Invent. math. 93, 35–76 (1988)

Inventiones mathematicae

Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups

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Summary. The notion of concrete monoidal W^*-category is introduced and investigated. A generalization of the Tannaka-Krein duality theorem is proved. It leads to new examples of compact matrix pseudogroups. Among them we have twisted $SU(N)$ groups denoted by $S_\mu U(N)$. It is shown that the representation theory for $S_\mu U(N)$ is similar to that of $SU(N)$: irreducible representations are labeled by Young diagrams and formulae for dimensions and multiplicity are the same as in the classical case.

0. Introduction

Two notions play the central role in this paper. The first one is the notion of compact matrix pseudogroup investigated in [8], the second introduced later in the next Section is the notion of concrete monoidal W^*-category. The two notions are linked in the following way. For any compact matrix pseudogroup G, the category of all unitary representations of G endowed with its natural structure is a concrete monoidal W^*-category. It satisfies certain conditions. All concrete monoidal W^*-categories satisfying these conditions can be obtained in this way. The last statement is a generalization of the Tannaka-Krein duality theorem. In special cases this result combined with Prop 2.4 and Theorem 1.5 of [8] gives the classical Tannaka-Krein theorem.

The generalized Tannaka-Krein duality gives us a large class of examples of compact matrix pseudogroups. The point is that there exists a procedure producing concrete monoidal W^*-categories which corresponds to the well known from the elementary algebra method of construction of semigroups (monoid) starting with given generators and relations. We shall describe this procedure in a special case leading to twisted $SU(N)$ groups.

The paper is organized in the following way. In Sect. 1 we remind the definition of compact matrix pseudogroup, list the properties of unitary representations and introduce the notion of concrete monoidal W^*-category. Then we

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formulate the duality theorem and give some applications. At this moment twisted $SU(N)$ groups are introduced. The proof of the duality theorem is presented in Sect. 3. Section 2 contains elementary facts concerning concrete monoidal W^*-categories that are used in the main proof. In Sect. 4 we investigate twisted $SU(N)$ groups and prove that their representation theory is fully similar to that of $SU(N)$.

Working with categories in a not completely trivial way one meets the logical problems related to the fact that there is no set containing all the objects of the category. In the context of the present paper these problems seem not to be serious. The objects related to any fixed Hilbert space form a set and we can restrict our considerations to Hilbert spaces belonging to some Hilbert Space Universe (i.e. to a set of Hilbert spaces containing all $\mathbb{C}^N$ and closed under direct sum, tensor product and passing to a subspace operations, cf. the Grothendieck notion of universal set [1]).

In the paper we use $\textcircled{\perp}$ and $\textcircled{\top}$ products introduced in [7] and [8]. Let A be an algebra and K, L be $f-d$. (finite dimensional) complex vector spaces. We remind that

$$\textcircled{\perp}: (B(K)\otimes A)\times(B(K)\otimes A)\to B(K)\otimes A\otimes A$$
$$\textcircled{\top}: (B(K)\otimes A)\times(B(L)\otimes A)\to B(K\otimes L)\otimes A$$

are bilinear mappings such that

$$(m_1\otimes a)\textcircled{\perp}(m_2\otimes b)=m_1m_2\otimes a\otimes b$$
$$(m\otimes a)\textcircled{\top}(n\otimes b)=m\otimes n\otimes ab$$

for any $m_1, m_2, m\in B(K)$, $n\in B(L)$ and $a, b\in A$.

Assume that A is unital and that v and w belonging to $B(L)\otimes A$ and $B(K)\otimes A$ resp. are invertible. One can easily check that $v\textcircled{\top}w$ is invertible and

$$(v\textcircled{\top}w)^{-1}=(\tau\otimes\mathrm{id})(w^{-1}\textcircled{\top}v^{-1}) \tag{0.1}$$

where τ: $B(K\otimes L)\to B(L\otimes K)$ is the C^*-isomorphism such that $\tau(m\otimes n)=n\otimes m$ for any $m\in B(K)$ and $n\in B(L)$.

1. Definitions and main results

Let A be a C^*-algebra with unity. The set $M_N(A)$ of all $N\times N$ matrices with entries belonging to A will be identified with $B(\mathbb{C}^N)\otimes A$. Let $u=(u_{kl})_{kl=1,2,\ldots,N}$ be such a matrix. We remind (cf. [8]) that the pair $G=(A,u)$ is said to be a compact matrix pseudogroup if the following three conditions are satisfied.

CMP I. The *-subalgebra $\mathscr{A}$ generated by matrix elements of u is dense in A.

CMP II. There exists C^*-algebra homomorphism

$$\phi:\ A\to A\otimes A$$

such that

$$(\mathrm{id}\otimes\phi)\,u = u \textcircled{T} u. \tag{1.1}$$

CMP III. u is invertible and there exists linear antimultiplicative mapping

$$\kappa\colon \mathscr{A}\to\mathscr{A}$$

such that $\kappa(\kappa(a^*)^*)=a$ for all $a\in\mathscr{A}$ and

$$(\mathrm{id}\otimes\kappa)\,u=u^{-1}.$$

It is known that ϕ and κ (called comultiplication and coinverse associated with G) are uniquely determined.

Any compact matrix pseudogroup $G=(A,u)$ gives rise to an interesting category denoted by Rep G. Objects of this category are finite-dimensional unitary representations of G, morphisms are intertwining operators. Let us remind that v is called a unitary representation of G acting on a $f-d$. Hilbert space H_v if v is a unitary element of the C^*-algebra $B(H_v)\otimes A$ such that

$$(\mathrm{id}\otimes\phi)\,v = v \textcircled{T} v. \tag{1.2}$$

In this case we write $v\in\mathrm{Rep}\,G$.

Comparing (1.1) and (1.2) we see that $u\in\mathrm{Rep}\,G$ (in general u is not a unitary element of $M_N(A)$, however in virtue of Theorem 5.2 of [8] one can always modify the scalar product in $\mathbb{C}^N$ in such a way that u becomes a unitary element of $B(\mathbb{C}^N)\otimes A$). We say that u is the fundamental representation of G. Another example of unitary representation acting on a $f-d$. Hilbert space H we get by setting $v=I_{B(H)}\otimes I$. Clearly condition (1.2) is satisfied. In this case v is called the trivial representation.

For any $v,w\in\mathrm{Rep}\,G$, the set of intertwining operators is introduced by the formula

$$\mathrm{Mor}(v,w)=\{t\in B(H_v,H_w)\colon\ w(t\otimes I)=(t\otimes I)\,v\}. \tag{1.3}$$

Clearly, $\mathrm{Mor}(v,w)$ is a linear subspace of $B(H_v,H_w)$. Moreover, taking into account the results of [8] one can easily see that $R=\mathrm{Rep}\,G$ has the following properties:

CMW* I. For any $r\in R$, the identity operator acting on H_r (denoted later by I_r) belongs to $\mathrm{Mor}(r,r)$.

CMW* II. For any $r,r',r''\in R$, $a\in B(H_r,H_{r'})$ and $b\in B(H_{r'},H_{r''})$

$$\begin{pmatrix} a\in\mathrm{Mor}(r,r')\\ b\in\mathrm{Mor}(r',r'')\end{pmatrix}\Rightarrow(b\,a\in\mathrm{Mor}(r,r'')).$$

CMW* III. For any $r,s\in R$ and $a\in B(H_r,H_s)$

$$(a\in\mathrm{Mor}(r,s))\Rightarrow(a^*\in\mathrm{Mor}(s,r)).$$

CMW* IV. For any $r,s\in R$

$$\begin{pmatrix} H_r=H_s\\ I_r\in\mathrm{Mor}(r,s)\end{pmatrix}\Rightarrow(r=s).$$

CMW* V. For any $r \in R$ and any unitary V mapping H_r onto a Hilbert space K there exists $s \in R$ such that $H_s = K$ and $V \in \mathrm{Mor}(r, s)$. We say that s is equivalent to r.

CMW* VI. For any $r \in R$ and any orthogonal projection $p \in \mathrm{Mor}(r, r)$ there exists $s \in R$ such that $H_s = p H_r$ and $i \in \mathrm{Mor}(s, r)$, where i denotes the embedding $H_s \to H_r$. We say that s is a subobject of r.

CMW* VII. For any $r, r' \in R$ there exists $s \in R$ such that $H_s = H_r \oplus H_{r'}$ and the canonical embeddings $H_r \to H_r \oplus H_{r'}$ and $H_{r'} \to H_r \oplus H_{r'}$ belong to $\mathrm{Mor}(r, s)$ and $\mathrm{Mor}(r', s)$ resp. We say that s is a direct sum of r and r': $s = r \oplus r'$.

If v and w are unitary representations of G acting on $f-d$. Hilbert spaces H_v and H_w, then $v \mathbin{\text{Ⓣ}} w$ is a unitary representation of G acting on $H_v \otimes H_w$. This is the tensor product of representations introduced in [8]. Therefore $R = \mathrm{Rep}\, G$ is equipped with the binary operation

$$R \times R \ni (v, w) \to v\, w \overset{\mathrm{df}}{=} v \mathbin{\text{Ⓣ}} w \in R.$$

In virtue of [8] this operation has the following properties.

CMW* VIII. For any $r, s \in R$, $H_{rs} = H_r \otimes H_s$. If $a \in \mathrm{Mor}(r, r')$ and $b \in \mathrm{Mor}(s, s')$ (where $r, s, r', s' \in R$) then $a \otimes b \in \mathrm{Mor}(r\, s, r'\, s')$.

CMW* IX. For any $p, r, s \in R$: $(p\, r)\, s = p(r\, s)$ (we always identify $(H_p \otimes H_r) \otimes H_s$ with $H_p \otimes (H_r \otimes H_s)$).

CMW* X. There exists $\mathbb{1} \in R$ such that $H_{\mathbb{1}} = \mathbb{C}$ and $\mathbb{1} r = r \mathbb{1} = r$ for any $r \in R$ (we always identify $\mathbb{C} \otimes H_r$ and $H_r \otimes \mathbb{C}$ with H_r).

One can easily see that $\mathbb{1}$ is uniquely determined. Clearly, it coincides with the trivial representation acting on $\mathbb{C}$.

The properties listed above play the important role in our theory. We shall consider the following structure:

$$R = (R, \{H_r\}_{r \in R}, \{\mathrm{Mor}(r, s)\}_{r, s \in R})$$

where R is a class (members of R will be called objects), $\{H_r\}_{r \in R}$ is a family of $f-d$. Hilbert spaces indexed by R and for any $r, s \in R$, $\mathrm{Mor}(r, s)$ is a linear subspace of $B(H_r, H_s)$.

R is called a concrete W^*-category if conditions CMW* I–IV are fulfilled. If in addition conditions CMW* V–VII hold then R is said to be complete. One can prove the following

Proposition 1.1. *A concrete W^*-category $(R, \{H_r\}_{r \in R}, \{\mathrm{Mor}(r, s)\}_{r, s \in R})$ is complete if and only if the following condition is satisfied:*

For any $f-d$. Hilbert space H and any finite family $\{b_k\}_{k \in \Delta}$ of linear mappings: $b_k \in B(H_{r_k}, H)$ where $r_k \in R$, $k \in \Delta$ such that $b_k^* b_l$ belong to $\mathrm{Mor}(r_l, r_k)$ for all $k, l \in \Delta$ and $\sum_k b_k b_k^* = I_{B(H)}$, there exists $s \in R$ such that $H_s = H$ and $b_k \in \mathrm{Mor}(r_k, s)$ for all $k \in \Delta$.

A concrete monoidal W^*-category is a concrete W^*-category R endowed with a binary operation

$$R \times R \ni (r, s) \to r\, s \in R$$

satisfying the conditions CMW* VIII–X.

The notions of W^*-category and monoidal W^*-category in the most general version were introduced by J. Roberts (see e.g. [3]). We refer to this paper for the most elementary properties of W^*-categories. The adjective "concrete" is added in order to stress that each object of R is related to some concrete Hilbert space. In general these Hilbert spaces may be infinite-dimensional (and then $\mathrm{Mor}(r, s)$ have to be weakly closed, cf. [6]), in this paper however we always assume that $\dim H_r < \infty$ for all $r \in R$.

Let H and H' be $f-d$. Hilbert spaces, $\dim H = \dim H'$ and j: $H \to H'$ be an invertible antilinear mapping. Any such mapping defines in a canonical way two linear mappings

$$t_j\colon\ \mathbb{C} \to H \otimes H' \tag{1.4}$$

$$\bar{t}_j\colon\ H' \otimes H \to \mathbb{C} \tag{1.5}$$

that will play the important role throughout this paper.

The mapping t_j is determined uniquely by its value at the point $1 \in \mathbb{C}$. We set

$$t_j(1) = \sum_i e_i \otimes j(e_i) \tag{1.6}$$

where (e_i) is an orthonormal basis in H (notice that the right hand side of (1.6) is independent of the particular choice of the basis).

In order to introduce $\bar{t}_j$ we notice that the mapping $H' \otimes H \ni (x', x) \to (j^{-1}(x') \mid x) \in \mathbb{C}$ is bilinear. Therefore there exists a unique functional (1.5) such that

$$\bar{t}_j(x' \otimes x) = (j^{-1}(x') \mid x) \tag{1.7}$$

for any $x' \in H'$ and $x \in H$.

Assume that

$$R = (R, \{H_r\}_{r \in R}, \{\mathrm{Mor}(r, s)\}_{r, s \in R}, \cdot) \tag{1.8}$$

is a concrete monoidal W^*-category (the dot $\cdot$ denotes the binary operation). Let $r, \bar{r} \in R$. We say that $\bar{r}$ is a complex conjugation of r if there exists an invertible antilinear mapping j: $H_r \to H_{\bar{r}}$ such that

$$t_j \in \mathrm{Mor}(\mathbb{1}, r\,\bar{r}) \tag{1.9}$$

and

$$\bar{t}_j \in \mathrm{Mor}(\bar{r}\, r, \mathbb{1}). \tag{1.10}$$

It is not difficult to show that a complex conjugation of any object of R (if it exists) is uniquely determined (up to equivalence). For details see Sect. 2. Moreover, one can easily check that $t_{j^{-1}} = (\bar{t}_j)^*$ and $\bar{t}_{j^{-1}} = (t_j)^*$. Therefore, the relation "to be a complex conjugation of" is symmetric.

In Sect. 2, we shall prove that for the category Rep G (where G is a compact matrix pseudogroup) the complex conjugation introduced above essentially coincides with the complex conjugation considered in [8].

Assume again that (1.8) is a concrete monoidal W^*-category. Let Q be a finite subset of R. We say that Q generates R if for any $s \in R$ there exists a finite family $\{b_k\}$ of morphisms: $b_k \in \mathrm{Mor}(r_k, s)$ where r_k is a product of a sequence of elements of Q such that $\sum_k b_k b_k^* = I_s$.

It is known (cf. [8]) that any unitary representation of a compact matrix pseudogroup G can be decomposed into the direct sum of irreducible representations. On the other hand any irreducible representation is contained in the tensor product of sufficiently many copies of the fundamental representation f and its complex conjugation $\bar{f}$. Therefore $\{f, \bar{f}\}$ generates $\mathrm{Rep}\, G$.

Summarizing we have the following theorem (it is essentially contained in [8]):

Theorem 1.2. *Let G be a compact matrix pseudogroup. Then the class of all finite-dimensional unitary representations* $\mathrm{Rep}\, G$ *endowed with its natural structure (morphisms are intertwining operators, the binary operation is the tensor product) is a complete concrete monoidal W^*-category. For any $r \in \mathrm{Rep}\, G$ there exists the complex conjugation $\bar{r} \in \mathrm{Rep}\, G$. The fundamental representation f is the distinguished object of* $\mathrm{Rep}\, G$. *$\{f, \bar{f}\}$ generate* $\mathrm{Rep}\, G$.

The main result of this paper states that any complete concrete monoidal W^*-category R containing a distinguished object f such that a complex conjugation $\bar{f}$ exists and $\{f, \bar{f}\}$ generates R is isomorphic to $\mathrm{Rep}\, G$, where G is a compact matrix pseudogroup. To formulate this result in a more concrete and precise way we have to introduce a few notions.

Let $R = (R, \{H_r\}_{r \in R}, \{\mathrm{Mor}(r, s)\}_{r, s \in R}, \cdot, f)$ be a concrete monoidal W^*-category with the distinguished object $f \in R$. We always assume that the Hilbert space H_f is one of the arithmetic spaces $\mathbb{C}^N$ (although the scalar product in H_f need not to coincide with the canonical one).

We shall consider pairs of the form $M = (B, \{v^r\}_{r \in R})$ where B is a C^*-algebra with unity I and $\{v^r\}_{r \in R}$ is a family of unitaries: for each $r \in R$, $v^r \in B(H_r) \otimes B$.

We say that M is a model of R if

$$v^{rs} = v^r \textcircled{\scriptsize T}\, v^s \tag{1.11}$$

$$v^r (t \otimes I) = (t \otimes I)\, v^s \tag{1.12}$$

for any $r, s \in R$ and $t \in \mathrm{Mor}(s, r)$.

Assume that $\bar{f}$ exists and that $\{f, \bar{f}\}$ generates R. Then (cf. Sect. 2) any model $M = (B, \{v^r\}_{r \in R})$ is uniquely determined by B and v^f. This fact leads to the following notion.

Let (B, v) be a pair such that B is a C^*-algebra with unity and v is a unitary element of $B(H_f) \otimes B$. We say that (B, v) is a R-admissible pair if there exists a model $M = (B, \{v^r\}_{r \in R})$ of R such that $v^f = v$.

