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Titel: Scattering frequencies and Gevrey 3 singularities.

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Jahr: 1987

PURL: https://resolver.sub.uni-goettingen.de/purl?356556735_0090|log9

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Invent. math. 90, 77–114 (1987)

Inventiones mathematicae
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Scattering frequencies and Gevrey 3 singularities

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1. Introduction

In this paper we study the scattering frequencies μ_k. For the Laplacian Δ, in the region $\Omega=\mathbb{R}^n\backslash\mathcal{O}$ exterior to a compact analytic obstacle $\mathcal{O}$. We suppose that n is odd and that Dirichlet boundary conditions are imposed on $\partial\Omega$. The set $\{\mu_k\}$, discrete in $\mathrm{Re}\,\mu<0$, arises in a variety of different disguises (see [L.P. 2]).

(1) They are the poles of the analytic continuation of the resolvent $(z^2-\Delta)^{-1}$ which is analytic for $\mathrm{Re}\,z>0$.

(2) They are the values μ such that the boundary value problem

$$(\mu^2-\Delta)u=0 \quad \text{in} \quad \Omega, \qquad u=0 \quad \text{in} \quad \partial\Omega,$$

where u satisfies the Sommerfeld radiation condition, has a non zero solution.

(3) Via an expansion $\sum e^{\mu}k^t u_k$, they describe the asymptotic behavior as $t\to\infty$ of solutions to

$$\partial_t^2 u-\Delta u=0 \quad \text{in} \quad \mathbb{R}_t\times\Omega, \qquad u=0 \quad \text{in} \quad \mathbb{R}_t\times\partial\Omega, \tag{1.1}$$

$$u(0,\cdot),\ u_t(0,\cdot) \text{ of compact support.} \tag{1.2}$$

(4) They are the poles of the analytic continuation of the scattering matrix.

(5) They form the spectrum of the infinitesimal generator B of the Lax-Phillips semigroup $Z(t)$.

Because of (3) and (4) a particularly important role is played by those μ with small real part, and this paper addresses the question of how fast $\mathrm{Re}\,\mu$ must grow as $|\mu|\to\infty$.

Lax and Phillips recognized that the response is related to the propagation of singularities for (1.1). Their reasoning together with the resolution of the problem of propagation of C^∞ singularities by Taylor, Melrose and Sjöstrand showed that for non trapping obstacles there is a constant $C>0$ such that:

$$\mathrm{Re}\,\mu_k \leqq -C\,\mathrm{Log}|\mu_k| \ \forall k. \tag{1.3}$$

The key to this is the fact that solutions to (1.1), (1.2) are, for large t, C^∞ on compact sets. Lebeau [L.2] has recently shown that for analytic, non trapping obstacles this can be improved to Gevrey 3. As in the earlier work of Goodhue [G], this allows us to improve the bound to

$$\mathrm{Re}\,\mu_k \leqq -C|\mu_k|^{1/3} \ \forall k. \tag{1.4}$$

For the case $\mathcal{O} = (|x| \leqq R)$, it can be shown by explicit calculation that there is an infinite number of k with $\mathrm{Re}\,\mu_k \geqq -C|\mu_k|^{1/3}$ so, in that case, the bound (1.4) is optimal. Our main result is that this optimality is valid for most analytic strictly convex obstacle. In broadest terms, our idea is to associate resonances with a closed non degenerate geodesic on $\partial\Omega$. Such a geodesic is an "analytic ray" in the sense of Sjöstrand and is therefore the carrier of a function of class exactly Gevrey 3 which remains trapped. To take advantage of this, we follow [L.P.2] and [B.G.R.] (and also [D.G.] and [C]) in considering:

$$\mathrm{Tr}(t) = \mathrm{Tr}\,Z(t) = \sum e^{\mu_k t}, \tag{1.5}$$

where the μ_k are repeated according to their algebraic multiplicity. When $\mathcal{O}$ is analytic and non trapping Tr is Gevrey 3 on $(0, \infty)$ and for strictly convex $\mathcal{O}$ we show that Tr is analytic except at the numbers $T \in \mathcal{L}$, the union of the lengths of closed geodesics in $\partial\Omega$. If T is isolated in L and is the length of exactly one geodesic which is, in addition, non degenerate, we obtain a lower bound

$$|\partial_t^n T(t)| \geqq AC^n(3n)! \qquad A, C > 0, \tag{1.6}$$

and it follows that with a constant $C' > 0$

$$\#\{\mu_k : \mathrm{Re}\,\mu_k \geqq -C'|\mu_k|^{1/3}\} = \infty \tag{1.7}$$

showing that the bound (1.3) is sharp. In fact our results are more precise than (1.6), (1.7). And finally it is reasonable to expect that the convexity can be replaced by convexity near the closed geodesic.

The paper is organized as follows. In the first section, we state precisely our three main theorems corresponding to the bound (1.3), the regularity assertions about $\mathrm{Tr}(t)$ which we call the Poisson relation, and the assertion (1.7). We prove the first and then show how the last follows from the Poisson relation. We then show how the study of Tr can be reduced to the analysis of the restriction to $\mathbb{R}_t \times \partial\Omega$ of elementary solutions in $\mathbb{R}_t \times \mathbb{R}^n$ and $\mathbb{R}_t \times \mathcal{O}$. The crux of our proof is a calculation of the constant C which appears in the Gevrey 3 regularity of Tr. This constant can be viewed as a measure of regularity within the class Gevrey 3 and it is related to the Gevrey 3 regularity of solutions to (1.1). In Sect. II, we recall the Fourier Bros Iagolnizer characterizations of microlocal regularity. The notions of Gevrey 3 Lagrangian distributions and finally, Lagrangian distributions Gevrey 3 with weight ψ. The weight ψ is a microlocal version of C. Section 3 (resp. 4) are devoted to a precise analysis of the elementary solution in $\mathbb{R}_t \times \mathbb{R}^n$ (resp. $\mathbb{R}_t \times \Omega$) as Lagrangian distributions. This done, the Poisson relation follows. The appendices contain two geometric results. The first computes a Maslov index and the second shows that, generically, obstacles do have closed geodesics with the properties we

require. The results contained in this paper were announced in [B.L.R.]. By a different method (estimation of Greens functions) G. Popov [P.] has obtained upper bounds like (1.4).

2. Main theorems

Suppose $n \geqq 3$ is odd and that $\mathcal{O} \subset \mathbb{R}^n$ is a compact set and $\Omega = \mathbb{R}^n \backslash \mathcal{O}$. We assume that Ω is an analytic connected manifold with boundary $\partial\Omega$. Let u be the solution of the wave equation with Dirichlet boundary conditions (1.1) and initial data

$$u(0, \cdot) = u_0(\cdot), \quad \partial_t u(0, \cdot) = u_1(\cdot). \tag{2.1}$$

We denote by $\underline{u}(t) \equiv (u(t, \cdot), \partial_t u(t, \cdot))$ the Cauchy data at time t and by $H(\Omega)$ the completion of $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ in norm

$$\|(u_0, u_1)\|^2 \equiv |(u_0, u_1)|_E^2 = \int_\Omega |D_x u_0|^2 + |u_t|^2 \, dx .$$

Then, $H(\Omega)$ is a Hilbert space and the map $\underline{u}(t) \equiv U(t)\underline{u}(0)$ defines a unitary group with generator.

$$A = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}.$$

With ϱ chosen so that $\mathcal{O} \subset \{|x| < \varrho\}$ we introduce the spaces $D_\pm^{\varrho\perp}$ Lax-Phillips [L.P.1]:

$$D_\pm^\varrho = \{\underline{u} \in H : U(t)\underline{u} = 0 \quad \text{if} \quad |x| \leqq \pm t + \varrho \quad \text{and} \quad \pm t > 0\}.$$

Then $D_\pm^\varrho$ are orthogonal, and if $P_\pm^\varrho$ are the orthogonal projections on $D_\pm^{\varrho\perp}$ then:

$$Z(t) \equiv P_+^\varrho U(t) P_-^\varrho$$

is a contraction semigroup on $K \equiv (D_+^\varrho \oplus D_-^\varrho)^\perp$. Z describes the behavior of U near $\mathcal{O}$ in the sense that if $f \in H$, $\operatorname{supp} f \subset \{|x| \leqq \varrho\}$ then for all $t > 0$

$$Z(t) f = U(t) f \quad \text{on} \quad \{|x| \leqq \varrho\}.$$

The infinitesimal generator, B, of $Z(t)$ has discrete spectrum in $\operatorname{Re}\mu < 0$ consisting of eigenvalues of finite multiplicity. As discussed in the introduction (1.3) holds for non trapping obstacles (see [M.1] for definition).

Theorem 1. *If the analytic manifold, $\partial\Omega$, is non trapping, then there is a constant $C > 0$ such that*

$$\sigma(B) \subset \{\mu \in \mathbb{C} | \operatorname{Re}\mu \leqq -c|\mu|^{1/3}\}. \tag{2.2}$$

Our main result is that this bound is optimal for convex analytic manifolds satisfying an additional geometric hypothesis which is generically satisfied (see Appendix 2).

In the Theorem 2, as in many problems of diffraction, the Airy function, Ai, intervenes. The Fourier transform of $\mathrm{Ai}(x)$ is $\exp(i\xi^3/3)$ and a brief discussion of its properties can be found on pages 203–205 of [H.2].

Theorem 2. *Suppose that the Obstacle $\mathcal{O}$ is convex with analytic boundary of strictly positive curvature. Suppose that γ is a geodesic or length T on $\partial\mathcal{O}$ and satisfies*

(1) *There is a neigborhood of T in $\mathbb{R}$ such that γ is the only closed geodesic on $\partial\mathcal{O}$ whose length is in that neigbourhood.*

(2) *γ is non degenerate in the sense that the first return map of the geodesic flow at time T has 1 as a simple eigenvalue.*

Then for any $\varepsilon>0$, the set

$$\{\mu\in\sigma(B)|\mathrm{Re}\,\mu\geqq-(C_\gamma+\varepsilon)|\mu|^{1/3}\} \tag{2.3}$$

is infinite, where

$$C_\gamma\equiv\omega\cos(\pi/6)(2^{1/3}T)^{-1}\int_0^T\varrho^{2/3}(s)ds\,, \tag{2.4}$$

$-\omega$ is the zero of smallest modulus of the Airy function, ϱ is the curvature in $\mathbb{R}^n$ of γ, and s is the arclength parameter.

Theorem 2 is a direct consequence of Melrose's [M.2] polynomial bound for the resonances and the following Poisson type relation in the Gevrey category.

Theorem 3 (Poisson Relation). *Suppose that $\partial\Omega$ is analytic and that $\mathcal{O}$ is strictly convex. Then:*

(i) *The distribution $\mathrm{Tr}(\cdot)$ defined in (1.5) lies in $G^3(]0,\infty])$.*

(ii) *If L is the set of lengths of closed geodesics on $\partial\Omega$, then Tr is analytic on $]0,\infty]\backslash L$.*

(iii) *If $T\in\mathcal{L}$ and T is the length of exactly one closed geodesic γ which in addition, is nondegenerate, then for any $\varepsilon>0$, one has:*

$$\limsup_{n\to\infty}\left[\sup_{|t-T|<\varepsilon}|((3n)!)^{-1}\partial_t^n\mathrm{Tr}(t)|^{1/3n}\right]\geqq(C_\gamma T)^{-1}\,. \tag{2.5}$$

with C_γ defined in (2.4).

In the remainder of this section, we derive Theorem 1 from Lebeau's G^3 propagation theorem and we deduce Theorem 2 from Theorem 3 and Melrose's polynomial bound. We then derive a representation for $\mathrm{Tr}(\cdot)$. The remaining sections are devoted to the analysis of the terms entering in this representation, culminating in a proof of Theorem 3.

Definition 2.1. *For $\Omega\subset\mathbb{R}^n$ open, $s\in[1,\infty[$ and $f\in C^\infty(\Omega)$, we say that f is of class Gevrey s, $f\in G^s(\Omega)$ symbolically, if for any compact $K\subset\Omega$, there are two constants A_K, B_K such that*

$$(\forall\alpha\in\mathbb{N}^n)\sup_K|\partial_x^\alpha f|\leqq A_K(B_K)^{|\alpha|}(|\alpha|!)^s\,.$$

The notion of G^5 on an open subset of a C^ω manifold or C^ω manifold with boundary is similar. In particular, $G^1(\Omega)$ is the space of real analytic functions on Ω.

For a distribution f, the set of points p with the property that f is Gevrey s on a neighborhood of p is open and its complement is called the Gevrey s singular support.

Proof of Theorem 1. For any $R>\varrho$, let $\Omega_R\equiv\Omega\cap\{|x|\leqq R\}$ and consider Cauchy data $\underline{u}(0)\in H(\Omega)$ with $\operatorname{supp}\underline{u}\in\Omega_R$. If $\mathcal{O}$ is nontrapping, then we know that there is a $t_1>0$ so that any "C^∞-ray" beginning in Ω_R lies in $\Omega\cap\{|x|>R\}$ for $t>t_1$. By [L.3] we know that the Gevrey 3 singular support is a union of C^∞ rays. Therefore the solution $\underline{u}(x,t)$ with Cauchy data $\underline{u}(0)$ is Gevrey 3 on $[t\geqq t_1]\cap\Omega_R$. Following a proof of Lax [L, Thm. 2.2] we show that this implies that there are constants A, C depending only in Ω, t_1, t_2, R so that

$$\forall 1\geqq 0\quad \sup_{[t_1,t_2]\times\Omega_R}[|\partial_t^1 u|+|\partial_t^1\nabla u|]\leqq AC^1(31)!\,\|u(0)\|_E\,. \tag{2.6}$$

Toward this end let $\Gamma=[t_1,t_2]\times\Omega_R$ and, for $N\in\mathbb{N}$

$$G_N^3=\left\{u\in C^\infty(\Gamma)\colon \sup_{\Gamma,\alpha}|D^\alpha u|/(\alpha!)^3N^{|\alpha|}\right\}<\infty\,.$$

Then $G_N^3(\Gamma)$ is a Banach space and $G^3(\Gamma)=U_NG_N^3(\Gamma)$. Let $A_N\subset H$ be the set of initial data with the property that the solution with data in A_N lies in $G_N^3(\Gamma)$. Then A_N is a linear subspace of H and $U_NA_N=H$. The Baire Category theorem implies that for some N_0, A_{N_0} is of the second category. The map $H\supset A_{N_0}\ni\underline{u}\to u|_\Gamma\in G_{N_0}^3(\Gamma)$ is defined on a subspace of second category and has closed graph, hence is continuous, proving (2.6) with $C=N_0$.

Choose $\chi\in C^\infty$ with $\chi=0$ on $[2,\infty]$ and $\chi=1$ on $[-\infty,1]$. Q_r is then the truncation operator defined by $Q_rf(x)=\chi(|x|/r)f(x)$. Following Lax and Phillips: [L.P.1], let $M\equiv U(2\varrho)-U_0(2\varrho)$ where U_0 is the unitary group defined by the wave equation on $\mathbb{R}^n$ and $U(2\varrho)$ is extended to $H(\mathbb{R}^n)$ as a linear map vanishing on $H(\Omega)^\perp$. Then formula (3.8) p. 155 of the above book asserts that for $t\geqq 4\varrho$

$$Z(t)=P_+^\varrho MQ_{5\varrho}U(t-4\varrho)Q_{3\varrho}MP_-^\varrho\,.$$

Then for $t_0\geqq 4\varrho$, and any $l\in\mathbb{Z}_+$,

$$B^lZ(t_0)=\partial_t^lZ(t_0)=P_+^\varrho MQ_{5\varrho}\partial_t^lU(t_0-4\varrho)Q_{3\varrho}P_-^\varrho\,. \tag{2.7}$$

Choosing $R=10\varrho$ and $t_0>t_1$, we see that (2.6) and (2.7) show that $B^lZ(t_0)$ is bounded from $H(\Omega)$ to itself with

$$\|B^lZ(t_0)\|\leqq A'(C')^l(3l)!\;\forall l\,.$$

For $\mu\in\sigma(B)$, $\mu^le^{\mu t_0}\in\sigma(B^lZ(t_0)$ and is therefore bounded by the norm of that operator:

$$|\mu|^l\,e^{t_0\operatorname{Re}\mu}\leqq A'(C')^l\cdot(3l)!\,.$$

Given μ, choose l so that $|3l-(|\mu|/C')^{1/3}|<1$ and use Stirling's formula to prove (2.2).

