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Rigid reparametrizations and cohomology for horocycle flows

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In this paper we establish further rigidity properties of the classical horocycle flows in addition to those found in [8–11]. Namely, we show that, for flows obtained from horocycle flows by C^1-time changes all their ergodic joinings as well as all their factors remain algebraic. Moreover, we show that every nontrivial joining of two smoothly time-changed horocycle flows is induced by certain cohomological relations between the time changes.

We begin with some basic concepts and definitions (see [10, 11]).

Let T and S be two measure preserving transformations (m.p.t.'s) on probability spaces $(X, \mathscr{B}_X, \mu)$ and $(Y, \mathscr{B}_Y, \nu)$ respectively. A $T\times S$-invariant measure m on $(X\times Y, \mathscr{B}_X\times\mathscr{B}_Y)$ is called a joining of T and S if m has marginals μ and ν, i.e., $m(A\times Y)=\mu(A)$ for all $A\in\mathscr{B}_X$ and $m(X\times B)=\nu(B)$ for all $B\in\mathscr{B}_Y$. It is clear that $\mu\times\nu=m$ is a joining of T and S called the trivial joining.

Let $J(T,S)$ denote the set of all ergodic joinings of T and S. We say that T and S are disjoint if $J(T,S)$ is either empty or consists of the trivial joining $\mu\times\nu$. (This notion of disjointness was introduced by Furstenberg in [2].)

Let T_t be an m.p. flow on a probability space $(X, \mathscr{B}_X, \mu)$ and let $T=T_p=S$, $p\neq 0$, be ergodic. For $s\in R$ let m_s be the probability measure on $(X\times X, \mathscr{B}_X\times\mathscr{B}_X)$ defined by $m\{(x, T_s x)\colon x\in A\}=\mu(A)$ for all $A\in\mathscr{B}_X$. It is clear that $m_s\in J(T,T)$ for all $s\in R$. The measures $m_s, s\in R$ are called off-diagonal self-joinings of T. T is said to have trivial self-joinings if every $m\in J(T,T)$ is either off-diagonal or the product $\mu\times\mu$. (This terminology is due to Rudolph [13].)

Sometimes the word "joining" will be used for the pair $(T\times S, m)$, meaning $T\times S$ acting on $(X\times Y, m)$. We say that $(T\times S, m)$ is a finite extension of S if there is a $T\times S$-invariant subset $\Omega\subset X\times Y$ with $m(\Omega)=1$ such that the intersection $\Omega(y)=\Omega\cap(X\times\{y\})$ is finite for ν-almost every (a.e.) $y\in Y$. If $(T\times S, m)$ is ergodic, the number of points in $\Omega(y)$ is the same for ν-a.e. $y\in Y$.

Similar definitions may be made for measure preserving flows and semiflows.

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Let T be an m.p.t. on (X, μ) and S an m.p.t. on (Y, ν). We say that S is a factor of T if there is a measure-preserving $\psi: X \to Y$ such that $\psi T(x) = S\psi(x)$ for μ-a.e. $x \in X$. The map ψ is called a factor map or a conjugacy between T and S. T and S are called isomorphic ($T \sim S$) if there is an invertible conjugacy between T and S called an isomorphism. The conjugacy ψ induces a T-invariant measurable partition η of X into sets $\psi^{-1}\{y\}$, $y \in Y$, called ψ-fibers. The m.p.t. T induces an m.p.t. on the quotient space $(X/\eta, \mu_\eta)$ which is isomorphic to S via the map $\psi_\eta: X/\eta \to Y$, $\psi_\eta(\eta(x)) = \psi(x)$, where $\eta(x)$ denotes the ψ-fiber containing $x \in X$. A factor S is called trivial if $\nu\{y\} = 1$ for some $y \in Y$. Henceforth the word "factor" means non-trivial factor.

Let T_t be an m.p. flow on (X, μ) and let τ be a positive integrable function on X. Set

$$\int_X \tau \, d\mu = \bar{\tau}.$$

We say that a flow T_t^τ is obtained from T_t by the time change τ if

$$T_t^\tau(x) = T_{w(x,t)}(x)$$

for μ-a.e. $x \in X$ and all $t \in R$, where $w(x, t)$ is defined by

$$\int_0^{w(x,t)} \tau(T_u x) \, du = t.$$

The flow T_t^τ preserves the probability measure μ_τ on X defined by $d\mu_\tau(x) = (\tau/\bar{\tau}) \, d\mu(x)$, $x \in X$.

We say that two integrable functions $\tau_1, \tau_2: (X, \mu) \to R$ are cohomologous along T_t if there is a measurable $v: X \to R$ such that

$$\int_0^t (\tau_1 - \tau_2)(T_u x) \, du = v(T_t x) - v(x)$$

for μ-a.e. $x \in X$ and all $t \in R$. One can check that two time changes τ_1 and τ_2 are cohomologous via v if and only if the map $\chi_v: X \to X$ defined by

$$\chi_v(x) = T_{v(x)}^{\tau_2} x$$

is an isomorphism between $T_t^{\tau_1}$ and $T_t^{\tau_2}$, i.e.

$$\chi_v T_t^{\tau_1}(x) = T_t^{\tau_2} \chi_v(x)$$

for a.e. $x \in X$. We shall call χ_v the isomorphism induced by v. If T_t is ergodic and τ_1, τ_2 are cohomologous along T_t via different measurable functions v_1 and v_2 then $v_1 - v_2$ is equal to a constant a.e.

Let $T_t^{\tau_i}$ be obtained from an m.p. flow $T_t^{(i)}$ on $(X_i, \mathscr{B}_i, \mu_i)$ by a time change τ_i, $i = 1, 2$, and let $\phi: (X_1, \mu_1) \to (X_2, \mu_2)$ be an m.p. conjugacy between $T_t^{(1)}$ and $T_t^{(2)}$. Suppose that τ_1 and $\tau_2 \circ \phi$ are cohomologous along $T_t^{(1)}$ via v. It is clear that the map $\psi = \phi \circ \chi_v$ is an m.p. conjugacy between $T_t^{\tau_1}$ and $T_t^{\tau_2}$.

Now let I be an index set and let $\mathbb{F} = \{T_t^{(i)}: i \in I\}$ be a family of m.p. flows $T_t^{(i)}$ on probability spaces (X_i, μ_i). Let K_i, $i \in I$, be a class of positive integrable functions on (X_i, μ_i) and let $\mathbb{K} = \bigcup_{i \in I} K_i$.

Definition 1. The class $\mathbb{K}$ is said to be conjugacy (isomorphism)-rigid for the family $\mathbb{F}$ if given $\tau_i \in K_i$, $\tau_j \in K_j$, $\bar{\tau}_i = \bar{\tau}_j$, $i, j \in I$, and an m.p. conjugacy (isomorphism) ψ between $T_t^{\tau_i}$ and $T_t^{\tau_j}$ there are a measurable $v\colon X_i \to R$ and an m.p. conjugacy (isomorphism) ϕ between $T_t^{(i)}$ and $T_t^{(j)}$ such that

(1) τ_i and $\tau_j \circ \phi$ are cohomologous along $T_t^{(i)}$ via v;

(2) $\psi = \phi \circ \chi_v$ where χ_v is the isomorphism between $T_t^{\tau_i}$ and $T_t^{\tau_j \circ \phi}$ induced by v.

Next we consider a pair of time changes for two m.p. flows, and define what is meant by saying that these time changes are "jointly cohomologous".

Let T^{τ_i} and $T^{(i)}$ on (X_i, μ_i), $i = 1, 2$, be as above. Let $X = X_1 \times X_2$ and let $f_i\colon X \to R^+$ be defined by

$$f_i(x_1, x_2) = \tau_i(x_i), \qquad i = 1, 2.$$

Definition 2. Let m be a joining of $T_t^{(1)}$ and $T_t^{(2)}$. We say that τ_1 and τ_2 are jointly cohomologous via m if f_1 and f_2 are cohomologous along $S_t = T_t^{(1)} \times T_t^{(2)}$ on (X, m).

More specifically, if f_1 and f_2 are cohomologous along S_t via a measurable function $v\colon X \to R$ then we say that τ_1 and τ_2 are jointly cohomologous via (m, v).

We have

$$v(T_t^{(1)} x_1, T_t^{(2)} x_2) - v(x_1, x_2) = \int_0^t (f_1 - f_2)(T_u^{(1)} x_1, T_u^{(2)} x_2)\, du \tag{1}$$

for m-a.e. $(x_1, x_2) \in X$ and all $t \in R$.

There exists a unique family $\{m_{x_1} : x_1 \in X_1\}$ of probability measures on X_2 such that

$$\int_X f\, dm = \int_{X_1} f^{(m)}(x_1)\, d\mu_1(x_1)$$

for every $f \in L_1(X, m)$, where

$$f^{(m)}(x_1) = \int_{X_2} f(x_1, x_2)\, dm_{x_1}(x_2)$$

and

$$m_{T_t^{(1)} x_1}(A) = m_{x_1}(T_{-t}^{(2)} A) \tag{2}$$

for every $t \in R$, $A \in \mathscr{B}_2$ and μ_1-a.e. $x_1 \in X_1$. Set

$$\tau_2^{(m)}(x_1) = f_2^{(m)}(x_1), \qquad x_1 \in X_1.$$

Similarly we define $\tau_1^{(m)}(x_2)$, $x_2 \in X_2$.

Expressions (1) and (2) show that if $v(x_1, \cdot) \in L_1(X_2, m_{x_1})$ for μ-a.e. $x_1 \in X_1$ then τ_1 and $\tau_2^{(m)}$ are cohomologous along $T_t^{(1)}$ via $v^{(m)}\colon X_1 \to R$. We have just proved the following.

Proposition 1. *Let τ_i be a time change for $T_t^{(i)}$ on (X_i, μ_i), $i = 1, 2$. Suppose that τ_1 and τ_2 are jointly cohomologous via (m, v) with $v(x_1, \cdot) \in L_1(X_2, m_{x_1})$ for μ_1-a.e. $x_1 \in X_1$. Then τ_1 and $\tau_2^{(m)}$ are cohomologous along $T_t^{(1)}$.*

If τ_1 and τ_2 are jointly cohomologous via (m, v) then the flows $S_t^{f_1}$ on (X, m_{f_1}) and $S_t^{f_2}$ on (X, m_{f_2}) are isomorphic via the isomorphism χ_v induced by

v. Also the map $\psi_v\colon X \to X$ defined by

$$\psi_v(x) = (x_1, T^{\tau_2}_{v(x)} x_2), \qquad x = (x_1, x_2)$$

is a measurable conjugacy between $S^{f_1}_t$ and $T^{\tau_1}_t \times T^{\tau_2}_t$; that is,

$$\psi_v \circ S^{f_1}_t = (T^{\tau_1}_t \times T^{\tau_2}_t) \circ \psi_v. \tag{3}$$

Let $\nu = \nu_{(m,v)}$ be the measure on $\mathscr{B}_1 \times \mathscr{B}_2$ defined by

$$\nu(A) = m_{f_1}(\psi_v^{-1}(A)), \qquad A \in \mathscr{B}_1 \times \mathscr{B}_2.$$

It follows from (3) that ν is a joining of $T^{\tau_1}_t$ and $T^{\tau_2}_t$. The flow $T^{\tau_1}_t \times T^{\tau_2}_t$ on (X, ν) is a factor of $S^{f_1}_t$ on (X, m_{f_1}) via ψ_v and a factor of $S^{f_2}_t$ on (X, m_{f_2}) via $\psi_v \circ \chi_v^{-1}$. If m is ergodic, then so is ν. We shall call $\nu_{(m,v)}$ the joining induced by (m, v).

Definition 3. Let I, $\mathbb{F}$ and $\mathbb{K}$ be as in Definition 1. The class $\mathbb{F}$ is said to be joining-rigid for the family $\mathbb{F}$ is given $\tau_i \in K_i$, $\tau_j \in K_j$, $\bar{\tau}_i = \bar{\tau}_j$, $i, j \in I$ and a nontrivial ergodic joining ν of $T^{\tau_i}_t$ and $T^{\tau_j}_t$, there are a measurable $v\colon X_i \times X_j \to R$ and a nontrivial ergodic joining m of $T^{(i)}_t$ and $T^{(j)}_t$ such that

(1) τ_i and τ_j are jointly cohomologous via (m, v);
(2) $\nu = \nu_{(m,v)}$ is induced by (m, v).

Next we introduce an analog of Definition 2 for factors.

Let T^τ_t be obtained from an m.p. flow T_t on (X, μ) by a time change τ and let ζ and η be two measurable partitions of (X, μ) respectively induced by a factor U_t of T_t and by a factor V_t of T^τ_t. We shall identify U_t with the flow on the quotient space $(X/\zeta, \mu_\zeta)$ induced by T_t.

There exists a unique family $\{\mu_C, C \in \zeta\}$, of probability measures μ_C on atoms of ζ such that

$$\int_X f\, d\mu = \int_{X/\zeta} f_\zeta(C)\, d\mu_\zeta(C)$$

for every $f \in L_1(X, \mu)$ where

$$f_\zeta(C) = \int_C f(x)\, d\mu_C(x), \qquad C \in \zeta$$

and

$$\mu_{C(T_t x)}(T_t A) = \mu_{C(x)}(A)$$

for all measurable $A \subset C(x)$, all $t \in R$ and μ-a.e. $x \in X$. Set

$$\tau_\zeta(x) = \tau_\zeta(C(x)), \qquad x \in C(x) \in \zeta.$$

Definition 4. The factor-induced partitions ζ and η are called shift-related along T_t if there is a measurable function $v(x, y) = v_x(y)$, $x \in X$, $y \in \zeta(x)$, $v(x, x) = 0$ such that for μ-a.e. $x \in X$,

(1) v_x is integrable on $(\zeta(x), \mu_{\zeta(x)})$,
(2) $\eta(x) = \{T^\tau_{v(x,y)} y\colon y \in \zeta(x)\}$.

The following proposition can be deduced from Proposition 1. It is proved in Sect. 5.

Proposition 2. *Let ζ and η be the factor-induced partitions of (X, μ) described above. Suppose that ζ and η are shift-related along T_t via $v(x, y)$, $x \in X$, $y \in \zeta(x)$. Then*

1) *τ_ζ is cohomologous to τ via $\tilde{v}: X \to R$ defined by*

$$\tilde{v}(x) = -\int_{\zeta(x)} v(x, y)\, d\mu_{\zeta(x)}(y).$$

2) *$\eta(x) = \chi_{\tilde{v}}(\zeta(\chi_{\tilde{v}}^{-1}(x)))$ for μ-a.e. $x \in X$, where $\chi_{\tilde{v}}$ is the isomorphism between $T_t^{\tau_\zeta}$ and T_t^τ induced by $\tilde{v}$.*

Consequently, $U_t^{\tau_\zeta} \sim V_t$.

Definition 5. Let I, $\mathbb{F}$ and $\mathbb{K}$ be as in Definition 1. The class $\mathbb{K}$ is said to be factor-rigid for the family $\mathbb{F}$ if given $\tau_i \in K_i$, $i \in I$, and a $T_t^{\tau_i}$-invariant partition η of (X_i, μ_i) there are a measurable function $v: X_i \to R$ and a $T_t^{(i)}$-invariant partition ζ such that

(1) $(\tau_i)_\zeta$ is cohomologous to τ_i via v;

(2) $\eta(x) = \chi_v(\zeta(\chi_v^{-1}(x)))$ for μ_i-a.e. $x \in X_i$, where χ_v is the isomorphism between $T_t^{(\tau_i)_\zeta}$ and $T_t^{\tau_i}$ induced by v.