We say that (A, u) is a universal R-admissible pair if the following conditions are satisfied

1. A is the smallest C^*-algebra containing the matrix elements of u.
2. (A, u) is R-admissible.

3. For any R-admissible pair (B, v) there exists C^*-homomorphism

$$\varphi: A \to B$$

such that $(\mathrm{id} \otimes \varphi) u = v$.

Clearly the universal R-admissible pair is defined uniquely (up to the identity relation introduced in [8]). The existence of the universal R-admissible pair can easily be established. Indeed, let $\mathbb{C}[\![\tau]\!]$ be the free (non-commutative) *-algebra generated by N^2 elements τ_{kl} $(k, l = 1, 2, \ldots, N)$. A *-representation π of $\mathbb{C}[\![\tau]\!]$ acting on a Hilbert space H is said to be admissible if $v = (\pi(\tau_{kl}))_{k,l=1,2,\ldots,N}$ is a unitary element of $B(H_f) \otimes B(H)$ and $(B(H), v)$ is a R-admissible pair. For any $a \in \mathbb{C}[\![\tau]\!]$ we set

$$\|a\| = \sup \|\pi(a)\| \tag{1.13}$$

where π runs over the set of all admissible representations of $\mathbb{C}[\![\tau]\!]$. Clearly $\|a\|$ is finite for any $a \in \mathbb{C}[\![\tau]\!]$ and $\|\cdot\|$ is a C^*-seminorm.

Therefore the set

$$N = \{a \in \mathbb{C}[\![\tau]\!] : \|a\| = 0\}$$

is a two-sided ideal in $\mathbb{C}[\![\tau]\!]$ and the seminorm (1.13) produces a C^*-norm on the quotient algebra

$$\mathscr{A} = \mathbb{C}[\![\tau]\!]/N.$$

Let A be the completion of $\mathscr{A}$ with respect to this norm, φ be the canonical mapping $\mathbb{C}[\![\tau]\!] \to \mathscr{A} = \mathbb{C}[\![\tau]\!]/N$ and $u = (\pi(\tau_{kl}))_{k,l=1,2,\ldots,N}$. Then (A, u) is the universal R-admissible pair. Another construction of the universal R-admissible pair can be found in Sect. 3.

The main result of the paper is contained in the following theorem

Theorem 1.3. *Let $R = (R, \{H_r\}_{r \in R}, \{\mathrm{Mor}(r, s)\}_{r, s \in R}, \cdot, f)$ be a concrete monoidal W^*-category with the distinguished object $f \in R$. Assume that $\bar{f}$ exists and that $\{f, \bar{f}\}$ generates R.*

Let $G = (A, u)$ be the universal R-admissible pair and $(A, \{u^r\}_{r \in R})$ be the model of R such that $u^f = u$. Then

1. *G is a compact matrix pseudogroup.*

2. *For any $r \in R$, u^r is a unitary representation of G acting on H_r. If R is complete then any $f-d$. unitary representation of G can be obtained in this way.*

3. *For any $r, s \in R$ and $t \in B(H_r, H_s)$: $(t \otimes I) u^r = u^s (t \otimes I)$ if and only if $t \in \mathrm{Mor}(r, s)$.*

4. *For any $r, s \in R$*

$$u^{rs} = u^r \textcircled{T} u^s.$$

Let

$$E_- = (E_{i_1, i_2, \ldots, i_N})_{i_1, i_2, \ldots, i_N = 1, 2, \ldots, N} \tag{1.14}$$

be N^N -element array of complex numbers. Fixing a number $k \in \{1, 2, \ldots, N\}$ and inserting in (1.14) $i_1 = k$ we obtain a N^{N-1}-element array denoted by E_{k-}. We say that (1.14) is left non-degenerate if arrays $E_{1-}, E_{2-}, \ldots, E_{N-}$ are linearly independent. Similarly inserting in (1.14) $i_N = k$ we get an N^{N-1}-element array denoted by E_{-k}. (1.14) is right non-degenerate if the arrays $E_{-1}, E_{-2}, \ldots, E_{-N}$ are linearly independent.

Using our main theorem we shall prove

Theorem 1.4. *Let A be the universal C^*-algebra generated by N^2 elements u_{kl} $(k, l=1, 2, \ldots, N)$ such that*

$$\sum_k u_{kl}^* u_{km} = \delta_{lm} I \tag{1.15}$$

$$\sum_k u_{mk} u_{lk}^* = \delta_{lm} I \tag{1.16}$$

$$\sum_{k_1, k_2, \ldots, k_N} u_{l_1 k_1} u_{l_2 k_2} \ldots u_{l_N k_N} E_{k_1 k_2 \ldots k_N} = E_{l_1 l_2 \ldots l_N} I \tag{1.17}$$

where $l, m, l_1, \ldots, l_N = 1, 2, \ldots, N$ and δ_{lm} is the Kronecker symbol. Let $u = (u_{kl})_{kl=1,2,\ldots,N}$.

Assume that E_- is left and right non-degenerate. Then $G=(A, u)$ is a compact matrix pseudogroup.

Equations (1.15) and (1.16) say that u is unitary. The relation (1.17) is called the twisted unimodularity condition.

Unfortunately, unless E_- is chosen in a very special way, the pseudogroup G introduced in Theorem 1.4 is very small (in the generic case A is an N-dimensional commutative C^*-algebra and G reduces to the N-element cyclic group).

However (cf. [2], formulae (16)–(19)), fixing a number $\mu \in]0, 1]$ and putting

$$E_{i_1, i_2 \ldots i_N} = \begin{cases} 0 & \text{if } i_k = i_l \text{ for some } k, l \\ (-\mu)^{I(i_1 i_2 \ldots i_N)} & \text{otherwise} \end{cases} \tag{1.18}$$

where $I(i_1, i_2, \ldots, i_N)$ denotes the number of inversed pairs in the sequence $(i_1, i_2, \ldots, i_N)$, we obtain an interesting pseudogroup. It will be denoted by $S_\mu U(N)$ and called the twisted $SU(N)$ group. For $N=2$ and 3 we obtain $S_\mu U(2)$ and $S_\mu U(3)$ considered in [7] and [8]. For $\mu=1$, $S_\mu U(N)$ coincides with the classical $SU(N)$ group.

In [7] we proved that the representation theory for $S_\mu U(2)$ is very similar to that of $SU(2)$. The following theorem generalizes this result:

Theorem 1.5. *Irreducible representations of $S_\mu U(N)$ are labeled by Young diagrams consisting at most of N rows. Let u_d be the irreducible representation corresponding to the Young diagram d. Any irreducible representation of $S_\mu U(N)$ is equivalent to one of the u_d. u_d is equivalent to $u_{d'}$ if and only if d' can be obtained from d by adding (or subtracting) a number of full* (i.e. *consisting of N boxes) columns. The one box Young diagram corresponds to the fundamental representation of $S_\mu U(N)$. The dimension of u_d and the multiplicity of u_d in $u_{d'} \textcircled{T} u_{d''}$ are given by the same formulae as in the classical* (i.e. *$SU(N)-$) case.*

2. Concrete monoidal W^*-categories

In this section we collect the results concerning concrete monoidal W^*-categories that will be used in the proof of our main theorem. We mainly deal with the notion of complex conjugate object. Throughout this section

$$R=(R, \{H_r\}_{r\in R}, \{\mathrm{Mor}(r,s)\}_{r,s\in R}, \cdot)$$

is a concrete monoidal W^*-category. We start with the following

Proposition 2.1. *Let $r,s\in R$; $\bar r$, $\bar s$ be their complex conjugations and j_r: $H_r\to H_{\bar r}$, j_s: $H_s\to H_{\bar s}$ be the corresponding invertible antilinear mappings. Then for any $a\in\mathrm{Mor}(r,s)$*

$$j_s\, a\, j_r^{-1}\in\mathrm{Mor}(\bar r,\bar s). \tag{2.1}$$

Proof. One can easily check (the details are left to the reader) that for any $a\in B(H_r,H_s)$:

$$j_s\, a\, j_r^{-1}=(\bar t_{j_r}\otimes I_{\bar s})(I_{\bar r}\otimes a^*\otimes I_{\bar s})(I_{\bar r}\otimes t_{j_s}) \tag{2.2}$$

and for $a\in\mathrm{Mor}(r,s)$ we immediately get (2.1). □

Two objects $r,s\in R$ are said to be equivalent if $\dim H_r=\dim H_s$ and $\mathrm{Mor}(r,s)$ contains an invertible element (this is the case if and only if $\mathrm{Mor}(r,s)$ contains a unitary element). Proposition 2.1 shows that the complex conjugations of equivalent objects are equivalent. In particular complex conjugation of any object is unique up to equivalence.

In the following Proposition we denote by R_{cc} the subclass of objects that admit complex conjugation. For any $r\in R_{cc}$, $\bar r$ will denote a complex conjugation of r. The corresponding invertible antilinear mapping will be denoted by j_r. It maps H_r onto $H_{\bar r}$.

Proposition 2.2. *Assume that R is complete. Then*

1. *R_{cc} is closed under the binary operation: for any $r,s\in R_{cc}$, $\bar s\bar r$ is a complex conjugation of rs.*
2. *R_{cc} is closed under direct sum: for any $r,s\in R_{cc}$, $\bar r\oplus\bar s$ is a complex conjugation of $r\oplus s$.*
3. *If $r\in R_{cc}$, $s\in R$ and there exists $b\in\mathrm{Mor}(r,s)$ such that $bb^*=I_s$ then $s\in R_{cc}$.*

Proof. Ad 1. Assume that $r,s\in R_{cc}$. Let σ: $H_{\bar r}\otimes H_{\bar s}\to H_{\bar s}\otimes H_{\bar r}$ be unitary mapping such that $\sigma(x\otimes y)=y\otimes x$ for all $x\in H_{\bar r}$ and $y\in H_{\bar s}$ and $j=\sigma(j_r\otimes j_s)$. Clearly, j: $H_{rs}\to H_{\bar s\bar r}$ is an invertible antilinear mapping. One can easily check (the details are left to the reader) that

$$t_j=(I_r\otimes t_{j_s}\otimes I_{\bar r})\,t_{j_r}\in\mathrm{Mor}(\mathbb{1}, rs\bar s\bar r)$$
$$\bar t_j=\bar t_{j_s}(I_{\bar s}\otimes\bar t_{j_r}\otimes I_s)\in\mathrm{Mor}(\bar s\bar r rs,\mathbb{1}).$$

Therefore $\bar s\bar r$ is a complex conjugation of rs.

Ad 2. Assume that $r,s\in R_{cc}$. We denote by i_r and i_s the canonical embeddings of H_r and H_s into $H_r\oplus H_s$. Similarly, $i_{\bar r}$ and $i_{\bar s}$ will denote the canonical embeddings of $H_{\bar r}$ and $H_{\bar s}$ into $H_{\bar r}\oplus H_{\bar s}$. We know (cf. CMW* VII) that $i_r\in\mathrm{Mor}(r,r\oplus s)$, $i_s\in\mathrm{Mor}(s,r\oplus s)$, $i_{\bar r}\in\mathrm{Mor}(\bar r,\bar r\oplus\bar s)$ and $i_{\bar s}\in\mathrm{Mor}(\bar s,\bar r\oplus\bar s)$.

Let $j=j_r\oplus j_s$. Clearly, j: $H_{r\oplus s}\to H_{\bar r\oplus\bar s}$ is an invertible antilinear mapping. One can easily check that

$$t_j=(i_r\otimes i_{\bar r})\,t_{j_r}+(i_s\otimes i_{\bar s})\,t_{j_s}\in\mathrm{Mor}(\mathbb{1},(r\oplus s)(\bar r\oplus\bar s))$$
$$\bar t_j=\bar t_{j_r}(i_{\bar r}^*\otimes i_r^*)+\bar t_{j_s}(i_{\bar s}^*\otimes i_s^*)\in\mathrm{Mor}((\bar r\oplus\bar s)(r\oplus s),\mathbb{1}).$$

Therefore $\bar r\oplus\bar s$ is a complex conjugation of $r\oplus s$.

Ad 3. Let $K=j_r b^*(H_s)$. Clearly K is a subspace of $H_{\bar r}$ and $j_r b^* b j_r^{-1}$ is a projection (not necessarily orthogonal) onto K. Let $p\in B(H_{\bar r})$ be the orthogonal projection onto K and $i\colon K\to H_{\bar r}$ be the embedding. We assumed that $b\in\mathrm{Mor}(r,s)$. Therefore $b^*b\in\mathrm{Mor}(r,r)$ and (cf. Prop. 2.1) $j_r b^* b j_r^{-1}\in\mathrm{Mor}(\bar r,\bar r)$.

Since $\mathrm{Mor}(\bar r,\bar r)$ is a W^*-algebra, we get $p\in\mathrm{Mor}(\bar r,\bar r)$. In virtue of CMW* VI there exists $s'\in R$ such that $H_{s'}=K$ and $i\in\mathrm{Mor}(s',\bar r)$. Let $j=i^* j_r b^*$. Then $j\colon H_s\to H_{s'}$ is an invertible antilinear mapping. One can easily check that

$$t_j=(b\otimes i^*)\,t_{j_r}\in\mathrm{Mor}(\mathbb{1},s\,s')$$
$$\bar t_j=\bar t_{j_r}(i\otimes b^*)\in\mathrm{Mor}(s'\,s,\mathbb{1})$$

(to prove the second equality one has to notice that $j^{-1}=b j_r^{-1} i$). Therefore s' is a complex conjugation of s. □

Now we can prove

Proposition 2.3. *Let $f\in R$. Assume that R is complete, $\bar f$ (the complex conjugation of f) exists and that the pair $\{f,\bar f\}$ generates R. Then for any object of R a complex conjugation exists.*

Proof. Let $s\in R$. Remembering that $\{f,\bar f\}$ generates R we can find a finite family $\{b_k\}$ of morphisms: $b_k\in\mathrm{Mor}(r_k,s)$, where r_k is a product of a certain number of f and $\bar f$ such that

$$\sum_k b_k b_k^*=I_s. \tag{2.3}$$

Let $r=\bigoplus_k r_k$, $H_r=\bigoplus_k H_{r_k}$ and i_k be the canonical embeddings of H_{r_k} into H_r. We know that $i_k\in\mathrm{Mor}(r_k,r)$. Therefore

$$b=\sum_k b_k i_k^*\in\mathrm{Mor}(r,s).$$

Taking into account (2.3) one immediately checks that $b b^*=I_s$. The existence of the complex conjugation of s follows now from Prop. 2.2. □

Let $\alpha\in R$. We say that α is irreducible if

$$\mathrm{Mor}(\alpha,\alpha)=\{\lambda I_\alpha\colon \lambda\in\mathbb{C}\}. \tag{2.4}$$

One can easily check that any object equivalent to an irreducible one is irreducible.

The proof presented in Sect. 3 is mainly based on the following properties of irreducible objects.

Proposition 2.4. *Let R_{irr} be a complete set of mutually non-equivalent irreducible objects of R (the adjective complete means that any irreducible object of R is equivalent to an object belonging to R_{irr}). Then*

1. *For any $\alpha, \beta \in R_{\text{irr}}$*

$$\operatorname{Mor}(\alpha,\beta)=\begin{cases}\mathbb{C} I_\alpha & \text{for } \alpha=\beta\\ \{0\} & \text{for } \alpha\neq\beta.\end{cases} \tag{2.5}$$

2. *Complex conjugation of any irreducible object (if it exists) is irreducible.*
3. *Let $\alpha, \beta, \bar{\alpha} \in R_{\text{irr}}$. Assume that $\bar{\alpha}$ is a complex conjugation of α. Then*

$$\operatorname{Mor}(\mathbb{1},\beta\bar{\alpha})=\begin{cases}\mathbb{C} t_{j_\alpha} & \text{for } \beta=\alpha\\ \{0\} & \text{for } \beta\neq\alpha.\end{cases} \tag{2.6}$$

4. *If R is complete then for any $r\in R$ there exists a finite family $\{p_k\}$ of morphisms: $p_k \in \operatorname{Mor}(\alpha_k, r)$, $\alpha_k \in R_{\text{irr}}$ such that $\sum_k p_k p_k^* = I_r$.*

Proof. Ad 1. For $\alpha=\beta$ formula (2.5) coincides with (2.4). Let $\alpha\neq\beta$ and $q\in\operatorname{Mor}(\alpha,\beta)$. Then $q^*q\in\operatorname{Mor}(\alpha,\alpha)$, $qq^*\in\operatorname{Mor}(\beta,\beta)$ and using (2.4) we have $q^*q=\lambda I_\alpha$ and $qq^*=\lambda' I'_\beta$. Clearly $\lambda=\lambda'$. Assume that $q\neq 0$. Then $\lambda>0$ and q is invertible ($q^{-1}=\lambda^{-1}q^*$). On the other hand $\operatorname{Mor}(\alpha,\beta)$ does not contain invertible elements (α is not equivalent to β). This contradiction shows that $q=0$.

Ad 2. Inserting in (2.1) $r=s=\bar{\alpha}$ and taking into account (2.4) we get

$$j_{\bar{\alpha}}\operatorname{Mor}(\bar{\alpha},\bar{\alpha})j_{\bar{\alpha}}^{-1}\subset\operatorname{Mor}(\alpha,\alpha)=\mathbb{C} I_\alpha.$$

Therefore $\operatorname{Mor}(\bar{\alpha},\bar{\alpha})\subset\mathbb{C} I_{\bar{\alpha}}$ and $\bar{\alpha}$ is irreducible.

