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Titel: Locally finite classical Tits chamber Systems of large order.

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Jahr: 1987

PURL: https://resolver.sub.uni-goettingen.de/purl?356556735_0087|log34

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Invent. math. 87, 603–641 (1987)

Inventiones mathematicae
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Locally finite classical Tits chamber systems of large order

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§ 1. Introduction

A *chamber system* $\mathfrak{C}=(\mathfrak{C},(\mathfrak{p}_i)_{i\in I})$ over some index set I is a set $\mathfrak{C}$ of "chambers" together with partitions $\mathfrak{p}_i$, $i\in I$ of $\mathfrak{C}$. If $J\subseteq I$ then $\mathfrak{p}_J$ is the join of all partitions $\mathfrak{p}_j$, $j\in J$. If $c\in\mathfrak{C}$ and $J\subseteq I$ then $\underline{\Delta_J(c)}$ is the element of $\mathfrak{p}_J$ containing c. Notice that $\Delta_J(c)$ is again a chamber system over J with partitions $\mathfrak{p}_j$, $j\in J$ restricted to $\Delta_J(c)$. $\mathfrak{C}$ is *connected* if and only if $\Delta_I(c)=\mathfrak{C}$ for some $c\in\mathfrak{C}$. The permutation group G of $\mathfrak{C}$ is an automorphism group of $\mathfrak{C}$ if it respects all the partitions $\mathfrak{p}_i$, $i\in I$. Let $\operatorname{rank}\mathfrak{C}=|I|$.

The canonical example of a chamber system with transitive automorphism group is obtained in the following way: Let G be a group, B a subgroup of G and X_i, $i\in I$ subgroups of G containing B. Then the chamber system $\mathfrak{C}=\mathfrak{C}(G;B;(X_i)_{i\in I})$ has as chambers the cosets Bg, $g\in G$, two cosets Bg, Bh being i-adjacent if and only if $X_i g=X_i h$. If now $\mathfrak{C}=(\mathfrak{C},(\mathfrak{p}_i)_{i\in I})$ is a chamber system with transitive automorphism group G, set $B=G_c$ for some $c\in\mathfrak{C}$ and $X_i=G_{\Delta_i(c)}$. Then the map

$$c^g\to Bg$$

is a canonical isomorphism of $\mathfrak{C}=(\mathfrak{C},(\mathfrak{p}_i)_{i\in I})$ onto $\mathfrak{C}(G;B;(X_i)_{i\in I})$. Further $\mathfrak{C}$ is connected if and only if $G=\langle X_i|_{i\in I}\rangle$.

It is obvious that, although they are very interesting to group theorists, arbitrary chamber systems with transitive automorphism group are too general. So one has to look for some conditions which relate them closer to interesting groups. Such conditions were introduced by Tits [21].

A rank 2 chamber system $\mathfrak{C}=(\mathfrak{C},\mathfrak{p}_i,\mathfrak{p}_j)$ is called a *classical generalized m_{ij}-gon* ($m_{ij}\geqq 3$) if $\mathfrak{C}$ is isomorphic to $\mathfrak{C}(G;B;P_i,P_j)$; where G is an (essentially[1]) simple rank 2 group of Lie-type (in the sense of [1]), B a Borel-subgroup of G and P_i, P_j the two maximal parabolic subgroups of G containing B. Here m_{ij} is given by the relations $w_i^2=w_j^2=(w_iw_j)^{m_{ij}}=1$ defined by the Weyl-group of G. A rank 2 chamber system $\mathfrak{C}=(\mathfrak{C},\mathfrak{p}_i,\mathfrak{p}_j)$ is called a *generalized digon*, if $\mathfrak{C}$ is

[1] One has to allow some degenerate case like $Sp(4,2)\cong\Sigma_6$

isomorphic to $\mathfrak{C}(G; B; P_i, P_j)$; where P_i, P_j are different subgroups of G satisfying $G = P_iP_j = P_jP_i$ and $B = P_i \cap P_j$. We say that the connected chamber system $\mathfrak{C}$ over I is a *classical Tits chamber system* if for each $c \in \mathfrak{C}$ and pair $i, j \in I$, $\Delta_{\{i,j\}}(c)$ is either a classical generalized m_{ij}-gon (for some $m_{ij} \in \mathbb{N}$) or a generalized digon. $\mathfrak{C}$ is called *locally finite*, if all $\Delta_{i,j}(c)$ are finite. The diagram (or type) $\Delta = \Delta(I)$ is defined by connecting the "nodes" i and j by a bond of strength $m_{ij} - 2$. (They will be drawn in the usual way as Coxeter-diagrams.)

Given any Coxeter-diagram Δ (with the proper restrictions on m_{ij} by the Feit-Higman-theorem!) there are classical Tits chamber systems of type Δ, namely the chamber systems of buildings of type Δ. But there exist many more, namely all types of projections of such buildings.

As Tits pointed out, even the determination of finite classical Tits chamber systems of affine type seems to be a senseless task. So we consider in this paper only classical, locally finite Tits chamber systems with a transitive automorphism group G. The main aim of this paper is to prove that these are essentially finite spherical buildings, if also a Borel subgroup $B = G_c$, $c \in \mathfrak{C}$ is finite.

In [21] Tits has shown that a building is uniquely determined by its chamber system. So, abusing notation, we call a chamber system a building if it is isomorphic to a chamber system of a building. To state our main results properly we fix the following notation: If $\mathfrak{C} = (\mathfrak{C}, (\mathfrak{p}_i)_{i \in I})$ is a locally finite classical Tits chamber system over I with diagram $\Delta = \Delta(I)$ and transitive automorphism group G, we fix a chamber c and set $B = G_c$. For each $J \subset I$ let $X_J = G_{\Delta_J(c)}$, $G_J = X_{I-J}$ and K_J the kernel of the action of G_J on $\Delta_{I-J}(c)$.

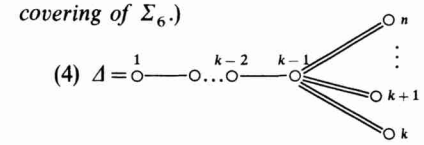
Then we have:

Theorem 1. *Suppose $\mathfrak{C}$ is a classical, locally finite Tits chamber system over I with connected diagram $\Delta = \Delta(I)$ of* rank $\geqq 3$ *with transitive automorphism group G and finite Borel-subgroup B. Suppose further that $|\Delta_i(c)| \geqq 6$ for all $i \in I$. Then one of the following holds:*

(1) *$\mathfrak{C}$ is a finite spherical building and G an extension of a group of Lie-type Δ by diagonal and field automorphisms.*

(2) *Δ is a complete graph, $|\Delta_i(c)| = 9$ and $X_{i,j}$ is a Frobenius-group of order $73 \cdot 9$ for all $i \neq j \in I$.*

(3) $\Delta = \overset{1}{\circ}\!\!-\!\!\!-\overset{2}{\circ}\!\!\equiv\!\!\overset{3}{\circ}$, *$G_1 \simeq G_2(5)$, G_3 is a non-split extension of $SL_3(5)$ by $\mathbb{Z}_5^3$ and $G_2 \simeq 5^{1+4}(\hat{\Sigma}_6 * \mathbb{Z}_4)$ ($*$ means central product and $\hat{\Sigma}_6$ is the perfect central 2-covering of Σ_6.)*

(4) $\Delta =$  *with $k - 1 \geqq 1$ and $n - k \geqq 1$.*

If $K = \{k, \ldots, n\}$ then the following holds:

(a) $|\Delta_i(c)| = 8$ *for each $i \in I$.*

(b) $K_J = 1$ *and* $K_{J,k-1} = K_{k-1}$ *for each $\emptyset \neq J \subsetneq K$.*

(c) $G_J \simeq PSp(2k, 7)$ *for each $J \subset K$ with $|K - J| = 1$.*

(d) $(G_{k-1}/K_{k-1})' \simeq PSL_{k-1}(7) \times (\mathbb{Z}_2^{3(n-k)} GL_3(2))$.

An example of type (3) coming from the Lyon's group is well-known. A description can be found in [6]. Thomas Meixner constructed in [8] examples of type (2) and (4). The existence of Type (4) comes from the fact that $\mathbb{Z}_2^3 \cdot GL_3(2)$ is the product of two $PSL_2(7)$'s. It is clear that the structure of K_{k-1} in (4) is also completely determined, since, for J as in (c) $K_{k-1} = K_{k-1,J}$ and G_J is as in (4) (c).

It has been shown in [21] that a chamber system with non-connected diagram is the "direct sum" of chamber systems with connected diagrams. So our Theorem 1 determines all such chamber systems if the rank 1 residues are sufficiently large.

The bulk of the proof of Theorem 1 consists of proving Theorem 2 below for hyperbolic[2] diagrams, which we prove for later applications without restrictions on $|\Delta_i(c)|$, $i \in I$.

Here a diagram $\Delta = \Delta(I)$ is *hyperbolic*, if it is non-spherical, connected, and all connected proper subdiagrams are spherical. To state Theorem 2 we need some further notation.

$\underline{\mathfrak{L}}_p^i$ = set of finite simple rank i groups of Lie-type in char p together with:

$$PSL_2(3) \quad \text{and} \quad {}^2G_2(3)$$
$$PSL_2(2), \quad PSU_3(2) \quad \text{and} \quad Sz(2) \simeq F_{20}$$

if $i=1$ and $p=3$ resp. 2 and

$$A_6, \ \Sigma_6 \simeq Sp(4,2), \ G_2(2)' \simeq U_3(3), \ G_2(2), \ {}^2F_4(2)' \quad \text{and} \quad {}^2F_4(2)$$

if $p=2$ and $i=2$.

A *parabolic system* $\mathfrak{p}$ of char p of the group G is a system $\mathfrak{p} = \{X_i \mid i \in I\}$ of subgroups satisfying:

(1) $G = \langle X_i \mid i \in I \rangle$.

(2) There exists a finite p-subgroup $S \leqq \bigcap_{i \in I} X_i$ such that $S \in \mathrm{Syl}_p(X_{i,j})$, $X_{i,j} = \langle X_i, X_j \rangle$ for all $i, j \in I$.

(3) $\bar{X}_i = O^{p'}(X_i/O_p(X_i))$ is a perfect central extension of a group in $\mathfrak{L}_p^1$.

(4) Either $\bar{X}_{i,j} = O^{p'}(X_{i,j}/O_p(X_{i,j}))$ is a perfect central extension[3] of a group in $\mathfrak{L}_p^2$ or

$$(\bar{Y}_i * \bar{Y}_j)\bar{S} \subseteq \bar{X}_{i,j} \cong \bar{X}_i * \bar{X}_j$$

where $\bar{Y}_i \simeq O^p(\bar{X}_i)$ and $*$ means central product. Further, if $\bar{X}_{i,j} \simeq \hat{A}_6$, then $N_{O^{2'}(X_{i,j})}(S) \leqq X_i \cap X_j$.

We call $\mathfrak{p}$ a *strong* parabolic system of G, when always $\bar{X}_{i,j} \simeq \bar{X}_i * \bar{X}_j$ in the second possibility for case (4).

So if all the $\bar{X}_i$, $i \in I$ are quasi-simple, then $\mathfrak{p}$ is always strong. The diagram (or type) $\Delta = \Delta(\mathfrak{p})$ is defined in the obvious way.

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[2] Our definition of hyperbolic diagrams differs from the usual one in which hyperbolic diagrams are not affine

[3] A perfect central extension $\hat{X}$ of X is a central extension with $Z(\hat{X}) \leqq \hat{X}'$. We use - convention all over this paper. I.e. if $S \leqq O^{p'}(X_{i,j})$ then $\bar{S}$. Is the image of S under the homomorphism

$$- : O^{p'}(X_{i,j}) \to \bar{X}_{i,j} = O^{p'}(X_{i,j}/O_p(X_{i,j}))$$

It has been shown in [14, §4] that, by slightly altering the groups X_i, one obtains from such a parabolic system[4] of G a locally finite, classical Tits chamber system with transitive automorphism group $\bar{G}=G/B_G$ and finite chamber stabilizer; here B_G is the intersection of all B^g, $g\in G$. So by classifying such chamber systems one also obtains a classification of parabolic systems. In proving Theorem 2 we establish a partial converse.

Theorem 2. *Let $\mathfrak{C}$ be a classical, locally finite Tits chamber system over I with hyperbolic diagram $\Delta=\Delta(I)$, transitive automorphism group G and finite chamber stabilizer B. Then one of the following holds:*

(1) *There exist $B^*\trianglelefteqq B$, $X_i^*\trianglelefteqq X_i$, $i\in I$ such that for $G^*=\langle X_i^*|i\in I\rangle$ the following holds:*

(a) *$G^*B=G$, $X_i^*B=X_i$ and $|X_i\colon X_i^*|=|B\colon B^*|$ for each $i\in I$,*

(b) *The map $B^*g\to Bg$, $g\in G^*$ is a 2-covering in the sense of* [21] *of $\mathfrak{C}(G^*; B^*; (X_i^*)_{i\in I})$ onto $\mathfrak{C}=\mathfrak{C}(G; B; (X_i)_{i\in I})$,*

(c) *$\mathfrak{p}=\{X_i^*|i\in I\}$ is a parabolic system of G^* of type Δ.*

(2) $\Delta=$ *and either $G_i\simeq F_{21}$ for each $i\in I$ or $G_i\simeq F_{73\cdot 9}$ for each $i\in I$.*

(3) $\Delta=\overset{1}{\circ}\!\!-\!\!\circ\!\!\equiv\!\!\overset{3}{\circ}$, *$G_1\simeq G_2(5)$, G_3 is the non-split extension of $SL_3(5)$ by $(\mathbb{Z}_5)^3$ and $G_2\simeq 5^{1+4}(\hat{\Sigma}_6 * \mathbb{Z}_4)$.*

(4) $\Delta=\overset{1}{\circ}\!\!=\!\!\overset{2}{\circ}\!\!=\!\!\overset{3}{\circ}$ *and $G_1\simeq G_3\simeq PSp(4, 7)$. Further*

$$O^{7'}(G_2)\simeq(7^{1+2}\times\mathbb{Z}_2^3)SL_2(7).$$

(5) *There exists a $j\in I$ such that $\Delta_{I-j}(c)$ is the $Sp(4, 3)$ or $U_4(3)$ generalized quadrangle. Further*

$$G_j/K_j\in\{2^4\cdot F_{20}, L_3(4)\cdot 2, 2^4\cdot A_5\}.$$

We prove Theorem 2 also for the case $|\Delta_i(c)|\leqq 5$ for some $i\in I$, since in [19] we determine all parabolic systems of rank 3. Hence by Theorem 2 above we obtain also a classification of all classical, locally finite Tits chamber systems of rank 3 with transitive automorphism group and finite Borel-subgroup.

If now in case 1 of Theorem 2 $\mathfrak{p}$ is strong and $|\Delta_i(c)|\geqq 6$ for all $i\in I$, then a theorem of Niles, see (2.4), shows that some normal subgroup of G^* has a BN-pair of type Δ with finite Borel-subgroup. But it is easy to see, (2.7), that for hyperbolic Δ the Borel-subgroup must be already infinite. So to obtain Theorem 1 as a corollary of Theorem 2 one must only study the extensions of case (2), (3) or (4) of Theorem 2.

For the proof of Theorem 2 one has, apart from the special cases (2)–(5), to find subnormal subgroups X_i^* of X_i and B^* of B such that $\mathfrak{p}=\{X_i^*|i\in I\}$ is a

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[4] Parabolic systems in this paper are weak parabolic systems of [14]. The further condition $G\neq\langle X_i|i\in J\rangle$ for proper subsets $J\subset I$ corresponds to the condition that the chamber system is not tight

parabolic system with common Sylow-normalizer B^* and the map

$$B^*g \to Bg$$

is a 2-covering from $\mathfrak{C}(G^*; B^*; (X_i^*)_{i\in I})$ on $\mathfrak{C}$. For this purpose we have to take three essentially different steps:

I. Abolish p-parts of field automorphisms of X_i/Q_i; where Q_J is the kernel of the action of X_J on $\Delta_J(c)$.

II. Show that the Q_i are essentially p-groups. ($[Q_i, O^{p'}(X_i)] \leqq O_p(Q_i)$).

III. Show, if Δ is linear and i, j are non-connected nodes of Δ, that $X_{i,j}/Q_{i,j}$ is essentially a direct product or that case (3) or (4) of Theorem 2 holds.

Since $X_i \leqq G_j$ for $j \in I - i$ these steps will in fact be taken for the corresponding "maximal parabolics". If we have accomplished step I, step II is relatively easy. It follows from (3.12), which is a more general proposition applying to this situation. The solution of step I is really the heart of this paper. It requires most of §4 ((4.1)–(4.7) and (4.15)) and also most of the auxilliary results about finite Lie-type groups in §3. The treatment of step III occupies the rest of §4 and §5.

It is interesting that one can take step I without any hypothesis on the finiteness of B. So I will state this as an independent theorem, though I do not know of any application.