Proof of Theorem 2 assuming Theorem 3. We show that if $\sigma(B)\cap\{\operatorname{Re}\mu\geqq -C|\mu|^{1/3}\}$ is finite, then $C\leqq C_\gamma$. To accomplish this, we use the supposed finiteness and

Melrose's polynomial bound [M.2]

$$\exists_{N,M} \text{ s.t. } \forall R \, \#\{\mu_k : |\mu_k| \leqq R\} \leqq M(R+1)^N \tag{2.8}$$

to derive upper bounds for $\partial_t^n \mathrm{Tr}(t)$ which are then compared with (2.5). In the following sums, we sum such $\mu \in \sigma(B)$ repeated according to multiplicity. Let $M \subset \sigma(B)$ be the finite set of M with $|\mu| \leqq 1$ or $\mathrm{Re}\,\mu \geqq -C|\mu|^{1/3}$. Write

$$\mathrm{Tr}(t) = \sum_{\mu \in \mathbf{M}} e^{\mu t} + \sum_{l=1}^{\infty} \left(\sum_{l \leqq |\mu| < l+1} e^{\mu t} \right) \equiv \sum_{l=0}^{\infty} f_l(t).$$

Then $f_0 \equiv \sum_{\mu \in \mathbf{M}} e^{\mu t}$ is entire analytic and

$$|\partial_t^n f_l| \leqq M(l+1)^{n+N} e^{-Cl^{1/3}t}.$$

Thus, comparing the sum to an integral of a function with one maximum on $(0, \infty)$ yields

$$|\partial_t^n(\mathrm{Tr}(t) - f_0(t))| \leqq \sum_{l=1}^{\infty} M(l+1)^{n+N} e^{-Cl^{1/3}t} \leqq M e^{ct} \int_{]0,\infty[} x^{n+N} e^{-Ctx^{1/3}} dx$$
$$+ e^{ct} M \max_{[0,\infty]} (x^{n+N} e^{-Ctx^{1/3}}) \leqq \text{const.}(Ct)^{-3n-3N-3}(3n+3N+2)!\, e^{ct}.$$

Thus, $\mathrm{Tr} \in G^3(0, \infty)$ and $\limsup_{n \to \infty} |(3n!)^{-1} \partial_t^n \mathrm{Tr}(t)|^{1/3n} \leqq (Ct)^{-1}$. Taking $t = T$ and comparing with the conclusion (2.5) of Theorem 3, we see that we must have $C \leqq C_\gamma$. □

In the remainder of this section, we will derive a representation formula for $\mathrm{Tr}Z(t)$. We begin with a formula of Melrose [M.2]

$$\mathrm{Tr}Z(t) = 2\partial_t \int_{\Omega} E(t, x, x) dx, \quad t > 0,$$

where E is the fundamental solution of $\Box$ on $\mathbb{R}_t \times \Omega$, that is for $y \in \Omega$

$$(\partial_t^2 - \Delta_x) E(t, x, y) = 0 \qquad E(t, x, y)|_{(t,x) \in \mathbb{R}_t \times \partial\Omega} = 0,$$
$$E(0, x, y) = 0 \qquad E_t(0, x, y) = \delta(x - y).$$

There are similar kernels $K^{\pm}(t, x, y)$, $(t, x, y) \in \mathbb{R} \times \Omega \times \partial\Omega$ associated with inhomogeneous Dirichlet conditions

$$(\partial_t^2 - \Delta_x) K^{\pm}(t, x, y) = 0 \qquad \mathrm{Supp}\, K^{\pm} \subset \{\pm t \geqq 0\} \qquad K^{\pm}(t, x, y)|_{(t,x) \in \mathbb{R}_t \times \partial\Omega} = \delta_{t=0, x=y}.$$

Then

$$F(t, x) = \iint_{\mathbb{R} \times \partial\Omega} K^{+}(t - s, x, y) f(s, y) ds dy$$

is the solution of $\Box F = 0$, equal to f on $\mathbb{R}_t \times \partial\Omega$ and vanishing for $t \leqq \min\{s : (\exists x)(s, x) \in \mathrm{supp}\, f\}$.

Define $k^{\pm}(t, x, y) \in \mathscr{D}'(\mathbb{R} \times \partial\Omega \times \partial\Omega)$ by

$$K^{\pm} = \partial_{n_x} K^{+}|_{\mathbb{R}_t \times \partial\Omega \times \partial\Omega}. \tag{2.9}$$

Then for $t, x \in \mathbb{R} \times \partial\Omega$,

$$\partial_{n_x} F(t,x) = \iint k^+(t-s,x,y) f(x,y) ds dy$$

so that k^+ is the kernel of the Neumann operator. Similar formulas hold for k^- and for $f_\pm \in \mathscr{E}(\mathbb{R} \times \partial\Omega)$, we can consider $F_\pm$ defined using $k^\pm$. Green's identity applied to these functions yields

$$\iint_{\partial\Omega \times \mathbb{R}} (F_+ \partial_{n_x} F_- - \partial_{n_x} F_+ F_-) dt dx = 0.$$

As this holds for arbitrary boundary data $f_\pm$, we have the symmetry relation

$$k^+(t,x,y) = k^-(-t,y,x). \tag{2.10}$$

Finally we denote by $e_0^+ \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$ the forward fundamental solution for $\Box$, $\Box e_0^+ = \delta$, $\operatorname{supp} e_0^+ \subset \{t \geqq 0\}$.

Restricting $e_0^+(t, x-y)$ to $\mathbb{R} \times \Omega \times \Omega$ yields E apart from a correction for the inhomogeneous Dirichlet data $e_0^+(t,x-y)|_{(t,x) \in \mathbb{R} \times \partial\Omega} \in \mathscr{E}(\mathbb{R}_t \times \partial\Omega)$:

$$E(t,x,y) = e_0^+(t,x-y) - E(t,x,y),$$

$$E(t,x,y) \equiv \iint_{\mathbb{R} \times \partial\Omega} K^+(t-s,x,z) e_0^+(s,z-y) ds dz.$$

From Huyghen's principle, we see that $e_0^+(t,x-y) = 0$ on a neighborhood of $x=y$ when $t>0$, so for $t>0$

$$-\int_\Omega E(t,x,x) dx = \int_\Omega \tilde{E}(t,x,x) dx, \qquad = \int_\Omega \left(\iint_{\mathbb{R} \times \partial\Omega} K^+(t-s,x,z) e_0^+(s,z-x) ds dz \right) dx. \tag{2.11}$$

Introduce $h_0^+(t,x)$ solution in $\mathbb{R} \times \mathbb{R}^n$ to $\Box h_0^+ = e_0^+$, $\operatorname{supp} h_0^+ \subset \{t \geqq 0\}$. Then continuing the equality (2.11),

$$= \int_{\partial\Omega} \int_\Omega \int_{\mathbb{R}} K^+(t-s,x,z) (\Box h_0^+)(s,z-x) ds dx dz.$$

Integration by parts in the $ds dx$ integral yields

$$= \int_{\partial\Omega} \iint_{\mathbb{R} \times \partial\Omega} (h_0^+ \partial_{n_x} K^+ - K^+ \partial_{n_x} h_0^+) ds dx dz.$$

Taking into account the boundary values of K^+, the second term is equal to $\int_{\partial\Omega} n_x \cdot \nabla_x h_0^+(t,0) dx$. For $t>0$, $h_0^+(t,\cdot)$ is smooth near zero so $\nabla_x h_0^+(t,0)$ is a fixed vector independent of $x \in \partial\Omega$ whence the above integral vanishes. Thus we have shown that

$$-\operatorname{Tr} Z(t) = 2\partial_t \int_{\partial\Omega} B(t,x,x) dx, \tag{2.12}$$

where

$$B(t,x,y) \equiv \int_{\mathbb{R}} \int_{\partial\Omega} h_0^+(t-s,z-x) k^+(s,z,y) dz ds. \tag{2.13}$$

The kernels that intervene here are the traces of the free propagation h_0^+ and the Neumann operator k^+. The proof of Theorem 3 rests on a detailed analysis of these kernels in sections four and five respectively. The description uses the language of microlocal Gevrey regularity and Lagrangian distributions of class Gevrey 3 developed in Sect. 3.

3. Fourier-Bros-Iagolnizer transform and conormal Gevrey distributions

Following [S.1] we introduce the Fourier-Bros-Iagolnizer (F.B.I.) transform of $f \in S'(\mathbb{R}^n)$:

$$Tf(y,\lambda) \equiv \int e^{-\lambda(y-x)^2/2} f(x)dx. \tag{3.1}$$

Tf is defined on $\mathbb{C}^n \times \mathbb{R}_+$ and is holomorphic with respect to y and bounded by $Ce^{\lambda|\mathrm{Im}\, y|^2/2}(\lambda+|y|)^N$. For $y=\alpha+i\beta$, the asymptotic behavior, for $\lambda \to +\infty$, measures the microlocal regularity of f at $(\alpha, -\beta) \in T^*(\mathbb{R}^n)\backslash\{0\}$. For example, it is not hard to show that

$$(\alpha_0,\beta_0) \notin WFf \Leftrightarrow (\forall N>0)(\exists C_N)\,\text{s.t.}\,|Tf(y,\lambda)| \leqq C_N e^{|\mathrm{Im}\, y|^2\lambda/2}\lambda^{-N}$$

holds uniformly for y in a neighborhood of $\alpha_0 - i\beta_0$ and $\lambda > 1$. The Gevrey s wave front set, WF^s, is measured similarly:

$$(\alpha_0,\beta_0) \notin WF^s \Leftrightarrow (\exists C)\,\text{s.t.}\,|Tf(y,\lambda)| \leqq Ae^{\lambda|\mathrm{Im}\, y|^2/2}e^{-C\lambda^{1/s}} \tag{3.2}$$

for all y in a neighborhood of $\alpha_0 - i\beta_0$ and $\lambda > 1$.

It is not obvious that the characterizations of WF in terms of Tf are homogeneous with respect to the action of $\mathbb{R}_+$ on the fibers. For the proof we recall Lebeau's inversion formula for T [L.1]

$$(2\pi)^d f(x) = 1/2 \int_{\mathbb{R}^d_\xi} e^{-R|\xi|/2}(1 - i(\xi/|\xi|^2)\cdot\nabla_y)\, Tf(x - iR\xi/|\xi|, |\xi|/R)d\xi,$$

where $R>0$ is at our disposal; then:

$$Tf(y,\lambda) = c\int\int e^{-\lambda(y-x)^2/2 - R|\xi|/2}(1 - i(\xi/|\xi|^2)\cdot\nabla_y)\, Tf(x - iR\xi/|\xi|, |\xi|/R)d\xi\, dx.$$

The crucial fact is that for any $\delta > 0$ the contribution from $|\xi/\lambda + \mathrm{Im}\, y| > \delta$ are dominated by $Ce^{-\varepsilon\lambda}e^{\lambda|\mathrm{Im}\, y|^2/2}$ for some $\varepsilon > 0$. The contribution from $|\xi/\lambda + \mathrm{Im}\, y| < \delta$ is then estimated using (3.1). To see that $|\xi/\lambda + \mathrm{Im}\, y| > \delta$ is negligible, one deforms the dx contour. Using the bound for Tf valid everywhere, the integrand has an exponential with exponent

$$-\lambda(y-x)^2/2 - R|\xi|/2 + |\mathrm{Im}\, x - R\xi/|\xi||^2|\xi|/2R$$

which has real part

$$-\lambda[\mathrm{Re}(y-x)]^2/2 + \lambda[\mathrm{Im}\, y]^2/2 - \mathrm{Im}\, x(\xi + \lambda\,\mathrm{Im}\, y) + (\mathrm{Im}\, x)^2(\lambda + |\xi|/R)/2.$$

Deforming to $\mathrm{Im}\, x = \eta(\xi + \lambda\,\mathrm{Im}\, y)/|(\xi + \lambda\,\mathrm{Im}\, y)|$ with $0 < \eta \ll 1$ yields the desired result.

We next recall Sjöstrand's [S1] intrinsic formulation. The holomorphic tangent space to $Y \equiv \mathbb{C}^n$ is spanned by $\partial/\partial y_1, \ldots, \partial/\partial y_n$, with

$$\partial/\partial y_j \equiv 1/2(\partial/\partial \alpha_j - i\partial/\partial \beta_j), \quad \alpha_j = \mathrm{Re}\, y_j, \quad \beta_j \equiv \mathrm{Im}\, y_j.$$

The holomorphic cotangent space is spanned by $d\zeta_j$ defined by $d\zeta_j(\partial/\partial y_k) = \delta_{jk}$ and the canonical two form σ_y is defined by $\sum d\zeta_j \wedge dy_j$. Following Schapira [S.], we consider the $\mathbb{R}$ multilinear forms $\mathrm{Re}\,\sigma_Y$ and $\mathrm{Im}\,\sigma_Y$, with $d\gamma_j \equiv \mathrm{Re}\, d\zeta_j$ and $d\delta_j \equiv \mathrm{Im}\, d\zeta_j$ we have:

$$\mathrm{Re}\,\sigma_Y = \sum_j d\gamma_j \wedge d\alpha_j - d\delta_j \wedge d\beta_j, \quad \mathrm{Im}\,\alpha_Y = \sum_j d\delta_j \wedge d\alpha_j + d\gamma_j \wedge d\beta_j.$$

Both are symplectic forms on T^*Y, the latter viewed as a C^ω manifold of dimension $4n$. (See [S.1, pp. 77–81].) If φ is a real valued C^ω function on Y, then

$$\Lambda_\varphi \equiv \{(y, -2i\sum \partial\varphi/\partial y_j(y) | y \in Y\}$$

is a C^ω submanifold of $T^*(\mathbb{C}^n)$ of real dimension $2n$ which is always $\mathrm{Im}\,\sigma_Y$ lagrangian. In addition Λ_φ is $\mathrm{Re}\,\sigma_Y$ symplectic if and only if $\partial\partial\varphi$ is non degenerate. In this case $T^*(\mathbb{C}^n)$ is its complexification. This holds in particular if φ is strictly plurisubharmonic (s.p.s.h.), that is

$$\sum \partial^2\varphi/\partial y_j \partial \bar{y}_k w_j \bar{w}_k > 0 \qquad \forall w \in \mathbb{C}^n \backslash 0.$$

Suppose M is a C^ω manifold of real dimension n, $p_0 \in T^*M\backslash 0$, $\varphi \in C^\omega(\mathbb{C}^n)$ s.p.s.h., and χ a C^ω canonical transformation from a neighborhood of p_0 to Λ_φ sending p_0 to the point of Λ_φ over y_0. Denote by X the complexification of M and $\chi^{\mathbb{C}}$ the (local) holomorphic extension of χ to a map $T^*X \to T^*Y$. Then there is a unique holomorphic g defined on a neighborhood of (y_0, x_0) $(x_0 \equiv \pi(p_0))$, such that

(i) $g(y_0, x_0) = -i\varphi(y_0)$

(ii) The graph of $\chi^{\mathbb{C}}$ in $T^*X \times T^*Y$ is the set of points $(x, y, -\partial g/\partial x, \partial g/\partial y)$

(iii) The function $x \to -\mathrm{Im}\, g(y, x)$ has a non degenerate critical point at

$$x = \pi\chi^{-1}\left(y, \frac{2}{i}\frac{\partial\varphi}{\partial y}(y)\right).$$

The critical value is $\varphi(y)$ and the signature is $(0+, n-)$.

In particular, if $M = \mathbb{R}^n$ and χ is the mapping $(x, \xi) \to (x - i\xi, \xi)$ and if φ is the function $\varphi(y) = 1/2(\mathrm{Im}\, y)^2$, the function $g(x, y)$ which satisfies the properties (i), (ii), and (iii) is

$$g(x, y) = i(x - y)^2/2$$

which appears in formula (3.1).

The above construction associates to any set $(M, \varphi, p_0, dx, \chi\colon T^*M \to \Lambda\varphi)$, an F.B.I. transform given by:

$$(T_\chi f)(y, \lambda) \equiv \int e^{i\lambda g(y, x)} f(x) dx.$$

In the right hand side, dx is a C^ω volume element on M, g is the function defined by the properties (i)–(iii); λ belongs to $\mathbb{R}_+$ and is assumed to be large; f is a distribution (or more generally a hyperfunction) defined in a neighborhood of x_0. The function $T_\chi f$ is holomorphic with respect to y and more precisely belongs to the H_φ space of Sjöstrand, that is, for y near y_0 it satisfies the estimate:

$$\forall \varepsilon > 0, \exists C_\varepsilon, \forall \lambda \geqq \lambda_0, \quad |T_\chi f(y, \lambda)| \leqq C_\varepsilon e^{\lambda(\varphi(y) + \varepsilon)}.$$

The F.B.I. transform is used to characterize the microlocal structure of the analytic singular support according to the

Definition 3.1. *For $s\in[1,\infty[$ and $f\in\mathscr{D}'(M)$, the Gevreys wave front set, $G^sWFf\subset T^*M\backslash 0$, is the complement of the set of points $p\in T^*M\backslash 0$ with the following property: there is an F.B.I. transform T_χ and a neighborhood ω of $\pi(\chi(p))$ such that*

$$(\exists C>0)\ (\forall z\in\omega, \lambda\geqq 1)\qquad |T_\chi f(z,\lambda)|\leqq A e^{\lambda\varphi((z)-C\lambda^{1/s}}. \tag{3.4}$$

Remarks. 1) The case $s=1$ yields the analytic wave front set

2) Functions Gevrey^s at $\pi(p)$ satisfy $|\partial_x^\alpha f|\leqq AB^{s|\alpha|}(s|\alpha|)|$. The constant C in (3.4) is a microlocal analogue of the constant $1/B$.

Lemma 3.1. *The Definition* (3.1) *is homogeneous with respect to the $\mathbb{R}_+$ action in the fibers. It is also independent of χ in the sense that* (3.4) *is satisfied for one χ iff it is satisfied for all.*

Proof. If χ_i are canonical transformations on Λ_{φ_i}, $i=1,2$, then there exists a complex canonical transformation Γ between the Sjöstrand spaces H_{φ_i} with $\Gamma\circ T_{\chi_2}=T_{\chi_i}$. This yields the desired independence. The homogeneity is a consequence of the fact that the definitions are homogeneous for the ordinary F.B.I. as indicated at the beginning of this section. □

In our analysis, it is important to know how the constant C in (3.4) varies from point to point. This leads to the notion of Gevrey s with weights. We consider $s=3$.