Ad 3. Let $t\in\operatorname{Mor}(\mathbb{1},\beta\bar{\alpha})$. Then $(I_\beta\otimes\bar{t}_{j_\alpha})(t\otimes I_\alpha)$ belongs to $\operatorname{Mor}(\alpha,\beta)$ and using (2.5) we get

$$(I_\beta\otimes\bar{t}_{j_\alpha})(t\otimes I_\alpha)=\begin{cases}\lambda I_\alpha & \text{for } \beta=\alpha\\ 0 & \text{for } \beta\neq\alpha\end{cases}$$

where $\lambda\in\mathbb{C}$. Tensoring both sides from the right by $I_{\bar{\alpha}}$ we get

$$(I_\beta\otimes\bar{t}_{j_\alpha}\otimes I_{\bar{\alpha}})(t\otimes I_{\alpha\bar{\alpha}})=\begin{cases}\lambda I_{\alpha\bar{\alpha}} & \text{for } \beta=\alpha\\ 0 & \text{for } \beta\neq\alpha.\end{cases} \tag{2.7}$$

Inserting in (2.2) $r=s=\alpha$ and $a=I_\alpha$ we get

$$(\bar{t}_{j_\alpha}\otimes I_{\bar{\alpha}})(I_{\bar{\alpha}}\otimes t_{j_\alpha})=I_{\bar{\alpha}}.$$

Therefore multiplying both sides of (2.7) by t_{j_α} from the right and using the obvious relation $(t\otimes I_{\alpha\bar{\alpha}})t_{j_\alpha}=t\otimes t_{j_\alpha}=(I_{\beta\bar{\alpha}}\otimes t_{j_\alpha})t$ we get

$$t=\begin{cases}\lambda t_{j_\alpha} & \text{for } \beta=\alpha\\ 0 & \text{for } \beta\neq\alpha.\end{cases}$$

Ad 4. Let $r \in R$. We know that $\mathrm{Mor}(r,r)$ is a finite-dimensional W^*-algebra. Therefore $I_r = \sum_k q_k$, where q_k are minimal projections in $\mathrm{Mor}(r,r)$. In virtue of CMW* VI, for each k there exists $s_k \in R$ such that $H_{s_k} = q_k H_r$ and the embedding i_k: $H_{s_k} \to H_r$ belongs to $\mathrm{Mor}(s_k, r)$. Since q_k is minimal, s_k is irreducible and one can find $\alpha_k \in R_{\mathrm{irr}}$ equivalent to s_k. Let O_k be a unitary element of $\mathrm{Mor}(\alpha_k, s_k)$ and $p_k = i_k O_k$. Then $p_k \in \mathrm{Mor}(\alpha_k, r)$ and $\sum_k p_k p_k^* = \sum_k i_k i_k^* = \sum_k q_k = I_r$. □

If H and K are $f-d$. Hilbert spaces ($\dim H = \dim K$) and j: $H \to K$ is an invertible antilinear mapping then for any $m \in B(H)$ we set

$$m^j = j\, m\, j^{-1}. \tag{2.8}$$

Clearly, $m^j \in B(K)$ and the mapping $B(H) \ni m \to m^j \in B(K)$ is antilinear and multiplicative.

Proposition 2.5. *Let H, K be $f-d$. Hilbert spaces, $\dim H = \dim K$, j: $H \to K$ be an invertible antilinear mapping and t_j: $\mathbb{C} \to H \otimes K$ and $\bar{t}_j$: $K \otimes H \to \mathbb{C}$ be linear mappings related to j in the way described in Sect. 1.*

Moreover, let A be an algebra with unity I, $*$: $A \to A$ be an antilinear map (the $*$-image of an element $a \in A$ will be denoted by a^*), $v \in B(H) \otimes A$ and $w \in B(K) \otimes A$.

Then any two of the following conditions imply the third one:

I. *v is invertible and $v^{-1} = v^{*\otimes *}$.*

II. *$w = v^{j \otimes *}$.*

III. *$(v \oplus w)(t_j \otimes I) = t_j \otimes I$ and $(\bar{t}_j \otimes I)(w \oplus v) = \bar{t}_j \otimes I$.*

Proof. Let

$$v = \sum_a m_a \otimes v_a,$$

$$w = \sum_b n_b \otimes w_b,$$

where $m_a \in B(H)$; $n_b \in B(K)$; $v_a, w_b \in A$; a and b run over finite sets. For any $n \in B(K)$ we set

$$\tau(n) = (j^{-1}\, n\, j)^*.$$

Clearly τ: $B(K) \to B(H)$ is an invertible linear map. Applying $\tau \otimes \mathrm{id}$ to both sides of the relation $w = v^{j \otimes *}$ we see that the Condition II is equivalent to the equation $\tilde{w} = v^{*\otimes *}$, where

$$\begin{aligned} \tilde{w} &= (\tau \otimes \mathrm{id})\, w \\ &= \sum_b (j^{-1}\, n_b\, j)^* \otimes w_b. \end{aligned} \tag{2.9}$$

The first equation of Condition III means that

$$\sum_{abl} m_a e_l \otimes n_b j(e_l) \otimes v_a w_b = \sum_l e_l \otimes j(e_l) \otimes I$$

where (e_l) is an orthonormal basis in H. It holds if and only if

$$\sum_{abl} (k|n_b j(e_l))\, m_a e_l \otimes v_a w_b = \sum_l (k|j(e_l))\, e_l \otimes I$$

for all $k \in K$. Replacing k by $(j^*)^{-1}(h)$ (where $h \in H$) we see that the first equation of Condition III is satisfied if and only if

$$\sum_{abl} ((j^{-1})^*(h)|n_b j(e_l))\, m_a e_l \otimes v_a w_b = \sum_l ((j^{-1})^*(h)|j(e_l))\, e_l \otimes I \tag{2.10}$$

for all $h \in H$. We remind that for an antilinear operator j the relation between j and its hermitian conjugate j^* is described by the formula $(j^*(k)|h) = (j(h)|k)$. Using this formula (with j replaced by j^{-1}) and remembering that (e_l) is an orthonormal basis in H we have

$$\begin{aligned}\sum_l ((j^{-1})^*(h)\,|\,n_b j(e_l))\, m_a e_l &\\ &= \sum_l (j^{-1} n_b j\, e_l\,|\,h)\, m_a e_l \\ &= \sum_l m_a e_l (e_l\,|\,(j^{-1} n_b j)^* h) = m_a (j^{-1} n_b j)^* h.\end{aligned}$$

Similarly, one can check that

$$\sum_l ((j^{-1})^*(h)\,|\,j(e_l))\, e_l = h.$$

Inserting these data into (2.10), making use of the freedom of h and comparing with (2.9) we see that the first equation of Condition III is equivalent to the relation $v\tilde{w} = I_{B(H)\otimes A}$.

The second equation of Condition III means that

$$\sum_{ab} (j^{-1}(n_b k)\,|\,m_a h)\, w_b v_a = (j^{-1}(k)\,|\,h)\, I$$

for all $k \in K$ and $h \in H$. Replacing k by $j(h')$ where $h' \in H$ we see that the second equation of Condition III is satisfied if and only if

$$\sum_{ab} (j^{-1} n_b j h'\,|\,m_a h)\, w_b v_a = (h'\,|\,h)\, I$$

for all $h, h' \in H$. Clearly, $(j^{-1} n_b j h'\,|\,m_a h) = (h'\,|\,(j^{-1} n_b j)^* m_a h)$.

Inserting this data into the above equation, making use of the freedom of h and h' and comparing with (2.9) we see that the second equation of Condition III is equivalent to the relation $\tilde{w} v = I_{B(H)\otimes A}$. This way we proved that Condition III means that v is invertible and $v^{-1} = \tilde{w}$. Summarizing we have

I $\Leftrightarrow$ v is invertible and $v^{-1} = v^{*\otimes *}$.

II $\Leftrightarrow$ $v^{*\otimes *} = \tilde{w}$.

III $\Leftrightarrow$ v is invertible and $v^{-1} = \tilde{w}$.

Now the statement of Prop. 2.5 is obvious. □

Remark. In the applications of Prop. 2.5 considered in this section, A equipped with the $*$-operation is a $*$-algebra. In this case $*^{\otimes}*$ is usually denoted by $*$ and Condition I means that u is unitary.

Let $G=(A,u)$ be a compact matrix pseudogroup and $v\in B(H)\otimes A$ be a unitary representation of G acting on a $f-d$. Hilbert space H. We remind the construction of the complex conjugate representation given in [8]. $\bar{v}$ acts on a vector space K related to H by an antilinear invertible map j: $H\to K$ and is given by the formula $\bar{v}=v^{j\otimes *}$. Next one introduces a scalar product on K such that K becomes a Hilbert space and $\bar{v}$ becomes a unitary element of $B(K)\otimes A$.

Now using the ((I and II)$\Rightarrow$III)-part of Prop. 2.5 we see that t_j and $\bar{t}_j$ are intertwining operators: $t_j\in\mathrm{Mor}(\mathbb{1}, v\mathbin{\textcircled{\small T}}\bar{v})$ and $\bar{t}_j\in\mathrm{Mor}(\bar{v}\mathbin{\textcircled{\small T}}v,\mathbb{1})$. Therefore $\bar{v}$ is a complex conjugation of v in the sense introduced in Sect. 1.

Now we again consider a concrete monoidal W^*-category

$$R=(R,\{H_r\}_{r\in R},\{\mathrm{Mor}(r,s)\}_{r,s\in R},\cdot).$$

Assume that $M=(B,\{v^r\}_{r\in R})$ is a model of R. Using (1.11) and the relation $\mathbb{1}\cdot\mathbb{1}=\mathbb{1}$ one easily checks that

$$v^{\mathbb{1}}=I_B. \tag{2.11}$$

Let $r,\bar{r}\in R$, $\bar{r}$ be a complex conjugation of r and j_r: $H_r\to H_{r'}$ be the corresponding invertible antilinear mapping. We know that $t_{j_r}\in\mathrm{Mor}(\mathbb{1}, r\bar{r})$ and $\bar{t}_{j_r}\in\mathrm{Mor}(\bar{r}r,\mathbb{1})$. Taking into account (1.11), (1.12) and (2.11) we get

$$(v^r\mathbin{\textcircled{\small T}}v^{\bar{r}})(t_{j_r}\otimes I)=t_{j_r}\otimes I$$
$$(\bar{t}_{j_r}\otimes I)(v^{\bar{r}}\mathbin{\textcircled{\small T}}v^r)=\bar{t}_{j_r}\otimes I.$$

Remembering that v^r is unitary and using ((I and III)$\Rightarrow$II)-part of Prop. 2.5 we obtain

$$v^{\bar{r}}=(v^r)^{j_r\otimes *}. \tag{2.12}$$

Proposition 2.6. *Let* $R=(R,\{H_r\}_{r\in R},\{\mathrm{Mor}(r,s)\}_{r,s\in R},\cdot,f)$ *be a concrete monoidal* W^**-category with the distinguished object* f. *Assume that* $\bar{f}$ *(a complex conjugation of* f*) exists and that* $\{f,\bar{f}\}$ *generates* R.

Let (B,v) *be an* R*-admissible pair. Then the model* $M=(B,\{v^r\}_{r\in R})$ *of* R *such that* $v^f=v$ *is uniquely determined. Moreover denoting by* $\mathscr{B}$ *the* $*$*-subalgebra of* B *generated by matrix elements of* v *we have*

$$v^s\in B(H_s)\otimes\mathscr{B}$$

for any $s\in R$.

Proof. Let $s\in R$. Remembering that $\{f,\bar{f}\}$ generates R we can find a finite family $\{b_k\}$ of morphisms: $b_k\in\mathrm{Mor}(r_k,s)$ where r_k is a product of a certain number of f and $\bar{f}$, such that $\sum_k b_k b_k^*=I_s$. Using (1.12) one can easily check that

$$v^s=\sum_k(b_k\otimes I)\,v^{r_k}(b_k^*\otimes I). \tag{2.13}$$

Clearly v^r is uniquely determined and v^r belongs to $B(H_r)\otimes\mathscr{B}$ for $r=f$. Formula (2.12) shows that the same statement holds for $r=\bar{f}$. In virtue of (1.11) it remains true for $r=r_k$. Now (2.13) shows that v^s is uniquely determined and $v^s\in B(H)\otimes\mathscr{B}$. □

The main proof presented in Sect. 3 works only for complete categories. Therefore we need

Proposition 2.7. *Let* $R=(R,\{H_r\}_{r\in R},\{\mathrm{Mor}(r,s)\}_{r,s\in R},\cdot)$ *be a concrete monoidal* W^**-category. Then there exists a complete concrete monoidal* W^**-category* $\tilde{\tilde{R}}$ $=(\tilde{\tilde{R}},\{\tilde{\tilde{H}}_r\}_{r\in\tilde{\tilde{R}}},\{\widetilde{\widetilde{\mathrm{Mor}}}(r,s)\}_{r,s\in\tilde{\tilde{R}}},\cdot)$ *such that* $R\subset\tilde{\tilde{R}}$ *and*

1. $\tilde{\tilde{H}}_r=H_r$ *for all* $r\in R$.
2. $\widetilde{\widetilde{\mathrm{Mor}}}(r,s)=\mathrm{Mor}(r,s)$ *for all* $r,s\in R$.
3. *The binary operation on* $\tilde{\tilde{R}}$ *extends that on* R.
4. *Every subset of* R *generating* R *generates* $\tilde{\tilde{R}}$.
5. *If* $M=(B,\{v^r\}_{r\in R})$ *is a model of* R *then there exists unique model* $\tilde{\tilde{M}}$ $=(B,\{\tilde{\tilde{v}}^r\}_{r\in\tilde{\tilde{R}}})$ *of* $\tilde{\tilde{R}}$ *such that* $\tilde{\tilde{v}}^r=v^r$ *for all* $r\in R$.

If R *contains the distinguished object* f *such that* $\bar{f}$ *exists and* $\{f,\bar{f}\}$ *generates* R *then distinguishing* f *in* $\tilde{\tilde{R}}$ *we have*

6. *A pair* (B,v) *is* $\tilde{\tilde{R}}$*-admissible if and only if it is* R*-admissible.*

Remark. We say that $\tilde{\tilde{R}}$ is the completion of R.

Proof. We use the way suggested by Prop. 1.1. Let us consider objects of the form

$$\tilde{a}=(H_a,\{(r_k,a_k)\}_{k\in\Delta_a}) \tag{2.14}$$

where H_a is a $f-d$. Hilbert space Δ_a is a finite set, $r_k\in R$, $a_k\in B(H_{r_k},H_a)$ satisfy the relations: $a_{k'}^*a_k\in\mathrm{Mor}(r_k,r_{k'})$ for all $k,k'\in\Delta_a$ and $\sum_k a_k a_k^*=I_a$.

The class of all such objects will be denoted by $\tilde{R}$.

If $\tilde{a}\in\tilde{R}$ is of the form (2.14) then we set $\tilde{H}_{\tilde{a}}=H_a$. Moreover if

$$\tilde{a}=(H_a,\{(r_k,a_k)\}_{k\in\Delta_a})$$
$$\tilde{b}=(H_b,\{(s_l,b_l)\}_{l\in\Delta_b})$$

are elements of $\tilde{R}$ then we set

$$\widetilde{\mathrm{Mor}}(\tilde{a},\tilde{b})=\left\{t\in B(H_a,H_b):\begin{array}{l}b_l^*\,t\,a_k\in\mathrm{Mor}(r_k,s_l)\\ \text{for all } k\in\Delta_a,\ l\in\Delta_b\end{array}\right\}$$

$$\tilde{a}\cdot\tilde{b}=(H_a\otimes H_b,\{(r_k s_l,a_k\otimes b_l)\}_{(k,l)\in\Delta_a\times\Delta_b}). \tag{2.15}$$

Clearly $\widetilde{\mathrm{Mor}}(\tilde{a},\tilde{b})$ is a linear subset of $B(\tilde{H}_{\tilde{a}},\tilde{H}_{\tilde{b}})$ and $\tilde{a}\tilde{b}\in\tilde{R}$. One can easily check that

$$\tilde{R}=(\tilde{R},\{\tilde{H}_{\tilde{a}}\}_{\tilde{a}\in\tilde{R}},\{\widetilde{\mathrm{Mor}}(\tilde{a},\tilde{b})\}_{\tilde{a},\tilde{b}\in\tilde{R}},\cdot)$$

where the $\cdot$ denotes the binary operation introduced by (2.15) satisfies all conditions CMW* I–X except CMW* IV. In order to pass over this difficulty we introduce the equivalence relation

$$\tilde{a} \simeq \tilde{b} \Leftrightarrow \begin{pmatrix} \tilde{H}_{\tilde{a}} = \tilde{H}_{\tilde{b}} \text{ and} \\ I_{\tilde{a}} \in \tilde{\mathrm{Mor}}(\tilde{a}, \tilde{b}) \end{pmatrix}$$

and consider $\tilde{\tilde{R}} = \tilde{R}/\simeq$. The class of $\tilde{a} \in \tilde{r}$ will be denoted by $\tilde{\tilde{a}}$. Clearly $\tilde{H}_{\tilde{a}}$, $\tilde{\mathrm{Mor}}(\tilde{a}, \tilde{b})$ and the class of $\tilde{a}\tilde{b}$ depend only on $\tilde{\tilde{a}}$ and $\tilde{\tilde{b}}$.

