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Titel: Mod p loop space homology.

Autor: Halperin, S.; Felix, Y.; Lemaire, J.-M.

Jahr: 1989

PURL: https://resolver.sub.uni-goettingen.de/purl?356556735_0095|log19

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Invent. math. 95, 247–262 (1989)

Inventiones mathematicae
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Mod *p* loop space homology

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§1. Introduction

Denote by $\mathbb{Z}_p$ the prime field of characteristic p. (Thus $\mathbb{Z}_0=\mathbb{Q}$.) Consider the following condition on topological spaces X:

$$X \text{ is simply connected, and each } H_i(X;\mathbb{Z}_p) \text{ is finite dimensional.} \qquad (*)_p$$

If X satisfies $(*)_p$ then the loop space homology, $H_*(\Omega X;\mathbb{Z}_p)$ is a connected co-commutative graded Hopf algebra.

In this paper we show how finiteness properties of X are reflected in $H_*(\Omega X;\mathbb{Z}_p)$. Specifically, let cat X denote the Lusternik-Schnirelmann category of X. Then we establish

Theorem A. *If X satisfies $(*)_p$ then*

$$\text{depth } H_*(\Omega X;\mathbb{Z}_p) \leqq \text{cat } X.$$

(The depth of an algebra is defined in section 2.)

This is the second example of a class of spaces whose loop space homology has finite depth. In fact there follows immediately from [5; Remark 1.3.2 and Proposition 1.6] the

This research was partially supported by the second named author's NSERC grant, and by a NATO travel grant.

* Chercheur qualifié au FNRS.

** U.A. au CNRS 168.

*** U.A. au CNRS 751.

Theorem B. *Let X satisfy $(*)_p$ and suppose $Y = X \bigcup_f e^n$ is obtained by adding an n-cell to $X (n \geqq 2)$. If e^n represents a nontrivial class in $H_n(Y; \mathbb{Z}_p)$* then *$H_*(\Omega Y; \mathbb{Z}_p)$ has finite depth.*

The usefulness of Theorems A and B resides in the fact that Hopf algebras of finite depth have very special properties. We need to recall ([8]) that if $H \subset G$ is a sub Hopf algebra of a graded connected co-commutative Hopf algebra, G, then H is *normal* if $H_+ G = G H_+$ and in this case the quotient $G/H_+ G$ inherits a Hopf algebra structure which we denote by $G//H$.

Our second main result reads

Theorem C. *Let G be a graded connected co-commutative Hopf algebra of finite depth, m, over a field,* lk. *Then*

(i) *Then union, R, of the normal solvable sub Hopf algebras is a nilpotent normal sub Hopf algebra. (R is called the* radical *of G.)*

(ii) *R is finitely generated as an algebra.*

(iii) *If* lk *is perfect and if $R(i)$ denotes the lower central series for R* ([9]) *then* gr $R = \bigotimes_i R(i)//R(i+1)$ *is (as an algebra) the tensor product of a finite dimensional algebra with a polynomial algebra in at most m variables.*

A theorem of Serre [10] asserts that a non-contractible finite simply connected CW complex has infinitely many non-trivial homotopy groups: in other words its Postnikov tower decomposition is infinite. Theorems A and C above, combined with a result of Moore and Smith, give a considerable generalization of this which we now describe.

Recall that a space, X, has finitely many nonvanishing homotopy groups precisely when its Postnikov decomposition is finite. Much more generally, we say X is an *n-stage Postnikov space* if it can be decomposed as a tower of n principal fibrations

$$X = X_n \xrightarrow{\rho_n} X_{n-1} \longrightarrow \ldots \xrightarrow{\rho_1} X_0 = pt$$

in which the fibre F_i of each ρ_i is a *(possibly infinite)* product:

$$F_i = \prod_{k=1}^{\infty} K(G_{i,k}; k)$$

of Eilenberg-Maclane spaces. We say X is *of finite type at p* if each $G_{i,k} (k \geqq 2)$ localizes at p to a finitely generated module over the ring $\mathbb{Z}_{(p)}$ of the integers localized at p.

This notion of n stage space was introduced by Moore and Smith [9]. It is dual to the notion of *m-cone*: Y is an m-cone if

$$Y = Y_m \supset Y_{m-1} \supset \ldots \supset Y_0 = pt$$

and each Y_{i+1} has the form $Y_{i+1} = Y_i \cup CA_i$ (for some continuous map $A_i \to Y_i$. Finite CW complexes Y are m-cones ($m = \dim Y$) just as spaces with finitely many non-vanishing homotopy groups are n-stage Postnikov spaces. Thus Serre's theorem is a special case of

Theorem D. *Let X be a simply connected space which is both an m-cone, and an n-stage Postnikov space of finite type at p, for some prime $p>0$. Then*

$$H_i(X;\mathbb{Z}_p)=0,\ i\geqq 1.$$

In particular if X is a p-local CW complex it is contractible.

This strengthening of Serre's theorem is in a different direction than that of McGibbon and Neisendorfer [7], who show that if X is finite simply connected complex with $H_+(X;\mathbb{Z}_p)\neq 0$ then $\pi_i(X)$ has p-torsion for infinitely many i. A possible third generalization is suggested by Theorem B:

Problem. Is it possible for a simply connected topological space Y with finite mod p betti numbers to satisfy simultaneously:

(a) $Y=X\bigcup_f e^n (n\geqq 2)$ with e^n homologically non trivial in $H_n(Y;\mathbb{Z}_p)$.

(b) There is a principal fibration $K(\mathbb{Z},q)\to Y\to Z$, with $\mathbb{Z}=\pi_q(K(\mathbb{Z},q))$ mapping nontrivially to $\pi_q(Y;\mathbb{Z}_p)$.

Theorems A and C were established in [4] for the case of characteristic zero via Sullivan's minimal models (as were the analogous results for commutative local rings over fields of arbitrary characteristic). These techniques, however, do not apply to loop space homology in positive characteristic.

Observe also that Theorem D is false in characteristic zero. Indeed any Lie group, once rationalized, is an n-cone and a 1-stage Postnikov space.

This paper is organized as follows: Theorem A is proved in Sect. 2, and after some preliminary work in Sect. 3, Theorem C is proved in Sect. 4. Theorem D is proved in Sect. 5.

