Invent. math. 92, 597–631 (1988)

Inventiones mathematicae

P-adic measures attached to automorphic representations of $GL(3)$

Dedicated to H.-W. Leopoldt on his 60th birthday

C.-G. Schmidt

Rijksuniversiteit Groningen, Subfaculteit Wiskunde en Informatica, Postbus 800, NL-9700 AV Groningen, The Netherlands

Contents

Introduction

Let π be a cuspidal automorphic representation of $GL(2)$ which corresponds to a primitive cusp form f of weight $k \geqq 2$ for the congruence subgroup $\Gamma_0(N)$, where N is some integer $\geqq 1$. For each prime number ℓ, let $M_\ell(\pi)$ be the ℓ-adic representation attached to π by Deligne. The symmetric squares $\mathrm{Sym}^2(M_\ell(\pi))$ of these ℓ-adic representations form a compatible system of 3-dimensional ℓ-adic representations for the Galois group of $\bar{\mathbb{Q}}$ over $\mathbb{Q}$, and we write $L(\mathrm{Sym}^2(M(\pi)), s)$ for the complex L-function attached to this compatible system. After earlier work by many authors, the analytic continuation and functional equation for $L(\mathrm{Sym}^2(M(\pi)), s)$ were finally established in general using work of Jacquet-Gelbart [5] and Carayol [1]. An important step in this work is the identification of $L(\mathrm{Sym}^2(M(\pi)), s)$ as the (twisted) L-function of a certain automorphic representation Π of $GL(3)$.

Now let p be a prime number which is *ordinary* for π in the sense of § 3. The aim of the present paper is to construct a p-adic analogue of each twisted L-function $L(\Pi \otimes \lambda, s)$ or of $L(\mathrm{Sym}^2(M(\pi)) \otimes \lambda, s)$, and to prove its p-adic holomorphy and functional equation. This strengthens results in our earlier paper [2] where only a weaker statement about the p-adic meromorphic nature of some of these functions was established. The main interest of this p-adic analogue of $L(\Pi\otimes\lambda, s)$ is its conjectural relationship with a certain Iwasawa module attached to π (see [2] for the definition of this module when f has weight $k=2$ and rational Fourier coefficients). However, we do not discuss this module at all in the present paper, and simply concentrate on establishing the analytic properties of the p-adic analogue of $L(\Pi\otimes\lambda, s)$.

Notation. Let $\mathbb{A}$ denote the adèle ring of $\mathbb{Q}$ and let $I=\mathbb{A}^\times$ be the idèle group. $\hat{\mathbb{Z}}$ denotes the formal completion $\varprojlim \mathbb{Z}/n\mathbb{Z}$ and $G_{\mathbb{Q}}$ is the Galois group of an algebraic closure $\bar{\mathbb{Q}}$ over $\mathbb{Q}$. For any prime number p we write Ω_p for the completion of an algebraic closure $\bar{\mathbb{Q}}_p$ of the field of p-adic numbers $\mathbb{Q}_p$. Sometimes we interprete a character of the Weil group $W_{\mathbb{Q}}$ as a größencharacter $\chi: I/\mathbb{Q}^\times \to \mathbb{C}^\times$ and write $\chi=\prod_{\ell}\chi_\ell$, where χ_ℓ is the restriction of χ to the local field $\mathbb{Q}_\ell$ for $\ell\neq\infty$ resp. $\mathbb{R}$ for $\ell=\infty$. If χ is of finite order we denote by χ_0 or $\tilde{\chi}$ the corresponding primitive Dirichlet character, and we let c_χ denote the conductor of χ_0 (which is equal to the conductor of χ). For higher dimensional representations Σ of the Weil group let $\operatorname{cond}(\Sigma)$ denote the Artin conductor of Σ. We use the convention that $\chi_0(n)=0$ if $(n, c_\chi)>1$.

$\Gamma_0(N)$ denotes the group of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{Z})$ where N divides c. The symbol $[r]$ stands for the integral part of $r\in\mathbb{R}$. We normalize the Petersson inner product of two forms f_1, f_2 of weight k for $\Gamma_0(N)$ by putting

$$\langle f_1, f_2\rangle = \langle f_1, f_2\rangle_N = \int_{\Gamma_0(N)\backslash\mathscr{H}} f_1(z)\overline{f_2(z)}\, y^{k-2}\, dx\, dy,$$

whenever this is defined. We decompose the group of p-adic units $\mathbb{Z}_p^\times$ by writing $\mathbb{Z}_p^\times=\Delta\cdot\Gamma$ where $\Gamma=1+p\mathbb{Z}_p$ and Δ denotes the group of $(p-1)$-th roots of unity in $\mathbb{Z}_p^\times$.

Acknowledgements. The research described in this article was supported by the Deutsche Forschungsgemeinschaft and was performed in part at the Max-Planck-Institut in Bonn and at the Mathematical Sciences Research Institute in Berkeley. The author thanks these institutions, and their directors, for their friendly hospitality.

§ 1. The Jacquet-Gelbart lift

The aim of this first section is to develop the complex-analytic properties of the L-function of each twist of Π or what comes to the same of each twist of $\mathrm{Sym}^2(M(\pi))$ following Gelbart and Jacquet [5]. We must do this because later on we will need some information about the ε-factors in the functional equation of these L-functions for rationality questions of special values and for the study of the p-adic analytic properties of the associated p-adic L-functions in the later sections. We also want to describe explicitly how these L-functions are related to the (imprimitive) symmetric square L-functions considered by Shimura [11] and Sturm [13].

We start out with a primitive normalized eigenform $f \in S_k(N, \psi)$, i.e. a cusp form of weight $k \geqq 1$, level N and nebentypus character ψ. Let $\pi_f = \pi = \otimes \pi_\ell$ denote the irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ attached to f. Suppose f has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n \cdot q^n, \qquad q = \exp(2\pi i z),$$

and for any prime number ℓ let

$$1 - a_\ell X + \psi(\ell)\,\ell^{k-1} X^2 = (1 - \alpha_\ell X)(1 - \beta_\ell X). \tag{1.1}$$

The L-function associated with f is defined by

$$L(s, f) = \sum_{n=1}^{\infty} a_n \cdot n^{-s}$$

and has an Euler product expansion of the form

$$L(s, f) = \prod_\ell ((1 - \alpha_\ell \ell^{-s})(1 - \beta_\ell \ell^{-s}))^{-1}.$$

In terms of the representation π one usually takes $L(s, \pi) = L\left(s + \frac{k-1}{2}, f\right)$, where the local L-function attached to π_ℓ contributes a factor

$$L(s, \pi_\ell) = (1 - \alpha_\ell \ell^{-\frac{k-1}{2}} \ell^{-s})^{-1} (1 - \beta_\ell \ell^{-\frac{k-1}{2}} \ell^{-s})^{-1}. \tag{1.2}$$

By the work of Eichler, Shimura and Deligne we know how to attach to π a compactible system $\sigma = \{\sigma^{(\lambda)}\}$ of 2-dimensional λ-adic representations of the Weil group $W_{\mathbb{Q}}$ of $\mathbb{Q}$, where λ runs over all finite places of the number field F generated over $\mathbb{Q}$ by the Fourier coefficients of f. The system σ is such that for all but finitely many prime numbers ℓ the restriction σ_ℓ of σ to the local Weil group $W_{\mathbb{Q}_\ell}$ has to correspond to π_ℓ in the sense of local Langlands correspondence. Since by the work of Kutzko the local Langlands correspondence for $GL(2)$ is fully established, one knows that even for the exceptional primes ℓ the representation π_ℓ corresponds to *some* 2-dimensional representation σ'_ℓ of $W_{\mathbb{Q}_\ell}$. That σ'_ℓ is in fact equivalent to σ_ℓ was one of the main results in Carayol's thesis [1]. So if for any prime number ℓ and any finite place $\lambda \nmid \ell$ of F we consider the representation $\sigma_\ell^{(\lambda)}\colon W_{\mathbb{Q}_\ell} \to GL(V_\lambda)$, we find that $L(s, \pi_\ell) = Z(V_\lambda, \ell^{-s})$, where

$$Z(V_\lambda, t) = \det(1 - \phi_\ell t;\, V_\lambda^{I_\ell})^{-1}, \tag{1.3}$$

with the standard notations as in [15]. Thus at all prime numbers ℓ we have $L(s, \pi_\ell) = L(s, \sigma_\ell)$. Moreover conductors and ε-factors coincide.

The action of $W_{\mathbb{Q}}$ on V_λ via $\sigma^{(\lambda)}$ also defines a canonical action of $W_{\mathbb{Q}}$ on the dual space V_λ^*. This defines the contragredient representation $\check{\sigma}^{(\lambda)}: W_{\mathbb{Q}} \to GL(V_\lambda^*)$. Note that for the natural pairing $\langle\ \rangle: V_\lambda^* \times V_\lambda \to F_\lambda$ we have

$$\langle \check{\sigma}^{(\lambda)}(w) v^*, v\rangle = \langle v^*, \sigma^{(\lambda)}(w^{-1}) v\rangle,$$

so that the pairing extends to a homomorphism of $W_{\mathbb{Q}}$-modules $V_\lambda^* \otimes V_\lambda \to F_\lambda$, where $W_{\mathbb{Q}}$ acts trivially on F_λ. The kernel U_λ therefore defines a 3-dimensional representation $\Sigma^{(\lambda)}: W_{\mathbb{Q}} \to GL(U_\lambda)$ and moreover $\Sigma = \{\Sigma^{(\lambda)}\}$ forms a compatible system of λ-adic representations. So we can define for each ℓ the usual local L-function

$$L(s, \Sigma_\ell) := Z(U_\lambda; \ell^{-s}) \tag{1.4}$$

with any $\lambda \nmid \ell$, where again $Z(U_\lambda, t) = \det(1 - \phi_\ell t, U_\lambda^{I_\ell})^{-1}$ is independent of the choice of λ. In order to twist Σ by an algebraic Hecke character $\chi: W_{\mathbb{Q}} \to \mathbb{C}^\times$ we fix an algebraic closure $\bar{\mathbb{Q}}_q = \bar{F}_\lambda$ of F_λ and an embedding $i_q: \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_q$ such that $i_q(F) \subset F_\lambda$. We have

$$\Sigma^{(\lambda)} \otimes \chi: W_{\mathbb{Q}} \to GL(U_\lambda \otimes \bar{\mathbb{Q}}_q), \qquad w \mapsto \chi(w) \cdot \Sigma^{(\lambda)}(w),$$

and we can define the local L-function $L(s, \Sigma_\ell \otimes \chi)$ as in (1.4) taking into account that $W_{\mathbb{Q}}$ now acts via $\Sigma \otimes \chi = \{\Sigma^{(\lambda)} \otimes \chi\}$.

Example. The symmetric square. A little exercise in linear algebra shows $\check{\sigma} \cong \sigma \otimes (\det \sigma)^{-1}$, where $(\det \sigma)^{(\lambda)} = \det(\sigma^{(\lambda)}): W_{\mathbb{Q}} \to F_\lambda^\times$ factors through $W_{\mathbb{Q}}^{ab} \cong I/\mathbb{Q}^\times \cong \mathbb{R}_{>0} \times \hat{\mathbb{Z}}^\times$. In fact by (1.1)–(1.3) $\det \sigma$ is the unique größencharacter of $\mathbb{Q}$ trivial on $\mathbb{R}_{>0}$ (in the decomposition above) whose restriction to $\hat{\mathbb{Z}}^\times$ factors through the finite quotient $(\mathbb{Z}/N\mathbb{Z})^\times$ such that $\det \sigma|_{(\mathbb{Z}/n\mathbb{Z})^\times} = \psi^{-1}$. Note that by the convention in [15] $\phi_\ell \in W_{\mathbb{Q}_\ell}^{ab}$ corresponds to $\ell \in \mathbb{Q}_\ell^\times$ under the isomorphism $W_{\mathbb{Q}_\ell}^{ab} \cong \mathbb{Q}_\ell^\times$. The action of $W_{\mathbb{Q}}$ on $\mathrm{Sym}^2(V_\lambda) \leqq V_\lambda \otimes V_\lambda$ defines a compatible system of λ-adic representations $\mathrm{Sym}^2(\sigma)$ where $\mathrm{Sym}^2(\sigma)^{(\lambda)} = \mathrm{Sym}^2(\sigma^{(\lambda)}): W_{\mathbb{Q}} \to GL(\mathrm{Sym}^2(V_\lambda))$, and we easily find

$$\mathrm{Sym}^2(\sigma) \cong \Sigma \otimes \psi^{-1}. \tag{1.5}$$

(Here we use the same notation for the Dirichlet character ψ and the corresponding größencharacter.)

We come back to our system of λ-adic representations $\Sigma \otimes \chi$, where χ is an arbitrary größencharacter of $\mathbb{Q}$. We will take care now of the infinite place of $\mathbb{Q}$. The component π_∞ of π is a twist of the discrete series representation with lowest weight k. The corresponding representation σ_∞ of the Weil group $W_{\mathbb{R}}$ is the irreducible 2-dimensional representation induced from a quasi-character of $W_{\mathbb{C}} = \mathbb{C}^\times$:

$$\sigma_\infty = \mathrm{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, \theta), \qquad \theta(z) = \frac{z^{k-1}}{(z\bar{z})^{(k-1)/2}}, \tag{1.6}$$

where $\det \sigma_\infty = \mathrm{sgn}^k$ and $\check{\sigma}_\infty \cong \sigma_\infty$. The analogue of Σ at ∞ is now defined as

$$\Sigma_\infty = \mathrm{Sym}^2(\sigma_\infty): W_{\mathbb{R}} \to GL_3(\mathbb{C}), \tag{1.7}$$

and we have (see (3.3.12) in [5])

$$\Sigma_\infty = \mathrm{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, \theta^2) \oplus \mathrm{sign}. \tag{1.8}$$

Lemma 1.1. *The L-function of* $\Sigma_\infty \otimes \mathrm{sgn}^\kappa$ $(\kappa = 0, 1)$ *is given by*

$$L(s, \Sigma_\infty \otimes \mathrm{sgn}^\kappa) = \Gamma_{\mathbb{R}}(s+1-\kappa) \cdot \Gamma_{\mathbb{C}}(s+k-1),$$

where as usual we put $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \cdot \Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \cdot \Gamma(s)$.

Proof. This follows immediately from Sect. 3 in [15] and (1.8), using the fact that $\mathrm{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, \theta^2) \cong \mathrm{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, \theta^2) \otimes \mathrm{sgn}$.

We can now formulate the main result in [5] in our situation. From now on let $\chi = \chi_\infty \cdot \prod_\ell \chi_\ell$ be an idèle character of finite order such that $\chi|_{\mathbb{Q}^\times} = 1$. Put

$$L(s, \Sigma \otimes \chi) := L(s, \Sigma_\infty \otimes \chi_\infty) \cdot \prod_\ell L(s, \Sigma_\ell \otimes \chi_\ell),$$

and

$$\varepsilon(s, \Sigma \otimes \chi) := \varepsilon(s, \Sigma_\infty \otimes \chi_\infty) \cdot \prod_\ell \varepsilon(s, \Sigma_\ell \otimes \chi_\ell).$$

Here the local ε-factors at finite places ℓ are those of (3.6) in [15] in Delignés convention [3], i.e.

$$\varepsilon(s, \Sigma_\ell \otimes \chi_\ell) = \varepsilon(\Sigma_\ell \otimes \chi_\ell \omega_s, \Psi_\ell, dx_\ell)$$

with an additive character $\Psi_\ell \neq 1$ of $\mathbb{Q}_\ell$ and the Haar measure dx_ℓ on $\mathbb{Q}_\ell$ self-dual with respect to Ψ_ℓ. The ε-factor at ∞ is normalized by taking $\Psi_\infty(x) = \exp(2\pi i x)$ such as in (3.2.4) in [15]. The Ψ_ℓ are chosen in such a way that $\Psi = \Psi_\infty \cdot \prod_\ell \Psi_\ell$ is an additive character of $\mathbb{A}$ trivial on $\mathbb{Q}$.

Theorem 1.2 (Gelbart-Jacquet). *The global L-function* $L(s, \Sigma \otimes \chi)$ *continues to a meromorphic function on the whole complex plane and satisfies the functional equation*

$$L(s, \Sigma \otimes \chi) = \varepsilon(s, \Sigma \otimes \chi) \cdot L(1-s, \Sigma \otimes \chi^{-1}).$$

There is an automorphic representation Π *of* $GL_3(\mathbb{A})$, *such that for any größencharacter* χ *of* $I/\mathbb{Q}^\times$ *we have* $L(s, \Pi \otimes \chi) = L(s, \Sigma \otimes \chi)$. *If* π *is not equivalent to any twist* $\pi \otimes \chi$ *with* $\chi \neq 1$, *then* Π *is cuspidal and* $L(s, \Pi \otimes \chi)$ *is entire for all* χ.