Definition 3.3. *Suppose m is a strictly positive function defined on a neighborhood of p in $T^*M\backslash 0$ which is Lipschitz continuous, and homogeneous of degree 1/3 in the fiber. A distribution $f\in\mathscr{D}'(M)$ is microlocally Gevrey 3 with weight m at p if for some F.B.I. transform T_χ there is a neighborhood ω of $\pi(\chi(p))$ such that* $(\forall\varepsilon>)$ $(\exists C_\varepsilon)$ $(\forall\lambda\geqq 1)$ $(\forall y\in w)$

$$|T_\chi f(y,\lambda)|\leqq C_\varepsilon e^{\lambda\varphi(y)-\lambda^{1/3}(m(y)-\varepsilon)},\quad \text{where}\quad m\equiv m\circ\pi_Y\circ\chi. \tag{3.5}$$

Remarks. 1) That this is homogeneous and independent of χ follows as in Lemma 3.1.

2) Since $m>0$, Gevrey^3 with weight m at p implies Gevrey^3 at p.

Next we define what is meant by a Lagrangian distribution of class Gevrey 3. Suppose that $\Lambda\subset T^*(M)$ is a C^ω homogeneous Lagrangian submanifold and $\psi:\Lambda\to\mathbb{C}$ is C^ω and homogeneous of degree 1/3 with $\mathrm{Re}\,\psi(p_0)>0$. Let $h(x,\theta)$ by a C^ω nondegenerate phase, which represents Λ (see [H.1]). The parameter θ is defined in a conic neighborhood Γ_0 of θ_0 in $\mathbb{R}^N$. Let

$$C_h\equiv\{(x,\theta)|\partial_\theta h(x,\theta)=0\},$$

and $j:C_h\to\Lambda$ the isomorphism defined by $j(x,\theta)\equiv(x,\partial_x h(x,\theta))$ with $p_0=j(x_0,\theta_0)$. Let $l(x,\theta)$ be C^ω, homogeneous of degree 1/3 with $l|_{C_h}=\psi\circ j$. We assume that Γ_0 is so small that $\mathrm{Re}\,l>0$.

Definition 3.4. *A distribution $f\in\mathscr{D}(M)$ lies in $I^3_{\Lambda,\psi}$ at p_0 if for x near $x_0=\pi(p_0)$, one has $f=f_1+f_2$ where f_2 is microlocally Gevrey^3 with weight m at p_0 such that $m(p_0)>\mathrm{Re}\,\psi(p_0)$ and*

$$f_1(x)=\int_{\Gamma_0} e^{ih(x,\theta)-l(x,\theta)}a(x,\theta)d\theta, \tag{3.6}$$

where a is an analytic symbol in the sense of Boutet de Monvel [BdM.] *with an expansion*

$$a \sim \sum_{k=0}^{\infty} a_k(x,\theta), \tag{3.7}$$

where the a_k are C^ω, homogeneous of degree $d-N/2-k/3$ in θ. The expansion (3.7) means that there are constants A, B and a complex conic neighborhood N of $\pi(p_0)$ in $\mathbb{C}^n \times \mathbb{C}^N$ such that for all x, $\theta \in N$ and $k \in \mathbb{N}$

$$\left| a - \sum_{k=0}^{k=K} a_k \right| \leqq AB^K K! |\theta|^{d-N/2-(K+1)/3}.$$

Remarks. 1) Since $\mathrm{Re}\, l > 0$, the integral is absolutely convergent and $f_1 \in G^3$.

2) This definition is clearly homogeneous in the fibers.

3) The fact that the definition is independent of the choice of h and l is proved in Proposition 3.1 below.

Preparing for Proposition 3.1, we introduce a canonical transformation χ as above and denote φ_Λ the unique holomorphic function defined by (cf. [L.1]):

$$\left(y, \frac{\partial \varphi_\Lambda}{\partial y}(y)\right) = \chi^{\mathbb{C}}(\Lambda^{\mathbb{C}}); \qquad i\varphi_\Lambda(y_0) = \varphi(y_0).$$

Here, $\chi^{\mathbb{C}}$, $\Lambda^{\mathbb{C}}$ denote the natural complexifications. We recall that we have $-\mathrm{Im}\,\varphi_\Lambda(y) \leqq \varphi(y)$ and that equality holds if and only if $\left(y, \frac{2}{i}\frac{\partial \varphi_\Lambda}{\partial y}(y)\right) \in \chi(\Lambda)$. Let $\psi(y)$ be the holomorphic function on Y defined by the relation: $\psi \circ \pi_\gamma \circ \chi^{\mathbb{C}} = \psi^{\mathbb{C}}$ where $\psi^{\mathbb{C}}$ denotes the holomorphic extension of ψ to $\Lambda^{\mathbb{C}}$. Then we have the

Proposition 3.1. *The following assertions are equivalent:*

(i) *f belongs to $l^3_{\Lambda,\psi}$.*

(ii) *There exists a positive constant ε such that, for $|y-y_0|<\varepsilon$, $T_\chi f$ admits the decomposition*

$$T_\chi f = e^{i\lambda\varphi_\Lambda(y) - \lambda^{1/3}\psi(y)} b(y,\lambda) + r(y,\lambda), \tag{3.8}$$

where the functions $r(y,\lambda)$ and $b(y,\lambda)$ have the following properties:

(a) *$r(y,\lambda)$ satisfies the estimate*

$$|r(y,\lambda)| \leqq e^{\lambda\varphi(y) - C_0\lambda^{1/3}}, \tag{3.9}$$

with a constant C_0 strictly greater than $\mathrm{Re}\,\psi(p_0)$.

(b) *$b(y,\lambda)$ is holomorphic with respect to y and has, for $|y-y_0|<\varepsilon$ an asymptotic expansion of the form:*

$$\forall K \in \mathbb{N}, \sup_{|y-y_0|<\varepsilon} \left| b(y,\lambda) - \sum_{k=0}^{K} b_k(y)\lambda^{d-n/2-k/3} \right| \leqq AB^K K! \lambda^{d-n/2-(K+1)/3}. \tag{3.10}$$

Proof. First we prove that (i) implies (ii). Since f_2 is microlocally Gevrey 3 with weight m, the function $r(y,\lambda) = T_\chi f_2$ satisfies the estimate (3.5), and we have:

$$T_\chi f_1 = \lambda^N \int e^{i\lambda(g(y,x)+h(x,\alpha)) - \lambda^{1/3} l(x,\alpha)} a(x,\lambda\alpha)\, dx\, d\alpha.$$

Since $e^{-\lambda^{1/3} l(x,\alpha)} a(x, \lambda\alpha)$ is an analytic symbol, we can use standard contour deformations to reduce the problem to the use of the stationary phase method at the unique critical point of the mapping

$$(x, \alpha) \to g(y, x) + h(x, \alpha)$$

which is given by the relation

$$(x, \alpha) = (x(y), \alpha(y)) = j^{-1} \chi_{\mathbb{C}}^{-1}(y, (\partial_y \varphi_\Lambda)(y)).$$

The critical value is $\varphi_\Lambda(y)$, and we denote by $\psi(y)$ the expression $\psi(y) = l(x(y), \alpha(y))$. Using the variables of the Morse lemma, we have:

$$T_\chi f_1 \sim \lambda^N e^{i\lambda\varphi_\Lambda(y) - \lambda^{1/3}\psi(y)} \int_{|s| \leqq \varrho} e^{-\lambda s^2/2 - \lambda^{1/3}(l(x(s),\alpha(s)) - \psi(y))} a(y(s), \lambda\alpha(s)) J(s) ds. \tag{3.11}$$

In this integral $J(s)$ denotes the Jacobian related to the change of variables. As it is well known we have, at the stationnary point

$$J(s)|_{s=0} = J(y, x(y), \alpha(y)) = (\det \operatorname{Hess}(g+h)/i)^{-1/2},$$

where the Hessian is taken with respect to both variables x and α and where he square root is determined from the square of unity equal to 1 by the homotopy $s \to (1-s) \operatorname{Hess}(g+h)/i + s\,\mathrm{Id}$. A formal application of the method of the stationnary phase would give (see [Sj. 1, p. 9])

$$T_\chi f_1 \sim e^{i\lambda\varphi_\Lambda(y) - \lambda^{1/3}\psi(y)} \cdot \lambda^{(N-n)/2} \sum_k \lambda^{-k}/k!\,(1/2\Delta)^k (e^{-\lambda^{1/3}(l(x,\alpha) - l(x(y),\alpha(y))} \cdot Ja(x, \lambda\alpha)),$$

where Δ is the Laplace operator in the variables from the Morse lemma.

To show that the above expression is not formal but satisfies indeed the estimates (3.10) we proceed as follow. (3.11), yields an integral of the form

$$I = \int_{|s| \leqq \varrho} e^{-\lambda s^2/2 + \lambda^{1/3} s F(y,s)} \sigma(y, s, \lambda) ds,$$

where σ is a classical analytical symbol in $\lambda^{-1/3}$. We split I into two integrals I_1 and I_2 over $|s| \leqq \lambda^{-1/3} C_0$ and $\lambda^{-1/3} C_0 \leqq |s| \leqq \varrho$ respectively. For $s \geqq \lambda^{-1/3} C_0$ we have $-\lambda s^2/2 + \lambda^{1/3} s|F| \leqq -\lambda^{1/3} C_0^2/2 + C_0|F|$ therefore, $I_2(y, \lambda)$ satisfies the estimate (3.9) (Note that C_0 can be taken arbitrarly large and that this type of estimate works for terms of the form λ^α only for $\alpha < 1/2$.)

On the other hand, I_1 is given by the expression

$$I_1 = \int_{|t| < C_0} e^{-\lambda^{1/3} t^2} e^{tF(y, \lambda^{-1/3} t)}\, \sigma(y, \lambda^{-1/3} t, \lambda) \lambda^{-1/3} dt,$$

which can be computed by the complex method of stationary phase with large parameter $\lambda^{1/3}$ which gives the same result as the formal expansion and which is of form (3.10).

To prove that (ii) implies (i), we use the microlocal nature of the problem; the property (i) is invariant under any real homogeneous canonical transform and (ii) is invariant under any canonical transform between Sjöstrand's spaces. We can assume that we have $\Lambda = T_0^* \mathbb{R}^n$ near $(0, \xi^0)$, $|\xi^0| = 1$ and that T_χ is the F.B.I.

transform given by (3.1). In this situation, we have:

$$\varphi(y)=1/2(\mathrm{Im}\, y)^2\,,\qquad \varphi_A(y)=iy^2/2;\qquad \psi(y)=\psi(x=0,\xi=iy)\,.$$

If we assume that (ii) is satisfied, there exists an analytical symbol (in the sense of Boutet de Monvel, i.e. holomorphic in λ in a conic neighborhood of $\mathbb{R}_+$) $\tilde{b}(y,\lambda)$ which has the same formal asymptotic expansion as $b(y,\lambda)$. Since we have, for $\lambda\geqq 1$, $|\tilde{b}-b|\leqq e^{-c\lambda^{1/3}}(c>0)$, we can replace b by $\tilde{b}$, with only a change on the term $r(y,\lambda)$. Then we use the inversion formula, with $R=1$. We introduce a small conic neighborhood Γ_0 of ξ_0 and, microlocally near $(0,\xi_0)$, we can use (3.8); f is microlocally the sum of the two following functions:

$$f_2(x)=(1/2)(2\pi)^{-n}\int_{\Gamma_0} e^{-|\xi|/2}(1-i(\xi/|\xi|^2)\cdot\partial_x)r)(x-i\xi/|\xi|,|\xi|)d\xi\,,$$

$$f_1(x)=(2\pi)^{-n}\int_{\Gamma_0} e^{ix\cdot\xi-(x^2/2)|\xi|-|\xi|^{1/3}\psi(ix+\xi/|\xi|)}\mathscr{L}(b)d\xi\,.$$

The term $\mathscr{L}(b)$ is given by the formula:

$$\mathscr{L}(b)=(1/2)\,[(1-i(\xi/|\xi|^2)\cdot\partial_y)b+i\xi/|\xi|^2(|\xi|(x-i\xi/|\xi|)+|\xi|^{1/3}\partial_y\hat{\psi})b]\,.$$

As in the proof of the Lemma 3.1, the function f_2 is microlocally Gevrey 3 with a weight m, satisfying the estimate $m(0,\xi^0)>\mathrm{Re}\,\psi(\xi^0)$. Finally, since $\mathscr{L}(b)(x,\xi)$ is a symbol in the sense of Boutet de Monvel, one can introduce, for $|x|$ small, the change of variables $\xi\to\xi+i(x/2)|\xi|$, and obtain for f_1 an expression in terms of the real phase $x\cdot\xi$ corresponding to right hand side of (3.6). The proof of the Proposition (3.1) is complete. □

4. Traces of the fundamental solutions for $\Box$, $\Box^2$

Our goal is to analyse $\mathrm{Tr}(t)$ using formulas (2.12) and (2.13). In B, two kernels intervene, h_0^+ and k^+. The latter is the Neumann operator on $\mathbb{R}_t\times\Omega$ and the former is the trace on $\mathbb{R}_t\times\partial\Omega$ of a fundamental solution to $\Box^2$ on $\mathbb{R}\times\mathbb{R}^n$. In this section, we study this second type of kernel.

The fundamental solutions e_0, h_0 are defined by:

$$\Box e_0^+=\delta\,,\qquad \Box^2 h_0^+=\delta\,,\qquad \mathrm{supp}\, e_0^+, h_0^+\subset\{t\geqq 0\}\,.$$

In this section, we study the operators

$$\mathbb{T}^j:\mathscr{E}'(\mathbb{R}_t\times\partial\Omega)\to\mathscr{D}(\mathbb{R}_t\times\partial\Omega)$$

$$\mathbb{T}^1 f\equiv[e_0^+ * (f\otimes\delta_{\partial\Omega})]_{|\mathbb{R}_t\times\partial\Omega}$$

and $\mathbb{T}^2$ is defined with h_0^+ in place of e_0^+. Here $\partial\Omega$ is the boundary of the open set $\mathbb{R}^n\backslash 0$ and the traces are taken from the Ω side. From the definition, we see that the kernels of $\mathbb{T}^i$ which are functions of $t,x,s,z\in(\mathbb{R}_t\times\partial\Omega)^2$ depend only on $t-s,x,z$ so

we write as $\mathbb{T}^i(t-s,x,y)$. Again from the definition, we see that

$$\begin{aligned}\mathbb{T}^1(t,x,z)&=e_0^+(t,x-z)|_{x\in\partial\Omega,\,z\in\partial\Omega},\\ \mathbb{T}^2(t,x,z)&=h_0^+(t,x-z)|_{x\in\partial\Omega,\,z\in\partial\Omega}.\end{aligned}\tag{4.1}$$

$\partial\Omega\times\mathbb{R}_t$ is non caracteristic for the wave equation and therefore, with the exception of $t=0$ and $x=z$ the trace of e_0^+ and h_0^+ on $\partial\Omega\times\mathbb{R}_t$ is well defined and we have:

$$WF(e_0^+|_{\partial\Omega X\partial\Omega X\mathbb{R}_t})\subset\pi WF(e_0^+)\cap T^*(\partial\Omega X\partial\Omega X\mathbb{R}_t)$$

and

$$WF(h_0^+|_{\partial\Omega X\partial\Omega X\mathbb{R}_t})\subset\pi WF(h_0^+)\cap T^*(\partial\Omega X\partial\Omega X\mathbb{R}_t),$$

where π is the projection on $T^*(\partial\Omega X\partial\Omega X\mathbb{R}_t)$. The only difficulty for the definition of the trace arises near $t=0$ and $x=z=0$ where we will, with a change of variables, make an explicit computation. These computations are classical for elliptic problems and like in the elliptic case they give differents results when the trace is taken from the interior or from the exterior of the obstacle due to a change of sign in the determination of the square root.

We denote by $\hat{\mathbb{T}}^j(\tau,x,z)$ the partial Fourier transforms with respect to t, so $\hat{\mathbb{T}}^j$ is holomorphic in $\operatorname{Im}\tau<0$, and we have

$$\hat{\mathbb{T}}^j(\tau,x,y)=(2\pi)^{-n-1}\int(\xi^2-\tau^2)^{-j}e^{i(x-y)\xi}\,d\xi|_{x\in\partial\Omega,\,y\in\partial\Omega},\qquad \operatorname{Im}\tau<0.\tag{4.2}$$

If $x^0\in\partial\Omega$, we may relabel the coordinates so that near x^0, Ω is given by $\{x_n>S(x')\}$, where $x'\equiv(x_1,\ldots,x_{n-1})$. With x' as coordinates for $\partial\Omega$, the canonial projection $T_x^*\mathbb{R}^n\to T_x^*\partial\Omega$ for $x\in\partial\Omega$ is given by

$$\xi',\xi_n\to\xi'+\xi_n\nabla_{x'}S(x').\tag{4.4}$$

Choose a $C^\omega(\mathbb{R}^{n-1},\mathbb{R}^{n-1})$ function H such that

$$S(x')-S(y')=(x'-y')H(x',y'),\qquad H(x',x')=\partial_{x'}S(x').\tag{4.5}$$

Then, the metric form Θ on $T^*(\partial\Omega)$ induced by the Euclidean metric is given by $\theta(x',x',\cdot)$ where

$$\Theta^2(x',y',v')=v'^2-(v'\cdot H(x',y')^2/(1+H^2),$$

where the scalar products on the right are $\mathbb{R}^{n-1}$ Euclidean. Then Θ is a positive definite form on $T_x^*(\partial\Omega)$ provided $|y'-x'|$ is small. The glancing set is given by $\tau^2=\Theta^2(x',x',v')$, and the second fundamental form, R, by

$$R(x',v')=-\operatorname{Hess}S(\xi',\xi')\cdot[(1+|\nabla S|^2+|\nabla S|^4)/(1+|\nabla S|^2)^2]^{1/2},$$

where ξ' is defined by

$$\xi'\equiv v'-\xi_n\nabla_{x'}S,\qquad \xi_n\equiv v'\cdot\nabla S/(1+|\nabla S|^2).$$

The quadratic form R gives the curvature in $\mathbb{R}^n$ of the $\partial\Omega$ geodesic passing through (x',v') when $\Theta(x',x',v')=1$.