Throughout the paper we work over an arbitrary field lk and we adopt the convention: *Hopf algebra* means a graded co-commutative connected Hopf algebra $G_*=\mathrm{lk}\oplus\bigoplus_{i\geqq 1} G_i$.

§2. The depth theorem

We work in the category of graded modules over graded lk-algebras, R. In particular we use $\mathrm{Hom}_R(-,-)$ and $\mathrm{Ext}^p_R(-,-)$ to denote the functors "graded Hom" and "graded Ext"; thus $\mathrm{Hom}_R(M,N)=\bigoplus_q \mathrm{Hom}^q_R(M,N)$ with $\mathrm{Hom}^q_R(M,N)$ consisting of the R-linear maps sending each M_r to N_{r-q}. The "internal" grading induced in Ext^p_R will be written $\bigoplus_q \mathrm{Ext}^{p,q}_R$.

Now suppose $R=\bigoplus_{k\geqq 0} R_k$ with $R_0=\mathrm{lk}$. The projection $R\to R/R_+=\mathrm{lk}$ makes lk into a graded R-module, and we have the classical

Definition 2.1. The *global dimension* of R (possibly ∞) is the least n such that $\mathrm{Ext}^i_R(\mathrm{lk},\mathrm{lk})=0,\ i>n$.

The *depth* of R (possibly ∞) is the largest n such that $\mathrm{Ext}^i_R(\mathrm{lk},R)=0,\ i<n$.

Now recall that our first main theorem reads

Theorem A. *Let X be a simply connected topological space,* lk *a field, and assume that each $H_i(X; \mathrm{lk})$ is finite dimensional. Then*

$$\operatorname{depth} H_*(\Omega X; \mathrm{lk}) \leqq \operatorname{cat}(X).$$

The first step in the proof is a reduction to a problem in differential algebra. Let $C^*(X; \mathrm{lk})$ be the DGA of singular cochains on X. According to [6] there is a DGA morphism of the form

$$\phi: (T(V), d) \xrightarrow{\simeq} (C^*(X; \mathrm{lk}), d)$$

in which $\simeq$ means that $H(\phi)$ is an isomorphism, and $T(V)$ denotes the tensor algebra on the vector space V. Moreover we may suppose that

$$\begin{aligned} &V = \bigoplus_{j \geq 2} V^j, \\ &\text{each } V^j \text{ has finite dimension,} \\ &d: V \to T^{\geq 2}(V), \end{aligned} \tag{2.2}$$

where $T^k(V)$ denotes the k^{th} tensor power $V^{\otimes k}$.

Next we recall ([6]) the definition of the invariant left $M\mathrm{cat}(T(V), d)$. Any DGA morphism $(T(V), d) \xrightarrow{\alpha} (A, d_A)$ factors as $(T(V), d) \to (T(V \oplus W), d) \xrightarrow{\simeq} (A, d_A)$ where, of course, the middle DGA depends on (A, d_A). In particular, for any m factor the quotient $\rho_m: (T(V), d) \to (T(V)/T^{>m}(V), d)$ as

$$(T(V), d) \xrightarrow{j} (T(V \oplus W_m), d) \xrightarrow{\simeq} (T(V)/T^{>m}(V), \bar{d}).$$

Definition. Left $M\mathrm{cat}(T(V), d)$ is the least m such that there is a morphism $r: (T(V \oplus W_m), d) \to (T(V), d)$ of left $(T(V), d)$ differential modules such that $rj = \mathrm{id}_{T(V)}$.

According to [6]

$$\text{left } M\mathrm{cat}(T(V), d) \leqq \operatorname{cat} X.$$

On the other hand, the algebra $H_*(\Omega X; \mathrm{lk}) = R$ can be computed from $(T(V), d)$ as follows ([6]); Appendix). Filter $T(V)$ by the ideals $T^{\geq k}(V)$. In the resulting spectral sequence the E_1-term is the DGA $(T(V), d_2)$ defined by:

$$d_2: T \to T^2(V), \qquad d - d_2: V \to T^{>2}(V);$$

explicitly $E_1^{p,q} = [T^p(V)]^{p+q}$. The corresponding bigrading of $H(T(V), d_2)$ is written $H^{p,q}$.

Now note that $(T(V), d_2)$ is necessarily the cobar construction (reduced) on a graded coalgebra $C = \mathrm{lk} \oplus sV$ with co-multiplication given by $d_2: V \to V \otimes V$ (sV is the suspension of V defined by $(sV)^k = V^{k+1}$). The dual graded algebra

$R=\mathrm{lk}\oplus(sV)^*$ is exactly $H_*(\Omega X;\mathrm{lk})$ – [6, Prop. A. 8]. Thus Theorem A is a corollary of the following result, which is the non-commutative analogue of Theorem A of [4].

Theorem A′. *Let* $(T(V), d)$ *be a* DGA *over* lk *satisfying* (2.2), *and let* R *be the graded algebra defined from* $(T(V), d)$ *as described above. Then*

(1) *depth* $R\leqq$ *left* $\mathcal{M}\mathrm{cat}(T(V), d)\leqq \mathrm{gl\,dim}(R)$.

(2) *If depth* $R=$ *left* $\mathcal{M}\mathrm{cat}(T(V), d)$, *then depth* $R=\mathrm{gl\,dim}(R)$.

Proof. Step 1. *We express* $\mathrm{Ext}_R(\mathrm{lk},\mathrm{lk})$ *and* $\mathrm{Ext}_R(\mathrm{lk}, R)$ *in terms of* $(T(V), d_2)$.