Let

$$\tilde{\tilde{R}} = (\tilde{\tilde{R}}, \{\tilde{\tilde{H}}_{\tilde{\tilde{a}}}\}_{\tilde{\tilde{a}} \in \tilde{\tilde{R}}}, \{\tilde{\tilde{\mathrm{Mor}}}(\tilde{\tilde{a}}, \tilde{\tilde{b}})\}_{\tilde{\tilde{a}}, \tilde{\tilde{b}} \in \tilde{\tilde{R}}}, \cdot)$$

where $\tilde{\tilde{H}}_{\tilde{\tilde{a}}} = \tilde{H}_{\tilde{a}}$, $\tilde{\tilde{\mathrm{Mor}}}(\tilde{\tilde{a}}, \tilde{\tilde{b}}) = \tilde{\mathrm{Mor}}(\tilde{a}, \tilde{b})$ and the dot $\cdot$ is the binary operation introduced by $\tilde{\tilde{a}}\tilde{\tilde{b}}$ = the class of $\tilde{a}\tilde{b}$. Then $\tilde{\tilde{R}}$ is a complete concrete monoidal W^*-category (one can easily verify CMW* I–X).

The embedding $R \to \tilde{\tilde{R}}$ is defined in the following way. For any $r \in R$ we set $\tilde{r} = (H_r, \{(r, I_r)\})$ (Δ is now a one-point set) and $\tilde{\tilde{r}}$ = the class of $\tilde{r}$. Statements 1–4 are obvious.

Let $M = (B, \{v^r\}_{r \in R})$ be a model of R. For any $\tilde{\tilde{a}} \in \tilde{\tilde{R}}$ we choose a representative $\tilde{a} = (H_a, \{(r_k, a_k)\}_{k \in \Delta})$ and set (cf. (2.13))

$$\tilde{\tilde{v}}^{\tilde{\tilde{a}}} = \sum_k (a_k \otimes I_B)\, v^{r_k} (a_k^* \otimes I_B)$$

(notice that the right hand side does not depend on the choice of $\tilde{a}$) and

$$\tilde{\tilde{M}} = (B, \{\tilde{\tilde{v}}^{\tilde{\tilde{a}}}\}_{\tilde{\tilde{a}} \in \tilde{\tilde{R}}}).$$

One can easily check that $\tilde{\tilde{M}}$ is the model of $\tilde{\tilde{R}}$ satisfying the requirement of Statement 5. Statement 6 follows immediately from 5. □

3. The proof of the duality theorem

In this section we complete the proof of our main result. Let $G = (A, u)$ be the universal R-admissible pair. In order to show that G satisfies condition CMP III and in order to prove the "only if" part of Statement 3 of Theorem 1.3 we need another construction of G giving better control of its properties than the one described in Sect. 1.

Throughout this section

$$R = (R, \{H_r\}_{r \in R}, \{\mathrm{Mor}(r, s)\}_{r, s \in R}, \cdot, f)$$

is a concrete monoidal W^*-category with distinguished object. We assume that $\bar{f}$ exists and that $\{f, \bar{f}\}$ generates R. In virtue of Prop. 2.7 we may assume that R is complete. Then R contains sufficiently many irreducible objects (cf. Prop. 2.4.4).

Let R_{irr} be a complete set of mutually non-equivalent irreducible objects of R. We include $\mathbb{1}$ into R_{irr} (clearly $\mathbb{1}$ is irreducible). For any $\alpha \in R_{\text{irr}}$ we consider the space $B(H_\alpha)'$ of linear functionals defined on $B(H_\alpha)$. We set

$$\mathscr{A} = \bigoplus_{\alpha \in R_{\text{irr}}} B(H_\alpha)'. \tag{3.1}$$

At the moment $\mathscr{A}$ is a complex vector space. In the following we endow $\mathscr{A}$ with a $*$-algebra structure. On the other hand each $B(H_\alpha)'$ is endowed with a natural $*$-conjugation (induced by hermitian conjugation of operators acting on H_α) which also will be used and which does not coincide with the $*$-conjugation that will be introduced on $\mathscr{A}$. In order to avoid possible misunderstandings we shall use the following notation: for any $\alpha \in R_{\text{irr}}$ and any $\rho \in B(H_\alpha)'$, the corresponding element of $\mathscr{A}$ will be denoted by u^α_ρ. With this notation any element $a \in \mathscr{A}$ can be written in the following form:

$$a = \sum_{\alpha \in R_{\text{irr}}} u^\alpha_{\rho_\alpha} \tag{3.2}$$

where $\rho_\alpha \in B(H_\alpha)'$ are uniquely determined and the sum is finite (i.e. $\rho_\alpha = 0$ for all but finite number of elements $\alpha \in R_{\text{irr}}$).

For any $\alpha \in R_{\text{irr}}$ the embedding

$$B(H_\alpha)' \ni \rho \to u^\alpha_\rho \in \mathscr{A}$$

is linear. Therefore there exists unique $u^\alpha \in B(H_\alpha) \otimes \mathscr{A}$ such that

$$(\rho \otimes \text{id})\, u^\alpha = u^\alpha_\rho \tag{3.3}$$

for any $\rho \in B(H_\alpha)'$.

Let $\mathscr{A}'$ denote the space of all linear functionals defined on $\mathscr{A}$. It follows immediately from (3.1) that for any family $(m^\alpha)_{\alpha \in R_{\text{irr}}}$ of elements $m^\alpha \in B(H_\alpha)$ there exists a unique $f \in \mathscr{A}'$ such that $f(u^\alpha_\rho) = \rho(m^\alpha)$ for any $\alpha \in R_{\text{irr}}$ and $\rho \in B(H_\alpha)'$. Taking into account (3.3) we get $(\text{id} \otimes f)\, u^\alpha = m^\alpha$ for any $\alpha \in R_{\text{irr}}$.

In particular, if $\alpha, \beta \in R_{\text{irr}}$ and $\alpha \neq \beta$ then there exists $f \in \mathscr{A}'$ such that

$$(\text{id} \otimes f)\, u^\alpha = 0 \quad \text{and} \quad (\text{id} \otimes f)\, u^\beta = I_\beta. \tag{3.4}$$

Moreover for any $\alpha \in R_{\text{irr}}$ and any $m \in B(H_\alpha)$ there exists $f \in \mathscr{A}'$ such that

$$(\text{id} \otimes f)\, u^\alpha = m. \tag{3.5}$$

We shall also use the functional $h \in \mathscr{A}'$ such that

$$(\text{id} \otimes h)\, u^\alpha = \begin{cases} 1 & \text{if } \alpha = \mathbb{1} \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Let $M = (B, \{v^r\}_{r \in R})$ be a model of R. Then for any $\alpha \in R_{\text{irr}}$ and any $\rho \in B(H_\alpha)'$ we set

$$\varphi_M(u^\alpha_\rho) = (\rho \otimes \text{id})\, v^\alpha. \tag{3.7}$$

Clearly this formula defines a linear map

$$\varphi_M\colon \mathscr{A} \to B. \tag{3.8}$$

In order to have the short formulation of the next statements we introduce the symbol I that can multiply elements of $\mathscr{A}$ from both sides and such that $I\,a = a\,I = a$ for all $a \in \mathscr{A}$. Later, when the algebraic structure on $\mathscr{A}$ is introduced, I will be identified with the unity of $\mathscr{A}$.

Proposition 3.1. *There exists a unique family* $\{u^r\}_{r\in R}$, *where for each* $r \in R$, $u^r \in B(H_r) \otimes \mathscr{A}$ *such that*

1. *For* $r = \alpha \in R_{\text{irr}}$, u^r *coincides with* u^α *introduced by* (3.3).
2. *For any* $r, s \in R$ *and any* $t \in \operatorname{Mor}(s, r)$

$$(t \otimes I)\, u^s = u^r (t \otimes I). \tag{3.9}$$

Moreover we have

3. *For any* $r, s \in R$ *and any* $t \in B(H_s, H_r)$

$$((t\otimes I)\, u^s = u^r (t \otimes I)) \Rightarrow (t \in \operatorname{Mor}(s, t)). \tag{3.10}$$

4. *For any model* $M = (B, \{v^r\}_{r\in R})$ *of* R

$$(\mathrm{id} \otimes \varphi_M)\, u^r = v^r \tag{3.11}$$

for all $r \in R$.

Proof. For each $r \in R$ we choose the decomposition

$$I_r = \sum_k p_k\, p_k^* \tag{3.12}$$

where k runs over a finite set, $p_k \in \operatorname{Mor}(\alpha_k, r)$ and $\alpha_k \in R_{\text{irr}}$ (cf. Prop. 2.4.4). Taking into account (3.9) (with t and s replaced by p_k and α_k resp.) we compute

$$\begin{aligned} u^r = u^r (I_r \otimes I) &= \sum_k u^r (p_k \otimes I)(p_k^* \otimes I) \\ &= \sum_k (p_k \otimes I)\, u^{\alpha_k} (p_k^* \otimes I). \end{aligned} \tag{3.13}$$

Using Condition 1 we see that the family $\{u^r\}_{r\in R}$ is uniquely defined. To prove the existence we have to show that $\{u^r\}_{r\in R}$ introduced by (3.13) satisfies Conditions 1 and 2.

For $r = \alpha \in R_{\text{irr}}$ the decomposition (3.12) consists of one term: $I_r = I_\alpha I_\alpha^*$ and the equality $u^r = u^\alpha$ follows immediately.

Let $s \in R$. Then

$$u^s = \sum_l (q_l \otimes I)\, u^{\beta_l} (q_l^* \otimes I) \tag{3.14}$$

where l runs over a finite set, $q_l \in \operatorname{Mor}(\beta_l, s)$, $\beta_l \in R_{\text{irr}}$ and

$$\sum_l q_l\, q_l^* = I_s. \tag{3.15}$$

Assume that $t \in \mathrm{Mor}(s, r)$. Then $p_k^* t q_l \in \mathrm{Mor}(\beta_l, \alpha_k)$. If $\beta_l \neq \alpha_k$ then $p_k^* t q_l = 0$. If $\beta_l = \alpha_k$ then $p_k^* t q_l$ is a multiple of I_{α_k} (cf. Prop. 2.4.1). In both cases $(p_k^* t q_l \otimes I) u^{\beta_l} = u^{\alpha_k}(p_k^* t q_l \otimes I)$. Taking into account this relation and using (3.12), (3.14), (3.15) and (3.13) we have

$$\begin{aligned}(t\otimes I)\, u^s &= \sum_{kl} (p_k p_k^* t q_l \otimes I)\, u^{\beta_l}(q_l^* \otimes I)\\ &= \sum_{kl} (p_k \otimes I)\, u^{\alpha_k}(p_k^* t q_l q_l^* \otimes I)\\ &= u^r (t \otimes I).\end{aligned}$$

Condition 2 is satisfied. This proves the first part of our proposition.

Assume now that $t \in B(H_s, H_r)$ and that

$$(t\otimes I)\, u^s = u^r (t \otimes I).$$

Multiplying both sides by $p_k^* \otimes I$ from the left and by $q_l \otimes I$ from the right and using formula (3.9) we get

$$(p_k^* t q_l \otimes I)\, u^{\beta_l} = u^{\alpha_k}(p_k^* t q_l \otimes I).$$

Therefore for any $f \in \mathscr{A}'$ we have

$$p_k^* t q_l (\mathrm{id} \otimes f)\, u^{\beta_l} = (\mathrm{id} \otimes f)\, u^{\alpha_k} \cdot p_k^* t q_l.$$

If $\alpha_k \neq \beta_l$ then using (3.4) we get $p_k^* t q_l = 0$. If $\alpha_k = \beta_l$ then using (3.5) we see that $p_k^* t q_l$ commutes with all elements of $B(H_{\alpha_k})$. Hence $p_k^* t q_l = \lambda_{kl} I_{\alpha_k}$ where $\lambda_{kl} \in \mathbb{C}$. Taking into account (3.12) and (3.15) we have

$$t = \sum_{kl} p_k p_k^* t q_l q_l^* = \sum_{kl}{}' \lambda_{kl}\, p_k q_l^*$$

where $\sum'$ denotes the sum over all pairs (k, l) such that $\alpha_k = \beta_l$. Now the conclusion of (3.10) follows immediately: $p_k \in \mathrm{Mor}(\alpha_k, r)$ and $q_l \in \mathrm{Mor}(\beta_l, s)$. Therefore for $\alpha_k = \beta_l$ we have $q_l^* \in \mathrm{Mor}(s, \alpha_k)$, $p_k q_l^* \in \mathrm{Mor}(s, r)$ and $t \in \mathrm{Mor}(s, r)$.

To complete the proof we have to check (3.11). For $r = \alpha \in R_{\mathrm{irr}}$, (3.11) follows immediately from (3.7). In the general case, taking into account (3.13), (1.12) with s and t replaced by α_k and p_k resp. and (3.12) we obtain

$$\begin{aligned}(\mathrm{id} \otimes \varphi_M)\, u^r &= \sum_k (p_k \otimes I)(\mathrm{id} \otimes \varphi_M)\, u^{\alpha_k}(p_k^* \otimes I)\\ &= \sum_k (p_k \otimes I)\, v^{\alpha_k}(p_k^* \otimes I)\\ &= \sum_k v^r (p_k p_k^* \otimes I) = v^r. \quad \square\end{aligned}$$

Now, using the binary operation on R we introduce the multiplicative structure on $\mathscr{A}$. For any $\alpha, \beta \in R_{\mathrm{irr}}$ and any $\rho \in B(H_\alpha)'$, $\sigma \in B(H_\beta)'$ we set

$$u_\rho^\alpha u_\sigma^\beta = (\rho \otimes \sigma \otimes \mathrm{id})\, u^{\alpha\beta}. \tag{3.16}$$

Clearly this formula defines a unique bilinear map

$$\cdot : \mathscr{A} \times \mathscr{A} \to \mathscr{A}. \tag{3.17}$$

Proposition 3.2. *The vector space $\mathscr{A}$ equipped with* (3.17) *is an algebra with unity. For any $r, s \in R$*

$$u^{rs} = u^r \textcircled{T} u^s. \tag{3.18}$$

Moreover for any model $M = (B, \{v^r\}_{r \in R})$ of R, the mapping $\varphi_M: \mathscr{A} \to B$ is multiplicative and unital: $\varphi_M(I_{\mathscr{A}}) = I_B$.

Proof. Let $\alpha, \beta \in R_{\rm irr}$. It follows immediately from the definition of $\textcircled{T}$ that for any $\rho \in B(H_\alpha)'$ and $\sigma \in B(H_\beta)'$

$$(\rho \otimes \sigma \otimes \mathrm{id})(u^\alpha \textcircled{T} u^\beta) = (\rho \otimes \mathrm{id})\, u^\alpha (\sigma \otimes \mathrm{id})\, u^\beta.$$

Taking into account (3.3), comparing with (3.16) and using the freedom of ρ and σ we get

$$u^{\alpha\beta} = u^\alpha \textcircled{T} u^\beta.$$

In order to prove (3.18) in full generality we compute using formulae (3.13) and (3.14):

$$\begin{aligned} u^r \textcircled{T} u^s &= \sum_{kl} [(p_k \otimes I)\, u^{\alpha_k} (p_k^* \otimes I)] \textcircled{T} [(q_l \otimes I)\, u^{\beta_l} (q_l^* \otimes I)] \\ &= \sum_{kl} (p_k \otimes q_l \otimes I)(u^{\alpha_k} \textcircled{T} u^{\beta_l})(p_k^* \otimes q_l^* \otimes I) \\ &= \sum_{kl} (p_k \otimes q_l \otimes I)\, u^{\alpha_k \beta_l} (p_k^* \otimes q_l^* \otimes I). \end{aligned}$$

We know (cf. CMW* VIII) that the tensor product of morphisms is a morphism. Therefore $p_k \otimes q_l \in \mathrm{Mor}(\alpha_k \beta_l, r\,s)$ and using (3.9) with r, s and t replaced by $r\,s$, $\alpha_k \beta_l$ and $p_k \otimes q_l$ resp. we obtain

$$u^r \textcircled{T} u^s = \sum_{kl} u^{rs} (p_k\, p_k^* \otimes q_l\, q_l^* \otimes I) = u^{rs}.$$

In the last step we used (3.12) and (3.15). Formula (3.18) is proved.

We know (cf. CMW* IX) that the binary operation in R is associative. In particular $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for any $\alpha, \beta, \gamma \in R_{\rm irr}$. Using (3.18) we obtain

$$(u^\alpha \textcircled{T} u^\beta) \textcircled{T} u^\gamma = u^\alpha \textcircled{T} (u^\beta \textcircled{T} u^\gamma).$$

Applying $\rho \otimes \sigma \otimes \eta \otimes \mathrm{id}$ (where $\rho \in B(H_\alpha)'$, $\sigma \in B(H_\beta)'$ and $\eta \in B(H_\gamma)'$) to both sides of the above equation and using (3.3) we get

$$(u^\alpha_\rho\, u^\beta_\sigma)\, u^\gamma_\eta = u^\alpha_\rho (u^\beta_\sigma\, u^\gamma_\eta).$$

This relation shows that the product (3.17) is associative i.e. $\mathscr{A}$ is an algebra.

We know that $H_{\mathbb{1}} = \mathbb{C}$. Therefore $B(H_{\mathbb{1}}) \otimes \mathscr{A} = \mathscr{A}$. In particular $u^{\mathbb{1}} \in \mathscr{A}$. Remembering that $\alpha \mathbb{1} = \mathbb{1} \alpha = \alpha$ and using (3.16) one can easily check that $u^\alpha_\rho u^{\mathbb{1}} = u^{\mathbb{1}} u^\alpha_\rho = u^\alpha_\rho$ for any $\alpha \in R_{\rm irr}$ and any $\rho \in B(H_r)'$. It means that $u^{\mathbb{1}}$ is the unity of the algebra $\mathscr{A}$.