Theorem 3. *Let $\mathfrak{C}$ be a classical locally finite Tits chamber system over I of char. p with hyperbolic diagram $\Delta = \Delta(I)$ and transitive automorphism group G. Then one of the following holds:*

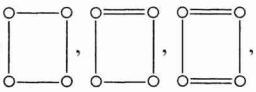
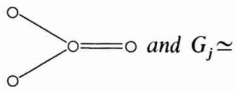
(1) *There exist subgroups $B^* \trianglelefteqq B$, $X_i^* \trianglelefteqq X_i = G_{I-i}$ such that for $G^* = \langle X_i^* | i \in I\rangle$ the following holds:*

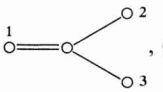
(a) *$G^*B = G$, $X_i^* B = X_i$ and $|X_i : X_i^*| = |B : B^*|$ is finite for all $i \in I$.*

(b) *The map $B^*g \to Bg$, $g \in G^*$ is a 2-covering of $\mathfrak{C}^* = \mathfrak{C}(G^*; B^*, (X_i^*)_{i\in I})$ onto $\mathfrak{C}$.*

(c) *Let G_i^*, K_i^* be defined for $\mathfrak{C}^*$ as G_i and K_i for $\mathfrak{C}$. Then for each $j \in I$ for which $\Delta - j$ is connected, $O^{p'}(G_j^*/K_j^*)$ is a finite simple group of Lie-type $\Delta - j$. (Or Σ_6, $G_2(2)$, ${}^2F_4(2)$.)*

(2) $\Delta =$, *G_i is a Frobenius-group of order 21 or $73 \cdot 9$ for each $i \in I$.*

(3) $\Delta =$  , *or*  *and $G_j \simeq A_7$ for each "non-middle" node of Δ.*

(4) $\Delta =$  , *$G_1/K_1 \simeq A_7$ and $G_i/K_i \simeq Sp(6, 2)$ for $i = 2, 3$.*

(5) *$\Delta_{I-j}(c)$ is the $Sp(4, 3)$ or $U_4(3)$ generalized quadrangle and $G_j/K_j \in \{2^4 \cdot F_{20}, L_3(4) \cdot 2, 2^4 \cdot A_5\}$ for some $j \in I$.*

Of course in case (4) and (5) we have very faint information. But since the K_j might be infinite, the usual group-theoretic arguments to determine the

structure of K_j do not work. It is interesting that, though they might be infinite, the K_j are all 1 in case (3).

Finally a word about geometries. $\mathfrak{C}(\Gamma)$ is a classical locally finite Tits chamber system with diagram $\Delta(I)$ if Γ is a classical locally finite Tits geometry with diagram Δ(I). (For definition see §2!) Further, if G is a flag-transitive automorphism group of Γ, then G is also a chamber-transitive automorphism group of $\mathfrak{C}(\Gamma)$. So our Theorems 1, 2 and 3 always imply similar theorems for geometries. But it is easy to see that they are more general.

§ 2. Properties of classical Tits chamber systems

(2.1) *Notation.* Classical, locally finite Tits chamber systems and parabolic systems have been defined already in the introduction. So we do not repeat the definition.

If now $\mathfrak{C}$ is such a classical, locally finite Tits chamber system over I with diagram $\Delta=\Delta(I)$, $c\in\mathfrak{C}$ and $J\subseteq I$ then

$$\underline{\Delta_J(c)} = \text{the residue of type } J \text{ containing } c$$
$$= \text{the element of } \mathfrak{p}_J \text{ containing } c.$$

If G is a transitive automorphism group of $\mathfrak{C}$ and $c\in\mathfrak{C}$ is a fixed chamber we set

$$B := G_c \text{ - a Borel-subgroup of } G.$$
$$X_J := G_{\Delta_J(c)},\ J\subseteq I \quad \text{with } B=X_\phi.$$
$$G_J := X_{I-J}.$$

Finally Q_J resp. K_J is the kernel of the action of X_J resp. G_J on $\Delta_J(c)$ resp. $\Delta_{I-J}(c)$ and $\bar{X}_J=X_J/K_J$, $\bar{G}_J=G_J/K_J$.

A *geometry* $\underline{\Gamma}$ over the index set I is a triple $\Gamma=(V,*,\tau)$ such that V is a set of "objects" or "vertices", τ the type map a surjective map from V onto I and $*$ an incidence relation on V satisfying: $v*w$ and $\tau(v)=\tau(w)$ if and only if $v=w$.

Γ is *connected* if the incidence graph of Γ is connected. A *flag* F of Γ is a subset of pairwise incident objects of V. The *residue* Γ_F of a flag F is the triple $(V_F, *|_{V_F}, \tau|_{V_F})$ where

$$V_F=\{v\in V \mid v*f \text{ for all } f\in F\}-F.$$

Obviously Γ_F is again a geometry over $\tau(V_F)$. $\tau(F)$ resp. $\tau(V_F)$ is the type of F resp. of Γ_F. $|I|=\operatorname{rank}\Gamma$. Finally Γ is *residually connected* if Γ is connected and:

(1) Γ_F is connected for each flag F of Γ with $|I-\tau(F)|\geqq 2$.

(2) $\Gamma_F\neq\emptyset$ for each flag F of Γ with $|I-\tau(F)|=1$.

The standard examples of such geometries are the *group geometries* $\Gamma(G;(G_i)_{i\in I})$, where G is a group and G_i, $i\in I$ are subgroups of G. Here $V=\cup G_i x$, $i\in I$, $x\in G$; $\tau(G_i x)=i$ and $G_i x * G_j y$ if and only if $G_i x\cap G_j y\neq\emptyset$. On the other hand, if Γ is a geometry over I with an automorphism group G acting transitively on the rank 2 flags of each type and if Γ has a flag $F=\{f_i\mid i\in I\}$ of type I, then $\Gamma\simeq\Gamma(G;(G_i)_{i\in I})$ if we set $G_i=G_{f_i}$. Further Γ is connected if and only if $G=\langle G_i\mid i\in I\rangle$.

Using this fact we call the rank 2 geometry Γ a *classical generalized* m_{ij}-gon, $m_{ij} \geqq 3$, if and only if

$$\Gamma \simeq \Gamma(G, G_1, G_2)$$

where $G \in \mathfrak{Q}_p^2$ and G_1, G_2 are the two maximal parabolic subgroups containing some Borel-subgroup of G. Here m_{ij} is the number defined by the Weyl-group of G.

Γ is a generalized 2-*gon*, if and only if $\Gamma \simeq \Gamma(G, G_1, G_2)$ where $G = G_1 G_2 = G_2 G_1 \neq G_i$.

Finally the residually connected geometry Γ is called a *classical Tits geometry* if and only if for each pair $i \neq j \in I$ and each residue Γ_F of type $\{i, j\}$ either Γ_F is a classical generalized m_{ij}-gon (for some fixed $m_{ij} \geqq 3$) or a generalized 2-gon. The diagram $\Delta(I)$ is defined in the same way as for chamber systems. Also Γ is *locally finite* if all the rank 2 residues are finite.

(2.2) *Connection: chamber system - geometry.* Suppose Γ is a geometry over I. Then one can define the chamber system $\mathfrak{C}(\Gamma)$ in the following way: The "chambers" are the flags of Γ of type I, two such chambers being i-adjacent if and only if they coincide in a flag of type $I - i$. Then $\mathfrak{C}(\Gamma)$ is connected if Γ is residually connected. Furthermore one has the following trivial:

Lemma. *Suppose Γ is a classical, locally finite Tits geometry over I with diagram $\Delta = \Delta(I)$ and flag-transitive automorphism group G. Then $\mathfrak{C}(\Gamma)$ is a classical, locally finite Tits chamber system over I with diagram Δ and G is a chamber-transitive automorphism group of $\mathfrak{C}(\Gamma)$.*

This is essentially contained in [21].

To come back from chamber systems to geometries is not so easy. There is a way to construct a geometry $\Gamma(\mathfrak{C})$ from a chamber system $\mathfrak{C}$ by setting:

The objects of type i are the residues $\Delta_{I-i}(c)$, $c \in \mathfrak{C}$ two such objects $\Delta_{I-i}(c)$ and $\Delta_{I-j}(d)$ being incident if and only if $\Delta_{I-i}(c) \cap \Delta_{I-j}(d) \neq \emptyset$. But in general one has:

(1) $\Gamma(\mathfrak{C})$ need not be residually connected if $\mathfrak{C}$ is connected.

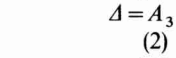
(2) $\Gamma(\mathfrak{C})$ need not be a classical Tits geometry if $\mathfrak{C}$ is a classical Tits chamber system.

A useful criterium for (1) and (2), which we will use later on, is given in [9].

Next we state some known classifications which we need subsequently.

(2.3) **Theorem.** *Suppose $\mathfrak{C}$ is a classical, locally finite Tits chamber system over I of* rank $\geqq 3$ *with spherical diagram $\Delta = \Delta(I)$ and transitive automorphism group G and finite chamber stabilizer. Then one of the following holds:*

(1) *$\mathfrak{C}$ is a finite spherical building of type Δ and G is an extension of a simple group of Lie-type Δ by diagonal and field automorphisms or $G \simeq A_7$ and $\Delta = A_3$.*

(2) *$\Delta =$*  , *$\mathfrak{C}$ is the chamber system of type C_3 obtained from A_7 and $G \simeq A_7$.*

This is Theorem (3.1) of [15].

(2.4) **Theorem.** *Suppose $\mathfrak{p} = \{X_i \mid i \in I\}$ is a strong parabolic system of the group G of type Δ. Suppose further that all the $\bar{X}_i = O^{p'}(X_i / O_p(X_i))$ are defined over*

fields of order at least 5. Let $G_0=\langle O^{p'}(X_i)|i\in I\rangle$. *Then* $G_0\trianglelefteq G$ *and* G_0 *has a BN-pair of type* Δ *with finite Borel-subgroup.* (Δ *is the type of the Weyl-group*!)

This is Theorem 2 of [11]. Actually Niles formulates (2.4) only for finite groups, but all he needs for the proof is the "local" finiteness, which is part of the definition of parabolic systems.

We will show later on, see (2.7), that under the hypothesis of (2.4) all connected components of Δ are spherical. Hence G is "known" by Tits classification of buildings of spherical type [20]. (For the exact statement see [18, (2.4)].)

(2.5) **Lemma.** *Suppose* $\mathfrak{C}$ *is a connected chamber system with transitive automorphism group* G *over the index set* I. *Let* $B=G_c$, $c\in\mathfrak{C}$ *a chamber, and* $X_J=G_{\Delta_J(c)}$ *for* $J\subseteq I$. *Suppose there exists a subgroup* B^* *of* B *such that for each* $J\subseteq I$ *with* $|J|\leqq 2$ *there exists a subgroup* X_J^* *of* X_J *satisfying*

(a) $B^*=X_J^*\cap B$

(b) $X_J^*B=X_J$.

Let $G_0=\langle X_i^*|i\in I\rangle$. *Then, identifying* $\mathfrak{C}$ *with* $\mathfrak{C}(G,B,(X_i)_{i\in I})$, *the map:*

$$\chi\colon B^*g\to Bg,\quad g\in G_0$$

is a 2-covering of $\mathfrak{C}(G_0,B^*,(X_i^*)_{i\in I})$ *on* $\mathfrak{C}$.

Proof. We have $G_0B=\langle X_i^*B|i\in I\rangle=\langle X_i|i\in I\rangle=G$ by connectedness of $\mathfrak{C}$. So χ is surjective. As $X_i^*B=X_i$ for $i\in I$, χ is a morphism. Let $J=\{i,j\}\subseteq I$. To show that χ is a 2-covering it obviously suffices to show that the restriction φ of χ on $\mathfrak{C}(X_J^*,B^*,X_i^*,X_j^*)$ is a bijection onto $\mathfrak{C}(X_J,B,X_i,X_j)$ for each such J. Obviously φ is surjective. But (a) implies that φ is also injective. This proves (2.5).

(2.6) **Proposition.** *Suppose* $\mathfrak{C}$ *is a classical, locally finite Tits chamber system with hyperbolic linear diagram* Δ, *transitive automorphism group* G *and* rank $n=|I|$; *satisfying:*

(*) $F^*(\bar{G}_j)\in\mathfrak{Q}_p^{n-1}$ *for each end node* j *of* Δ.

Then the following holds:

(1) $\Gamma(\mathfrak{C})$ *is a locally finite, classical Tits geometry of type* Δ. *Further* $\mathfrak{C}\simeq\mathfrak{C}(\Gamma(\mathfrak{C}))$ *and* G *acts flag-transitively on* $\Gamma(\mathfrak{C})$.

(2) *If we identify* I *with a maximal flag of* $\Gamma(\mathfrak{C})$, *then* G_J *is the stabilizer of the flag* $J\subseteq I$ *of* Γ *and acts flag-transitively on the residue of* J.

Proof. By the main theorem of [9] it suffices to verify the fundamental condition

$$X_J\cap X_K=X_{J\cap K}\quad\text{for } J,K\subseteq I\ (X_\phi=B!)$$

of [9] to prove (2.6).

Now this condition is obviously satisfied if we can show:

(1) If K' is a maximal subset of $K\subsetneqq I$, then $X_{K'}$ is a maximal subgroup of X_K.

Now (1) holds if Δ_K is connected by the assumption (*). To prove (1) for disconnected Δ_K we need the structure of the corresponding X_K. This is given by

(2) Suppose $J \cup L = K \subseteq I$ such that no node in J is connected with a node in L in Δ. Then $X_K = X_J X_L = X_L X_J$.

If $j \in J$ and $l \in L$, then $\Delta_{j,l}(c)$ is the chamber system of a generalized digon on which $X_{j,l}$ acts flag-transitively. Hence $X_{j,l} = X_j \cdot X_l$. It is obvious that this proves (2).

Now to the proof of (1). Suppose $K = K_1 \cup \ldots \cup K_k$ such that Δ_K splits in the connected components Δ_{K_i}, $i = 1, \ldots, k$. Since $|K - K'| = 1$ we may assume $L = K_1 \cup \ldots \cup K_{k-1} \subseteq K'$ and $J = K' \cap K_k$ is maximal in K_k. Suppose $X_K \geqq Y > X_{K'}$. Let $y \in Y - X_{K'}$. Then $y = x_l \cdot x'$ with $x_l \in X_L$ and $x' \in X_{K_k} - X_J$ by (2). Hence $x' = x_l^{-1} y \in (Y \cap X_{K_k}) - X_J$. Now, as (1) holds for connected subsets, X_J is maximal in X_{K_k}. Hence

$$X_{K_k} = \langle X_J, x' \rangle \leqq Y.$$

Since $X_L \leqq Y$ this implies $Y \geqq X_L \cdot X_{K_k} = X_K$. This proves (1) and thereby (2.6).

(2.7) **Lemma.** *Suppose Γ is a thick infinite building (of finite rank $n = |I|$.) Then the number of apartsments containing a given chamber is infinite.*

Proof. If Γ is spherical this is obvious. So assume Γ is not spherical. Then there exists by Proposition 4 of [21] an infinite strictly increasing chain of reduced types R_m, $m \in \mathbb{N}$ in the corresponding Coxeter-group.

As Γ is thick there exists a $q \geqq 2$ such that $|\Delta_i(d)| \geqq q+1$ for each $i \in I$ and each chamber d of Γ. Hence there exist at least q^m simple galleries of type R_m starting with c in Γ. Now by Proposition 3 of [21] each such gallery is contained in an apartment, and each such apartment contains a unique such gallery. Hence

$$A_m(c) \geqq q^m,$$

where $A_m(c)$ is the number of apartments containing c, which contain a simple gallery of type R_m. Hence $A_m(c) > M$ for each $M \in \mathbb{N}$ if m is big enough.

§ 3. Properties of the finite Lie-type groups

In this section we prove some special properties of the finite groups of Lie-type. So let for the following lemmata G be an extension of an element of $\mathfrak{L}_p^i$, $i \geqq 2$ by a p'-group of diagonal and field automorphisms.

Consider G as a group of automorphisms of its spherical building $\mathfrak{C}$ over I with diagram $\Delta = \Delta(I)$. (Here the building is considered as a chamber system to use the notation introduced in (2.1).) For $J \subseteq I$ let $G_J = M_J \cdot L_J$ be the "Levi-decomposition" of G_J. ($G_J = P_{I-J}$ and $M_J = U_{I-J}$ in the usual notation. Further $M_J \leqq K_J$ and K_J/M_J is a p'-group with $[K_J, L_J] \leqq M_J$!) Then we have:

(3.1) **Lemma.** (a) $M_J \subseteq M_K$ *if and only if* $J \subseteq K$. (b) K_J *is p-closed for each* $J \subseteq I$ (i.e. *has a normal p-Sylow-subgroup* M_J). (c) $M_J M_K = M_{J \cup K}$.

(a) *is a direct consequence of the definition of the* U_L *in* [1, (8.5)] *and the fact that each element of U is expressible in a unique way as a product of root elements corresponding to positive roots.* (b) *follows from* $K_J \subseteq B$. (c) *follows again from the definition of* U_L *and* $\Phi_J \cap \Phi_K = \Phi_{J \cap K}$ *by* [1, (2.5.6)].

In the next Lemma we describe, without proof, the structure of the maximal parabolic subgroups of the rank 2 Lie-type groups different from $A_2(q)$, $C_2(q)$ and $B_2(q)$. (Since their structure is well-known!) A proof can be obtained from [3].