In the coordinates l, x', $l \equiv x_n - S(x')$, we have $\delta_{\partial\Omega} = (1+|\nabla S(x')|)^{1/2} \otimes \delta(l)$. Thus (4.2) yields

$$\hat{\mathbb{T}}^j(\tau, x, y) = (2\pi)^{-n-1} \int e^{i\xi'(x'-y') + i\xi_n(x_n - S(x'))}(1+|\nabla S(y')|^2)^{1/2}(\xi^2 - \tau^2)^{-j} d\xi .$$

Following a calculation which is standard in the elliptic case, we take for new phase variables $v' = \xi' + \xi_n H(x', y')$, $v_n = \xi_n$ and integrating with respect to v_n and then letting $l = x_n - S(x')$ decrease to zero, we find that

$$\hat{\mathbb{T}}^j(\tau, x', y') = 2^{-j}(2\pi)^{-n} \int e^{iv'(x'-y')}(\Theta^2 - \tau^2)^{(1-2j)/2}[(1+|\nabla S(y')|^2)/(1+H(x'y')^2)]^{1/2} dv' . \tag{4.6}$$

The square root of $\Theta^2 - \tau^2$ is chosen with positive real part. Next, let $Q(x', y')$ be the positive definite matrix such that

$$\Theta^2(x', y', v') = |Q^{-1}v'|^2$$

and introduce the new variable $\xi' \equiv Q^{-1}v'$ to find

$$\hat{\mathbb{T}}^j(\tau, x', y') = f(x', y') \int e^{i\xi' \cdot Q(x', y')(x'-y')}(\xi'^2 - \tau^2)^{\alpha_j} \cdot d\xi' .$$

Thus, the analytic spectrum of $\mathbb{T}^j$ can be computed in terms of the inverse Fourier transform of the distribution $(\xi'^2 - (\tau - i0)^2)^{\alpha_j}$ which is well known. In particular, the singular spectrum of $\mathbb{T}^j$ is, for $t > 0$, the restriction to $\partial\Omega$ of the propagation for the wave equation; therefore we have the following:

Lemma 4.1. *The analytic wave front of the kernels $\mathbb{T}^j(t, x, z)$ is contained in the set of the points $(t, x, z; \tilde{t}, \tilde{x}, \tilde{z})$ which satisfy one of the two following relations*

$$t = 0, \quad x = z, \quad \tilde{x} + \tilde{z} = 0, \tag{4.7}$$

or,

$$t > 0, \ \exists \xi \in \mathbb{R}^n, \quad \tilde{z} = -p_z(\xi), \quad \tilde{x} = p_x(\xi), \quad \tilde{t} = \mp t\xi/|\xi|, \quad x = z \pm t\xi/|\xi| . \tag{4.8}$$

Here p_x denotes the canonical projection of $T_x^*(\mathbb{R}^n)$ on $T_x^*(\partial\Omega)$.

Next, we use the results of Sect. 3 to study the propagation of Gevrey 3 singularities on $\partial\Omega$. Consider the C^ω, $2n-1$ manifold $M \equiv \mathbb{R}_t \times \partial\Omega \times \partial\Omega$. In T^*M, we introduce the lagrangian manifold $\Lambda = \Lambda^+ \cup \Lambda^-$ where

$$\Lambda^\pm = \{(t, x, y; \tau, \xi, \eta) | \pm\tau = |\xi| = |\eta|, (x, \xi) = \Phi_{\pm t}(y, -\eta)\} . \tag{4.9}$$

The metric on $\partial\Omega$ is induced by the Euclidian metric in $\mathbb{R}^n$ and Φ denotes the geodesic flow, or more precisely the flow induced on $T^*(\partial\Omega)$ by the Hamiltonian $H_{|\xi|}$. We will denote by $\varrho(x, \xi)$ the curvature of the geodesic through the point $(x, \xi/|\xi|)$ and by $-\omega$ the zero of smallest modulus of the Airy function. Since $\tau + |\xi| = 0$ on $\Lambda^\pm$, we have $H_{\tau+|\xi|}$ tangent to $\Lambda^\pm$ and we define ψ on $\Lambda^\pm$ by

$$H_{\tau+|\xi|}\psi = e^{\pm i\pi/6}(\pm\tau\varrho^2/2)^{1/3}\omega, \quad \psi|_{t=0} = 0 . \tag{4.10}$$

Our analysis turns on the fact that several distributions entering in the description of $\mathrm{Tr} Z$ lie in $I^3_{\Lambda,\psi}$ and have leading terms which can be explicitly calculated. The two propositions of this section are results of that form.

Proposition 4.1. *If $f \in D'(\mathbb{R}_t \times \partial\Omega \times \partial\Omega)$, $\operatorname{supp} f \subset (t \geqq 0\}$, has analytic wave front set in the union $(t=0$ and $x=y) \cup (\Lambda \cap (t>0))$ and lies in $I^3_{\Lambda,\psi}$ for $t>0$ then on a neighborhood of the set $(t>0, |x-z|<t)$ the distributions g^j,*

$$g^j(\ , x, z) \equiv \iint_{\mathbb{R}_s \times \partial\Omega} \mathbb{T}^j(t-s, x, y) f(s, y, z) ds dy$$

have analytic wave front set in Λ and lie in $I^3_{\Lambda,\psi}$.

Proof. Since $\mathbb{T}^j$ propagates at speed exactly equal to one and $|x-z|<t$, the singularities of f at $t=0$ do not influence those of $g(t, x, z)$. In addition, as Λ lies in the glancing set and $\partial\Omega$ is strictly convex, singularities in f for $t>0$ are propagated along glancing rays, and therefore away from $\partial\Omega$. This shows that the analytic wave front of g^k is in Λ and in addition that for $p_0=(t_0, x_0, z_0, \tau_0, \xi_0, \eta_0) \in \Lambda$ the microlocal behavior of g^j near p_0 is determined by that of f near p_0 and the microlocal behavior of $\mathbb{T}^k$ at $(0, x_0, x_0, \tau_0, \xi_0, -\xi_0)$. For the rest of the proof, we suppose that $p_0 \in \Lambda^+$. The case $p_0 \in \Lambda^-$ is similar. We compute in coordinates x', x_n with $\Omega = \{x_n > S(x')\}$ as above. We write $f = f_1 + f_2$ with f_2 Gevrey 3 with weight m, $m(p) > \operatorname{Re}\psi(p_0)$ and

$$f_1(s, y, z) = \int e^{i\varphi(s, y, z, \theta) - l(s, y, z, \theta)} a(s, y, z, \theta) d\theta, \tag{4.11}$$

where the phase φ is defined by the eikonal equation on $\partial\Omega$:

$$\partial\varphi/\partial s = \Theta(y, y, \partial_y\varphi), \qquad \varphi|_{s=t_0} = \varphi_0(y, z, \theta),$$

where φ_0 is a non degenerate phase describing Λ^+ near $\exp(-t_0 H_{|\xi|}(z_0, -\eta_0))$.

In the integrals (4.6) defining $\mathbb{T}^k(\tau, x, y)$, we can take the inverse Fourier transform to find expressions for $\mathbb{T}^k(t, x, y)$. In the resulting expression, there are two possible sources of trouble, the vanishing of the denominator when $\tau^2 = \Theta^2$ and contributions from $v, \tau \to \infty$, $\tau^2 \neq \Theta^2$. The latter is the transform of classical symbol and does not affect $\mathbb{T}^k$ microlocally at $(0, x_0, x_0, \tau_0, \xi_0, -\xi_0)$. Thus we may restrict attention to a neighborhood of $\tau^2 = \Theta^2$ where we introduce the new variables σ, v' by $\tau = \Theta(x, y, v)(1-\sigma)$, $|\sigma| \leqq \sigma_0$ small, and modulo a microlocally analytic correction we have:

$$\mathbb{T}^k = \int_{-\sigma_0}^{\sigma_0} (\sigma + i0)^{1/2-k} \int_{\Gamma_0} e^{iv\cdot(x-y) + (t-s)(1-\sigma)\Theta(x, y, v)} H^k(x, y, \sigma, v) dv d\sigma, \tag{4.12}$$

where Γ_0 is a conic neighborhood of ξ_0, $H^k \in C^\omega$ is homogeneous of degree $2-2k$ with respect to v, and we have suppressed the prime in v. Thus,

$$\mathbb{T}^k = \int_{-\sigma_0}^{\sigma_0} (\sigma + i0)^{+1/2-k} \mathbb{T}^k_\sigma d\sigma,$$

where $\mathbb{T}^k_\sigma$ are Fourier integral operators with canonical relations C_σ determined by the phase $\Phi_\sigma(x, t, y, s, v) \equiv v\cdot(x-y) + (t-s)(1-\sigma)\Theta(x, y, v)$.

We do not need to take into account the boundary of Γ_0, because the point $(t_0, x_0, \tau_0, \xi_0; t_0, x_0, \tau_0, \xi_0)$ does not belong to the canonical relation of $\Phi_{\sigma|\partial\Gamma_0}$. Notice also that for (t, s) near (t_0, t_0), Φ_σ induces a canonical isomorphism of $T^*(\partial\Omega)$ into itself, which is close to the identity. (The relation C_σ itself is not an isomorphism

but one can check that it induces a ramified canonical transformation with two sheets; the branching points occur at $\sigma=0$ along the glancing hypersurface $\mathscr{G}$.)

To compute $(\mathbb{T}^k f_1)(t,x,z)$, we superpose the integrals (4.11) and (4.12) and use the classical stationary phase method to eliminate the variables (y,v). The critical points are given by the equations:

$$\begin{aligned} y_c &= x+(t-s)(1-\sigma)\partial_v\Theta(x,y_c,v_c) \\ v_c &= \partial_y\varphi(s,y_c,z,\theta)+(t-s)(1-\sigma)\partial_y\Theta(x,y_c,v_c)\,. \end{aligned} \tag{4.13}$$

We will denote by v_0 the vector $\partial_x\varphi(t,x,z,\theta)$. To compute the critical value, we assume that the function S which describes $\partial\Omega$ near x, satisfies

$$S(x+u)=-{}^t u A u+O(|u|^3)\,,$$

where A is a non degenerate positive symmetric matrix. In this situation, we have the relations:

$$\Theta(x,x,v)\equiv|v|\,,\qquad \partial_x\Theta(x,x,v)\equiv\partial_y\Theta(x,x,v)\equiv 0\,,\qquad \partial_v\Theta=v/|v| \tag{4.14}$$

and the Hessians $\partial^2_{xx}\Theta$, $\partial^2_{yy}\Theta$, $\partial^2_{xy}\Theta$ are all equal to $-1/|v|({}^t\cdot Av)^2$. Finally the curvature of the geodesic through $(x,v/|v|)$ is given by the relation

$$\varrho(v,x)=2{}^t v\cdot Av/|v|^2\,.$$

Using these formulas, we obtain the expression:

$$\mathbb{T}^k f_1(t,x,z)=\int e^{i\varphi(t,x,z,\theta)-l(t,x,z,\theta)}b^k(t,x,z,\theta)d\theta\,, \tag{4.15}$$

where the b^k are given by the relations

$$\begin{aligned} \cdot\, b^k(t,x,z,\theta)= &\int_{]-\sigma_0,\sigma_0[}(\sigma+i0)^{1/2-k}d\sigma\int e^{ig\Theta(x,x,v_0)+l(t,x,z,\theta)} \\ &\times Q(H^k a\, e^{-l(s,y,z,\theta)})|_{y=y_c,\,v=v_c}ds\,. \end{aligned} \tag{4.16}$$

Q denotes the operator from the asymptotic expansion in the stationary phase method; H^k is the function defined by the formula (4.12) and $g\in C^\omega$ is homogeneous of degree zero with respect to θ with:

$$g(s,t,x,z,\sigma,\theta)=\sigma(s-t)-(1/3)(1/8)(s-t)^3\varrho^2(x,v_0)+O((s-t)^4+\sigma(t-s)^2)\,. \tag{4.17}$$

With (4.13), we have:

$$\begin{aligned} l(t,x,z,\theta)-l(s,y_c,z,\theta)=&(t-s)(\partial_t l(t,x,z,\theta)-\partial_v\Theta(x,x,v_0)\partial_x l(t,x,z,\theta)) \\ &+O((t-s)^2+\sigma(t-s))\,, \end{aligned}$$

and using (4.10), we obtain:

$$\begin{aligned} l(t,x,z,\theta)-l(s,y_c,z,\theta)=&(t-s)(\Theta(x,x,v_0))^{1/3}\varrho^{2/3}(x,v_0)e^{i\pi/6}2^{-1/3}\omega \\ &+O((t-s)^2+\sigma(t-s))\,. \end{aligned}$$

Finally, the relation (4.15) is valid mirolocally near p_0 if the integration with respect to s in (4.16) is done in a neighborhood of t_0 which is small compared to σ_0, that is $|\sigma_0|\gg(t-s)^2$ (see (4.17)).

We denote by λ the expression $\Theta(x, x, v_0)$ and instead of $(\sigma, t-s)$, we introduce the new variables $(\hat{\sigma}, \hat{s})$ with the following properties: $(\hat{\sigma}, \hat{s})$ depend analytically on $z, \theta/|\theta|, t, x, \lambda^{-2/3}$, and satisfy the relations:

$$\hat{\sigma}|_{\sigma=0} \equiv 0, \quad \hat{s}|_{t=s} \equiv 0, \quad \hat{s} > 0$$

if and only if $t > s$, and

$$g(s,t,x,z,\theta) - i\lambda^{-2/3}[l(t,x,z,\theta) - l(s,y_c,z,\theta)] = -\hat{\sigma}\hat{s} + \hat{s}^3/3 - i\lambda^{-2/3}\hat{s}a(t,x,z,\theta/|\theta|;\hat{s}).$$

This reduces the study of the right hand side of (4.15) to the asymptotic evaluations of the integrals:

$$I_k(\lambda) = \iint_C e^{i\lambda(-\hat{\sigma}\cdot\hat{s} + \hat{s}^3/3 - i\lambda^{-2/3}\hat{s}a(\hat{s})} f(\hat{\sigma}, \hat{s}, \lambda)(\hat{\sigma} + i0)^{1/2-k} d\hat{\sigma}\, d\hat{s}, \tag{4.18}$$

where f denotes an analytic symbol in λ which is classical in $\lambda^{1/3}$ and where C is the $\mathbb{R}^2$ neighborhood of the critical point $(\sigma, s) = (0,0)$ defined by:

$$|\hat{\sigma}| \leqq \hat{\sigma}_0; \quad |\hat{s}| \leqq \hat{s}_0 \quad \text{with} \quad \hat{\sigma}_0 \gg \hat{s}_0^2.$$

We denote by Φ the function

$$\Phi(\hat{\sigma}, \hat{s}) = -\hat{\sigma}\cdot\hat{s} + \hat{s}^3/3,$$

and we notice that we have $\partial_{\hat{\sigma}}\Phi \neq 0$ for $\hat{s} = \pm\hat{s}_0$, and $\partial_{\hat{s}}\Phi \neq 0$ if $\hat{\sigma} = 0$ or $\hat{\sigma} = \pm\hat{\sigma}_0$; finally on every vertex of ∂C, we have $\Phi \neq 0$. Therefore the boundary of C does not contribute microlocally near p_0. The only contribution comes from the critical point $(\hat{\sigma}, \hat{s}) = (0,0)$. With an integration by part, we can reduce the case $k=2$ to the case $k=1$, which is considered below. The change of variable $\hat{\sigma} = u^2$, $2du = \hat{\sigma}^{-1/2} d\hat{\sigma}$, yields an integral with a holomorphic symbol and a phase given by

$$\Phi(u, \hat{s}, \lambda) = -u^2\hat{s} + \hat{s}^3/3 - i\lambda^{-2/3}\hat{s}a. \tag{4.19}$$

This phase has an isolated critical point at $(u, \hat{s}) = (0,0)$, for $\lambda^{-2/3} = 0$. We use the division theorem for symbols [BdM.] with the Jacobian ideal of (4.19), spanned by

$$-2u\hat{s}, \; -u^2 + \hat{s}^2 - i\lambda^{-2/3}\frac{\partial \hat{s}a}{\partial \hat{s}}.$$

Near zero $z = \lambda^{-2/3}$ is an holomorphic parameter. For $z = 0$ the quotient is spanned by the four functions $l, u, \hat{s}, \hat{s}^2$ since the right hand side of (4.19) depends only on u^2 there are no terms in u. Therefore we reduce the study of (4.18) to the case where f is a linear combination of the functions $1, s, s^2$ with coefficients which are analytic with respect to $z = \lambda^{-1/3}$.