In the exceptional case (see (3.7) in [5]) let η be a non-trivial größencharacter such that $\pi \cong \pi \otimes \eta$. Then $\eta^2 = 1$ and η therefore defines a quadratic extension field $K/\mathbb{Q}$. Moreover there is a größencharacter θ of K such that $\sigma_\ell = \mathrm{Ind}(W_{\mathbb{Q}_\ell}, W_{K_{\mathscr{L}}}, \theta_{\mathscr{L}})$ in case $\eta_\ell \neq 1$, i.e. when ℓ does not split in K and $\mathscr{L} | \ell$; if $\eta_\ell = 1$ then $\ell = \mathscr{L}\mathscr{L}'$ splits and $\sigma_\ell = \theta_{\mathscr{L}} \oplus \theta_{\mathscr{L}'}$ in this case. As a consequence we have

$$\Sigma_\ell = \begin{cases} \mathrm{Ind}(W_{\mathbb{Q}_\ell}, W_{K_{\mathscr{L}}}, \theta_{\mathscr{L}} \theta'^{-1}_{\mathscr{L}}) \oplus \eta_\ell & \text{if } \eta_\ell \neq 1, \\ \theta_{\mathscr{L}} \theta_{\mathscr{L}'}^{-1} \oplus \theta_{\mathscr{L}'} \theta_{\mathscr{L}}^{-1} \oplus \eta_\ell & \text{if } \eta_\ell = 1, \end{cases} \tag{1.9}$$

where the conjugate größencharacter θ' is defined by $\theta'(x)=\theta(x^\rho)$ with the non-trivial automorphism $\rho\in\mathrm{Gal}(K/\mathbb{Q})$. Hence we get an L-function with größen-characters

$$L(s,\Sigma\otimes\chi)=L(s,\theta\cdot\theta'^{-1}\cdot\chi_K)\cdot L(s,\eta\chi), \tag{1.10}$$

where $\chi_K=\chi\circ\mathcal{N}_{K/\mathbb{Q}}$ (see Remark 9.9 in [5]). It is now an easy exercise to classify those χ for which $L(s,\Sigma\otimes\chi)$ is not entire and which poles actually occur. The result is

Proposition 1.3. a) *If $\theta\neq\theta'$ then $L(s,\Sigma\otimes\chi)$ is entire for all finite characters $\chi\neq\eta$. $L(s,\Sigma\otimes\eta)$ has only simple poles at $s=0,1$.*

b) *If $\theta=\theta'$ then*

$$L(s,\Sigma\otimes\chi)=L(s,\chi)\cdot L(s,\eta\chi)^2$$

is entire for $\chi\neq 1,\eta$ and for $\chi=1$ or η the poles are the obvious ones of the Riemann zeta function at $s=0,1$.

We now gather properties of the global ε-factor in Theorem 1.2 for later reference. For an automorphism $\tau\in G_{\mathbb{Q}}$ let $a_\tau=(a_{\tau,\ell})$ in $\hat{\mathbb{Z}}^\times=\prod_\ell\mathbb{Z}_\ell^\times$ be such that on roots of unity τ acts like taking the a_τ-th power. We denote by $\tilde{\chi}$ the associated primitive Dirichlet character defined by restricting the idèle character χ to $\hat{\mathbb{Z}}^\times$. Note that each ℓ-adic representation Σ_ℓ $(\ell\neq\infty)$ is in fact defined over $\mathbb{Q}$ and hence Σ_ℓ^τ is well defined. We put $\Sigma_\infty^\tau=\Sigma_\infty$.

Lemma 1.4. a) *For any integer $m\in\mathbb{Z}$ the ε-factor $\varepsilon(m,\Sigma\otimes\chi)$ is algebraic and for $\tau\in G_{\mathbb{Q}}$ we have*

$$\varepsilon(m,\Sigma\otimes\chi)^\tau=\tilde{\chi}(-1)^{(a_{\tau,2}-1)/2}\,\chi^3(a_\tau)^\tau\cdot\varepsilon(m,\Sigma^\tau\otimes\chi^\tau).$$

b) *Let λ be another character of conductor c_λ such that $(c_\lambda,c_\chi)=1$, and assume $(c_\chi,N)=1$. Then we have*

$$\varepsilon(s,\Sigma\otimes\lambda\chi)=\varepsilon(s,\Sigma\otimes\lambda)\,\frac{\sqrt{\tilde{\lambda}(-1)}}{\sqrt{\tilde{\lambda}\tilde{\chi}(-1)}}\,c_\chi^{-3s}\,\frac{G(\tilde{\lambda}\tilde{\chi}^{-1})^3}{G(\tilde{\lambda}^{-1})^3}\cdot\tilde{\chi}(c_\lambda^3\cdot\mathrm{cond}(\Sigma\otimes\lambda)^{-1}),$$

where $\sqrt{\tilde{\lambda}(-1)}=1$ or i and the Gauß sum $G(\tilde{\lambda})$ is defined by

$$G(\tilde{\lambda})=\sum_{x=1}^{c_\lambda}\tilde{\lambda}(x)\cdot\exp(2\pi i x/c_\lambda).$$

Proof. a) Since in the non-archimedean case our measures dx_ℓ are such that $\mathbb{Z}_\ell$ gets measure 1, the local ε-factors do behave well under automorphisms of $\mathbb{C}$ (see (3.6.10) in [15]). Namely for any $\tau\in\mathrm{Aut}(\mathbb{C})$ and for $m\in\mathbb{Z}$ we have

$$\varepsilon(m,\Sigma_\ell\otimes\chi_\ell)^\tau=\varepsilon(\Sigma_\ell\otimes\chi_\ell\omega_m,\Psi_\ell,dx_\ell)^\tau=\varepsilon(\Sigma_\ell^\tau\otimes\chi_\ell^\tau\omega_m,\Psi_\ell^\tau,dx_\ell),$$

since $\omega_m(x)=|x|_\ell^m\in\mathbb{Q}$. Since $\Psi_\ell(x)^\tau=\Psi_\ell(x\cdot a_{\tau,\ell})$ we get from (3.4.4) in [15]:

$$\varepsilon(m,\Sigma_\ell\otimes\chi_\ell)^\tau=\chi_\ell^3(a_{\tau,\ell})^\tau\cdot\varepsilon(m,\Sigma_\ell^\tau\otimes\chi_\ell^\tau). \tag{1.11}$$

In the archimedean case we know $\Sigma_\infty \otimes \chi_\infty = \mathrm{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, \theta^2) \otimes \chi_\infty \oplus \mathrm{sgn} \cdot \chi_\infty$, where we can omit the χ_∞-twist in the first term. Since ε is inductive in degree 0 we use $\mathrm{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, 1) = 1 \oplus \mathrm{sgn}$ to get

$$\begin{aligned}\varepsilon(s, \Sigma_\infty \otimes \chi_\infty) &= \varepsilon(\mathrm{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, \theta^2) \otimes \omega_s) \cdot \varepsilon(\mathrm{sgn} \cdot \chi_\infty \omega_s) \\ &= \varepsilon(\mathrm{Ind}(W_{\mathbb{R}}, W_{\mathbb{C}}, \theta^2 \omega_s - 1)) \cdot \varepsilon(\mathrm{sgn} \oplus \mathrm{sgn} \cdot \chi_\infty \omega_s) \\ &= \varepsilon_{\mathbb{C}}(\theta^2 \omega_s) \cdot \varepsilon_{\mathbb{R}}(\mathrm{sgn} \oplus \mathrm{sgn} \cdot \chi_\infty \omega_s).\end{aligned}$$

Hence by (1.6) and (3.2.4), (3.2.5) in [15] we have

$$\varepsilon(s, \Sigma_\infty \otimes \chi_\infty) = (-1)^k / \sqrt{\tilde{\chi}(-1)}, \tag{1.12}$$

and therefore we find in particular

$$\varepsilon(m, \Sigma_\infty \otimes \chi_\infty)^\tau = \tilde{\chi}(-1)^{(a_{\tau,2}-1)/2} \cdot \varepsilon(m, \Sigma_\infty \otimes \chi_\infty), \tag{1.13}$$

which completes the proof of a).

b) As in a) our proof works locally. In case ℓ divides the conductor $\mathrm{cond}(\Sigma \otimes \lambda)$, χ_ℓ is unramified by assumption. Thus we know (see for instance [6], p. 23)

$$\varepsilon(s, \Sigma_\ell \otimes \lambda_\ell \chi_\ell) = \chi_\ell(\mathrm{cond}(\Sigma_\ell \otimes \lambda_\ell)) \cdot \varepsilon(s, \Sigma_\ell \otimes \lambda_\ell). \tag{1.14}$$

If ℓ does not divide $\mathrm{cond}(\Sigma \otimes \lambda)$ and if in addition χ_ℓ is unramified, then $\varepsilon(s, \Sigma_\ell \otimes \lambda_\ell \chi_\ell) = 1$. In the remaining case $\ell | c_\chi$ our assumption implies that $\Sigma_\ell \otimes \lambda_\ell = \alpha_1 \otimes \alpha_2 \otimes \alpha_3$, where the α_i are unramified 1-dimensional representations of $W_{\mathbb{Q}_\ell}$. Now proceeding as in Tate's thesis we get

$$\varepsilon(s, \alpha_i \chi_\ell) = c_{\chi_\ell}^{-s} G(\bar{\chi}_\ell) \cdot \alpha_i \chi_\ell(c_{\chi_\ell}), \tag{1.15}$$

where we point out that the additive character Ψ_ℓ here is normalized such that $\Psi_\ell(\ell^{-n}) = \exp(-2\pi i/\ell^n)$ for $n \geq 0$. (Since $\Psi_\infty(x) = \exp(2\pi i x)$ this is necessary to have the global Ψ trivial on $\mathbb{Q}$.) Since $\alpha_1 \alpha_2 \alpha_3 = \lambda_\ell^3$ the formula in b) follows by the well known product formula for Gauß sums and by (1.12). This finishes the proof of Lemma 1.4.

In the remainder of this section we shortly want to indicate the connection to Shimura's work [11]. Given an arbitrary primitive Dirichlet character χ_0 Shimura studies the Euler product $D(s, f, \chi_0) = \prod_\ell D_\ell(\chi_0(\ell) \ell^{-s})^{-1}$, where for each prime number ℓ

$$D_\ell(X) = D_\ell(f, X) = (1 - \alpha_\ell^2 X)(1 - \alpha_\ell \beta_\ell X)(1 - \beta_\ell^2 X) \tag{1.16}$$

with α_ℓ, β_ℓ as in (1.1). Comparing Euler factors it becomes clear that $D(s, f, \chi_0)$ coincides with $L(s-k+1, \Sigma \otimes (\chi\psi)^{-1})$ up to the Euler factors at ∞ and at the bad primes $\ell \mid N$. Here χ denotes the größencharacter with $\tilde{\chi}=\chi_0$. Recall that for $\ell \mid N$ we have $\alpha_\ell \beta_\ell = 0$ and that moreover

$$\begin{array}{lll} \text{a)} & |a_\ell| = \ell^{\frac{k-1}{2}} & \text{if } \operatorname{ord}_\ell c_\psi = \operatorname{ord}_\ell N, \\ \text{b)} & a_\ell^2 = \psi(\ell)\cdot \ell^{k-2} & \text{if } c_\psi \mid N/\ell \text{ and } \ell^2 \nmid N, \\ \text{c)} & a_\ell = 0 & \text{otherwise.} \end{array} \tag{1.17}$$

We may and we will assume that the conductor N of the automorphic representation $\pi = \pi_f$ is minimal among all twists $\pi \otimes \lambda$ with finite größencharacters λ. Note that the Jacquet-Gelbart lift Π of π_f and that of $\pi_f \otimes \lambda$ are the same. Thus by comparing local L-functions one immediately sees that the three possible cases a), b), c) have the following representation theoretical meaning:

a) $\pi_\ell = \pi(\mu, \nu)$ is principal series representation with ν unramified,
b) $\pi_\ell = \sigma(\mu, \nu)$ is special representation with μ, ν unramified and $\mu\nu^{-1} = |\,|$,
c) π_ℓ is a supercuspidal representation.

In these terms we will describe the local L-function (see [5] (3.11))

$$L(s, \Pi_\ell \otimes (\psi\chi)_\ell^{-1}) = \frac{L(s, \pi_\ell \otimes (\psi\chi)_\ell^{-1} \times \check{\pi}_\ell)}{L(s, (\psi\chi)_\ell^{-1})}, \tag{1.18}$$

using (1.2) and (1.4) in [5]. We first dispose of the harmless cases a) and b). By (1.4) in [5] and Proposition 3.5 in [8] we have

$$\begin{aligned} &L(s, \Pi_\ell \otimes (\psi\chi)_\ell^{-1}) \\ &\quad = L(s, \nu\mu^{-1}(\psi\chi)_\ell^{-1}) \cdot L(s, \mu\nu^{-1}(\psi\chi)_\ell^{-1}) \cdot L(s, (\psi\chi)_\ell^{-1}) \end{aligned} \tag{1.19}$$

if $\pi_\ell = \pi(\mu, \nu)$, and if $\pi_\ell = \sigma(\mu, \nu)$ we have $L(s, \Pi_\ell \otimes (\psi\chi)_\ell^{-1}) = L(s+1, (\psi\chi)_\ell^{-1})$. Note that $\nu(\ell) = a_\ell \cdot \ell^{(1-k)/2}$ and $\mu = \nu^{-1}\psi_\ell^{-1}$. So we get

Lemma 1.5. *If $\pi_\ell = \sigma(\mu, \nu)$ is special, then we have*

$$D_\ell(\chi_0(\ell)\cdot \ell^{-s})^{-1} = L(s-k+1, \Sigma_\ell \otimes (\psi\chi)_\ell^{-1}).$$

If $\pi_\ell = \pi(\mu, \nu)$ is principal and μ is ramified, ν unramified, then we have

$$\begin{aligned} &L(s-k+1, \Sigma_\ell \otimes (\psi\chi)_\ell^{-1}) \\ &\quad = D_\ell(\chi_0(\ell)\,\ell^{-s})^{-1}(1-(\psi^2\chi)_0(\ell)\,\bar{a}_\ell^2\,\ell^{-s})^{-1}(1-(\psi\chi)_0(\ell)\,\ell^{k-1}\,\ell^{-s})^{-1}. \end{aligned}$$

Now suppose π_ℓ is supercuspidal. Let η_ℓ denote the unramified quadratic character of $\mathbb{Q}_\ell^\times$. In this case clearly $D_\ell(X) = 1$ by (1.16) and (1.17)c). We must generalize (1.3) in [5].

Lemma 1.6. *Let μ be any finite character of $\mathbb{Q}_\ell^\times$. If μ^2 is ramified, then $L(s, \Sigma_\ell \otimes \mu) = 1$. If μ is unramified, then we have*

$$L(s, \Sigma_\ell \otimes \mu) = \begin{cases} (1+\mu(\ell)\,\ell^{-s})^{-1} & \text{if } \pi_\ell \cong \pi_\ell \otimes \eta_\ell, \\ 1 & \text{if } \pi_\ell \not\cong \pi_\ell \otimes \eta_\ell. \end{cases}$$

If μ is ramified and μ^2 is unramified, let $\lambda_1, \lambda_2 = \lambda_1 \eta_1$ denote the two quadratic ramified characters such that $\mu\lambda_i$ is unramified for $i=1, 2$. Then we have

$$L(s, \Sigma_\ell \otimes \mu) = \begin{cases} 1 & \text{if } \pi_\ell \not\cong \pi_\ell \otimes \lambda_i \text{ for } i=1,2, \\ (1-\lambda\mu(\ell)\,\ell^{-s})^{-1} & \text{if } \pi_\ell \cong \pi_\ell \otimes \lambda \text{ for exactly one } \lambda_i = \lambda, \\ (1-\mu^2(\ell)\,\ell^{-2s})^{-1} & \text{if } \pi_\ell \cong \pi_\ell \otimes \lambda_i \text{ for } i=1,2. \end{cases}$$

Proof. By (1.2) in [5] a complex number s_0 is a pole of $L(s, \pi_\ell \otimes \mu \times \check{\pi}_\ell)$ if and only if $\pi_\ell \otimes |\;|^{s_0}\mu$ is equivalent to π_ℓ. Also the poles are simple. Note that in general a necessary condition for such an equivalence to hold is, that the twisting character be quadratic, i.e. $|\;|^{s_0} = \mu^{-1}, \mu^{-1}\eta_\ell$ or $\mu^{-1}\lambda$ with some ramified quadratic character λ. If μ^2 is ramified then such a s_0 obviously doesn't exist, hence $L(s, \Sigma_\ell \otimes \mu) = 1$ by (1.18). Here we always keep in mind that $L(s, \pi_\ell \otimes \mu \times \check{\pi}_\ell)^{-1}$ is a polynomial $P(X)$ in $X = \ell^{-s}$ such that $P(0)=1$. The explicit formulas now follow from the characterization of poles given before. This proves the lemma.

§ 2. Galois properties of special values

In this section we want to discuss the algebraicity properties of special values of the finite part of the L-functions considered in the previous section. We put

$$\mathscr{D}(s, f, \chi_0) = \prod_\ell L(s, \Sigma_\ell \otimes \chi_\ell), \tag{2.1}$$

where χ_0 denotes a primitive Dirichlet character and χ is the idèle character with $\tilde{\chi} = \chi_0$. Following Deligne [4] we say that an integer $m \in \mathbb{Z}$ is *critical* for the motive $\Sigma \otimes \chi$ if $L(m, \Sigma_\infty \otimes \chi_\infty)$ and $L(1-m, \Sigma_\infty \otimes \chi_\infty)$ both are finite.

Lemma 2.1. *Let $\kappa = 0$ or 1 according as $\chi_0(-1) = (-1)^\kappa$. The critical m for $\Sigma \otimes \chi$ are given by*

$$m \in \{2-k, 3-k, \dots, 0\} \quad \text{such that } m \equiv \kappa(2)$$

and

$$m \in \{1, 2, \dots, k-1\} \quad \text{such that } m \not\equiv \kappa(2).$$

Proof. The poles of $L(s, \Sigma_\infty \otimes \chi_\infty)$ are by Lemma 1.1 the poles of $\Gamma\left(\frac{s+1-\kappa}{2}\right) \cdot \Gamma(s+k-1)$. Since the (simple) poles of $\Gamma(s)$ are exactly at $s=0, -1, -2, \dots$ we easily find the desired list of critical m.