(3.2) **Lemma.** *Let $G \in \mathfrak{Q}_p^2$ different from $A_2(q)$, $C_2(q)$ and $B_2(q)$, $P_i = U_i \cdot L_i$, $i = 1, 2$ the two maximal parabolic subgroups of G with unipotent radical U_i and Levi-complement L_i. Let $R_i = O^{p'}(L_i)$.*

Then one of the following holds:

(1) *$G \simeq U_4(q)$, $R_1 \simeq SL_2(q^2) \simeq \Omega^-(4, q)$ and U_1 is a natural $\Omega^-(4, q)$-module. $R_2 \simeq SL_2(q)$, U_2 special of order q^5 with $U_2/Z(U_2)$ an irreducible L_2-module which is the direct sum of 2 natural R_2-modules.*

(2) *$G \simeq U_5(q)$, $R_1 \simeq SL_2(q^2) \simeq \Omega^-(4, q)$, $Z(U_1) = \Phi(U_1)$ is a natural $\Omega^-(4, q)$-module, while $U_1/Z(U_1)$ is a natural $SL_2(q^2)$-module. $R_2 \simeq U_3(q)$, U_2 special of order q^7 and $U_2/Z(U_2)$ is an irreducible R_2-module.*

(3) *$G \simeq G_2(q)$, $q \geqq 4$ and $q \neq 3^n$. $R_1 \simeq R_2 \simeq SL_2(q)$, U_1 special of order q^5 and $U_1/Z(U_1)$ is an irreducible R_1-module if $q > 4$. (If $q = 4$ then $U_1/Z(U_1)$ is the direct sum of natural $\Omega^-(4, 2)$-modules.) $Z(U_2)$ is a natural R_2-module. Further there exists an $A \lhd P_2$, with $|A| = q^3$, $Z(U_2) < A$, $\Phi(A) = 1$ and U_2/A a natural R_2-module.*

(4) *$G \simeq {}^3D_4(q)$, U_1 special of order q^9 with $|Z(U_1)| = q$ and $U_1/Z(U_1)$ an irreducible $R_1 \simeq SL_2(q^3)$-module. $L_2 \simeq SL_2(q) * \mathbb{Z}_{q^3-1}$, $Z(U_2)$ is a natural R_2-module. There exists an elementary abelian normal subgroup A of P_2 of order q^5 with $Z(U_2) < A$ and $[A, U_2 R_2] \leqq Z(U_2)$. Further U_2/A is an irreducible L_2-module which is the direct sum of three natural R_2-modules.*

(5) *$G \simeq {}^2F_4(q)$, $q = 2^n$, $R_1 \simeq Sz(q)$ (resp. F_{20} if $q = 2$) and $U_1/\Phi(U_1)$ the natural R_1-module. $R_2 \simeq SL_2(q)$, $|U_2/\Phi(U_2)| = q^3$ and $U_2/\Phi(U_2)$ is the extension of a natural by a trivial R_2-module. (If $q > 2$ then $U_2/\Phi(U_2)$ is an indecomposable R_2-module!)*

(6) *$G \simeq G_2(3^n)$, $P_1 \simeq P_2$, $R_i \simeq SL_2(q)$, $|\Phi(U_i)| = q$, $|Z(U_i)| = q^3$ and $U_i/Z(U_i)$ as well as $Z(U_i)/\Phi(U_i)$ are natural R_i-modules.*

(7) *$G \simeq G_2(2)$, $U_1 \simeq Q_8 * Q_8$, $L_i \simeq L_2(2)$ and $[U_1, L_1'] \simeq Q_8$. $U_2 \simeq (\mathbb{Z}_4 \times \mathbb{Z}_4)\langle t \rangle$ with $t^2 = 1$ and t inverting $\mathbb{Z}_4 \times \mathbb{Z}_4 \simeq [U_2, L_2]$. (${}^2F_4(2)'$ resp. $G_2(2)'$ are subgroups of index 2 of ${}^2F_4(2)$ resp. $G_2(2)$!)*

(3.3) **Lemma.** *On of the following holds:*

(1) *$[M_i, G_i] = M_i$ for each $i \in I$.*

(2) *$G \simeq Sp(4, 2)$, ${}^2F_4(2)$ or $G_2(2)$ resp. $G_2(2)'$.*

Especially, if G is simple, either (1) *holds or $G \simeq G_2(2)'$.*

Proof. Let H be a fixed Cartan subgroup. Then the proof of the simplicity of Chevalley-groups in [1, p. 174 resp. 263-264] shows that either $[M_i, H] = M_i$ for each $i \in I$ or G is defined over $GF(2)$ or $G \simeq C_l(3)$, $l = |I|$. In the latter case we have $[\mathscr{X}_r, H] = \mathscr{X}_r$ for all short roots v of the root system of G by the same argument. Let now

$$\Delta = \overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{2}{\circ}\ldots\overset{l-1}{\circ}\!\!=\!\!\overset{l}{\circ}.$$

Since G_1 is the stabilizer of a point in the natural action of G it is easy to see that

$$\mathscr{X}_r = M_1' = Z(M_1)$$

is the only long root subgroup contained in M_1.

Hence

$$\mathscr{X}_r \leqq [M_1, M_i] \leqq M_i \quad \text{for all } i \in I.$$

But as G_i acts transitively on the long root subgroups in M_i, as well as on the points in the totally isotropic subspace stabilized by it, this together with the above implies

$$M_i \leqq [M_i, G_i] \quad \text{for all } i \in I$$

where $M_i = [M_i, G_i]$ in this case as well.

So the case where G is defined over $GF(2)$ remains to be treated. Let Φ be the root system of G with fundamental system $\pi = \{r_i | i \in I\}$. Suppose that all roots in Φ have the same length. For $i \in I$ and $I' = I - i$. Let $\bar{\Phi}_{I'} = \{r \in \Phi | r \notin \Phi_{I'}\}$. Then by [1, (8.5)]

$$M_i = \langle \mathscr{X}_r | r \in \Phi^+ \cap \bar{\Phi}_{I'} \rangle.$$

Let $r \in \Phi^+ \cap \bar{\Phi}_{I'}$. Then $r = r_{k_1} + \ldots + r_{k_{s-1}} + r_{k_s}$ with $r_{k_i} \in \pi$ and $\sum_{i=1}^{a} r_{k_i} \in \Phi^+$ for all $a \leqq s$. Suppose $s > 1$ and let $\rho = r_{k_1} + \ldots + r_{k_{s-1}}$. Then $r = \rho + r_{k_s}$ and $\mathscr{X}_r = [\mathscr{X}_\rho, \mathscr{X}_{r_{k_s}}]$ by the Chevalley-commutator relations, since all roots have the same length. If $r_{k_s} = r_i$, then $\mathscr{X}_{r_{k_s}} \leqq M_i$ and $\mathscr{X}_r \leqq [M_i, U] \leqq [M_i, G_i]$. If $r_{k_s} \neq r_i$ then $r_{k_s} \in \Phi_{I'}$ and so $\rho = r - r_{k_s} \in \bar{\Phi}_{I'} \cap \Phi^+$. But then again $\mathscr{X}_r \leqq [M_i, U] = [M_i, G_i]$.

So it remains to be shown $\mathscr{X}_r \leqq [M_i, G_i]$ for fundamental roots $r \notin \Phi^+ \cap \bar{\Phi}_{I'}$. But r_i is the only such root. Since r_i is conjugate to $r_i + r_j$ by w_{r_j}, when j is a node of Δ connected to i, $\mathscr{X}_{r_i}$ is conjugate to $\mathscr{X}_{r_i + r_j} \leqq [M_i, G_i]$ by an element of L_i. This shows $M_i \leqq [M_i, G_i]$ in case $\Delta = A_l$, D_l or E_l, $l \geqq 2$.

In the remaining cases $C_n(2)$, $n \geqq 3$, $U_n(2)$, $\Omega^-(2n, 2)$, $n \geqq 4$ and ${}^2E_6(2)$, $F_4(2)$ one can prove (3.3) inductively in the following way. There exists a $j \in I$ such that L_j is isomorphic to $C_{n-1}(2)$, $U_{n-2}(2)$, $\Omega^-(2n-2, 2)$, $U_6(2)$ resp. $C_3(2)$. Further either $P_j = N_G(\mathscr{X}_r)$ for some long root r and $M_j/\mathscr{X}_r$ is elementary abelian and (except in case $F_4(2)$) an irreducible L_j-module or $\Phi(M_j) = 1$ and M_j is an irreducible L_j-module (in case $\Omega^-(2n, 2)$). In any case $[M_j, L_j] = M_j$. Now by (3.1) (c) $M_{i,j} = M_i M_j$ and by induction assumption

$$(*) \qquad M_{i,j} = M_j [M_i, L_{ij}].$$

In case L_j acts irreducibly on $M_j/\mathscr{X}_r$ resp. on M_j [17] implies that $M_j/[M_i, M_j]$ is an irreducible L_{ij}-module. Hence $[M_i, M_j] = M_i \cap M_j$. Since in case of $F_4(2)$ $M_j/\mathscr{X}_r$ is the direct sum of a natural and spin $Sp(6, 2)$-module we get in any case $M_i \cap M_j = [M_i, M_j]$ and by $(*)$

$$M_i \leqq (M_i \cap M_j)[M_i, L_{ij}] \leqq [M_i, G_i].$$

If $G \simeq {}^3D_4(2)$ then (3.3) is a consequence of (3.2).

(3.4) **Lemma.** *Suppose G is simple and $G_i = N(Z)$ where Z is a "long root subgroup" of G for some $i \in I$ such that $\Delta - i$ is connected. ($Z = \Omega_1(Z(R))$, R a long root subgroup in the twisted cases.) Let $R_i = O^{p'}(L_i)$. Then one of the following holds:*

(1) $G \simeq C_n(q)$, $U_n(q)$; $R_i \simeq C_{n-1}(q)$ *resp.* $U_{n-2}(q)$ *and* M_i/Z *is the natural module for* R_i. *Further* $Z = \Phi(M_i) = Z(M_i)$, *except in case* $C_n(2^m)$ *in which* $\Phi(M_i) = 1$ *and* M_i *is an indecomposable* R_i*-module. (Except for* $C_2(2) \simeq \Sigma_6$!)

(2) $G \simeq E_n(q)$, $6 \leqq n \leqq 8$; $R_i \simeq A_5(q)$, $D_6(q)$ *resp.* $E_7(q)$, $Z = \Phi(M_i) = Z(M_i)$ *and* M_i/Z *is an irreducible* R_i*-module.*

(3) $G \simeq F_4(q)$, ${}^2E_6(q)$; $R_i \simeq C_3(q)$ *resp.* $U_6(q)$. $Z = \Phi(M_i)$ *and* M_i/Z *is an irreducible* R_i*-module in case of* ${}^2E_6(q)$, *while* M_i/Z *contains exactly two* R_i*-composition factors which are both non-trivial irreducible* R_i*-modules in case of* $F_4(q)$.

(4) $G \simeq G_2(q)$, $G_2(2)'$, ${}^3D_4(q)$; $R_i \simeq A_1(q)$ *resp.* $A_1(q^3)$, $\Phi(M_i) = Z$ *and* $M_i/Z = [M_i/Z, R_i]$ *except in case* $G_2(2)$ *and* $G_2(2)'$.

(5) $G \simeq {}^2F_4(2^n)$ *or* ${}^2F_4(2)'$; $R_i \simeq Sz(2^m)$ *resp.* F_{20}. $Z \leqq \Phi(M_i)$ *and* $[M_i/\Phi(M_i), R_i] = M_i/\Phi(M_i)$.

Proof. The structure of a Levi-complement of the normalizer of a long root subgroup can be read from the extended Dynkin-diagram. The structure of M_i and the action of L_i on M_i is described in the cases (1)–(3) in the root involution paper or in Thompson's quadratic pair paper. In case (4) and (5) the structure of G_i is described in (3.2).

(3.5) **Corollary.** *Let* G, Z *and* G_i *be as in* (3.4). *Suppose* R *is another simple Lie-type group of the same characteristic and the same rank, having a maximal parabolic subgroup* P *which has a quotient* P/X *such that* $O^{p'}(P/X) \simeq O^{p'}(G_i/Z)$. *Then* $G \simeq R$ *and* $P \simeq G_i$ *or* $G \simeq C_2(q)$ *and* $R \simeq A_2(q)$, $G_2(q)$, $U_5(q^{1/2})$ *or* ${}^2F_4(q)$.

Proof. Since P has the same rank as G_i, we have $X \leqq O_p(P)$. If now rank $G \geqq 3$ then simple order arguments and consideration of the Dynkin-diagram imply $G \simeq R$ and $G_i \simeq P$. (If for example $G \simeq E_6(q)$ then $O^{p'}(L_i) \simeq A_5(q)$. Hence $R \in \{A_6(q), B_6(q), C_6(q), D_6(q), E_6(q), U_{12}(q^{1/2}), U_{13}(q^{1/2}), \Omega^-(14, q)\}$. Now in all cases except $E_6(q)$, $U_{13}(q^{1/2})$ and $\Omega^-(14, q)$ we have $|R|_p < |G_i/Z|_p$. The last two cases are out by a closer look at the corresponding parabolic of these groups!)

So suppose rank $G = 2$. If $G \simeq {}^3D_4(q)$ then $|G_i/Z|_p = q^{11}$, $|M_i/Z| = q^8$ and M_i/Z is an irreducible $L_2(q^3)$-module. But by (3.2) it is easy to see that neither in $G_2(q^3)$, $U_4(q^3)$, $U_4(q^{3/2})$, nor in $U_5(q^{3/2})$ there is a section of a parabolic subgroup having such an $L_2(q^3)$-module. Next suppose $G \simeq G_2(q)$. Then by the above $R \not\simeq {}^3D_4(q)$. Hence by (3.2) it is easy to see that either $R \simeq G_2(q)$ and (3.5) holds or $R \simeq U_4(q)$ and P is also the normalizer of a root subgroup in R. In the latter case $O_p(P)/Z(O_p(P))$ is the direct sum of two natural $SL_2(q)$-modules, which is not the case for M_i/Z when $G \simeq G_2(q)$. If $G \simeq U_5(q)$, then $O^{p'}(L_i) \simeq U_3(q)$ and so $G \simeq R$, $G_i \simeq P$.

So we are left with the cases $G \simeq U_4(q)$ or $C_2(q)$. In the first case $|M_i/Z| = q^4$ and M_i/Z is the direct sum of two natural $SL_2(q)$-modules. But by (3.2) P does not have such quotient if $R \simeq G_2(q)$ or ${}^3D_4(q)$. Hence $G \simeq R$ in this case again.

Suppose finally $G \simeq C_2(q)$. Then we only have to show $R \not\simeq U_4(q)$ or ${}^3D_4(q)$. Suppose R is isomorphic to one of these. As $O^{p'}(L_i) \simeq SL_2(q)$, P is the normalizer of a root subgroup in case of $U_4(q)$ while P is not the normalizer of a root subgroup if $R \simeq {}^3D_4(q)$. Hence $O_p(P)/\Phi(O_p(P))$ is an irreducible P-module, which is the direct sum of two resp. three natural $SL_2(q)$-modules. This shows P does not have such a quotient.

This proves (3.5) in case $G \simeq C_2(q)$.

(3.6) **Lemma.** *Let $i \in I$ such that $\Delta - i$ is connected. Suppose $F_i \leqq K_i$, $R_i \leqq G_i$ satisfying:*

(1) $F_i \trianglelefteqq G_i$, $R_i \trianglelefteqq G_i$.

(2) $[F_i, R_i] = 1$.

(3) $O^{p'}(G_i) \leqq F_i R_i$.

Then either $F_i = 1$ or F_i is a "long root subgroup" of G and $G_i = N_G(F_i)$. In any case $O^{p'}(G_i) \leqq R_i$ or $G \simeq \Sigma_6$, $R_i \simeq \Sigma_4$ and $F_i \simeq \mathbb{Z}_2$.

Proof. Let G be a minimal counter example. Since by [1] neither a diagonal nor a field automorphism centralizes $O^{p'}(G_i)$ and since, intersecting F_i and R_i with $O^{p'}(G)$, the hypothesis of (3.6) is also satisfied for $O^{p'}(G)$, we obtain $G = O^{p'}(G) \in \mathfrak{L}_p^i$ for some $i \geqq 2$.

Suppose now $F_i \cap M_i = 1$ or $F_i \cap M_i$ is a long root subgroup. Then by (2) and (3) $[F_i, O^{p'}(G_i)] \leqq F_i' \leqq F_i \cap M_i$ and so by (3.4) $F_i = F_i \cap M_i$. Now by (3.4) $F_i \leqq M_i'$ in all cases except $C_n(2^m)$. As $M_i = F_i(M_i \cap R_i)$ this implies $M_i \leqq R_i$ and $O^{p'}(G_i) \leqq R_i$ in case of $F_i \leqq M_i'$. Hence (3.6) holds.

Now in case $C_n(2^m)$ different from $C_2(2) \simeq \Sigma_6$, $\Phi(M_i) = 1$ but M_i is an indecomposable $GF(2)O^{2'}(L_i)$-module with M_i/F_i the natural module. Hence again

$$F_i \leqq M_i = [M_i, R_i] \leqq R_i$$

and (3.6) holds in this case as well.