Let $M=(B,\{v^r\}_{r\in R})$ be a model of R. Then using (3.16), (3.11) with r replaced by $\alpha\,\beta$, (1.11) and (3.7) we get

$$\begin{aligned}\varphi_M(u^\alpha_\rho\, u^\beta_\sigma)&=(\rho\otimes\sigma\otimes\varphi_M)\,u^{\alpha\beta}\\&=(\rho\otimes\sigma\otimes\mathrm{id})\,v^{\alpha\beta}\\&=(\rho\otimes\sigma\otimes\mathrm{id})\,(v^\alpha ⓣ v^\beta)\\&=[(\rho\otimes\mathrm{id})\,v^\alpha]\,[(\sigma\otimes\mathrm{id})\,v^\beta]\\&=\varphi_M(u^\alpha_\rho)\,\varphi_M(u^\beta_\sigma).\end{aligned}$$

Therefore φ_M is multiplicative.

To end the proof we remind (cf. (2.11)) that $v^{\mathbf{1}}=I_B$. Therefore using (3.11) we get

$$\varphi_M(u^{\mathbf{1}})=v^{\mathbf{1}}=I_B. \quad \square$$

At this moment we may reinterpret the meaning of the symbol I. Before it was used as formal multiplier acting trivially on $\mathscr{A}$. Since now $I=u^{\mathbf{1}}$ is the unity of the algebra $\mathscr{A}$.

We shall introduce $*$-algebra structure on $\mathscr{A}$. To this end we use the notion of complex conjugate object. In virtue of Prop. 2.3 any object of R admits complex conjugation. Moreover (cf. Prop. 2.4.2) the complex conjugation of any object $\alpha\in R_{\mathrm{irr}}$ is irreducible.

For any $\alpha\in R_{\mathrm{irr}}$ we denote by $\bar\alpha$ the complex conjugation of α belonging to R_{irr} and by $j_\alpha\colon H_\alpha\to H_{\bar\alpha}$ the corresponding invertible antilinear mapping. Clearly $\bar{\bar\alpha}=\alpha$ and $j_{\bar\alpha}=j_\alpha^{-1}$.

For any $\alpha\in R_{\mathrm{irr}}$ and any $m\in B(H_\alpha)$ we set (cf. (2.8))

$$m^j=j_\alpha\, m\, j_\alpha^{-1}.$$

Clearly $m^j\in B(H_{\bar\alpha})$. Moreover for any $\rho\in B(H_\alpha)'$ we denote by ρ^j the linear functional on $B(H_{\bar\alpha})$ such that

$$\rho^j(m^j)=\overline{\rho(m)}$$

for all $m\in B(H_\alpha)$. Remembering that $j_{\bar\alpha} j_\alpha=I_\alpha$ we get

$$m^{jj}=m, \qquad \rho^{jj}=\rho$$

for any $m\in B(H_\alpha)$ and $\rho\in B(H_\alpha)'$.

For any $\alpha\in R_{\mathrm{irr}}$ and any $\rho\in B(H_\alpha)'$ we set

$$(u^\alpha_\rho)^*=u^{\bar\alpha}_{\rho^j}. \tag{3.19}$$

Clearly this formula defines unique antilinear involution

$$*\colon \mathscr{A}\to\mathscr{A}. \tag{3.20}$$

Proposition 3.3. *The algebra $\mathscr{A}$ equipped with* (3.20) *is a $*$-algebra. For any $r\in R$, u^r is a unitary element of $B(H_r)\otimes\mathscr{A}$. Moreover for any model $M=(B,\{v^r\}_{r\in R})$ of R the mapping $\varphi_M\colon\mathscr{A}\to B$ is an unital $*$-algebra homomorphism.*

Proof. Let $\alpha\in R_{\mathrm{irr}}$. Clearly (3.19) is equivalent to the relation

$$u^{\bar\alpha}=(u^\alpha)^{j\otimes *}.$$

We know that $t_{j_\alpha}\in\mathrm{Mor}(\mathbb{1},\alpha\bar\alpha)$ and $\bar t_{j_\alpha}\in\mathrm{Mor}(\bar\alpha\alpha,\mathbb{1})$. Using Prop. 3.1.2, formula (3.18) and remembering that $u^{\mathbb{1}}=I$ we obtain

$$(u^\alpha \mathbin{\bigcirc\!\!\!\!\top} u^{\bar\alpha})(t_{j_\alpha}\otimes I)=t_{j_\alpha}\otimes I$$
$$(\bar t_{j_\alpha}\otimes I)(u^{\bar\alpha}\mathbin{\bigcirc\!\!\!\!\top} u^\alpha)=\bar t_{j_\alpha}\otimes I.$$

Applying now Prop. 2.5 we see that u^α is invertible and

$$(u^\alpha)^{-1}=(u^\alpha)^{*\otimes *}. \tag{3.21}$$

This formula holds for any $\alpha\in R_{\mathrm{irr}}$. We have to prove the similar result for all $r\in R$. According to (3.13)

$$u^r=\sum_k (p_k\otimes I)\,u^{\alpha_k}(p_k^*\otimes I)$$

where $p_k\in\mathrm{Mor}(\alpha_k,r)$, $\alpha_k\in R_{\mathrm{irr}}$ and $\sum_k p_k p_k^*=I_r$. Remembering that the hermitian conjugation (i.e. the first star in $*\otimes*$) is anti-multiplicative we get

$$(u^r)^{*\otimes *}=\sum_l (p_l\otimes I)\,(u^{\alpha_l})^{*\otimes *}\,(p_l^*\otimes I).$$

Therefore

$$(u^r)^{*\otimes *}\,u^r=\sum_{kl}(p_l\otimes I)\,(u^{\alpha_l})^{*\otimes *}\,(p_l^*\,p_k\otimes I)\,u^{\alpha_k}(p_k^*\otimes I).$$

Clearly $p_l^*\,p_k\in\mathrm{Mor}(\alpha_k,\alpha_l)$. If $\alpha_k\neq\alpha_l$ then $p_l^*\,p_k=0$. If $\alpha_k=\alpha_l$ then (cf. (2.5)) $p_l^*\,p_k$ is a multiple of I_{α_k}. In both cases $(p_l^*\,p_k\otimes I)\,u^{\alpha_k}=u^{\alpha_l}(p_l^*\,p_k\otimes I)$. Taking into account (3.21) we get

$$(u^r)^{*\otimes *}\,u^r=\sum_{kl}p_l\,p_l^*\,p_k\,p_k^*\otimes I=I_{B(H_r)\otimes\mathscr{A}}.$$

Similarly one can check that $u^r(u^r)^{*\otimes *}=I_{B(H_r)\otimes\mathscr{A}}$. This way we proved that for any $r\in R$, u^r is invertible and

$$(u^r)^{-1}=(u^r)^{*\otimes *}. \tag{3.22}$$

We still have to show that the involution (3.20) is antimultiplicative. Let $\alpha,\beta\in R_{\mathrm{irr}}$. Inserting $r=\alpha\beta$ in (3.22), using (0.1) and (3.18) we get

$$\begin{aligned}(u^{\alpha\beta})^{*\otimes *}&=(u^{\alpha\beta})^{-1}=(u^\alpha\mathbin{\bigcirc\!\!\!\!\top} u^\beta)^{-1}\\ &=(\tau\otimes\mathrm{id})\,\{(u^\beta)^{*\otimes *}\mathbin{\bigcirc\!\!\!\!\top}(u^\alpha)^{*\otimes *}\}.\end{aligned} \tag{3.23}$$

For any $r\in R$ and any $\rho\in B(H_r)'$ we denote by ρ^* the linear functional on $B(H_r)$ such that $\rho^*(m^*)=\overline{\rho(m)}$ for any $m\in B(H_r)$. Then $(\rho^*\otimes\mathrm{id})((u^r)^{*\otimes *})=((\rho\otimes\mathrm{id})\,u^r)^*$.

Let $\rho\in B(H_\alpha)'$ and $\sigma\in B(H_\beta)'$. Clearly $(\rho^*\otimes\sigma^*)\,\tau=\sigma^*\otimes\rho^*$. Therefore applying $\rho^*\otimes\sigma^*\otimes\mathrm{id}$ to both sides of (3.23) and taking into account (3.16) we get

$$(u^\alpha_\rho\,u^\beta_\sigma)^*=(u^\beta_\sigma)^*\,(u^\alpha_\rho)^*.$$

This formula shows that (3.20) is antimultiplicative and $\mathscr{A}$ is a $*$-algebra. From now on we shall use the commonly accepted notation: For $v \in B(H) \otimes \mathscr{A}$ we write v^* instead of $v^{*\otimes *}$.

To end the proof we notice that the formula (2.12) implies that

$$(\rho^j \otimes \mathrm{id})\, v^{\bar{r}} = ((\rho \otimes \mathrm{id})\, v^r)^*$$

for any $\rho \in B(H_r)'$. Therefore

$$\begin{aligned}\varphi_M((u_\rho^\alpha)^*) &= \varphi_M(u_{\rho j}^{\bar{\alpha}}) = (\rho^j \otimes \mathrm{id})\, v^{\bar{\alpha}} \\ &= ((\rho \otimes \mathrm{id})\, v^\alpha)^* = \varphi_M(u_\rho^\alpha)^*. \quad \square\end{aligned}$$

For any $\alpha \in R_{\mathrm{irr}}$ and any $\rho \in B(H_\alpha)'$ we set

$$\kappa(u_\rho^\alpha) = (\rho \otimes \mathrm{id})\, ((u^\alpha)^*). \tag{3.24}$$

Clearly this formula defines unique linear map

$$\kappa \colon \mathscr{A} \to \mathscr{A}. \tag{3.25}$$

Proposition 3.4. 1. *κ is antimultiplicative.*

2. *For any $a \in \mathscr{A}$*

$$\kappa(\kappa(a^*)^*) = a. \tag{3.26}$$

3. *For any $r \in R$*

$$(\mathrm{id} \otimes \kappa)\, (u^r) = (u^r)^{-1}. \tag{3.27}$$

Proof. Formula (3.24) means that $(\mathrm{id} \otimes \kappa)\, u^\alpha = (u^\alpha)^*$. Let $r \in R$. Using (3.13) and performing standard computations we get $(\mathrm{id} \otimes \kappa)\, u^r = (u^r)^*$ and formula (3.27) follows.

Let us compute $\kappa(u_\rho^\alpha u_\sigma^\beta)$, where $\alpha, \beta \in R_{\mathrm{irr}}$, $\rho \in B(H_\alpha)'$ and $\sigma \in B(H_\beta)'$. Using (3.16), (3.27), (3.18) and (0.1) we obtain

$$\begin{aligned}\kappa(u_\rho^\alpha u_\sigma^\beta) &= (\rho \otimes \sigma \otimes \kappa)\, u^{\alpha\beta} \\ &= (\rho \otimes \sigma \otimes \mathrm{id})\, ((u^{\alpha\beta})^{-1}) \\ &= ((\rho \otimes \sigma)\, \tau \otimes \mathrm{id})\, ((u^\beta)^{-1} \textcircled{\scriptsize T} (u^\alpha)^{-1}) \\ &= (\sigma \otimes \rho \otimes \mathrm{id})\, ((u^\beta)^{-1} \textcircled{\scriptsize T} (u^\alpha)^{-1}) = \kappa(u_\sigma^\beta)\, \kappa(u_\rho^\alpha).\end{aligned}$$

Statement 1 is proved.

To prove Statement 2 we notice that

$$\kappa((u_\rho^\alpha)^*) = \kappa(u_{\rho j}^{\bar{\alpha}}) = (\rho^j \otimes \mathrm{id})\, ((u^{\bar{\alpha}})^*) = (u_{\rho j *}^{\bar{\alpha}})^* = u_{\rho j * j}^\alpha.$$

Therefore

$$\kappa(\kappa((u_\rho^\alpha)^*)^*) = u_{\rho j * j j * j}^\alpha = u_\rho^\alpha$$

and (3.26) follows. $\square$

We shall also use the following

Proposition 3.5. *The functional $h \in \mathscr{A}'$ introduced by* (3.6) *is positive and faithful:*

$$h(a\,a^*) > 0 \tag{3.28}$$

for any non-vanishing $a \in \mathscr{A}$.

Proof. Assume that a is given by (3.2). Then

$$h(a\,a^*) = \sum_{\alpha\beta} h(u^{\beta}_{\rho_\beta}(u^{\alpha}_{\rho_\alpha})^*). \tag{3.29}$$

Using (3.19), (3.16) and (3.13) we have

$$\begin{aligned} h(u^{\beta}_{\rho_\beta}(u^{\alpha}_{\rho_\alpha})^*) &= h(u^{\beta}_{\rho_\beta}\, u^{\bar\alpha}_{\rho^j_\alpha}) \\ &= (\rho_\beta \otimes \rho^j_\alpha \otimes h)\, u^{\beta\bar\alpha} \\ &= (\rho_\beta \otimes \rho^j_\alpha \otimes h) \sum_k (p_k \otimes I)\, u^{\alpha_k} (p^*_k \otimes I) \end{aligned}$$

where $p_k \in \mathrm{Mor}(\alpha_k, \beta\bar\alpha)$, $\alpha_k \in R_{\mathrm{irr}}$ and $\sum p_k p^*_k = I_{\beta\bar\alpha}$. Taking into account (3.6) we see that

$$h(u^{\beta}_{\rho_\beta}(u^{\alpha}_{\rho_\alpha})^*) = (\rho_\beta \otimes \rho^j_\alpha)\,(\sum_k{}' p_k p^*_k),$$

where $\sum_k{}'$ denotes the sum restricted to $k's$ satisfying condition $\alpha_k = \mathbb{1}$. For these $k's$, $p_k \in \mathrm{Mor}(\mathbb{1}, \beta\bar\alpha)$. If $\beta \neq \alpha$ then in virtue of (2.6) $p_k = 0$ and $h(u^{\beta}_{\rho_\beta}(u^{\alpha}_{\rho_\alpha})^*) = 0$. If $\beta = \alpha$ then using again (2.6) we see that $\sum_k{}' p_k p^*_k = \lambda_\alpha t_{j_\alpha} t^*_{j_\alpha}$; where λ_α is a non-negative numerical factor. In fact λ_α is strictly positive. To see this we notice that $p^*_k t_{j_\alpha}$ belongs to $\mathrm{Mor}(\mathbb{1}, \alpha_k)$ and (cf. (2.5)) $p^*_k t_{j_\alpha} = 0$ if $\alpha_k \neq \mathbb{1}$. Therefore

$$\begin{aligned} t_{j_\alpha} &= \sum_k p_k p^*_k t_{j_\alpha} = \sum_k{}' p_k p^*_k t_{j_\alpha} \\ &= \lambda_\alpha t_{j_\alpha} t^*_{j_\alpha} t_{j_\alpha} \end{aligned}$$

and $\lambda_\alpha = (t^*_{j_\alpha} t_{j_\alpha})^{-1} > 0$.

Using (1.6) and performing rather simple computations one may check that

$$\begin{gathered} t^*_{j_\alpha} t_{j_\alpha} = \mathrm{Tr}(F_\alpha) \\ (\rho_\alpha \otimes \rho^j_\alpha)\,(t_{j_\alpha} t^*_{j_\alpha}) = \mathrm{Tr}(F_\alpha \hat\rho_\alpha \hat\rho^*_\alpha) \end{gathered}$$

where $\hat\rho_\alpha$ is the density corresponding to the functional ρ_α (i.e. $\hat\rho_\alpha \in B(H_\alpha)$ and $\rho_a(m) = \mathrm{Tr}(\hat\rho_\alpha m)$ for all $m \in B(H_\alpha)$) and $F_\alpha = j^*_\alpha j_\alpha$. Inserting these date into (3.29) we get

$$h(a\,a^*) = \sum_\alpha \frac{\mathrm{Tr}(F_\alpha \hat\rho_\alpha \hat\rho^*_\alpha)}{\mathrm{Tr}(F_\alpha)}$$

and (3.28) follows. □

Now let $\|\cdot\|$ be the largest C^*-seminorm on the $*$-algebra $\mathscr{A}$. In virtue of Prop. 3.5, $\|\cdot\|$ is a norm. Let A be the completion of $\mathscr{A}$ with respect to this norm. Then A is a C^*-algebra with unity, $\mathscr{A}$ is a dense $*$-subalgebra of A and any $*$-homomorphism of $\mathscr{A}$ into a C^*-algebra B can be extended to a C^*-homomorphism of A into B. For each $r \in R$, u^r is a unitary element of $B(H_r) \otimes A$ and formulae (3.9) and (3.18) show that $(A, \{u^r\}_{r \in R})$ is a model of R. Therefore (A, u) where $u = u^f$ is an R-admissible pair.

We shall prove that $\mathscr{A}$ coincides with the $*$-subalgebra of A generated by matrix elements of u. Indeed denoting the latter algebra by $\mathscr{A}_0$ we obviously have $\mathscr{A}_0 \subset \mathscr{A}$. On the other hand in virtue of Prop. 2.6, $u^s \in B(H_s) \otimes \mathscr{A}_0$ for any $s \in R$. In particular $u^\alpha_\rho \in \mathscr{A}_0$ for any $\alpha \in R_{\text{irr}}$ and $\rho \in B(H_\alpha)'$. Therefore $\mathscr{A} \subset \mathscr{A}_0$ and the conclusion follows.

