either Φ or is an extension of $\mathbb{Z}$ by a finitely generated free group, it is easy to see that it has a presentation of deficiency 1, and the final assertion then follows as in [22]. □

Corollary. $Wh(\pi) = 0$.

Proof. Since all such groups are HNN extensions of free groups, they are in Waldhausen's class Cl, and so have trivial Whitehead group [36]. (Note that Φ can be given by the HNN construction $\mathbb{Z}*_\phi$ where ϕ maps $\mathbb{Z}$ isomorphically to its subgroup $2\mathbb{Z}$). □

The Whitehead group of a classical knot group is also trivial [36]. (In fact there is no known example of a finitely presentable group with finite cohomological dimension which has nontrivial Whitehead group).

Let $\Pi=\mathbb{Z}[\pi]$ and for any left Π-module N let e^qN be the left Π-module which has the same underlying abelian group as the right Π-module $\mathrm{Ext}^q_\Pi(N,\Pi)$ with the conjugate left Π-module structure. If c.d. $\pi=2$ then Π has global dimension 3, so $e^qN=0$ for $q>3$.

Theorem 8. *Let K be a 2-knot whose group π has a finite 2-dimensional $K(\pi,1)$ complex and one end. Then $\pi_2(M)\simeq e^2\mathbb{Z}$, and the isomorphism is unique up to sign.*

Proof. Let $C_*=0\to C_4\to C_3\to C_2\to C_1\to C_0\to 0$ be the cellular chain complex of the universal covering space of a finite cell complex homotopy equivalent to M. Then C_* is a complex of finitely generated free left Π-modules whose homology is $H_*=H_*(M;\Pi)=H_*(\tilde{M};\mathbb{Z})$. Since π has a presentation of deficiency 1 the augmentation Π-module $\mathbb{Z}$ has a resolution of the form

$$0\to\Pi^a\to\Pi^{a+1}\to\Pi\to\mathbb{Z}\to 0. \tag{1}$$

Since c.d. $\pi=2$ and π has one end, $e^0\mathbb{Z}=e^1\mathbb{Z}=e^3\mathbb{Z}=0$, and transposing (1) gives a resolution

$$0\to\Pi\to\Pi^{a+1}\to\Pi^a\to e^2\mathbb{Z}\to 0 \tag{2}$$

for $e^2\mathbb{Z}$. We then see that $e^0e^2\mathbb{Z}=e^1e^2\mathbb{Z}=e^3e^2\mathbb{Z}=0$ and $e^2e^2\mathbb{Z}=\mathbb{Z}$. In particular $e^2\mathbb{Z}$ is nonzero, and by the functoriality of $e^2(-)$ we have $\mathrm{Aut}(e^2\mathbb{Z})\simeq\mathrm{Aut}(\mathbb{Z})=\pm 1$. By equivariant Poincaré duality and the universal coefficient spectral sequence $H_j=0$ if $j\neq 0$ or 2, while $e^1H_2=e^3H_2=0$, $e^2H_2=\mathbb{Z}$ and there is an exact sequence of left Π-modules $0\to e^2\mathbb{Z}\to H_2\to e^0H_2\to 0$. Applying the functor $e^*(-)$ to this exact sequence, and writing P for e^0H_2, we obtain a twelve term exact sequence $0\to e^0P\to\ldots e^3e^3\mathbb{Z}\to 0$. Using the above observations, we find that $e^1P=0$, and that there is an exact sequence

$$0\to e^2P\to e^2H_2\to e^2e^2\mathbb{Z}\to e^3P\to 0. \tag{3}$$

But $e^3P=0$ since P is a submodule of a free module and Π has global dimension 3. Since $e^2H_2\simeq e^2e^2\mathbb{Z}=\mathbb{Z}$, it follows that $e^2P=0$ also. Hence P is projective, and so $H_2\simeq P\oplus e^2\mathbb{Z}$. Therefore we have exact sequences

$$0\to C_4\to C_3\to Z_2\to H_2=P\oplus e^2\mathbb{Z}\to 0 \tag{4}$$

and

$$0\to Z_2\to C_2\to C_1\to C_0\to\mathbb{Z}\to 0 \tag{5}$$

where Z_2 is the module of 2-cycles, which is projective, since (from (5)) it is a third syzygy. Now $P \oplus e^2\mathbb{Z}$ also has a resolution of the form

$$0 \to \Pi \to \Pi^{a+1} \to \mathrm{P} \oplus \Pi^a \to \mathrm{P} \oplus \mathrm{e}^2\mathbb{Z} \to 0. \tag{5}$$