This shows that it remains to be shown that $Z = F_i \cap M_i$ is a "long root subgroup" or $Z = 1$. Suppose $Z \neq 1$. Then $Z_1 = Z(Z) \neq 1$. As F_i/Z is a p'-group we have $O^{p'}(G_i) \leqq ZR_i$ and $Z_1 \leqq Z(ZR_i)$. Now it is well-known (and follows from (3.4)) that a long root subgroup is a maximal p-subgroup centralized by $O^{p'}(G_i)$, where G_i is a maximal parabolic with connected diagram, or $G \simeq F_4(2^m)$ or $G_2(3^m)$ in which case root subgroups corresponding to long and short roots have the same property. Since in the latter case long and short root subgroups are interchanged by a diagram automorphism and so their normalizers have the same structure, we may in any case assume that Z_1 is contained in a long root subgroup. Hence Z_1 is a long root subgroup, since a Cartan-subgroup acts irreducibly on such a long root subgroup.

Now by (3.4) $[M_i, R] = M_i \leqq R_i$ in all cases except Σ_6, $G_2(2)$ and $G_2(2)'$. Hence $Z = Z_1$ is a root subgroup and (3.6) holds. Since (3.6) holds in case of $G_2(2)$ or $G_2(2)'$ by (3.2) this completes the proof.

(3.7) **Lemma.** *Suppose $A \leqq \mathrm{Aut}(G)$ is a p-subgroup such that $AG \leqq \mathrm{Aut}(\mathfrak{C})$. (I.e. the elements of $A - G$ induce field automorphisms on G.) Then the following holds:*

(1) *A fixes a chamber $c \in \mathfrak{C}$.*

(2) *$A \cap G \leqq A \cap E_j$ for each $j \in I$ for which $\bar{X}_j$ is non-solvable; where E_j is the preimage of $E(AX_j/Q_j^*)$ and Q_j^* the kernel of the action of AX_j on $\Delta_j(c)$.*

(3) *$A \cap G = A \cap E_i$ for some $i \in I$.*

Proof. Since A normalizes a p-Sylow subgroup of G, it normalizes a Borel-subgroup and it stabilizes the unique chamber $c \in \mathfrak{C}$ fixed by B. Hence A normalizes all residues $\Delta_j(c)$, $j \in I$.

Now $A \cap G = A \cap X_i$ for each $i \in I$. If now $\bar{X}_j$ is non-solvable, then the structure of parabolic subgroups of Lie-type groups, see [1, (8.5)], implies $|\bar{X}_j : E(\bar{X}_j)|$ is a p'-group. Hence $A \cap G \leqq A \cap E_j$. (In general one does not have equality, since $A \cap Q_j^*$ might not be contained in G!)

As $AG \leqq \mathrm{Aut}\,\mathfrak{C}$, AG is an extension of a Lie-type group by diagonal and field automorphisms. Hence we may identify $A/A \cap G \simeq AG/G$ by [1, (12.5.1)] with a subgroup of $\mathrm{Gal}(K)$, K the field of definition of G. ($K = GF(q^2)$ or $GF(q^3)$ if $GF(q)$ is the fixed field for the twisted types!) But even for the twisted types there is a minimal parabolic subgroup X_i such that $\bar{X}_i$ is a rank 1 group defined over K. Hence no element of $A - A \cap G$ induces an inner automorphism on $E(\bar{X}_i)$.

This implies $|A : A \cap G| \leqq |A : A \cap E_i|$ and so $A \cap G = A \cap E_i$ by (2) for such an $i \in I$.

(3.8) **Lemma.** *Suppose G is of rank 2 and defined over $GF(7)$ such that $G_i/K_i \simeq PSL_2(7)$ for $i = 1$ or 2. Then $G \simeq PSp(4, 7)$.*

Proof. Suppose $O^{p'}(G) \simeq PSp(4, 7)$. Then G is an extension of $PSp(4, 7)$ by diagonal automorphisms. But as $M_i \simeq 7^{1+2}$ it is easy to see that an outer diagonal automorphism cannot centralize $O^{p'}(G_i/K_i) \simeq PSL_2(7)$. Thus $G = O^{p'}(G) \simeq PSp(4, 7)$.

So to prove (3.8) it suffices to show that for each other simple rank 2 group G defined over $GF(7)$ we have $PGL_2(7) \not\leqq G_i/K_i$ if $PSL_2(7) \leqq G_i/K_i$. But this is easy to see using for example [1, p. 122 and 251].

(3.9) **Lemma.** *Let $G \simeq PSP(2n, q)$, $n \geqq 2$ and q odd. Let $G_i = M_i L_i$ be the stabilizer of a maximal totally isotropic subspace U in the natural (projective) representation of G. Then M_i is elementary abelian and L_i acts irreducibly on M_i.*

Proof. Let $q = p^r$ and $S \in \mathrm{Syl}_p(G_i)$. Then it is well-known that $Z = Z(S)$ has order q and consists of symplectic transvections. Now (*) $M_i = \langle Z^G \cap M_i \rangle$ as can be easily shown by induction on n.

But (3.9) is a direct consequence of (*), since each proper L_i-invariant subspace of M_i must contain Z.

(3.10) **Proposition.** *Suppose G is an extension of a group in $\mathfrak{L}_p^i$, $i \geqq 2$ by diagonal and field automorphisms. Let B be a Borel-subgroup of G and P_1, P_2 two (different) maximal parabolic subgroups of G. Suppose X_1, X_2 are subgroups of G satisfying:*

(1) $X_i \lhd P_i$, $|P_i : X_i| = p$, $i = 1, 2$

(2) $B \cap X_1 = B^* = B \cap X_2$.

Let $G^* = \langle X_1, X_2 \rangle$. Then $F^*(G) \leqq G^* \lhd G$, $G = G^* \cdot B$ and $|G : G^*| = p$. Especially G does not contain such a pair X_1, X_2 if $G/F^*(G)$ is a p'-group.

Proof. Let $G_0 = F^*(G)$. Then, as G_0 acts already chamber-transitively on the corresponding building, $G = G_0 B$. Let $R_i = P_i \cap G_0$, $Y_i = X_i \cap G_0$, $Q = B \cap G_0$ and $Q^* = B^* \cap G_0$. Then Q is a Borel-subgroup of G_0 and R_1, R_2 are two different maximal parabolic subgroups of G_0. Now $Y_i = X_i \cap R_i$, whence $|R_i : Y_i| \leqq p$. Moreover,

$$Q^* = B^* \cap Y_i = Q \cap Y_i, \qquad i = 1, 2.$$

As Q contains a p-Sylow subgroup S of G_0 this shows that $S \leqq Y_1$ implies $S \leqq Y_2$ so that we obtain

(a) $R_1 = Y_1$ and $R_2 = Y_2$, $Q = Q^*$ or

(b) $|R_i : Y_i| = |Q : Q^*| = p$ for $i = 1, 2$.

We show that in case (a) the conclusion of (3.10) holds while case (b) is impossible. Now in case (a) we have

$$Y_i B^* = (X_i \cap G_0) B^* \leqq X_i \leqq P_i = Y_i B, \quad \text{as } Y_i = R_i.$$

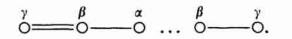
Since $|Y_i B : Y_i B^*| \leqq |B : B^*| = p$ and $X_i \neq P_i$, this implies $X_i = Y_i B^*$ for $i = 1, 2$. Hence $G^* = G_0 B^* \trianglelefteq G_0 B = G$ and

$$G/G^* \simeq B/B \cap G_0 B^* = B/B^*(B \cap G_0) = B/B^*.$$

This shows that (3.10) holds in case (a).

Assume now that (b) holds. Then the simple group of Lie-type G_0 satisfies the same hypothesis as G. As $|R_i/Y_i| = p$, $R_i' \leqq Y_i$. Let $U_i = O_p(R_i)$, $i = 1, 2$. Then in all cases except $G_0 \simeq G_2(2)'$ we have by (3.3) $U_i = [U_i, R_i] \leqq R_i' \leqq Y_i$. If now rank $G_0 = 2$, then $S = U_1 U_2 \leqq Q^*$ which is impossible. (If $G_0 = G_2(2)'$ we have $[U_1, R_1] = U_1$ and $S = U_1 [U_2, R_2]$ as can be easily seen. Hence again $S \leqq Q^*$!)

So rank $G_0 \geqq 3$. If $O^{p'}(R_i/U_i)$ is semi-simple for $i = 1$ or 2, then again $S \leqq R_i' \leqq Y_i$, contradicting $|R_i : Y_i| = p$. So $O^{p'}(R_i/U_i)$ contains either a solvable Levi-factor or a Levi-factor isomorphic to $\Sigma_6 \simeq Sp(4, 2)$ for $i = 1$ and 2. So in any case G_0 is defined over $GF(2)$ or $GF(3)$. Suppose $O^{p'}(R_1/U_1)$ contains a Levi-factor isomorphic to Σ_6. Then $G_0 \simeq C_n(2)$ for $n \geqq 3$ and R_1 is the maximal parabolic corresponding to the node α in Δ:



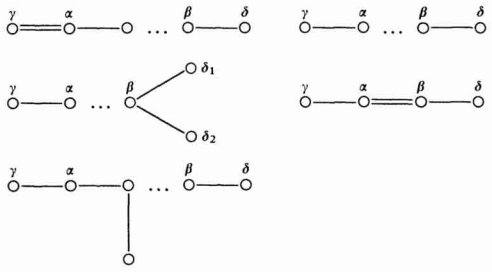
Suppose R_2 corresponds to the node β. Then β is as above, since $O^{p'}(R_2/U_2)$ must contain a solvable direct factor. As $S \not\leqq Y_2$, $Y_2 \cap O^{p'}(R_2) \lhd O^{p'}(R_2)$ and is of index 2. Hence $Y_2 \cap O^{p'}(R_2)/U_2 \simeq \mathbb{Z}_3 \times R$; where R is the Chevalley-group corresponding to $\Delta - \{\beta, \gamma\}$ in the above diagram.

But then $S \cap Y_2 = Q_\gamma =$ the normal 2-subgroup of the minimal parabolic corresponding to γ. But it is obvious that $O^{p'}(R_1) \leqq Q_\gamma Y_1$, whence

$$S \leqq S \cap Q_\gamma Y_1 \leqq Q^*,$$

a contradiction.

So in the final case $O^{p'}(R_i/U_i)$ contain a solvable Levi-factor for $i = 1$ and 2. Hence R_1, R_2 correspond to the nodes α, β in the list below:



(Interchanging R_1 and R_2 if necessary).

But then the above argument implies $S \cap Y_1 = Q_\gamma$, $S \cap Y_2 = Q_\delta$ using the same notation as above, since G_0 is defined over $GF(2)$ or $GF(3)$. ($\delta = \delta_1$ or $\delta = \delta_2$ in case D_n!). But then $S = Q_\gamma Q_\delta \leqq B^*$. This finally shows that case (b) is impossible.

(3.11) **Corollary.** *Suppose* $\hat{G} = G \times Z$, G *as in* (3.10) *and* $Z \simeq \mathbb{Z}_p$. *Let* B *and* P_1, P_2 *be as in* (3.10). *Suppose* X_1, X_2 *are subgroups of* $\hat{G}$ *satisfying:*

(1) $X_i \lhd P_i Z$, $|P_i Z : X_i| = p$ *for* $i = 1, 2$

(2) $BZ \cap X_1 = B^* = BZ \cap X_2$.

Let $G^* = \langle X_1, X_2 \rangle$. *Then* $F^*(G) \leqq G^* \lhd \hat{G}$, $\hat{G} = G^*(BZ)$ *and* $|\hat{G} : G^*| = p$.

Proof. As in (3.10) we have $\hat{G} = G_0 BZ$, $G_0 = F^*(G)$. Moreover, $|P_i \cap G_0 : X_i \cap G_0| \leqq p$ as $P_i Z \cap G_0 = P_i \cap G_0$. If $X_1 \cap G_0 = P_1 \cap G_0$ then

$$B \cap G_0 \leqq X_1 \cap G_0 \cap BZ \leqq G_0 \cap B^* \leqq X_2 \cap G_0$$

so that also $X_2 \cap G_0 = P_2 \cap G_0$ and $B \cap G_0 = B^* \cap G_0$. Since $|BZ : B^*| = p$ and

$$(X_i \cap G_0) B^* \leqq X_i < P_i Z = (P_i \cap G_0) BZ$$

we obtain $X_i = (X_i \cap G_0) B^*$ for $i = 1, 2$. Hence $G^* = G_0 B^* \trianglelefteqq G_0 BZ = \hat{G}$ and $\hat{G}/G^* \simeq BZ/B^* \simeq \mathbb{Z}_p$.

So, to prove (3.11), we may assume $|P_i \cap G_0 : X_i \cap G_0| = p$ for $i = 1, 2$. As $BZ \cap G_0 = B \cap G_0$ we also have

$$B \cap (X_1 \cap G_0) = B^* \cap G_0 = B \cap (X_2 \cap G_0).$$

But by (3.10) G_0 does not contain such a pair.

(3.12) **Proposition.** *Suppose* G *is a group generated by two finite subgroups* G_1 *and* G_2 *satisfying:*

(a) G_i/K_i *is an extension of a group in* $\mathfrak{Q}_p^n$, $n \geqq 2$ *by a* p'*-group of diagonal and field automorphisms.*

(b) $G_1 \cap G_2/K_i$ *is a maximal parabolic subgroup with connected diagram in* G_i/K_i *for* $i = 1$ *and* 2.

(c) $K_1 K_2/K_i \leqq B_i/K_i$ *for some Borel-subgroup* B_i/K_i *of* G_i/K_i.

(d) *There does not exist a* $1 \neq N \lhd G$ *with* $N \leqq G_1 \cap G_2$.

Then K_i *is p-closed with* $[K_i, \tilde{G}_i] \leqq O_p(K_i)$ *for* $i = 1$ *and* 2; *where* $\tilde{G}_i = O^{p'}(G_i)$.

Proof. Suppose (3.12) is false. We derive a contradiction in several steps. Let for the whole proof $F_i = F^*(K_i)$ and $\bar{G}_i = \tilde{G}_i K_i/K_i$, $i = 1, 2$. As $K_1 K_2 \lhd G_1 \cap G_2$ we may by (b) and (c) assume

$$K_1 K_2/K_i \leqq B_i/K_i \leqq G_1 \cap G_2/K_i.$$

But then by (a) $B_i/K_1 K_2$ is the normalizer of a p-Sylow subgroup of $G_1 \cap G_2/K_1 K_2$ for $i = 1$ and 2, whence $B_1 \sim B_2$ in $G_1 \cap G_2$. But then we may assume $B_1 = B_2 = B$ and $K_1 K_2 \leqq K_{1,2} = \bigcap_{g \in G_1 \cap G_2} B^g$. Now hypothesis (a) together with (3.1) imply

(1) $K_{1,2}/K_i$ is p-closed for $i = 1$ and 2.

Next we show:

(2) $F_i = O_p(K_i)$, $i = 1, 2$.

As $E(K_i) \leqq K_{1,2}$ we have $E(K_i) \trianglelefteqq K_j$ for $\{1,2\} = \{i,j\}$. This implies $E(K_i) \leqq E(K_j)$ and so by symmetry and (d) $E(K_i) = E(K_j) = 1$. The same argument shows $O_s(K_1) = O_s(K_2) = 1$ for each prime $s \neq p$. This proves (2).

(3) $K_i \not\trianglelefteq K_j$ for $\{i,j\} = \{1,2\}$.

Suppose $K_1 \leqq K_2$. Then (d) implies $K_1 < K_2$.

Now by (2) $F_2 \not\leqq K_1$, since otherwise by (d) $F_1 = F_2 = 1$, a contradiction. Now

$$[F_1, K_1] \leqq F_2 \cap K_1 \leqq F_1 \leqq O^{p'}(K_1)$$

Let $\hat{G}_i = \langle F_j^{G_i} \rangle$, $\{i,j\} = \{1,2\}$. Then $\tilde{G}_1 \leqq K_1 \hat{G}_1$. Let $P \in \mathrm{Syl}_p(O^{p'}(K_2))$. Then $K_1 P = K_1 O^{p'}(K_2) \leqq K_1 \tilde{G}_1$, since by (1) $K_{1,2}/K_1$ is p-closed. Hence by the above $[K_1, P] \leqq O^{p'}(K_1)$. But it is obvious that this implies $O^{p'p}(K_1) = 1$ by (d). Hence the K_i are p-closed. Moreover, as $F_1 < F_2$, we have $\tilde{G}_1 = \hat{G}_1$ and $[K_1, \tilde{G}_1] \leqq F_1 = O_p(K_1)$. Since a Sylow p-subgroup of $G_1 \cap G_2$ centralizes $K_1 K_2 / O^{p'}(K_1) K_1$ this shows that also $[K_2, \tilde{G}_2] \leqq O_p(K_2)$ and so (3.12) holds, a contradiction.

(4) $F_i \not\leqq K_j$ for $\{i,j\} = \{1,2\}$.