Now let G denote the group $SL(2, R)$ equipped with a left invariant Riemannian metric and let $\mathscr{T}$ be the set of all discrete subgroups Γ of G such that the quotient space $M = \Gamma\backslash G = \{\Gamma g : g \in G\}$ has finite volume. Recall that the horocycle flow h_t and the geodesic flow g_t on M are defined by

$$h_t(\Gamma g) = \Gamma g \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad g_t(\Gamma g) = \Gamma g \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

$g \in G$, $t \in R$. The flows h_t and g_t preserve the normalized volume measure μ on M, are ergodic and mixing on (M, μ) and obey the commutation relation

$$g_t \circ h_s = h_{s \exp(2t)} \circ g_t \tag{4}$$

for all $s, t \in R$.

It was shown in [11] that there is a large class $K(M)$ of positive square-integrable functions on (M, μ) such that for $\tau \in K(M)$ the time-changed horocycle flow h_t^τ inherits the rigidity property [8] from h_t. The class $K(M)$ specifically contains all positive bounded functions τ such that τ^{-1} is also bounded and s.t. τ is Hölder continuous in the direction of the rotation subgroup of G (see [5, 12]).

In the present paper we consider the more restricted class $B^1(M)$ of all continuously differentiable positive functions on M, which are bounded with bounded reciprocals. We will show that for time changes $\tau \in B^1(M)$, the flows h_t^τ, in addition to the above mentioned rigidity property for conjugates, also inherit the factor and joining rigidity properties [9, 10].

The flows h_t^τ with $\tau \in B^1(M)$ are known [4] to be mixing in (M, μ_τ).

Restricting the time change τ to $B^1(M)$ enables us to show that h_t^τ inherits a fundamental property of h_t, the "H-property", which we introduced in [10].

An important geometrical fact about the horocycle flow h_t is that two initially close trajectories diverge polynomially. The H-property states that this divergence develops much faster in the direction of the flow than in any other direction. (The exact description of the H-property is rather technical; it will be given in the proof of the first theorem.)

Theorem 1. *Let h_t be the horocycle flow on $(M=\Gamma\backslash G, \mu)$, $\Gamma\in\mathscr{T}$, and let h_t^τ be obtained from h_t by a time change $\tau\in B^1(M)$. Then h_t^τ possesses the H-property.*

The significance of the H-property is reflected in the following corollaries, proved in [10].

Corollary 1. *Let h_t^τ be as in Theorem 1 and let $h^\tau=h_1^\tau$, (the "time 1" map).*

(a) *If v is a nontrivial ergodic joining of h^τ and some m.p.t U then $(h^\tau\times U, v)$ is a finite extension of U.*

(b) *If $U=U_1$ for some measure preserving flow U_t then $v\in J(h^\tau, U)$ if and only if $v\in J(h_t^\tau, U_t)$ for all $t\in R$.*

Corollary 2. *Let h_t^τ be as in Theorem 1 and let $h^\tau=h_1^\tau$.*

(a) *If an m.p.t V is a factor of h^τ via a factor map ψ then a.e. fiber of ψ is finite.*

(b) *If $V=V_1$ for some m.p. flow V_t and V is a factor of h^τ via a factor map ψ then the flow V_t is a factor of h_t^τ via ψ.*

Let $h_t^{(i)}$ be the horocycle flow on $(M=\Gamma_i\backslash G, \mu_i)$, $\Gamma_i\in\mathscr{T}$ and let $h_t^{\tau_i}$ be obtained from $h_t^{(i)}$ by a time change τ_i, $i=1,2$.

Consider $\kappa(\Gamma_1,\Gamma_2)=\{\alpha\in G: \Gamma_1\cap\alpha\Gamma_2\alpha^{-1}\in\mathscr{T}\}$. For $\alpha\in\kappa(\Gamma_1,\Gamma_2)$, let $\Gamma_\alpha=\Gamma_1\cap\alpha\Gamma_2\alpha^{-1}$, $M_\alpha=\Gamma_\alpha\backslash G$ and let $h_t^{(\alpha)}$ be the horocycle flow on (M_α, μ_α). Let $\psi_i: M_\alpha\to M_i$, $i=1,2$, be defined by

$$\psi_1(\Gamma_\alpha g)=\Gamma_1 g, \quad \psi_2(\Gamma_\alpha g)=\Gamma_2\alpha^{-1}g, \quad g\in G.$$

The flows $h_t^{(1)}$ and $h_t^{(2)}$ are factors of $h_t^{(\alpha)}$ via ψ_1 and ψ_2 respectively. Let $\tau_{\alpha,i}: M_\alpha\to R^+$ be defined by

$$\tau_{\alpha,i}(x)=\tau_i(\psi_i(x)),$$

$x\in M_\alpha$, $i=1,2$. We call $\tau_{\alpha,i}$ the time changes associated to α.

Definition 6. τ_1 and τ_2 are said to be algebraically cohomologous via $\alpha\in\kappa(\Gamma_1,\Gamma_2)$ if $\tau_{\alpha,1}$ and $\tau_{\alpha,2}$ are cohomologous along $h_t^{(\alpha)}$.

The following theorem is a consequence of the joinings classification theorem for horocycle flows [10, Theorem 6].

Theorem 2. *Let $h_t^{\tau_i}$ be obtained from the horocycle flow $h_t^{(i)}$ on $(M_i=\Gamma_i\backslash G, \mu_i)$, $\Gamma_i\in\mathscr{T}$ by a time change τ_i, $i=1,2$. The functions τ_1 and τ_2 are jointly cohomologous via a nontrivial $m\in J(h_t^{(1)}, h_t^{(2)})$ if and only if τ_1 and τ_2 are algebraically cohomologous via some $\alpha\in\kappa(\Gamma_1,\Gamma_2)$.*

Theorem 3 (Main Joinings Theorem). *Let $h_t^{\tau_i}$ be as in Theorem 2. Assume that $\tau_i\in B^1(M_i)$, $i=1,2$ and $\bar\tau_1=\bar\tau_2$. Let v be a nontrivial ergodic joining of $h_1^{\tau_1}$ and $h_1^{\tau_2}$. Then $v\in J(h_t^{\tau_1}, h_t^{\tau_2})$ for all $t\in R$. Moreover, there exist a measurable $v: M_1\times M_2\to R$ and a nontrivial $m\in J(h_t^{(1)}, h_t^{(2)})$ such that*

(1) τ_1 *and* τ_2 *are jointly cohomologous via* (m, v);

(2) $v = v_{(m,v)}$ *is induced by* (m, v);

(3) $(h_t^{\tau_1} \times h_t^{\tau_2}, v) \sim (S_t^{f_1}, m_{f_1}) \sim (S_t^{f_2}, m_{f_2})$, *where* $S_t = h_t^{(1)} \times h_t^{(2)}$ *and* $f_i\colon M_1 \times M_2 \to R$ *is defined by* $f_i(x_1, x_2) = \tau_i(x_i)$, $i = 1, 2$.

Corollary 3. *The class* $\mathbb{B}^1 = \bigcup_{\Gamma \in \mathcal{T}} B^1(\Gamma \backslash G)$ *is joining-rigid for the family* $\mathbb{H}$ *of all horocycle flows on* $\Gamma \backslash G$, *where* Γ *ranges over all lattices in* G.

We showed in [10] that if m is a nontrivial ergodic joining of $h_t^{(1)}$ and $h_t^{(2)}$ then there are $\sigma \in R$ and $\alpha \in \kappa(\Gamma_1, \Gamma_2)$ such that the set

$$\{(x_1, h_\sigma^{(2)} x_2) : (x_1, x_2) \in \Omega_\alpha\}$$

has m-measure 1, where

$$\Omega_\alpha = \{(\Gamma_1 g, \Gamma_2 \alpha^{-1} \gamma_k g) : g \in G, k = 1, \dots, n\} \subset M_1 \times M_2,$$

$n = |\Gamma_\alpha \backslash \Gamma_1|$ is the cardinal number of $\Gamma_\alpha \backslash \Gamma_1 = \{\Gamma_\alpha \gamma_k : k = 1, \dots, n\}$.

We obtain the following algebraic version of Theorem 3.

Theorem 4 (Joinings Classification Theorem). *Let* τ_i *and* $h_t^{\tau_i}$, $i = 1, 2$, *be as in Theorem* 3. *Let* v *be a nontrivial ergodic joining of* $h_1^{\tau_1}$ *and* $h_1^{\tau_2}$. *Then there exist* $\alpha \in \kappa(\Gamma_1, \Gamma_2)$ *and a measurable* $v\colon M_1 \times M_2 \to R$ *such that*

(1) τ_1 *and* τ_2 *are algebraically cohomologous via* α;

(2) $(h_t^{\tau_1} \times h_t^{\tau_2}, v) \sim h_t^{\tau_{\alpha,1}} \sim h_t^{\tau_{\alpha,2}}$, *where* $h_t^{\tau_{\alpha,i}}$ *is obtained from the horocycle flow* $h_t^{(\alpha)}$ *on* (M_α, μ_α) *by the time change* $\tau_{\alpha,i}$ *associated to* α, $i = 1, 2$.

(3) $v(\psi_v(\Omega_\alpha)) = 1$, *where* $\psi_v\colon M_1 \times M_2 \to M_1 \times M_2$ *is defined by*

$$\psi_v(x) = (x_1, h_{v(x)}^{\tau_2} x_2), \qquad x = (x_1, x_2).$$

Theorems 3, 4 and Proposition 1 imply the following

Corollary 4. *Let* τ_i *and* $h_t^{\tau_i}$, $i = 1, 2$, *be as in Theorem* 3.

(1) *If* $h_1^{(1)}$ *and* $h_1^{(2)}$ *are disjoint, then so are* $h_1^{\tau_1}$ *and* $h_1^{\tau_2}$;

(2) *If* $h_t^{(1)}$ *has only trivial self-joinings, then so does* $h_t^{\tau_1}$;

(3) *If* τ_1 *and* τ_2 *are not algebraically cohomologous then* $h_1^{\tau_1}$ *and* $h_1^{\tau_2}$ *are disjoint. If, in addition,* M_1 *and* M_2 *are compact then* $h_1^{\tau_1} \times h_1^{\tau_2}$ *is uniquely ergodic;*

(4) *If* $\bar{\tau}_2 = 1$ *and* τ_2 *is not cohomologous to* 1 *along* $h_t^{(2)}$ *then* $h_1^{(1)}$ *and* $h_1^{\tau_2}$ *are disjoint. In this case* $h_1^{(1)} \times h_1^{\tau_2}$ *is uniquely ergodic, if* M_1 *and* M_2 *are compact.*

The following theorem is really a particular case of Theorem 3 (or Theorem 4) (see [10], where a similar situation is discussed in detail).

Theorem 5 (Rigidity of Time Changes Theorem). *Let* τ_i *and* $h_t^{\tau_i}$, $i = 1, 2$, *be as in Theorem* 3. *Suppose that there is an m.p.* $\psi\colon (M_1, \mu_{\tau_1}) \to (M_2, \mu_{\tau_2})$ *such that*

$$\psi h_t^{\tau_1}(x) = h_t^{\tau_2} \psi(x)$$

for μ_{τ_1}*-a.e.* $x \in M_1$ *and all* $t \in R$. *Then there are* $\alpha \in G$ *and a measurable* $v\colon M_2 \to R$ *such that*

(1) $\alpha^{-1}\Gamma_1\alpha\subset\Gamma_2$;
(2) $\psi(x)=h^{\tau_2}_{v(\psi_\alpha(x))}(\psi_\alpha(x))$ *a.e. where* $\psi_\alpha(\Gamma_1 g)=\Gamma_2\alpha^{-1}g$, $g\in G$;
(3) τ_1 *and* $\tau_2\circ\psi_\alpha$ *are cohomologous along* $h_t^{(1)}$.

In [11] we proved this theorem for τ_i belonging to the class $K(M_i)$ mentioned above. Since $K(M_i)$ is much larger than $B^1(M_i)$ the proof in [11] is necessarily rather technical.

In Sect. 3 we give a proof of Theorem 5 different from that in [11] and independent of Theorems 1–4. In fact, this proof combined with some arguments from [10] yields the proof of the Main Theorem 3.

To prove part (1) of Theorem 5 we use none of the techniques needed in [11], such as the combinatorial Lemma 2.2 or the estimate of the decay rate of the correlation function of τ_i for $h_t^{(i)}$. (It is only the cohomological parts (2) and (3) of the theorem that need these techniques.) It seems plausible that this method of proof can be extended to deal with various geometrical and algebraic generalizations.

Corollary 5. *The class* $\mathbb{K}=\bigcup_{\Gamma\in\mathcal{T}}K(\Gamma\backslash G)\supset\mathbb{B}^1$ *is conjugacy and isomorphism-rigid for* $\mathbb{H}=\{h_t$ *on* $\Gamma\backslash G\colon\Gamma\in\mathcal{T}\}$. *In particular if* $\tau\in K(M)\supset B^1(M)$ $M=\Gamma\backslash G$, *then* $h_t^\tau\sim h_t^{\bar\tau}$ *if and only if* τ *is cohomologous to* $\bar\tau$ *along* h_t.

Next we classify factors of h_t^τ, $\tau\in B^1(M)$, $M=\Gamma\backslash G$, $\Gamma\in\mathcal{T}$.

We showed in [9] that if U_t is a factor of h_t on (M,μ) then there exists $\Gamma'\supset\Gamma$, $\Gamma'\in\mathcal{T}$ such that the h_t-invariant partition ζ of M induced by U_t has the form

$$\zeta(\Gamma g)=\psi^{-1}_{\Gamma,\Gamma'}\{\Gamma' g\},$$

where

$$\psi_{\Gamma,\Gamma'}(\Gamma g)=\Gamma' g,\qquad g\in G.$$

We shall denote ζ by $\zeta(\Gamma,\Gamma')$.

Theorem 6 (Factors Theorem). *The class* $\mathbb{B}^1$ *is factor-rigid for the family* $\mathbb{H}=\{h_t$ *on* $\Gamma\backslash G\colon\Gamma\in\mathcal{T}\}$. *More precisely, let* h_t^τ *be obtained from the horocycle flow* h_t *on* $(M=\Gamma\backslash G,\mu)$, $\Gamma\in\mathcal{T}$ *by a time change* $\tau\in B^1(M)$. *Let* V *be a factor of* h_1^τ *and* η *be the* h_1^τ*-invariant partition of* M *induced by* V. *Then* η *is invariant under* h_t^τ *for all* $t\in R$. *Moreover, there exist* $\Gamma'\supset\Gamma$, $\Gamma'\in\mathcal{T}$ *and a measurable* $v\colon M\to R$ *such that*

(1) *the function* τ_ζ *defined by* $\tau_\zeta(x)=\big(\sum_{y\in\zeta(x)}\tau(y)\big)/|\Gamma\backslash\Gamma'|$ *is cohomologous to* τ *via* v, *where* $\zeta=\zeta(\Gamma,\Gamma')$;

(2) $\eta(x)=\chi_v(\zeta(\chi_v^{-1}(x))$ *for* μ*-a.e.* $x\in M$, *where* χ_v *is the isomorphism between* $h_t^{\tau_\zeta}$ *and* h_t^τ *induced by* v;

(3) $V_t\sim h_t^{\tau'}$, *where* $h_t^{\tau'}$ *is obtained from the horocycle flow* h'_t *on* $(M'=\Gamma'\backslash G,\mu')$ *by the time change* $\tau'(\Gamma' g)=\tau_\zeta(\psi^{-1}_{\Gamma,\Gamma'}\{\Gamma' g\})$, $g\in G$, *and* V_t *is the flow on the quotient space* M/η *induced by* h_t^τ, $V_1\sim V$.

Corollary 6. *Let* h_t^τ *be as in Theorem* 6.

(1) *The number of nonisomorphic factors of* h_1^τ *is finite.*
(2) *If* h_1 *has only trivial factors then so does* h_1^τ.