Since $\mathscr{D}(s, f, \chi_0)$ differs from $L(s, \Sigma \otimes \chi)$ only by a product of Γ-functions, we see from Theorem 1.2 that $\mathscr{D}(s, f, \chi_0)$ continues to a meromorphic function on $\mathbb{C}$. Moreover Lemma 1.1, Theorem 1.2 and Proposition 1.3 show

Remark 2.2. If m is critical, then $\mathscr{D}(m, f, \chi_0)$ is finite.

For a systematic study of these special values we introduce some more notation. Let χ_0^+ denote the primitive even Dirichlet character such that for all $a \in \mathbb{Z}$ with $(a, 4c_\chi) = 1$ we have

$$\chi_0^+(a) = \chi_0(a) \cdot \left(\frac{\chi_0(-1)}{a}\right). \tag{2.2}$$

As in the previous section $G(\chi_0^+)$ denotes the Gauß sum

$$G(\chi_0^+)=\sum_{a=1}^{r}\chi_0^+(a)\cdot\exp(2\pi i a/r),$$

where r is the conductor of χ_0^+. For critical m let $\delta=\delta(m,\chi_0)=0$ or 1 according as $\chi_0(-1)=(-1)^{m+\delta}$, and put

$$\mathscr{L}(m,f,\chi_0):=\frac{G(\chi_0^+)^{1+\delta}}{\pi^{(1+\delta)m+k-1}\cdot\langle f,f\rangle}\cdot\mathscr{D}(m,f,\chi_0). \tag{2.3}$$

Theorem 2.3. *If m is critical for $\Sigma\otimes\chi$ then $\mathscr{L}(m,f,\chi_0)$ is algebraic and for any $\tau\in G_{\mathbb{Q}}$ we have*

$$\mathscr{L}(m,f,\chi_0)^\tau=\mathscr{L}(m,f^\tau,\chi_0^\tau).$$

Remarks 2.4. a) This establishes in particular the conjecture of Deligne in [4] for the motives $\Sigma\otimes\chi$, i.e. the motives $\mathrm{Sym}^2(M(f))\otimes\chi$ in Deligne's notation (see (7.8.4) in [4]).

b) For non-critical $m\leqq 0$ we have $\mathscr{L}(m,f,\chi_0)=0$, so the statement of the theorem trivially holds.

c) The analogue of the theorem for the imprimitive symmetric square L-function defined via (1.16) was proven by Sturm [13], [14] and in special cases by Zagier [16]. Note that our L-function differs from that one by finitely many Euler factors which may vanish at a critical $s=m$.

Proof. We may and will suppose that f has even level N. If not we might twist f by a character ε of conductor $c_\varepsilon=4$. This won't affect the function $\mathscr{D}(s,f,\chi_0)=\mathscr{D}(s,f_\varepsilon,\chi_0)$. In addition we know by Proposition 1 in [12] that the quotient $J(f)=\langle f_\varepsilon,f_\varepsilon\rangle/\langle f,f\rangle$ only depends on f and the conductor of ε, and that we have $J(f)^\tau=J(f^\tau)$ for any $\tau\in\mathrm{Aut}(\mathbb{C})$. So suppose N is even. Under this condition Sturm [13, 14] has shown that the statement of the theorem holds if we replace $\mathscr{D}(m,f,\chi_0)$ in (2.3) by the imprimitive L-value $D(m+k-1,f,(\chi\psi)_0^{-1})$. Remember that by Lemma 1.5 and 1.6

$$\mathscr{D}(s,f,\chi_0)=\mathscr{P}(s,f,\chi_0)^{-1}\cdot D(s+k-1,f,(\chi\psi)_0^{-1}), \tag{2.4}$$

where $\mathscr{P}(s,f,\chi_0)$ is a finite Euler product defining an entire complex function whose possible zeros are on the line $\mathrm{Re}(s)=0$. Note that in Lemma 1.5, 1.6 we assume f has minimal level among all twists, whereas here we want f to have even level (which might destroy minimality at 2). So if the minimal level of all twists is odd we replace f by f_ε in (2.4), so that the whole Euler factor at the prime $\ell=2$ is a factor of $\mathscr{P}(s,f_\varepsilon,\chi_0)$. Again the zeros of this (good) Euler factor are on the line $\mathrm{Re}(s)=0$. By the explicit formulas in Lemma 1.5, 1.6 we know that for any integer m the value $\mathscr{P}(m,f,\chi_0)$ is algebraic and that for $\tau\in G_{\mathbb{Q}}$ we have $\mathscr{P}(m,f,\chi_0)^\tau=\mathscr{P}(m,f^\tau,\chi_0^\tau)$. Hence the statement of our theorem becomes obvious for $m\neq 0$ by Sturm's results.

Case $m=0$. If $m=0$ is critical then χ_0 is even and $m=1$ is also critical. So by what we proved before we know that $\mathscr{L}(1,f,\bar{\chi}_0)^\tau=\mathscr{L}(1,f^\tau,\bar{\chi}_0^\tau)$ for $\tau\in G_{\mathbb{Q}}$. On the other hand the functional equation in Theorem 1.2 tells us that

$$\mathscr{L}(0,f,\chi_0)=\frac{G(\chi_0)}{G(\bar{\chi}_0)^2}\cdot\varepsilon(0,\Sigma\otimes\chi)\cdot\frac{k-1}{2}\cdot\mathscr{L}(1,f,\bar{\chi}_0),$$

hence by Lemma 1.4a) our proof is complete.

Remark 2.5. Although it seems that we prove the theorem only for a form f of minimal level among all twists, the statement holds in fact for any f by the same reasoning as at the beginning of the proof.

Corollary 2.6. *Modifying* (2.3) *we put*

$$I(m,f,\chi_0):=\left(\frac{G(\chi_0)}{(2\pi i)^m}\right)^{1+\delta}\frac{\mathscr{D}(m,f,\chi_0)}{\pi^{k-1}\cdot\langle f,f\rangle}.$$

If m is critical for $\Sigma\otimes\chi$, *then for any* $\tau\in G_{\mathbb{Q}}$ *we have*

$$I(m,f,\chi_0)^\tau=I(m,f^\tau,\chi_0^\tau).$$

For later reference we work out a "functional equation" for the numbers

$$\tilde{I}(m,f,\lambda_0\chi_0):=\Gamma(k+m-1)\cdot(-c_{\lambda\chi})^{(m-1)(1+\delta)+1}\cdot I(m,f,\lambda_0\chi_0),$$

where λ,χ is a pair of finite größencharacters such that the conductor c_χ of χ is prime to Nc_λ and $\delta=\delta(m,\lambda_0\chi_0)$. Let

$$C(\Sigma,\lambda):=\varepsilon(0,\Sigma\otimes\lambda)\cdot\left(\frac{G(\lambda_0)}{\sqrt{\lambda_0(-1)}\,c_\lambda}\right)^3$$

and put $M_\lambda:=-c_\lambda^{-3}\cdot\operatorname{cond}(\Sigma\otimes\lambda)$.

Proposition 2.7. *If* $m\leq 0$ *is critical for* $\Sigma\otimes\lambda\chi$, *then we have*

$$\tilde{I}(m,f,\lambda_0\chi_0)=C(\Sigma,\lambda)\cdot M_\lambda^{-m}\,\bar{\chi}_0(M_\lambda)\cdot 2\cdot\Gamma(1-m)\cdot\tilde{I}(1-m,f,\overline{\lambda_0\chi_0}).$$

Proof. By Lemma 1.1 we easily get

$$\frac{L(1-m,(\Sigma\otimes\lambda\chi)_\infty)}{L(m,(\Sigma\otimes\lambda\chi)_\infty)}=(2\pi i)^{3m-2}\cdot 2i^{\kappa+2}\,\frac{\Gamma(k-m)\cdot\Gamma(1-m)}{\Gamma(m+k-1)},\tag{2.5}$$

where $\lambda_0\chi_0(-1)=(-1)^\kappa$. Now specialize the functional equation in Theorem 1.2 to $s=m$ and use Lemma 1.4b) to control the contribution by ε-factors. Together with (2.5) this proves the proposition.

§ 3. *P*-adic interpolation of half the critical values

We fix a prime number $p>2$ and an embedding $i_p: \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. Throughout the rest of this article we make the following

Hypothesis. p does not divide N and a_p is a p-adic unit, i.e. $|i_p(a_p)|_p=1$. We say p is *"ordinary"* for π.

Thus in the decomposition

$$1-a_p X+\psi(p)\, p^{k-1} X^2=(1-\alpha_p X)(1-\beta_p X)$$

we may choose α_p to be a p-adic unit.

Remember that a distribution μ on $\mathbb{Z}_p^\times$ is determined by the integrals of all finite order characters χ of $\mathbb{Z}_p^\times$, i.e. any family (A_χ) of values defines a distribution μ by demanding that

$$\int_{\mathbb{Z}_p^\times} \chi\, d\mu=A_\chi \qquad \text{for all } \chi.$$

We fix a primitive Dirichlet character λ_0 with corresponding größencharacter λ of conductor c_λ prime to p, and we define for each $m=2-k, \ldots, 0$ a complex valued distribution $\mu_m=\mu_m(\Sigma \otimes \lambda)$ as follows. Define an Euler factor at p by

$$E_p(X):=(1-\bar{\psi}_0(p)\cdot \beta_p^2 \cdot p^{1-k} X)(1-X)$$

and put $\gamma_m:=p^{m+k-2}\cdot \psi(p)\cdot \alpha_p^{-2}$. Let $Q_{m,\lambda}:=\Gamma(k+m-1)\cdot(-c_\lambda)^m$. Then μ_m is given by

(i) $$\int_{\mathbb{Z}_p^\times} d\mu_m=Q_{m,\lambda}\cdot(1-\lambda_0(p)\,\gamma_m)\cdot E_p(\bar{\lambda}_0(p)\, p^{-m})\cdot I(m,f,\lambda_0),$$

and for all primitive Dirichlet characters χ_0 of conductor $c_\chi=p^{m_\chi}$, $m_\chi>0$

(ii) $$\int_{\mathbb{Z}_p^\times} \chi_0\, d\mu_m=Q_{m,\lambda}\cdot \gamma_m^{m_\chi}\cdot I(m,f,\lambda_0\chi_0).$$

Our aim is to prove the existence of a p-adic measure on $\mathbb{Z}_p^\times$ which roughly speaking coincides with μ_m at almost all characters χ_0 (see § 5). For the moment we deal with a slightly modified distribution for technical reasons. Let μ'_m denote the distribution defined by the formulas (i) and (ii) where we have replaced $\mathscr{D}(m,f,\lambda_0\chi_0)$ by the special values of the imprimitive symmetric square $D(m+k-1, f, (\lambda\chi\psi)_0^{-1})$. In the following we do not suppose that $f\in S_k(N,\psi)$ has minimal level among all twists. Let F denote the field generated over $\mathbb{Q}$ by adjoining all Fourier coefficients of f and the values of λ_0. Let $F_\mathfrak{p}$ denote the completion of F with respect to the place $\mathfrak{p}\,|\,p$ in F which corresponds to the embedding $i_p: \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. Note that by our hypothesis we have $\alpha_p \in F_\mathfrak{p}$.

Theorem 3.1. *Suppose N is even. Then each distribution μ'_m for $m=2-k, \ldots, 0$ has values in $F_\mathfrak{p}$. If λ_0 is not imaginary-quadratic then these values are p-adically* *bounded, i.e. μ'_m is a measure. If λ_0 is imaginary-quadratic then for any topological generator u of $\mathbb{Z}_p^\times$ there is a unique measure $\mu'_{m,u}$ such that we have*

$$(1-\bar{\chi}_0(u)\,u^{1-m})\int_{\mathbb{Z}_p^\times}\chi_0\,d\mu'_m=\int_{\mathbb{Z}_p^\times}\chi_0\,d\mu'_{m,u}\quad \textit{for all } \chi_0.$$

Proof. The algebraicity results of Sturm [13] easily imply that the values of μ'_m are in F_p. The crucial point of the proof is to show that these distributions have bounded p-adic absolute value for λ_0 not imaginary-quadratic.

As in [9] we define the modified form $f_0(z)=f(z)-\beta_p\cdot f(pz)$ of level Np and note the following properties:

$$D(s,f_0,(\lambda\psi\chi)_0^{-1})=D(s,f,(\lambda\psi\chi)_0^{-1})\quad \text{for } m_\chi\geqq 1, \tag{3.1}$$

$$\int_{\mathbb{Z}_p^\times} d\mu'_m=Q_{m,\lambda}(1-\lambda_0(p)\cdot\gamma_m)\cdot I'(m,f_0,\lambda_0), \tag{3.2}$$

Where for any character δ_0 of parity $\delta_0(-1)=(-1)^m$ we put

$$I'(m,f_0,\delta_0):=\frac{G(\delta_0)}{(2\pi i)^m}\,\frac{D(m+k-1,f_0,(\delta\psi)_0^{-1})}{\pi^{k-1}\cdot\langle f,f\rangle},$$

$$f_0\,|\,T(p)=\alpha_p\cdot f_0\quad \text{for the Hecke operator } T(p). \tag{3.3}$$

For any subset $U=y+p^r\mathbb{Z}_p\subseteq\mathbb{Z}_p^\times$ we therefore can write

$$\mu'_m(U)=\frac{Q_{m,\lambda}}{\varphi(p^r)}\cdot((1-\gamma_m\cdot\lambda_0(p))\cdot I'(m,f_0,\lambda_0)$$
$$+\sum_{\substack{\chi\neq 1\\ m_\chi\leqq r}}\bar{\chi}_0(y)\cdot\gamma_m^{m_\chi}\cdot I'(m,f_0,\lambda_0\chi_0)) \tag{3.4}$$

and we must show the existence of a constant $C>0$ such that $|\mu'_m(y+p^r\mathbb{Z}_p)|_p\leqq C$ for all y and r. It is obviously sufficient to show this for $r\geqq 2$.

The starting point of our proof is the following integral formula of Shimura [11]. Let $\rho_\chi=0$ or 1 such that $(\lambda\psi\chi)_0(-1)=(-1)^{\rho_\chi}$ and recall from §2 that $(\chi\lambda)_0^+(a)=(\chi\lambda)_0(a)\cdot\left(\frac{\chi_0\lambda_0(-1)}{a}\right)$. Then by (1.5) in [11] we have

$$(4\pi)^{-s/2}\cdot\Gamma(s/2)\cdot D(s-\rho_\chi,f_0,(\lambda\psi\chi)_0^{-1})$$
$$=\int_{\Gamma_0(N_\chi)\backslash\mathscr{H}} f_0(z)\cdot\overline{\theta_{(\lambda\psi\chi)_0}(z)}\,y^{\rho_\chi+1/2}\cdot H(s,z)\,y^{-2}\,dx\,dy, \tag{3.5}$$

where we put $N_\chi:=\text{l.c.m.}\,(Np,4c^2_{\lambda\psi\chi})$,

$$H(s,z):=L_{Np}(2(s-\rho_\chi)-2k+2,(\lambda\chi)_0^{-2})\cdot E(z,s+1-2\rho_\chi,2k-2\rho_\chi-1,\overline{(\lambda\chi)_0^+})$$

and

$$\theta_{(\lambda\psi\chi)_0}(z)=\frac{1}{2}\sum_{n=-\infty}^{\infty}(\lambda\psi\chi)_0(n)\cdot n^{\rho_\chi}\cdot q^{n^2},$$

$$E(z,s,t,\omega)=y^{s/2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N_\chi)}\omega(d_\gamma)\cdot j(\gamma,z)^t\,|j(\gamma,z)|^{-2s},$$

where the $\gamma=\begin{pmatrix}a_\gamma & b_\gamma\\ c_\gamma & d_\gamma\end{pmatrix}$ run over a system of representatives for $\Gamma_0(N_\chi)$ modulo $\Gamma_\infty:=\left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix};\ n\in\mathbb{Z}\right\}$. If necessary we do replace f by a twisted form in order to make sure that we have

$$c_{\lambda\psi}^2 \text{ divides } N/2, \text{ i.e. } N_\chi=\text{l.c.m.}\,(Np,4c_\chi^2), \tag{3.6}$$

without changing the function $D(s,f,(\lambda\psi\chi)_0^{-1})$. This is possible since twisting f by a character ξ of conductor $c_\xi=c_{\lambda\psi}^n$ will for large n create a factor c_ξ^2 in the level of f_ξ, whereas the conductor of $\lambda\psi\xi^2$ is c_ξ and

$$D(s,f,(\lambda\psi\chi)_0^{-1})=D(s,f_\xi,(\lambda\psi\xi^2\chi)_0^{-1})$$

with $f_\xi\in S_k(M,\psi\xi^2)$ for some M. In particular we have $N_1=p\cdot\text{l.c.m.}\,(N,4)$. We replace s by $t=s+k-1+\rho_\chi$ in (3.5) and put

$$\mathscr{H}(z,\chi,s)=\theta_{(\lambda\psi\chi)_0}(z)\cdot L_{Np}(2s,(\lambda\chi)_0^2)\cdot E_0(z,s,\chi_0), \tag{3.7}$$

where $E_0(z,s,\chi_0):=E(z,s+\rho_\chi+1-k,1-2(k-\rho_\chi),(\lambda\chi)_0^+)$, so that for real s in terms of the Petersson inner product we get

$$(4\pi)^{-t/2}\,\Gamma(t/2)\,D(t-\rho_\chi,f_0,(\lambda\psi\chi)_0^{-1})=\langle f_0(z),\mathscr{H}(z,\chi,s)\rangle_{N_\chi}. \tag{3.8}$$