Assume first $F_1 \leqq K_2$ and $F_2 \leqq K_1$. Then $F_1 = F_2 = 1$ by (d), a contradiction. So to prove (4) we may assume $F_1 \leqq K_2$ but $F_2 \not\leqq K_1$. As in (3) $\tilde{G}_1 \leqq K_1 \hat{G}_1$ and

$$[K_1, \hat{G}_1] \leqq F_1 = O_p(K_1) \leqq F_2.$$

Let $\bar{G}_{1,2} = G_1 \cap G_2 / K_2$. Then $O^{p'}(\bar{G}_{1,2}) \leqq \bar{K}_1 (\overline{\hat{G}_1 \cap G_2})$, since $O^{p'}(G_{1,2}) \leqq K_1 \tilde{G}_1 = K_1 \hat{G}_1$. Moreover, $\overline{\hat{G}_1 \cap G_2} \trianglelefteqq \bar{G}_{1,2}$, $\bar{K}_1 \lhd \bar{G}_{1,2}$ and $[\bar{K}_1, \overline{\hat{G}_1 \cap G_2}] = 1$. Hence by (3.6) either $\bar{K}_1 = 1$ or $\bar{K}_1$ is a long root subgroup of G_2/K_2. The first case contradicts (3). In the second case the hypothesis of (3.5) is satisfied. Hence by (3.6) either $K_1 K_2/K_1$ is also a long root subgroup or $O^{p'}(G_2/K_2) \simeq C_2(q)$ and $O^{p'}(G_1/K_1) \in \{A_2(q), G_2(q), U_5(q^{1,2}), {}^2F_4(q)\}$. In the first case we have

$$O^p(K_1) = O^p(K_1 K_2) = O^p(K_2) = 1 \quad \text{by (d)}$$

and so (3.12) holds, a contradiction. If $O^{p'}(G_1/K_1) \simeq A_2(q)$, then $K_2 \leqq K_1$, contradicting (3). If $O^{p'}(G_1/K_1) \simeq U_5(q^{1,2})$ or ${}^2F_4(q)$, then by (3.2) $K_1 K_2/K_1 \cap O_p(K_{1,2}/K_1) \leqq \Phi(O_p(K_{1,2}/K_1))$. Hence a p'-element of $K_1 K_2/K_1$ would centralize $O^{p'}(G_1 \cap G_2/K_1)$ and so again $K_1 K_2/K_1$ is a p-group, a contradiction as above. Finally, if $O^{p'}(G_1/K_1) \simeq G_2(q)$, we obtain again that $K_1 K_2/K_1$ is a p-group, since the full diagonal group of $G_2(q)$ has order $(q-1)^2$ and is acting already non-trivially on $K_{1,2}/K_1 K_2$. This proves (4).

Now we have $\tilde{G}_i \leqq K_i \hat{G}_i$ and $[K_i, \hat{G}_i] \leqq F_i \leqq O^{p'}(K_i)$. Since $O^{p'}(K_j) \leqq O^{p'}(G_i) \leqq \tilde{G}_i$ this implies:

(5) $O^{p'}(K_j) \leqq O^{p'}(K_i) \hat{G}_i$ for $\{i,j\} = \{1,2\}$.

Next we show:

(6) $O^{p'p}(K_2) \leqq O^{p'p}(K_1)$.

Let $A = O^{p'p}(K_1)$, $B = O^{p'p}(K_2)$. By (1) we have $AB \leqq K_1 \cap K_2$. We claim

$$(*) \qquad [O^{p'}(K_2), K_1 \cap K_2] \leqq C = O^{p'}(K_1) \cap K_2.$$

Suppose our claim (*) holds. Let $S \in \mathrm{Syl}_p(K_2)$ and $-$ be the natural homomorphism from K_2 on K_2/C. Then $O^{p'}(K_2) \leqq (K_1 \cap K_2)S$ and so $\overline{O^{p'}(K_2)}$

$=(\overline{K_1 \cap O^{p'}(K_2)}) \times \bar{S}$. Hence $\bar{K}_2$ is p-closed. Since C/A is a p-group this shows that K_2/A is p-closed so that $B \leqq A$.

Now to the proof of (*). By (5) we have

$$O^{p'}(K_2) \leqq O^{p'}(K_1)\hat{G}_1 \cap G_2 = O^{p'}(K_1(\hat{G}_1 \cap G_2)).$$

Hence

$$\begin{aligned}[O^{p'}(K_2), K_1 \cap K_2] &\leqq [O^{p'}(K_1)(\hat{G}_1 \cap G_2), K_1 \cap K_2] \\ &\leqq C[\hat{G}_1 \cap G_2, K_1 \cap K_2] \leqq C(F_1 \cap K_2) \leqq C,\end{aligned}$$

since $[K_1, K_2] \leqq F_1$ and so $[\hat{G}_1, K_1] \leqq F_1$. This proves (6).

Now (6) and symmetry imply by (d) $O^{p'p}(K_1) = O^{p'p}(K_2) = 1$. Since $\tilde{G}_i \leqq K_i \hat{G}_i$ for $i = 1$ and 2 and $[K_i, \hat{G}_i] \leqq F_i$ this final contradiction proves (3.12).

We also need similar statements as (3.10), (3.11) for direct products.

(3.13) **Lemma.** *Suppose $R = (G_1 \times G_2)B$; where*

(1) *G_1 and G_2 are either both simple finite rank-1 groups of Lie-type of the same characteristic p or $G_i \simeq \mathbb{Z}_3 \cong PSL_2(2)$, $G_i \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \cong PSU_3(2)$, $G_i \simeq \mathbb{Z}_5 \cong Sz(2)$ and $p = 2$ or $G_i \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2) \cong PSL_2(3)$ and $p = 3$ and either G_j is simple of rank 1 and the same characteristic, or belongs to the same list (and has the same characteristic!) for $\{i, j\} = \{1, 2\}$.*

(2) *$C_B(G_1 \times G_2) = 1$.*

(3) *$B \cap G_i$ is a Borel-subgroup of G_i if G_i is simple, while $B \cap G_i = 1$ if G_i is solvable.*

Let $P_i = G_i B$, $i = 1, 2$. Suppose there exist normal subgroups X_i of P_i satisfying:

(a) *$|P_i : X_i| = p$, $i = 1, 2$*

(b) *$B \cap X_1 = B^* = B \cap X_2$.*

Let $R_0 = \langle X_1, X_2 \rangle$. Then $G_1 \times G_2 \leqq R_0 \lhd R$, $R = R_0 B$ and $|R : R_0| = p$.

Proof. Let $G = G_1 \times G_2$. Then again $|P_i \cap G : X_i \cap G| \leqq p$. Further, if $X_1 \cap G = P_1 \cap G$, then $B \cap G \leqq X_2$ and thus $P_2 \cap G = X_2 \cap G$. Hence $G = G_1 G_2 \leqq R_0 \lhd R$ and $R = R_0 B$, since $P_i = X_i B$.

Moreover, as $R = GB$, we have $P_i = (P_i \cap G)B$ and $(P_i \cap G)B^* \leqq X_i$. Since B obviously contains a p-Sylow subgroup of R, this implies $X_i = (P_i \cap G)B^*$, $i = 1, 2$. Hence $R_0 = GB^*$ and

$$R/R_0 \simeq B/B \cap GB^* = B/B^* \simeq \mathbb{Z}_p$$

since $B \cap G = (B \cap G_1) \times (B \cap G_2) \leqq \langle B \cap P_1 \cap G,\ B \cap P_2 \cap G \rangle \leqq B^* \cap G$. So in this case (3.13) holds.

Hence, as in (3.10), we may assume $|P_i \cap G : X_i \cap G| = p$ for $i = 1, 2$. But as $P_i \cap G = G_i \times (B \cap G_j)$, $\{1, 2\} = \{i, j\}$, it is obvious that G_j must be defined over $GF(p)$, $B \cap G_j \neq 1$ and $G_i \leqq X_i \cap G$. By the same reason $G_j \leqq X_j \cap G$ and $B \cap G_i \neq 1$. Hence

$$B \cap G_1 = B \cap X_1 \cap G = B \cap X_2 \cap G = B \cap G_2$$

which is obviously impossible.

(3.14) **Corollary.** *Suppose $\hat{R} = R \times Z$, R as in (3.13) and $Z \simeq \mathbb{Z}_p$. Let P_i be as in (3.13). Suppose X_1, X_2 are subgroups of $\hat{R}$ satisfying:*

(1) $X_i \lhd P_i Z$, $|P_i Z : X_i| = p$ *for* $i = 1, 2$
(2) $BZ \cap X_1 = B^* = BZ \cap X_2$.
Let $R_0 = \langle X_1, X_2 \rangle$. *Then* $G_1 \times G_2 \leqq R_0 \lhd \hat{R}$, $\hat{R} = R_0(BZ)$ *and* $|\hat{R} : R_0| = p$.

Proof. Let $G = G_1 \times G_2$. Then $\hat{R} = G(BZ)$. Moreover as $P_i \cap G = P_i Z \cap G$ since $GB \cap Z = 1$, we obtain again $|P_i \cap G : X_i \cap G| \leqq p$, $i = 1, 2$. Now by the same argument as in (3.13) one shows that either
(a) $|P_i \cap G : X_i \cap G| = p$, $i = 1, 2$ or
(b) $P_i \cap G = X_i \cap G$, $i = 1, 2$.
Now by (3.13) applied to G case (a) is impossible. In case (b) $BZ \cap G = B \cap G = B^* \cap G$. Since $|BZ : B^*| = p$, as $|P_i Z : X_i| = p$, and since

$$(X_i \cap G)B^* \leqq X_i < P_i Z = (P_i \cap G)BZ = (X_i \cap G)BZ$$

we obtain $X_i = (X_i \cap G)B^*$ for $i = 1, 2$. Hence

$$R_0 = GB^* \trianglelefteqq G(BZ) = \hat{R}$$

and $\hat{R}/R_0 \simeq BZ/B^* \simeq \mathbb{Z}_p$, since $BZ \cap G \leqq B^*$.

An *FF-module* (failure of factorization module) for a quasi-simple group G of Lie-type in char. p is a faithful $GF(p)G$-module V satisfying

$$|V : C_V(A)| \leqq |A|$$

for some elementary abelian p-subgroup A of G. Such subgroups are called *offending subgroups.*

We need the following:

(3.15) **Proposition.** *Suppose G is a perfect central extension of a simple rank 2 group of Lie-type defined over $GF(5)$ and different from $PSU(5, 5)$ by a 5′-group and V an irreducible failure of factorization module for G. Then one of the following holds:*
(1) $G \simeq SL_3(5)$ *and V a natural module*
(2) $G \simeq Sp(4, 5)$ *and V a natural module*
(3) $G \simeq SU(4, 5)$ *and V is the natural or orthogonal $GF(5)G$-module.*

Proof. Suppose (3.15) is false. We will derive a contradiction in several steps. Since the proof is much shorter if one uses the representation theory of Chevalley-groups, we sometimes just indicate the arguments. Choose for the whole proof an offending subgroup E of G satisfying:
(a) $[V, E] \leqq C_V(E)$
(b) $|C_V(E)|$ is maximal with (a)
(c) E is maximal with (a) and (b).
Since minimal offending subgroups act by the Thompson-replacement-theorem [4, chapter 8, 2.5] quadratically on V, such an E exists. We have
(1) $|V : C_V(x)| \geqq 5^2$ for each 5-element x of G.
Otherwise by [7] (3.7) holds.
(2) $C_V(E) = C_V(H)$ for each hyperplane H of E.
Since $|V : C_V(E)| \leqq |E|$ we have $|E| \geqq 5^2$ by (1). Suppose $C_V(H) > C_V(E)$ for a hyperplane H of E. Then

$$|V : C_V(H)| \leqq 1/5|E| = |H|.$$

Since $[V, H] \leqq C_V(E) \leqq C_V(H)$ condition (a) holds for H instead of E. Hence we obtain a contradiction to the maximality of $|C_V(E)|$.

(3) $N(U) \leqq N(E)$ for each subgroup $U \leqq E$ with $C_V(U) = C_V(E)$.

Especially $N(H) \leqq N(E)$ for each hyperplane H of E.

Let $x \in N(U)$. Then $[V, E^x] \leqq C_V(U) = C_V(E) \cap C_V(E^x)$. Hence $F = \langle E, E^x \rangle$ is elementary abelian by the 3-subgroup lemma. Further $C_V(F) = C_V(E)$ and $[V, F] \leqq C_V(E)$ so that F is an offending subgroup satisfying (a) and (b). But then the maximality of E implies $E = F = E^x$.

Let now $R = \langle r \rangle$ be a long root subgroup of G centralized by E. (Exists since $R \leqq Z(S)$, $S \in \mathrm{Syl}_5(G)$!) Let $P = N(R)$. Then P is a parabolic subgroup of G. Let $P = U \cdot L$ with unipotent radical U and Levi-complement L. Next we show

(4) $R \leqq E \leqq U$.

If $E \leqq U$ then (4) holds since U is extraspecial and $|E| \geqq 5^2$. So suppose $E \not\leqq U$. Then $G \not\simeq SL_3(5)$. If $G \simeq Sp(4, 5)$ or $SU(4, 5)$ then (3) and the action of L on U imply $C_U(E) \leqq E$. Hence E is elementary abelian of order 5^3 or 5^4 and $N(E)$ is the other maximal parabolic containing the same Borel-subgroup. Now it is easy to see that $G = \langle E, E^g \rangle$ for some $g \in G$. Hence $|V| \leqq 5^6$ resp. 5^8. It is now easy to show that this implies that (3.15) holds, a contradiction.

Next assume $G \simeq G_2(5)$. Then by (3) $N(E \cap U) \leqq N(E)$. But this is impossible by the action of L on $U/Z(U)$. ($U/Z(U)$ is the unique 4-dimensional irreducible $GF(5)SL_2(5)$-module!) So it remains to be shown that $G \not\simeq {}^3D_4(5)$. Now since $5^8 + 5^4 + 1 = \dfrac{5^{12} - 1}{5^4 - 1} \nmid |SL_{11}(5)|$ we have $|V| \geqq 5^{12}$. Moreover, $|E| \leqq 5^5 = m_5({}^3D_4(5))$. If $|E| = 5^5$, then $R \leqq E$ and $P' = \langle E, E^x \rangle$ for some $x \in E$. Since $C_V(E) + C_V(E^x) \leqq C_V(R)$ (1) implies $|V : C_V(P')| \leqq 5^8$ and equality holds if and only if $|V : C_V(R)| = 5^2$. But $G = \langle P', R^g \rangle$ for some $g \in G$ and so $C_V(G) \neq 0$, a contradiction.

Since $P' = \langle E, E^x \rangle$ in any case if $E \not\leqq U$, as $|E| \geqq 5^2$, and since $C_V(E \cap U) > C_V(E)$ by (3) if $|E| = 5^4$ the same arguments lead always to a contradiction. This proves (4).

We now come to the final contradiction. If $G \simeq SL_3(5)$ then (3.15) holds by [16, (2.5)]. If $G \simeq Sp(4, 5)$ then $C_V(E) = C_V(R) = C_V(U)$ by (2). But this is a contradiction to the 3-subgroup lemma. The same argument shows $|E| = 5^3$ if $G \simeq SU(4, 5)$ or $G_2(5)$. As in both cases $|V : C_V(E)| \leqq 5^3$ and $|V : C_V(R)| \geqq 5^2$ we obtain $|V : C_V(U)| = 5^4$, $|V : C_V(R)| = 5^2$ and $|V : C_V(E)| = 5^3$ by (3) and since $U = \langle E, E^x \rangle$ for some $x \in P$.

But $G = \langle U, R_- \rangle$ for an opposite root subgroup R_-. Hence $|V| \leqq 5^6$ and so $G \not\simeq G_2(5)$ and V is the orthogonal module if $G \simeq SU(4, 5) \simeq \Omega^-(6, 5)$.

So it finally remains to be shown that $G \not\simeq {}^3D_4(5)$ also in case $E \leqq U$. If $|E| = 5^5$ then $U = \langle E, E^x \rangle$ for some $x \in P$. So $|V : C_V(U)| \leqq 5^8$ and equality holds if and only if $|V : C_V(R)| = 5^2$. Since $G = \langle U, R_- \rangle$ and $|V| \geqq 5^{12}$ this is again impossible.

Now (2), (3) and (4) imply $|E| \geqq 5^3$. If $|E| = 5^3$, then $|C_V(R) : C_V(E)| = 5$ and $|V : C_V(R)| = 5^2$. But then we get a contradiction as before. So we finally have $|E| = 5^4$. Then either $|C_V(R) : C_V(E)| = 5$ or 5^2. As Q can be generated by 3 conjugates of E it follows in any case $|V : C_V(Q)| \leqq 5^8$. As $C_V(R) > C_V(E)$ this implies again $C_V(G) \neq 0$, a contradiction. This proves (3.15).