Applying Schanuel's lemma to (4) and (6) and to (1) and (5), we find that $C_4 \oplus \Pi^{a+1} \oplus Z_2 = \Pi \oplus C_3 \oplus P \oplus \Pi^a$ and $Z_2 \oplus \Pi^a \oplus C_1 \oplus \Pi = C_2 \oplus \Pi^{a+1} \oplus C_0$ and so $C_4 \oplus \Pi^{a+1} \oplus C_2 \oplus \Pi^{a+1} \oplus C_0 = \Pi \oplus C_3 \oplus \Pi^a \oplus P \oplus \Pi^a \oplus C_1 \oplus \Pi$. But $\chi(M)=0$, so $C_4 \oplus C_2 \oplus C_0 = C_3 \oplus C_1$. Thus there is a finitely generated free Π-module $F = C_3 \oplus C_1 \oplus \Pi^{2a+2}$ such $F = F \oplus P$. By a lemma of Kaplansky (cf. [32]) this implies that $P=0$, so $H_2 = e^2\mathbb{Z}$ and the first assertion follows from the Hurewicz theorem. The isomorphism is unique up to sign since $\mathrm{Aut}(e^2 Z) \simeq (\pm 1)$. □

Corollary. *If K_1 is another 2-knot with group π, then there is a map f_0: $M(K_1)\backslash \mathrm{int}\, D^4 \to M(K)$ inducing isomorphisms on π_1 and π_2. If f_0 extends to a map f: $M(K_1) \to M(K)$, then f is a homotopy equivalence.*

Proof. In each case $H^3(\pi; \pi_2(M))$ is trivial, since c.d. $\pi = 2$, and so the first k-invariant is 0. The first assertion now follows as in Theorem 4. If f extends f_0 then it must induce isomorphisms on the homology of the universal covering spaces, since $H_j(M(K_1); \Pi) = H_j(M(K); \Pi) = 0$ for $j \geqq 3$. Therefore f is a homotopy equivalence. □

Thus the major task in determining the homotopy type of M is to decide when f_0 extends. In contrast to the cases treated above where M was 2-connected, we must now expect the equivariant intersection pairing on $\pi_2(M)$ to play a role. The essential uniqueness of the Poincaré duality isomorphism implies that this pairing is also unique up to sign. Once the homotopy type is known we may hope to apply surgery to find the topological type.

Theorem 9. *Let K be a 2-knot with group Φ. Then any homotopy equivalence f: $N \to M(K)$ of closed manifolds is homotopic to a homeomorphism.*

Proof. Although $\Phi = \mathbb{Z} *_\phi$ is not square root closed accessible. Cappell's splitting theorem holds for it, by the remark on page 167 of [5]. Therefore there is still an exact sequence

$$L_5(\mathbb{Z}) \to L_5(\mathbb{Z}) \to L_5(\Phi) \to L_4(\mathbb{Z}) \to L_4(\mathbb{Z})$$

where the extreme maps are essentially $1 - \phi_*$ (cf. [2; page 498 et seq.]). Since the map induced on $L_5(\mathbb{Z}) \simeq \mathbb{Z}$ by ϕ is multiplication by 2 (cf. [38; page 138] or [39; Proposition 1]), the left hand map is an isomorphism. The abelianization homomorphism from Φ to $\mathbb{Z}$ induces a map to the corresponding exact sequence for $\mathbb{Z}$, considered as an HNN extension of the trivial group. A diagram chase now shows that the induced map from $L_5(\Phi)$ to $L_5(\mathbb{Z}) \simeq L_4(1) = \mathbb{Z}$ is an isomorphism. Similarly $L_4(\Phi) \simeq L_4(\mathbb{Z}) \simeq L_4(1)$. Now by the corollary to Theorem 3 the map f is s-cobordant to id_M, and so f is homotopic to a homeomorphism. □

Corollary. *A ribbon knot K with group Φ is determined up to orientation by the homotopy type of $M(K)$.*

Proof. Since Φ is metabelian, there is an unique weight class up to inversion, so the knot exterior is determined by $M(K)$, and since K is a ribbon knot it is determined by its exterior. □

This corollary applies in particular to Examples 10 and 11 of Fox [14]. Note also that ribbon 2-knots are -amphicheiral [33; page 266], but that no 2-knot with an asymmetric Alexander polynomial can be invertible. Thus as oriented knots these examples are distinct. Is there a 2-knot with group Φ which is not a ribbon knot?

A high dimensional knot group whose commutator subgroup is free has a finite 2-dimensional Eilenberg Mac Lane complex, and so by Lemma 3 is the group of some 2-knot. In a separate paper we shall show that if K is such a 2-knot with $\pi(K)'$ free of rank r, then $M(K)$ is homotopy equivalent to a smooth 4-manifold which fibres over S^1 with fibre $\#^r(S^1 \times S^2)$. Moreover $Wh(\pi)=0$ and π is square root closed accessible, so abelianization gives an isomorphism of $L_*(\pi)$ with $L_*(\mathbb{Z})$ (cf. [4]). Thus if 5-dimensional s-corbordisms with group π were products, such a knot K would be determined up to orientations and Gluck reconstruction by π together with a weight class. (If K is a spun 1-knot then it is ribbon [35], hence -amphicheiral, and $K=K^*$, and so a spun invertible fibred 1-knot would be determined completely by its group together with the conjugacy class of a meridian). Cochran has shown that, modulo the 3-dimensional Poincaré conjecture, a fibred 2-knot with π' free is homotopy ribbon [8]. Is every such group the group of a ribbon knot?

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