We must study $\mathscr{H}^*(z,\chi,s):=\mathscr{H}(z,\chi,s)|_k W_{N_\chi}$ where $W_{N_\chi}=\begin{pmatrix}0 & -1\\ N_\chi & 0\end{pmatrix}$. By definition we have

$$\mathscr{H}^*(z,\chi,s)=\theta_{(\lambda\psi\chi)_0}\left(\frac{-1}{zN_\chi}\right)\cdot L_{Np}(2s,(\lambda\chi)_0^2)\cdot E_0\left(\frac{-1}{zN_\chi},s,\chi_0\right)\quad (z\sqrt{N_\chi})^{-k}. \tag{3.9}$$

Following Sturm [13] the specialisations to $s=m\in\{2-k,\dots,0\}$ of $\mathscr{H}(z,\chi,s)$ and hence also those of $\mathscr{H}^*(z,\chi,s)$ are generalized modular forms (in general non-holomorphic) under the assumption that m be critical for $\Sigma\otimes\lambda\chi$, i.e. $m\equiv k+\rho_\chi(2)$. In terms of their holomorphic projection $\mathscr{H}_0^*(z,\chi,m)$ we have

$$\langle f_0(z),\mathscr{H}(z,\chi,m)\rangle_{N_\chi}=\langle f_0(z),\mathscr{H}_0^*(z,\chi,m)|_k W_{N_\chi}\rangle_{N_\chi}.$$

Now we proceed as on p. 217 in [9] applying the trace $\mathrm{tr}=\mathrm{tr}(\Gamma_0(N_\chi)\backslash\Gamma_0(N_1))$ to the form $\mathcal{H}_0^*(z,\chi,m)\,|\,W_{N_\chi}$, which leads to the following formulas for any integer $r\geqq m_\chi$:

$$\alpha_p^{2(r-m_\chi)}\,\Gamma_m\,D(m+k-1,f_0,\overline{(\lambda\psi\chi)_0})$$
$$=p^{-(2m_\chi-1)\left(\frac{k}{2}-1\right)}\langle f_0(z)|\,W_{N_1},\mathcal{H}_0^*(z,\chi,m)\,|\,T(p)^{2r-1}\rangle_{N_1} \qquad (3.10)$$

if $m_\chi\geqq 1$, and for the trivial character $\chi=1$ we have

$$\alpha_p^{2r}\cdot(1-\gamma_m\,\lambda_0(p))\cdot\Gamma_m\cdot D(m+k-1,f_0,(\lambda\psi)_0^{-1})$$
$$=\langle f_0(z)|\,W_{N_1},R(z,m)\,|\,T(p)^{2r-2}\rangle_{N_1}, \qquad (3.11)$$

where $\Gamma_m:=(4\pi)^{-\left[\frac{m+k-1}{2}\right]-\frac{1}{2}}\,\Gamma\left(\left[\frac{m+k-1}{2}\right]+\frac{1}{2}\right)$ and where we let

$$R(z,m):=\mathcal{H}_0^*(z,1,m)\,|\,T(p)^2-\overline{(\lambda\psi)}_0(p)\cdot p^{m+k-2}\,\mathcal{H}_0^*(z,1,m).$$

So if we define the holomorphic form $\mathcal{F}_{r,y}(z):=\mathcal{F}_{r,y}^{(m)}(z)$ by

$$\mathcal{F}_{r,y}(z):=\overline{G(\lambda_0)}\cdot R(z,m)\,|\,T(p)^{2r-2}$$
$$+\sum_{\substack{\chi\neq 1\\ m_\chi\leqq r}}\chi_0(y)\cdot c_\chi^m\,p^{\frac{k}{2}-1}\,\bar{\psi}(c_\chi)\cdot\overline{G(\lambda_0\chi_0)}\,\mathcal{H}_0^*(z,\chi,m)\,|\,T(p)^{2r-1},$$

we easily get via (3.10), (3.11)

$$\langle f_0\,|\,W_{N_1},\mathcal{F}_{r,y}\rangle_{N_1}$$
$$=\alpha_p^{2r}((1-\gamma_m\,\lambda_0(p))\cdot G(\lambda_0)\,\Gamma_m\,D(m+k-1,f_0,\overline{(\lambda\psi)}_0)$$
$$+\sum_\chi\bar{\chi}_0(y)\cdot\gamma_m^{m_\chi}\cdot G(\lambda_0\chi_0)\cdot\Gamma_m\cdot D(m+k-1,f_0,\overline{(\lambda\psi\chi)}_0)). \qquad (3.12)$$

Note that by Remark 2.4b) for non-critical m we can put $\mathcal{H}^*(z,\chi,m)=\mathcal{H}(z,\chi,m)=0$, i.e. it suffices to consider those χ where $\rho_\chi\equiv m+k(2)$. The comparison with (3.4) immediately shows

Lemma 3.2. *Let* $\mathcal{P}_m=\mathcal{P}_{m,\lambda}=Q_{m,\lambda}\cdot(-2\pi i)^{-m}\,\pi^{1-k}\,\Gamma_m^{-1}$ *and put*

$$\tilde{\mathcal{F}}_{r,y}(z)=\tilde{\mathcal{F}}_{r,y}^{(m)}(z)=\mathcal{P}_m\cdot\varphi(p^r)^{-1}\cdot\mathcal{F}_{r,y}^{(m)}(z).$$

Then we have

$$\mu'_m(y+p^r\,\mathbb{Z}_p)=\alpha_p^{-2r}\,\frac{\langle f_0\,|\,W_{N_1},\tilde{\mathcal{F}}_{r,y}\rangle_{N_1}}{\langle f,f\rangle}.$$

The problem of finding a uniform bound for the p-adic absolute values of these numbers amounts to showing that the Fourier coefficients of the form $\tilde{\mathcal{F}}_{r,y}$

have p-adic absolute values bounded independently of r and y. This will be carried out by using the Fourier development of the generalized modular forms $\mathscr{H}^*(z,\chi,m)$ in an analogous way as in [9]. We will just quote the results that we need here and refer to that article for further details. The forms $\mathscr{H}^*$ have a Fourier development of the form

$$\mathscr{H}^*(z,\chi,m)=\sum_{j=0}^{[k/2]}(4\pi y)^{-j}\sum_{n=0}^{\infty}c_{n,j}\cdot q^n,$$

where $c_{n,j}=c_{n,j}^{(m)}(\chi)$. Write the holomorphic projection as $\mathscr{H}_0^*(z,\chi,m)=\sum_n c_n q^n$, where $c_n=c_n^{(m)}(\chi)$. Then we have the following relationship between the coefficients of $\mathscr{H}^*$ and those of $\mathscr{H}_0^*$ (see Lemma 4.1 in [9]). Let $k_0:=[k/2]$.

Lemma 3.3. *There is a positive integer R and there are linear forms $F_n(X_0,\ldots,X_{k_0})$ in $\mathbb{Z}[X_0,\ldots,X_{k_0}]$ which only depend on R and n, such that $F_n(\underline{X})\equiv R\cdot X_0 \bmod n$ and*

$$R\cdot\mathscr{H}_0^*(z,\chi,m)=\sum_{n=0}^{\infty}F_n(c_{n,0},\ldots,c_{n,k_0})\cdot q^n.$$

Recall that the action of the Hecke operator $T(p)^t$ on $\mathscr{H}_0^*(z,\chi,m)$ is given in terms of the Fourier coefficients by

$$\mathscr{H}_0^*(z,\chi,m)\,|\,T(p)^t=\sum_{n=0}^{\infty}c_{np^t}\cdot q^n.$$

Since $\mathscr{F}_{r,y}(z)=\sum_{n=0}^{\infty}a_{r,y}(n)\cdot q^n$ is a linear combination of these forms we can easily describe the $a_{r,y}(n)$ as follows.

Lemma 3.4. *The Fourier coefficients of $R(z,m)=\sum_{n=0}^{\infty}r_n\cdot q^n$ are given by $r_n=c_{np^2}(1)-(\overline{\lambda\psi})_0(p)\cdot p^{m+k-2}\cdot c_n(1)$, and we have*

$$a_{r,y}(n)=\overline{G(\lambda_0)}\cdot r_{np^{2r-2}}+\sum_{\substack{\chi\neq 1\\ m_\chi\leq r}}\chi_0(y)\,c_\chi^m\,p^{\frac{k}{2}-1}\,\overline{\psi}(c_\chi)\cdot\overline{G(\lambda_0\chi_0)}\cdot c_{np^{2r-1}}(\chi).$$

Next we want to reduce the proof of Theorem 3.1 in case λ_0 is not imaginary-quadratic to the proof of

Proposition 3.5. *There is a numberfield K_0 of finite degree over $\mathbb{Q}$ containing the Fourier coefficients of the holomorphic forms $\tilde{\mathscr{F}}_{r,y}(z)=\sum_{n=0}^{\infty}\tilde{a}_{r,y}(n)\cdot q^n$ for all residue classes $y+p^r\mathbb{Z}_p\subseteq\mathbb{Z}_p^*$. There is a universal bound $C>0$ such that we have $|\tilde{a}_{r,y}(n)|_p<C$ for $r\geq 2$ if λ_0 is not imaginary-quadratic.*

Proof of Theorem 3.1. Let us assume for the moment that Proposition 3.5 holds. Let $F_0 := FK_0$ and denote by $M_k(N_1, F_0)$ the space of (holomorphic) modular forms of weight k and level N_1 whose Fourier coefficients belong to F_0. Then according to Proposition 4.5 in [7] and Lemma 3 in [12] we have a linear form

$$\mathscr{L}: M_k(N_1, F_0) \to F_0, \quad \mathscr{F} \mapsto \frac{\langle f_0|_k W_{N_1}, \mathscr{F}^\rho\rangle_{N_1}}{\langle f, f\rangle_N},$$

which by base change induces a homomorphism of $\mathbb{Q}_p$-vector spaces. Note that under such maps the image of a $\mathbb{Z}_p$-lattice must be p-adically bounded. In particular we find a universal bound C' such that $|\mathscr{L}(\tilde{\mathscr{F}}_{r,y})|_p < C'$, since by Proposition 3.5 the forms $\tilde{\mathscr{F}}_{r,y}$ are contained in a certain $\mathbb{Z}_p$-lattice. This completes the proof of the theorem.

We now start with the proof of Proposition 3.5. Put $m' := \left[\frac{m+k}{2}\right] - 1$ and define rational numbers

$$B_j = B_{j,m} = \frac{\Gamma\left(\left[\frac{m-k+1}{2}\right] + \frac{1}{2} + j\right) \cdot \binom{m'}{j}}{\Gamma(m'+1) \cdot \Gamma\left(\left[\frac{m-k+1}{2}\right] + \frac{1}{2}\right)} \quad \text{for } j = 0, \ldots, m'.$$

Recall that $N' := \text{l.c.m.}\,(N, 4c_{\lambda\psi}^2)/(4c_{\lambda\psi}^2) = \text{l.c.m.}\,(N, 4)/(4c_{\lambda\psi}^2)$ by (3.6) and put

$$\mathscr{C}(m, j) := \pi^{m'+1}(-2i)^{k-\rho_\chi-1}\, 2^{-m} \sqrt{N'}^{\rho_\chi - m} \cdot B_j \cdot c_{\lambda\psi}^{-(m+1)},$$

where we recall that we always assume $\rho_\chi \equiv k + m(2)$. For every integer $n > 0$ let ε_n denote the quadratic character attached to the extension $\mathbb{Q}(\sqrt{-nN'})/\mathbb{Q}$. For each character χ we put

$$\beta_\chi(n) := \sum_{a,b} \mu(a) \cdot (\lambda\chi)_0\, \varepsilon_n(a) \cdot (\lambda\chi)_0^2(b) \cdot a^{-m} \cdot b^{1-2m},$$

where (a, b) runs over all pairs of positive integers prime to N_1 such that $(ab)^2$ divides n, and μ here denotes the Möebius function.

Lemma 3.6. *For $\chi \neq 1$ we have*

$$c_{n,j}(\chi) = \mathscr{C}(m, j)\, \frac{G((\lambda\psi\chi)_0)}{c_\chi^{m+1}} \sum_{n_1, n_2} \overline{(\lambda\psi\chi)_0}(n_1) \cdot n_1^{\rho_\chi} \cdot n_2^{m'-j}$$

$$\cdot \begin{cases} L_{N_1}(2m-1, (\lambda\chi)_0^2) & \text{if } n_2 = 0, \\ L_{N_1}(m, (\lambda\chi)_0\, \varepsilon_{n_2}) \cdot \beta_\chi(n_2) & \text{if } n_2 \neq 0, \end{cases}$$

where the sum runs over all pairs of integers $n_i \geq 0$ satisfying $n_1^2 N' + n_2 = n$. If $\chi = 1$ then the same formula holds except that we must replace N' by pN' and ε_{n_2} by ε_{pn_2} (also in $\mathscr{C}(m, j)$ and $\beta_1(n_2)$).

Proof. By Proposition 1 in [11] we see analogously to Lemmas 4.2 and 4.3 in [9] that we have the following Fourier expansion of Eisenstein series for $\chi \neq 1$:

$$L_{N_1}(2m, (\lambda\chi)_0^2)\, N_\chi^{\frac{2m+1}{4}}\, y^{\frac{k-1-m-\rho_\chi}{2}}\, E_0(z, m, \chi_0)|_{k-\rho_\chi-\frac{1}{2}} W_{N_\chi}$$
$$= \tau_0(y)\cdot L_{N_1}(2m-1, (\lambda\chi)_0^2)$$
$$+ \sum_{n=1}^{\infty} \tau_n(y)\cdot L_{N_1}(m, (\lambda\chi)_0\,\varepsilon_n)\cdot \beta_\chi(n)\cdot e^{2\pi i n x}. \tag{3.13}$$

For trivial χ the formula remains true if we replace ε_n by ε_{pn}. Here the τ_n are given by the Fourier coefficients of the function

$$\sum_{m=-\infty}^{\infty} (z+m)^{-\alpha}(\bar{z}+m)^{-\beta} = \sum_{n=-\infty}^{\infty} \tau_n(y, \alpha, \beta)\cdot e^{2\pi i n x},$$

where we specialize for $n \geqq 0$ to $\tau_n(y) := \tau_n\left(y, \left[\frac{m+k}{2}\right], \left[\frac{m-k+1}{2}\right]+\frac{1}{2}\right)$. These functions can be calculated explicitly (see [11]). We find

$$\tau_0(y) = \frac{\Gamma(m-\frac{1}{2})}{\Gamma\left(\left[\frac{m+k}{2}\right]\right)\cdot\Gamma\left(\left[\frac{m-k+1}{2}\right]+\frac{1}{2}\right)}\, i^{\rho_\chi-k+\frac{1}{2}}\, 2\pi(2y)^{\frac{1}{2}-m},$$

and for $n>0$

$$\tau_n(y) = n^{m-\frac{1}{2}}\, i^{\rho_\chi-k+\frac{1}{2}}(2\pi)^{m+\frac{1}{2}}\, e^{-2\pi n y} \sum_{j=0}^{m'} B_j(4\pi n y)^{-j-\left[\frac{m-k+1}{2}\right]-\frac{1}{2}}.$$

Thus the function $E^*(z, s, \chi_0) := L_{N_1}(2s, (\lambda\chi)_0^2)\cdot E_0(z, s, \chi_0)|_{k-\rho_\chi-\frac{1}{2}} W_{N_\chi}$ has at $s=m$ a Fourier expansion of the form

$$E^*(z, m, \chi_0) = \sum_{j=0}^{m'} \sum_{n=0}^{\infty} d_{n,j}(4\pi y)^{-j}\cdot q^n,$$

where by (3.13) the coefficients $d_{n,j} = d_{n,j}^{(m)}(\chi)$ are easily described as follows. With the convention $O^0 = 1$ we have

$$d_{n,j} = N_\chi^{-(2m+1)/4}(-2i)^{k-\rho_\chi-\frac{1}{2}}\,\pi^{m'+1}\, B_j$$
$$\cdot\, n^{m'-j}\begin{cases} L_{N_1}(2m-1, (\lambda\chi)_0^2) & \text{if } n=0,\\ L_{N_1}(m, (\lambda\chi)_0\,\varepsilon_n)\cdot\beta_\chi(n) & \text{if } n>0,\end{cases} \tag{3.14}$$

for $\chi \neq 1$, and for trivial χ we must replace ε_n by ε_{pn}. On the other hand the Fourier expansion of the theta functions is well-known (see Proposition 2.2 in [10]). We have

$$\theta_{(\lambda\psi\chi)_0}(z)|_{\rho_\chi+\frac{1}{2}} W_{N_\chi} = i^{\frac{1}{2}}\, \frac{G((\lambda\psi\chi)_0)}{\sqrt{c_{\lambda\psi\chi}}}\, \sqrt{N'}^{\rho_\chi+\frac{1}{2}}\, \theta_{(\lambda\psi\chi)_0}(zN') \tag{3.15}$$

for $\chi \neq 1$, where now the formula remains true for $\chi = 1$ if we replace N' by pN'. Hence writing

$$\mathscr{H}^*(z, \chi, m) = \theta_{(\lambda\psi\chi)_0}(z)|_{\rho_\chi + \frac{1}{2}} W_{N_\chi} \cdot E^*(z, m, \chi_0),$$

we find the formula of Lemma 3.6 by multiplying together the two developments above.