By construction $(T(V), d_2)$ is the dual of the reduced bar construction $(BR,\bar{V})$ on R, with the grading $T^k(V)$ corresponding to the resolution degree $B_k(R)$. Consider the unreduced bar construction $(R\otimes BR,\bar{V})$; it is an R-free graded resolution of lk. Since $(T(V), d_2)=\mathrm{Hom}_R((R\otimes BR,\bar{V}),\mathrm{lk})$ it follows that

$$\mathrm{Ext}_R^{p,q}(\mathrm{lk},\mathrm{lk})=H^{p,q}(T(V), d_2). \tag{2.3}$$

We next consider $\mathrm{Ext}_R(\mathrm{lk}, R)$. In the category of graded differential modules over a DGA, A, we can use the functor $\mathrm{Hom}_A(-,-)$ to define a graded space $\mathrm{Ext}_A^*(,)$ as described, for instance, in [5; Appendix]. If we consider, now, bigraded differential modules $(M^{*,*}, d_M)$ over $(T(V), d_2)$ with d_M of bidegree $(1, 0)$ then we can use the functor "bigraded hom" to define "bigraded Ext" in exactly the same way; these functors will be denoted by $\mathrm{Hom}^{p,q}_{(T(V),d_2)}(-,-)$ and $\mathrm{Ext}^{p,q}_{(T(V),d_2)}(-,-)$. If we regard a graded algebra A as a bigraded differential algebra concentrated in bidegrees $(0,*)$ and with zero differential then we can identify $\mathrm{Ext}_A^{p,q}(-,-)$ with $\mathrm{Ext}^{p,q}_{(A,0)}(-,-)$. Moreover, the argument of [5, Prop. 2.1] establishes

$$\mathrm{Ext}_R^{p,q}(\mathrm{lk}, R)=\mathrm{Ext}^{p,q}_{(T(V),d_2)}(\mathrm{lk},(T(V), d_2)). \tag{2.4}$$

Step 2. *Left* $\mathcal{M}\mathrm{cat}(T(V), d_2)\leqq \mathrm{gl\,dim}(R)$. Suppose $\mathrm{gl\,dim}(R)=n<\infty$. Write $T^n(V)=(\ker d_2)\oplus S$. Then $I=S\oplus T^{>n}(V)$ is a d_2-acyclic ideal in $T(V)$. Hence it is a d-acyclic ideal and $T(V)\xrightarrow{\simeq} T(V)/I$. This factors as

$$(T(V), d)\xrightarrow{\rho_n} T(V)/T^{>n}(V)\longrightarrow T(V)/I,$$

whence the diagram

$$\begin{array}{ccc}
(T(V), d) & \xrightarrow{\mathrm{id}} & (T(V), d)\\
\downarrow & \overset{r}{\nearrow} & \downarrow\simeq\\
(T(V\oplus W_n), d) & \longrightarrow & (T(V)/I, \bar{d}).
\end{array}$$

The existence of a DGA morphism r making both triangles above commute is automatic, and so left $\mathcal{M}\mathrm{cat}(T(V), d)\leqq n$.

Step 3. *Depth* $R \leqq \text{gl dim}(R)$. (This is classical.) Because $R_0 = \text{lk}$ there is an R-free resolution (F_*, ∂) of lk such that $\partial: F_k \to R_+ \cdot F_{k-1}$. It follows that $\text{Ext}_R^{p,*}(\text{lk}, \text{lk}) = \text{Hom}_R(F_p, \text{lk})$; in particular $F_p = 0$, $p > n = \text{gl dim}(R)$, and $F_n \neq 0$. Write $F_n = \bigoplus_i Rv_i$. Then the projection $F_n \to R \cdot v_1 = R$ defines a non-trivial element of $\text{Ext}_R^{n,*}(\text{lk}, R)$; i.e. depth $R \leqq n$.

Step 4. *Restatement of the problem.* In view of Steps 2 and 3 the entire theorem will be proved if we assume

$$\text{depth } R \geqq \text{ left } M\text{cat}(T(V), d) = m \tag{2.5}$$

and deduce that left $M\text{cat}(T(V), d) \geqq \text{gl dim}(R)$.

We first sketch the proof. We begin by constructing a differential $(T(V), d)$-module of the form $(T(V) \otimes (M \oplus \text{lk}), D)$ which models $T(V)/T^{>m}(V)$ and we use the fact $m = \text{left } M\text{cat}(T(V), d)$ to construct a retraction of this module onto $(T(V), d)$.

Then we use the hypothesis (2.5) to show that this retraction can be taken to be filtration preserving with respect to the standard filtration in $T(V)$ and a suitable filtration in $T(V) \otimes (M \oplus \text{lk})$. At the E_1-level this gives morphisms

$$(T(V), d_2) \underset{i}{\overset{\tau}{\leftrightarrows}} (T(V) \otimes (M \oplus \text{lk}), D_2) \xrightarrow{\simeq} (T(V)/T^{>m}(V), \bar{d}_2),$$

$H(\tau) H(i) = \text{id}$. This implies that $H(\rho_m): H(T(V), d_2) \to H(T(V)/T^{>m}(V), \bar{d}_2)$ is injective. In particular $H^{p,*}(T(V), d_2) = 0$, $p > m$ and so $\text{gl dim}(R) \leqq m$.

It remains to provide the details.

Step 5. *The model* $(T(V) \otimes (M \oplus \text{lk}), D)$. Put $M = M^{m,*} = sT^{m+1}(V)$; thus $M^{m,q} = s(T^{m+1}(V)^{m+1+q})$. Define an isomorphism

$$\omega: T(V) \otimes M \xrightarrow{\cong} T^{>m}(V)$$

by $\omega(\Phi \otimes s\Psi) = (-1)^{deg \Phi} \Phi \cdot \Psi$ and put

$$D' = -\omega^{-1} d\omega: T(V) \otimes M \to T(V) \otimes M.$$

Finally, define an operator D in $T(V) \otimes (M \oplus \text{lk})$ by

$$D|_{T(V)} = d \quad \text{and} \quad D|_{T(V) \otimes M} = \omega + D'.$$

A short and simple calculation shows that $(T(V) \otimes (M \oplus \text{lk}), D)$ is a differential $(T(V), d)$ module, and that an equivalence

$$\psi: (T(V) \otimes (M \oplus \text{lk}), D) \xrightarrow{\simeq} (T(V)/T^{>m}(V), \bar{d})$$

is defined by $\psi = \rho_m$ in $T(V)$ and $\psi = 0$ in $T(V) \otimes M$.