We assumed in Theorems 3–5 that $\bar{\tau}_1 = \bar{\tau}_2$. The case $\bar{\tau}_1 \neq \bar{\tau}_2$ can be derived from the case $\bar{\tau}_1 = \bar{\tau}_2$ using the commutation relation (4), just as in [11].

Indeed, let $a = \bar{\tau}_1 \neq \bar{\tau}_2 = b$ and let $s = \dfrac{(\log(b/a))}{2}$. Let σ_2 on M_2 be defined by $\sigma_2(x) = \dfrac{a\tau_2(g^{(2)}_{-s}x)}{b}$. We have $\bar{\sigma}_2 = a = \bar{\tau}_1$.

Relation (4) shows that $h^{\tau_2}_t$ and $h^{\sigma_2}_t$ are isomorphic via $g^{(2)}_s$, that is,

$$g^{(2)}_s \circ h^{\tau_2}_t = h^{\sigma_2}_t \circ g^{(2)}_s.$$

Let $\psi_s \colon M_1 \times M_2 \to M_1 \times M_2$ be defined by $\psi_s(x_1, x_2) = (x_1, g^{(2)}_s x_2)$. It is clear that a measure ν on $M_1 \times M_2$ is a nontrivial ergodic joining of $h^{\tau_1}_t$ and $h^{\tau_2}_t$ if and only if the measure $\nu_s = \psi^*_s \nu$ is a nontrivial ergodic joining of $h^{\tau_1}_t$ and $h^{\sigma_2}_t$.

Thus in order to state Theorem 3 for ν one should replace τ_2 by σ_2 in (1) and say "ν_s is induced by (m, v)" in (2).

Similar changes must be made in Corollary 4 and Theorems 4 and 5 for the case $\bar{\tau}_1 \neq \bar{\tau}_2$.

Finally we pose the following problems.

Problem 1. Does there exist a function $\tau \in B^1(M)$ which is not cohomologous to $\bar{\tau}$ along h_t?

This problem is motivated by a theorem of Kolmogorov [3], stating that there is a subset $A \subset R$ of full Lebesgue measure such that if an irrational α belongs to A then every C^∞-function τ on the 2-torus is cohomologous to $\bar{\tau}$ along the irrational flow induced by α. (In fact, τ is cohomologous to $\bar{\tau}$ via a C^∞-function v.) This, however, is not true for many irrational $\alpha \notin A$.

Problem 2. Suppose that $\tau \in B^1(M)$ and τ is cohomologous to $\bar{\tau}$ along h_t via v. Is v integrable? continuous? smooth?

Problem 3. Are there other naturally arising families of ergodic m.p. flows (possibly consisting of just one m.p. flow) for which the class of smooth functions is conjugacy-rigid? joining-rigid? factor-rigid? Do unipotent flows on finite volume homogeneous spaces of semi-simple Lie groups form such a family? (see [14]). What about uniformly parameterized horocycle flows for surfaces of variable negative curvature? (see [1]). Is the class of smooth functions conjugacy-rigid for irrational flows on the 2-torus or more generally for nilflows on compact nilmanifolds? (see [6, 7] for rigidity properties of nilflows).

1. Basic lemma

Let h_t and g_t be the horocycle and the geodesic flows on $(M = \Gamma \backslash G, \mu)$, $\Gamma \in \mathscr{T}$ respectively. We shall also consider the flow k_t on (M, μ) defined by

$$k_t(\Gamma g) = \Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$g \in G$, $t \in R$.

One has

$$g_s \circ k_t = k_{t \exp(-2s)} \circ g_s$$

for all $s, t \in R$.

The flows k_t and h_t satisfy the following commutation relation

$$h_t \circ k_b = k_{c(b,t)} \circ g_{r(b,t)} \circ h_{\sigma(b,t)} \tag{1.1}$$

for all $b, t \in R$ with $bt > -1$, where

$$\begin{aligned} r(b,t) &= \log(1+bt), \\ \sigma(b,t) &= t e^{-r(b,t)}, \\ c(b,t) &= b e^{-r(b,t)}. \end{aligned} \tag{1.2}$$

Denote $\Delta(b,t) = \sigma(b,t) - t = \dfrac{-bt^2}{1+bt}$.

For $a, b \in R$, $t > 0$, $bt > -1$ we have

$$h_t \circ k_b \circ g_a = k_{c(b,t)} \circ g_{r(a,b,t)} \circ h_{\sigma(a,b,t)}$$

where

$$\begin{aligned} r(a,b,t) &= a + r(b,t), \\ \sigma(a,b,t) &= e^{-2a} \sigma(b,t). \end{aligned}$$

It follows from (1.2) that there exists $0 < r_0 \leqq \frac{1}{2}$ such that if $|a|$, $|bt| \leqq r_0$ then

$$\begin{gathered} |e^{-2a} - 1| \leqq 0.1 \\ |\Delta(b,t)|/2t \leqq |r(b,t)| \leqq 2|\Delta(b,t)|/t \\ |\Delta(b,t)|/2t^2 \leqq |b| \leqq 2|\Delta(b,t)|/t^2 \\ |\Delta(b,s) - s^2 \Delta(b,t)/t^2| \leqq 0.01\,|\Delta(b,t)| \end{gathered} \tag{1.3}$$

for all $0 \leqq s \leqq t$.

Denote

$$\Delta(a,b,t) = \sigma(a,b,t) - t.$$

We have

$$\Delta(a,b,t) = e^{-2a} \Delta(b,t) - t(1 - e^{-2a}). \tag{1.4}$$

Lemma 1.1. *Let $a, b \in R$, $t > 0$ be such that $|a|$, $|b| \leqq r_0$. Then*

$$|\Delta(b,t)| \leqq 20 \max\{|\Delta(a,b,s)| : 0 \leqq s \leqq t\}.$$

Proof. Denote $\Delta_0 = \max\{|\Delta(a,b,s)| : 0 \leqq s \leqq t\}$. We have from (1.4) and (1.3),

$$\begin{aligned} \Delta(a,b,t/2) &= e^{-2a} \Delta(b,t/2) - t(1 - e^{-2a})/2, \\ e^{-2a} 0.1\,|\Delta(b,t)| &\leqq e^{-2a}(|\Delta(b,t)|/2 - |\Delta(b,t/2)|) \\ &\leqq |\Delta(a,b,t)/2 - \Delta(a,b,t/2)| \leqq 3\Delta_0/2 \end{aligned}$$

since $|\Delta(b,t/2)| \leqq 0.4\,|\Delta(b,t)|$ by (1.3). This implies that $|\Delta(b,t)| \leqq 20\Delta_0$ by (1.3). $\square$

Corollary 1.1. *If* $|a|$, $|bt|\leqq r_0$ *and*

$$|\Delta(a, b, t)| = \max\{|\Delta(a, b, s)| : 0\leqq s\leqq t\}$$

then

$$|\Delta(a, b, \beta t)| \geqq 0.4\,|\Delta(a, b, t)|$$

for all $\beta\in[0.99, 1]$.

Proof. We have

$$\begin{aligned}\Delta(a, b, \beta t) &= e^{-2a}\Delta(b, \beta t) + \beta t(e^{-2a}-1)\\ &= e^{-2a}(\Delta(b, \beta t) - \beta\Delta(b, t)) + \beta\Delta(a, b, t)\end{aligned}$$

and

$$|\Delta(a, b, \beta t)| \geqq \beta|\Delta(a, b, t)| - e^{-2a}|\Delta(b, \beta t) - \beta\Delta(b, t)|. \tag{1.5}$$

Also,

$$\begin{aligned}|\Delta(b, \beta t) - \beta\Delta(b, t)| &\leqq |\Delta(b, \beta t) - \beta^2\Delta(b, t)| + |\Delta(b, t)|\,|\beta-\beta^2|\\ &\leqq 0.02\,|\Delta(b, t)| \leqq 0.4\,|\Delta(a, b, t)|\end{aligned}$$

by Lemma 1.1. Thus (1.3) and (1.5) imply

$$|\Delta(a, b, \beta t)| \geqq 0.4\,|\Delta(a, b, t)|$$

if $\beta\in[0.99, 1]$. □

Let τ be a positive bounded function on (M, μ) and let $x\in M$, $y = g_{-a}k_b x$ for some $a, b\in R$. For $t > -1/|b|$, denote

$$\xi(x; a, b, t) = \int_0^{\sigma(a, b, t)} \tau(h_u y)\,du,$$

$$\xi(x; 0, 0, t) = \xi(x, t) = \int_0^t \tau(h_u x)\,du,$$

$$\xi(x; 0, b, t) = \xi(x; b, t).$$

Also let

$$\Delta_\tau(x; a, b, t) = \xi(x; a, b, t) - \xi(x, t).$$

We have

$$\Delta_\tau(x; a, b, t) = \Delta_\tau(x; b, t) + \Delta_\tau(g_a y; a, 0, \sigma(b, t)). \tag{1.6}$$

Lemma 1.2 (Basic). *Assume* $\tau\in B^1(M)$. *Then given* $\varepsilon>0$ *there are* $\gamma=\gamma(\varepsilon)>0$, $l=l(\varepsilon)>0$ *and a subset* $E=E(\varepsilon)\subset M$ *with* $\mu(E)>1-\varepsilon$ *such that if* $x, y\in E$, $x=k_b g_a y$ *for some* $|a|\leqq\gamma$, $|b|\leqq\gamma/l$ *then*

$$|\Delta_\tau(x; a, b, t) - \bar\tau\Delta(a, b, t)| \leqq \varepsilon\max\{|\Delta(a, b, s)| : s\in[0, t]\} \tag{1.7}$$

for all $l\leqq t\leqq\gamma/|b|$.

Proof. Denote

$$\tau_g(x) = \lim_{t\to 0}\frac{\tau(g_t x) - \tau(x)}{t},$$

$$\tau_k(x) = \lim_{t\to 0}\frac{\tau(k_t x) - \tau(x)}{t}.$$

The functions τ_g and τ_k are continuous on M and

$$|\tau(x)|, \quad |\tau_g(x)|, \quad |\tau_k(x)| \leqq L_\tau \tag{1.8}$$

for some $L_\tau \geqq 1$ and all $x \in M$. We have

$$\int_M \tau_g \, d\mu = \int_M \tau_k \, d\mu = 0.$$

Let $\varepsilon > 0$ be given and let $0 < \theta = \theta(\varepsilon) < 0.01\,\varepsilon$ be chosen later. Let $K \subset M$ be an open subset of M such that $\bar{K}$ is compact and

$$\mu(K) > 1 - \theta, \qquad \mu(\partial K) = 0$$

where ∂K denotes the boundary of K.

Let $0 < \bar{\gamma} = \bar{\gamma}(\theta) < \min\{r_0, \theta\}$ be such that $\mu(O_{\bar{\gamma}}(\partial K)) \leqq \theta$ and if $u, v \in \bar{K}$, $d(u, v) \leqq \bar{\gamma}$ then

$$\begin{aligned} &|\tau_g(u) - \tau_g(v)| \leqq \theta, \\ &|\tau_k(u) - \tau_k(v)| \leqq \theta, \end{aligned} \tag{1.9}$$

where $O_{\bar{\gamma}}(\partial K)$ denotes the $\bar{\gamma}$-neighborhood of ∂K.

Denote

$$K_{\bar{\gamma}} = K - O_{\bar{\gamma}}(\partial K), \qquad \mu(K_{\bar{\gamma}}) \geqq 1 - 2\theta.$$

It follows from (1.2) that there is $0 < \gamma \leqq 0.01\,\bar{\gamma}$ such that if $|bt| \leqq \gamma$ then

$$\begin{aligned} &|e^{-2\gamma} \sigma(b, t)| \geqq 0.99\,t \\ &\left| \frac{e^{-2r(b,s)} - 1}{\Delta(b, t)/t} - \frac{2s}{t} \right| \leqq \theta \end{aligned} \tag{1.10}$$

for all $0 \leqq s \leqq t$.

Since h_t is ergodic, there exist $t_1 = t_1(\varepsilon) > 0$ and a subset $E = E(\varepsilon) \subset M$ with $\mu(E) > 1 - \varepsilon$ such that if $u \in E$, $t \geqq t_1$ then the relative length measure of $K_{\bar{\gamma}}$ on the orbit interval $[u, h_t u]$ is at least $1 - 3\theta$ and

$$\begin{aligned} &\left| \frac{1}{t} \int_0^t \tau(h_s u)\, ds - \bar{\tau} \right| \leqq \theta, \\ &\left| \frac{1}{t} \int_0^t \tau_g(h_s u)\, ds \right| \leqq \theta. \end{aligned} \tag{1.11}$$

Choose $l = l(\varepsilon) > t_1$ such that

$$t_1 / l \leqq \theta. \tag{1.12}$$

Let us show that if $x, y \in E$, $x = k_b g_a y$ for some $|a| \leqq \gamma$, $|b| \leqq \gamma/l$, and $l \leqq t \leqq \gamma |b|$ then (1.7) holds if θ is sufficiently small.

Indeed we have from (1.6),

$$\Delta_\tau(x; a, b, t) = \Delta_\tau(g_a y; a, 0, \sigma(b, t)) + \Delta_\tau(x; b, t) = \bar{\Delta}_\tau + \tilde{\Delta}_\tau.$$

Also,

$$\Delta(a, b, t) = \sigma(b, t)(e^{-2a} - 1) + (\sigma(b, t) - t) = \bar{\Delta} + \tilde{\Delta}.$$

We shall estimate $\bar{\Delta}_\tau / \bar{\Delta}$ and $\tilde{\Delta}_\tau / \tilde{\Delta}$.

1) First let us estimate $\bar{\Delta}_\tau/\bar{\Delta}$. We have

$$\bar{\Delta}_\tau = \int_0^{\sigma(a,b,t)} \tau(h_s y)\,ds - \int_0^{\sigma(b,t)} \tau(h_s g_a y)\,ds$$

$$\int_0^{\sigma(b,t)} \tau(h_s g_a y)\,ds = \int_0^{\sigma(b,t)} \tau(g_a h_{se^{-2a}} y)\,ds = e^{2a} \int_0^{\sigma(a,b,t)} \tau(g_a h_s y)\,ds$$

$$= e^{2a}\left[\int_0^{\sigma(a,b,t)} \tau(h_s y)\,ds + a \int_0^{\sigma(a,b,t)} \tau_g(y_s)\,ds\right]$$

where $y_s = g_{a(s)} h_s y$ for some $a(s)\in[0, a]$. This implies that

$$\bar{\Delta}_\tau = (1-e^{2a}) \int_0^{\sigma(a,b,t)} \tau(h_s y)\,ds - ae^{2a} \int_0^{\sigma(a,b,t)} \tau_g(y_s)\,ds = I_1 + I_2.$$

We have

$$I_1 = \bar{\Delta}\left(\int_0^{\sigma(a,b,t)} \tau(h_s y)\,ds\right) \Big/ \sigma(a,b,t) = \bar{\Delta}(\bar{\tau}+\theta_1) \tag{1.13}$$

where $|\theta_1| \leqq \theta$ by (1.11), since $\sigma(a,b,t) \geqq t_1$ by (1.10).