In view of Lemma 3.4 we define linear forms $L_{r,y} = L_{r,y}^{(m)}$ in the indeterminates X_χ, Y and Z where χ runs over all non-trivial characters such that c_χ divides p^r and $\rho_\chi \equiv k+m(2)$. Let

$$L_{r,y}(\ldots, X_\chi, \ldots; Y, Z) := \sum_{\chi \neq 1} \alpha_\chi X_\chi + \alpha_1 \cdot \begin{cases} Y - (\overline{\lambda\psi})_0(p)\, p^{m+k-2} Z & \text{if } \lambda(-1) = (-1)^m, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_\chi := p^{\frac{k}{2}-1} \cdot \chi_0(y) \cdot c_\chi^m \cdot \bar{\psi}(c_\chi) \cdot \overline{G(\lambda_0 \chi_0)}$ for $\chi \neq 1$ and $\alpha_1 = \overline{G(\lambda_0)}$. With these notations Lemma 3.4 says

$$a_{r,y}(n) = L_{r,y}(\ldots, c_{np^{2r-1}}(\chi), \ldots; c_{np^{2r}}(1), c_{np^{2r-2}}(1)).$$

We must consider the analogous quantities $a_{r,y}(n,j) := a_{r,y}^{(m)}(n,j)$ given by

$$a_{r,y}(n,j) := L_{r,y}(\ldots, c_{np^{2r-1},j}(\chi), \ldots; c_{np^{2r},j}(1), c_{np^{2r-2},j}(1)).$$

Recall that eventually we want a bound for the p-adic absolute value of the algebraic numbers $\tilde{a}_{r,y}(n) = \mathscr{P}_m \cdot \varphi(p^r)^{-1} a_{r,y}(n)$. We first show

Lemma 3.7. a) *The numbers $\mathscr{P}_m \cdot a_{r,y}(n,j)$ are contained in the number field $K_0 = \mathbb{Q}(i, \sqrt{N'}, \sqrt{p}, G(\psi_0), G(\lambda_0)$, all $\lambda_0(t), \psi_0(t))$. In addition the algebraic numbers $\mathscr{P}_m \cdot \alpha_\chi \cdot c_{n,j}^{(m)}(\chi)$ have bounded p-adic absolute value.*

b) *There is a constant $\delta \in \mathbb{Q}^*$ such that for any residue class ($r \geqq 2$) $y + p^r \mathbb{Z}_p \subseteq \mathbb{Z}_p^*$ and all n we have $\delta \cdot \mathscr{P}_m \cdot a_{r,y}^{(m)}(n, 0) \equiv 0$ modulo p^r, if λ_0 is not imaginary-quadratic.*

We first finish the proof of Proposition 3.5, whose statement is equivalent to the existence of a constant $\delta' \in \mathbb{Q}^\times$ such that we have $\delta' \cdot \mathscr{P}_m \cdot a_{r,y}^{(m)}(n) \equiv 0 \bmod p^r$ for all r, y, n. Note that by Lemma 3.3 these numbers all belong to K_0. Also by that lemma there are linear forms $F_n, L_n \in \mathbb{Z}[\underline{X}]$ in $\underline{X} = (\ldots, X_j, \ldots)$ such that $F_n(\underline{X}) = R \cdot X_0 + n \cdot L_n(\underline{X})$ and $R \cdot c_n(\chi) = F_n(c_{n,0}(\chi), \ldots, c_{n,k_0}(\chi))$. Hence we have $a_{r,y}(n) = a_{r,y}(n, 0) + p^{2r} \cdot B(n, r)$, where

$$B(n,r) := \frac{n}{Rp^2}\big(p \cdot L_{np^{2r-1}}(\underline{x}) + p^2 L_{np^{2r}}(\underline{y}) - (\overline{\lambda\psi})_0(p) \cdot p^{m+k-2} \cdot L_{np^{2r-2}}(\underline{z})\big)$$

with $x_j = \sum_{\chi \neq 1} \alpha_\chi \cdot c_{np^{2r-1},j}(\chi)$, $y_j = \alpha_1 \cdot c_{np^{2r},j}(1)$ and $z_j = \alpha_1\, c_{np^{2r-2},j}(1)$.

By Lemma 3.7 this completes the proof of Proposition 3.5.

Proof of Lemma 3.7. a) The rationality statement follows immediately from the explicit description of the $c_{n,j}(\chi)$ in the preceding lemma. In addition we see at once that the terms in question have bounded p-adic absolute value.

b) A little calculation as in the proof of Lemma 4.4 in [9] shows

$$\begin{aligned} c_{np^3,0}(1)-\overline{(\lambda\psi)}_0(p)\cdot p^{m+k-2}\cdot c_{np,0}(1) &= G((\lambda\psi)_0)\cdot\mathscr{C}(m,0)\,p^{\frac{k}{2}-1} \\ &\cdot\sum_{n_1,n_2}{}' \overline{(\lambda\psi)}_0(n_1)\cdot n_1^{\rho_1}\cdot n_2^{m'}\cdot L_{N_1}(m,\lambda_0\,\varepsilon_{n_2})\cdot\beta_1(n_2), \end{aligned} \tag{3.16}$$

where the sum $\sum'$ runs over all pairs of integers $n_i>0$ such that $n_1^2 N'+n_2=np^2$ and p does not divide n_i. Note that for $\chi\neq 1$ the formula for $c_{np^2,0}(\chi)$ in Lemma 3.6 simplifies to

$$\begin{aligned} c_{np^2,0}(\chi) &= G((\lambda\psi\chi)_0)\cdot\mathscr{C}(m,0) \\ &\cdot\sum_{n_1,n_2}{}' \overline{(\lambda\psi\chi)}_0(n_1)\cdot n_1^{\rho_\chi}\cdot n_2^{m'}\cdot L_{N_1}(m,(\lambda\chi)_0\,\varepsilon_{n_2})\cdot\beta_\chi(n_2), \end{aligned} \tag{3.17}$$

hence we get for $r\geqq 2$

$$\begin{aligned} a_{r,y}(n,0) &= p^{\frac{k}{2}-1}\cdot\mathscr{C}(m,0)\cdot\overline{G(\lambda_0)}\cdot G((\lambda\psi)_0) \\ &\cdot\sum_{(n_1,n_2)\in W_n}\overline{(\lambda\psi)}_0(n_1)\,n_1^{\rho}\,n_2^{m'} \\ &\cdot\sum_{\substack{\chi,\,m_\chi\leqq r\\ \rho_\chi=\rho}}\chi_0(y c_{\lambda\psi}\,n_1^{-1}\,c_\lambda^{-1})\cdot\beta_\chi(n_2)\cdot L_{N_1}(m,(\lambda\chi)_0\,\varepsilon_{n_2}) \end{aligned} \tag{3.18}$$

where W_n denotes the set of pairs (n_1,n_2) of positive integers prime to p, satisfying $n_1^2 N'+n_2=np^{2r-1}$ and $\rho=\rho_m=0$ or 1 according as $\rho\equiv m+k(2)$. Let V_n denote the set of pairs (a,b) of positive integers prime to N_1 such that $(ab)^2$ divides n. The product of missing Euler factors in the L-function above is (over primes r)

$$\prod_{r\mid N_1}(1-(\lambda\chi)_0\,\varepsilon_{n_2}(r)\,r^{-m})=\sum_{d\mid N}\mu(d)\cdot(\lambda\chi)_0\,\varepsilon_{n_2}(d)\,d^{-m}(1-(\lambda\chi)_0\,\varepsilon_{n_2}(p)\,p^{-m})$$

so that by (3.18) we have

$$\begin{aligned} a_{r,y}^{(m)}(n,0) &= \mathscr{K}_{m,\lambda}\cdot\sum_{(n_1,n_2)\in W_n}\ \sum_{(a,b)\in V_{n_2}}\ \sum_{d\mid N}\alpha_{m,\lambda}(n_1,n_2,\mathrm{ad},b) \\ &\cdot M_r^{(m)}(ab^2\,d y c_{\lambda\psi}\,n_1^{-1}\,c_\lambda^{-1}), \end{aligned} \tag{3.19}$$

where we put

$$\mathscr{K}_{m,\lambda}=p^{\frac{k}{2}-1}\cdot\mathscr{C}(m,0)\cdot\overline{G(\lambda_0)}\cdot G((\lambda\psi)_0),$$

$$\alpha_{m,\lambda}(n_1,n_2,\mathrm{ad},b)=n_1^{\rho}\cdot n_2^{m'}\cdot(\mathrm{ad})^{-m}\cdot b^{1-2m}\,\overline{\psi}_0(n_1)\cdot\lambda_0(ab^2\,dn_1^{-1})\cdot\varepsilon_{n_2}(\mathrm{ad})$$

and where for any integer prime to p we have defined

$$M_r^{(m)}(x) := \sum_{\substack{\chi,\, m_\chi \leqq r \\ \rho_\chi = \rho_m}} \chi_0(x) \cdot L(m, (\lambda\chi)_0\, \varepsilon_{n_2}) \cdot (1 - (\lambda\chi)_0\, \varepsilon_{n_2}(p)\, p^{-m}).$$

So by (3.19) everything is reduced to show the congruence $M_r^{(m)}(x) \equiv 0 \bmod p^{r-1}$ for any x prime to p. But this congruence is a well-known integrality and holomorphy statement about the Kubota-Leopoldt L-functions $\mathscr{L}_p((\lambda\chi)_0\, \varepsilon_{n_2}\, \omega^{1-m}, s)$, where ω denotes the Teichmüller character mod p. A perhaps more familiar formulation of these properties is the existence of a measure ν_m on $\mathbb{Z}_p^\times$, with values in $\mathbb{Z}_p$ such that we have

$$\int_{\mathbb{Z}_p^\times} \bar{\chi}_0\, d\nu_m = \mathscr{L}_p((\lambda\chi)_0\, \varepsilon_{n_2}\, \omega^{1-m}, m).$$

Here it is important that the tame component of the character $(\lambda\chi)_0\, \varepsilon_{n_2}\, \omega^{1-m}$ never becomes trivial by the contribution of the nontrivial character ε_{n_2} of conductor prime to p, which can't be cancelled by λ_0 if λ_0 is not imaginary-quadratic. (For more details see the proof of Lemma 3.9.) This finishes the proof of Lemma 3.7.

Now we suppose $\lambda_0^2 = 1$ and $\lambda_0(-1) = -1$. Let u be a topological generator of $\mathbb{Z}_p^\times$ and define a distribution $\mu'_{m,u}$ by demanding that

$$\int_{\mathbb{Z}_p^\times} \chi_0\, d\mu'_{m,u} = (1 - \bar{\chi}_0(u)\, u^{1-m}) \int_{\mathbb{Z}_p^\times} \chi_0\, d\mu'_m \qquad \text{for all } \chi_0.$$

By the same reasoning as before the proof that $\mu'_{m,u}$ is a measure, can be reduced to show the congruence $M_{r,u}^{(m)}(x) \equiv 0 \bmod p^{r-1}$, where

$$M_{r,u}^{(m)}(x) := \sum_{\substack{\chi,\, m_\chi \leqq r \\ \rho_\chi = \rho_m}} (1 - \chi_0(u)\, u^{1-m}) \cdot \chi_0(x) \cdot L_p(m, (\lambda\chi)_0\, \varepsilon_{n_2}).$$

Now it is again a standard result in Iwasawa theory that there is a measure $\nu_{m,u}$ on $\mathbb{Z}_p^\times$ with values in $\mathbb{Z}_p$ such that $M_{r,u}^{(m)}(x) = \varphi(p^r) \cdot \nu_{m,u}(x + p^r \mathbb{Z}_p)$, which eventually completes the proof of Theorem 3.1.

Theorem 3.8. *The measures μ'_m respectively $\mu'_{m,u}$ in Theorem 3.1 are related by the formula*

$$d\mu'_m(x) = x^m \cdot d\mu'_0(x) \text{ resp. } d\mu'_{m,u}(x) = x^m \cdot d\mu'_{0,u}(x).$$

The proof is based on

Lemma 3.9. *For $m = 2-k, \ldots, 0$ and any $r \geqq 1$ we have*

$$M_r^{(m)}(x) \equiv x^m \cdot M_r^{(0)}(x) \bmod p^{2r-1}$$

for all integers x prime to p. Also the analogue for $M_{r,u}^{(m)}$ holds.

Proof of Lemma 3.9. Using the p-adic L-function of Kubota and Leopoldt we express the $M_r^{(m)}(x)$ as

$$M_r^{(m)}(x) = \sum_{\chi_0,\, m_\chi \leqq r} \chi_0(x) \cdot \mathscr{L}_p((\lambda\chi)_0\, \varepsilon_{n_2}\, \omega^{1-m}, m),$$

where we recall that we put $\mathscr{L}_p(\phi, s) = 0$ if ϕ is an odd character. Thus in terms of the measures ν_m introduced above, we must show the congruence $\nu_m(x + p^r \mathbb{Z}_p) \equiv x^m \cdot \nu_0(x + p^r \mathbb{Z}_p)$ modulo p^r. From Iwasawa theory we know that there are formal power series $G(\lambda_0\, \varepsilon_{n_2}\, \chi_{0t}\, \omega^{1-m}, T) \in \mathbb{Z}_p[\lambda_0][\![T]\!]$ such that

$$\mathscr{L}_p((\lambda\chi)_0\, \varepsilon_{n_2}\, \omega^{1-m}, s) = G(\lambda_0\, \varepsilon_{n_2}\, \chi_{0t}\, \omega^{1-m}, \chi_0(u_1)\, u_1^{1-s} - 1), \tag{3.20}$$

where we have decomposed the fixed generator u of $\mathbb{Z}_p^\times$ according to the splitting $\mathbb{Z}_p^\times = \Delta \cdot \Gamma$ in $u = \omega(u) \cdot u_1$ and where $\chi_{0t} = \chi_0|_\Delta$. Recall that (3.20) holds as long as $\lambda_0\, \varepsilon_{n_2}\, \chi_{0t}\, \omega^{1-m} \neq 1$. In the excluded case one has $\lambda_0 = \varepsilon_{n_2}$, $\chi_{0t} = \omega^{m-1}$ and

$$\mathscr{L}_p(\chi_0|_\Gamma, s) = \frac{G(1, \chi_0(u_1)\, u_1^{1-s} - 1)}{\chi_0(u_1)\, u_1^{1-s} - 1}$$

with a certain formal power series $G(1, T) \in \mathbb{Z}_p[\![T]\!]$. Now in the case of the measures μ'_m (where we assume $\lambda_0 \neq \varepsilon_{n_2}$) we have

$$\int_{\mathbb{Z}_p^\times} \bar{\chi}_0(x)\, d\nu_0(x) = G(\lambda_0\, \varepsilon_{n_2}\, \chi_{0t}\, \omega, \chi_0(u_1)\, u_1 - 1),$$

hence

$$\int_{\mathbb{Z}_p^\times} \bar{\chi}_0(x)\, x^m\, d\nu_0(x) = G(\lambda_0\, \varepsilon_{n_2}\, \chi_{0t}\, \omega^{1-m}, \chi_0(u_1)\, u_1^{1-m} - 1) = \int_{\mathbb{Z}_p^\times} \bar{\chi}_0(x)\, d\nu_m(x),$$

which proves the desired congruence $x^m \cdot d\nu_0(x) = d\nu_m(x)$. In the exceptional case of an imaginary-quadratic λ_0 we work with

$$M_{r,u}^{(m)}(x) = \sum_{\chi_0,\, m_\chi \leqq r} (1 - \chi_0(u)\, u^{1-m}) \cdot \chi_0(x) \cdot \mathscr{L}_p((\lambda\chi)_0\, \varepsilon_{n_2}\, \omega^{1-m}, m)$$

and the measures $\nu_{m,u}$ introduced above. The same argument as before now yields $M_{r,u}^{(m)}(x) \equiv x^m \cdot M_{r,u}^{(0)}(x) \bmod p^{2r-1}$, which completes the proof of Lemma 3.9.