The bigrading of M, and that of $T(V)$ defined above makes this module into a filtered module, where we filter as usual by the left hand degree. The

differential raises the filtration degree by at least one, and the E_1-term of the spectral sequence has the form $(T(V)\otimes(M\oplus \mathrm{lk}), D_2)$ where D_2 is constructed from d_2 by the identical procedure used to construct D from d. In particular, ψ may be regarded as an equivalence

$$\psi\colon (T(V)\otimes(M\oplus \mathrm{lk}), D_2)\xrightarrow{\simeq}(T(V)/T^{>m}(V), \bar{d}_2).$$

Step 6. *The retraction,* σ. Because $T(V)^1=0$ and $T(V)\otimes(M\oplus \mathrm{lk})$ is a free $T(V)$-module, it follows that $(T(V)\otimes(M\oplus \mathrm{lk}), D)$ is $(T(V), d)$-semifree in the sense of [2] and [5]. This implies that the lifting problem represented by the dotted arrow in the diagram

$$\begin{array}{ccc} T(V) & \hookrightarrow & T(V\oplus W_m) \\ \downarrow & \nearrow & \downarrow \simeq \rho_m \\ T(V)\otimes(M\oplus \mathrm{lk}) & \xrightarrow[\psi]{} & T(V)/T^{>m}(V) \end{array}$$

can be solved, yielding a morphism $(T(V)\otimes(M\oplus \mathrm{lk}), D)\to(T(V)\oplus W_m), d)$ of $(T(V), d)$-modules making the diagram commute.

We compose this morphism with the retraction $(T(V\oplus W_m), d)\xrightarrow{r}(T(V), d)$ which exists because left $M\,\mathrm{cat}(T(V), d)=m$, and we obtain a morphism

$$\sigma\colon (T(V)\otimes(M\oplus \mathrm{lk}), D)\to(T(V), d)$$

of $(T(V), d)$-modules such that $\sigma(\Phi\otimes 1)=\Phi$.

Now because M is concentrated in bidegrees $(m, *)$ it follows that σ has the form

$$\sigma=\sigma_{-m}+\sigma_{-m+1}+\ldots+\sigma_0+\ldots$$

with σ_i of bidegree $(i, -i)$.

On the other hand, consider an arbitrary morphism of degree 0,

$$\alpha=\alpha_{-k}+\ldots\colon (T(V)\otimes(M\oplus \mathrm{lk}), D)\to(T(V), d),$$

where $1\leqq k\leqq m$. Clearly α_{-k} is a morphism

$$\alpha_{-k}\colon (T(V)\otimes(M\oplus \mathrm{lk}), D_2)\to(T(V), d_2)$$

of $(T(V), d_2)$-modules; in other words α_{-k} is a cocycle in

$$\mathrm{Hom}^{-k,k}_{T(V)}(T(V)\otimes(M\oplus \mathrm{lk}), T(V))$$

with respect to the differential $\delta_2\alpha=d_2\alpha-\alpha D_2$.

But as we observed in Step 5, we have an equivalence

$$(T(V)\otimes(M\oplus \mathrm{lk}), D_2)\xrightarrow{\simeq}(T(V)/T^{>m}(V), d_2)$$

homogeneous of bidegree (0, 0). And, as we observed at the start of this step, $(T(V)\otimes(M\oplus \mathrm{lk}), D_2)$ is $(T(V), d_2)$-semi free. Thus by definition ([5])

$$\begin{aligned}&\mathrm{Ext}^{p,q}_{(T(V),d_2)}((T(V)/T^{>m}(V), d_2), (T(V), d_2))\\ &\quad = H^{p,q}(\mathrm{Hom}_{T(V)}(T(V)\otimes(M\oplus \mathrm{lk}), T(V), \delta_2).\end{aligned}$$

In particular we may identify the class of α_{-k} as an element in $\mathrm{Ext}^{-k,k}$.

To simplify our notation we write

$$E(-)=\mathrm{Ext}_{(T(V),d_2)}(-,(T(V), d_2)).$$

In our seventh (and final) step below we use the hypothesis depth $R\geqq m$ to deduce that $E^{p,*}(T(V)/T^{>m}(V), \bar{d}_2)=0$, $p<0$. It follows that $[\alpha_{-k}]\in E^{-k,*}$ is zero, so that $\alpha_{-k}=[\delta_2, \beta]$. Put

$$\alpha'=\alpha-(d\beta+\beta D)\colon (T(V)\otimes(M\oplus \mathrm{lk}), D)\to(T(V), d).$$

Then $\alpha'=\alpha'_{-(k-1)}+\cdots$.

Applying this procedure m times to σ, we deduce a $(T(V), d)$-morphism

$$\sigma'\colon (T(V)\otimes(M\oplus \mathrm{lk}), D)\to(T(V), d)$$

of the form $\sigma'=\sigma-(d\beta+\beta D)$ and such that

$$\sigma'=\sigma'_0+\sigma'_1+\cdots.$$

Let $i\colon T(V)\to T(V)\otimes(M\oplus \mathrm{lk})$ be the inclusion. Then $H(\sigma'\circ i)=H(\sigma')\circ H(i)$ $=H(\sigma)\circ H(i)=\mathrm{id}_{H(T(V),d)}$. It follows from (2.2) that $\sigma'\circ i$ is an isomorphism. Put $\phi=(\sigma'\circ i)^{-1}\circ\sigma'$:

$$\phi\colon (T(V)\otimes(M\oplus \mathrm{lk}), D)\to(T(V), d).$$

Then $\phi\circ i=id$, and

$$\phi=\phi_0+\phi_1+\cdots.$$

In particular the identity of $H(T(V), d_2)$ factors as

$$H(T(V), d_2)\to H(T(V)\otimes(M\oplus \mathrm{lk}), D_2)\xrightarrow{H(\phi_0)} H(T(V), d_2),$$

and since $H^{p,q}(T(V)\otimes(M\oplus \mathrm{lk}), D_2)\cong H^{p,q}(T(V)/T^{>m}(V), \bar{d}_2)$, we conclude that

$$\mathrm{Ext}^{p,*}_R(\mathrm{lk}, \mathrm{lk})=H^{p,*}(T(V), d_2)=0,\qquad p>m;$$

i.e. $\mathrm{gl\,dim}(R)\leqq m$.

We are now reduced to proving

Step 7. $E^{p,*}(T(V)/T^{>k}(V), \bar{d}_2)=0$, $\quad p<m-k$.