(3.16) **Corollary.** *Let G be as in* (3.15), *$V=\langle Z^G\rangle$ a failure of factorization module for G with $Z\leqq C_V(S)$, $S\in \mathrm{Syl}_5(G)$ and $C_V(G)=0$. Then one of the following holds:*

(1) *$G\simeq SL_3(5)$ and V has at most two $GF(5)G$-composition factors which are both equivalent natural modules.*

(2) *$G\simeq Sp(4,5)$ and V is a natural $GF(5)G$-module.*

(3) *$G\simeq SU(4,5)$ and V is either the natural or the orthogonal $GF(5)G$-module.*

Proof. If $G\simeq SL_3(5)$ this is [16, (2.8)]. In the other cases let V_0 be an irreducible $GF(5)G$-*submodule of* V. As $C_V(G)=0$, V_0 is non-trivial. So (3.15) applies to V_0 and G. But then either $|A|=|V_0:C_{V_0}(A)|$ or $|A|=5^3$, $|V_0:C_{V_0}(A)|=5^2$ and $G\simeq Sp(4,5)$ for each offending subgroup A of G. Hence either $V=V_0C_V(A)$ or $|V:V_0C_V(A)|\leqq 5$ in the latter case. In any case (3.15) shows that V/V_0 is a trivial $GF(5)G$-module. As $V=\langle Z^G\rangle$ this implies $V=V_0+Z$. But then $V=V_0$ by Gaschütz's theorem and $C_V(G)=0$. This proves (3.16).

§ 4. Proof of Theorem 2 and 3; the generic case

We assume in this section that $\mathfrak{C}$ is a locally finite, classical Tits chamber system of hyperbolic type Δ over I with transitive automorphism group G. We fix the following notation: Let $p=\mathrm{char.}(\mathfrak{C})$. Fix a chamber $c\in\mathfrak{C}$. Let $\Delta_J(c)$ be the residue of type J containing c for $J\subseteq I$. Let $B=G_c$ and $X_J=G_{\Delta_J(c)}$. Then

$$\mathfrak{C}\simeq\mathfrak{C}(G;\,B,(X_i)_{i\in I})$$

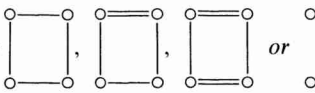
and we often identify $\mathfrak{C}$ with the latter. Let Q_J be the kernel of the action of X_J on $\Delta_J(c)$, $G_J=X_{I-J}$ ($X_\phi=B$!) and $K_J=Q_{I-J}$. Finally $\bar{X}_J=X_J/Q_J$ $\bar{G}_J=G_J/K_J$.

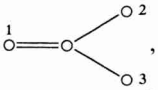
In the first proposition we treat the cases in which a spherical residue exists on which not a full Chevalley group of the corresponding type is induced.

(4.1) **Proposition.** *One of the following holds:*

(a) *$F^*(\bar{G}_i)$ is a finite simple Chevalley-group of char. p and the type $\Delta-i$ or $F^*(\bar{G}_i)\simeq Sp(4,2)'$, $G_2(2)'$ or ${}^2F_4(2)'$ for each "non-middle" node i of Δ.*

(b) $\Delta=$, *$\mathfrak{C}$ is defined over $GF(2)$ or $GF(8)$ and G_j is a Frobenius-group of order* 21 *or* $73\cdot 9$ *for each $i\in I$.*

(c) $\Delta=$  , *$\mathfrak{C}$ is defined over $GF(2)$ and $G_j\simeq A_7$ for each "non-middle" node j of Δ*

(d) $\Delta=$  , *$\bar{G}_1\simeq A_7$ and $\bar{G}_2\simeq\bar{G}_3\simeq Sp(6,2)$. Also $\mathfrak{C}$ is defined over $GF(2)$.*

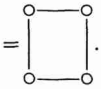
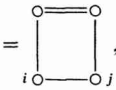
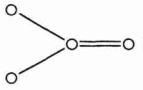
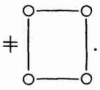
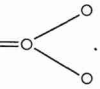
(e) *There exists a $j\in I$ such that $\Delta_{I-j}(c)$ is the $Sp(4,3)$ or $U_4(3)$ generalized quadrangle. Further $\bar{G}_j\in\{2^4F_{20},\ L_3(4)\cdot 2,\ 2^4A_5\}$.*

Proof. Assume (a) does not hold. Then by (2.3) and [12] rank $\mathfrak{C}=3$ or 4.

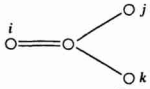
Suppose first rank $\mathfrak{C}=3$. Then by [12] either some $\bar{G}_i \simeq F_{21}$ or $F_{73\cdot 9}$ or char. $\mathfrak{C}=3$. In the first case $\bar{G}_{ij} \simeq \mathbb{Z}_3$ or $\mathbb{Z}_9$ so that $\bar{G}_j$ is also F_{21} or $F_{73\cdot 9}$ and $K_i=K_{ij}=K_j=1$ for each $j \in I$. Hence (b) holds.

Next suppose that char. $\mathfrak{C}=3$. Then again by [12] $\bar{G}_j \in \{2^4 F_{20}, L_3(4)\cdot 2, 2^4 A_5\}$ and $\Delta_{I-j}(c)$ is the $Sp(4,3)$ or $U_4(3)$ generalized quadrangle for some $j \in I$. So (e) holds.

Next assume rank $\mathfrak{C}=4$. Then by (2.3) $\bar{G}_i \simeq A_7$ and $\Delta - i =$ o——o——o or o——o═══o for some $i \in I$. Since Δ is hyperbolic, Δ must be one of the diagrams in (c) or (d). We need to show that the statements about the groups in (c) or (d) are satisfied.

Suppose first $\Delta =$  . Then it was shown in [14, (10.8)] that $K_i=1$ and $G_j \simeq G_i \simeq A_7$ for all $j \in I$. If $\Delta =$  , then $\bar{G}_{ij} \simeq A_6$. Hence $\bar{G}_j \simeq A_7$, since otherwise $\bar{G}_{ij} \simeq Sp(4,2) \simeq \Sigma_6$. But then $K_i=K_{ij}=K_j=1$, as $G=\langle G_i, G_j \rangle$. Hence $B \simeq D_8$ and so $G_l \simeq A_7$ for all $l \in I$. It is easy to see that the same argument applies in case or  if $\Delta - i =$ o——o═══o. So assume finally $\Delta - i =$ o——o——o, but $\Delta \neq$  . Then $\Delta =$ or  .

In the first case we have $\bar{G}_{ij} \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3)\mathbb{Z}_2$ by the structure of A_7. Hence $\bar{G}_j \neq Sp(6,2)$ or $\Omega^-(8,2)$ and thus $\bar{G}_j \simeq A_7$. But then we are in the case which has been treated already.

So we finally have  and by the same reason $\bar{G}_j \simeq A_7 \neq \bar{G}_k$. But then easily $\bar{G}_j \simeq \bar{G}_k \simeq Sp(6,2)$ or $\{\bar{G}_j, \bar{G}_k\} \subseteq \{\Omega^-(8,2), O^-(8,2)\}$. Since in the first case (d) holds it suffices to show that the second case is impossible.

As $\bar{G}_{jk} \simeq \Omega^-(6,2)$ or $O^-(6,2)$ we have $\bar{G}_j \simeq G_k \simeq \Omega^-(8,2)$ or $O^-(8,2)$. Now $K_k \leqq K_{i,k}=K_i$ and $K_j \leqq K_i$ by the structure of A_7. Since $|K_{j,k}/K_k|=2^6$ with $\bar{G}_{j,k}$ acting irreducibly we obtain either $K_{jk}=K_jK_k$ or $K_j \leqq K_k$. In the latter case by symmetry $K_j=K_k=1$. Hence $|K_i|=|K_{ik}/K_k|=2^9\cdot 3$ with $O_2(K_i)$ special of order 2^9 with $|Z(O_2(K_i))|=2^3$. As G_{ik} acts non-trivially on $Z(O_2(K_i))$ this is impossible.

So we have $K_{jk}=K_jK_k \leqq K_i$. As $|B:K_{jk}|=2^6\cdot 3$ or $2^7\cdot 3$ and $|B:K_i|=8$ we obtain $|K_i:K_{jk}|=2^3\cdot 3$ resp. $2^4\cdot 3$. Now the structure of $\Omega^-(6,2)$ implies $G_{ijk}/K_{jk} \simeq 2^{1+4}(\Sigma_3 \times \mathbb{Z}_3)$ resp. $2^{1+4}(\Sigma_3 \times \Sigma_3)$ (in case of $O^-(6,2)$!) Further $\Sigma_3 \times \mathbb{Z}_3$ acts irreducibly on the centre factor group of the extraspecial group 2^{1+4}. As $K_{jk} < K_i < K_{ijk}$ and K_i is G_{ijk}-invariant this is impossible. This proves (4.1).

(4.2) **Corollary.** *Suppose B is finite. Then in case* (4.1) (c) *and* (d) $\mathfrak{p}=\{(X_i), i \in I\}$ *is a parabolic system of G.*

Proof. In case (c) this is obvious. In case (d) we have $O^2(K_2) \leqq K_3$ and $O^2(K_3) \leqq K_2$ so that $O^2(K_2) = O^2(K_3) = 1$. Hence B is a 2-group. But then (4.2) is also obvious in this case.

(4.3) *Notation.* In view of (4.1) and (4.2) we assume now for the rest of this section that $F^*(\bar{G}_i)$ is a finite simple Chevalley-group of type $\Delta - i$ and char. p for each $i \in I$ for which $\Delta - i$ is connected.

We define parameters $m_i(G, \mathfrak{C})$ and $m(G, \mathfrak{C})$ in the following way:

$$m_i(G, \mathfrak{C}) = 0 \quad \text{if } X_i/Q_i \text{ is solvable}$$
$$m_i(G, \mathfrak{C}) = j \quad \text{if } p^j = |\bar{X}_i : E(\bar{X}_i)|_p$$
$$m(G, \mathfrak{C}) = \sum_{i \in I} m_i(G, \mathfrak{C}).$$

We will prove Theorem 2 and 3 by induction on $m(G, \mathfrak{C})$. As $m_i(G, \mathfrak{C})$ is the p-part of the field automorphism group induced by $\bar{X}_i$ on $E(\bar{X}_i) = F^*(\bar{X}_i)$, this seems sensible, since to obtain a parabolic system we must eliminate field automorphisms of p-power order.

We have the following elementary fact:

(4.4) **Lemma.** *Suppose $m(G, \mathfrak{C}) = 0$. Then for each $i \in I$ for which $\Delta - i$ is connected $O^{p'}(\bar{G}_i) \in \mathfrak{L}_p^n$ for some $n \geqq 2$.*

Since a field automorphism of a finite group of Lie-type induces a field automorphism on some minimal parabolic (4.4) is obvious.

Next we show:

(4.5) **Proposition.** *Suppose Δ is not linear, $m(G, \mathfrak{C}) = 0$ and B is finite. Then $\mathfrak{p} = \{X_i | i \in I\}$ is a strong parabolic system of G.*

Proof. Choose two "non-middle" nodes i, j of Δ such that $\Delta - \{i, j\}$ is connected. If $N \leqq G_i \cap G_j$ with $N \lhd G$, then $N \leqq B$ and so $N \leqq B_G = 1$. Hence the pair G_i, G_j satisfies the hypothesis of (3.12). Let $S \in \mathrm{Syl}_p(B)$. Then $S \leqq \tilde{G}_k$, $k = i, j$ by (4.4). Hence $[S, \tilde{K}_k] \leqq O_p(K_k) \leqq S$ by (3.12). This implies $S \trianglelefteqq B$ and $B = N_{G_k}(S)$ by the action of the spherical group of Lie-type $\bar{G}_k$ on its building $\Delta_{I-k}(c)$; $k = i, j$. Hence $S \in \mathrm{Syl}_p(X_l)$ and $B = N_{X_l}(S)$ for each $l \in I$, since $X_l \leqq G_i$ or $X_l \leqq G_k$. As $Q_l \leqq B$, also obviously $O^{p'}(X_l/O_p(X_l))$ is a finite rank 1 group of Lie-type.

Furthermore, if $\{l, s\} \neq \{i, j\}$, $S \in \mathrm{Syl}_p(X_{l,s})$ and $O^{p'}(X_{l,s}/O_p(X_{l,s}))$ has the desired structure, since $X_{l,s} \leqq G_i$ or $X_{l,s} \leqq G_j$ and by (4.4) $O^{p'}(\bar{G}_i)$ resp. $O^{p'}(\bar{G}_j) \in \mathfrak{L}_p^n$.

Now, since Δ is non-linear, there exists a non-middle node $k \in I - \{i, j\}$ of Δ. As $S \in \mathrm{Syl}_p(X_l)$ for some $l \in I - k$ the action of $\bar{G}_k$ on its building $\Delta_{I-k}(c)$ implies $S \in \mathrm{Syl}_p(G_k)$ and $B = N_{G_k}(S)$, since $K_k \leqq B$. But then $S \in \mathrm{Syl}_p(X_{i,j})$ and $O^{p'}(X_{i,j}/O_p(X_{i,j}))$ has the desired structure, since $O^{p'}(\bar{G}_k)$ is a Chevalley-group. This proves (4.5).

We next show:

(4.6) **Theorem.** *Suppose Δ is not linear. Then there exist $B^* \leqq B$, $X_i^* \leqq X_i$, $i \in I$ satisfying:*

(1) *$B^* \trianglelefteqq \trianglelefteqq B$, $X_i^* \trianglelefteqq \trianglelefteqq X_i$ for $i \in I$.*

(2) *$B^* \leqq X_i^*$ and $X_i^* B = X_i$ for $i \in I$.*

(3) *$|B : B^*| = |X_i : X_i^*| = p^m$, $m \in \mathbb{N} \cup \{0\}$ for each $i \in I$.*

(4) *If* $G^*=\langle X_i^*|i\in I\rangle$ *then* $G^*B=G$ *and the map* $B^*g\to Bg$, $g\in G^*$ *is a 2-covering of* $\mathfrak{D}=\mathfrak{C}(G^*;B^*,(X_i^*)_{i\in I})$ *on* $\mathfrak{C}$.

(5) $m(G^*,\mathfrak{D})=0$.

Proof. We prove (4.6) by induction on $m(G,\mathfrak{C})$. If $m(G,\mathfrak{C})=0$ there is nothing to show. So assume $m(G,\mathfrak{C})\geqq 1$ and (4.6) holds for all groups G^0 acting on chamber systems $\mathfrak{C}^0$ which satisfy the hypothesis of (4.6) and $m(G^0,\mathfrak{C}^0)<m(G,\mathfrak{C})$.

Choose $i\in I$ such that $m_i(G,\mathfrak{C})$ is maximal. Then $m_i(G,\mathfrak{C})\geqq 1$. Since Δ is not linear, there exist two non-middle nodes $j,k\in I$ with $i\in I-\{j,k\}$ and $\Delta-\{j,k\}$ is connected. (As Δ is hyperbolic, Δ is either of circular type or has at least 3 end nodes!)

Now $\tilde{B}=B/\bigcap_{i\in I}K_i$ is finite. Let S be the coimage in B of a Sylow p-subgroup of $\tilde{B}$. Then the action of $\bar{G}_j$ and $\bar{G}_k$ on their buildings shows that SK_j/K_j and SK_k/K_k are Sylow p-subgroups of $\bar{G}_j$ and $\bar{G}_k$. Further $\bar{S}=SQ_i/Q_i$ is a Sylow p-subgroup of X_i/Q_i. Now, as $m_i(G,\mathfrak{C})>0$, $E(\bar{X}_i)\neq 1$ and $\bar{S}/\bar{S}\cap E(\bar{X}_i)$ is cyclic, since it can be identified with a subgroup of the field automorphism group of $E(\bar{X}_i)$. Hence there exists a unique subgroup T of S satisfying $\bar{S}>\bar{T}\geqq\bar{S}\cap E(\bar{X}_i)$ and $|\bar{S}:\bar{T}|=p=|S:T|$. ($\bar{S}>\bar{S}\cap E(\bar{X}_i)$ as $m_i(G,\mathfrak{C})\geqq 1$!) Since $m_i(G,\mathfrak{C})$ is maximal, (3.7) implies

$$(*)\qquad S\cap E_i=S\cap F_j^*=S\cap F_k^*$$

where E_i is the coimage of $E(\bar{X}_i)$ in X_i and F_j^* resp. F_k^* are the coimages of $F^*(\bar{G}_j)$ resp. $F^*(\bar{G}_k)$ in G_j resp. G_k.

Set now $\bar{X}_i^*=O^p(\bar{X}_i)\bar{T}$ and $B^*=B\cap X_i^*$. Then $|X_i:X_i^*|=p=|B:B^*|$. Similarly define G_j^* and G_k^* by $\bar{G}_l^*=O^p(\bar{G}_l)\bar{T}$, $l\in\{j,k\}$. Then by (*) $\bar{T}\in\mathrm{Syl}_p(\bar{G}_l^*)$ and $\bar{B}^*$ is a Borel-subgroup of $\bar{G}_l^*$, $l=j$ or k. For arbitrary $l\in I$ set now $X_l^*=X_l\cap G_j^*$ resp. $X_l^*=X_l\cap G_k^*$. To show that X_l^* is well defined we need to show that $X_l\cap G_j^*=X_l\cap G_k^*$ if $l\in I-\{j,k\}$. Now $X_l=(X_l\cap G_j^*)B=(X_l\cap G_k^*)B$ and $|X_l:X_l\cap G_j^*|=|B:B^*|=|X_l:X_l\cap G_k^*|=p$. Since $TK_{j,k}/K_{j,k}$ is a p-Sylow subgroup of $X_l\cap G_j^*/K_{jk}$ and $X_l\cap G_k^*/K_{j,k}$. This implies $X_l\cap G_j^*=X_l\cap G_k^*=X_l^*$.