For I_2 we have

$$I_2 = ae^{2a} \int_0^{\sigma(a,b,t)} \tau_g(h_s y)\,ds + ae^{2a} \int_0^{\sigma(a,b,t)} (\tau_g(y_s) - \tau_g(h_s y))\,ds = I_2' + I_2''.$$

Also

$$I_2' = a\sigma(b,t)\,\theta_2 = \frac{1}{e^{-2a}-1}\,\bar{\Delta}\cdot\theta_2$$

where $|\theta_2| \leqq \theta$ by (1.11). Thus

$$|I'| \leqq |\bar{\Delta}|\,\theta/2. \tag{1.14}$$

We have

$$\begin{aligned} |I_2''| &\leqq |a|\,e^{2a} \int_0^{\sigma(a,b,t)} |\tau_g(y_s) - \tau_g(h_s y)\,ds \\ &\leqq |a|\,e^{2a} \int_{K_y} |\tau_g(y_s) - \tau_g(h_s y)|\,ds + 6|a|\,e^{2a}\,\sigma(a,b,t)\,L_\tau\theta \\ &\leqq |\bar{\Delta}|\,\theta(1-3\theta+6L_\tau)/2 \end{aligned} \tag{1.15}$$

by (1.8) and (1.9) since $y\in E$, $d(h_s y, y_s) \leqq |a| \leqq \gamma$ and $y_s \in \bar{K}$, if $h_s y \in K_{\bar{\gamma}}$, where $K_y = \{s\in[0, \sigma(a,b,t)]: h_s y \in K_{\bar{\gamma}}\}$.

It follows now from (1.13), (1.14) and (1.15) that we can choose $\theta = \theta(\varepsilon) > 0$ such that

$$|\bar{\Delta}_\tau - \bar{\tau}\bar{\Delta}| \leqq \varepsilon\bar{\Delta}. \tag{1.16}$$

2) Now let us estimate $\tilde{\Delta}_\tau/\tilde{\Delta}$. For $s\in[0, t]$ denote

$$c(s) = c(b,s), \qquad r(s) = r(b,s), \qquad \sigma(s) = \sigma(b,s).$$

We have from (1.2),

$$c(s) = be^{-r(s)}, \qquad \sigma'(s) = e^{-2r(s)}, \qquad \tilde{\Delta} = t(e^{-r(t)}-1). \tag{1.17}$$

Thus

$$\tilde{\Delta}_\tau = \int_0^{\sigma(t)} \tau(h_s z)\, ds - \int_0^t \tau(h_s x)\, ds$$

where $z = k_{-b} x$. We have

$$\begin{aligned}\int_0^{\sigma(t)} \tau(h_s z)\, ds &= \int_0^t \tau(h_{\sigma(s)} z)\, \sigma'(s)\, ds \\ &= \int_0^t \tau(g_{-r(s)} k_{-c(s)} h_s x)\, e^{-2r(s)}\, ds \\ &= \int_0^t \tau(h_s x)\, e^{-2r(s)}\, ds - b \int_0^t e^{-3r(s)}\, \tau_k(x_s)\, ds \\ &\quad - \int_0^t r(s) \cdot \tau_g(z_s)\, e^{-2r(s)}\, ds\end{aligned}$$

where $x_s = k_{c_s} h_s x$ for some $c_s \in [-c(s), 0]$ and $z_s = g_{r_s} h_{\sigma(s)} z$ for some $r_s \in [0, r(s)]$. This implies

$$\begin{aligned}\tilde{\Delta}_\tau = \int_0^t (e^{-2r(s)} - 1)\, \tau(h_s x)\, ds + b \int_0^t e^{-3r(s)}\, \tau_k(x_s)\, ds \\ + \int_0^t r(s)\, \tau_g(z_s)\, e^{-2r(s)}\, ds = J_1 + J_2 + J_3.\end{aligned} \tag{1.18}$$

Using (1.10) we have

$$J_1 = \frac{2\tilde{\Delta}(t)}{t^2} \int_0^t s\tau(h_s x)\, ds + \frac{\tilde{\Delta}(t)}{t} \int_0^t \theta_3(s)\, \tau(h_s x)\, ds = J_1' + J_1'' \tag{1.19}$$

where $|\theta_3(s)| \leqq \theta$ for all $0 \leqq s \leqq t$. Thus

$$|J_1''| \leqq |\tilde{\Delta}|\, \theta(\bar{\tau} + \theta) \tag{1.20}$$

by (1.11), since $x \in A_0$.

For J_1' we have integrating by parts

$$\begin{aligned}\frac{1}{t^2} \int_0^t s\tau(h_s x)\, ds &= \frac{1}{t} \int_0^t \tau(h_s x)\, ds - \frac{1}{t^2} \int_0^t \left(\int_0^s \tau(h_u x)\, du \right) ds \\ &= \bar{\tau} + \theta_1(t) - \frac{1}{t^2} \int_{t_1}^t \left(\int_0^s \tau(h_u x)\, du \right) ds + \theta_4(t) \\ &= \bar{\tau} + \theta_1(t) - \frac{1}{t^2} \int_{t_1}^t s(\bar{\tau} + \theta_1(s))\, ds + \theta_4(t)\end{aligned}$$

where $|\theta_1(s)| \leqq \theta$ for all $t_1 \leqq s \leqq t$ and $|\theta_4(t)| \leqq \theta \cdot L_\tau$ by (1.11), (1.8) and (1.12). This, (1.19) and (1.20) imply that

$$\left| \frac{J_1}{\tilde{\Delta}} - \bar{\tau} \right| \leqq \theta(L_\tau + \bar{\tau} + 10). \tag{1.21}$$

For J_2 we have, using (1.3),

$$|J_2| \leqq \frac{4|\tilde{\Delta}|}{t} L_\tau \leqq \theta L_\tau \tag{1.22}$$

by (1.17), since $|r(t)| \leqq \gamma \leqq 0.01\,\theta$.

For J_3 we have

$$\begin{aligned}\left| J_3 - \int_0^t r(s)\,\tau_g(h_s x)\,e^{-2r(s)}\,ds \right| &\leqq 2|r(t)| \Big[\int_{K_x} |\tau_g(z_s) - \tau_g(h_s x)|\,ds + 6L_\tau t\theta \Big] \\ &\leqq 2t|r(t)|(\theta(1-3\theta) + 6L_\tau \theta) \\ &\leqq 2|\tilde{\Delta}|\,\theta(4+6L_\tau)\end{aligned} \tag{1.23}$$

by (1.8) and (1.9), since $d(h_s x, z_s) \leqq 2\gamma < \bar{\gamma}$, $z_s \in \bar{K}$ if $h_s x \in K_{\bar{\gamma}}$, where

$$K_x = \{s \in [0,t] : h_s x \in K_{\bar{\gamma}}\}.$$

We also have

$$\begin{aligned}\left| \int_0^t r(s)\,\tau_g(h_s x)\,e^{-2r(s)}\,ds - \int_0^t r(s)\,\tau_g(h_s x)\,ds \right| &\leqq |\tilde{\Delta}/t|(2+\theta)\cdot|r(t)|\cdot t\cdot L_\tau \\ &\leqq 3|\tilde{\Delta}|\,\theta L_\tau\end{aligned} \tag{1.24}$$

by (1.10).

Finally by using integration by parts we get

$$\int_0^t r(s)\,\tau_g(h_s x)\,ds = r(t)\int_0^t \tau_g(h_s x)\,ds - \int_0^t r'(s)\left(\int_0^s \tau_g(h_u x)\,du\right)ds = Q_1 + Q_2$$

where $r'(s) = b e^{-r(s)}$ by (1.2). We have

$$|Q_1| \leqq 2\theta|\tilde{\Delta}| \tag{1.25}$$

by (1.3) and (1.11). Also,

$$|Q_2| \leqq |b| \left[\int_0^{t_1} e^{-r(s)} \left| \int_0^s \tau_g(h_u x)\,du \right| ds + 2\theta \int_{t_1}^t s\,ds \right] \leqq 2|\tilde{\Delta}|\,\theta(L_\tau + 1)$$

by (1.3), (1.8) and (1.12). This and (1.18), (1.21)–(1.25), imply that

$$|\tilde{\Delta}_\tau - \bar{\tau}\tilde{\Delta}| \leqq 100\,\theta|\tilde{\Delta}|(L_\tau + \bar{\tau} + 1).$$

This, (1.16) and Lemma 1.1 show that

$$|\Delta_\tau(x; a, b, t) - \bar{\tau}\Delta(a, b, t)| \leqq \varepsilon \max\{|\Delta(a, b, s)| : s \in [0, t]\}$$

if $\theta = \theta(\varepsilon)$ is sufficiently small. This completes the proof of the lemma. □

Remark 1.1. It follows from the proof of Lemma 1.2 that we can set $E = M$ in the lemma if M is compact, since in this case h_t is uniquely ergodic.

2. The H-property

In this section we shall prove Theorem 1. Let us state this theorem in full.

Theorem 2.1 (The H-property, see [10]). *Suppose $\tau \in B^1(M)$, $M = \Gamma \backslash G$, $\Gamma \in \mathcal{T}$. Then the flow h_t^τ possesses the following property: given $p > 0$, $\varepsilon > 0$, $N > 1$, there are $\alpha = \alpha(p, \varepsilon) > 0$, $\delta = \delta(p, \varepsilon, N) > 0$ and a subset $B = B(p, \varepsilon, N) \subset M$ with $\mu(B) > 1 - \varepsilon$ such that if $x, y \in B$, $d(x, y) \leqq \delta$ and y is not on the h_t^τ-orbit of x, then there are $L = L(x, y) > 0$ and $Q = Q(x, y) \geqq N$ with $Q/L \geqq \alpha$ such that either $d(h_{np}^\tau, h_{(n+1)p}^\tau) \leqq \varepsilon$ for all integers $n \in [L, L+Q]$ or $d(h_{np}^\tau, h_{(n-1)p}^\tau) \leqq \varepsilon$ for all integers $n \in [L, L+Q]$, where d denotes the distance in M.*

Proof. First let us show that Theorem 2.1 follows from the following property: given $p, \varepsilon, N > 0$ there are $\tilde{\alpha} = \tilde{\alpha}(p, \varepsilon) > 0$, $\tilde{\delta} = \tilde{\delta}(p, \varepsilon, N) > 0$ and a subset $\tilde{B} = \tilde{B}(p, \varepsilon, N) \subset M$ with $\mu(\tilde{B}) > 1 - \varepsilon$ such that if $x, y \in \tilde{B}$, $x = h_c k_b g_a y$ for some $|c|$, $|a|$, $|b| \leqq \delta$, $|a| + |b| \neq 0$ then there are $\tilde{L} = \tilde{L}(x, y) > 0$ and $\tilde{Q} = \tilde{Q}(x, y) > N$ with $\tilde{Q}/\tilde{L} = \tilde{\alpha}$ such that

$$|b|(\tilde{L} + \tilde{Q}) \leqq \varepsilon$$

and

$$\begin{aligned} &\text{either} \quad |\Delta_\tau(x; a, b, t) - p| \leqq \varepsilon \quad \text{for all } t \in [\tilde{L}, \tilde{L} + \tilde{Q}] \\ &\text{or} \quad |\Delta_\tau(x; a, b, t) + p| \leqq \varepsilon \quad \text{for all } t \in [\tilde{L}, \tilde{L} + \tilde{Q}], \end{aligned} \tag{2.1}$$

where $\Delta_\tau(x; a, b, t)$ is as in Sect. 1.

Indeed, let $p, \varepsilon > 0$, $N > 1$, be given and let $\varepsilon_1 = \min\{\varepsilon/3, \tilde{\alpha}(p, \varepsilon/3)\}$.

Since h_t is ergodic, there are $t_1 \geqq 2Np/\bar{\tau}$ and $B_1 \subset M$, $\mu(B_1) > 1 - 0.4\,\varepsilon_1/1 + \varepsilon_1$ such that if $u \in B_1$, $t \geqq t_1$ then

$$\bar{\tau} t/2 \leqq \int_0^t \tau(h_s u)\, ds \leqq 3\bar{\tau} t/2. \tag{2.2}$$

Also there are $t_2 \geqq 2t_1$ and $B_2 \subset M$, $\mu(B_2) > 1 - \varepsilon/3$ such that if $u \in B_2$, $t \geqq t_2$ then the relative length measure of B_1 on the orbit interval $[u, h_t u]$ is at least $1 - 0.5\,\varepsilon_1/1 + \varepsilon_1$.

Now set $\alpha = \tilde{\alpha}(p, \varepsilon/3)/6 = \tilde{\alpha}/6$, $\delta = \tilde{\delta}(p, \varepsilon/3, t_2)/3 = \tilde{\delta}/3$, $B = B_1 \cap B_2 \cap \tilde{B}(p, \varepsilon/3, t_2)$, $\mu(B) > 1 - \varepsilon$, and show that α, δ and B satisfy Theorem 2.1.

Indeed, let $x, y \in B$, $d(x, y) \leqq \delta$ and y is not on the h_t^τ-orbit of x. Then $x = h_c k_b g_a y$ for some $|c|$, $|b|$, $|a| \leqq \tilde{\delta}$ and $|a| + |b| \neq 0$.

Since $x, y \in \tilde{B}(p, \varepsilon/3, t_2)$ there are $\tilde{L} > 0$, $\tilde{Q} \geqq t_2$ with $\tilde{Q}/\tilde{L} = \tilde{\alpha}$ such that (2.1) holds with $\varepsilon/3$ instead of ε. We have

$$\alpha \tilde{L} = \tilde{Q} \geqq t_2 \geqq 2t_1, \qquad \tilde{L} \geqq t_2. \tag{2.3}$$

Since $x \in B_2$ there is $s \in [\tilde{L}, \tilde{L} + \tilde{Q}]$ such that $h_s x \in B_1$. This and (2.3) imply that

$$\xi(h_{\tilde{L}} x, \tilde{Q}) = \int_{\tilde{L}}^{\tilde{L}+\tilde{Q}} \tau(h_s x)\, ds \geqq \tilde{Q}\bar{\tau}/4 \geqq t_1\, \bar{\tau}/2 \geqq Np.$$

Also

$$\xi(x, \tilde{L}) \leqq 3\bar{\tau}\tilde{L}/2 \tag{2.4}$$

by (2.2), since $x \in B_1$.

Set $L=\xi(x,\tilde{L})/p$, $Q=\xi(h_{\tilde{L}}x,\tilde{Q})/p$. We have from (2.3) and (2.4)

$$Q/L \geqq \tilde{Q}/6\tilde{L}=\tilde{\alpha}/6=\alpha, \qquad Q \geqq N.$$

It is clear that if $\xi(x,t)=np$ for some integer $n\in[L,L+Q]$ then $t\in[\tilde{L},\tilde{L}+\tilde{Q}]$ and either $d(h^{\tau}_{np},h^{\tau}_{(n+1)p})\leqq\varepsilon$ or $d(h^{\tau}_{np},h^{\tau}_{(n-1)p})\leqq\varepsilon$ by (1.2) and (2.1) with $\varepsilon/3$ instead of ε.

Thus (2.1) implies Theorem 2.1. Now we shall prove (2.1).

It was shown in [9] that the horocycle flow h_t satisfies the following stronger form of (2.1): given $p,\varepsilon,N>0$ there are $0<\bar{\alpha}=\bar{\alpha}(p,\varepsilon)<1$ and $\bar{\delta}=\bar{\delta}(p,\varepsilon,N)>0$ such that if $|a|$, $|b|\leqq\delta$, $|a|+|b|\neq 0$ then there are $\bar{L}=\bar{L}(a,b)>0$, $\bar{Q}=\bar{Q}(a,b)\geqq N$ with $\bar{Q}/\bar{L}=\bar{\alpha}$ such that

$$|b|(\bar{L}+\bar{Q})\leqq\varepsilon$$
$$\max\{|\Delta(a,b,s)|: s\in[0,\bar{L}]\}\leqq p$$

and

$$\begin{aligned} &\text{either} \quad |\Delta(a,b,t)-p|\leqq\varepsilon \quad \text{for all } t\in[\bar{L},\bar{L}+\bar{Q}]\\ &\text{or} \quad |\Delta(a,b,t)+p|\leqq\varepsilon \quad \text{for all } t\in[\bar{L},\bar{L}+\bar{Q}]. \end{aligned}$$

We use this and Lemma 1.2 to prove (2.1). Indeed, let $p,\varepsilon,N>0$ be given. Let $q=p/\bar{\tau}$ and let

$$\begin{aligned} &\gamma=\gamma(\varepsilon/4q)<\min\{\varepsilon/4q,\varepsilon/4\bar{\tau},\varepsilon/4\},\\ &l=l(\varepsilon/4q),\\ &E=E(\varepsilon/4q)\subset M, \qquad \mu(E)>1-\varepsilon \end{aligned} \tag{2.5}$$

be as in Lemma 1.2.