In fact we have $E^{p,*}(\mathrm{lk})=\mathrm{Ext}^{p,*}_R(\mathrm{lk}, R)$ (cf. (2.4)), and so when $k=0$ Step 7 is exactly the hypothesis (2.5). Next, apply E to the short exact sequence

$$0\to(T^k(V), 0)\to(T(V)/T^{>k}(V), \bar{d}_2)\to(T(V)/T^{\geqq k}(V), \bar{d}_2)\to 0,$$

to obtain an exact sequence

$$E^{p,q}(T(V)/T^{\geq k}(V), \bar{d}_2) \to E^{p,q}(T(V)/T^{>k}(V), \bar{d}_2) \to E^{p,q}(T^k(V), 0).$$

Because $(T^k(V), 0)$ is, as a $(T(V), d_2)$-module, a direct sum of copies of lk concentrated in bidegree $(k, *)$, our hypothesis (2.5) implies $E^{p,q}(T^k(V), 0) = 0$, $p < m-k$. Step 7 follows by induction on k. □

§3. Gorenstein Hopf algebras

A Hopf algebra, H, is *Gorenstein* if for some pair of integers (n, k):

$$\mathrm{Ext}_H^{p,q}(\mathrm{lk}, H) \cong \begin{cases} \mathrm{lk} & (p, q) = (n, k) \\ 0 & \text{otherwise.} \end{cases}$$

We call (n, k) the *injective bidegree* of H; n is the *injective dimension* of H and k is the *internal injective dimension.*

In this section we establish

Proposition 3.1. *Let $H \subset G$ be a normal sub Hopf algebra and suppose H is Gorenstein of injective bidegree (n, k). Then*

$$\mathrm{Ext}_G^{p,q}(\mathrm{lk}, G) \cong \mathrm{Ext}_{G//H}^{p-n,q-k}(\mathrm{lk}, G//H).$$

For the proof we require three lemmas which apply to an arbitrary normal sub Hopf algebra.

Lemma 3.2. *Let $H \subset G$ be a normal sub Hopf algebra. Then for $g \in G_p$, $h \in H_q$.*

$$gh - (-1)^{pq} hg \in H_{>q} \cdot G.$$

Consequently, for any q, $H_{>q} G = G H_{>q}$.

Proof. Let E be a graded complement for the ideal $I = GH_+G$. Then the projection $\pi: G \to G/I = G//H$ maps $E \xrightarrow{\cong} G//H$. Multiplication in G restricts to an isomorphism $H \otimes E \xrightarrow{\cong} G$, and with the aid of this it is easy to see that it is enough to prove the lemma for $g \in E$.

Consider the map $\rho: G \xrightarrow{\Delta} G \otimes G \xrightarrow{\mathrm{id} \otimes \pi} G \otimes G//H = G \otimes E$. A short calculation shows that $\rho(gh) \equiv (-1)^{pq} h \otimes g \bmod (G_{>q} \otimes E)$. On the other hand, writing $gh = \Sigma h_i \otimes g_i (h_i \in H, g_i \in E)$ we find $\rho(gh) = \Sigma (h_i \otimes 1) \rho g_i$. Suppose $\deg h_i = s$ $(1 \leqq i \leqq t)$ and $\deg h_i > s (i > t)$. Then clearly $\rho(gh) \equiv \sum_1^t h_i \otimes g_i \bmod (G_{>s} \otimes E)$.

These two congruences for $\rho(gh)$ imply $s = q$, $t = 1$, $h_1 = h$ and $g_1 = (-1)^{pq} g$. The first assertion of the lemma follows.

The identical argument also shows that $gh-(-1)^{pq}hg\in G\cdot H_{>q}$. It follows that $h_{>q}G=GH_{>q}$. □

Again, suppose $H\subset G$ is an arbitrary normal sub Hopf algebra. Let

$$\xrightarrow{\partial} F(m)\xrightarrow{\partial} F(m-1)\xrightarrow{\partial}\dots\xrightarrow{\partial} H\xrightarrow{\varepsilon}\mathrm{lk}$$

be an H-free graded resolution of lk. As above, choose a graded complement, E, for GH_+G – thus $E\otimes H\xrightarrow{\cong} G$. In particular, this gives $(E\otimes F(*), \mathrm{id}\otimes\partial)\cong G\otimes_H(F(*),\partial)$, and so this complex is a G-free resolution of $G/\!/H$. Thus right multiplication in $G/\!/H$ by $g\in G$ lifts to an endomorphism of this G-complex which we denote by $\Omega\mapsto(\Omega)\tilde{g}$.

Lemma 3.3. *Suppose $g\in E_q$. Then $\tilde{g}$ can be chosen so that*

$$(v)\tilde{g}-(-1)^{q(p+m)}g\otimes v\in E_{<q}\otimes F(m),\qquad v\in F_p(m).$$

Proof. When $m=0$ we let $\tilde{g}$ be right multiplication by g in $G\otimes_H H=G$. The equation above is exactly Lemma 3.2.

Suppose the lemma established for $F(k)$, $k\leqq m$. Then for $v\in F_p(m+1)$,

$$\begin{aligned}(\partial v)\tilde{g}&=(-1)^{q(p+m)}g\otimes\partial v+\Omega\\&=(-1)^{q(p+m+1)}\partial(g\otimes v)+\Omega\qquad\text{some }\Omega\in E_{<q}\otimes F(m).\end{aligned}$$

Since $(E\otimes F(*),\partial)$ is a direct sum of the subcomplexes $E_i\otimes F(*)$, it follows that $\Omega=\partial\Omega'$, $\Omega'\in E_{<q}\otimes F(m+1)$. In particular, if v_i is an H-basis for $F(m+1)$ we have $(\partial v_i)\tilde{g}=(-1)^{q(p+m+1)}\partial(g\otimes v_i)+\partial\Omega_i'$, $\Omega_i'\in E_{<q}\otimes F(m+1)$, and we lift $\tilde{g}$ by setting

$$(v_i)\tilde{g}=(-1)^{q(p+m+1)}g\otimes v_i+\Omega_i'.$$

The equation above thus holds for the v_i. It follows at once from Lemma 3.2 that it holds for all $v\in F(m+1)$. □

With the notation above, we have an isomorphism

$$\mathrm{Hom}_G(G\otimes_H F(*), G)=\mathrm{Hom}_H(F(*), G),$$

which identifies

$$\mathrm{Ext}_G(G/\!/H, G)=\mathrm{Ext}_H(\mathrm{lk}, G).$$

The right $G/\!/H$-module structure in $G/\!/H$ makes $\mathrm{Ext}_G(G/\!/H, G)$ into a left $G/\!/H$-module.