By definition of X_l^* we have $|X_l:X_l^*|=|B:B^*|=p$ for each $l\in I$. Further $X_l^*\lhd X_l$, $B^*\lhd B$. Let $G^*=\langle X_i^*|i\in I\rangle$. To show that $B^*g\to Bg$ $g\in G^*$ is a 2-covering of $\mathfrak{D}=\mathfrak{C}(G^*;B^*,(X_i^*)_{i\in I})$ on $\mathfrak{C}$ we only need by (2.5) to show $B^*=B\cap X_J^*$ for each $J\subseteq I$, $|J|=2$; where $X_J^*=\langle X_j^*|j\in J\rangle$. If now $J\subseteq I-j$ or $J\subseteq I-k$ this is obvious since $X_J^*\leqq G_j^*$ or $X_J^*\leqq G_k^*$. So it remains to be shown that $B\cap X_{k,j}^*=B^*$.

To do this pick another non-middle node $r\in I-\{k,j\}$ and let $G_r^*=\langle X_l^*|l\in I-r\rangle$. Then $B^*\leqq G_r^*$ and to show $B^*=B\cap X_{k,j}^*=B\cap G_r^*$ it suffices to show that $|G_r:G_r^*|=p$. Let $K_r^*=K_r\cap B^*$. Then $K_r^*=K_r\cap X_l^*$ for each $l\in I-r$. Hence $K_r^*\lhd G_r$ and $|K_r:K_r^*|\leqq p$.

Let $\wedge$ be the natural homomorphism from G_r on G_r/K_r^*. Then $\hat{B}^*\cap\hat{K}_r=1$ and if $\hat{K}_r\neq 1$, then $\hat{B}=\hat{K}_r\times\hat{B}^*$. As $\hat{B}$ contains a p-Sylow subgroup of $\hat{G}_r$, Gaschütz's theorem now implies that either $\hat{G}_r=\bar{G}_r$ or $\bar{G}_r=\hat{K}_r\times\hat{R}_r$ with $\hat{R}_r\simeq\bar{G}_r$. Now $|\widehat{G_r\cap G_k}:\widehat{G_r\cap G_k^*}|=p$ and the same holds with G_j, G_j^*, since $B\cap G_r\cap G_k^*\leqq B\cap G_k^*=B^*$. Hence (3.10) resp. (3.11) implies $|\hat{G}_r:\hat{G}_r^*|=p$, as

$$\hat{G}_r^*=\langle\widehat{G_r\cap G_k^*},\ \widehat{G_r\cap G_j^*}\rangle.$$

This finally shows $B \cap X_J^* = B^*$ for $J \subseteq I$ with $|J| = 2$. Now we obviously have $m(G^*, \mathfrak{D}) < m(G, \mathfrak{C})$, so that by induction assumption (4.6) holds for $(G^*, \mathfrak{D})$. But since the composition of two 2-covering maps is again a 2-covering, this obviously implies that (4.6) holds.

(4.7) **Corollary.** *Suppose the same hypothesis as in* (4.6) *holds and that B is finite. Then* $\mathfrak{p} = \{X_i^* | i \in I\}$ *is a strong parabolic system of* G^* *of type* Δ. (X_i^*, G^* *as in* (4.6)!)

This is a direct consequence of (4.5) and (4.6).

In view of (4.6), (4.7) we assume for the rest of this section that Δ is a hyperbolic linear diagram. Let $\Gamma(\mathfrak{C})$ be as in Sect. 2. Then by (4.3) and (2.6) $\Gamma(\mathfrak{C})$ is actually a Tits geometry of type Δ.

(4.8) **Notation.** Using (2.6), we assume for the rest of this section that G is a flag-transitive automorphism group of the locally finite, classical Tits geometry $\Gamma = \Gamma(\mathfrak{C})$ over I with hyperbolic linear diagram Δ. We identify $c \in \mathfrak{C}$ with a maximal flag I of $\Gamma(\mathfrak{C})$. Assume the enumeration of the nodes of Δ is chosen in such a way that $i, j \in I$ are connected in Δ if and only if $|i-j| = 1$. For subsets $J, L \subseteq I$ with $J \cap L = \emptyset$ let

$$\Delta_J(L) := \text{set of vertices of type } J \text{ incident with all vertices in } L, \text{ considered as a Tits geometry over } J.$$

By (2.6) G_L is the stabilizer of $L \subseteq I$. K_L is the kernel of the action of G_L on $\Delta_{I-L}(L)$ and $\bar{G}_L = G_L/K_L$ is a flag-transitive automorphism group of $\Delta_{I-L}(L)$.

The additional hypothesis in (4.3) implies, that if $L \neq \emptyset$ and $J - L$ is connected in Δ then:

$\bar{G}_L$ is an extension of a Chevalley-group of type $\Delta|_{J-L}$ of char. p by diagonal and field automorphism.

Namely by (4.3) we assume this for $\bar{G}_1$ and $\bar{G}_n$ and so it holds for $\bar{G}_L$, since $1 \in L$ or $n \in L$.

$\bar{R}_{2,1}$ is the kernel of the action of G_2/K_2 on $\Delta_1(2)$

$\bar{R}_{2,n}$ is the kernel of the action of G_2/K_2 on $\Delta_{3,\ldots,n}(2)$.

Similarly

$\bar{R}_{n-1,n}$ is the kernel of the action of G_{n-1}/K_{n-1} on $\Delta_{1,\ldots,n-2}(n-1)$.

Further $\bar{Y}_{2,1} = F^*(\bar{R}_{2,1})$, $\bar{Y}_{2,n} = F^*(\bar{R}_{2,n})$ and so on.

As Δ is linear we have:

$$\Delta_1(2) = \Delta_1(2, \ldots, n); \quad \Delta_{3,\ldots,n}(2) = \Delta_{3,\ldots,n}(1,2)$$
$$G_2 = G_{2,1} G_{2,\ldots,n} = G_{2,\ldots,n} G_{2,1} \quad \text{with } G_{2,1} \cap G_{2,\ldots,n} = B.$$

Moreover, the same holds for $\Delta(n-1)$ and G_{n-1}. Hence we have:

(4.9) **Lemma.** (a) $\bar{R}_{2,1} \cap \bar{R}_{2,n} = 1$ *and* $\bar{R}_{n-1,n} \cap \bar{R}_{n-1,1} = 1$.

(b) $\bar{R}_{2,1} \cap \bar{K}_{2,1} = 1 = \bar{R}_{2,n} \cap \bar{K}_{2,\ldots,n}$ *and the same holds for* $\bar{G}_{n-1}$.

Next we show:

(4.10) **Proposition.** *Suppose* $|I| \geqq 4$. *Then the following holds:*

(1) $\bar{G}_2 = (\bar{R}_{2,1} \times \bar{R}_{2,n})\bar{B} = (\bar{Y}_{2,1} \times \bar{Y}_{2,n})\bar{B}$ *and*
$\bar{G}_{n-1} = (\bar{R}_{n-1,1}\bar{R}_{n-1,n})\bar{B} = (\bar{Y}_{n-1,1} \times \bar{Y}_{n-1,n})\bar{B}$.

(2) $C_{\bar{B}}(Y_{2,1}) \leqq \bar{R}_{2,n}$ *and* $C_{\bar{B}}(\bar{Y}_{2,n}) \leqq \bar{R}_{2,1}$, *and the same holds in* $\bar{G}_{n-1}$.

(3) $\bar{Y}_{2,1}$ resp. $\bar{Y}_{n-1,n}$ *is a simple Chevalley-group of type* $\Delta(1,2) = \Delta - \{1,2\}$ resp. $\Delta(n, n-1) = \Delta - \{n, n-1\}$.

(4) $\bar{Y}_{2,n}$ resp. $\bar{Y}_{n-1,1}$ *is either a simple* rank 1 *Chevalley group of* char. p *or is isomorphic to* $\mathbb{Z}_3 \leqq L_2(2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \leqq PGL_2(3)$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \leqq P\Gamma U_3(2)$. *Moreover,* $\bar{Y}_{2,n}$ resp. $\bar{Y}_{n-1,1}$ *act already transitively on* $\Delta_1(2)$ resp. *on* $\Delta_n(n-1)$.

Proof. (1) We have $\bar{G}_{2,\ldots,n}/\bar{R}_{2,n} \leqq \bar{G}_2/\bar{R}_{2,n}$ and acts faithfully on the building $\Delta_{I-\{1,2\}}(2)$ and fixes the chamber $\{3, \ldots, n\}$. Since $\bar{G}_{1,2}\bar{R}_{2,n}/\bar{R}_{2,n}$ acts already as a Chevalley-group of type $\Delta(I-\{1,2\})$ on this building (and thus is a normal subgroup of its automorphism group) with chamber stabilizer $\overline{BR}_{2,n}/\bar{R}_{2,n}$ we either have $\bar{G}_{2,\ldots,n} = \bar{R}_{2,n}\bar{B}$ or

$$\bar{G}_{2,\ldots,n} \cap \bar{R}_{2,n}\bar{B} = \bar{G}_{2,\ldots,n} \cap \bar{G}_{1,2}\bar{R}_{2,n}$$

is a proper normal subgroup of $\bar{G}_{2,\ldots,n}$ containing $\bar{B}$. But, as may be seen from the structure of $\bar{G}_n$, $G_{2,\ldots,n}/K_{2,\ldots,n}$ is an extension of a rank 1 group of Lie-type by diagonal and field automorphisms and therefore does not contain such a proper normal subgroup.

Hence $\bar{G}_{2,\ldots,n} = \bar{R}_{2,n}\bar{B} = \bar{Y}_{2,n}\bar{B}$ and $\bar{Y}_{2,n}$ is transitive on $\Delta_1(2)$. As $\bar{Y}_{2,n} \lhd \bar{G}_2$ and since a Chevalley group of type $\Delta(I-\{1,2\})$ cannot act non-trivially on $\bar{Y}_{2,n}$ we have $\bar{G}_{1,2} = C_{G_{1,2}}(\bar{Y}_{2,n})\bar{B}$. Obviously $C_{G_{1,2}}(\bar{Y}_{2,n}) \leqq \bar{R}_{2,1}$ since $\bar{Y}_{2,n}$ is transitive on $\Delta_1(2)$. This implies

$$\bar{G}_2 = (\bar{Y}_{2,n} \times \bar{R}_{2,1})\bar{B}.$$

As $\bar{R}_{2,1} \cap \bar{K}_{2,1} = 1$, $\bar{R}_{2,1}$ is an extension of a Chevalley-group of type $\Delta(1,2)$ by diagonal and field automorphism. Hence $\bar{R}_{2,1} = \bar{Y}_{2,1}\bar{B}$ and so (1) holds.

If $\bar{R}_{2,n}$ is non-solvable then (4) is obvious. If $\bar{R}_{2,n}$ is solvable then $\bar{R}_{2,n}$ is a flag-transitive subgroup of $L_2(2)$, $PGL_2(3)$ or $P\Gamma U_3(2)$. So (4) holds in any case.

(3) is obvious by definition of $\bar{Y}_{2,1}$ resp. $\bar{Y}_{n-1,n}$.

So (2) remains to be shown.

Now the automorphism group of the building $\Delta_{I-\{1,2\}}(2)$ is the extension of $\bar{Y}_{2,1}$ by diagonal and field automorphisms. So $C_{\bar{B}}(\bar{Y}_{2,1})$ acts trivially on $\Delta_{I-\{1,2\}}(2)$ and thus $C_{\bar{B}}(\bar{Y}_{2,1}) \leqq \bar{R}_{2,n}$ by (4.9). Finally, if $|\Delta_1(2)| = q+1$, $q = p^m$ then either $\bar{Y}_{2,n} \simeq PSL_2(q)$ or we are in one of the solvable cases. Hence in any case $\bar{Y}_{2,n}$ is a selfcentralizing subgroup of Σ_{q+1} and so $C_{\bar{B}}(\bar{Y}_{2,n})$ acts trivially on $\Delta_1(2)$. This proves (2).

(4.11) Let $|\Delta_j(I-j)| - 1$ be the *parameter* of the rank 1 residue of type j. (By flag transitivity these are well defined!) Then the following holds:

(1) All parameters of the rank 1 residues are powers of the same prime p.

(2) Suppose $q = p^n = |\Delta_i(I-i)| - 1$ is a smallest rank 1 parameter. Then one of the following holds:

(α) $|\Delta_j(I-j)| - 1 \in \{q, q^2, q^3\}$ for all $j \in I$,

(β) $n = 2m$, and for some j connected to i in Δ and $J = \{i,j\}$, $k = I - J$ we have $|\Delta_j(I-j)| - 1 = p^{3m}$ and $\Delta_J(k)$ is the $U_5(p^m)$ generalized quadrangle.

(Here we write $G(q)$ for a twisted Lie-type group if $GF(q)$ is the fixed field.) This follows immediately from the structure of the finite rank 2 Lie-type group.

(4.12) **Lemma.** *Suppose* $|I|=3$ *and* $\bar{G}_{1,2}$ *and* $\bar{G}_{2,3}$ *are solvable. Then* $\bar{G}_2=(\bar{R}_{2,1}\times\bar{R}_{2,3})\bar{B}$. *Moreover,* $C_{\bar{B}}(\bar{R}_{2,i})\leqq\bar{R}_{i,j}$, $\{i,j\}=\{1,3\}$ *and* $F^*(\bar{R}_{2,i})=O_{p'}(\bar{R}_{2,i})=\bar{Y}_{2,i}$ *acts already transitively on* $\Delta_j(2)$; *where* $p=\mathrm{char.}(\Gamma)$.

Proof. By (4.11) one easily obtains for the parameters $(|\Delta_1(2)|-1, |\Delta_3(2)|-1)$ just the possibilities (2, 2), (3, 3), (2, 4), (2, 8), (4, 4), (8, 8); where in the latter two cases $\Delta_{2,3}(1)$ is either the $U_5(2)$ generalized quadrangle or the ${}^2F_4(2)$ generalized octagon. In the first two cases it is obvious that $\bar{Y}_{2,i}$ acts already transitively on $\Delta_j(2)$ for $\{i,j\}=\{1,2\}$. Hence (4.12) holds.

In case (2, 4) resp. (2, 8) $\bar{Y}_{2,1}$ acts transitively on $\Delta_3(2)$, since $\mathbb{Z}_5=O_{2'}(D_{10})$ or $O_{2'}(Sz(2))$ and $\mathbb{Z}_3\times\mathbb{Z}_3=O_{2'}(P\Gamma U(3,2))$ cannot act on two symbols. This implies $C_{G_2}(\bar{Y}_{2,1})\leqq\bar{R}_{2,3}$. It is now obvious that in this case also (4.12) holds.

Now in case (4, 4) the same argument applies. So finally case (8, 8) remains to be considered. Here $\bar{G}_{2,1}$ acts on the eight elements of $\Delta_1(2)-1$. As $PSU(3,2)\cong G_{2,1}/K_{2,1}\cong P\Gamma U(3,2)$, since $G_{2,1}\leqq G_1$, $\bar{G}_{2,1}$ cannot act faithfully. Since $\bar{K}_{2,1}\cap\bar{R}_{2,1}=1$ and $\bar{K}_{2,1}\leqq\bar{B}$ this implies $\bar{G}_{2,1}=\bar{R}_{2,1}\bar{B}=\bar{Y}_{2,1}\bar{B}$, since $\mathbb{Z}_3\times\mathbb{Z}_3$ acts already transitively on $\Delta_3(2)$. Similarly $\bar{G}_{2,3}=\bar{Y}_{2,3}\bar{B}$. This shows that (4.12) also holds in this case.

(4.13) **Proposition.** *Suppose* $|I|=3$. *Then one of the following holds:*

(1) $\bar{G}_2=(\bar{R}_{2,1}\times\bar{R}_{2,3})\bar{B}=(\bar{Y}_{2,1}\times\bar{Y}_{2,3})\bar{B}$; *where* $\bar{Y}_{2,i}=F^*(\bar{R}_{2,i})$ *is either a simple* rank 1 *group of Lie-type or* $\bar{Y}_{2,i}=O_{p'}(\bar{R}_{2,i})$, $i=1,3$. *Moreover,* $C_{\bar{B}}(\bar{Y}_{2,i})\leqq\bar{R}_{2,j}$, $\{i,j\}=\{1,2\}$.

(2) $|\Delta_1(2)|=6=|\Delta_3(2)|$ *and* $A_6\cong\bar{G}_2\cong\mathrm{Aut}(A_6)$.

(3) $|\Delta_1(2)|=8=|\Delta_3(2)|$ *and* $\bar{G}_2\simeq\mathbb{Z}_2^3GL_3(2)$.

Proof. By (4.12) we may assume that $\bar{G}_{2,1}$ or $\bar{G}_{2,3}$ is non-solvable. Suppose $|\Delta_i(2)|\geqq|\Delta_j(2)|$, $\{i,j\}=\{1,3\}$. Then by (4.11) in all cases except $\bar{G}_{2,j}\cong P\Gamma U(3,2)$ and $\bar{G}_{2,i}\cong P\Gamma L_2(4)$ we may assume that $G_{2,j}$ is non-solvable. In the latter case, as $G_{2,j}/K_2$ acts on $\Delta_j(2)-j$ we obtain $G_{2,j}/K_{2,j}R_{2,j}\cong\Sigma_4$. But then $\bar{Y}_{2,j}\simeq\mathbb{Z}_3\times\mathbb{Z}_3$ acts transitively on $\Delta_i(2)$. Hence arguing as in (4.12) it is easy to see that (1) holds in this case.