In order to finish the proof of Theorem 3..8 we must show the congruence $\mu'_m(x + p^r \mathbb{Z}_p) \equiv x^m \cdot \mu'_0(x + p^r \mathbb{Z}_p)$ modulo $p^{r-\kappa}$ and its analogue for $\mu'_{m,u}$ for some constant $\kappa \in \mathbb{N}$ which is independent of x, r and m. By Lemma 3.2 we have

$$\mu'_m(x + p^r \mathbb{Z}_p) = \frac{\langle f_0(z) \mid W_{N_1}, \mathscr{G}_{r,x}^{(m)}(z) \rangle_{N_1}}{\langle f, f \rangle_N},$$

where we put $\tilde{\mathscr{G}}^{(m)}_{r,x}(z)=\alpha_p^{-2r}\cdot\tilde{\mathscr{F}}^{(m)}_{r,x}(z)$ with Fourier expansion $\tilde{\mathscr{G}}^{(m)}_{r,x}(z)=\sum_{n=0}^{\infty} b^{(m)}_{r,x}(n)\cdot q^n$. The Fourier coefficients $b^{(m)}_{r,x}(n)$ are in $K_0(\alpha_p)$ by Proposition 3.5. Using the linear form

$$\mathscr{L}: M_k(N_1\cdot F_0)\to F_0,\ \mathscr{F}\mapsto \langle f_0\,|\,W_{N_1},\mathscr{F}^\rho\rangle_{N_1}/\langle f,f\rangle_N,$$

it suffices to show the following congruence of Fourier coefficients of $\tilde{\mathscr{G}}^{(m)}_{r,x}$ which corresponds to the congruence stated in the theorem:

$$b^{(m)}_{r,x}(n)\equiv x^m\cdot b^{(0)}_{r,x}(n) \bmod p^{r-\kappa'} \tag{C}$$

for a suitable constant $\kappa'\in\mathbb{N}$. By Lemmas 3.3 and 3.7a) we have

$$R\cdot\mathscr{P}_m\cdot a^{(m)}_{r,x}(n)\equiv R\cdot\mathscr{P}_m\cdot a^{(m)}_{r,x}(n,0) \bmod p^{2r-1}$$

for a suitable positive constant $R\in\mathbb{N}$, hence

$$R\cdot b^{(m)}_{r,x}\equiv R\,\frac{\mathscr{P}_m}{\alpha_p^{2r}\cdot\varphi(p^r)}\,a^{(m)}_{r,x}(n,0) \bmod p^r.$$

To verify (C) it is therefore sufficient to show the congruence

$$\mathscr{P}_m\cdot a^{(m)}_{r,x}(n,0)\equiv x^m\cdot\mathscr{P}_0\cdot a^{(0)}_{r,x}(n,0) \bmod p^{2r-1}. \tag{C'}$$

We easily compute $\mathscr{P}_m\cdot\mathscr{C}(m,0)=i(2i)^{3k-4}(-N')^{(\rho-m)/2}\,c_{\lambda\psi}^{-m-1}\,c_\lambda^m$, so that by (3.19) we find

$$\mathscr{P}_m\cdot a^{(m)}_{r,x}(n,0)=x^m\,t(\lambda,\psi,k)\cdot\sum_{n_i,a,b,d} c(n_1,n_2,a,b,d)\cdot(-n_1^2N'/n_2)^{\frac{\rho-k-m}{2}}$$

$$\cdot\,(ab^2dx\,c_{\lambda\psi}\,n_1^{-1}c_\lambda^{-1})^{-m}\cdot M^{(m)}_r(ab^2dx\,c_{\lambda\psi}\,n_1^{-1}c_\lambda^{-1})$$

with the abbreviations

$$t(\lambda,\psi,k):=i\,p^{\frac{k}{2}-1}\cdot\overline{G(\lambda_0)}\,G((\lambda\psi)_0)\,\frac{c_\lambda}{c_{\lambda\psi}^2}\,N'^{\frac{k}{2}}\,2^{3k-4},$$

$$c(n_1,n_2,a,b,d):=n_1^k\,n_2^{-1}\,b\,c_{\lambda\psi}\,c_\lambda^{-1}\,\bar{\psi}_0(n_1)\cdot\lambda_0(ab^2dn_1^{-1})\cdot\varepsilon_{n_2}(\mathrm{ad}).$$

This immediately implies (C′) by Lemma 3.9 and by the fact that the choice of (n_1,n_2) is such that $-n_1^2N'/n_2\equiv 1 \bmod p^{2r-1}$.

This proves Theorem 3.9 for the measures μ'_m. Again the measures $\mu'_{m,u}$ can be handeled completely analogous, hence the proof is complete.

Remark 3.10. Results similar to those of this section have been obtained independently by Hida (unpublished).

§ 4. *P*-adic interpolation of the other critical values

The following last two sections contain the really new ideas of this article. The measures μ'_m resp. $\mu'_{m,u}$ constructed in the preceding section provide us as usual with a package of p-adic formal power series which roughly speaking interpolate half the critical values of the imprimitive symmetric square. The aim of this section is to construct measures ν'_m resp. $\nu'_{m,u}$ for the other m's, i.e. such that the corresponding power series interpolate the remaining critical values of the imprimitive symmetric square. In the last section we will then relate the pairs of power series attached to the pair (μ'_m, ν'_{1-m}) resp. $(\mu'_{m,u}, \nu'_{1-m,u})$ via the functional equation of the primitive complex L-function. This requires of course multiplication by some p-adic Euler factors, i.e. by power series which interpolate the missing Euler factors in the imprimitive symmetric square. A priori one ends up with pairs of quotients of power series interpolating the critical values at m and $1-m$ of the primitive L-function. However, a careful analysis of the zeros of the p-adic Euler factors mentioned before eventually will enable us to conclude p-adic holomorphy and hence the existence of the measures alluded to at the beginning of Sect. 3, with a few exceptions. As an aside we get a functional equation for measures and p-adic L-functions.

As in the previous section our starting point is again Shimura's integral representation (3.5) which we now want to specialize at $s=1, \ldots, k-1$. We do not suppose that f has minimal level among all twists. As in (3.2) we define analogous to Corollary 2.6 for any character δ_0 of parity $\delta_0(-1)=(-1)^{m+1}$

$$I'(m, f_0, \delta_0) := \left(\frac{G(\delta_0)}{(2\pi i)^m}\right)^2 \frac{D(m+k-1, f_0, (\delta\psi)_0^{-1})}{\pi^{k-1}\langle f, f\rangle}. \tag{4.1}$$

Also we extend the definition of the factors γ_m to the whole critical strip by putting $\gamma_m := p^{(m-1)(\delta+1)+k-1}\,\psi(p)\cdot\alpha_p^{-2}$, where $\delta=0$ or 1 is as in Sect. 2, i.e. for instance $\delta=(1+\operatorname{sgn}(m-\frac{1}{2}))/2$ for critical m. The aim of this section is to show for N not necessarily even:

Theorem 4.1. *Let $m=1, \ldots, k-1$. If λ_0 is not imaginary-quadratic then there is a unique p-adic measure ν'_m on $\mathbb{Z}_p^\times$ such that for any Dirichlet character χ_0 with $\chi_0^2 \neq 1$ of conductor $c_\chi = p^{m_\chi}$ we have*

$$\int_{\mathbb{Z}_p^\times} \chi_0\, d\nu'_m = \gamma_m^{m_\chi}\cdot I'(m, f_0, \lambda_0\chi_0) \quad \text{if } \lambda_0\chi_0(-1)=(-1)^{m+1},$$

whereas the integral vanishes if $\lambda_0\chi_0(-1)=(-1)^m$. If λ_0 is imaginary-quadratic then for any topological generator $u\in\mathbb{Z}_p^\times$ there is a unique measure $\nu'_{m,u}$ such that we have either

$$\int_{\mathbb{Z}_p^\times} \chi_0\, d\nu'_{m,u} = (1-\chi_0(u)\,u^m)\cdot\gamma_m^{m_\chi}\cdot I'(m, f_0, \lambda_0\chi_0)$$

or the integral vanishes according to the previous cases.

Proof. For $\chi_0^2 \neq 1$ and $m=1, \ldots, k-1$ again by [13] the function $\mathscr{H}(z, \chi, m)$ from (3.7) becomes a generalized modular form, and (3.10) remains true for these m if we put

$$\Gamma_m := \Gamma\left(\left[\frac{m+k}{2}\right]\right) \cdot (4\pi)^{-\left[\frac{m+k}{2}\right]}.$$

Now fix a generator u of $\mathbb{Z}_p^\times$ and define an auxiliary distribution $\tilde{\nu}_{m,u}$ by demanding that for any character χ_0 of conductor p^{m_χ} we have

$$\int_{\mathbb{Z}_p^\times} \chi_0 \, d\tilde{\nu}_{m,u} = (1-\chi_0^{-2}(u)) \cdot \gamma_m^{m_\chi} \cdot I'(m, f_0, \lambda_0 \chi_0) \tag{4.2}$$

if $\rho_\chi \equiv k+m+1(2)$, and that the integral vanishes otherwise. Recall that we defined $\rho_\chi = 0$ or 1 according as $(\lambda\psi\chi)_0(-1) = (-1)^{\rho_\chi}$. As in (3.12) we consider for a given residue class $x + p^r\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$ the modular form $\mathscr{F}(z) = \mathscr{F}_{r,x}^{(m,u)}(z)$ defined by

$$\mathscr{F}(z) := p^{\frac{k}{2}-1} \sum_{\chi_0} \chi_0(x) \cdot (1-\chi_0^2(u)) \cdot c_\chi^{2m-1} \overline{\psi}(c_\chi) \overline{G(\lambda_0\chi_0)}^2 \mathscr{H}_0^*(z, \chi_0, m)|_k T(p)^{2r-1},$$

where the sum runs over all characters χ_0 of conductor c_χ dividing p^r such that $\rho_\chi \equiv k+m+1(2)$. With $\Omega_m := (2\pi i)^{-2m} \pi^{1-k} \Gamma_m^{-1}$ we can now easily reformulate (4.2) by

$$\tilde{\nu}_{m,u}(x + p^r\mathbb{Z}_p) = \left\langle f_0 \,\middle|\, W_{N_1}, \frac{\Omega_m}{\varphi(p^r)} \mathscr{F} \right\rangle_{N_1} / \langle f, f \rangle. \tag{4.3}$$

As in the previous section we have to show that the Fourier coefficients in the q-expansion of the modular forms $\Omega_m \varphi(p^r)^{-1} \mathscr{F}_{r,x}^{(m,u)}(z)$ have a universally bounded p-adic absolute value (for λ_0 not imaginary-quadratic). The Fourier coefficients $c_n = c_n^{(m)}(\chi)$ of the $\mathscr{H}_0^*(z, \chi, m)$ and the Fourier coefficients $a_n = a_{r,x}^{(m,u)}(n)$ of $\mathscr{F}$ are obviously related as follows.

Lemma 4.2. *For $m=1, \ldots, k-1$ we have*

$$a_n = p^{\frac{k}{2}-1} \sum_{\chi_0} \chi_0(x) \cdot (1-\chi_0^2(u)) \cdot c_\chi^{2m-1} \overline{\psi}(c_\chi) \cdot \overline{G(\lambda_0\chi_0)}^2 c_{np^{2r-1}},$$

where the sum runs as in the definition of $\mathscr{F}$.

Our first aim is to show that there is a constant $C>0$ such that $|\Omega_m \varphi(p^r)^{-1} \cdot a_{r,x}^{(m,u)}(n)|_p < C$ for all $r, n \geqq 0$, $m=1, \ldots, k-1$ and x prime to p, at least for λ_0 not imaginary-quadratic. Again we must evaluate the Fourier coefficients $c_{n,j}^{(m)}(\chi)$ of the generalized modular form $\mathscr{H}^*(z, \chi, m)$ from (3.9), now for $m=1, \ldots, k-1$. Let $m_0 := \left[\frac{m+k-1}{2}\right]$.

Lemma 4.3. *Suppose* $\chi_0^2 \neq 1$ *and* $\rho_\chi = \rho \equiv k+m+1\,(2)$. *Then there exist algebraic numbers* $\mathscr{C}_{m,j} \in \bar{\mathbb{Q}}$ *such that we have*

$$c_{n,j}^{(m)}(\chi) = \pi^{m_0} \cdot \mathscr{C}_{m,j} \frac{G((\lambda\psi\chi)_0)}{c_\chi^{m+1}} \sum_{n_1, n_2} (\lambda\psi\chi)_0^{-1}(n_1)$$
$$\cdot\, n_1^\rho \cdot n_2^{m_0-j-\frac{1}{2}} \cdot L_{N_1}(m, (\lambda\chi)_0\, \varepsilon_{n_2}) \cdot \beta_\chi(n_2).$$

Proof. By Proposition 1 in [11] the same type of formula (3.13) also holds for $m=1, \ldots, k-1$, where the terms at $n<0$ can be omitted since the $\tau_n(y, \ldots)$ vanish and $L_{N_1}(s, (\lambda\chi)_0\, \varepsilon_{n_2})$ never has a pole at m because $(\lambda\chi)_0\, \varepsilon_{n_2} \neq 1$ (cf. Lemma 6 in [13]). We just have to modify the $\tau_n(y)$. By the formula on p. 225 in [13] we have $\tau_0(y, \ldots) = 0$, and by Lemma 5 in [13] the right $\tau_n(y)$ for $n>0$ are given by

$$\tau_n(y) := n^{m-\frac{1}{2}}\, i^{\rho+\frac{1}{2}-k} (2\pi)^{m+\frac{1}{2}} \exp(-2\pi n y)$$
$$\cdot \sum_{j=0}^{m''} \binom{m''}{j} \cdot \Gamma\left(\frac{m+k-\rho}{2} - j\right)^{-1} \cdot (-1)^j \cdot (4\pi n y)^{m''-j},$$

where $m'' := \left[\frac{k-1-m}{2}\right]$. In these terms we have

$$E^*(z, m, \chi_0) = N_\chi^{-\frac{2m+1}{4}}\, y^{-m''} \sum_{n=1}^{\infty} \tau_n(y)\, e^{2\pi i n x} \cdot L_{N_1}(m, (\lambda\chi)_0\, \varepsilon_n) \cdot \beta_\chi(n). \tag{4.4}$$

It is now clear how to determine the coefficients $d_{n,j} = d_{n,j}^{(m)}(\chi)$ in the expansion

$$E^*(z, m, \chi_0) = \sum_{j=0}^{m''} \sum_{n=0}^{\infty} d_{n,j} \cdot (4\pi n y)^{-j} \cdot q^n.$$

We have $d_{0,j} = 0$ for all j, and for $n>0$ we find

$$d_{n,j} = \pi^{m_0}\, \mathscr{C}'_{m,j} \cdot n^{m_0-j-\frac{1}{2}}\, c_\chi^{-m-\frac{1}{2}} \cdot \beta_\chi(n) \cdot L_{N_1}(m, (\lambda\chi)_0\, \varepsilon_n), \tag{4.5}$$

where $\mathscr{C}'_{m,j}$ is an algebraic number independent of n and χ. Finally taking the product of the Fourier expansions of E^* and the theta function in (3.15) delivers the asserted formula, hence the proof of Lemma 4.3 is complete.

We now define similar as in the previous section linear forms $L_{r,x} = L_{r,x}^{(m,u)}(\underline{X})$ in the indeterminates X_χ where χ runs over all characters such that $\chi^2 \neq 1$, c_χ divides p^r and $\rho_\chi \equiv k+m+1\,(2)$. We let $L_{r,y}$ be the function given by Lemma 4.2 such that we have $a_{r,x}^{(m,u)}(n) = L_{r,x}(\underline{t})$ where $t_\chi = c_{np^{2r-1}}^{(m)}(\chi)$, and we put $a(n,j) := a_{r,x}^{(m,u)}(n,j) := L_{r,x}(\underline{t}(j))$ with $t_\chi(j) = c_{np^{2r-1},j}^{(m)}(\chi)$.

Lemma 4.4. *There is a universal constant $C'>0$ such that we have*

$$\left|\frac{\Omega_m}{\varphi(p^r)}\cdot a_{r,x}^{(m,u)}(n,0)\right|_p < C'$$

for all occurring r, x, m and n, if we assume that λ_0 is not imaginary-quadratic.

Proof. We will show the existence of some fixed $\delta\in\bar{\mathbb{Q}}^\times$ such that $\delta\cdot\Omega_m\cdot a_{r,x}^{(m,u)}(n,0)$ becomes a p-integral algebraic number divisible by p^{r-1} for all r, x and n. Collecting in $A\in\bar{\mathbb{Q}}^\times$ algebraic factors which only depend on k, m, j we get

$$\Omega_m\cdot a_{r,x}^{(m,u)}(n,0)=A\cdot\sum_{n_1,n_2} C(n_1,n_2)\sum_{\chi_0}\chi_0(x n_1^{-1})\cdot(1-\chi_0^2(u))$$
$$\cdot\,\beta_\chi(n_2)\,\overline{G(\chi_0)}\,\bar{\lambda}_0(c_\chi)\,c_\chi^{m-1}\,\pi^{-m}\cdot L_{N_1}(m,(\lambda\chi)_0\,\varepsilon_{n_2}),$$

where $C(n_1,n_2)$ are p-adic units in $\bar{\mathbb{Q}}^\times$. By the functional equation of Dirichlet L-series we replace $\pi^{-m}\cdot L_{N_1}(m,(\lambda\chi)_0\,\varepsilon_{n_2})$ by a corresponding multiple of $L(1-m,(\lambda\chi)_0^{-1}\,\varepsilon_{n_2})$, so that we have

$$\Omega_m\cdot a_{r,x}^{(m,u)}(n,0)=A'\cdot\sum_{n_1,n_2}\sum_{d\mid N} B(n_1,n_2,d)\sum_{\chi_0}\chi_0(x' d n_1^{-1})$$
$$\cdot\,\beta_\chi(n_2)(1-\chi_0^2(u))\cdot L(1-m,(\lambda\chi)_0^{-1}\,\varepsilon_{n_2}),$$

where $A'\in\bar{\mathbb{Q}}^\times$ is independent of r, x, n, $B(n_1,n_2,d)$ is a p-adic unit in $\bar{\mathbb{Q}}^\times$ and $x':=x c_\lambda c_{\varepsilon_{n_2}}$. Therefore it suffices to show for any integer $y\in\mathbb{Z}$ prime to p that we have

$$\sum_{\substack{\chi_0,\, m_\chi\le r\\ \rho_\chi\equiv m+k+1\,(2)}}(1-\chi_0^2(u))\cdot\chi_0(y)\cdot L_p(1-m,(\lambda\chi)_0^{-1}\,\varepsilon_{n_2})\equiv 0 \bmod p^{r-1}.$$

But this follows exactly as in the proof of Lemma 3.7b) from the existence of the Kubota-Leopoldt L-functions and their holomorphy properties, assuming that λ_0 is not imaginary-quadratic. Thus the proof of Lemma 4.4 is finished.