Finally the right hand side of (4.19) is holomorphic with respect to the zero order term of the Taylor expansion of a. Deriving with respect to this parameter we reduce the study of (4.18) to the case where f is constant. Then we write

$$\int_{]-\sigma_0, \sigma_0[} e^{-i\lambda\sigma s}(\sigma + i0)^{-1/2} d\sigma = \text{Constant}(\lambda s_+)^{-1/2} + e^{-i\lambda\sigma_0 s} T^+(\lambda s) - e^{i\lambda\sigma_0 s} T^-(\lambda s).$$

Here, $T^+(\lambda s)$ is given by:

$$T^+(\lambda s) = \int_0^\infty (\lambda + \sigma_0)^{-1/2} e^{-i\lambda x s} dx,$$

which is holomorphic in the sector $-\pi-\varepsilon \leqq \operatorname{Arg} s \leqq \varepsilon$, $s \neq 0$ and which satisfies in this region the estimate

$$|T^+(\lambda s)| \leqq \text{Constant}(1+|\lambda s|^{-1/2}).$$

Therefore, a contour deformation with respect to s in the half plane $\operatorname{Im} s<0$ shows that the contribution of T^+ is microlocally zero near p_0; the same remark holds for the term T^-. Finally, it is enough to study the integral

$$I(\lambda)=\lambda^{-1/2}\int_{[0,s_1]} e^{i\lambda(s^3/3-i\lambda^{-2/3}sa)}s^{-1/2}ds.$$

In the right hand side, we replace a by $a_0+a_1 s$ using a change of variable which leaves the origin unchanged and which is analytic with respect to $\lambda^{-2/3}$; then we replace s_1 by $+\infty$. We remark that:

$$\int_0^\infty e^{iz\xi+i\xi^3/3}\xi^{-1/2}d\xi=\alpha\,\mathrm{Ai}(2^{-2/3}e^{-2i\pi/3}z)\,\mathrm{Ai}(2^{-2/3}z) \tag{4.20}$$

with $\alpha=3^{-5/6}e^{i\pi/12}\Gamma(1/6)(\mathrm{Ai}(0))^{-2}$, because the two sides of (4.20) satisfy the differential equation:

$$-f''+zf'+f/2=0, \tag{4.21}$$

whose solutions are products of Airy functions and the identification is completed by noticing that the left hand side of (4.20) is bounded for $\varepsilon<\operatorname{Arg} z<\pi-\varepsilon$. Therefore, up to a function with exponential decay in λ, $I(\lambda)$ is given by the series:

$$S(\lambda)=\lambda^{-2/3}\sum_{0\leqq k\leqq\infty}((a_1\lambda^{-1/3})^k/k!)\cdot(-i\partial_z)^{2k}[\alpha\,\mathrm{Ai}(2^{-2/3}e^{-2i\pi/3}z)$$
$$\times\,\mathrm{Ai}(2^{-2/3}z)]|_{z=-ia_0}.$$

The series converges because the derivatives of order $2k$ are bounded by $C^k(2k/3)!$ $\leqq(2k/3)!\cdot(k!)\leqq e^{k/3}k^{-k/3}$.

It remains to show that $\mathbb{T}^k f_2$ is microlocally Gevrey 3 with weight m', $m'(p_0)$ $>\operatorname{Re}\psi(p_0)$. The analytic singular support of f is, by hypothesis, contained in $\{t=0, x=y\}\cup\{\Lambda\cap(t>0)\}$, we deduce from (4.7) and (4.8) that we can assume that analytic spectrum of f_2 is arbitrarily close to p_0. The result is then a consequence of the following lemma applied to the distribution kernel $\mathbb{T}^j(t-s,x,y)\cdot\delta_{z=\omega}$.

Lemma 4.2. *Let $k(x,y)$ be a distribution kernel with the following properties:*

(i) *The analytic singular support $SS(k)$ satisfies the relation:*

$$\{SS(k)\cap\xi=0\}\cup\{SS(k)\cap\eta=0\}\subset\{\xi=\eta=0\}. \tag{4.22}$$

(ii) *For any $\varepsilon>0$ there exists a neighborhood V of (x_0,ξ_0) such that the relations $(x,\xi,y,\eta)\in SS(k)$ and $(y,-\eta)\in V$ imply*

$$|(x,\xi)-(y,-\eta)|\leqq\varepsilon.$$

Then for any $\alpha>0$, there exists a neighborhood W of (x_0,ξ_0) such that for any function with analytic singular support containted in W, and which is microlocally

Gevrey 3 with weight $m_0|\eta|^{1/3}$ *on W, the function:*

$$g(x)=\int k(x,y)f(y)dy$$

is microlocally Gevrey 3 at (x_0,ξ_0) *with weight* $m_0(1-\alpha)|\xi|^{1/3}$.

Proof. We just sketch the proof. Using (4.22), we can assume that $k(x,y)$ is the boundary value of a function h which is holomorphic in the region:

$$(\mathrm{Im}\,x-\mathrm{Im}\,y)\cdot\xi_0>\delta_0[|\mathrm{Im}\,x|+|\mathrm{Im}\,y|];\ |\mathrm{Im}\,x|,|\mathrm{Im}\,y|<\delta_1,$$

with δ_0 as small as we wish and, replacing k by $\Delta^{-N}k$, we may assume that k is bounded. Then we have:

$$\int e^{-\lambda(z-x)^2/2}k(x,y)dx=F(z,y,\lambda),$$

where F is a holomorphic function in z and y which satisfies the estimate:

$$|F(z,y,\lambda)|\leqq e^{\lambda(\mathrm{Im}\,z-\theta\,\mathrm{Im}\,y)^2/2},$$

provided z is close to $x_0-i\xi_0$, $\mathrm{Re}\,y$ is close to x_0, $\theta<1$ close to 1 and $\mathrm{Im}\,y\cdot\xi_0(1-\theta)\leqq-\delta_0|\mathrm{Im}\,y|$, $|\mathrm{Im}\,y|\leqq\delta_0/2$.

Then the ordinary F.B.I. transform of g is given by

$$\int\int e^{-\lambda(z-x)^2/2}k(x,y)f(y)dydx=\int F(z,y,\lambda)f(y)dy.$$

The demonstration is completed by expressing f using the inversion formula for the F.B.I. transform, and then deforming the contour in y. □

The next proposition describes the action of the operators $\mathbb{T}^k$ on the symbol of $f\in I^3_{\Lambda,\psi}$.

Proposition 4.2. *Let* $f=f_1+f_2\in I^3_{\Lambda,\psi}$ *with* f_2 *microlocally Gevrey 3 with weight larger than* $\mathrm{Re}\,\psi(p_0)$ *and with* f_1 *having symbol* a

$$f_1(t,x,z)=\int e^{i\varphi(t,x,z,\theta)-l(t,x,z,\theta)}a(t,x,z,\theta)d\theta$$

and let $b^k(t,x,z,\theta)$ *be the symbol of* $\mathbb{T}^k f$

$$(\mathbb{T}^k f)_1(t,x,z)=\int e^{i\varphi(t,x,z,\theta)-l(t,x,z,\theta)}b^k(t,x,z,\theta)d\theta.$$

Then, the symbol b^k *can be computed from the symbol* a *according to the formula*

$$b^k=\widehat{\mathbb{T}^k}a=\Theta^{2/3-4k/3}\sum_{n=0}^{\infty}\Theta^{-n/3}Q^k_n(a),$$

where

$$Q^k_n(t,x,z,\theta/|\theta|,D_t,D_x)=\sum_{|\beta|\leqq n}Q^k_{n,\beta}(t,x,z,\theta/|\theta|))\partial^\beta_{t,x}$$

is a differential operator of degree less than or equal to n, *and, the coefficient* $Q^k_{n,\beta}$ *are holomorphic functions satisfying the estimates:*

$$|Q^k_{n,\beta}|\leqq C^n(n-|\beta|)!. \tag{4.23}$$

Furthermore Q_0^1 is identically equal to zero and for the principal symbol denoted by σ, we have the relations

$$\sigma(\mathbb{T}^1 a)=C_1\Theta^{-1}\varrho^{-1}\{\tau-\Theta,\sigma(a)\},\qquad \sigma(\mathbb{T}^2 a)=C_2\Theta^{-2}\varrho^{-1}\sigma(a),\tag{4.24}$$

where C_1 and C_2 are universal constants in $\mathbb{C}\backslash 0$ and where $\Theta=\Theta(x,x,v_0)$.

Proof. We follow the idea of the proof of the Proposition 4.1 using coordinates as in (4.13). With $\mathbb{T}^k$ given by (4.12), we have:

$$H^k(x,y,\sigma,v)=2^{-k}(2\pi)^{-n}[\Theta(x,y,v)]^{2-2k}(2-\sigma)^{1/2-k}[(1+(\partial_y\cdot S)^2)/(1+(H(x,y))^2)]^{1/2}.\tag{4.25}$$

Let

$$f^k(t,x,z,\theta,\sigma,s)\equiv e^{l(s,y_c,z,\theta)}Q(H^k a\, e^{-l(s,y,z,\theta)})_{|y=y_c,\,v=v_c}$$

the function which appears in (4.16) and we have:

$$f^k=\sum_{0\leqq m<\infty}|\theta|^{-m/3}F_m(H^k a),$$

where F_m is a differential operator of degree $2m/3$. Since $\Theta(x,x,v)=|v|$ and $\det[\Theta''_{v,v}(x,x,v)]=0$, we have:

$$F_0=(2\pi)^{n-1}(1+O((t-s)^2)).\tag{4.26}$$

The coefficients $b^k(t,x,z,\theta)$ are obtained from the asymptotic expansion, with respect to $\Theta(x,x,v_0)$ of the expression

$$\iint_{\mathrm{Im}\,\sigma>0}\exp\{i\Theta[\sigma(s-t)-\varrho^2(s-t)^3/24+O((s-t)^4+\sigma(s-t)^2)]$$
$$+\Theta^{1/3}[(t-s)\varrho^{2/3}e^{i\pi/6}2^{-1/3}\omega+O((s-t)^4+\sigma(s-t)^2)]\}\sigma^{1/2-k}\tilde{f}^k(\sigma,s)d\sigma\,ds.$$

To obtain the asymptotic expansion, it is convenient to introduce the change of scale:

$$\sigma=\Theta^{-2/3}\beta,\qquad (t-s)=\Theta^{-1/3}\gamma.$$

The terms $O[(s-t)^4+\sigma(s-t)^2]$ of the phase then contribute only lower order terms to the symbol f^k and we denote $\tilde{f}^k$ the new symbol which is obtained in this way. We have to compute the principal term of:

$$I(\Theta)=\Theta^{2k/3-4/3}\iint_{\mathrm{Im}\,\beta>0}e^{i[\gamma^3\varrho^2/24+\gamma(-i\varrho^{2/3}e^{i\pi/6}2^{-1/3}\omega)-\beta\gamma]}$$
$$\times\beta^{1/2-k}\tilde{f}^k(\Theta^{-2/3}\beta,t-\Theta^{-1/3}\gamma)d\gamma\,d\beta.$$

We obtain the formal expansion

$$I(\Theta)=\Theta^{2k/3-4/3}\sum_{j_1,j_2}\Theta^{-(2j_1+j_2)/3}((-1)^{j_2}/(j_1!)(j_2!)$$
$$\times(\partial_\sigma^{j_1}\partial_s^{j_2}f^k(\sigma,s))_{\sigma=0,s=t}(2\varrho^{-2/3})^{j_2-j_1+k-1/2}\cdot C^k_{j_1j_2}(e^{-i\pi/3}2^{2/3}\omega).$$

Here $C^k_{j_1j_2}(z)$ denotes the entire function of z given by:

$$C^k_{j_1j_2}(z)=\int e^{i(\xi^3/3+z\xi)}\xi^{j_2}d\xi\int_{\mathrm{Im}\,x>0}e^{-ix\xi}x^{1/2-k+j_1}dx.$$

From the formula (4.27), we deduce that the Q_n^k are differential operators of degree at most n, because n is the maximum of $2m/3+j_1+j_2$ under the condition $2j_1+j_2+m=n$; it is obtained for $m=j_1=0$ and $j_2=n$. The estimations (4.23) are consequences of the Proposition 4.1 which implies that $\mathbb{T}^k$ is a continuous mapping of the space of regular symbols with respect to $\Theta^{1/3}$ to itself. The fact that Q_0^1 is identically zero is a consequence of the formula (4.20) which implies that one has:

$$C_{0,0}^1(z)=\text{Constant}\cdot \text{Ai}(e^{-2i\pi/3}2^{-2/3}z)\,\text{Ai}(2^{-2/3})z)$$

and with $z=2^{2/3}e^{-i\pi/3}\omega$ we have:

$$\text{Ai}(e^{-2i\pi/3}2^{-2/3}(z))=\text{Ai}(-\omega)=0\,.$$

The leading term of $\mathbb{T}^1 a$ is obtained for $j_1=0$ and $j_2=1$

$$C_{0,1}^1=\text{Constant}\,\partial_z(\text{Ai}(e^{-2i\pi/3}2^{-2/3}z)\,\text{Ai}(2^{-2/3}z))|_{z=2^{2/3}e^{-i\pi/3}\omega}\,.$$

Similarly, the leading term of $\mathbb{T}^2 a$ is obtained for $j_1=j_2=0$ and it is also of the form:

$$C_{0,0}^2=\text{Constant}\,\partial_z(\text{Ai}(e^{-2i\pi/3}2^{-2/3}z)\,\text{Ai}(2^{-2/3}z))|_{z=2^{2/3}e^{-i\pi/3}\omega}\,.$$

The relations (4.24) follow form (4.25), (4.26), and (4.27).

5. Proof that $k^+(t,x,y)\in I^3_{\Lambda,\psi}$

This section is devoted to proving that $k^+\in I^3_{\Lambda,\psi}$ in $t>0$ with Λ,ψ defined in Eqs. (4.9), (4.10). The proof has two major steps, first the result is obtained for small t (Proposition 5.2), using a parametrix construction of Lebeau [L.2]. The result is extended to all times using the methods of second microlocalization as in [L.2, §4], [L.2, §5], [S.K.K.] and [S.1] and a transport equation involving the operators $\mathbb{T}^k$ of the last section.

The parametrix is constructed using families $f(t,x,\lambda)$, $\lambda\geqq\lambda_0$ of functions holomorphic with respect to t,x.

Suppose $f:\omega\times(\lambda_0,\infty)$, ω a neighborhood in $\mathbb{C}_t\times\mathbb{C}_x$ of (t_0,x_0), is holomorphic in t,x and satisfies:

$$(\exists C>0)\qquad |f(t,x,\lambda)|\leqq e^{C\lambda},\qquad t,x,\lambda\in\omega\times[\lambda_0,\infty]\,, \tag{5.1}$$

$$(\exists N_0)\ (\forall\alpha)\ (\exists C_\alpha)\qquad |\partial_{t,x}^\alpha f(t,x,\lambda)|\leqq C_\alpha\lambda^{N_0+|\alpha|}\qquad t,x,\lambda\in(\omega\cap\mathbb{R}\times\Omega)\times[\lambda_0,\infty]\,. \tag{5.2}$$

(The exponential versus polynomial growth in these relations comes from the fact that in (5.1) x and t are complex while in (5.2) they are real.)

Definitions 5.1. *The asymptotic singular spectrum of $f(t,x,\lambda)$, denoted by $SS_\partial f$ is the closed set of $T^*(\mathbb{R}_t\times\partial\Omega)\cup T^*(\mathbb{R}_t\times\Omega)$ defined as follows: An interior point $p_0\in T^*(\mathbb{R}_t\times\Omega)$ does not belongs to SS_∂ i, there exists a F.B.I. transform:*

$$f\to\int e^{i\lambda g(z,t,x)}f(t,x,\lambda)dx\,,$$

with value in the space H_φ, related to the canonical transform:

$$\chi:(T^*(\mathbb{R}_t\times\Omega),p_0)\to(\Lambda_\varphi,z_0),$$

for which f satisfies in some complex neighborhood ω_0 of z_0, and with some strictly positive constant C_0, the following estimate:

$$\forall z\in\omega_0,\ \forall\lambda\geqq\lambda_0,\qquad |\int e^{i\lambda g(z,t,x)}f(t,x,\lambda)dt\,dx|\leqq Ce^{\lambda(\varphi(z)-C_0)}. \tag{5.3}$$

A boundary point $q_0\in T^*(\mathbb{R}_t\times\partial\Omega)$ does not belong to $SS_\partial f$ if in a system of C^ω local coordinates x',l with $\Omega=\{l>0\}$ there is a tangential F.B.I.

$$f\to\int e^{i\lambda g(z',t',x')}f(t,x',l,\lambda)dx'\,dt,$$

with values in $H_{\varphi'}$ related to the canonical transformation

$$\chi':(T^*(\mathbb{R}_t\times\partial\Omega),q_0)\to(\Lambda_{\varphi'},z_0')$$

and $C,C_0>0$ such that for all $z'\in\omega_0\subset\mathbb{C}^{n-1}$ a complex neighborhood of z_0', $0\leqq l<l_0$, and $\lambda>\lambda_0$

$$|\int e^{i\lambda g(z',t,x')}f(t,x',l,\lambda)dt\,dx'|\leqq Ce^{\lambda(\varphi'(z')-C_0)}. \tag{5.4}$$

We say that f is an asymptotic solution if there exists constants $C,C_0>0$ so that

$$|(\partial_t^2-\Delta_x)f(t,x,\omega)|\leqq Ce^{-C_0\lambda} \tag{5.5}$$

for all $t,x,\in\omega$, $\lambda\geqq\lambda_0$.