Assume that (B, v) is another R-admissible pair. Let $M = (B, \{v^r\}_{r \in R})$ be the model of R such that $v^f = v$, φ_M be the $*$-homomorphism of $\mathscr{A}$ into B related to M and $\tilde{\varphi}_M$ be the extension of φ_M to A. Then $\tilde{\varphi}_M$ is a C^*-homomorphism and (cf. (3.11)) $(\mathrm{id} \otimes \tilde{\varphi}_M) u = v$. It shows that (A, u) is the universal R-admissible pair.

Let $G = (A, u)$. We shall prove that G is a compact matrix pseudogroup. To this end we have to verify conditions CMP I–III. We already know that CMP I holds. To prove CMP II we notice that $(A \otimes A, \{u^r \mathbin{ⓣ} u^r\}_{r \in R})$ is a model for R. Therefore $(A \otimes A, u \mathbin{ⓣ} u)$ is an R-admissible pair. Existence of Φ follows now from the universality of (A, u). CMP III follows immediately from Prop. 3.4 (insert $r = f$ in (3.27)). This proves the Statement 1 of Theorem 1.3.

The "if" part of Statement 3 and Statement 4 of Theorem 1.3 follows immediately from the fact that $(A, \{u^r\}_{r \in R})$ is a model of R (cf. (1.12) and (1.11)). The "only if" part of Statement 3 coincides with Prop. 3.1.3.

To prove Statement 2 we notice that the comultiplication Φ coincides with the C^*-homomorphism $\tilde{\varphi}_M$ associated with the model $M = (A \otimes A, \{u^r \mathbin{ⓣ} u^r\}_{r \in R})$. Therefore (cf. (3.11))

$$(\mathrm{id} \otimes \phi)\, u^r = u^r \mathbin{ⓣ} u^r$$

and u^r is a unitary representation of G for any $r \in R$.

Assume that v is a unitary representation of G acting on a $f-d$. Hilbert space H. Let $\bar{u}$ be a complex conjugation of the fundamental representation. Clearly the tensor product of a certain number of u and $\bar{u}$ is of the form u^r, where $r \in R$. Since $\{u, \bar{u}\}$ generate $\operatorname{Rep} G$, one can find a finite family $\{b_k\}$ of intertwiners (b_k intertwins u^{r_k} with $v, r_k \in R$) such that $\sum_k b_k b_k^* = I_{B(H)}$. Then $b_l^* b_k$ intertwins u^{r_k} with u^{r_l} and in virtue of Theorem 1.3.3 $b_l^* b_k \in \operatorname{Mor}(r_k, r_l)$ for all k, l. Using Prop. 1.1 one can find $s \in R$ such that $H_s = H$ and $b_k \in \operatorname{Mor}(r_k, s)$ for all k. Now we have

$$\begin{aligned} v &= \sum_k v(b_k \otimes I)(b_k^* \otimes I) = \sum_k (b_k \otimes I)\, u^{r_k} (b_k^* \otimes I) \\ &= \sum_k u^s (b_k \otimes I)(b_k^* \otimes I) = u^s. \end{aligned}$$

It shows that any unitary representation of G is of the form u^r for some $r \in R$ and Statement 2 holds.

The proof of Theorem 1.3 is complete.

4. Twisted $SU(N)$ groups

It is well known that the algebra of operators intertwining the n-th power of the fundamental representation of $SU(N)$ group with itself is generated by the representation of the permutation group of n elements acting naturally on $H_f^{\otimes n}$. This is the place where the Young diagrams (which originally were invented to classify the representations of the permutation group) enter into the representation theory of $SU(N)$. We shall prove the analogous result for a twisted $SU(N)$ group. In this case the group algebra of the permutation group should be replaced by the Hecke algebra (see e.g. [4]).

At first however, we have to present

Proof of Theorem 1.4. It is sufficient to construct a concrete monoidal W^*-category R with the distinguished object f satisfying the assumptions of Theorem 1.3 such that a pair (A, u) is R-admissible if and only if the matrix elements of u satisfy the relations (1.15)–(1.17).

At first we describe a smaller category R_0. R will be obtained by completion (cf. Prop. 2.7) of R_0.

All objects of R_0 are powers of the distinguished object: $R_0=\{f^0=\mathbb{1}, f, f^2, \ldots\}\cdot H_f=\mathbb{C}^N$ (the scalar product on $\mathbb{C}^N$ is the standard one) and consequently $H_{f^n}=(\mathbb{C}^N)^{\otimes n}$. In order to define $\mathrm{Mor}(f^n, f^m)$ we use the linear mapping

$$E\colon \mathbb{C}\to H_{f^N}$$

introduced by the formula

$$E(1)=\sum_{k_1,\ldots,k_N} E_{k_1k_2\ldots k_N}\,\varepsilon_{k_1}\otimes\varepsilon_{k_2}\otimes\ldots\otimes\varepsilon_{k_N}$$

where $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ is the canonical basis in $\mathbb{C}^N$. An element of $B(H_{f^n}, H_{f^m})$ is said to be a monomial if it is a composition of mappings of the form $I_{f^k}\otimes E^{\#}\otimes I_{f^l}$, where k, l are non-negative integers, I_{f^k} and I_{f^l} are identity mappings acting on H_{f^k} and H_{f^l} and $E^{\#}$ denotes either E or E^*. If $n=m$ then I_{f^n} is also included into the set of monomials. $\mathrm{Mor}(f^n, f^m)$ is by definition the set of all linear combinations of monomials belonging to $B(H_{f^n}, H_{f^m})$. Clearly, $\mathrm{Mor}(f^n, f^m)$ contains non-zero elements if and only if $n\equiv m \bmod N$.

One can easily check that

$$R_0=(R_0, \{H_r\}_{r\in R_0}, \{\mathrm{Mor}(r,s)\}_{r,s\in R_0}, \cdot, f)$$

described above satisfies the conditions CMW* I–IV and CMW* VIII–X i.e. R_0 is a concrete monoidal W^*-category with distinguished object.

Let R be the completion of R_0 (cf. Prop. 2.7). We shall prove that R contains a complex conjugation of f. To this end we consider the morphism $q\in\mathrm{Mor}(f^{N-1}, f^{N-1})$ introduced by the formula

$$q=(E^*\otimes I_{f^{N-1}})(I_{f^{N-1}}\otimes E). \tag{4.1}$$

One can easily check that for any $z\in H_{f^{N-1}}$

$$q\,z=\sum_{k=1}^{N} x_k(y_k\,|\,z) \tag{4.2}$$

where x_k and y_k $(k=1,2,\ldots,N)$ are elements of $H_{f^{N-1}}$ introduced by the formulae

$$x_k=\sum_{i_2,\ldots,i_N} E_{ki_2\ldots i_N}\,\varepsilon_{i_2}\otimes\ldots\otimes\varepsilon_{i_N}$$

$$y_k=\sum_{i_1,\ldots,i_{N-1}} E_{i_1\ldots i_{N-1}k}\,\varepsilon_{i_1}\otimes\ldots\otimes\varepsilon_{i_{N-1}}.$$

Since E is left (right resp.) non-degenerate, vectors $x_1,x_2,\ldots,x_N$ ($y_1,y_2,\ldots,y_N$ resp.) are linearly independent. Therefore q is a linear mapping of rank N and the space

$$\bar{H}=q(H_{f^{N-1}})$$

is N-dimensional. Clearly $(x_1,x_2,\ldots,x_N)$ is a basis in $\bar{H}$ (not orthonormal in general).

Let $p\in B(H_{f^{N-1}})$ be the orthogonal projection onto $\bar{H}$ and $i\in B(\bar{H},H_{f^{N-1}})$ be the embedding. Remembering that $q\in\mathrm{Mor}(f^{N-1},f^{N-1})$ and that $\mathrm{Mor}(f^{N-1},f^{N-1})$ is a W^*-algebra one easily concludes that $p\in\mathrm{Mor}(f^{N-1},f^{N-1})$. Moreover there exists $q'\in\mathrm{Mor}(f^{N-1},f^{N-1})$ such that

$$q\,q'=p. \tag{4.3}$$

In virtue of CMW* VI there exists an object $s\in R$ such that $H_s=\bar{H}$ and $i\in\mathrm{Mor}(s,f^{N-1})$.

Let j: $H_f\to H_s$ be the antilinear mapping such that $j(\varepsilon_k)=i^*x_k$. Since $(x_k)_{k=1,2,\ldots,N}$ is a basis in $\bar{H}=H_s$, j is invertible. We have

$$\begin{aligned} t_j&=(I_f\otimes i^*)\,E\in\mathrm{Mor}(\mathbb{1},fs)\\ \bar{t}_j&=E^*(q'i\otimes I_f)\in\mathrm{Mor}(sf,\mathbb{1}). \end{aligned} \tag{4.4}$$

The first equation follows immediately from definition (1.6) with (e_l) replaced by (ε_l). To prove the second we notice that $E^*(z\otimes y)=\sum_k(y_k|z)(\varepsilon_k|y)$ for any $z\in H_{f^{N-1}}$ and $y\in H_f$. Therefore for any $x\in H_s$ and $y\in H_f$ we have

$$\begin{aligned} E^*(q'i\otimes I_f)(x\otimes y)&=E^*(q'ix\otimes y)=\sum_k(y_k|q'ix)(\varepsilon_k|y)\\ &=\sum_k(y_k|q'ix)(j^{-1}(i^*x_k)|y)=(j^{-1}(i^*(\sum_k x_k(y_k|q'ix)))|y). \end{aligned}$$

Now using (4.2), (4.3) and remembering that p is the projection onto $\bar{H}=i(H_s)$ we get

$$E^*(q'i\otimes I_f)(x\otimes y)=(j^{-1}(x)|y)$$

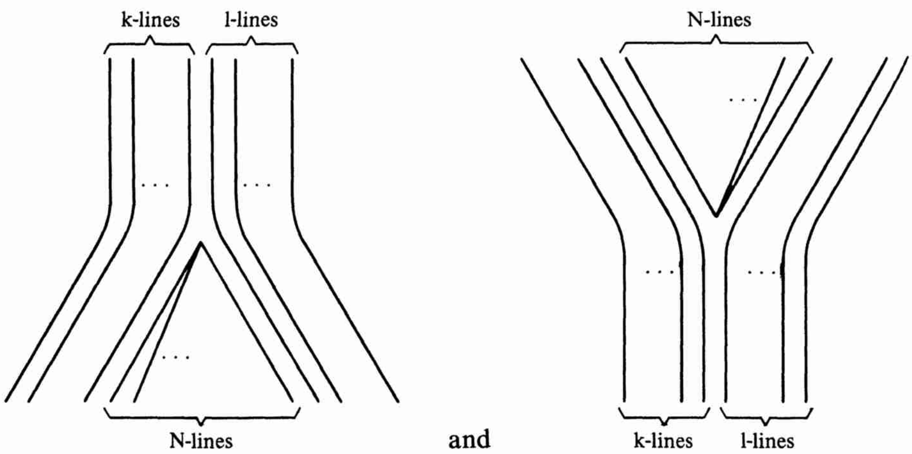
and Eq. (4.4) follows (cf. (1.7)). This way we proved that s is a complex conjugation of f.

One can easily construct a model of R_0. Indeed, if u_{kl} $(k,l=1,2,\ldots,N)$ are elements of a C^*-algebra A with unity I satisfying (1.15)–(1.17) then setting $u=(u_{kl})_{k,l=1,\ldots,N}$, $u^{f^n}=u^{\textcircled{T}n}$ $(n=1,2,\ldots;\ u^{\textcircled{T}n}$ denotes the $\textcircled{T}$-product of n copies of u) and $u^{\mathbb{1}}=I$ we define a model $(A,\{u^r\}_{r\in R_0})$ of R_0. Clearly, any model of R_0 is of this form. Therefore a pair (A,u) (where A is a C^*-algebra with unity and $u \in B(\mathbb{C}^N) \otimes A)$ is R-admissible if and only if the matrix elements of u satisfy relations (1.15)–(1.17). Now the statement of Theorem 1.4 follows from Theorem 1.3.1. □

In order to prove Theorem 1.5 we have to analyse more in detail the structure of the morphism sets $\mathrm{Mor}(f^n, f^m)$. In this analysis we deal with complicated monomials being products of many factors of the form $I_{f^k} \otimes E^{\#} \otimes I_{f^l}$ (typically the number of factors will be of the order of N^2). For this reason we have to use the graphic notation described below.

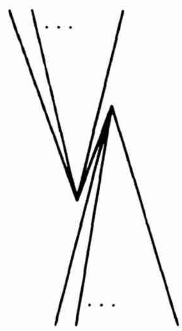
In this notation monomials will be represented by diagrams consisting of vertices and lines of approximately vertical direction. Each vertex is either the end point of N lines coming from up (then we say that the vertex points down) or the start point of N lines going down (the vertex points up). Each line either has a free upper (lower resp.) end or starts from (ends up at resp.) a vertex. Lines having free upper (lower resp.) end are called incoming (outgoing resp.). Lines must not intersect (later we admit this possibility giving the special meaning to the intersections).

The monomial represented by a diagram α will be denoted by $[\alpha]$. $[\alpha] \in \mathrm{Mor}(f^n, f^m)$ if and only if α has n incoming and m outgoing lines. I_{f^n} is represented by the diagram consisting of n vertical lines (and no vertex). Elementary monomials $I_{f^k} \otimes E \otimes I_{f^l}$ and $I_{f^k} \otimes E^* \otimes I_{f^l}$ are represented by the diagrams



respectively.

The diagram representing the composition of monomials $[\alpha][\beta]$ is obtained by placing the diagram α below the diagram β and connecting the lower ends of β with the corresponding upper ends of α (notice that $[\alpha][\beta]$ is well defined only if the number of outgoing lines of β equals the number of incoming lines of α). For example the morphism (4.1) is represented by the diagram



The above rules give the complete description of our graphic notation. To any given monomial one can easily assign the representing diagram. Conversely cutting a given diagram α with horizontal lines into elementary pieces (each consisting one vertex) one finds all factors of the form $I_{f^k}\otimes E^{\#}\otimes I_{f^l}$ entering into the decomposition of $[\alpha]$. Let us notice that the diagram representing $[\alpha]^*$ can be obtained by mirroring α with respect to a horizontal line. Moreover $[\alpha]\otimes[\beta]=[\alpha\beta]$, where $\alpha\beta$ is a diagram obtained by drawing α side by side β (α on the left).

In what follows we shall need an explicit formula for matrix elements of the monomial $[\alpha]$ corresponding to a given diagram α. Let $V(\alpha)$ and $L(\alpha)$ be the set of all vertices and the set of all lines of the diagram α. For each $w\in V(\alpha)$, $w_1, w_2, \dots, w_N$ will denote the lines starting from (or ending at) the vertex w listed in the natural order (w_1 is the most left line). Moreover $\alpha_1^{\text{in}}, \alpha_2^{\text{in}}, \dots, \alpha_n^{\text{in}}$ ($\alpha_1^{\text{out}}, \alpha_2^{\text{out}}, \dots, \alpha_m^{\text{out}}$ resp.) will denote the incoming (outgoing resp.) lines of diagram α listed in the natural order. Using the induction with respect to the number of vertices of α one can easily show that for any $i_1, i_2, \dots, i_m, j_1, j_2, \dots, j_n = 1, 2, \dots, N$ we have

$$(\varepsilon_{i_1}\otimes\varepsilon_{i_2}\otimes\dots\otimes\varepsilon_{i_m}|[\alpha]\,\varepsilon_{j_1}\otimes\varepsilon_{j_2}\otimes\dots\otimes\varepsilon_{j_n})=\sum_{\lambda}\Big(\prod_{w\in V(\alpha)} E^w_\lambda\Big) \tag{4.5}$$

where E^w_λ equals $E_{\lambda(w_1),\dots,\lambda(w_N)}$ ($\bar{E}_{\lambda(w_1),\dots,\lambda(w_N)}$ resp.) if the vertex w points up (down resp.) and $\sum_\lambda$ denotes the sum over all mappings

$$\lambda\colon L(\alpha)\to\{1, 2, \dots, N\} \tag{4.6}$$

such that $\lambda(\alpha_k^{\text{in}})=j_k$ ($k=1, 2, \dots, n$) and $\lambda(\alpha_k^{\text{out}})=i_k$ ($k=1, 2, \dots, m$). Each of the mappings (4.6) is called a labelling. To any line $l\in L(\alpha)$ it assigns the label $\lambda(l)$. The labelling is said to be regular if the corresponding term in (4.5) does not vanish.

Starting from this place we assume that the array E_- is given by the formula (1.18). One can easily check that E_- is left and right non-degenerate.

Lemma 4.1. *Let α be a diagram having no incoming lines and N outgoing lines $\alpha_1^{\text{out}}, \alpha_2^{\text{out}}, \ldots, \alpha_N^{\text{out}}$ and $i_1, i_2, \ldots, i_N = 1, 2, \ldots, N$. Then*

$$\sum_{\lambda} \Big(\prod_{w \in V(\alpha)} E_{\lambda}^{w} \Big) = c_\alpha E_{i_1, i_2, \ldots, i_N} \tag{4.7}$$

where the summation runs over all labellings λ such that $\lambda(\alpha_k^{\text{out}}) = i_k$ $(k = 1, 2, \ldots, N)$ and c_α is the value of the left hand side for $(i_1, i_2, \ldots, i_N) = (1, 2, \ldots, N)$.