Completely similar to the proof of how Lemma 3.7 implies Proposition 3.5 we find

Lemma 4.5. *If λ_0 is not imaginary-quadratic, then the p-adic absolute values $|\Omega_m\,\varphi(p^r)^{-1}\,a_{r,x}^{(m,u)}(n)|_p$ are bounded, hence $\tilde{\nu}_{m,u}$ is a measure on $\mathbb{Z}_p^\times$.*

We now come to the end of the proof of Theorem 4.1. Let ν'_m denote any distribution satisfying the conditions in the theorem. The fact that $\tilde{\nu}_{m,u}$ is a p-adic measure simply means that for each character ϕ of Δ there is a $\delta_\phi\in\bar{\mathbb{Q}}^\times$ and a formal power series $G_\phi^{(m,u)}(T)$ in $\mathcal{O}[[T]]$ with coefficients in the ring of integers $\mathcal{O}$ of some finite extension field over $\mathbb{Q}_p$ such that we have

$$\delta_\phi\cdot\int_{\mathbb{Z}_p^\times}\chi\, d\tilde{\nu}_{m,u}=G_\phi^{(m,u)}(\chi(u_1)-1)$$

for all characters χ of $\mathbb{Z}_p^\times$ whose restriction to Δ coincides with ϕ. In case $\phi^2 \neq 1$ we find that $1-\chi^2(u)$ is a p-adic unit for all χ with $\chi|_\Delta = \phi$. Hence there exists a power series $G_\phi^{(m)}(T) \in \mathcal{O}[\![T]\!]$ such that for all these χ we have

$$\delta_\phi \cdot \int_{\mathbb{Z}_p^\times} \chi \, d\nu'_m = G_\phi^{(m)}(\chi(u_1)-1).$$

In case $\phi^2 = 1$ we know by definition of $\tilde{\nu}_{m,u}$ that $G_\phi^{(m,u)}(0)=0$. Hence we find in this case that $T^{-1} G_\phi^{(m,u)}(T)$ is in $\mathcal{O}[\![T]\!]$, and that there is a power series $G_\phi^{(m)}(T) \in \mathcal{O}[\![T]\!]$ such that for every $\chi \neq \phi$, where $\chi|_\Delta = \phi$, we have

$$\delta_\phi \cdot \int_{\mathbb{Z}_p^\times} \chi \, d\nu'_m = G_\phi^{(m)}(\chi(u_1)-1).$$

So all together the power series $G_\phi^{(m)} (\phi \in \hat{\Delta})$ define a distribution ν'_m with the required integrals, and in addition by its definition now ν'_m is a measure. This completes the proof of Theorem 4.1 if λ_0 is not imaginary-quadratic. In case λ_0 is imaginary-quadratic the proof works completely analogous if we make the appropriate modifications, i.e. if we replace the factor $1-\chi_0^2(u)$ in the definition of the form $\mathcal{F}(z)$ and in all subsequent quantities by the factor $(1-\chi_0^2(u)) \cdot (1-\chi_0^{-1}(u)\, u^m)$.

§ 5. Holomorphy and functional equation

In this section we return to the study of the distributions $\mu_m = \mu_m(\Sigma \otimes \lambda)$ $(m = 2-k, \ldots, 0)$ attached to the system of ℓ-adic representations $\Sigma \otimes \lambda$ or what comes to the same attached to the twists of the automorphic representation Π of $GL(3)$ as considered in the first section. Recall that we assume that the conductor c_λ of λ is prime to p and that by definition we have

$$\int_{\mathbb{Z}_p^\times} \chi_0 \, d\mu_m = Q_{m,\lambda} \cdot \gamma_m^{m_\chi} \, I(m, f, \lambda_0 \chi_0) \tag{5.1}$$

for all non-trivial Dirichlet characters χ_0 of conductor $c_\chi = p^{m_\chi}$. Since the L-function $\mathcal{D}(s, f, \chi_0)$ does not change when we replace f by a twisted form, we may and will assume that f has minimal level among all twists. Thus the comparison with the imprimitive L-function becomes more pleasant. Let P resp. S denote the set of primes ℓ dividing N where the local representation π_ℓ is principal resp. supercuspidal. The finite Euler product

$$E(s, \lambda_0 \chi_0) = \prod_{\ell \in P} \left(1 - (\bar{\lambda} \psi \bar{\chi})_0(\ell) \frac{\ell^{k-1}}{a_\ell^2} \ell^{-s}\right) (1 - (\overline{\lambda \chi})_0 \, \ell^{-s}) \prod_{\ell \in S} L(s, \Sigma_\ell \otimes \lambda_\ell \chi_\ell)^{-1}$$

obviously has all its zeros on the line $\mathrm{Re}(s) = 0$ and we have

$$\mathcal{D}(s, f, \lambda_0 \chi_0) = D(s+k-1, f, (\lambda \psi \chi)_0^{-1}) \, E(s, \lambda_0 \chi_0)^{-1} \tag{5.2}$$

by Lemmas 1.5 and 1.6. We are still free to vary f by twists which do not raise the level N. This won't change $\mathscr{D}(s,f,\lambda_0\chi_0)$ as remarked earlier. Depending on λ we twist π_f by the finite größencharacter $\delta=\prod_\ell \delta_\ell$ which is given by $\delta_\ell=\psi_\ell^{-1}$ if $\ell\in P$ with ψ_ℓ^2 ramified and $(\bar\lambda\psi)_\ell$ unramified, and where δ_ℓ is unramified otherwise. Let $\tilde f$ denote the (minimal) newform such that $\pi_{\tilde f}=\pi_f\otimes\delta$ with central character $\tilde\psi$. On the right hand side of (5.2) we replace f by $\tilde f$ and we modify $E(s,\lambda_0\chi_0)=E(s,f,\lambda_0\chi_0)$ analogously so that the new $E(s,\lambda_0\chi_0):=E(s,\tilde f,\lambda_0\chi_0)$ is given by

$$E(s,\lambda_0\chi_0)=\prod_{\ell\in P_\lambda}\left(1-(\bar\lambda\tilde\psi\bar\chi)_0(\ell)\,\frac{\ell^{k-1}}{\tilde a_\ell^2}\,\ell^{-s}\right)\cdot\prod_{\ell\in P\setminus P_\lambda}(1-(\overline{\lambda\chi})_0(\ell)\,\ell^{-s})\cdot\prod_{\ell\in S}L(s,\Sigma_\ell\otimes\lambda_\ell\chi_\ell)^{-1},$$

where P_λ is the set of primes in P such that ψ_ℓ^2 and $(\lambda\psi)_\ell$ are both unramified. For later application we remark that by the same argument we can moreover modify $\tilde f$ and E for each fixed $\ell\in P_\lambda$ by replacing the Euler factor at ℓ by $1-(\lambda\tilde\psi\chi)_0^{-1}(\ell)\,\tilde a_\ell^{-1}\,\ell^{k-1-s}$. We will specify the choice of this sort of modification in the proof of Proposition 5.2. Note that the sets P, $P_\lambda=P_{\lambda^{-1}}$ are the same for f and $\tilde f$. So for the rest of this article we assume that f equals $\tilde f$.

Since the level N might be odd we cannot apply Theorem 3.1 directly to the imprimitive symmetric square attached to f. So in case N is odd we twist f by the quadratic character ε of conductor $c_\varepsilon=4$ and put $h:=f_\varepsilon$. We get $D(s,f)=D_2(f,2^{-s})^{-1}\cdot D(s,h)$ where $D_2(h,X)=1$ and $D_2(f,X)$ is as in (1.16). For even N we let $h=f$. By Theorem 3.1 for each $m=2-k,\ldots,0$ there is a measure $\mu'_m=\mu'_{m,h}$, resp. $\mu'_{m,u}=\mu'_{m,u,h}$ such that for $\chi_0\neq 1$ we have

$$\int_{\mathbf{Z}_p^\times}\chi_0\,d\mu'_m=Q_{m,\lambda}\cdot\gamma_m^{m_\chi}\cdot I'(m,h,\lambda_0\chi_0) \tag{5.3}$$

for λ_0 not imaginary-quadratic, resp.

$$\int_{\mathbf{Z}_p^\times}\chi_0\,d\mu'_{m,u}=(1-\bar\chi_0(u)\,u^{1-m})\,Q_{m,\lambda}\cdot\gamma_m^{m_\chi}\cdot I'(m,h,\lambda_0\chi_0) \tag{5.3.u}$$

otherwise. Note that ψ_0 also defines the nebentypus character of h and that $\alpha_{p,f}^2=\alpha_{p,h}^2$. Furthermore the quotient $J:=\langle h,h\rangle/\langle f,f\rangle$ is algebraic by Proposition 1 in [12]. By Theorem 4.1 also for each $m=1,\ldots,k-1$ there exists a measure ν'_m or $\nu'_{m,u}$ such that for $\chi_0^2\neq 1$ we have $\int\chi_0\,d\nu'_m=0$ if $\lambda_0\chi_0(-1)=(-1)^{m+1}$ and

$$\int_{\mathbf{Z}_p^\times}\chi_0\,d\nu'_m=\gamma_m^{m_\chi}\cdot I'(m,f,\overline{\lambda_0\chi_0})\qquad\text{otherwise,} \tag{5.4}$$

or we have $\int\chi_0\,d\nu'_{m,u}=0$ if $\lambda_0\chi_0(-1)=(-1)^{m+1}$ and

$$\int_{\mathbf{Z}_p^\times}\chi_0\,d\nu'_{m,u}=(1-\bar\chi_0(u)\,u^m)\cdot\gamma_m^{m_\chi}\cdot I'(m,f,\overline{\lambda_0\chi_0})\qquad\text{otherwise,} \tag{5.4.u}$$

according as λ_0 is not imaginary-quadratic or it is. Note that the notation slightly differs from that in Theorem 4.1. The fact that these distributions μ'_{1-m}, ν'_m resp. $\mu'_{1-m,u}$, $\nu'_{m,u}$ are measures can also be expressed by the existence of power series $G^{(\phi)}_{1-m}(T)$, $H^{(\phi)}_m(T)$ resp. $G^{(\phi)}_{1-m,u}(T)$, $H^{(\phi)}_{m,u}(T)$ in $\Omega_p[\![T]\!]$ for each character ϕ of Δ, with bounded coefficients, such that for all characters χ_0 of $\mathbb{Z}_p^\times$ with $\chi_0|_\Delta=\phi$, $\chi_0^2 \neq 1$ we have

$$G^{(\phi)}_{1-m}(\chi_0(u_1)-1)=\int_{\mathbb{Z}_p^\times} \chi_0\, d\mu'_{1-m}, \quad H^{(\phi)}_m(\chi_0(u_1)-1)=\int_{\mathbb{Z}_p^\times} \chi_0\, d\nu'_m \tag{5.5}$$

if λ_0 is not imaginary-quadratic, resp.

$$G^{(\phi)}_{1-m,u}(\chi_0(u_1)-1)=\int_{\mathbb{Z}_p^\times} \chi_0\, d\mu'_{1-m,u}, \quad H^{(\phi)}_{m,u}(\chi_0(u_1)-1)=\int_{\mathbb{Z}_p^\times} \chi_0\, d\nu'_{m,u} \tag{5.5.u}$$

otherwise, where $m=1, \ldots, k-1$. In order to take care of the omitted Euler factors we define further pairs of power series $E^{(\phi)}_{1-m}(T)$, $F^{(\phi)}_m(T)$ with p-integral coefficients by demanding that we have

$$\begin{aligned} &E^{(\phi)}_{1-m}(\chi_0(u_1)-1)\\ &\quad = E(1-m, \lambda_0\chi_0)\cdot D_2(f, (\lambda\psi\chi)_0(2)^{-1}2^{m-k}) \quad \text{or} \quad E(1-m, \lambda_0\chi_0) \end{aligned} \tag{5.6}$$

according as N is odd or N is even, and

$$F^{(\phi)}_m(\chi_0(u_1)-1)=E(m, \overline{\lambda_0\chi_0}) \tag{5.7}$$

for $m=1, \ldots, k-1$ and $\chi_0|_\Delta=\phi$. Now we are going to exploit the functional equation of $L(s, \Sigma\otimes\lambda\chi)$ at $m=1, \ldots, k-1$ in order to describe the relationship between the formal power series given by (5.5) resp. (5.5.u).

Lemma 5.1. *For each $m=1, \ldots, k-1$ and for each character ϕ of Δ there is a constant $C_m \in \bar{\mathbb{Q}}^\times$ and a unit $U^{(\phi)}(T)$ in $\mathbb{Z}_p[\![T]\!]^\times$ such that we have*

$$G^{(\phi)}_{1-m}(T)\cdot F^{(\phi)}_m(T)=C_m\cdot U^{(\phi)}(T)\cdot H^{(\phi)}_m(T)\cdot E^{(\phi)}_{1-m}(T)$$

for λ_0 not imaginary-quadratic, and

$$G^{(\phi)}_{1-m,u}(T)\cdot F^{(\phi)}_m(T)=C_m\cdot U^{(\phi)}(T)\cdot H^{(\phi)}_{m,u}(T)\cdot E^{(\phi)}_{1-m}(T)$$

otherwise.

Proof. It suffices to show that both sides coincide when specialized to $T=\chi_0(u_1)-1$ for almost all χ_0 such that $\chi_0|_\Delta=\phi$. We evaluate the functional equation of Theorem 1.2 at $m=2-k, \ldots, 0$ for all characters $\lambda\chi$ with $\chi^2 \neq 1$ such that m is critical for $\Sigma\otimes\lambda\chi$. Extending the definition of $Q_{m,\lambda}$ to the whole critical strip analogous to γ_m (cf. Theorem 4.1) by

$$Q_{m,\lambda}:=\Gamma(m+k-1)\cdot(-c_\lambda)^{(m-1)(1+\delta)+1}, \tag{5.8}$$

we get by Proposition 2.7 for $m=2-k, \ldots, 0$

$$Q_{m,\lambda}\cdot\gamma_m^{m\chi}\cdot I(m, f, \lambda_0\chi_0)=C_{m,\lambda}\cdot\bar{\chi}_0(M_\lambda)\cdot\gamma_{1-m}^{m\chi}\cdot I(1-m, f, \overline{\lambda_0\chi_0}), \tag{5.9}$$

where $C_{m,\lambda} := C(\Sigma, \lambda) \cdot M_\lambda^{-m} \cdot 2\Gamma(1-m) \cdot Q_{1-m,\lambda}$ with the notations of Sect. 2. Now writing

$$\mathscr{D}(m, f, \lambda_0 \chi_0) = D(m+k-1, h, (\lambda \psi \chi)_0^{-1}) \cdot E_m^{(\phi)}(\chi_0(u_1)-1)^{-1}$$

and

$$\mathscr{D}(1-m, f, \overline{\lambda_0 \chi_0}) = D(k-m, f, (\lambda \chi \psi^{-1})_0) \cdot F_{1-m}^{(\phi)}(\chi_0(u_1)-1)^{-1}$$

whenever $E_m^{(\phi)}(\chi_0(u_1)-1) \cdot F_{1-m}^{(\phi)}(\chi_0(u_1)-1) \neq 0$ (which is true for almost all χ), we conclude by (5.3)–(5.7) and (5.9) that for λ_0 not imaginary-quadratic we have

$$\begin{aligned} &G_m^{(\phi)}(\chi_0(u_1)-1) \cdot F_{1-m}^{(\phi)}(\chi_0(u_1)-1) \\ &\quad = C_{m,\lambda} \cdot \bar{\chi}_o(M_\lambda) \cdot H_m^{(\phi)}(\chi_0(u_1)-1) \cdot E_m^{(\phi)}(\chi_0(u_1)-1), \end{aligned}$$

hence the lemma follows in this case. The proof is similar for imaginary-quadratic λ_0, so we omit it.

Call p an *exceptional prime* for $\Sigma \otimes \lambda$ if the set P'_λ of primes $\ell \in P_\lambda$ such that $0 < |1 - \psi_0^2(\ell) \ell^{2k-2} a_\ell^{-4}|_p < 1$, is not empty. The set of exceptional primes for $\Sigma \otimes \lambda$ is obviously finite.