Set $\tilde{\alpha}=\bar{\alpha}(q,\gamma)$, $\tilde{\delta}=\min\{\gamma/l,\bar{\delta}(q,\gamma,l/\tilde{\alpha})\}$, $\tilde{B}=E(\varepsilon/4q)$, $\mu(\tilde{B})>1-\varepsilon$. We claim that $\tilde{\alpha},\tilde{\delta},\tilde{B}$ satisfy (2.1).

Indeed, let $x,y\in\tilde{B}$, $x=h_c k_b g_a y$ for some $|c|$, $|a|$, $|b|\leqq\tilde{\delta}$, $|a|+|b|\neq 0$. Set $\tilde{L}=\tilde{L}(x,y)=\bar{L}(a,b)>0$, $\tilde{Q}=\tilde{Q}(x,y)=\bar{Q}(a,b)>0$.

We have, by our choice of $\tilde{\delta}$,

$$\tilde{Q}\geqq t_0/\tilde{\alpha}\geqq N, \qquad \tilde{\alpha}\tilde{Q}=\tilde{L}\geqq t_0. \tag{2.6}$$

Also,

$$|b|(\tilde{L}+\tilde{Q})\leqq\gamma\leqq\varepsilon/4$$
$$\max\{|\Delta(a,b,s|: s\in[0,\tilde{L}]\}\leqq q$$

and

$$\begin{aligned} &\text{either} \quad |\Delta(a,b,t)-q|\leqq\gamma \quad \text{for all } t\in[\tilde{L},\tilde{L}+\tilde{Q}]\\ &\text{or} \quad |\Delta(a,b,t)+q|\leqq\gamma \quad \text{for all } t\in[\tilde{L},\tilde{L}+\tilde{Q}]. \end{aligned} \tag{2.7}$$

It follows from (2.6), (2.7) and Lemma 1.2 that

$$|\Delta_\tau(x;a,b,t)-\bar{\tau}\Delta(a,b,t)|\leqq\varepsilon/4$$

for all $t\in[\tilde{L},\tilde{L}+\tilde{Q}]$. This, (2.5) and (2.7) imply (2.1). This completes the proof of Theorem 2.1. □

3. The rigidity of time changes theorem

In this section we give a proof of Theorem 5 independent of Theorems 1–4.

So let $\tau_i \in B^1(M)$, $i=1,2$, $\bar\tau_1=\bar\tau_2$. We can assume without loss of generality that $\bar\tau_1=\bar\tau_2=1$. Also,

$$Q^{-1} \leqq |\tau_i(x)| \leqq Q \tag{3.1}$$

for some $Q \geqq 1$, $i=1,2$.

Let $\psi: (M_1, \mu_{\tau_1}) \to (M_2, \mu_{\tau_2})$ be m.p. and let

$$\psi h_t^{\tau_1}(x) = h_t^{\tau_2} \psi(x)$$

for all $t \in R$ and μ_{τ_1}-a.e. $x \in M_1$.

Let $p: G \to M_2$ be the covering projection $p(g)=\Gamma_2 g$, $g \in G$. Since Γ_2 is discrete, there are a compact $K \subset M_2$ with $\mu_1(\psi^{-1}(K)) > 0.999$ and $\rho=\rho(K)>0$ such that if $g \in p^{-1}(K)$ then p is an isometry on the ball of radius ρ centered at g.

Let $\varepsilon_n = 10^{-n}$, $n=4,5,\ldots$ and let $0<\bar\varepsilon_n \leqq 0.1\,\varepsilon_n$ be such that

$$\mu_1(\psi^{-1}(B)) \leqq 0.1\,\varepsilon_n$$

whenever $\mu_2(B) \leqq \bar\varepsilon_n$, $B \subset M_2$.

Let $\gamma_n^{(i)} = \gamma^{(i)}(\bar\varepsilon_n)$, $l_n^{(i)} = l^{(i)}(\bar\varepsilon_n)$ and $E_n^{(i)} = E^{(i)}(\bar\varepsilon_n)$ be as in Lemma 1.2 for τ_i, $i=1,2$. Denote

$$\gamma_n = \min\{\gamma_n^{(1)}, \gamma_n^{(2)}, 0.1\,\varepsilon_n\},$$
$$l_n = \max\{l_n^{(1)}, l_n^{(2)}\},$$
$$E_n = E_n^{(1)} \cap \psi^{-1}(E_n^{(2)}).$$

We have

$$\mu_1(E_n) > 1 - 0.2\,\varepsilon_n.$$

Since ψ is measurable, there is $\Lambda_n \subset M_1$, $\mu_1(\Lambda_n) > 1-0.1\,\varepsilon_n$ such that ψ is uniformly continuous on Λ_n. Given $\varepsilon>0$ let $\delta(\varepsilon)>0$ be such that $d(\psi(u), \psi(v)) \leqq \varepsilon/4$ whenever $u, v \in \Lambda_n$, $d(u,v) \leqq \delta(\varepsilon)$.

Denote

$$D = \rho/2Q.$$

Since $h_t^{(i)}$ is ergodic, $i=1,2$, there are $t_n \geqq \max\{l_n, 20D\gamma_n^{-1}\}$ and a subset $A_n \subset M_1$ with $\mu_1(A_n) > 1-0.1\,\varepsilon_n$ such that if $u \in A_n$, $t \geqq t_n$ then the relative length measure of $\Lambda_n \cap \psi^{-1}(K)$ on the orbit interval $[u, h_t^{(1)}u]$ is at least 0.998 and

$$\begin{aligned} \left|\int_0^t \tau_1(h_s^{(1)}u)\,ds - t\right| &\leqq 0.1\,\varepsilon_n t, \\ \left|\int_0^t \tau_2(h_s^{(2)}\psi(u))\,ds - t\right| &\leqq 0.1\,\varepsilon_n t. \end{aligned} \tag{3.2}$$

Denote

$$V_n = \Lambda_n \cap A_n \cap E_n, \qquad \mu_1(V_n) \geqq 1-\varepsilon_n = 1-10^{-n}. \tag{3.3}$$

For $a, b \in R$, $|a|+|b| \neq 0$, let $L(a,b)$ denote the first $t>0$ with $\Delta(a,b,t)=D/4$, where $\Delta(a,b,t)$ is defined in (1.4).

Let $0<\omega_n<\gamma_n$ be so small that if $|a|$, $|b|\leqq\omega_n$, $|a|+|b|\neq 0$, then

$$L(a, b)\geqq \max\{5t_n, 2D(\delta(\varepsilon_n))^{-1}\}. \tag{3.4}$$

Denote

$$\begin{aligned}\delta_n &= \min\{\omega_n, \delta(\omega_n)\}/4,\\ r_n &= \max\{-(\log\delta_n), \log t_n\},\\ V &= \bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} g^{(1)}_{-r_n}(V_n).\end{aligned} \tag{3.5}$$

It follows from (3.3) that

$$\mu_1(V)=1.$$

Lemma 3.1. *Let $x, y\in V$ and $x=k_b^{(1)}y$ for some $b\in R$. Then $d(\bar{x}_n, k_b^{(2)}\bar{y}_n)\to 0$ when $n\to\infty$, where $\bar{x}_n=g^{(2)}_{-r_n}\psi g^{(1)}_{r_n}x$, $\bar{y}_n=g^{(2)}_{-r_n}\psi g^{(1)}_{r_n}y$, $n=1, 2, \ldots$.*

Proof. Denote

$$x_n^{(1)}=g_{r_n}^{(1)}x_n, \qquad y_n^{(1)}=g_{r_n}^{(1)}y.$$

Since $x, y\in V$, there is $m\geqq 4$ such that $x_n^{(1)}, y_n^{(1)}\in V_n$ for all $n\geqq m$.

Henceforth we assume that $n\geqq m$. Denote

$$b_n=e^{-2r_n}b, \qquad x_n^{(2)}=\psi(x_n^{(1)}), \qquad y_n^{(2)}=\psi(y_n^{(1)}).$$

We have

$$x_n^{(1)}=k_{b_n}^{(1)}y_n^{(1)}$$
$$|b_n|\leqq|b|\,\delta_n^2\leqq\delta_n\leqq\omega_n,$$

if n is sufficiently large. Also,

$$x_n^{(2)}=k_{\beta_n}^{(2)}g_{\alpha_n}^{(2)}h_{c_n}^{(2)}y_n^{(2)}$$

for some $|\alpha_n|$, $|\beta_n|$, $|c_n|\leqq\eta_n\leqq\gamma_n$. We can assume without loss of generality that $c_n=0$. Denote

$$L_n=\min\{L(0, b_n), L(\alpha_n, \beta_n)\}\geqq 5t_n.$$

Since $x_n^{(1)}, y_n^{(1)}\in V_n$ and $L_n\geqq 5t_n$ there is

$$2t_n\leqq 0.99\,L_n\leqq\lambda_n\leqq 0.995\,L_n \tag{3.6}$$

such that

$$\begin{aligned}&h_{\lambda_n}^{(1)}x_n^{(1)}\in\Lambda_n\cap\psi^{-1}(K),\\ &h_{\sigma(b_n,\lambda_n)}^{(1)}y_n^{(1)}\in\Lambda_n\cap\psi^{-1}(K)\end{aligned} \tag{3.7}$$

if δ_n is sufficiently small, where $\sigma(b_n, \lambda_n)$ is defined in (1.2). Also

$$d(h_{\lambda_n}^{(1)}x_n^{(1)}, h_{\sigma(b_n,\lambda_n)}^{(1)}y_n^{(1)})\leqq\frac{2|\Delta(b_n, L_n)|}{L_n}+2|b_n|\leqq\frac{D}{L_n}+2\delta_n\leqq\delta(\varepsilon_n),$$
$$|b_n\lambda_n|\leqq 2|\Delta(b_n, \lambda_n)|/\lambda_n\leqq D/\lambda_n\leqq\gamma_n$$

by (1.3), (3.4) and (3.5), since $t_n\leqq\lambda_n\leqq L_n$.

This implies via Lemma 1.2 that

$$|\Delta_{\tau_1}(x_n^{(1)}; b_n, \lambda_n) - \Delta(b_n, \lambda_n)| \leqq 0.1\, D\varepsilon_n \tag{3.8}$$

since $x_n^{(1)} \in E_n$ and $\lambda_n \geqq t_n \geqq l_n$.

Denote

$$z_n = h_{s_n}^{(2)} x_n^{(2)}, \qquad w_n = h_{u_n}^{(2)} y_n^{(2)},$$

where s_n and u_n are defined by

$$\xi^{(1)}(x_n^{(1)}, \lambda_n) = \xi^{(2)}(x_n^{(2)}, s_n)$$
$$\xi^{(1)}(y_n^{(1)}, \sigma(b_n, \lambda_n)) = \xi^{(2)}(y_n^{(2)}, u_n)$$

and

$$\xi^{(i)}(v^{(i)}, t) = \int_0^t \tau_i(h_s^{(i)} v^{(i)})\, ds, \qquad v^{(i)} \in M_i,\ i = 1, 2.$$

We have using (3.2) and (3.6),

$$\begin{gathered} 2t_n \leqq 0.98\, L_n \leqq s_n \leqq 0.996\, L_n \leqq L(\alpha_n, \beta_n), \\ |s_n - \lambda_n| \leqq 0.8\, \varepsilon_n s_n. \end{gathered} \tag{3.9}$$

Also,

$$|\beta_n s_n| \leqq \frac{20D}{s_n} \leqq \frac{20D}{t_n} \leqq \gamma_n$$

by Lemma 1.1, (1.3) and the definition of t_n. This implies via Lemma 1.2 that

$$|\Delta_{\tau_2}(x_n^{(2)}; \alpha_n, \beta_n, s_n) - \Delta(\alpha_n, \beta_n, s_n)| \leqq 0.1\, D\varepsilon_n$$

since $|\alpha_n| \leqq \omega_n \leqq \gamma_n$, $l_n \leqq t_n \leqq s_n$. This and (3.8) imply that

$$\begin{aligned} &|(\xi^{(2)}(y_n^{(2)}; \sigma(\alpha_n, \beta_n, s_n)) - \xi^{(2)}(y_n^{(2)}; u_n)) \\ &\quad - (\Delta(\alpha_n, \beta_n, s_n) - (\Delta(b_n, \lambda_n))| \leqq 0.1\, D\varepsilon_n \end{aligned} \tag{3.10}$$

and therefore

$$\int_{u_n}^{\sigma(\alpha_n, \beta_n, s_n)} \tau_2(h_s^{(2)} y_n^{(2)})\, ds = |\xi^{(2)}(y_n^{(2)}; \sigma(\alpha_n, \beta_n, s_n)) - \xi^{(2)}(y_n^{(2)}; u_n)| \leqq 2D.$$

This implies that

$$|u_n - \sigma(\alpha_n, \beta_n, s_n)| \leqq 2DQ = \rho. \tag{3.11}$$

We have

$$d(h_{\sigma(\alpha_n, \beta_n, s_n)}^{(2)} y_n^{(2)}, z_n) \leqq 3\gamma_n \leqq \varepsilon_n/3. \tag{3.12}$$

Also,

$$\begin{gathered} z_n = \psi(h_{\lambda_n}^{(1)} x_n^{(1)}) \in K, \\ w_n = \psi(h_{\sigma(b_n, \lambda_n)}^{(1)} y_n^{(1)}) \in K, \\ d(z_n, w_n) \leqq \varepsilon_n/4. \end{gathered} \tag{3.13}$$

This and (3.12) show that

$$d(h_{\sigma(\alpha_n, \beta_n, s_n)}^{(2)} y_n^{(2)}, w_n) \leqq \varepsilon_n$$

and therefore

$$|u_n - \sigma(\alpha_n, \beta_n, s_n)| \leqq \varepsilon_n \tag{3.14}$$

by (3.11) and the definition of ρ. This implies that

$$|\xi^{(2)}(y_n^{(2)}; \sigma(\alpha_n, \beta_n, s_n)) - \xi^{(2)}(y_n^{(2)}; u_n)| \leqq Q\varepsilon_n$$

and therefore

$$|\Delta(\alpha_n, \beta_n, s_n) - \Delta(b_n, \lambda_n)| \leqq \varepsilon_n(D+Q) \tag{3.15}$$

by (3.10). Also

$$|\Delta(b_n, s_n) - \Delta(b_n, \lambda_n)| \leqq 10\,\varepsilon_n\, \Delta(b_n, \lambda_n) \leqq 5\,\varepsilon_n D$$

by (1.2) and (3.9). This and (3.15) give

$$|\Delta(\alpha_n, \beta_n, s_n) - \Delta(b_n, s_n)| \leqq \varepsilon_n(6D+Q). \tag{3.16}$$

We have

$$x_n^{(2)} = k_{b_n}^{(2)}(k_{p_n}^{(2)}\, g_{\alpha_n}^{(2)}\, y_n^{(2)})$$

where $p_n = \beta_n - b_n$. Expression (3.16) says that

$$|\Delta(\alpha_n, p_n, \sigma(b_n, s_n))| \leqq \varepsilon_n(6D+Q)$$

and therefore

$$\begin{aligned} |p_n| &\leqq \frac{2\varepsilon_n(6D+Q)}{[\sigma(b_n, s_n)]^2} \leqq c_1\, \varepsilon_n / L_n^2, \\ |\alpha_n| &\leqq \frac{2\varepsilon_n(6D+Q)}{|\sigma(b_n, s_n)|} \leqq c_1\, \varepsilon_n / L_n \end{aligned} \tag{3.17}$$

by (1.2), (1.3), (3.6) and (3.9), where $c_1 > 0$ is a constant.