On the other hand multiplication $H\otimes E\xrightarrow{\cong} G$ is an isomorphism of H-modules. If e_i is a basis for E this isomorphism induces an isomorphism

$$\mathrm{Hom}_H(-, G)\xrightarrow{\cong}\prod_i\mathrm{Hom}_H(-, H)\cdot e_i$$

whence we obtain in particular an identification

$$\mathrm{Ext}_H(\mathrm{lk}, G) \xrightarrow{\cong} \prod_i \mathrm{Ext}_H(\mathrm{lk}, H)\cdot e_i.$$

In particular, we have the inclusion

$$\mathrm{Ext}_H(\mathrm{lk}, H)\otimes E \subset \mathrm{Ext}_H(\mathrm{lk}, G) \tag{3.4}$$

which maps $\mathrm{Ext}_H(\mathrm{lk}, H)\otimes e_i$ to $\mathrm{Ext}_H(\mathrm{lk}, H)\cdot e_i$.

Lemma 3.5. *The left action of $G//H$ on $\mathrm{Ext}_H(\mathrm{lk}, G)$ restricts to an isomorphism*

$$A\colon G//H \otimes \mathrm{Ext}_H(\mathrm{lk}, H) \xrightarrow{\cong} \mathrm{Ext}_H(\mathrm{lk}, H)\otimes E \subset \mathrm{Ext}_H(\mathrm{lk}, G).$$

Proof. Let α denote the inclusion $\mathrm{Ext}_H(\mathrm{lk}, H) \to \mathrm{Ext}_H(\mathrm{lk}, H)\cdot 1 \subset \mathrm{Ext}_H(\mathrm{lk}, G)$. Then A is given by $\bar{g}\otimes\omega \mapsto \bar{g}\cdot\alpha(\omega)$. Represent $\alpha(\omega)$ by a G-linear map $\phi\colon G\otimes_H F(m) \to G$ such that $\phi\colon F(m)\to H$, and fix $g\in E_q$. Then $\bar{g}\cdot\alpha(\omega)$ is represented by the G-linear map $g\cdot\phi\colon G\otimes_H F(m)\to G$ given by

$$(g\cdot\phi)(v) = \pm\phi(v\cdot\tilde{g}), \qquad v\in F(m),$$

where $\tilde{g}$ is as described in Lemma 3.3. In particular it follows that

$$(g\cdot\phi)(v) = g\otimes\phi v + \Phi, \qquad \Phi\in E_{<q}\otimes H.$$

It follows now from Lemma 3.2 that

$$(g\cdot\phi)(v) = \pm\phi v\otimes g + \Phi', \qquad \Phi'\in H\otimes E_{<q}.$$

This shows that A restricts to linear maps.

$$A_q\colon (G//H)_q \otimes \mathrm{Ext}_H(\mathrm{lk}, H) \to \mathrm{Ext}_H(\mathrm{lk}, H)\otimes E_{\leq q}$$

such that $A_q(\bar{g}\otimes\omega) \equiv \pm\omega\otimes g \bmod \mathrm{Ext}_H(\mathrm{lk}, H)\otimes E_{<q}$. The lemma follows. □

Proof of Proposition 3.1. Since by hypothesis $\mathrm{Ext}_H(\mathrm{lk}, H)$ is one-dimensional the inclusion (3.4) is an isomorphism. Hence the injection A of Lemma 3.5 is also an isomorphism.

It follows that $\mathrm{Ext}_H(\mathrm{lk}, G)$ is a free $G//H$-module on a single generator of bidegree (n, k). The Hochschild-serre spectral sequence, converging from $\mathrm{Ext}_{G//H}(\mathrm{lk}, \mathrm{Ext}_H(\mathrm{lk}, G))$ to $\mathrm{Ext}_G(\mathrm{lk}, G)$ therefore collapses at E_2, and gives the isomorphism claimed in the proposition. □

§4. Solvable Hopf algebras

Let G be a Hopf algebra and write $[a, b] = ab - (-1)^{\deg a \deg b} ba$, $a, b\in G$. If H' and H'' are normal sub Hopf algebras then the ideal I generated by the elements

$[h', h'']$, $h' \in H'$, $h'' \in H''$ is a Hopf ideal. As in [9] we denote by $[H', H''] \subset H' \cap H''$ the Hopf algebra kernel of the projection $G \to G/I$.

In particular we have [9] the *derived series* $G[n]$ for G defined by $G[0] = G$ and $G[n+1] = [G[n], G[n]]$, and the *lower central series* $G(n)$ defined by $G(0) = G$ and $G(n+1) = [G, G(n)]$. G is called *solvable* of *solv length* n if $G[n+1] = \mathrm{lk}$ and $G[n] \neq \mathrm{lk}$; it is *nilpotent* of *nil length* n if $G(n+1) = \mathrm{lk}$ and $G(n) \neq \mathrm{lk}$.

Let $\mathscr{R}$ be the collection of normal solvable sub Hopf algebras of a Hopf algebra G. Since if $R', R'' \in R$ so does $R' \cdot R''$, it follows that

$$R = \bigcup_{R^\alpha \in \mathscr{R}} R^\alpha$$

is a normal sub Hopf algebra (not necessarily solvable).

Definition 4.1. R is the *radical* of the Hopf algebra G.

In this section we establish

Theorem 4.2. *Let R be the radical of a Hopf algebra G over a field* lk. *If depth $R = m < \infty$ then*

(i) *R is a nilpotent Gorenstein Hopf algebra, finitely generated as an algebra.*

(ii) *If* lk *is perfect then the commutative algebra* $\mathrm{gr}\, R = \bigotimes_{1 \geq 0} R(i)/\!/R(i+1)$ *is the tensor product of a polynomial algebra on m generators and a finite dimensional algebra, F:*

$$\mathrm{gr}\, R = \mathrm{lk}[x_1, \ldots, x_m] \otimes F.$$

(iii) *If G is a finitely generated algebra then G acts nilpotently on R;* i.e. *the sequence G, $[G, R]$, $[G, [G, R]]$, ... terminates at* lk *at at some finite stage.*

Remark 4.3. If depth $G < \infty$ then depth $R \leqq$ depth $G < \infty$, by the Hochschild-Serre spectral sequence. Thus Theorem 4.2 implies Theorem C of the introduction.