Proof. Let $V^{\text{up}}(\alpha)$ ($V^{\text{down}}(\alpha)$ resp.) be the set of all vertices of the diagram α pointing up (down resp.). Clearly

$$\# V^{\text{up}}(\alpha) = 1 + (\# V^{\text{down}}(\alpha)).$$

($\# A$ denotes the number of elements of A). Let λ be a regular labelling. Then at each vertex we have precisely one line with a given label i. Therefore denoting by L_i^{out} the number of outgoing lines with label i we have

$$\# V^{\text{up}}(\alpha) = L_i^{\text{out}} + (\# V^{\text{down}}(\alpha)).$$

Comparing the two relations we get $L_i^{\text{out}} = 1$ for all $i = 1, 2, \ldots, N$. If in the sequence $(i_1, i_2, \ldots, i_N)$ a number i occurs more than once, then $L_i^{\text{out}} > 1$ and the above reasoning shows that in the sum (4.7) there is no non-vanishing term. In this case both sides of (4.7) vanish and the formula holds. Therefore we may assume that $(i_1, i_2, \ldots, i_N)$ is a permutation of $(1, 2, \ldots, N)$.

Let i be one of the numbers $1, 2, \ldots, N-1$ such that $i, i+1$ occur in the sequence $(i_1, i_2, \ldots, i_N)$ in the reverse order (i.e. $i+1$ precedes i) and λ be a regular labelling such that $\lambda(\alpha_k^{\text{out}}) = i_k$ $(k = 1, 2, \ldots, N)$. We shall consider the following walk on α. We start at the lowest point of the outgoing line with label i and move alternately up and down using the lines with labels i and $i+1$ resp., until we reach the lowest point of the outgoing line with label $i+1$. At this moment we end our walk.

Let C be the path passed this way. Clearly, C is a non-self-intersecting (there is no vertex visited more than once) broken line (open polygon). At each vertex belonging to C we make a left or right turn of almost 180°. Remembering that $i+1$ precedes i in the sequence $(i_1, i_2, \ldots, i_N)$ so the ending point of our walk is placed on the left with respect to the starting point one can easily show that the number of left turns exceeds by one the number of right turns:

$$\# V_L = \# V_R + 1 \tag{4.8}$$

where V_L (V_R resp.) denotes the set of all these vertices belonging to C, where we make a left (right resp.) turn.

Let λ' be the labelling obtained from λ by exchanging the labels i and $i+1$ on the lines belonging to C:

$$\lambda'(k) = \begin{cases} i & \text{if } k \subset C \text{ and } \lambda(k) = i+1 \\ i+1 & \text{if } k \subset C \text{ and } \lambda(k) = i \\ \lambda(k) & \text{if } k \not\subset C. \end{cases}$$

Then $\lambda'(\alpha_k^{\mathrm{out}})=j_k$, where $(j_1, j_2, \ldots, j_N)$ is the sequence obtained from $(i_1, i_2, \ldots, i_N)$ by exchanging i with $i+1$. Notice that $i, i+1$ occur in $(j_1, j_2, \ldots, j_N)$ in the natural order.

It follows immediately from (1.18) that

$$E_\lambda^w = \begin{cases} -\mu E_{\lambda'}^w & \text{if } w \in V_L \\ -\dfrac{1}{\mu} E_{\lambda'}^w & \text{if } w \in V_R \\ E_{\lambda'}^w & \text{otherwise.} \end{cases}$$

Taking into account (4.8) we get

$$\prod_{w \in V(\alpha)} E_\lambda^w = -\mu \prod_{w \in V(\alpha)} E_{\lambda'}^w. \tag{4.9}$$

Denote the left hand side of (4.7) by $\tilde{E}_{i_1, \ldots, i_N}$. Summing both sides of (4.9) over all labellings λ such that $\lambda(\alpha_k^{\mathrm{out}})=i_k$ $(k=1, 2, \ldots, N)$ we get

$$\tilde{E}_{i_1 \ldots i_N} = -\mu \tilde{E}_{j_1 \ldots j_N}. \tag{4.10}$$

Let $I(i_1, i_2, \ldots, i_N)$ be the number of inversed pairs in the sequence $(i_1, i_2, \ldots, i_N)$. Using $I(i_1, i_2, \ldots, i_N)$ times the formula (4.10) we obtain

$$\tilde{E}_{i_1 \ldots i_N} = (-\mu)^{I(i_1, \ldots, i_N)} \tilde{E}_{1, 2, \ldots, N}.$$

This formula coincides with (4.7). $\square$

Taking into account (4.5) we get

$$\mathrm{Mor}(\mathbb{1}, f^N) = \{c E: c \in \mathbb{C}\}. \tag{4.11}$$

Let $p \in \mathrm{Mor}(f, f)$ and $x \in H_f$. Since E_- is left nondegenerate, there exists a linear functional φ defined on $H_{f^{N-1}}$ such that $x=(I_f \otimes \varphi) E(1)$. Therefore

$$p x = (p \otimes \varphi) E(1) = (I_f \otimes \varphi)(p \otimes I_{f^{N-1}}) E(1).$$

Clearly $(p \otimes I_{f^{N-1}}) E \in \mathrm{Mor}(\mathbb{1}, f^N)$. In virtue of (4.11) there exists $c \in \mathbb{C}$ such that $(p \otimes I_{f^{N-1}}) E = c E$. Therefore $p x = (I_f \otimes \varphi) c E(1) = c x$ and

$$\mathrm{Mor}(f, f) = \{c I_f: c \in \mathbb{C}\}. \tag{4.12}$$

It means that the fundamental representation of $S_\mu U(N)$ is irreducible.

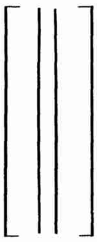
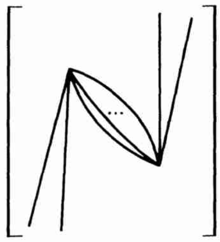
For any natural number n we set

$$\mathrm{Fact}_\mu(n) = \sum_i \mu^{2I(i)} = \prod_{k=1}^{n} \frac{1-\mu^{2k}}{1-\mu^2}$$

where the summation runs over all permutations i of the set $\{1, 2, \ldots, n\}$ and $I(i)$ denotes the number of inversed pairs in the sequence $(i_1, \ldots, i_N)$. Clearly $\mathrm{Fact}_1(n)=n!$. The twisted factorial $\mathrm{Fact}_\mu(n)$ will be used in the remaining part of this section.

Let

$$\sigma = I_{f^2} - \frac{\mu^{-2N+4}}{\mathrm{Fact}_\mu(N-2)} (I_{f^2} \otimes E^*)(E \otimes I_{f^2})$$

$=$  $- \dfrac{\mu^{-2N+4}}{\mathrm{Fact}_\mu(N-2)}$ 

Clearly $\sigma \in \mathrm{Mor}(f^2, f^2)$. Using (4.5) one can easily check that

$$\sigma(\varepsilon_a \otimes \varepsilon_b) = \begin{cases} \mu\, \varepsilon_b \otimes \varepsilon_a & \text{for } a<b \\ \varepsilon_b \otimes \varepsilon_a & \text{for } a=b \\ \mu\, \varepsilon_b \otimes \varepsilon_a + (1-\mu^2)\, \varepsilon_a \otimes \varepsilon_b & \text{for } a>b. \end{cases} \tag{4.13}$$

If $\mu=1$ then $\sigma(x \otimes y) = y \otimes x$ for all $x, y \in H_f$. It shows that in this case the fundamental representation u commutes with itself. According to the proof of Theorem 1.4 (see beginning of this section) the complex conjugation $\bar{u}$ is contained in the $(N-1)$-th power of u. Therefore $\bar{u}$ commutes with u and (cf. [8] Prop. 2.4) the algebra of the continuous functions on $S_1U(N)$ is commutative. In this case $S_1U(N)$ may be identified with a group of $N \times N$ matrices and using formula (1.21) of [8] we get $S_1U(N) = SU(N)$.

The formula (4.13) can be used to check the following relations

$$\sigma^* = \sigma \tag{4.14}$$

$$\sigma^2 = (1-\mu^2)\,\sigma + \mu^2 I_{f^2} \tag{4.15}$$

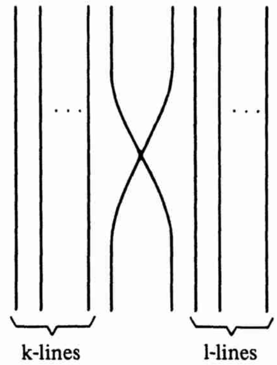
$$(\sigma \otimes I_f)(I_f \otimes \sigma)(\sigma \otimes I_f) = (I_f \otimes \sigma)(\sigma \otimes I_f)(I_f \otimes \sigma) \tag{4.16}$$

$$(I_{f^k} \otimes \sigma \otimes I_{f^{N-k-2}})\, E = -\mu^2 E \tag{4.17}$$

$$E^*(I_{f^k} \otimes \sigma \otimes I_{f^{N-k-2}}) = -\mu^2 E^*. \tag{4.18}$$

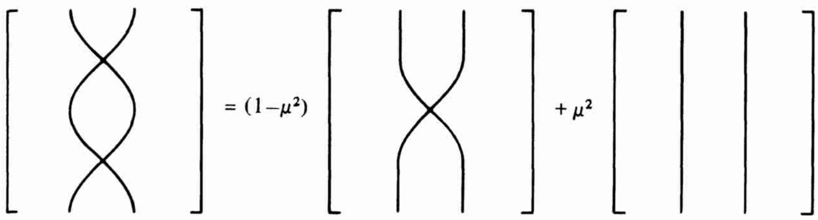
In the two last equations $k = 0, 1, \ldots, N-2$.

Now we have to extend our graphic notation. We say that an element of $B(H_{f^n}, H_{f^m})$ is a quasi-monomial if it is a composition of monomials and mappings of the form $I_{f^k} \otimes \sigma \otimes I_{f^l}$, where k, l are non-negative integers. Completing the list of elementary diagrams with the diagram

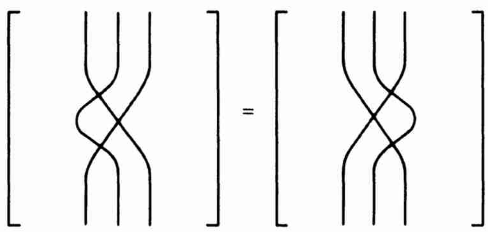


(4.19)

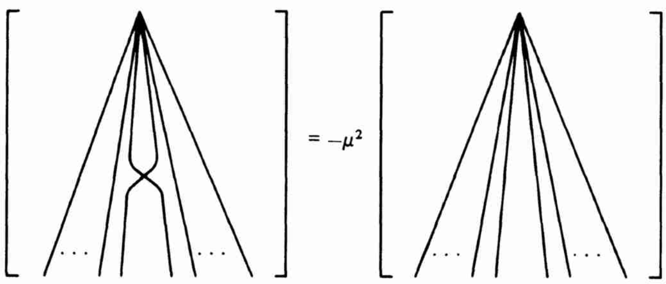
representing the morphism $I_{f^k}\otimes\sigma\otimes I_{f^l}\in\mathrm{Mor}(f^{k+l+2}, f^{k+l+2})$ we can represent graphically any quasi-monomial. The symmetry of (4.19) with respect to a horizontal line reminds us that σ is selfadjoint. Relations (4.15)–(4.18) mean that



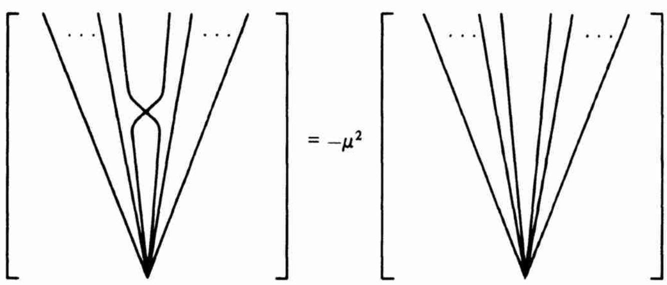
(4.20)



(4.21)



(4.22)



(4.23)

It follows immediately from (4.15) that $\mathrm{Sp}\,\sigma=\{1, -\mu^2\}$. An element $x\in H_{f^2}$ is said to be symmetric (antisymmetric resp.) if $\sigma x=x$ ($\sigma x=-\mu^2 x$ resp.). Using (4.13) one can easily check that an element $x=\sum_{ab} x_{ab}\,\varepsilon_a\otimes\varepsilon_b$ is antisymmetric if and only if

$$x_{ab}=\begin{cases}-\mu\, x_{ba} & \text{for } a>b\\ 0 & \text{for } a=b\\ -\dfrac{1}{\mu}x_{ba} & \text{for } a<b.\end{cases} \tag{4.24}$$

Let $x \in H_{f^n}$. We say that x is completely antisymmetric if

$$(I_{f^k}\otimes\sigma\otimes I_{f^{n-k-2}})\,x=-\mu^2 x$$

for $k=0,1,\dots,n-2$. The subspace of completely antisymmetric elements of $H_{f^n} = H_f^{\otimes n}$ will be denoted by $H_f^{\wedge n}$. Using (4.24) one can easily check that $\dim H_f^{\wedge n} = \binom{N}{n}$ and that $x \in H_f^{\wedge n}$ ($n \leqq N$) if and only if there exists a linear functional $\varphi\colon H_{f^{N-n}} \to \mathbb{C}$ such that

$$x=(I_{f^n}\otimes\varphi)\,E(1). \tag{4.25}$$

Lemma 4.2. *Let $q\in \mathrm{Mor}(f^n, f^n)$, $q^*=q$ and $q\,H_{f^n}\subset H_f^{\wedge n}$. Then q is proportional to the orthogonal projection onto $H_f^{\wedge n}$.*

Proof. It is sufficient to show that there exists a complex number c such that

$$q\,x=c\,x$$

for all $x\in H_f^{\wedge n}$. Clearly $(q\otimes I_{f^{N-n}})\,E\in \mathrm{Mor}(\mathbb{1}, f^N)$. Therefore (cf. (4.11)) $(q\otimes I_{f^{N-n}})\,E=c\,E$ and using (4.25) we get

$$\begin{aligned} q\,x&=(q\otimes\varphi)\,E(1)=(I_{f^n}\otimes\varphi)\,(q\otimes I_{f^{N-n}})\,E(1)\\ &=c\,(I_{f^n}\otimes\varphi)\,E(1)=c\,x. \quad \square\end{aligned}$$

Let $\gamma\in\mathrm{Perm}(n)$ ($\mathrm{Perm}(n)$ denotes the group of all permutations of the set $\{1,2,\dots,n\}$). With the same letter γ we denote a diagram consisting of n (intersecting) lines (and no vertex) such that there is no pair of lines intersecting more than once and $\gamma_k^{\mathrm{in}}=\gamma_{\gamma(k)}^{\mathrm{out}}$ for $k=1,2,\dots,n$. We remind that $\gamma_1^{\mathrm{in}},\gamma_2^{\mathrm{in}},\dots,\gamma_n^{\mathrm{in}}$ ($\gamma_1^{\mathrm{out}},\gamma_2^{\mathrm{out}},\dots,\gamma_n^{\mathrm{out}}$ resp.) denote the incoming (outgoing resp.) lines of the diagram γ listed in the natural order: γ_1^{in}, (γ_1^{out} resp.) is the most left incoming (outgoing resp.) line. For a given permutation γ there exist many diagrams γ satisfying the above conditions, however, due to (4.21) the morphism $[\gamma]\in\mathrm{Mor}(f^n,f^n)$ is uniquely determined. Let us notice that $[\gamma^*]=[\gamma^{-1}]$.