It follows now from Corollary 1.1, expressions (3.6), (3.9) and the definition of L_n that

$$\max\{|\Delta(b_n, \lambda_n)|, |\Delta(\alpha_n, \beta_n, s_n)|\} \geqq 0.1\, D.$$

This implies via (3.15) that

$$\min\{|\Delta(b_n, \lambda_n)|, |\Delta(\alpha_n, \beta_n, s_n)|\} \geqq 0.05\, D$$

if n is sufficiently large. This gives

$$\frac{0.05\, D}{2L_n^2} \leqq \frac{|\Delta(b_n, \lambda_n)|}{2\lambda_n^2} \leqq |b_n|$$

by (1.3) and therefore

$$\begin{aligned} |p_n| &\leqq c_2\, \varepsilon_n\, |b_n| \\ |\alpha_n| &\leqq c_2\, \varepsilon_n\, |b_n|^{\frac{1}{2}} \end{aligned} \tag{3.18}$$

for some $c_2 > 0$ by (3.17). We have

$$\bar{x}_n = g_{-r_n}^{(2)}\, x_n^{(2)} = k_b^{(2)}(k_{p_n e^{2r_n}}^{(2)}\, g_{\alpha_n}^{(2)}\, \bar{y}_n)$$

where

$$|p_n|\, e^{2r_n} \leqq c_1\, \varepsilon_n\, |b_n|\, e^{2r_n} = c_1\, \varepsilon_n\, |b|.$$

This and (3.18) show that

$$d(\bar{x}_n, k_b^{(2)}\bar{y}_n) \to 0 \quad \text{when} \quad n \to \infty.$$

This completes the proof of the lemma. □

Lemma 3.2. *If $x \in V$ and $y = h_p^{(1)}x$ for some $p \in R$ then $d(\bar{y}_n, h_p^{(2)}\bar{x}_n) \to 0$ when $n \to \infty$, where $\bar{x}_n$, $\bar{y}_n$ are as in Lemma 2.1.*

Proof. Let $x_n^{(i)}$, $y_n^{(i)}$, $i=1,2$, be as in the proof of Lemma 2.1. We have

$$y_n^{(1)} = h_{pe^{2r_n}}^{(1)}(x_n^{(1)}), \qquad y_n^{(2)} = h_{q_n}^{(2)}(x_n^{(2)})$$

where q_n is given by

$$\xi^{(1)}(x_n^{(1)}, pe^{2r_n}) = \xi^{(2)}(x_n^{(2)}, q_n)$$

and

$$pe^{2r_n} \geqq pt_n^2 \geqq t_n$$

if n is sufficiently large. It follows from (3.2) that

$$|pe^{2r_n} - q_n| \leqq \varepsilon_n pe^{2r_n} \tag{3.19}$$

if n is sufficiently large. We have

$$\bar{y}_n = g_{-r_n}^{(2)} y_n^{(2)} = h_{q_n e^{-2r_n}}^{(2)}(\bar{x}_n)$$

where

$$|p - q_n e^{-2r_n}| \leqq \varepsilon_n p$$

by (3.19). This completes the proof. □

Lemma 3.3. *There is an $h_t^{(1)}$-invariant subset $U \subset M_1$ with $\mu_1(U)=1$ and a subsequence $\{n_k: k=1,2,\ldots\} \subset \{n: n=1,2,\ldots\}$ such that if $u \in U$ then $\lim_{k\to\infty} \bar{u}_{n_k} = \zeta(u) \in M_2$ exists and $\zeta(h_p^{(1)}u) = h_p^{(2)}\zeta(u)$ for all $p \in R$, $u \in U$.*

Proof. Let $M_2 = \bigcup_{n=1}^{\infty} K_n$, where K_n are compact and $\mu_2(M_2 - K_n) < 2^{-n}$, $n = 1, 2, \ldots$. Denote

$$\tilde{K}_n = M_2 - K_n$$

$$F_n = g_{-r_n}^{(1)} \psi^{-1} g_{r_n}^{(2)} \tilde{K}_n, \qquad n = 1, 2, \ldots.$$

We have

$$\sum_{n=1}^{\infty} \mu_1(F_n) < \infty.$$

Let

$$F = \{u \in M_1: u \text{ belongs to finitely many } F_n\}.$$

By the Borel-Cantelli lemma,

$$\mu_1(F) = 1.$$

If $u \in F$ then $\bar{u}_n \in \tilde{K}_n$ for finitely many n. This implies that there is a subsequence $n_k(u)$, $k=1,2,\ldots$, such that $\bar{u}_{nk(u)}$ converges in M_2.

For $x \in M_1$ let $W(x)=\{g_t^{(1)} k_s^{(1)} x: s, t \in R\}$ be the stable leaf of x for $g_t^{(1)}$. Let $I(x)$ denote the orbit interval $[x, k_{-1}^{(1)} x]$ and let $\pi_x: I(x) \to W(h_1^{(1)} x)$ be defined by

$$\pi_x(y)=h_{\sigma(b,\,1)}^{(1)} y$$

if $y=k_{-b}^{(1)} x$, $0 \leqq b \leqq 1$, where $\sigma(b, 1)$ is defined in (1.1). The map π_x is a diffeomorphism from $I(x)$ onto $\pi_x(I(x))$.

Denote $\tilde{x}=h_1^{(1)} x$. We have from (1.1),

$$\tilde{x}=k_{c(b,\,1)}^{(1)} g_{r(b,\,1)}^{(1)} \pi_x(y)$$

if $y=k_{-b}^{(1)} x \in I(x)$. Define $q_x: I(x) \to [\tilde{x}, g_{-r(1,\,1)}^{(1)} \tilde{x}]=J(\tilde{x})$ by

$$q_x(y)=g_{-r(b,\,1)}^{(1)} \tilde{x}, \qquad y=k_{-b}^{(1)} x.$$

The map q_x is a diffeomorphism from $I(x)$ onto $J(\tilde{x})$. Let

$$\begin{aligned} T(x)=\{z \in J(x)&: \text{the length measure of } M_1-V \text{ on the } k_t^{(1)}\text{-orbit} \\ &\text{of } z \text{ is } 0\}, \\ S'=\{x \in M_1&: \text{the relative length measure of } T(x) \text{ on } J(x) \text{ is } 1\}, \\ S=\{x \in V&: \tilde{x} \in S' \text{ and the relative length measure of the set} \\ &\tilde{J}(\tilde{x})=T(\tilde{x}) \cap q_x(V \cap \pi_x^{-1}(V) \cap I(x)) \text{ on } J(\tilde{x}) \text{ is } 1\}. \end{aligned}$$

Standard measure theoretic arguments show that $\mu_1(S')=\mu_1(S)=1$.

Now pick $x \in S \cap F$ and let $n_k=n_k(x)$, $k=1, 2, \ldots$ be such that $\bar{x}_{n_k}$ converges in M_2 when $k \to \infty$.

Let $\tilde{J}=\tilde{J}(\tilde{x})$ and let $u \in \tilde{J}$. Then $q_x^{-1}(u) \in V \cap I(x)$ and therefore $(\overline{q_x^{-1}(u)})_{n_k}$ converges in M_2 by Lemma 2.1 and $(\overline{\pi_x(q_x^{-1}(u))})_{n_k}$ converges in M_2 by Lemma 2.2. This implies via Lemma 2.1 that $\bar{z}_{n_k}$ converges in M_2 for every $z \in V$ on the $k_t^{(1)}$-orbit of u, since $\pi_x(q_x^{-1}(u))$ lies on this orbit and belongs to V.

Thus $\bar{z}_{n_k}$ converges for every z in the set

$$\tilde{V}=V \cap \Big(\bigcup_{u \in \tilde{J}} I(u)\Big).$$

It follows from the definition of $\tilde{J}$ that the Riemannian volume of $\tilde{V}$ on the leaf $W(\tilde{x})$ is positive. Let

$$U=\{h_t^{(1)} v: v \in \tilde{V}\}.$$

The set U is $h_t^{(1)}$-invariant, $\mu_1(U)=1$ and $\lim_{k \to \infty} \bar{z}_{n_k}=\zeta(z) \in M_2$ exists for every $z \in U$ by Lemma 2.2. It also follows from that lemma that $\zeta(h_p^{(1)} z)=h_p^{(2)} \zeta(z)$ for every $z \in U$. This completes the proof. $\square$

Proof of Theorem 5. Let $U \subset M_1$, $\mu_1(U)=1$ and a subsequence $\{n_k\} \subset \{n\}$ be as in Lemma 3.3. We can assume without loss of generality that $U=M_1$ and $\{n_k\}=\{n\}$. Thus

$$\lim_{n \to \infty} \bar{u}_n=\zeta(u) \in M_2$$

exists for all $u \in M_1$ and

$$\zeta(h_p^{(1)} u)=h_p^{(2)} \zeta(u)$$

for all $p \in R$, $u \in M_1$. This says that the map

$$\zeta: (M_1, \mu_1) \to (M_2, \mu_2)$$

is a measurable conjugacy between $h_t^{(1)}$ and $h_t^{(2)}$. It follows then from the rigidity theorem [8] that there are $C \in G$, $a \in R$ such that

$$\begin{gathered} C\Gamma_1 C^{-1} \subset \Gamma_2 \quad \text{and} \\ \zeta(u) = h_a^{(2)} \psi_C(u) \end{gathered} \tag{3.20}$$

for μ_1-a.e. $u \in M_1$, where $\psi_C(\Gamma_1 g) = \Gamma_2 C g$, $g \in G$. It follows from Lemma 3.1 that if $u, v \in V$, $v = k_b^{(1)} u$ then

$$\zeta(v) = k_b^{(2)} \zeta(u).$$

This implies that $a = 0$ in (3.20) and therefore

$$\zeta(u) = \psi_C(u)$$

for μ_1-a.e. $u \in M_1$. This completes the proof of part (1) of the theorem.

Part (3) follows from part (2). In order to prove part (2) one has to repeat the proof of the corresponding part of the main theorem in [11].

Let us outline the main point of this proof. The proof makes essential use of the decay rate of the correlation function of $\tau_i - 1$ for $h_t^{(i)}$, $i = 1, 2$, found in [5] and [12].

Indeed, using this decay rate we proved in [11] (Lemma 3.1) that given $\varepsilon > 0$ there are $P = P(\varepsilon) \subset M_1$, $\mu_1(P) > 1 - \varepsilon$ and $m = m(P) > 0$ such that if $u \in P$, $t \geqq m$, then

$$\begin{gathered} \left| \int_0^t \tau_1(h_s^{(1)} u)\, ds - t \right| \leqq t^\alpha \\ \left| \int_0^t \tau_2(h_s^{(2)} \psi(u))\, ds - t \right| \leqq t^\alpha \end{gathered} \tag{3.21}$$

for some $0 < \alpha < 1$ and all $t \geqq m$.

Now let $\bar{X} \subset M_1$, $\mu(\bar{X}) > 1 - \varepsilon$ and $n_0 > 0$ be such that if $u \in \bar{X}$ and $n \geqq n_0$ then

$$d(\bar{u}_n, \zeta(u)) \leqq \varepsilon.$$

Fix $n \geqq n_0$ with $e^{2r_n} \geqq 2Q^2 m$, where $Q \geqq 1$ is as in (3.1), and denote

$$v(u) = \bar{u}_n.$$

Let $\tilde{X}$ be the generic set of $\bar{X} \cap g_{-r_n}^{(1)}(P)$ for $h_t^{(1)}$, $\mu_1(\tilde{X}) = 1$. This means that if $u \in \tilde{X}$ then the relative length measure of $\bar{X} \cap g_{-r_n}^{(1)}(P)$ on $[u, h_t^{(1)} u]$ tends to $\mu_1(\bar{X} \cap g_{-r_n}^{(1)}(P)) \geqq 1 - 2\varepsilon$ when $t \to \infty$.

Let

$$X = \tilde{X} \cap \bar{X} \cap g_{-r_n}^{(1)}(P), \qquad \mu_1(X) > 0$$

and for $u \in X$ let

$$A(u) = \{s \in R^+ : h_s^{(1)} u \in \bar{X} \cap g_{-r_n}^{(1)}(P)\}.$$

For $s \in R^+$ define $t(s) > 0$ by

$$h^{(2)}_{t(s)} v(u) = v(h^{(2)}_s u).$$

We have for $s \in A(u)$,

$$g^{(1)}_{r_n} h^{(1)}_s u \in P$$
$$\psi(g^{(1)}_{r_n} h^{(1)}_s u) = g^{(2)}_{r_n} v(h^{(1)}_s u) = g^{(2)}_{r_n} h^{(2)}_{t(s)} v(u).$$

It follows from (3.21) that if $s, s' \in A(u)$ and $|s - s'| \geqq 1$ then

$$e^{2r_n} |(t(s) - t(s')) - (s - s')| \leqq |s - s'|^\alpha e^{2\alpha r_n}$$

and therefore

$$|(t(s) - t(s')) - (s - s')| \leqq |s - s'|^\alpha. \tag{3.22}$$

Also,

$$d(h^{(2)}_{t(s)} v(u), h^{(2)}_s \zeta(u)) \leqq \varepsilon \tag{3.23}$$

for all $s \in A(u)$ and

$$l(A(u) \cap [0, t])/t \geqq 1 - 2\varepsilon$$

for all sufficiently large t, where $l(A)$ denotes the length measure of A. This, (3.22) and (3.23) are the main conditions of the basic Lemma 2.1 in [11]. It follows then from that lemma that $v(u) = \bar{u}_n$ lies on the $h^{(2)}_t$-orbit of $\zeta(u)$ for every $u \in X$, if $\varepsilon > 0$ is chosen sufficiently small.

We have

$$\zeta(g^{(1)}_{r_n} u) = g^{(2)}_{r_n} \zeta(u)$$

for μ_1-a.e. $u \in M_1$. This implies that if we denote

$$X_n = g^{(1)}_{r_n} X, \qquad \mu(X_n) > 0,$$

then

$$\psi(u) = h^{(2)}_{\sigma(u)} \zeta(u)$$

for some $\sigma(u) \in R$ and all $u \in X_n$. The set

$$\hat{X} = \{u \in M_1 \colon \psi(u) = h^{(2)}_{\sigma(u)} \zeta(u) \text{ for some } \sigma(u) \in R\}$$

is $h^{(1)}_t$-invariant and contains X_n. Therefore $\mu_1(\hat{X}) = 1$, since $h^{(1)}_t$ is ergodic and $\mu_1(X_n) > 0$. This completes the proof of the theorem. □

4. The Joinings theorem

In this section we prove Theorem 3. The proof follows that of Theorem 5 in the previous section using the techniques from [10].

So let $\tau_i \in B^1(M)$, $i = 1, 2$, $\bar{\tau}_1 = \bar{\tau}_2 = 1$ be as above and let ν be a nontrivial ergodic joining of $h^{\tau_1}_1$ and $h^{\tau_2}_1$. By Corollary 1(b), ν is an ergodic joining of $h^{\tau_1}_t$ and $h^{\tau_2}_t$ for all $t \in R$.