The following example shows that when lk has positive characteristic the hypothesis "G finitely generated" is necessary in part (iii) of Theorem 4.2.

Example 4.4. Fix variables $h, x_1, x_2, \ldots$ and a field lk of characteristic $p > 0$. Put $\deg h = 2$ and $\deg x_i = 2(p^i - 1)$. Let $T(h, x_1, x_2, \ldots)$ be the tensor algebra over lk and make it into a Hopf algebra by making h primitive and setting $\Delta x_n = \sum_{i+j=n} x_i \otimes x_j$ $(x_0 = 1)$.

A calculation shows that the ideal I generated by the elements $[x_i, h] - h^p \cdot x_{i-1}$, is a Hopf ideal. Put $G = T(h, x_1, x_2 \ldots)/I$. We observe that $\mathrm{gl\,dim}(G) = 2$, as follows at once from the results of Anick [1]. Thus Theorem A′ implies that depth $G \leqq 2 < \infty$. In fact it is easy to see that depth $G = 2$.

Now in G let H denote the polynomial algebra $\mathrm{lk}[h]$ and note that $H_+ G = G H_+$; i.e. H is a normal abelian sub Hopf algebra. On the other hand, by construction

$$h^{p^k} \in [\underbrace{G, [G, \ldots [G, H]}_{k \text{ factors}} \ldots]$$

and so G does not act nilpotently on H. □

We now turn to the proof of Theorem 4.2. We require three preliminary propositions.

Proposition 4.5. *Let R be a solvable Hopf algebra with depth $R=m<\infty$. Then R is Gorenstein and, if* lk *is perfect, there is an isomorphism of commutative algebras*

$$\bigotimes_{i\geqq 0} R[i]//R[i+1]\cong \mathrm{lk}[x_1,\ldots,x_m]\otimes F$$

with F an algebra of finite dimension. In particular, R is finitely generated as an algebra.

Proof. It is a classical and obvious fact that if A and A' are graded connected algebras of finite type then $\mathrm{Ext}^*_{A\otimes A'}(\mathrm{lk}, A\otimes A')=\mathrm{Ext}^*_A(\mathrm{lk}, A)\otimes \mathrm{Ext}^*_{A'}(\mathrm{lk}, A')$ and so depth $(A\otimes A')=$ depth $A+$ depth A'. There is, moreover an obvious isomorphism $\mathrm{Ext}^A_0(\mathrm{lk}, A)\cong\{a\in A\,|\,a\cdot A_+=0\}$.

From these observations it follows that if A is an infinite tensor product then depth $A=\infty$. Indeed clearly $\{a\in A\,|\,a\cdot A_+=0\}=0$ and so depth $A\geqq 1$. But then for any n, A can be written $A^{(1)}\otimes\ldots\otimes A^{(n)}$ with each $A^{(i)}$ itself an infinite tensor product; thus depth $A=\sum_i$ depth $A^{(i)}\geqq n$ for all n.

Now consider R. If R is abelian and lk is perfect a theorem of Borel [3] asserts that, as an algebra, R is the tensor product of algebras of the form $\mathrm{lk}[x]$, $\mathrm{lk}[x]/x^{p^k}(p=\mathrm{char}\,\mathrm{lk})$ and $\mathrm{lk}[x]/x^2$(if deg x is odd). Since depth $R<\infty$ this is a finite tensor product. A straightforward, classical calculation shows that the algebras $\mathrm{lk}[x]$, (resp. $\mathrm{lk}[x]/x^n$) are Gorenstein and of depth 1 (resp. 0), and the proposition follows in this case. For general lk we simply tensor with the algebraic closure $\bar{\mathrm{lk}}$ of lk and note that all the properties claimed in the proposition, except for the decomposition as a tensor product, descend from $R\otimes_{\mathrm{lk}}\bar{\mathrm{lk}}$ to R.

We now argue by induction on the solv length of R. Indeed, since $R[1]$ satisfies the conclusion of the proposition by induction, it is Gorenstein and we can apply Proposition 3.1 to get

$$\mathrm{Ext}^{r,q}_R(\mathrm{lk}, R)\cong \mathrm{Ext}^{r-n,q-k}_{R//R[1]}(\mathrm{lk}, R//R[1]).$$

It follows that depth $R[1]+$depth $R//R[1]=$depth R. In particular, $R//R[1]$ is an abelian Hopf algebra of finite depth and since the proposition is proved for these it follows in general. □

Next we need

Proposition 4.6. *Let A be a normal abelian sub Hopf algebra of a Hopf algebra G. Assume A and G are finitely generated as algebras. Then the sequence G, $[G, A]$, ... $[G, [G, \ldots [G, A]\ldots]$ terminates at* lk *at some finite stage.*

Proof. We may again tensor with the algebraic closure of lk to reduce to the case lk is perfect. *If* char $\mathrm{lk}=0$ then by Milnor-Moore [8], G is the enveloping algebra of a graded Lie algebra L and A is the enveloping algebra of an abelian

Lie ideal $I \subset L$. Because A is finitely generated I is finite dimensional and so the sequence $[L, I]$, $[L, [L, I]]$ ... is eventually zero.

Next suppose char $\mathrm{lk} = p > 2$. Put $A\{0\} = A$ and $A\{n\} = [G, A\{n-1\}]$. We show first that for n sufficiently large, $A\{n\}$ is zero in odd degrees. Indeed, because A is finitely generated, by the result of Borel [3] it is an exterior algebra E on finitely many generators of odd degree tensored with an algebra S such that $S_{\mathrm{odd}} = 0$. The ideal $E_+ \cdot A = E_+ \otimes S$ is obviously a Hopf ideal I and since $A/I \cong S$ it follows that the Hopf kernel K of $A \to A/I$ is isomorphic (as a graded space) with E.

But for some N, $E_{>N} = 0$. Choose n so that $A\{n\}_+$ is concentrated in degrees $> N$. Then $A\{n\} \cap K = \mathrm{lk}$ and so $A\{n\} \to A/\!/K \cong S$ is injective. Thus $A\{n\}_{\mathrm{odd}} = 0$. Since we may obviously replace A by $A\{n\}$ in proving the proposition we lose no generality in assuming $A_{\mathrm{odd}} = 0$, and we suppose this henceforth. (Note that as $A\{n\}$ is normal in A, depth $A\{n\} \leqq$ depth $A < \infty$, and so $A\{n\}$ is finitely generated by 4.4.)