Let

$$A_n=\sum_{\gamma\in\mathrm{Perm}(n)}(\mathrm{sign}\,\gamma)\,[\gamma]. \tag{4.26}$$

Then $A_n^*=A_n$ and using (4.15) one can check that

$$(I_{f^k}\otimes\sigma\otimes I_{f^{n-k-2}})\,A_n=-\mu^2 A_n$$

for $k=0,1,\dots,n-2$. Therefore $A_n H_{f^n}\subset H_f^{\wedge n}$ and using Lemma 4.2 we get

$$A_n=c_n P_n^{\wedge} \tag{4.27}$$

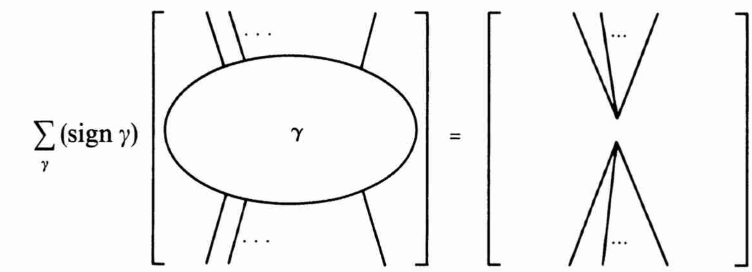
where $P_n^\wedge$ is the orthogonal projection onto $H_f^{\wedge n}$ and $c_n \in \mathbb{C}$. Computing $A_n \varepsilon_1 \otimes \varepsilon_2 \otimes \ldots \otimes \varepsilon_n$ one easily finds that

$$c_n = \mathrm{Fact}_\mu(n). \tag{4.28}$$

In the special case $n=N$ we have $\dim H_f^{\wedge N}=1$, $H_f^{\wedge N}=\mathbb{C}E$, $P_N^\wedge = c_N^{-1} EE^*$ where $c_N = E^*E = \mathrm{Fact}_\mu(N)$ and

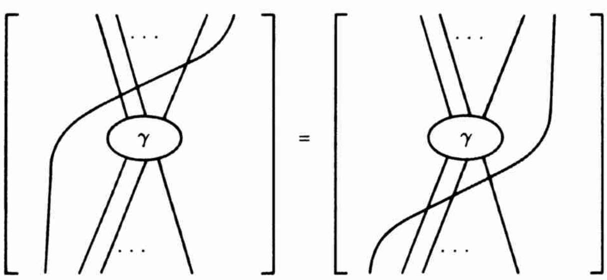
$$A_N = EE^*. \tag{4.29}$$

In diagram notation

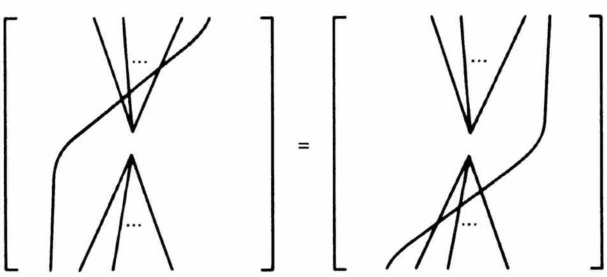


(4.30)

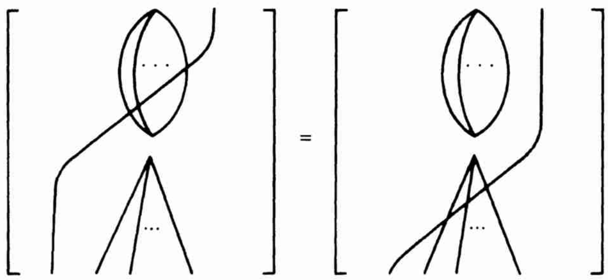
In virtue of (4.21)



Therefore

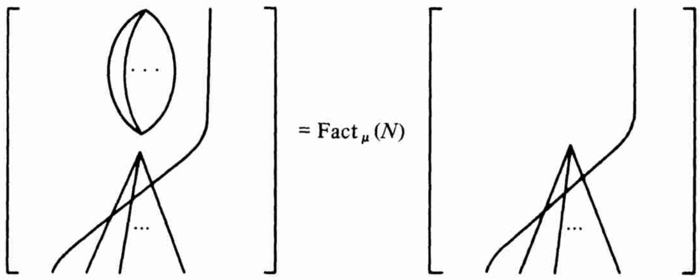


and



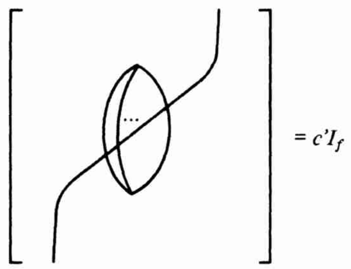
(4.31)

Clearly



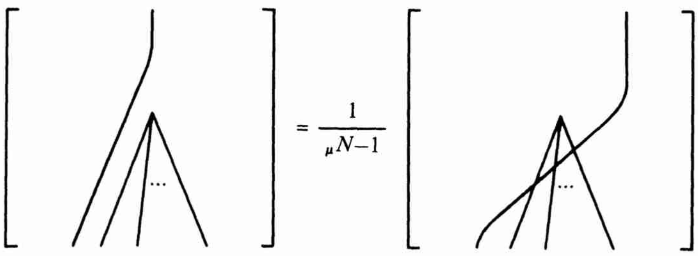
(4.32)

Moreover in virtue of (4.12)



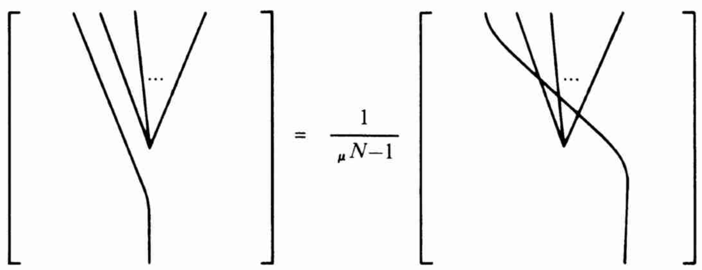
(4.33)

where $c' \in \mathbb{C}$. Computing the action of (4.33) on the vector ε_N we get $c' = \mu^{N-1}\,\mathrm{Fact}_\mu(N)$. Inserting (4.32) and (4.33) into (4.31) we get



(4.34)

Taking the hermitian conjugation of both sides we obtain



(4.35)

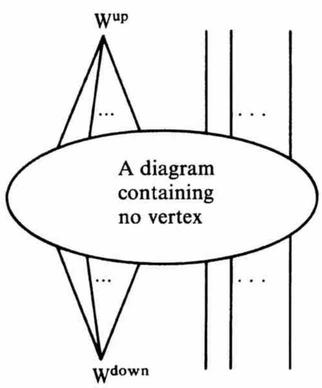
Now we are ready to prove the theorem revealing the structure of the morphism sets $\mathrm{Mor}(f^n, f^n)$.

Theorem 4.3. *$\mathrm{Mor}(f^n, f^n)$ is the subalgebra of $B(H_{f^n})$ generated by elements $I_{f^k} \otimes \sigma \otimes I_{f^{n-k-2}}$ $(k=0, 1, \ldots, n-2)$.*

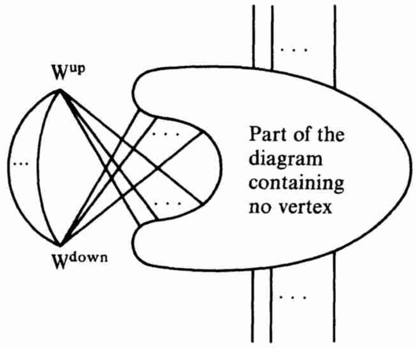
Proof. Let α be a diagram having the same number of incoming and outgoing lines. We have to show that

$$[\alpha] \text{ is a linear combination of quasimonomials represented by diagrams having no vertices.} \tag{4.36}$$

We shall use the induction with respect to the number of pairs of vertices. Assume that α contains two vertices (w^{up} pointing up and w^{down} pointing down). In virtue of (4.34) and (4.35) we may assume that α is of the form

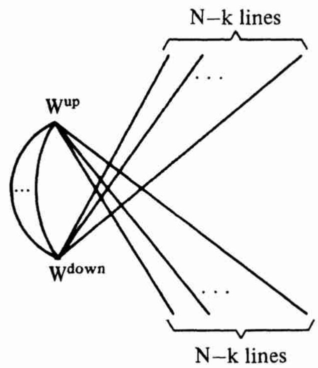


Moreover taking into account (4.22) and (4.23) we may assume that the lines $w_1^{\mathrm{up}}, w_2^{\mathrm{up}}, \ldots, w_N^{\mathrm{up}}$ starting at w^{up} are pairwise non-intersecting and that the lines $w_1^{\mathrm{down}}, w_2^{\mathrm{down}}, \ldots, w_N^{\mathrm{down}}$ ending at w^{down} have the same property. Let k be the number of lines connecting w^{up} and w^{down}. Then any of the lines $w_{k+1}^{\mathrm{up}}, w_{k+2}^{\mathrm{up}}, \ldots, w_N^{\mathrm{up}}$ intersects each of the lines $w_{k+1}^{\mathrm{down}}, w_{k+2}^{\mathrm{down}}, \ldots, w_N^{\mathrm{down}}$. Moving these intersection points to the left (this is possible due to (4.21)) we may assume that α is of the form



Let us consider the left part of this diagram

$\beta_{N-k} =$



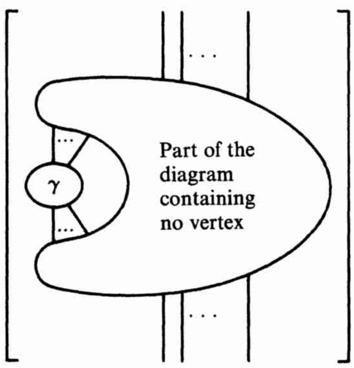
Clearly $[\beta_{N-k}]$ is selfadjoint and using (4.21) and (4.22) one can easily show that $[\beta_{N-k}]\,H_{f^{N-k}}$ is contained in $H_f^{A(N-k)}$. Therefore (cf. Lemma 4.2 and (4.27))

$$[\beta_{N-k}] = c\,A_{N-k}$$

where A_{N-k} is given by (4.26) and c is a numerical constant.

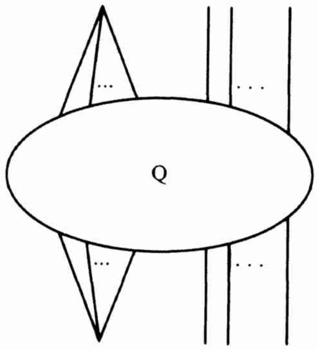
Therefore

$$[\alpha] = c \sum_{\gamma \in \mathrm{Perm}(N-k)} (\mathrm{sign}\ \gamma)$$



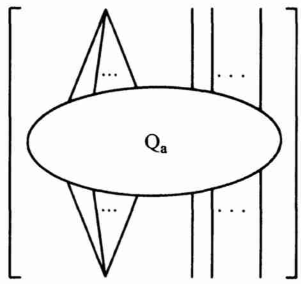
and (4.36) follows.

Assume now that (4.36) holds for any diagram α having $2k$ vertices. Let α be a diagram having $2k+2$ vertices. In virtue of (4.34) and (4.35) we may assume that α is of the form



where Q is a diagram having $2k$ vertices. Using the induction assumption we see that

$[\alpha] = \sum_{\alpha} c_a$



where Q_a are diagrams having no vertices. Each of the diagram on the right hand side of the above equation has two vertices. Therefore (cf. the first part of the proof) (4.36) holds. □

Let us remind [4] that the Hecke algebra $H_{q,n}$ (where $q \in \mathbb{C}$) is the universal associative complex algebra with unity generated by $n-1$ elements $g_1, g_2, \ldots, g_{n-1}$ satisfying the following relations

$$\left.\begin{array}{ll} g_i^2 = (1-q)\, g_i + q\, I & i=1,2,\ldots,n-1 \\ g_i\, g_{i+1}\, g_i = g_{i+1}\, g_i\, g_{i+1} & i=1,2,\ldots,n-2 \\ g_i\, g_j = g_j\, g_i & |i-j| \geqq 2,\ i,j=1,2,\ldots,n-1. \end{array}\right\} \tag{4.37}$$

Elements $g_1, g_2, \ldots, g_{n-1}$ and their products can be (in the obvious way) represented by diagrams consisting of n lines and no vertex (g_{k+1} is represented by the diagram (4.19) where $l=n-k-2$). In particular, for any $\gamma \in \mathrm{Perm}(n)$ we denote by $[\gamma]_q$ the element of $H_{q,n}$ represented by the diagram γ corresponding to the permutation γ. It is known that $\dim H_{q,n} = n!$ and that

$$\{[\gamma]_q\}_{\gamma \in \mathrm{Perm}(n)} \tag{4.38}$$

is a basis in $H_{q,n}$. In the following we assume that $q>0$. Then $H_{q,n}$ is a C^*-algebra (the star is introduced by the formula $q_i^*=q_i$, $i=1,2,\ldots,n-1$).

In [5] (see also [4]) for each Young diagram d consisting of n boxes an irreducible representation π_q^d of $H_{q,n}$ is constructed. The carrier Hilbert space K^d of the representation is the same for all q and for any $\gamma\in\mathrm{Perm}(n)$ the mapping

$$\mathbb{R}_+\ni q\to\pi_q^d([\gamma]_q)\in B(K^d) \tag{4.39}$$

is continuous. Considering all Young diagrams with n boxes we obtain the complete set of mutually non-equivalent irreducible representations of $H_{q,n}$.

Let c_q^d be the projection in $H_{q,n}$ supporting π_q^d (i.e. $I-c_q^d$ is the largest projection killed by π_q^d). Clearly c_q^d is a minimal central projection in $H_{q,n}$ and

$$\sum_d c_q^d=I. \tag{4.40}$$

Using the basis (4.38) we have

$$c_q^d=\sum_{\gamma\in\mathrm{Perm}(n)} t_q^d(\gamma)\,[\gamma]_q. \tag{4.41}$$

Remembering that the mappings (4.39) are continuous one can show that the coefficients $t_q^d(\gamma)$ depend continuously on q.

It follows immediately form (4.14)–(4.16) that operators $g_k'=I_{f^{k-1}}\otimes\sigma\otimes I_{f^{n-k-1}}$ acting on H_{f^n} are selfadjoint and satisfy the relations (4.37) with $q=\mu^2$. Therefore there exists an unital representation π of $H_{\mu^2,n}$ acting on H_{f^n} such that

$$\pi(g_k)=I_{f^{k-1}}\otimes\sigma\otimes I_{f^{n-k-1}}.$$

Clearly $\pi([\gamma]_{\mu^2})=[\gamma]$ for any $\gamma\in\mathrm{Perm}(n)$.

Let d be a Young diagram consisting of n boxes. We consider the orthogonal projection

$$c^d=\pi(c_{\mu^2}^d). \tag{4.42}$$

It depends continuously on μ. Indeed using (4.41) we have

$$c^d=\sum_{\gamma\in\mathrm{Perm}(n)} t_{\mu^2}^d(\gamma)\,[\gamma]$$

and all quantities on the right hand side are continuous with respect to μ (continuity of $[\gamma]$ follows immediately from (4.13)). $c^d=0$ if and only if d contains more than N rows (for $\mu=1$ this fact is well known, for $\mu<1$ it follows from the continuity argument).

Let d be a Young diagram consisting of n boxes with number of rows smaller or equal to N. In virtue of Theorem 4.3, $\pi(H_{\mu^2,n})=\mathrm{Mor}(f^n,f^n)$ and c^d is a minimal central projection belonging to $\mathrm{Mor}(f^n,f^n)$. Therefore $c^d(H_{f^n})$ is u^n-invariant (To simplify the notation we write u^n instead of u^{f^n}.) and the subrepresentation of u^n acting on $c^d(H_{f^n})$ is factorial i.e. is equivalent to the direct sum of $(\dim K^d)$-copies of an irreducible representation of $S_\mu U(N)$. The latter representation will be denoted by u^d. The dimension count shows that

$$\dim u^d = \frac{\dim c^d}{\dim K^d}. \tag{4.43}$$

Clearly representations u^d (where d runs over the set of all Young diagrams with n boxes and number of rows smaller or equal to N) are mutually non-equivalent. Moreover (cf. (4.40)) any irreducible subrepresentation of u^n is equivalent to one of u^d.

Let d (d', d'' resp.) be a Young diagram consisting of n (n', $n''=n+n'$ resp.) boxes with number of rows smaller or equal to N and $m_{d''}^{dd'}$ be the multiplicity of $u^{d''}$ in $u^d \textcircled{T} u^{d'}$. Then $c^d(H_{f^n}) \otimes c^{d'}(H_{f^{n'}}) = (c^d \otimes c^{d'})(H_{f^{n''}})$ is $u^{n''}$-invariant and the corresponding subrepresentation is equivalent to the direct sum of $(\dim K^d)(\dim K^{d'})$-copies of $u^d \textcircled{T} u^{d'}$. Therefore, the multiplicity of $u^{d''}$ in this subrepresentation equals $m_{d''}^{dd'}(\dim K^d)(\dim K^{d'})$. Remembering that $c^{d''}$ selects the subrepresentations equivalent to $u^{d''}$ we get

$$\dim(c^{d''}(c^d \otimes c^{d'})) = m_{d''}^{dd'}(\dim K^d)(\dim K^{d'})(\dim u^{d''})$$

and (cf. (4.43))

$$m_{d''}^{dd'} = \frac{(\dim K^{d''})(\dim(c^{d''}(c^d \otimes c^{d'})))}{(\dim K^d)(\dim K^{d'})(\dim c^{d''})}. \tag{4.44}$$

The continuity argument applied to (4.43) and (4.44) shows that $\dim u^d$ and $m_{d''}^{dd'}$ are independent of μ. Therefore these quantities are expressed by the same formulae as in the $SU(N)$ case. It means that the decomposition

$$u^d \textcircled{T} u^{d'} = \sum_{d''}^{\oplus} m_{d''}^{dd'} u^{d''}$$

has the same form as in $SU(N)$ case. In particular if d' is the diagram consisting of N boxes placed in one column then

$$u^d \textcircled{T} u^{d'} = u^{d''}$$

where d'' is the diagram obtained from d by adding one full column. Using the continuity argument one can show that $u^{d'}$ is trivial. Therefore

$$u^d = u^{d''}. \tag{4.45}$$

Let d (d' resp.) be a Young diagram consisting of n (n' resp. $n<n'$) boxes with number of rows smaller or equal to N. Assume that u^d is equivalent to $u^{d'}$. Then $n'-n \equiv 0 \mod N$ (otherwise $\mathrm{Mor}(f^n, f^{n'}) = \{0\}$). Let d'' be the Young diagram obtained from d by adding $(n'-n)/N$ full columns. Then (cf. (4.45)) $u^{d''}$ is equivalent to $u^{d'}$ and $d'=d''$ (d' and d'' have the same number of boxes). This ends the proof of Theorem 1.5.

The following problem (for $N \geqq 3$) is still open: Find all compact matrix pseudogroups that have the property of $S_\mu U(N)$ described in Theorem 1.5.

For $N=2$ the pseudogroups presented in [7] are the only ones having this property.

Acknowledgement. This paper has been written down during the author's visits to the Institute of Mathematics "Guido Castelnuovo" University of Rome and to the Institute of Theoretical Physics, University of Leuven. The author would like to thank Prof. S. Doplicher and Prof. A. Verbeure for their kind hospitality creating the perfect atmosphere for the work. The here presented approach to the twisted $SU(N)$ groups is the result of stimulating discussions with Prof. S. Doplicher. The author is grateful to Dr. B. Nachtergaele and Mr. P. Podleś for the proofreading of the manuscript.

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Oblatum 15-I-1988