We denote by $q_0\in T^*(\mathbb{R}_t\times\partial\Omega)$ a point of the form $q_0=(t_0,x_0,-\tau_0,-\xi_0')$ with $\tau_0=|\xi_0'|$ and, for $q\in T^*(\mathbb{R}_t\times\partial\Omega)$ near q_0, we define the incoming flow $F^-(q)$. The definition is different in the elliptic ($|\tau|<|\xi'|$), hyperbolic ($|\tau|>|\xi'|$), and glancing ($|\tau|=|\xi'|$) sets.

If q is elliptic, then $F^-(q)=\{q\}$. $(5.6)_{\mathrm{i}}$

If q is hyperbolic or glancing, then $F^-(q)=\{q\}\bigcup\limits_{s\geqq0}\exp sH_{\tau^2-|\xi|^2}(\tilde q)$, $(5.6)_{\mathrm{ii}}$

where $\tilde q=(t_0,x_0,\tau,\xi',\xi_n)$ is the characteristic point projecting to q such that the bicharacteristic $\exp sH_{\tau^2-|\xi|^2}(q)$ lies over Ω for $s>0$. Notice that on bicharacteristics $dt/ds\simeq-2|\xi|<0$ so $s>0$ corresponds to the past.

Definition 5.2. *We say that $f(t,x,\lambda)$ is an incoming asymptotic solution if it is an asymptotic solution such that $f_0(t,x',\lambda)\equiv f(t,x',0,\lambda)$ satisfies*

$$SS(f_0)\subset\omega_0,\ \textit{a small neighborhood of } q_0 \textit{ in } T^*(\mathbb{R}_t\times\partial\Omega) \tag{5.7}$$

$$SS_\partial(f)\subset F^-\left[SS(f_0)\cup\left(\bigcup_{s>0}\exp sH_{\tau^2-|\xi'|^2}(SS(f_0)\cap G)\right)\right], \tag{5.8}$$

where $G\equiv$ the glancing set.

In Fig. 1, we sketch the projection on (t,x) of the singularities which correspond to the relation (5.8) and which come from the points (t_0,x_0,τ,ξ), for (τ,ξ) near

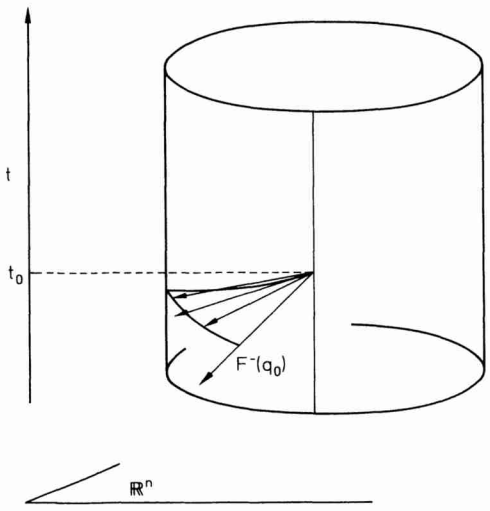

Fig. 1

$(\tau_0, \xi_0) \in G$. This set is the union of a part of the wave cone coming from (t_0, x_0) and of the rays which are glancing along the backward geodesic through x_0.

One can find (cf. [S2] a system of coordinates and a positive function $a(x)$ such that, near x_0, one has:

$$a^{-1} \circ \Delta_x \circ a = \partial_t^2 + B(x', l, D_{x'}), \tag{5.9}$$

where l denotes the variable normal to $\partial\Omega$ and where B is a second order elliptic operator with

$$\sigma(B)(x', 0, \xi') = -|\xi'|^2 . \tag{5.10}$$

Therefore, after a conjugation with the function a, the wave operator becomes:

$$P = a^{-1}(\Delta_x - \partial_t^2)a = (\partial_t^2) + ((B(x', l, D_x) - (\partial_t^2) \tag{5.11}$$

with symbol $p = -\xi_n^2 + (B(x', l, \xi') + \tau^2)$.

After this reduction, we make a tangential F.B.I. transform

$$Tf(z, l, \lambda) \equiv \int e^{i\lambda g_0(z,t,x')} f(t, x', l, \lambda) dx' dt \tag{5.12}$$

for z near z_0 in $\mathbb{C}^n$. The transform is related to a canonical transform

$$\chi_0 : T^*(\mathbb{R}_t \times \partial\Omega) \to (\mathbb{C}^n, -2i\bar{\partial}\partial\varphi_0),\ \varphi_0 \equiv 1/2(\mathrm{Im}\, z)^2,\ \chi_0(G) = \{z | \mathrm{Im}\, z_1 = 0\} .$$

As in [L.2], the operator P is transformed to

$$P = \partial_t^2 + \lambda^2 R(l, z, \xi, \lambda), \tag{5.13}$$

where R is a classical C^ω symbol of degree zero,

$$R \sim \sum_{j \geqq 0} R_j(l, z, \lambda) \lambda^{-j} .$$

The characteristic variety, $0=\sigma(P)=R_0-\tilde{l}^2$ ($\tilde{l}$ dual to l), is given by

$$l=-q(z,\xi,\tilde{l}^2)=-\xi_1'+e(z,\xi)\tilde{l}^2+O(\tilde{l}^4) \tag{5.14}$$

with $e(0,0)>0$.

Incoming asymptotic solutions to the wave equation are constructed in two steps, following [L.2, Prop. 3.3 and 4.2.3.]. First incoming asymptotic solutions of $P=0$ are constructed, then an inverse tangential F.B.I. is applied. The next result lifted from [L.2] accomplishes the first step. Note that $T^*(\mathbb{R}_t\times\partial\Omega)$ and $\mathbb{C}_z^n$ are identified using χ_0.

Proposition 5.1. *There exists a linear operator K which maps the set of function f_0 satisfying the relations $|f_0(z,\lambda)|\leqq e^{\lambda\varphi_0(z)}$, and, for $|z|>\delta/2$, $\delta>0$ small $|f_0(z,\lambda)|\leqq e^{\lambda(\varphi_0(z)-C)}$, $C>0$. To the set or incoming asymptotic solutions of $P=0$ with the following properties:*

$$(Kf)(0,z,\lambda)=f_0(z,\lambda)+O(e^{-C\lambda}),$$

and

$$\partial_l(Kf)(0,z,\lambda)=(\lambda/2\pi)^{n-1}\int dy'\wedge d\xi'$$
$$\times\int_{-2\delta/3}^{2\delta/3} e^{i\lambda(z'-y')\cdot\xi'-i\lambda^{1/3}(H(z,\xi')-H(y,\xi'))}k(z_1,z',y_1,\xi',\lambda)\,f(y,\lambda)\,dy_1+O(e^{-c\lambda}).$$

for z near $(z_1^0,0)$, $z_1^0\in\mathbb{R}$, $z_1^0\geqq r_0\geqq 2\delta$. The holomorphic function $H(z,\xi')$ satisfies the relation

$$\partial_{z_1}H(z,\xi')=\omega\,e^{-i\pi/3}(e(z,\xi_1=0,\xi'))^{1/3},$$

where $e(z,\xi)$ is given by (5.14). *k is an analytic symbol in a small neighborhood W (independent of δ and r_0) of $z_1=0$, $y_1=0$, $z'=0$, $\xi'=0$. In addition, there exists a family k^j of holomorphic functions defined in W which satisfy the following estimate:*

$$|k^j|\leqq AB^j j!,\qquad k_0|_{x_1=y_1}\equiv i,$$
$$\left|k(z_1,z',y_1,\xi',\lambda)-\sum_{0\leqq l\leqq j}k^l(z_1,z',y_1,\xi')\lambda^{1-l/3}\right|\leqq A(r_0)B(r_0)^j j!\,\lambda^{1-j/3}.$$

Remark 5.1. Near $q_0=(t_0,x_0',-\tau_0,-\xi_0')$, the incoming asymptotic solutions propagate singularities in the direction of negative times; they coincide with the outgoing solutions in the sense of [L.2]. Indeed, with P given by (5.13), they propagate the singularities to the right along the bicharacteristic. We have used the minus sign in front of τ_0 to recover the definition given in [L.2].

There exists an analytic, elliptic, classical symbol, σ_0, of degree zero such that the F.B.I.

$$f\to T^{-1}f=\lambda^n\int e^{-i\lambda g_0(z,t,x')}\sigma_0(t,x,z,\lambda)f(l,z,\lambda)dz$$

is a formal inverse of the F.B.I. defined by (5.12).

For $(z,-2i\partial_z\varphi_0(z))\in\chi(T^*_{(t,x)}(\mathbb{R}_t\times\partial\Omega))$, the function $z\to\operatorname{Im}g_0(z,t,x')+\varphi_0(z)$ has its critical points transversally non degenerate, and the critical value is zero with signature $(n^+,0^-)$. Then if $\tilde{f}(l,z,\lambda)$ is an incoming asymptotic solution with

$SS_\partial f(0,z,\lambda)\subset|z-z_0|<\varepsilon_0/4$, the function given by

$$f(t,x,\lambda)=a(x)\lambda^n\int\limits_{z\in\Sigma_0} e^{-i\lambda g_0(z,t,x')}\sigma_0\tilde{f}(l,z,\lambda)dz \tag{5.16}$$

with

$$\Sigma_0\equiv\{z|(z,-2i\partial_z\varphi_0(z))=\chi(t,x;\tau,\xi),|(\tau,\xi)-(-\tau_0,-\xi_0)|\leqq\varepsilon_0\}$$

is an incoming asymptotic solution of the wave equation, for

$$|(t,x)-(t_0,x_0)|\leqq\varepsilon_1\,,\qquad \varepsilon_1\ll\varepsilon\,.$$

Let $q_0^{(a)}\equiv(t_0,x_0';\tau_0,\xi_0)$ be the antipodal of q_0 and let

$$f\to T_1f\equiv\int e^{i\lambda g_1(z,t,x')}f(t,x',\lambda)dx'dt$$

be a F.B.I. transform related to the canonical transform

$$\chi_1:(T^*(\mathbb{R}_t\times\partial\Omega);q_0^{(a)})\to(\mathbb{C}^n,(2/i)\partial\partial\varphi_1)\,.$$

For z near z_1 (considered as a parameter), and (t,x) near (t_0,x_0) we denote by $f(t,x;z,\lambda)$ the incoming asymptotic solution, described above, which satisfies the boundary condition:

$$f(t,0,x';z,\lambda)=a(0,x')e^{i\lambda g_1(z,t,x')-\lambda\varphi_1(z)}+O(e^{-C\lambda})\,. \tag{5.17}$$

For $|z-z_1|\ll\varepsilon_2$, f decays exponentially with respect to λ for $t\geqq t_0+\varepsilon_2$. The kernel $K^+(t-s,x,y)$ from (2.9) has its support contained in the cone $|x-y|\leqq|t-s|$, therefore, if $|t_0-s|$ and $|x_0-y|$ are small enough, we have

$$O(e^{-C\lambda})=\int\limits_\Omega dx\int\limits_{-\infty}^{t_0+\varepsilon_2}dt[(\partial_t^2-\Delta_x)f(t,x,z,\lambda)]K^+(t-s,x,y)\,.$$

Integrating by parts in the right hand side yields

$$\int\limits_{\partial\Omega}dx'\int dt\,a(0,x')e^{i\lambda g_1(z,t,x')-\lambda\varphi_1(z)}k^+(t-s,x',y')=\partial_n f(s,y',z,\lambda)+O(e^{-C\lambda})\,. \tag{5.18}$$

This duality argument leads to

Lemma 5.1. *The analytic singular spectrum of $k^+(t,x',y')$ is, for $t>0$, contained in Λ.*

Proof. Since k^+ is real, it is enough to prove the relation: $\{SS(k^+\cap t>0\cap\tau>0\}\subset\Lambda^+$. For t small, one applies to both sides of the formula (5.18), a F.B.I. transform with respect to the variables (s,y'), and uses the result of Sjöstrand [S.2] describing the propagation of analytic singularities. □

Proposition 5.2. *For $t>0$ small enough, the kernel k^+ belongs to the class $I^3_{\Lambda,\Psi}$.*

Proof. It suffices to consider Λ^+. For $s<t_0$ and $|z-z_1|\ll t_0-s$, we deduce from the formulas (5.15) and (5.17) that for $(w,-2i\partial_w\varphi_0(w))$ near $\chi_0(\exp((t_0-s)H_{\tau-|\xi'|}(q_0))$,

one has

$$\int (e^{i\lambda g_0(w,s,y')}(a(0,y'))^{-1}\partial_n f(s,y',z,\lambda)ds\,dy'$$
$$=(\lambda/2\pi)^{n-1}\int du'\wedge d\xi'\int du_1\, e^{i\lambda(w'-u')\xi'-i\lambda^{1/3}[H(w,\xi')-H(u,\xi')]}$$
$$\times k(w_1,w',u_1,\xi',\lambda)\int e^{i\lambda g_0(u,t,x')+i\lambda g_1(z,t,x')-\lambda\varphi_1(z)}dt\,dx'. \tag{5.19}$$

The phase g_0+g_1 which appears in the right hand side defines an F.B.I. transform related to the canonical transform.

$$\chi_1\otimes\chi_0: T^*(\mathbb{R}_t\times\partial\Omega_{x'})\times T^*(\mathbb{R}_t\times\partial\Omega_{y'})\to\Lambda_{\varphi_1}\times\Lambda_{\varphi_0}\subset T^*(\mathbb{C}^n_z)\times T^*(\mathbb{C}^n_z).$$

Therefore we can use the results of the Sect. 3. First notice that the phase

$$(u',\xi';u_1,t,x')\to(w'-u')\cdot\xi'+g_0(u,t,x')+g_1(z,t,x')$$

has, for each (z,w), a non degenerate critical point. The critical value $\Phi(z,w)$ is the p.s.h. function which is canonically related to the Lagrangian manifold:

$$\chi_1\otimes\chi_0[\{(t,x',\tau,\xi')=\exp lH_{\tau-|\xi'|}(s,y',-\sigma,-\eta');\tau=|\xi'|,\sigma=-|\eta'|\}]. \tag{5.20}$$

This follows from the fact that the phase

$$(t,x')\to g_0(u,t,x')+g_1(z,t,x')$$

has a non degenerate critical point, and the critical value $G(u,z)$ satisfies

$$\{(z,u;G'_z,G'_u)\}=(\chi_1\otimes\chi_0)(\Lambda_{Id}). \tag{5.21}$$

Here Λ_{Id} is the Lagrangian manifold $t=s$, $x'=y'$, $\tau+\sigma=0$, $\xi'+\eta'=0$. This Lagrangian intersects transversally the manifold $\tau=|\xi'|$.

The weight function ψ, homogeneous of degree 1/3, is computed from the right hand side of (5.19) as is done in Theorem 1 of [L.2]. That is, ψ is obtained upon integrating the 1-form $e^{i\pi/6}2^{-1/3}\omega J$ along the bicharacteristics following G. Here J satisfies:

$$\langle J,H_{|\xi|-\tau^2}\rangle=\{\{x_n-S(x'),\xi^2-\tau^2\},\xi^2-\tau^2\}^{2/3}/\{\{x_n-S(x'),\xi^2-\tau^2\},x_n-S(x')\}^{1/3}.$$

That this yields the ψ defined in (4.10) is verified by performing the above computation in coordinates satisfying (4.13). This completes the proof Proposition 5.1. □

To extend this result to all positive times, we will use the notion of partial Gevrey 3 regularity on a C^ω involutive submanifold $V\subset T^*M\backslash 0$. Toward this end suppose that $V\subset T^*\mathbb{R}^N\backslash 0$ is an involutive manifold with its natural Hamilton foliation. In our example, V will be

$$V^+=[(t,x,y,\tau,\xi,\eta)\in T^*(\mathbb{R}_t\times\partial\Omega\times\partial\Omega)\backslash 0|\tau=|\xi|\},$$

and the leaves of the foliation are the integral curves of $H_{\tau-|\xi|}$. Let $\tilde{V}$ be the manifold obtained upon complexifying the leaves. Suppose $m=\tilde{V}\to R$ is pluri-harmonic on the leaves of $\tilde{V}$ and homogeneous of degree 1/3. In our example, m will

be the function $\mathrm{Re}\psi: \tilde{V}^+ \to R$ defined by (4.10). Note that $V^+ \supset \Lambda^+$ and the restriction of ψ to Λ is the weight defining $I^3_{\Lambda,\psi}$.

Definition 5.3. *Suppose that V and m are as above and that T is an F.B.I. transform associated with $M, \chi, \varphi, dx, p_0 \in T^*M$ and let φ_V be the p.s.h. defined by $\chi(\tilde{V})$ cf.* [L.1, § II.5]. *Then $f \in D(M)$ is partially Gevrey 3 on V with weight m at p_0 if for z in a neighborhood of $z_0 = \pi \circ \chi(p_0)$ and $\lambda > \lambda_0$, $T_\chi f$ satisfies:*

$$(\forall \varepsilon > 0)(\exists C_\varepsilon) \qquad |Tf(z,\lambda)| \leqq C_\varepsilon e^{\lambda \varphi_V(z) - \lambda^{1/3} m(z) + \varepsilon \lambda^{1/3}},$$

where m is defined by $\hat{m} \circ \pi_z \circ \chi^{\mathbb{C}} = m$.