Denote by ad $a: G \to G$ the map $g \mapsto [a, g] = ag - ga$, $a \in A$. According to Lemma 3.2, ad $a: G_q \to A \cdot G_{<q}$. Thus $(\mathrm{ad}\, a)^{n(q)}: G_q \to 0$. Let $g_1, \ldots, g_r$ be a finite set of generators for G, and let $a_1, \ldots, a_s$ be a finite set of generators for A. We now have that for some integer k, $(\mathrm{ad}\, a_i)^{p^k}(g_j) = 0$ for all i, j.

On the other hand, according to [9; Proposition 2.9] $(\mathrm{ad}\, a)^p = \mathrm{ad}\,(a^p)$ and $(A)^p = \mathrm{im}\{a \mapsto a^p\}$ is a sub Hopf algebra. It follows that $A' = (A)^{p^k}$ is a central sub Hopf algebra of G. Since A is finitely generated $A/\!/A'$ is finite dimensional, and so for large enough n, $A\{n\} \subset A'$. Since A' is central $A\{n+1\} = \mathrm{lk}$.

Finally, if char $\mathrm{lk} = 2$ the same argument applies except that we may treat the odd elements the same as even elements, so that we need not eliminate them as we did for $p > 2$. □

Finally, we need the

Proposition 4.7. *If a Hopf algebra, R, of finite depth is the union of normal solvable sub Hopf algebras then R is solvable.*

Proof. Let N be a normal solvable sub Hopf algebra of maximal depth. By Proposition 4.5 N is Gorenstein and so it follows from Proposition 3.1 that depth $R/\!/N =$ depth R-depth N, and that for any normal solvable sub Hopf algebra $N' \supset N$, depth $N'/\!/N =$ depth N'-depth N. Since depth N is maximal, this implies that depth $N'/\!/N = 0$. By Proposition 4.5, $N'/\!/N$ is finite dimensional.

Thus $R/\!/N$ is the union of finite dimensional normal sub Hopf algebras, and depth $R/\!/N$ is finite. Now let A denote the union of the abelian normal sub Hopf algebras of $R/\!/N$. It is an abelian normal sub Hopf algebra of $R/\!/N$ and hence finite dimensional. Let $n = \max\{i \mid A_i \neq 0\}$.

Let S be any normal sub Hopf algebra of $R/\!/N$ of solv length k. Then $S[k]$ is a normal abelian sub Hopf algebra, and $S[k]_j = 0$, $1 \leqq j < 2^k$. Thus $k < \log_2 n$. Since the solv lengths of all the normal sub Hopf algebras of $R/\!/N$ are bounded by $\log_2 n$ it follows that $R/\!/N$ is itself solvable. Thus R is solvable. □

Proof of 4.2. According to Proposition 4.7 R is solvable. By Proposition 4.4 it is therefore Gorenstein, and finitely generated as an algebra.

Next we show that if G is a finitely generated algebra, then the sequence $R\{n\}$ defined by $R\{0\} = R$ and $R\{n+1\} = [G, R\{n\}]$ satisfies $R\{n\} = \mathrm{lk}$, some

n. Indeed since $R[1]$ is normal in G we can consider the sequence $(R/\!/R[1])\{n\}$ in $G/\!/R[1]$.

Because R and G are finitely generated algebras so are $G/\!/R[1]$ and $R/\!/R[1]$. Thus Proposition 4.5 shows that for some n, $(R/\!/R[1])\{n\}=\mathrm{lk}$; i.e. $R\{n\}\subset R[1]$. An obvious iteration of this argument completes the proof that some $R\{n\}=\mathrm{lk}$.

In particular we can apply this assertion with $G=R$ to conclude that R itself is nilpotent. Thus

$$\mathrm{gr}\, R=\bigotimes_{i\geqq 0} R(i)/\!/R(i+1)$$

is a finite tensor product, and if lk is perfect the number of polynomial variables in $R(i)/\!/R(i+1)$ is just depth $R(i)-\mathrm{depth}\, R(i+1)$. It follows that $\tilde{R}=\mathrm{lk}[x_1, \ldots, x_m]\otimes F$ with $m=\mathrm{depth}\, R$ and F finite dimensional. □

§5. Finite-stage Postnikov decompositions

An n-cone has LS category $\leqq n$. Thus Theorem D of the introduction follows from

Theorem 5.1. *Suppose X is simply connected,* cat X *is finite and for some prime $p>0$ each $H_i(X;\mathbb{Z}_p)$ is finite dimensional. If X admits an n-stage Postnikov decomposition at p then*

$$H_i(X;\mathbb{Z}_p)=0, \qquad i\geqq 1.$$

Proof. Because X admits a finite Postnikov decomposition at p, a theorem of Moore and Smith [9; Theorem 6.2] asserts that $H=H_*(\Omega X;\mathbb{Z}_p)$ is solvable and that for each $x\in H_+$, $x^{n(x)}=0$.

On the other hand, because cat X is finite, Theorem A implies that so is depth H. Since H is solvable it is its own radical, and so Theorem B implies that each $H(i)/\!/H(i+1)$ is a polynomial algebra tensored with a finite dimensional algebra. Since in H_+ each $x^{n(x)}=0$ this is also true in $H(i)/\!/H(i+1)$; it follows that $H(i)/\!/H(i+1)$ is finite dimensional. Moreover, Theorem B asserts that some $H(i)=\mathrm{lk}$ and so H itself is finite dimensional.

Finally, Theorem 4.2 asserts that H is Gorenstein. Since $H_*(\Omega X;\mathbb{Z}_p)$ is a Gorenstein algebra, [5, Proposition 3.2 (iv)] asserts that X is Gorenstein over $\mathbb{Z}_p$, and since cat X is finite [5, Theorem 3.1 (iii)] asserts that $H_*(X;\mathbb{Z}_p)$ has finite dimension.

Thus both $H_*(X;\mathbb{Z}_p)$ and $H_*(\Omega X;\mathbb{Z}_p)$ are finite dimensional, and the classical corner argument on the Serre spectral sequence for the fibration $\Omega X\to PX\to X$ shows that $H_i(X;\mathbb{Z}_p)=0$, $i\geqq 1$. □

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Oblatum 3-III-1988