By Corollary 1(a), $(h^{\tau_1}_t \times h^{\tau_2}_t, \nu)$ is a finite extension of $h^{\tau_1}_t$. This means that there are an integer $N \geqq 1$ and an $(h^{\tau_1}_t \times h^{\tau_2}_t)$-invariant subset $\bar{\Omega} \subset M_1 \times M_2$, $\nu(\bar{\Omega})$

$=1$ such that for μ_{τ_1}-a.e. $x \in M_1$ the set $\bar{\eta}(x)=\bar{\Omega}\cap(\{x\}\times M_2)$ consists of exactly N points $\{(x, z_1), \ldots, (x, z_N)\}$ with $\nu_x\{(x, z_j)\}=1/N$, $j=1, \ldots, N$, where ν_x is the probability measure on $\{x\}\times M_2$ obtained from the Fubini theorem applied to ν. We can assume without loss of generality that this holds for all $x \in M_1$.

Let $\bar{\eta}$ denote the partition of $\bar{\Omega}$ into sets $\bar{\eta}(x)$, $x \in M_i$. For $A \subset \bar{\Omega}$ we write $A<\bar{\eta}$ if A consists of atoms of $\bar{\eta}$.

Let π_i: $M_1\times M_2 \to M_i$, $i=1, 2$, be the projection $\pi_1(x, z)=x$, $\pi_2(x, z)=z$.

It follows from the Fubini theorem that if $A\subset\Omega$, $\nu(A)>1-\alpha/N^2$ for some $0<\alpha\leqq 1$ then there is $A'\subset\Omega$ such that $\nu(A')>1-\alpha/N$ and $A'<\bar{\eta}$.

This implies that if $B\subset M_2$, $\mu_2(B)>1-\alpha/N^2$, then there is $\tilde{B}\subset M_1$, $\mu_1(\tilde{B})>1-Q^2\alpha$ such that $\pi_2(\bar{\eta}(x))\subset B$ for all $x\in\tilde{B}$, where Q is as in (3.1). We say that $\tilde{B}$ is induced by B.

Let a compact $K\subset M_2$, $\mu_2(K)>1-10^{-4}/Q^2 N^2$ and $0<\rho=\rho(K)<1$ be as in Sect. 3 and let $\tilde{K}\subset M_1$, $\mu_1(\tilde{K})>1-10^{-4}$ be induced by K.

Since atoms of $\bar{\eta}$ are finite there are $\rho_1>0$ and $K_1<\bar{\eta}$ with $\nu(K_1)>1-10^{-4}$ such that if $w_1 \neq w_2$ and $w_1, w_2 \in K_1\cap\bar{\eta}(x)$ for some $x\in M_1$ then $d(w_1, w_2)\geqq 4\rho_1$.

Let

$$\hat{K}=\pi_1(K_1)\cap\tilde{K}, \qquad \mu_1(\hat{K})>0.999, \qquad \hat{\rho}=\min\{\rho, \rho_1\}.$$

Let $\varepsilon_n=10^{-n}$, $n=5, 6, \ldots$ and let $\bar{\varepsilon}_n=\min\{0.1\,\varepsilon_n, \varepsilon_n/Q^2 N^2\}$.

Let $\gamma_n^{(i)}=\gamma_n^{(i)}(\bar{\varepsilon}_n)$, $l_n^{(i)}=l_n^{(i)}(\bar{\varepsilon}_n)$ and $E_n^{(i)}=E^{(i)}(\bar{\varepsilon}_n)$ be as in Lemma 1.2 for τ_i, $i=1, 2$. Denote

$$\begin{aligned}
&\gamma_n=\min\{\gamma_n^{(1)}, \gamma_n^{(2)}, 0.1\,\varepsilon_n\},\\
&l_n=\max\{l_n^{(1)}, l_n^{(2)}\},\\
&E_n=E_n^{(1)}\cap\tilde{E}_n^{(2)}, \qquad \mu_1(E_n)\geqq 1-0.2\,\varepsilon_n
\end{aligned}$$

where $\tilde{E}_n^{(2)}$ is induced by $E_n^{(2)}$.

As in [10] there are $\Omega'<\bar{\eta}$, $\nu(\Omega')=1$ and pairwise disjoint measurable subsets $\bar{\Omega}_j\subset\Omega'$, $\nu(\bar{\Omega}_j)=1/N$, $j=1, \ldots, N$ such that for every $x\in\pi_1(\Omega')$, the intersection $\bar{\eta}(x)\cap\bar{\Omega}_j$ consists of exactly one point and the map ψ_j: $\pi_1(\Omega')\to\bar{\Omega}_j$ defined by $\psi_j(x)=\bar{\eta}(x)\cap\bar{\Omega}_j$ is measurable, $j=1, \ldots, N$. We can assume without loss of generality that $\Omega'=\bar{\Omega}$.

Let $\Lambda_n\subset M_1$, $\mu_1(\Lambda_n)>1-0.1\,\varepsilon_n$ be such that ψ_j are uniformly continuous on Λ_n for $j=1, \ldots, N$. Given $\varepsilon>0$ let $\delta(\varepsilon)>0$ be such that $d(\psi_j(u), \psi_j(v))\leqq\varepsilon/4$ for all $j=1, \ldots, N$, whenever $u, v\in\Lambda_n$, $d(u, v)\leqq\delta(\varepsilon)$.

Denote

$$D=\hat{\rho}/2Q.$$

Since the flows $h_t^{(i)}$, $i=1, 2$, are ergodic, there are $t_n\geqq\max\{l_n, 20D\gamma_n^{-1}\}$ and a subset $A_n\subset M_1$ with $\mu_1(A_n)>1-0.1\,\varepsilon_n$ such that if $u\in A_n$, $t\geqq t_n$ then the relative length measure of $\Lambda_n\cap\hat{K}$ on the orbit interval $[u, h_t^{(1)}u]$ is at least 0.998 and

$$\left|\int_0^t \tau_1(h_s^{(1)}u)\,ds-t\right|\leqq 0.1\,\varepsilon_n t$$

$$\left|\int_0^t \tau_2(h_s^{(2)}\pi_2\psi_j(u))\,ds-t\right|\leqq 0.1\,\varepsilon_n t$$

for all $j=1, \ldots, N$. Denote

$$V_n = \Lambda_n \cap A_n \cap E_n, \qquad \mu_1(V_n) \geqq 1-\varepsilon_n = 1-10^{-n}.$$

Now let $L(a, b)$, ω_n, δ_n, r_n and V, $\mu_1(V)=1$, be as in the proof of Theorem 5 in Sect. 3.

For $x \in M_1$ denote

$$\bar{x}_{n,j} = g^{(2)}_{-r_n} \pi_2 \psi_j g^{(1)}_{r_n} x, \qquad j=1, \ldots, N.$$

Lemma 4.1. *Let $x, y \in V$ and $x = k^{(1)}_b y$ for some $b \in R$. Then $d(\bar{x}_{n,j}, k^{(2)} \bar{y}_{n,j}) \to 0$ when $n \to \infty$ for all $j=1, \ldots, N$.*

Proof. We shall prove the lemma for $j=1$. For $j=2, \ldots, N$, the proof is the same. Denote

$$\pi_2 \psi_1 = \psi, \qquad \bar{x}_{n,1} = \bar{x}_n$$

and repeat the proof of Lemma 3.1 up to expression (3.13), substituting $\psi^{-1}(K)$ in (3.7) by $\hat{K}$ and replacing ρ in (3.11) by $\hat{\rho}$. Instead of expression (3.13) we now have

$$z_n = \pi_2 \psi_k(h^{(1)}_{\lambda_n} x^{(1)}_n) \in K$$
$$w_n = \pi_2 \psi_j(h^{(1)}_{\sigma(b_n, \lambda_n)} y^{(1)}_n) \in K$$

for some $k, j \in \{1, \ldots, N\}$. Also

$$d(z_n, \pi_2 \psi_k(h^{(1)}_{\sigma(b_n, \lambda_n)} y^{(1)}_n)) \leqq \varepsilon_n/4 \tag{4.1}$$

since $h^{(1)}_{\lambda_n} x^{(1)}_n$, $h^{(1)}_{\sigma(b_n, \lambda_n)} y^{(1)}_n \in \Lambda_n$. This, (3.7) and (3.12) imply

$$d(w_n, \pi_2 \psi_k(h^{(1)}_{\sigma(b_n, \lambda_n)} y^{(1)}_n)) \leqq \hat{\rho} + \varepsilon_n \leqq 2\rho_1$$

if n is sufficiently large. This implies by the definition of ρ_1 that

$$w_n = \pi_2 \psi_k(h^{(1)}_{\sigma(b_n, \lambda_n)} y^{(1)}_n)$$

and

$$d(w_n, h^{(2)}_{\sigma(\alpha_n, \beta_n, s_n)} y^{(2)}_n) \leqq \varepsilon_n$$

by (3.12) and (4.1). This implies

$$|u_n - \sigma(\alpha_n, \beta_n, s_n)| \leqq \varepsilon_n$$

by (3.11) and the definition of ρ. Thus we obtained expression (3.14).

Now we repeat the proof of Lemma 3.1 from (3.14) to the end. This completes the proof of the lemma. □

Denote

$$\eta(x) = \pi_2 \bar{\eta}(x), \qquad \eta_n(x) = g^{(2)}_{-r_n} \eta(g^{(1)}_{r_n} x).$$

For $A, B \subset M_2$ define

$$d(A, B) = \max\{d_{A,B}, d_{B,A}\}$$

where

$$d_{A,B} = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

Corollary 4.1. *If* $x, y \in V$ *and* $x = k_b^{(1)} y$ *for some* $b \in R$ *then* $d(\eta_n(x), k_b^{(2)} \eta_n(y)) \to 0$ *when* $n \to \infty$.

Lemma 4.2. *If* $x \in V$ *and* $y = h_p^{(1)} x$ *for some* $p \in R$, *then* $d(\eta_n(y), h_p^{(2)} \eta_n(x)) \to 0$ *when* $n \to \infty$.

Proof. The proof is completely analogous to that of Lemma 3.2.

Lemma 4.3. *There are an* $h_t^{(1)}$*-invariant subset* $U \subset M_1$ *with* $\mu_1(U) = 1$ *and a subsequence* $\{n_k: k = 1, 2, \ldots\} \subset \{n: n = 1, 2, \ldots\}$ *such that if* $u \in U$ *then* $\lim_{k \to \infty} \eta_{n_k}(u) = \zeta(u) \subset M_2$ *exists and* $\zeta(h_p^{(1)} u) = h_p^{(2)} \zeta(u)$ *for all* $p \in R$, $u \in U$.

Proof. Let $\tilde{K}_n \subset M_2$, $\mu_2(\tilde{K}_n) \leqq 2^{-n}$, $n = 1, 2, \ldots$ be as in the proof of Lemma 3.3 and let

$$F_n = \{u \in M_1: \eta_n \cap g_{r_n}^{(2)}(\tilde{K}_n) \neq \emptyset\},$$

$$F = \{u \in M_1: g_{r_n}^{(1)} u \in F_n \text{ for finitely many } n\}.$$

We have, using the Fubini theorem,

$$\mu_1(F_n) \leqq 2^{-n+1}$$

if n is sufficiently large. This implies by the Borel-Cantelli lemma that $\mu_1(F) = 1$. Thus if $u \in F$ then there is a subsequence n_k such that $\eta_{n_k}(u)$ converges to a finite subset of M_2 when $k \to \infty$.

From now on we proceed exactly as in the proof of Lemma 3.3 to construct an $h_t^{(1)}$-invariant subset $U \subset M_1$ as required in the lemma. □

Let $U \subset M_1$, $\mu_1(U) = 1$, and a subsequence $\{n_k\} \subset \{n\}$ be as in Lemma 4.3. We can assume without loss of generality that $U = M_1$ and $\{n_k\} = \{n\}$. Thus

$$\lim_{n \to \infty} \eta_n(u) = \zeta(u) \subset M_2$$

exists for all $u \in M_1$ and

$$\zeta(h_p^{(1)} u) = h_p^{(2)} \zeta(u) \tag{4.2}$$

for all $p \in R$, $u \in M_1$.

The set $\zeta(u)$ is a finite subset of M_2 and the cardinal number $|\zeta(u)|$ of $\zeta(u)$ is the same for μ_1-a.e. $u \in M_1$, since $h_t^{(1)}$ is ergodic. It follows from Corollary 4.1 that

$$\zeta(k_b^{(1)} u) = k_b^{(2)} \zeta(u) \tag{4.3}$$

for all $b \in R$, μ_1-a.e. $u \in M_1$. This implies via (1.1) that

$$\zeta(g_s^{(1)} u) = g_s^{(2)} \zeta(u) \tag{4.4}$$

for μ_1-a.e. $u \in M_1$ and all $s \in R$. Let

$$\Omega = \{(u, z): u \in M_1, z \in \zeta(u)\}$$

and let m be the probability measure on Ω defined by

$$m(A \times B) = \int_A |B \cap \zeta(u)| / \ell \, d\mu_1(u)$$

for any measurable subsets $A \subset M_1$, $B \subset M_2$, where $\ell = |\zeta(u)|$.

Expressions (4.2), (4.3) and (4.4) show that m is invariant under $h_t^{(1)} \times h_t^{(2)}$, $g_t^{(1)} \times g_t^{(2)}$, $k_t^{(1)} \times k_t^{(2)}$. It is clear that the M_1-marginal of m is μ_1. Let m_2 be the M_2-marginal of m. Then m_2 is preserved by $h_t^{(2)}$, $g_t^{(2)}$ and $k_t^{(2)}$ and therefore $m_2 = \mu_2$. Thus m is a nontrivial G-invariant joining of $h_t^{(1)}$ and $h_t^{(2)}$.

We shall show that τ_1 and τ_2 are jointly cohomologous via m and that m is an ergodic joining of $h_t^{(1)}$ and $h_t^{(2)}$. To do so we follow the proof of part (1) of Theorem 5, which makes essential use of combinatorial Lemma 2.2 in [11] and the decay rate of the correlation function of τ_i for $h_t^{(i)}$, $i=1, 2$, found in [5] and [12].

Indeed, using these decay rates we were able to prove Lemma 3.1 in [11] which implies for the case in question that there is $0<\alpha<1$ such that given $\omega>0$ there are $P \subset M_1$ with $\mu_1(P)>1-\omega$ and $\bar{t}=\bar{t}(P)$ such that if $u \in P$, $t \geqq \bar{t}$. Then

$$\left|\int_0^t \tau_1(h_s^{(1)} u)\, ds - t\right| \leqq t^\alpha$$
$$\left|\int_0^t \tau_2(h_s^{(2)} z)\, ds - t\right| \leqq t^\alpha \tag{4.5}$$

for all $z \in \eta(u)$.

The following lemma is analogous to the Basic Lemma 2.1 in [11].

Lemma 4.4. *Given $0<\alpha$, $\omega<1$ there are $Y \subset M_1$ with $\mu_1(Y)>1-\omega$, $\theta=\theta(\alpha)>0$ and $\varepsilon=\varepsilon(Y)>0$ such that if $u \in Y$, $z \in \zeta(u)$, $w \in M_2$ and $d(z, w) \leqq \varepsilon$ then w lies on the $h_t^{(2)}$-orbit of z whenever there is a subset $A \subset R^+$ with the following properties:*

(1) $0 \in A$;

(2) *If $s \in A$ then $h_s^{(1)} u \in Y$ and there is $t(s)>0$ increasing in s such that $d(h_{t(s)}^{(2)} w, z_s) \leqq \varepsilon$ for some $z_s \in \zeta(h_s^{(1)} u)$;*

(3) *$|(t(s')-t(s))-(s'-s)| \leqq |s-s'|^\alpha$ for all $s', s \in A$ with $|s'-s| \geqq 1$;*

(4) *$l(A \cap [0, \lambda])/\lambda \geqq 1-\theta$ for all $\lambda \in A$ with $\lambda \geqq \lambda_0$, where $l(A)$ denotes the length measure of A.*

Proof. We begin as in the proof of the Basic Lemma 2.1 in [11]. Given $0<\alpha<1$ let $0<\gamma=\gamma(\alpha)<\alpha/2$ and $\theta=\theta(\gamma)>0$ be as in (2.1) of [11], where the combinatorial Lemma 2.2 has been used.