We write f is V.m.p.G^3 at p_0. To show that $k \in I^3_{\Lambda,\psi}$ for all $t > 0$, we must show that k is in this space microlocally at all points

$$p(t; y', \eta') \equiv (t, x'(t), y'; \tau, \xi'(t), \eta'), (x'(t), \xi'(t)) = \exp -tH_{|\xi'|}(y', -\eta'), \tag{5.22}$$

where $y', \eta' \in T^*(\partial\Omega)$ and $t > 0$ is arbitrary. It is sufficient to show that if $k \in I^3_{\Lambda,\psi}$ microlocally at all such points for $0 < t < t_0$ then the same is true for $t = t_0$. The first step is partial Gevrey 3 regularity.

Lemma 5.2. *If for all $0 < t < t_0$, and $y', \eta' \in T^*(\partial\Omega)$, we have $k \in I^3_{\Lambda^+,\psi}$ at $p(t; y', \eta')$ then for all $(y', \eta') \in T^*(\partial\Omega)$, k is partially Gevrey 3 on V with weight* $\mathrm{Re}\psi$ *at $p(t_0, y', \eta')$.*

Proof. Choose $\theta \in G^{3/2}(\mathbb{R})$ with $\theta = 0$ on $[-\infty, -1]$ and $\theta = 1$ on $[1, \infty]$. For $q_0^a \equiv (t_0, x_0'; \tau_0, \xi_0') \in T^*(\mathbb{R}_t \times \partial\Omega)$ with $(x_0', \xi_0') = \exp -tH_{|\xi_0|}(y_0', -\eta_0')$ let $f(t, x, z, \lambda)$ be the incoming asymptotic solution (5.16) with boundary values given by (5.17). For δ small and $0 < \varepsilon \ll \delta$ define $w_\varepsilon(t, x, y')$, $t, x, y' \in \mathbb{R}_t \times \mathbb{R}^n \times \partial\Omega$, by

$$(\partial_t^2 - \Delta_x) w_\varepsilon = \theta((t - t_0 + \delta)/\varepsilon) k^+(t, x', y') \otimes \delta_{\partial\Omega}$$

$w_\varepsilon = 0$ for $t < t_0 - \delta - \varepsilon$.

The asymptotic incoming solution $f(t, x, z, \lambda)$ decays exponentially in λ away from the projection on $\mathbb{R}_t \times \mathbb{R}^n_x$ of a small neighborhood of $F^-\Big[(q_0) \cup \Big(\bigcup_{s \leqq 0} \exp sH_{\tau^2 - |\xi|^2}(q_0)\Big)\Big]$. Therefore, an integration by parts in a small neighborhood of x_0 in Ω, yields (cf. formula (5.18)):

$$\begin{aligned} &\iint dx\,dt\, a(0, x') e^{i\lambda g_1(z,t,x') - \lambda \varphi_1(z)} \partial_{n_x} w_\varepsilon(t, x, y')|_{x \in \partial\Omega} \\ &= \iint \partial_{n_x} f(t, x; z, \lambda) w_\varepsilon(t, x, y')|_{x \in \partial\Omega} dt\, dx' + O(e^{-C\lambda}). \end{aligned} \tag{5.23}$$

Since for $t > t_0 - \delta + \varepsilon$ or $x \in \Omega$

$$(\partial_t^2 - \Delta_x)[K_+(t, x, y') - w_\varepsilon(t, x, y')] \equiv 0,$$

and K^+ and w_ε have no singularities for $t < 0$, we deduce from the strict convexity of $\partial\Omega$ that, for $t > t_0 - \delta + \varepsilon$, we have

$$\begin{aligned} p(t; y_0', \eta_0') &\notin SS[(K^+ - w_\varepsilon)|_{x \in \partial\Omega^+}] \\ p(t; y_0', \eta_0') &\notin SS[\partial_{n_x}(K^+ - w_\varepsilon)|_{x \in \partial\Omega^+}]. \end{aligned} \tag{5.24}$$

Therefore in the left hand side of (5.23), we can replace $\partial_n w_\varepsilon$ by $k^+(t, x', y')$. Since

$$w_\varepsilon|_{x\in\partial\Omega} = \mathbb{T}^1(\theta_\varepsilon k^+),$$

the relation

$$k^+ \in I^3_{\Lambda^+,\psi} \quad \text{for} \quad t < t_0$$

implies that $w_\varepsilon|_{x\in\partial\Omega}$ is microlocally Gevrey 3 with weight $\psi + o(1)$ near $p(t_0-\delta_0, y'_0, \eta'_0)$, with $o(1)$ going to zero with ε (use Lemma 4.2). From (5.24), we deduce that w_ε is microlocally zero near the points $p(t; y'_0, \eta'_0)$ if $t > t_0 - \delta_0 + \varepsilon$. To conclude the proof, we estimate $\partial_n f(t, x', z, \lambda)$ for z near $z_1 = \pi \circ \chi_1(q_0^{(a)})$, t near $t_0 - \delta_0$ and x' near $\chi'(t_0 - \delta_0; y'_0, -\eta'_0)$. Using (5.19) and (5.20), we obtain:

$$e^{i\lambda\varphi_1(z)}\partial_n f(s, y', z, \lambda) = e^{i\lambda\Phi(s,y',z) - \lambda^{1/3}\psi(s,y',z)}\sigma(s, y', z, \lambda).$$

In the right hand side, $\Phi(s, y', z)$ denotes the holomorphic phase which describes the lagrangian manifold

$$(\chi_1 \otimes \mathrm{Id})[(t, x', \tau, \xi') = \exp lH_{\tau-|\xi'|}(s, y', -\sigma, -\eta'); \quad \tau = |\xi'|, \sigma = -|\eta'|].$$

ψ is given by integrating the one-form $e^{i\pi/6}2^{-1/3}\,J$ from α to β along a leaf of $G^{\mathbb{C}}$ where α and β are given by

$$\alpha = (z, \Phi'_z(s, y', z)), \qquad \beta = (s, y', -\Phi'_{s,y}(s, y', z))$$

with an implicit identification by the isomorphism $\chi_1 \otimes \mathrm{Id}$. The symbol $\sigma(s, y', z, \lambda)$ is C^ω and classical with respect to $\lambda^{1/3}$. To complete the proof, one estimates the right hand side of (5.23). This is done using the fact that w_ε is microlocally Gevrey 3 with weight $\mathrm{Re}\,\psi + o(1)$ near the point $p(t_0 - \delta; y_0, \eta_0)$ and the inversion formula for the F.B.I. transform (cf. Fig. 2). □

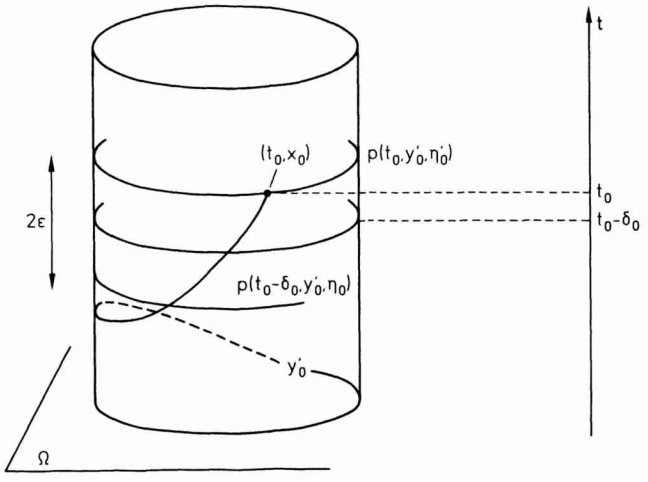

Fig. 2

Lemma 5.3. *With notations from Lemma 5.2, we suppose that f is V^+.m.p.G^3 on V^+ and microlocally Gevrey 3 with weight m' at p_0 with $m'(p_0)>m(p_0)$. Let $F_0 \subset V^+$ be the bicharacteristic through p_0. Then there is a weight $m''>m$ such that f is microlocally Gevrey 3 on F_0 with weight m''.*

Proof. Since the Definition 5.2 does not depend on the F.B.I. transform choosen, we can work in the standard setting:

$$z=(z',z''), \varphi=(\mathrm{Im}\, z)^2/2, \qquad V^+=\{\mathrm{Im}\, z'=0\}, \varphi_V=(\mathrm{Im}\, z'')^2/2,$$
$$F_0=\{(z',0)|\mathrm{Im}\, z'=0\}.$$

We consider a real C^ω weight function $m(z',z'')$ pluriharmonic with respect to z'.

Let $f(z;\lambda)$ be a holomorphic function which satisfies the two estimates

$$\forall \varepsilon>0, \exists C_\varepsilon, |f(z,\lambda)| \leqq C_\varepsilon e^{\lambda(\mathrm{Im}\, z'')^2/2-\lambda^{1/3}m(z',z'')+\varepsilon\lambda^{1/3}}.$$

For

$$|z| \leqq \varrho_0, |f(z,\lambda)| \leqq e^{\lambda(\mathrm{Im}\, z)^2/2-C_0\lambda^{1/3}}, \quad \text{with} \quad C_0>m(0,0).$$

The maximum principle implies the existence of two positive constants $\varrho_1>0$ and $C_1>m(0,0)$, such that:

$$|f(z,\lambda)| \leqq e^{\lambda(\mathrm{Im}\, z'')^2/2-C_1\lambda^{1/3}} \ \forall z, |z| \leqq \varrho_1.$$

Let $G(z',z'')$ be a function holomorphic with respect to z' which satisfies $\mathrm{Re}\, G=m$. Let F be the function

$$F(z',z'',\lambda)=f(z',z'';\lambda)\, e^{\lambda^{1/3}G(z',z'')}.$$

Then,

$$\forall \varepsilon>0, \exists C_\varepsilon, \qquad |F| \leqq C_\varepsilon e^{\lambda(\mathrm{Im}\, z'')^2/2+\varepsilon\lambda^{1/3}}, \tag{5.25}$$

and

$$|F| \leqq e^{\lambda(\mathrm{Im}\, z'')^2/2-C_2\lambda^{1/3}} \quad \text{with} \quad C_2>0 \quad \text{for} \quad |z|<\varrho_2. \tag{5.26}$$

Since F is holomorphic with respect to the variable z' a second application of the maximum principle shows that (5.25) and (5.26) suffice to complete the proof of the Lemma 4.3. □

The next proposition passing from local to global in time is a major step in our proof.

Proposition 5.3. *For any positive t the kernel k^+ belongs to the class $I^3_{\Lambda^+,\psi}$.*

Proof. We have to show that k^+ belongs to the class $I^3_{\Lambda^+,\psi}$ near the points $p(t_0;y',\eta')$, $t>0$, $(y',\eta') \in T^*\partial\Omega$.

Near $p(t;y_0,\eta_0)$, $t<t_0$, we have $k^+=k^{+,1}+k^{+,2}$ where $k^{+,2}$ is microlocally Gevrey 3 with weight $m>\mathrm{Re}\,\psi$ and $k^{+,1}$ has a representation as in (3.6). From the definitions of k^+, $\mathbb{T}^1$, we see that k^+ satisfies the hypothesis of the Proposition 4.1 and that $\mathbb{T}^1(k^+)=0$ microlocally near Λ for $t>0$ so if $\hat{k}^+$ is the symbol of $k^{1,+}$ we have

$$\hat{\mathbb{T}}^1(\hat{k}^+) \equiv 0.$$

From Proposition 4.2, we deduce that the formal operator $\hat{\mathbb{T}}^1$ has simple characteristics on V^+ (it is analytic-classical with large parameter $\lambda^{1/3}$ instead of λ), therefore $\hat{k}^+$ can be extended to a neighborhood of t_0 and therefore we may assume that $k^{+,1}$ is also defined in a neighborhood of t_0. By Lemma 5.2, k^+ is V$^+$.m.p.G^3 near $p(t_0; y'_0, \eta'_0)$. Since $k^{+,1}$ belongs to $I^3_{\Lambda^+,\psi}$, it is also V.m.p.G^3 near the same point. Therefore, the same property holds for $r=k-k^{+,1}$.

Applying Lemma 5.3 to r completes the proof of Proposition 5.3. □

Remark 5.3. Except for the formula (2.13), where they are used to reduce the problem to $\partial\Omega$, the proof of the Proposition 5.3 is the only place where the operators $\mathbb{T}^i$ play an essential role.

6. Proof of the Theorem 3

We recall that $\mathrm{Tr}(t)$ is for $t>0$ a C^∞ function, given by the formulas (2.12), (2.13):

$$-\mathrm{Tr}(t)=2(\partial/\partial t)\int B(t,x,x)dx$$

with $B(t,x,y)$ given by

$$B(t,x,y)=\int_{-\infty}^{\infty} ds \int_{\partial\Omega} \mathbb{T}^2(t-s,x,z)k^+(s,z,y)dz\,.$$

Since we have proven, in Proposition 5.2, that k^+ belongs to $I^3_{\Lambda,\psi}$ in $t>0$, we deduce from Proposition 4.1 that in $t>0$, $|x-y|<t$

$$SS(B)\subset\Lambda \tag{6.1}$$

and

$$B\in I^3_{\Lambda,\psi}\,. \tag{6.2}$$

Choose ε so that $\{|t-T|<\varepsilon\}$ contains only T from the length spectrum of $\partial\Omega$. We deduce from (4.2), and part (i) of Theorem 3 that in a neighborhood of $t=T$ we that we have.

$$SS(\mathrm{Tr}(t))\subset\{(t,\tau),\exists(x,\xi),(t,\tau,x,\xi,x,-\xi)\in\Lambda\}$$

and this proves part (ii).

Now suppose that γ is a closed geodesic of length T_0 which satisfies the hypothesis of part (iii). We use the arc length s as a parameter on γ. The lift of γ in $T^*(\partial\Omega)$ is the union of the two bicharacteristics curves

$$\Gamma^\pm=\{x=\gamma(s),\xi=\pm k(d\gamma/ds),k>0\}\,.$$

Proposition 6.1. *For t near T_0, one has $\mathrm{Tr}(t)\in I^3_{\Lambda_0,\psi_0}$ with*

$$\Lambda_0=T^*_{T_0}(\mathbb{R});\qquad \psi_0(T_0,\tau)=|\tau|^{1/3}\,e^{i\pi\,\mathrm{sign}(\tau)/6}2^{-1/3}\omega\int_0^{T_0}\varrho^{2/3}(s)ds$$

in addition, microlocally near $(T_0, +1)$, *one has*

$$\mathrm{Tr}(t) = \int_1^\infty e^{i(t-T_0)-\psi_0(T_0,\tau)} a(z)dz + r(t), \tag{6.3}$$

where a *is an analytic symbol in* $\lambda^{-1/3}$ *of degree zero and* $r(t)$ *is microlocally Gevrey 3 with weight strictly greater than* $\mathrm{Re}\psi_0(T_0, z)$ *near* $(T_0, +1)$.

Proof. Since $\mathrm{Tr}(t)$ is real, it is enough to consider the case $\tau > 0$. Furthermore, for any $(x, \xi) \in \Gamma^{\pm}$, one has $\psi(T, x, \xi) = \psi_0(T, |\xi|)$. Since γ is non degenerate and since T_0 is isolated, the fixed points of the mapping

$$\exp - T_0 H_{|\xi|} : T^*\partial\Omega \backslash 0 \to T^*\partial\Omega \backslash 0$$

are the points of $\Gamma^{\pm}$ and these fixed points are clean in the sense of [D.G.]. Therefore the principal part of the right hand side of (2.12) can be computed, for (t, τ) near $(T, 1)$, by the classical method of [D.G.], [C.], provided the computation of (7.5) in [D.G.] is replaced by the following lemma.

Lemma 6.1. *Let* $f(s, x)$ *be an analytic function defined for* $s \in S^1$ *and* $x \in \mathbb{R}^p$, $|x| < r$ *and let* $l(s, x)$ *be an analytic function which satisfies*

$$l(s, 0) = C^{te} \equiv l_0 .$$

Suppose that f *satisfies*

$$\mathrm{Re} f(s, x) > 0 \quad \text{for} \quad |x| \neq 0, f(s, 0) \equiv 0, \nabla f(s, 0) \equiv 0 . \tag{6.4}$$

$$f \text{ is transversally non degenerate on } S^1 \times \{0\} . \tag{6.5}$$

Then, for any analytic symbol, $a(s, x, \lambda)$, *classical in* $\lambda^{1/3}$ *and of degree* d, *the integral*

$$I_a(\lambda) \equiv \int_{S^1 \times \{|x| \leq r\}} e^{-\lambda f(s,x) - \lambda^{1/3} l(s,x)} a(s, x, \lambda) ds dx ,$$

is equal to $e^{-\lambda^{1/3} l_0} b(\lambda)$ *where* $b(\lambda)$ *is an analytic symbol, classical in* $\lambda^{1/3}$, *of degree* $d - p/2$. *Furthermore, the principal symbol is given by*

$$b^0(\lambda) = (2\pi)^{p/2} \int_{S^1} \theta(s) a^0(s, 0) ds, \theta(s) \equiv (\det f''_{xx}(s, 0))^{-1/2} ,$$

where the square root is chosen to converge to 1 *when* u *goes to zero in the homotopy*

$$u \to (1-u)|d + u f''_{xx} .$$

The proof of this lemma, using the Morse lemma and the method of steepest descent is ommitted. Proposition 6.1 is a straightforward consequence. □

Proposition 6.2. *The principal part* a_0 *of the symbol* $a(\tau)$ *in* (6.3) *is not equal to zero.*

Proof. The principal symbol a_0 is the sum of two terms $a_0 = a_0^+ + a_0^-$, where $a_0^{\pm}$ corresponds to the contribution of $\Gamma^{\pm}$. We know that for $t > 0$ the kernel $k^+(t, x, y)$ belongs to $I^3_{\Lambda, \psi}$. We denote by $k^{+,0}(t, x, \xi)$ the leading term of its symbol (which is of

degree 1), when it is written in the form (4.11).

$$k^+(t,x,y) \simeq \int e^{i\varphi(t,x,y,\xi)-l(t,x,y,\xi)} \hat{k}^+(t,x,\xi)d\xi,$$

with phase φ defined by

$$\partial_t \varphi = \Theta(x, x, \partial\varphi/\partial x), \qquad \varphi(0,x,y) = (x-y)\cdot\xi.$$

From Proposition 5.1, we deduce that the mapping $t \to k^{+,0}(t,x,\xi)$ can be extended to $t=0$ and that we have $k^{+,0}(0,x,\xi) \in i\mathbb{R}_+^*$.