Proposition 5.2. *For $m = 1, \ldots, k-1$ and any character ϕ of Δ the two power series $E_{1-m}^{(\phi)}$ and $F_m^{(\phi)}$ are relatively prime in $\mathbb{Q}_p[\![T]\!]$ if p is not exceptional for $\Sigma \otimes \lambda$ and f is suitably modified.*

Proof. We fix m and ϕ. The formal power series under consideration have their coefficients in the ring of integers $\mathcal{O}$ of some finite extension field of $\mathbb{Q}_p$. By the Weierstraß preparation theorem there exist polynomials $P(T)$, $Q(T) \in \mathcal{O}[T]$ with all zeros in Ω_p of negative valuation, and invertible power series $U(T)$, $V(T)$ in $\mathcal{O}[\![T]\!]^\times$, such that $E_{1-m}^{(\phi)} = P \cdot U$ and $F_m^{(\phi)} = Q \cdot V$. Therefore it suffices to show that $P(T)$ and $Q(T)$ have no common zero. By Lemma 1.6 and by the remarks following (5.2) we have

$$\begin{aligned} E(s, \lambda_0 \chi_0) = &\prod_{\ell \in S \cup P \setminus P_\lambda} (1 - \xi_\ell \bar{\chi}_0(\ell) \ell^{-s})(1 - \xi'_\ell \bar{\chi}_0(\ell) \ell^{-s}) \\ &\cdot \prod_{\ell \in P_\lambda} \left(1 - (\bar{\lambda} \psi \bar{\chi})_0(\ell) \frac{\ell^{k-1}}{a_\ell^2} \ell^{-s}\right), \end{aligned}$$

where ξ_ℓ, ξ'_ℓ is a root of unity or zero. Hence writing each prime $\ell \in P \cup S$ p-adically in the form $\ell = u_1^{e_\ell} \omega(\ell)$ with $e_\ell \in \mathbb{Z}_p$, $e_\ell \neq 0$, we have $F_m^{(\phi)}(T) = \prod_{\ell \in P \cup S} \mathscr{F}_\ell(T)$ where

$$\mathscr{F}_\ell(T) = (1 - \xi_\ell \phi(\ell) \ell^{-m} (1+T)^{e_\ell})(1 - \bar{\xi}'_\ell \phi(\ell) \ell^{-m} (1+T)^{e_\ell})$$

for $\ell \in S \cup P \setminus P_\lambda$ and

$$\mathscr{F}_\ell(T) = \left(1 - (\phi \lambda \psi)_0(\ell) \frac{\ell^{k-1-m}}{a_\ell^2} (1+T)^{e_\ell}\right) \quad \text{for } \ell \in P_\lambda.$$

So the possible zeros t of $F_m^{(\phi)}$ are of the form $t=\zeta u_1^m-1$ with $\zeta\in\mu_{p^\infty}$, coming from the first type of factors $\mathscr{F}_\ell(T)$, or t is of the form $t=\zeta\vartheta_\ell u_1^m-1$ with $\zeta\in\mu_{p^\infty}$, where $\vartheta_\ell\in\bar{\mathbb{Q}}_p$ with $\vartheta_\ell^{e_\ell}=i_p(\lambda\psi\phi\omega^{-m})_0^{-1}(\ell)\cdot a_\ell^2\cdot\ell^{1-k})\in i_p(\bar{\mathbb{Q}})$ such that $|i_p^{-1}(\vartheta_\ell^{e_\ell})|=1$. Note that a zero of the latter type can only occur if $|\vartheta_\ell^{e_\ell}-1|_p<1$. We now want to specify the modification of f (and E) with respect to the primes $\ell\in P_\lambda$ alluded to in the discussion following (5.2). If for some $\ell\in P_\lambda$ a zero t of $\mathscr{F}_\ell(T)$ occurs, not of the form $t=\zeta u_1^m-1$, then we modify f such that $\mathscr{F}_\ell(T)$ becomes

$$\mathscr{F}_\ell^*(T):=1-(\phi\lambda\bar{\psi})_0(\ell)\,\frac{\ell^{k-1-m}}{\bar{a}_\ell^2}\,(1+T)^{e_\ell}.$$

But $\mathscr{F}_\ell^*(T)$ is a unit in $\bar{\mathbb{Q}}_p[\![T]\!]$, since otherwise we get $|1-(\phi\lambda\bar{\psi})_0(\ell)\cdot\ell^{1-k-m}\cdot a_\ell^2|_p<1$, with together with $|\vartheta_\ell^{e_\ell}-1|_p<1$ implies that we have $|1-\psi_0^2(\ell)\,\ell^{2k-2}\,a_\ell^{-4}|_p<1$ in contradiction to the assumptions $P'_\lambda=\varnothing$ and $\vartheta_\ell\notin\mu_{p^\infty}$. Hence we can suppose that all zeros of $F_m^{(\phi)}$ are of the first type $t=\zeta u_1^m-1$.

On the other hand we have $E_{1-m}^{(\phi)}(T)=\prod_\ell \mathscr{E}_\ell(T)$, where the product runs over $S\cup P\cup\{2\}$, and where we put

$$\mathscr{E}_\ell(T)=(1-\xi_\ell\,\bar{\phi}(\ell)\,\ell^{m-1}(1+T)^{-e_\ell})(1-\xi'_\ell\,\bar{\phi}(\ell)\,\ell^{m-1}(1+T)^{-e_\ell})$$

for $\ell\in S\cup P\backslash P_\lambda$ and

$$\mathscr{E}_\ell(T)=1-(\bar{\phi}\lambda\psi)_0(\ell)\,\frac{\ell^{k+m-2}}{\alpha_\ell^2}\,(1+T)^{-e_\ell}\qquad\text{for }\ell\in P_\lambda,$$

where ψ and α_ℓ belong to the previously modified f. For even N we put $\mathscr{E}_2(T)=1$. If N is odd we put

$$\mathscr{E}_2(T)=D_2(f,\overline{(\lambda\psi\phi)}_0(2)\cdot 2^{m-k}\cdot(1+T)^{-e_\ell}).$$

The possible zeros of $E_{1-m}^{(\phi)}(T)$ are of the form $t=\zeta u_1^{m-1}-1$, or $t=\zeta\vartheta'_\ell u_1^{m-1}-1$, where $\vartheta'_\ell\in\bar{\mathbb{Q}}_p$ and $\vartheta_\ell'^{e_\ell}=i_p(\overline{(\phi\lambda}\psi\omega^{m-1})_0(\ell)\cdot\bar{\alpha}_\ell^2\cdot\ell^{1-k})$ in $i_p(\bar{\mathbb{Q}})$ with the possibility for $\ell=2$ to replace α_2 by β_2 in case N is odd. Here again $\zeta\in\mu_{p^\infty}$.

A zero $t=\zeta u_1^m-1$ of $F_m^{(\phi)}$ is obviously never a zero of $E_{1-m}^{(\phi)}$ of first type $t'=\zeta' u_1^{m-1}-1$. If t were a zero of the second type $t=\zeta'\vartheta'_\ell u_1^{m-1}-1$ then we get $u_1=\zeta\vartheta'_\ell$ for some ζ and therefore $\ell\cdot\omega(\ell)^{-1}=u_1^{e_\ell}=\zeta^{e_\ell}\vartheta_\ell'^{e_\ell}$ in $i_p(\bar{\mathbb{Q}})$, whence a contradiction comparing archimedean absolute values of inverse images in $\bar{\mathbb{Q}}\subseteq\mathbb{C}$. This completes the proof of Proposition 5.2.

Remark 5.2.1. Since P'_λ is a subset of

$$P'(\Sigma,p):=\{\ell\in P;\,0<|1-\psi_0^2(\ell)\cdot\ell^{2k-2}\cdot a_\ell^{-4}|_p<1\},$$

which is empty for all but finitely many p (for fixed Σ), we see that up to finitely many p: for all twists $\Sigma\otimes\lambda\,p$ is not exceptional.

Theorem 5.3. *Suppose p is not exceptional for $\Sigma\otimes\lambda$. If λ_0 is not imaginary-quadratic, then there is a unique measure $\mu=\mu(\Sigma\otimes\lambda)$ on $\mathbb{Z}_p^\times$ such that for almost all finite characters χ_0 of $\mathbb{Z}_p^\times$ and for $m=2-k, \ldots, 0$ we have*

$$\int_{\mathbb{Z}_p^\times} \chi_0(x)\, x^m \, d\mu(x) = Q_{m,\lambda}\cdot \gamma_m^{m\chi}\cdot I(m, f, \lambda_0\chi_0). \tag{5.10}$$

If λ_0 is imaginary-quadratic then for any topological generator $u\in\mathbb{Z}_p^\times$ there is a unique measure $\mu_{(u)}=\mu(\Sigma\otimes\lambda, u)$ such that for almost all finite characters χ_0 of $\mathbb{Z}_p^\times$ and for $m=2-k, \ldots, 0$ we have

$$\int_{\mathbb{Z}_p^\times} \chi_0(x)\, x^m \, d\mu_{(u)}(x) = (1-\bar{\chi}_0(u)\, u^{1-m})\, Q_{m,\lambda}\cdot \gamma_m^{m\chi}\cdot I(m, f, \lambda_0\chi_0). \tag{5.10.u}$$

Proof. By Lemmas 5.1 and 5.2 each quotient $\mathscr{G}_m^{(\phi)} := G_m^{(\phi)}/E_m^{(\phi)}$ is a power series in $\Omega_p[\![T]\!]$ with bounded coefficients. In particular for $m=0$ we can define a measure μ by putting

$$\int_{\mathbb{Z}_p^\times} \chi_0(x)\, d\mu(x) := \mathscr{G}_0^{(\phi)}(\chi_0(u_1)-1) \tag{5.11}$$

for all characters χ_0 of $\mathbb{Z}_p^\times$ with $\chi_0|_\Delta=\phi$. Obviously this measure satisfies (5.10) for $m=0$ and for almost all finite χ_0, and it remains to show the identity (5.10) for $m<0$. Now (5.11) implies that we have

$$\int_{\mathbb{Z}_p^\times} \chi_0(x)\, x^m\, d\mu(x) = \mathscr{G}_0^{(\phi\omega^m)}(\chi_0(u_1)\cdot u_1^m - 1),$$

and by Theorem 3.8 we know

$$G_m^{(\phi)}(\chi_0(u_1)-1) = G_0^{(\phi\omega^m)}(\chi_0(u_1)\cdot u_1^m - 1).$$

Furthermore one easily verifies the analogous relationship for $E_m^{(\phi)}$, namely

$$E_m^{(\phi)}(\chi_0(u_1)-1) = E_0^{(\phi\omega^m)}(\chi_0(u_1)\cdot u_1^m - 1),$$

so that we eventually find

$$\mathscr{G}_m^{(\phi)}(\chi_0(u_1)-1) = \mathscr{G}_0^{(\phi\omega^m)}(\chi_0(u_1)\, u_1^m - 1) = \int_{\mathbb{Z}_p^\times} \chi_0(x)\, x^m\, d\mu(x),$$

which proves the theorem for λ_0 not imaginary-quadratic. In case λ_0 is imaginary-quadratic, the same proof works for $d\mu_{(u)}$ with the obvious modifications. This finishes the proof of Theorem 5.3.

Corollary 5.4. *If λ_0 is not imaginary-quadratic, then there is a unique measure $\nu=\nu(\Sigma\otimes\lambda)$ on $\mathbb{Z}_p^\times$ such that for almost all finite characters χ_0 of $\mathbb{Z}_p^\times$ and for $m=1, \ldots, k-1$ we have*

$$\int_{\mathbb{Z}_p^\times} \bar{\chi}_0(x)\, x^m\, d\nu(x) = 2\cdot\Gamma(m)\cdot Q_{m,\lambda}\cdot \gamma_m^{m\chi}\cdot I(m, f, \overline{\lambda_0\chi_0}), \tag{5.12}$$

if $\lambda_0\chi_0(-1)=(-1)^{m+1}$, *and the integral vanishes otherwise. If* λ_0 *is imaginary-quadratic then for any topological generator* $u\in\mathbb{Z}_p^\times$ *there is a unique measure* $\nu_{(u)}$ *such that we have either*

$$\int_{\mathbb{Z}_p^\times} \bar{\chi}_0(x)\,x^m\,d\nu_{(u)}(x)=(1-\bar{\chi}_0(u)\,u^m)\cdot 2\cdot\Gamma(m)\,Q_{m,\lambda}\cdot\gamma_m^{m\varkappa}\cdot I(m,f,\overline{\lambda_0\chi_0}) \tag{5.12.u}$$

or the integral vanishes according to the previous cases.

Proof. By (5.9) the measure μ in Theorem 5.3 can also be described by the identity

$$\int_{\mathbb{Z}_p^\times} \chi_0(x)\,x^{1-m}\,d\mu(x)=C(\Sigma,\lambda)\cdot M_\lambda^{m-1}\,\bar{\chi}_0(M_\lambda)\cdot 2\cdot\Gamma(m)\,Q_{m,\lambda}\,\gamma_m^{m\varkappa}\,I(m,f,\overline{\lambda_0\chi_0})$$

for $m=1,\ldots,k-1$ and for almost all χ such that $\lambda_0\chi_0(-1)=(-1)^{m-1}$. This tells us at once that the measure ν that we are looking for, is given by

$$d\nu(x)=C(\Sigma,\lambda)^{-1}\cdot x^{-1}\,d\mu(x^{-1}\,M_\lambda^{-1}), \tag{5.13}$$

thus proving the corollary for non imaginary-quadratic λ_0. Again the proof works analogous for imaginary-quadratic λ_0, so the proof is complete.

Finally we want to reformulate our results in a way which exhibits the p-adic interpolation process. Let $\mathcal{T}$ denote the set of continuous homomorphisms $\theta\colon\mathbb{Z}_p^\times\to\Omega_p^\times$ and define the "*p-adic contragredient*" $\check{\theta}:=\theta^{-1}\cdot\iota$, where $\iota\in\mathcal{T}$ is the canonical embedding.

Theorem 5.5. *Suppose p is not exceptional for* $\Sigma\otimes\lambda$ *and that* λ_0 *is not imaginary-quadratic.*

a) *There exists a unique function* $\mathfrak{C}_\lambda\colon\mathcal{T}\to\Omega_p$ *with the following properties:*

(i) *For each character* ϕ *of* Δ *there is a power series* $g(\phi,T)$ *in* $\Omega_p[\![T]\!]$ *with bounded coefficients such that for any* $\theta\in\mathcal{T}$ *with* $\theta|_\Delta=\phi$ *we have* $\mathfrak{C}_\lambda(\theta)=g(\phi,\theta(u_1)-1)$.

(ii) *For all but finitely many* θ *of the form* $\theta=\chi_0\,\iota^m$ *with* $\chi_0\in\mathcal{T}_{\text{torsion}}$ *such that m is critical for* $\Sigma\otimes\lambda\chi$, *we have*

$$\mathfrak{C}_\lambda(\theta)=(2\Gamma(m))^\kappa\cdot Q_{m,\lambda}\cdot\gamma_m^{m\varkappa}\cdot I(m,f,\lambda_0\chi_0),$$

where $\kappa=0$ *or* 1 *according as* $\lambda_0\chi_0(-1)=(-1)^\kappa$.

b) *The function* $\mathfrak{C}_\lambda$ *satisfies the functional equation*

$$\mathfrak{C}_\lambda(\theta)=C(\Sigma,\lambda)\cdot\theta^{-1}(M_\lambda)\cdot\mathfrak{C}_{\bar{\lambda}}(\check{\theta}).$$

For the proof define $\mathfrak{C}_\lambda(\theta)$ by the integral of θ against the appropriate measure. Then a) follows from Theorem 5.3 and Corollary 5.4 applied to $\mu(\Sigma\otimes\lambda)$, $\mu(\Sigma\otimes\lambda^{-1})$ and $\nu(\Sigma\otimes\lambda)$, $\nu(\Sigma\otimes\lambda^{-1})$, and the functional equation b) plainly follows from (5.13).

Remarks 5.6. a) The theorem is also true for imaginary-quadratic λ_0 if we make the appropriate modifications as previously.

b) Note that the assumption "p not exceptional" is automatically fulfilled if λ is trivial or if for instance λ only ramifies outside N. This covers in particular the case of the symmetric square of a modular elliptic curve (see Conjecture 3.13(ii) in [2] and Remark 4.7b in [9]).

c) In general the symmetric square by (1.5) is of the form $\mathrm{Sym}^2(\sigma) \cong \Sigma \otimes \psi^{-1}$. So avoiding the finitely many exceptional primes p our theorems deliver in particular measures $\mu(\mathrm{Sym}^2(\sigma))$ and $\nu(\mathrm{Sym}^2(\sigma))$ which are related by the functional equation (5.13).

References

1. Carayol, H.: Courbes de Shimura, formes automorphes et représentations galoisiennes. Thèse, Paris 1984
2. Coates, J., Schmidt, C.-G.: Iwasawa theory for the symmetric square of an elliptic curve. J. Reine Angew. Math. **375/376**, 104–156 (1987)
3. Deligne, P.: Les constantes des équations fonctionelles des fonctions *L*. (Lect. Notes Math., Vol. 349, pp. 501–595). Berlin – Heidelberg-New York: Springer 1973
4. Deligne, P.: Valeurs de fonctions *L* et périodes d'intégrales. Proc. Symp. Pure Math. **33**, (Part 2) 313–346 (1979)
5. Gelbart, S., Jacquet, H.: A relation between automorphic representations of *GL*(2) and *GL*(3). Ann. Sci. Ec. Norm. Super., IV. Ser. **11**, 471–542 (1978)
6. Henniart, G.: La conjecture de Langlands locale pour *GL*(3). Mém. Soc. Math. Fr. Nouv. Ser. **11–12**, 186 (1984)
7. Hida, H.: A *p*-adic measure attached to the zeta functions associated with two elliptic modular forms. Invent. Math. **79**, 159–185 (1985)
8. Jacquet, H., Langlands, R.P.: Automorphic forms on *GL*(2). (Lect. Notes Math., Vol. 114). Berlin-Heidelberg-New York: Springer 1970
9. Schmidt, C.-G.: The *p*-adic *L*-functions attached to Rankin convolutions of modular forms. J. Reine Angew. Math. **368**, 201–220 (1986)
10. Shimura, G.: On modular forms of half integral weight. Ann. Math. **97**, 440–481 (1973)
11. Shimura, G.: On the holomorphy of certain Dirichlet series. Proc. London Math. Soc. **31**, 79–98 (1975)
12. Shimura, G.: The special values of the zeta functions associated with cusp forms. Commun. Pure Appl. Math. **29**, 783–804 (1976)
13. Sturm, J.: Special values of zeta functions and Eisenstein series of half integral weight. Am. J. Math. **102**, 219–240 (1980)
14. Sturm, J.: Evaluation of the symmetric square at the near center point. Preprint 1987
15. Tate, J.: Number theoretic background. Proc. Symp. Pure Math. **33**, (Part 2) 3–26 (1979)
16. Zagier, D.: Modular forms whose Fourier coefficients involve zeta functions of quadratic fields. (Lect. Notes Math., Vol. 627, pp. 105–169). Berlin-Heidelberg-New York: Springer 1977

Oblatum 8-IV-1987 & 18-XII-1987