Denote $\bar{\gamma}=\min\{\gamma, \omega\}$. Let a compact $K \subset M_2$ with $\mu_2(K)>1-0.1\,\bar{\gamma}/k^2$ and $0<\rho(K)<1$ be as in Sect. 3. Let $\bar{K} \subset M_1$, $\mu_1(\bar{K})>1-0.1\,\bar{\gamma}$ be such that $\zeta(u) \subset K$ for all $u \in \bar{K}$.

Let $Y_0 \subset M_1$, $\mu_1(Y_0)>1-0.1\,\bar{\gamma}$ and $\rho(Y_0)>0$ be such that if $u \in Y_0$, $z_1, z_2 \in \zeta(u)$ and $z_1 \neq z_2$, then $d(z_1, z_2) \geqq 4\rho(Y_0)$.

Since $g_t^{(1)}$ is ergodic, there are $\bar{Y} \subset M_1$ with $\mu_1(\bar{Y})>1-0.1\,\bar{\gamma}$ and $t_0>0$ such that if $u \in \bar{Y}$, $t \geqq t_0$, then the relative length measure of $\bar{K} \cap Y_0$ on $[u, g_t^{(1)} u]$ is at least $1-0.2\,\bar{\gamma}$. Set

$$Y=\bar{Y} \cap \bar{K} \cap Y_0, \qquad \mu_1(Y)>1-\omega.$$

Now in order to show that $\varepsilon=\varepsilon(\rho(K), \rho(Y_0), t_0)>0$ can be chosen so small as to satisfy the requirements of the lemma, we just have to repeat the proof of Lemma 2.1 in [11]. □

Proof of Theorem 3. (1) Let us show that there exists an $h_t^{(1)}$-invariant subset $\hat{X} \subset M_1$ with $\mu_1(\hat{X})=1$ such that if $u \in \hat{X}$, $w \in \eta(u)$ then there is $z(w) \in \zeta(u)$ such that w lies on the $h_t^{(2)}$-orbit of $z(w)$.

Indeed, let $0<\alpha<1$ be as in (4.5) and $\theta=\theta(\alpha)>0$ be as in Lemma 4.4. For $0<\omega<\theta$ let $P=P(\omega)\subset M_1$, $\mu_1(P)>1-0.1\,\omega$ and $\bar t=\bar t(P)>0$ be as in (4.5). Also let $Y\subset M_1$, $\mu_1(Y)>1-0.1\,\omega$ and $\varepsilon=\varepsilon(Y)>0$ be as in Lemma 4.4.

Let $\bar X\subset M_1$, $\mu_1(\bar X)>1-0.1\,\omega$ and $n_0>0$ be such that if $u\in\bar X$, $n\geqq n_0$, then

$$d(\eta_n(u),\zeta(u))\leqq\varepsilon.$$

Fix $n\geqq n_0$ with $e^{2r_n}\geqq 2Q^2\,\bar t(P)$, where $Q\geqq 1$ is as in (3.1). Let $\tilde X\subset M_1$ be the generic set of $\bar X\cap g^{(1)}_{-r_n}P\cap Y$ for $h^{(1)}_t$, $\mu_1(\tilde X)=1$ and let

$$X=\tilde X\cap\bar X\cap g^{(1)}_{-r_n}(P)\cap Y,\qquad \mu_1(X)>0.$$

Let $u\in X$ and $w\in\eta_n(u)$. Then there exists $z\in\zeta(u)$ such that $d(w,z)\leqq\varepsilon$. Let

$$A=\{s\in R^+:\,h^{(1)}_s u\in Y\cap\bar X\cap g^{(1)}_{-r_n}(P)\}.$$

We have $0\in A$ and

$$l(A\cap[0,\lambda])/\lambda\geqq 1-\omega\geqq 1-\theta \tag{4.6}$$

for all $\lambda\geqq\lambda_0$, since $u\in\tilde X$.

For $s\in R^+$ let $t(s)>0$ be defined by

$$\xi^{(1)}(g^{(1)}_{r_n}u,\,e^{2r_n}s)=\xi^{(2)}(g^{(2)}_{r_n}w,\,e^{2r_n}t(s))$$

where $\xi^{(i)}$, $i=1,2$, are as in Sect. 3. We have

$$h^{(2)}_{t(s)}v\in\eta_n(h^{(1)}_s u)$$

since $\bar\eta$ is $h^{\tau_1}_t\times h^{\tau_2}_t$-invariant. This implies that if $s\in A$ then

$$d(h^{(2)}_{t(s)}v,\,z_s)\leqq\varepsilon \tag{4.7}$$

for some $z_s\in\zeta(h^{(1)}_s u)$. We have

$$g^{(2)}_{r_n}h^{(2)}_{t(s)}w\in\eta(g^{(1)}_{r_n}h^{(1)}_s u).$$

Also

$$g^{(1)}_{r_n}h^{(1)}_s u\in P$$

if $s\in A$. This implies via (4.5) and our choice of n that if $s,s'\in A$ and $|s-s'|\geqq 1$ then

$$e^{2r_n}|(t(s)-t(s'))-(s-s')|\leqq|s-s'|^\alpha e^{2\alpha r_n}$$

and therefore

$$|(t(s)-t(s'))-(s-s')|\leqq|s-s'|^\alpha.$$

This, (4.6) and (4.7) imply via Lemma 4.4 that w lies on the $h^{(2)}_t$-orbit of z or that $g^{(2)}_{r_n}w\in\eta(g^{(1)}_{r_n}u)$ lies on the $h^{(2)}_t$-orbit of $g^{(2)}_{r_n}z\in\zeta(g^{(1)}_{r_n}u)$.

Now let

$$X_n=g^{(1)}_{r_n}(X),\qquad \mu_1(X_n)>0$$

and let

$$\hat X=\{u\in M_1:\text{ for every }w\in\eta(u)\text{ there is }z(w)\in\zeta(u)\text{ such that }w\text{ lies on the }h^{(2)}_t\text{-orbit of }z(w)\}.$$

The set $\hat{X}$ is $h_t^{(1)}$-invariant and $X_n \subset \hat{X}$. This implies that $\mu_1(\hat{X})=1$, since $h_t^{(1)}$ is ergodic. This completes the proof of claim (1).

We can assume without loss of generality that $\hat{X}=M_1$.

(2) Now let us show that τ_1 and τ_2 are jointly cohomologous via m.

Indeed, it follows from (4.3) that for μ_1-a.e. $u \in M_1$ no two distinct points of $\zeta(u)$ can lie on the same orbit of $h_t^{(2)}$. This implies that the map $w \to z(w)$ from $\eta(u)$ to $\zeta(u)$ is well defined for μ_1-a.e. $u \in M_1$.

Also for μ_1-a.e. $u \in M_1$ no two distinct points of $\eta(u)$ can lie on the same orbit of $h_t^{(2)}$, since $\bar{\eta}$ is invariant under $h_t^{\tau_1} \times h_t^{\tau_2}$ and v is an ergodic joining of $h_t^{\tau_1}$ and $h_t^{\tau_2}$. This implies that the map $w \to z(w)$ is bijective and the map $\phi: (\bar{\Omega}, v) \to (\Omega, m_{f_1})$ defined by

$$\phi(u, w)=(u, z(w))$$

is an isomorphism between $h_t^{\tau_1} \times h_t^{\tau_2}$ and $S_t^{f_1}$, where $S_t = h_t^{(1)} \times h_t^{(2)}$ and $f_1(u, z) = \tau_1(u)$, $(u, z) \in \Omega$. This implies that m is an ergodic joining of $h_t^{(1)}$ and $h_t^{(2)}$.

Let $f_2(u, z)=\tau_2(z)$, $(u, z) \in \Omega$ and let $v(u, z)$ be such that $w = h^{\tau_2}_{v(u,z)} z$, where $z = z(w)$. It is now clear that f_1 and f_2 are cohomologous via v along S_t acting on (Ω, m) and that $v = v_{(m, v)}$ is induced by (m, v). This completes the proof of Theorem 3. □

5. The factors theorem

In this section we prove Proposition 2 and Theorem 6.

Proof of Proposition 2. Let ζ and η be shift-related partitions of (X, μ) invariant under T_t and T_t^τ respectively. We have

$$\eta(x) = \{T^\tau_{v(x,y)} y : y \in \zeta(x)\}$$

for μ-a.e. $x \in X$, where $v(x, \cdot) = v_x \in L_1(\zeta(x), \mu_{\zeta(x)})$. We shall assume without loss of generality that this holds for all $x \in X$. We have

$$\begin{aligned} v(x, y) &= -v(y, x) \\ v(x, z) &= v(x, y) + v(y, z) \end{aligned} \tag{5.1}$$

and

$$v(T_t x, T_t y) - v(x, y) = \int_0^t (\tau(T_s x) - \tau(T_s y))\, ds \tag{5.2}$$

for all $y, z \in \zeta(x)$, $x \in X$ and all $t \in R$, since η is T_t^τ-invariant.

We have

$$\begin{aligned} \tau_\zeta(x) &= \int_{\zeta(x)} \tau(y)\, d\mu_{\zeta(x)}(y), \\ \tilde{v}(x) &= -\int_{\zeta(x)} v(x, y)\, d\mu_{\zeta(x)}(y). \end{aligned}$$

Expression (5.2) gives

$$\tilde{v}(T_t x) - \tilde{v}(x) = \int_0^t (\tau_\zeta - \tau)(T_s x)\, ds \tag{5.3}$$

for all $x \in X$, $t \in R$. This shows that τ_ζ and τ are cohomologous along T_t via $\tilde{v}$. Also

$$\tilde{v}(x) = \tilde{v}(y) - v(x, y) \tag{5.4}$$

for all $y \in \zeta(x)$, $x \in X$ by (5.1).

Let $\chi_{\tilde{v}}$ be the isomorphism between $T_t^{\tau_\zeta}$ and T_t^τ induced by $\tilde{v}$, that is

$$\chi_{\tilde{v}}(x) = T^\tau_{\tilde{v}(x)} x, \qquad x \in X.$$

We have

$$\begin{aligned}\chi_{\tilde{v}}(\zeta(x)) &= \{T^\tau_{\tilde{v}(y)} y : y \in \zeta(x)\}\\ &= \{T^\tau_{\tilde{v}(x)+v(x,y)} y : y \in \zeta(x)\}\\ &= T^\tau_{\tilde{v}(x)} \eta(x) = \eta(T^\tau_{\tilde{v}(x)} x) = \eta(\chi_{\tilde{v}}(x))\end{aligned}$$

for all $x \in X$ by (5.4), since η is T_t^τ-invariant. This completes the proof of the proposition. $\square$

Proof of Theorem 6. Let $\tau \in B^1(M)$, V be a factor of h_1^τ and η be the h_1^τ-invariant partition of M induced by V. It follows from Corollary 2 that a.e. atom of η is finite and η is invariant under h_t^τ for all $t \in R$.

Let ν be the probability measure on $M \times M$ defined by

$$\nu(A \times B) = \int_{M/\eta} \mu_C(A)\, \mu_C(B)\, d\mu_{\tau,\eta}(C)$$

for every measurable $A, B \subset M$, where $(M/\eta, \mu_{\tau,\eta})$ is the quotient space induced by η, $\mu_C(A) = |A \cap C|/|C|$ for every $C \in \eta$ and $|C|$ denote the cardinal number of C. The set

$$\bar{\Omega} = \{(u, w) : w \in \eta(u), u \in M\}$$

is $h_t^\tau \times h_t^\tau$-invariant and $\nu(\bar{\Omega}) = 1$.

The measure ν is a nontrivial joining of h_t^τ with itself. It might not be ergodic. Since atoms of η are finite, ν is a finite convex sum of nontrivial ergodic joinings of h_t^τ with itself (see [10] for this).

It follows from Theorem 3 that there exists $\zeta(u) \subset M$, $u \in M$ and a measurable v on $M \times M$ such that

$$\begin{aligned}|\zeta(u)| &= |\eta(u)|\\ \eta(u) &= \{h^\tau_{v(u,w)} w : w \in \zeta(u)\}\end{aligned} \tag{5.5}$$

for μ-a.e. $u \in M$ and ζ is invariant under the action of G on M.

Henceforth we use without loss of generality the word "everywhere" instead of "almost everywhere".

We claim that

$$\zeta(w) = \zeta(u) \quad \text{if } w \in \zeta(u), \quad u \in M. \tag{5.6}$$

Inded, it follows from Lemma 4.3 that

$$\zeta(u) = \lim_{n \to \infty} g_{-r_n}(\eta(g_{r_n} u)) \tag{5.7}$$

for some sequence $r_n \to \infty$ when $n \to \infty$. This implies that $u \in \zeta(u)$ for all $u \in M$, since $u \in \eta(u)$, $u \in M$.

Denote $u_n = g_{r_n}(u)$. We have

$$u_n,\, w_n \in \zeta(u_n), \qquad \eta(u_n) = \{h^{\tau}_{v(u_n,\, z)}\, z\colon z \in \zeta(u_n)\}.$$

This implies that

$$\eta(w_n) = h^{\tau}_{-v(u_n,\, w_n)}\, \eta(u_n)$$

since η is h^{τ}_t-invariant. We have

$$g_{-r_n}\, \eta(u_n) = \{h_{e^{-2r_n} s(z_n)}\, z_n\colon z \in \zeta(u)\}$$

where $s(z_n)$ is defined by

$$\int_0^{s(z_n)} \tau(h_t\, z_n)\, dt = v(u_n, z_n)$$

and

$$e^{-2r_n}\, s(z_n) \to 0, \qquad \text{when } n \to \infty \tag{5.8}$$

for all $z \in \zeta(u)$, $u \in M$ by (5.7). Also

$$g_{-r_n}\, \eta(w_n) = \{h_{e^{-2r_n} \bar{s}(z_n)}\, z\colon z \in \zeta(u)\} \tag{5.9}$$

where

$$\bar{s}(z_n) = s(z_n) - \int_0^{v(u_n,\, w_n)} \tau^{-1}(h_t(h_{s(z_n)}\, z_n))\, dt = s(z_n) - \tilde{s}(z_n)$$

and

$$|\tilde{s}(z)| \leqq Q^2\, s(w_n)$$

for all $z \in \zeta(u)$, $u \in M$, where $Q \geqq 1$ is as in (3.1) for $\tau_i = \tau$, $i = 1, 2$. This, (5.8) and (5.9) imply that

$$\zeta(w) = \lim_{n \to \infty} g_{-r_n}\, \eta(w_n) = \zeta(u).$$

This proves (5.7).

Denote

$$\Gamma' = \{g \in G\colon \Gamma g \in \zeta(\Gamma e)\}$$

where e denotes the unity in G.

Let $g_1, g_2 \in \Gamma'$ and $z = g_1^{-1} g_2$. We have

$$\Gamma g_2 \in \zeta(\Gamma z)$$

since ζ is G-invariant. Also,

$$\Gamma z \in \zeta(\Gamma g_2) = \zeta(\Gamma e)$$

by (5.6). Therefore $z \in \Gamma'$.

This proves that Γ' is a discrete subgroup of G. Also $\Gamma \subset \Gamma'$ and hence $\Gamma' \in \mathscr{T}$. We have

$$\zeta(\Gamma g) = \psi^{-1}_{\Gamma, \Gamma'}\{\Gamma' g\}$$

where $\psi_{\Gamma, \Gamma'}(\Gamma g) = \Gamma' g$, $g \in G$.

Expression (5.5) shows that η and ζ are shift related along h_t. The theorem now follows from Proposition 2. □

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