Since we have, for $t>0$, $\hat{\mathbb{I}}^1 k = 0$, Proposition 4.2 (and in particular the formula (4.24)) implies that $k^{+,0}$ is constant along the geodesic flow. Finally, using formula (4.24), $B = \mathbb{T}^2 \cdot k$ and the method of [D.G.] for the computation of the principal symbol of the function $B = \int B(t,x,x)dx$, yields $a_0^{\pm} = \alpha^{\pm} i^{\sigma_\pm}$, where $\alpha^{\pm}$ denote two non zero complex numbers with the same argument while $\sigma_\pm$ are the Maslov indices related to the curves $\Gamma^{\pm}$. The proof is therefore a consequence of the equality $\sigma^+ = \sigma^-$ which is proved in the Appendix 1. □

We are now ready to prove the final assertion (2.5) of Theorem 3. If (2.5) were not true there would exist $\varepsilon>0$ and $\eta>0$ such that one has both:

$$\overline{\lim_{n\to\infty}} \sup_{|t-T|<\varepsilon} \left| \frac{1}{3n!} \frac{d^n \mathrm{Tr}(t)}{dt^n} \right|^{1/3n} \leqq (C_\gamma T + \varepsilon)^{-1}$$

and

$$\overline{\lim_{n\to\infty}} \sup_{|t-T|<\varepsilon} \left| \frac{1}{3n!} \frac{d^n r(t)}{dt^n} \right|^{1/3n} \leqq (C_\gamma T + \varepsilon)^{-1}.$$

Choose $\chi \in \mathrm{Gevrey}^2(\mathbb{R})$, $\mathrm{Supp}\,\chi \subset \{|t-T_0|<\varepsilon\}$, $\chi \equiv 1$ near T_0. Then $f(t) \equiv \chi(t)\mathrm{Tr}(t)$ satisfies

$$\overline{\lim_{n\to\infty}} \sup_{\mathbb{R}} \left| \frac{1}{3n!} \frac{d^n f(t)}{dt^n} \right|^{1/3n} \leqq (C_\gamma T + \varepsilon)^{-1}. \tag{6.6}$$

In addition, for $z \in \mathbb{C}$ near $T_0 - i$, one has with a $C>0$,

$$\tilde{f}(z,\lambda) \equiv \int e^{-\lambda(z-t)^2/2} f(t)dt = \int_{]T_0-\varepsilon, T_0+\varepsilon[} e^{-\lambda(z-t)^2/2} \mathrm{Tr}(t)dt + O(e^{-C\lambda}).$$

The integral defining $\tilde{f}$ is equal to

$$(2\pi\lambda)^{-1/2} \int e^{i\tau z - \tau^2/2\lambda} \hat{f}(\tau)d\tau$$

and using (6.6),

$$|\hat{f}(\tau)| \leqq A((3n)!)/(\tau^n (C_0 T_0 + \eta)^{3n})$$

which in turn implies

$$|\tilde{f}(z_0,\lambda) e^{-\lambda/2}| \leqq A'' e^{-\beta\lambda^{1/3}}, \qquad \beta > C_0 T_0. \tag{6.7}$$

On the other hand, from (6.3), one has as $\lambda \to \infty$

$$e^{-\lambda/2} \tilde{f}(z, \lambda) e^{\lambda^{1/3} \psi_0(T_0, l)} \sim a_0 \neq 0. \tag{6.8}$$

Since $\mathrm{Re}\psi_0(T_0, l) = C_0 T_0$, the contradiction between (6.7) and (6.8) completes the proof. □

Appendix 1

Proposition 1. *Let Γ be a closed geodesic on $M = \partial\Omega$ and let*

$$\Gamma^{\pm} = \{x = \gamma(s), \xi = \pm d\gamma/ds\} \subset T^*(M)$$

be the two bicharacteristics projecting on Γ, parametrised by the arclength. Then the Maslov indices σ_+ and σ_- related to the curves Γ^+ and Γ^- coincide.

Proof. Starting from $x_0 = \gamma(0) = \gamma(T)$, we define the phase function: $\varphi^+(x, y, \xi, t)$ by the eiconal equation:

$$\partial_t \varphi = |\varphi'_x| \tag{A.1}$$

and the initial condition:

$$\varphi(x, y, \xi, 0) = (x - y) \cdot \xi. \tag{A.2}$$

The Eq. (A.1) is defined in the complementary of the focal points $(x_1, x_2, \ldots, x_N)$ which are of finite number. At each focal point x_p, $\nu_p \pi/2$ is added to the phase φ. Since the Maslov index is an integer, it is enough to prove the result for a sligthly perturbed hamiltonian on ∂M. With this perturbation, we may assume that $\gamma(T)$ is not a focal point. We denote by φ_p the phase on the arc (x_p, x_{p+1}). φ_p is a real non degenerate phase; and from 0 to T, microlocally near $(\gamma(0), \gamma'(0))$, the identity

$$\int e^{i(x-y)\xi} d\xi,$$

is changed into the operator

$$i^{\nu_1 + \nu_2 + \ldots \nu_p} \int e^{i(\varphi_N(x, \xi) - y \cdot \xi)} b_+ d\xi,$$

where b_+, constructed by the transport equation, is a real positive symbol.

Now we notice that, since $\varphi_N(x, \xi, T)$ is a real non degenerate phase, a microlocal identity is given by

$$\mathbb{I} = \int e^{l(\varphi_N(x, \xi) - \varphi_N(y, \xi))} b \, d\xi, \tag{A.3}$$

microlocally near $(\gamma(0) = \gamma(T), \gamma'(0) = \gamma'(T))$, where b is a real positive symbol. We use this form of the identity to construct a microlocal parametrix on Γ^-, starting from $(\gamma(0), -\gamma'(0))$, this changes the exponential appearing in T into its conjugate,

$$\mathbb{I} = \int e^{i\varphi_N(y, \xi) - i\varphi_N(x, \xi))} b \, d\xi.$$

Reversing the time, we see that passing through any focal point x_p introduces a factor $i^{-\nu_p}$. Therefore

$$e^{-i\varphi_N(x,\xi)} = e^{i\varphi_N(x,\xi)}$$

is changed into $i^{\nu_1+\nu_2+\dots\nu_p}e^{-ix\xi}$.

Thus we have shown that on Γ^+ the identity is changed into:

$$\mathbb{T}^+ = i^{\nu_1+\nu_2+\dots\nu_p}\int e^{i\varphi_N(x,\xi)-iy\xi}b_+ d\xi \tag{A.4}$$

and that on Γ^-, it is changed into:

$$\mathbb{T}^- = i^{\nu_1+\nu_2+\dots\nu_p}\int e^{i\varphi_N(y,\xi)-ix\xi}b_- d\xi\,, \tag{A.5}$$

Taking $x=y$ in the right hand side of (A.4) and (A.5) and integrating over x gives the result.

Appendix 2

In this appendix, we briefly discuss the genericity of the geometric hypothesis of Theorem 2. Denote $M=\partial\mathcal{O}$, $S^*M=T^*M\setminus\mathcal{O}/\mathbb{R}_+$ the cosphere bundle of M and φ_t the geodesic flow on S^*M.

Proposition 1. *The condition of Theorem 2 is open in the sense that if γ is a closed geodesic satisfying the hypotheses* (1), (2) *of Theorem 2, then all C^∞ obstacles C^4 close to $\partial\mathcal{O}$ possess a closed geodesic γ close to γ satisfying the same conditions.*

Proposition 2. *The conditions of Theorem 2 are satisfied for a dense set of M in the following sense. Any M has a closed geodesic γ and for any such γ and any $k\geqq 4$ there are analytic manifolds arbitrarily close to M in the C^k topology with a closed geodesic close to γ satisfying the conditions of Theorem 2.*

Sketch of Proof of Proposition 1. Let $\gamma_\pm$ be the two liftings of γ to S^*M. Since 1 is a simple eigenvalue of the differential of the Poincaré map; a perturbation theorem of Poincaré implies that nearby obstacles have nearby closed geodesics $\tilde{\gamma}$ with period and first return map close to those of γ. In addition, there are no other geodesics of $\tilde{M}$ which are close to $\tilde{\gamma}$.

On the other hand, compactness implies that given an open neighborhood of $\gamma_+\cup\gamma_-$ in S^*M, there is a $\delta>0$ so that $|\varphi_T(p)-p|\geqq\delta$ for p outside that neighborhood and thus for $\tilde{M}$ close to M there are no closed geodesics of length near T which lie close to this neighborhood. Combining the above two observations yields the proposition. □

Sketch of proof of Proposition 2. It is well known that by a small perturbation of M we can insure that the non degeneracy condition (2) is satisfied. We therefore suppose that (2) is satisfied and consider (1).

We construct perturbed analytic manifolds M_+ as follows. For $f\in C^\infty(M,\mathbb{R})$ small in C^4 norm, one defines

$$M_f=\{x+f(x)n(x)\colon x\in M\}\,, \tag{A.6}$$

where $n(x)$ is the outward unit normal to M. We consider those f such that

$$\forall x \in \gamma,\ \forall |\alpha| \leqq 2,\ D^\alpha f(x) = 0. \tag{A.7}$$

Since $f|_\gamma = 0$, one sees that $\gamma \in M_f$. Since $df|_\gamma = 0$, we see that for $x \in \gamma$, $n(x)$ is also normal to M_f. Since γ is geodesic in M, the osculating plane of γ is spanned by its tangent and $n(x)$. As n is also the normal to M_f, γ is also geodesic in M_f, closed and of length T. Since $D^2 f|_\gamma = 0$, we see that γ satisfies the non degeneracy condition (2) of Theorem 2.

We denote φ_t^f the geodesic flow in S^*M_f. We identify S^*M_f and S^*M by the parametrization (A.6). Let B be the closed subspace of $C^k(M)$ consisting of functions satisfying (A.7), and let $B_\varepsilon = \{f \in B : \|f\| \leqq \varepsilon\}$.

We choose ε so small that M^f is a regular convex obstacle satisfying condition (1) for all $f \in B_\varepsilon$. We will show that there exist arbitrarily small $f \in B_\varepsilon$ such that M^f also satisfies condition (1).

As in the proof of Proposition 1, there is a neighborhood ω of $\gamma_+ \cup \gamma_-$ in S^*M, and a $\delta > 0$ such that for ε sufficiently small, $f \in B_\varepsilon$, $p \in \partial\omega$, $|t - T| < \delta$, we have $|\varphi_t^f(p) - p| > \delta$. We must find arbitrarily small f such that this is true for $p \in S^*M^f \setminus \omega \equiv K$ with, perhaps, a smaller δ.

The crucial step is to show that for any $q \in K$, the point $\varphi_T^0(q)$ can be perturbed in any direction along S^*M by suitable small f.

Lemma. *For $q \in K$ and any vector u tangent to S^*M at $\varphi_T^0(q)$ there is an $f \in B$ such that*

$$d_s \varphi_T^{sf}(q)|_{s=0} = u. \tag{4}$$

Assuming the lemma, we complete the proof of Proposition 2. Since K is compact, we can find a finite dimensional linear subspace F of B with the property that for any $q \in K$ the differential at $f = 0$ of the map $B \ni f \to \varphi_T^f(q) \in S^*M$ is surjective. Thus, shrinking ε if necessary, we see that in $S^* \times S^* \times F$ the subvarieties

$$A \equiv \{(q, q, f) : q \in K, f \in B_\varepsilon \cap F\} \quad \text{and} \quad C \equiv \{(q, \varphi_T^f(q), f) : q \in K, f \in B_\varepsilon \cap F\},$$

with $\dim A = \dim C = \dim S^* + \dim F$ intersect transversally. Thus either $A \cap C$ is empty or $A \cap C$ is a submanifold of dimension equal to $\dim F$. In the first case, $\gamma \in M = M_0$ satisfies the hypotheses of Theorem 2. In the second case, we consider the projection π of $A \cap C$ on F.

If $(q, q, f) \in A \cap C$, then the same is true of $(\varphi_t^f q, \varphi_t^f q, f)$ so if δq is the tangent to the geodesic $\varphi_t^f(q)$ at $t = 0$ the $(\delta q, \delta q, 0)$ is tangent to $A \cap C$. Thus $\pi_*(T_{(q,q,f)}(A \cap C))$ has dimension at most $\dim F - 1$ in $T_f(F)$. The theorem of Sard implies that $\pi(A \cap C)$ is nowhere dense in F. Thus, we may choose $f \in F$ arbitrarily small so that $f \notin \pi(A \cap C)$. For such f, $\gamma \in M_f$ satisfies condition (A.2) of Theorem 2. It remains to prove the lemma.

Sketch of Proof of Lemma. Fix $q \in K$ and let $\bar{x}, \bar{u} = \varphi_T^0(q) \in S^*M$. In local coordinates a product of balls in $\mathbb{R}^n$, $|\bar{x} - x| \leqq r$, $|u - \bar{u}| \leqq r$, defines a small neighborhood of $\bar{x}, \bar{u}$. We will construct f with support in $B(r)$, the subset of B consisting of functions supported in $|x - \bar{x}| \leqq r$. We must show that the differential

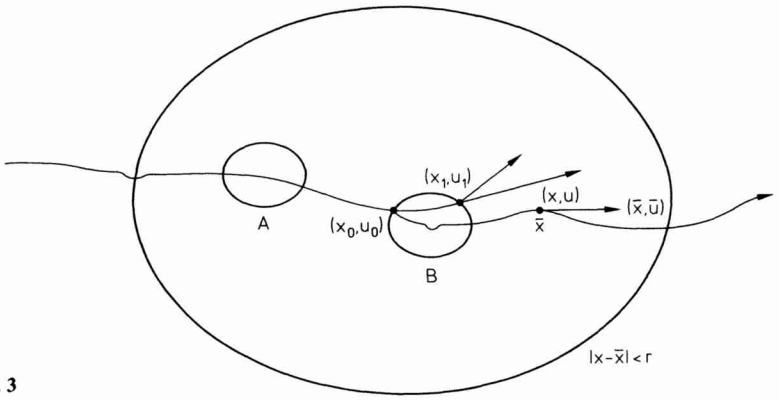

Fig. 3

of the map

$$B(r) \ni f \to \varphi_T^f(q) \in S^*M$$

is surjective. It suffices to show that there is an $r_1 \in]0, r]$ and $C > 0$ such that for all x, u with $|x-\bar{x}| < r_1$, $|u-\bar{u}| < r_1$, there is an $f \in B(r)$ with

$$\varphi_T^f(q) = (x, u), \qquad \|f\| \leqq C(|x-\bar{x}| + |u-\bar{u}|).$$

Choose discs in M as in Fig. 3. We will construct f with support in $A \cup B$. From (x, u) as above trace the backward geodesic till it intersects $\partial B \times |u-\bar{u}| \leqq r$ at (x_1, u_1). Let x_0, u_0 be the point in $\varphi_t(q)$ where γ enters B. Choose a smooth curve on M near γ joining x_0 to x_1 and with cotangent passing from u_0 to u_1. For any k this can be done with the C^k distance to γ being $O(|(x-\bar{x}, u-\bar{u})|)$. Choose f of the same order, supported in B so that this curve is a geodesic to M. To do this, one needs only deform M so that the normal to M is transformed to the principal normal of the curve. Having done this, the geodesic on the deformed M arrives at x, u in time $\tilde{T}$ with $\tilde{T} - T = O(|x-\bar{x}, u-\bar{u}|)$. Next deform M by choosing f in A so as to speed up or slow down the time spent through A to compensate for the time lag. Here it is crucial that M is strictly convex in A. The construction is complete. □

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Oblatum 30-IX-1986 & 17-XI-1986

Note added in proof

The authors want to thank Prof. V. Petkov for having pointed to them the following fact: The first indication of a polynomial bound (in the spirit of inequality (1.4)) for the resolvent set seems due Babic and Grigor'eva (cf. V. M. Babic and N. S. Grigor'eva: The analytic continuation of the resolvent of the exterior three dimensional problem for the Laplace operator to second sheet Funkcional Analysis i Prilozen. 8. (1974) n° 1, 71–74).