So we may assume that $\bar{G}_{2,j}$ is non-solvable for $j=1$ and 3. Suppose $\bar{K}_{2,j}\neq 1$. As $\bar{K}_{2,j}\cap\bar{K}_{2,i}=1$ and $\langle\bar{K}_{2j}^{G_{2,i}}\rangle\leqq\bar{R}_{2,i}$, the structure of $G_{2,i}/K_{2,i}$ implies $\bar{G}_{2,i}=\bar{R}_{2,i}\bar{B}$. Now $\bar{G}_2=C_{G_2}(\bar{R}_{2,i})\bar{R}_{2,i}\bar{B}$ and, as $\bar{R}_{2,i}$ acts already transitively on $\Delta_j(2)$ we obtain $\bar{G}_2=(\bar{R}_{2,j}\times\bar{R}_{2,i})\bar{B}=(\bar{Y}_{2,j}\times\bar{Y}_{2,i})\bar{B}$, which is to show.

So either (4.13) (1) holds or $\bar{K}_{2,j}=1$. Assume the latter. Then $F^*(\bar{G}_{2,j})$ is a simple rank 1 group of Lie-type, acting in the natural permutation representation on $\Delta_i(2)$. On the other hand $F^*(\bar{G}_{2,j})$ also acts on $\Delta_j(2)-j$ and $|\Delta_j(2)-j|<|\Delta_i(2)|$ by choice of i and j. Hence by [10], [13] and [5, II (8.28)] one of the following holds:

(a) $F^*(\bar{G}_{2,j})\leqq\bar{R}_{2,j}$,

(b) $F^*(\bar{G}_{2,j})\simeq PSL_2(5)$ and $|\Delta_j(2)-j|=5$,

(c) $F^*(\bar{G}_{2,j})\simeq PSL_2(7)$ and $|\Delta_j(2)-j|=7$,

(d) $F^*(\bar{G}_{2,j})\simeq PSL_2(9)$ and $|\Delta_j(2)-j|=9$, (since $|\Delta_j(2)|=3^2+1$!)

(e) $F^*(\bar{G}_{2,j})\simeq PSL_2(11)$ and $|\Delta_j(2)-j|=11$.

In case (a) we get by the above argument that case (1) of (4.13) holds. In case (b) and (c) we have $|\Delta_j(2)|=|\Delta_i(2)|=6$ or 8 and, by symmetry, $F^*(\bar{G}_{2,i})\simeq F^*(\bar{G}_{2,j})\simeq PSL_2(5)$ resp. $PSL_2(7)$. Now $\bar{G}_{2,i}$, $\bar{G}_{2,j}\cong PGL_2(5)$ resp. $PGL_2(7)$. Moreover, $\bar{B}=\bar{G}_{2,i}\cap\bar{G}_{2,j}$. As $\bar{G}_2=\bar{G}_{2,i}\cdot\bar{G}_{2,j}$ we obtain in the first case

$$2^3\cdot 3^2\cdot 5\leqq|\bar{G}_2|\leqq 2^4\cdot 3^2\cdot 5 \quad \text{and in the second case}$$
$$2^6\cdot 3\cdot 7\leqq|\bar{G}_2|\leqq 2^7\cdot 3\cdot 7.$$

In the first case obviously $S(\bar{G}_2)=1$ so that it is easy to see that (4.13) (2) holds. In the second case we obtain (4.13) (3), since $\bar{G}_2\cong\Sigma_8$ and since $PGL_2(7)$ cannot act non-trivially on a 2-group of order 8.

So to prove (4.13) it remains to be shown that in case (d) and (e) either (4.13) (1) holds or they do not occur. Now in case (d) the only permutation representation of $PSL_2(9)\simeq A_6$ of degree $\leqq 9$ has degree 6. Hence $F^*(\bar{G}_{2,j})=F^*(\bar{G}_{2,j'})$ for some $j'\in\Delta_j(2)-j$. Let $\bar{x}\in\bar{G}_{2,i}$ with $j^{\bar{x}}=j'$. Then

$$\bar{G}_2=\langle\bar{x},\bar{G}_{2,j}\rangle\leqq N_{\bar{G}_2}(F^*(\bar{G}_{2,j})),$$

since $\bar{B}$ is maximal in $\bar{G}_{2,i}$.

This implies $F^*(\bar{G}_{2,j})\leqq\bar{R}_{2,j}$ and we obtain again case (1) of (4.13).

In case (e) we obtain

$$PSL_2(11)\subseteq G_{2,i},G_{2,j}\subseteq PGL_2(11),$$

since by symmetry we also have $\bar{K}_{2,i}=1$. Hence

$$2^4\cdot 3^2\cdot 5\cdot 11\leqq|\bar{G}_2|\leqq 2^5\cdot 3^2\cdot 5\cdot 11, \quad \text{as } \bar{B}=\bar{G}_{2,i}\cap\bar{G}_{2,j}.$$

Now obviously $S(\bar{G}_2)=1$ and $|\bar{G}_2:\bar{G}_2^\infty|\leqq 2$, since $|PGL_2(11):PSL_2(11)|=2$. Now from the action of $PGL_2(11)$ on 12 symbols we have $PGL_2(11)\not\leqq A_{12}$. Hence in any case $|\bar{G}_2^\infty|=2^4\cdot 3^2\cdot 5\cdot 11$ and $\bar{G}_2^\infty$ is simple. Now it is well-known, see [22], that this implies to $\bar{G}_2^\infty\simeq M_{11}$. As M_{11} admits no outer automorphism we obtain $\bar{G}_2\simeq M_{11}$ and $\bar{G}_{2,i}\simeq\bar{G}_{2,j}\simeq PSL_2(11)$. Now all involutions in M_{11} are conjugate. But an involution in $\bar{G}_{2,i}$ does not fix an element of $\Delta_j(2)$, while an involution in $\bar{G}_{2,j}$ fixes $j\in\Delta_j(2)$. This contradiction shows that case (d) is also impossible, which proves (4.13).

We are now able to prove a similar proposition to (4.5) for linear diagrams.

(4.14) **Proposition.** *Suppose Δ is linear, B is finite and $m(G,\mathfrak{C})=0$. Then one of the following holds:*

(1) $\mathfrak{p}=\{X_i|i\in I\}$ *is a parabolic system of G.*

(2) $|I|=3$, $|\Delta_1(2)|=|\Delta_3(2)|=6$ resp. $|\Delta_1(2)|=|\Delta_3(2)|=8$ *and* $A_6\cong\bar{G}_2\subseteq\mathrm{Aut}(A_6)$ resp. $\bar{G}_2=\mathbb{Z}_2^3\cdot GL_3(2)$.

Proof. Assume that case (2) does not hold. Since $m(G,\mathfrak{C})=0$ and since each X_i is either contained in G_1 or G_n, (4.4) implies that $O^{p'}(\bar{G}_k)$ is a simple Chevalley-group of type $\Delta-k$ or Σ_6, $G_2(2)$ or ${}^2F_4(2)$ for $k=1,n$. Hence the pair G_1, G_n satisfies the hypothesis of (3.12) and thus K_k is p-closed with $[K_k,O^{p'}(G_k)]\leqq O_p(K_k)$ for $k=1,n$. Hence $S\leqq B$ for $S\in\mathrm{Syl}_p(B)$. Obviously $S\in\mathrm{Syl}_p(X_i)$ for each $i\in I$ and $O^{p'}(X_i/O_p(X_i))$ is a rank 1 Chevalley-group by the

action of $\bar{G}_1$ and $\bar{G}_n$ on their buildings $\Delta_{I-1}(1)$ or $\Delta_{I-n}(n)$. Also, if $X_{i,j} = \langle X_i, X_j \rangle \leqq G_1$ or G_n, then $S \in \mathrm{Syl}_p(X_{i,j})$ and $X_{i,j}$ has the desired structure.

So to show that (1) holds it remains to be shown that $S \in \mathrm{Syl}_p(X_{1,n})$ and $X_{1,n}$ has the desired structure. Now $X_{1,n} \leqq G_2$ and $|\Delta_1(2)| = q+1$, $q = p^\alpha$ so that a p-Sylow subgroup of $X_{1,n}$ is contained in G_1. This shows $S \in \mathrm{Syl}_p(X_{1,n})$. Since $S \trianglelefteqq B$ and since we assume that (2) does not hold, (4.10) and (4.13) show that $X_{1,n}$ has the desired structure.

By (4.10) and (4.13) it is now also possible to show that (4.6) holds in general. (By (4.3) we assume that $F^*(\bar{G}_i)$ is a Chevalley-group, if $\Delta - i$ is connected!)

(4.15) **Theorem.** *There exist subgroups $B^* \leqq B$ and $X_i^* \leqq X_i$ satisfying:*

(1) $B^* \trianglelefteqq \trianglelefteqq B$, $X_i^* \trianglelefteqq \trianglelefteqq X_i$ *for* $i \in I$.

(2) $B^* \leqq X_i^*$ *and* $X_i^* B = X_i$ *for* $i \in I$.

(3) $|X_i : X_i^*| = |B : B^*| = p^m$, $m \in \mathbb{N} \cup \{0\}$ ($p = \mathrm{char}.\, \mathfrak{C}$!)

(4) *If* $G^* = \langle X_i^* | i \in I \rangle$, *then* $G^* B = G$ *and the map* $B^* g \to Bg$, $g \in G^*$ *is a 2-covering of* $\mathfrak{D} = \mathfrak{C}(G^*, B^*, (X_i^*)_{(i \in I)})$ *on* $\mathfrak{C}$.

(5) $m(G^*, \mathfrak{D}) = 0$.

Proof. By (4.6) we may assume that Δ is linear. As in (4.6) we prove (4.15) by induction on $m(G, \mathfrak{C})$. Since (4.15) holds if $m(G, \mathfrak{C}) = 0$ we may assume $m(G, \mathfrak{C}) \geqq 1$ and (4.15) holds for all groups G^0 acting on chamber systems $\mathfrak{C}^0$, which satisfy the hypothesis and $m(G^0, \mathfrak{C}^0) < m(G, \mathfrak{C})$.

Since in case (2) and (3) of (4.13) all residues are defined over the prime field and thus $m(G, \mathfrak{C}) = 0$, case (1) of (4.13) holds if $|I| = 3$.

We now proceed exactly as in the proof of (4.6). Choose i such that $m_i(G, \mathfrak{C})$ is maximal. Then, as $m_i(G, \mathfrak{C}) > 0$, $E(X_i/Q_i) \neq 1$. Let j be an end node of Δ with $i \in \Delta - j$. If i is not an end node, let k be the other end node. If $i = 1$ resp. n let $k = 2$ resp. $n-1$. Let S be the maximal coimage of a p-Sylow subgroup of $B/\bigcap_{i \in I} K_i$. Then $SK_l/K_l \in \mathrm{Syl}_p(\bar{G}_l)$ for $l \in \{j, k\}$ by (4.10), (4.13), resp. by the action of $\bar{G}_l$ on its building $\Delta_{I-l}(l)$ in case l is an end node. Further $SQ_i/Q_i = \bar{S} \in \mathrm{Syl}_p(X_i/Q_i)$. Since $\bar{S}/\bar{S} \cap E(\bar{X}_i)$ can be identified with a subgroup of the field automorphism group of $E(\bar{X}_i)$ and since $m_i(G, \mathfrak{C}) \neq 0$, we have $1 \neq \bar{S}/\bar{S} \cap E(\bar{X}_i)$ is cyclic. Hence there exists again a unique subgroup T of S with $\bar{T} \geqq \bar{S} \cap E(\bar{X}_i)$ and $|\bar{S} : \bar{T}| = p = |S : T|$. We need to show

$$(*) \qquad S \cap E_i = S \cap F_j = S \cap F_k,$$

where E_i, F_j resp. F_k are the coimages of $E(\bar{X}_i)$, $F^*(\bar{G}_j)$ resp. $F^*(\bar{G}_k)$ in X_i, G_j resp. G_k, if k is an end node. If $k = 2$ let F_k be the coimage of $\bar{Y}_{2,n} \times \bar{R}_{2,1}$ and if $k = n-1$ let F_k be the coimage of $\bar{Y}_{n-1,1} \times \bar{R}_{n-1,n}$.

Then as in (4.6) $S \cap E_i = S \cap F_j$ and (*) holds if k is an end node. So suppose $i = 1$, $k = 2$. Let $-: G_2 \to \bar{G}_2$. As $E_1 = Y_{2,n} Q_1$ by (4.10), (4.13) we have

$$\overline{S \cap E_1} \leqq \bar{Y}_{2,n} C_{\bar{B}}(\bar{Y}_{2,n}) \leqq \bar{Y}_{2,n} \times \bar{R}_{2,1}.$$

On the other hand no element of $S \cap Y_{2,n} R_{2,1}$ induces a field automorphism on E_1/Q_1. Hence $S \cap E_1 = S \cap Y_{2,n} R_{2,1}$ and (*) holds.

Now define B^*, X_i^*, G_j^*, G_k^* and X_l^* for $l \in I$ in the same way as in (4.6). Since

$K_jK_k \leqq Q_i$ and $|SQ_i : TQ_i| = p$ we obtain as in (4.6)

$$|X_i : X_i^*| = |B : B^*| = |G_j : G_j^*| = |G_k : G_k^*| = |X_l : X_l^*| = p.$$

Let finally $G^* = \langle X_l^* | l \in I \rangle$. Then $G^*B = G$ and $G^* \trianglelefteqq G$, since B normalizes all X_l^*. Hence to show that the map

$$B^*g \to Bg, \quad g \in G^*$$

is a 2-covering from $\mathfrak{D} = \mathfrak{C}(G^*; B^*, (X_l^*)_{l \in I})$ on $\mathfrak{C}$ it suffices by (2.5) to show that $B \cap X_{r,s}^* = B^*$ for $r, s \in I$ and $X_{r,s}^* = \langle X_r^*, X_s^* \rangle$. If $X_{r,s} \leqq G_j$ or $X_{r,s} \leqq G_k$ this is obvious by definition of X_r^*, X_s^*. If $\{r,s\} \subseteq I-1$ or $\{r,s\} \subseteq I-n$ this might be shown as in (4.6), using (3.10), (3.11). So we may assume $\{r,s\} = \{1,n\} = \{j,k\}$. Then (4.10), (4.13) imply

$$\bar{X}_{1,n} = X_{1,n}/Q_{1,n} = (\bar{Y}_1 \times \bar{Y}_n)\bar{B}, \qquad C_{\bar{B}}(\bar{Y}_1 \times \bar{Y}_n) = 1,$$

where $\bar{Y}_1 = Y_{2,n}Q_{1,n}/Q_{1,n}$, $\bar{Y}_n = \bar{Y}_{n-1,1}Q_{1,n}/Q_{1,n}$ and so $\bar{Y}_1$, $\bar{Y}_n$ are either simple rank 1 groups of Lie-type or isomorphic to $\mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_5$. Moreover,

$$Q_{1,n} \cap B^* = Q_{1,n} \cap X_1^* = Q_{1,n} \cap X_n^* = Q_{1,n}^*$$

as in (4,6) and so $Q_{1,n}^* \lhd X_{1,n}$. Hence either $Q_{1,n} = Q_{1,n}^*$ or $|Q_{1,n}/Q_{1,n}^*| \simeq \mathbb{Z}_p$ and the extension of $\bar{X}_{1,n}$ by $Q_{1,n}/Q_{1,n}^*$ splits by Gaschütz's theorem. Hence in any case (3.13) or (3.14) implies $|X_{1,n}/Q_{1,n}^* : X_{1,n}^*/Q_{1,n}^*| = p$. Since $X_{1,n} = X_{1,n}^*B$ this shows $B^* = B \cap X_{1,n}^*$.

So $B^*g \to Bg$ is a 2-covering of $\mathfrak{D}$ on $\mathfrak{C}$. Obviously $m(G^*, \mathfrak{D}) < m(G, \mathfrak{C})$. So by induction assumption (4.15) holds for G^*. But then it also holds for G, since the composition of two 2-coverings is again a 2-covering map.

Collecting the results of this section we obtain

(4.16) **Corollary.** *Theorem* 3 *holds. This is a consequence of* (4.1) *and* (4.15).

(4.17) **Corollary.** *Suppose the hypothesis of this section and that B is finite. Then one of the following holds:*

(1) *Case* (1), (2) *or* (5) *of Theorem* 2 *holds.*

(2) $|I| = 3$, Δ *is linear and either*

(a) $|\Delta_1(2)| = |\Delta_3(2)| = 6$ *and* $A_6 \cong \bar{G}_2 \cong \mathrm{Aut}(A_6)$ *or*

(b) $|\Delta_1(2)| = |\Delta_3(2)| = 8$ *and* $\bar{G}_2 \simeq \mathbb{Z}_2^3 GL_3(2)$.

This is a consequence of (4.1), (4.2), (4.15), (4.5) *and* (4.14).

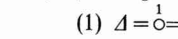
We will show in the next section that in case (2) of (4.17), case (3) or (4) of Theorem 2 hold.

§ 5. Proof of Theorem 2; the special cases

In this section we carry on with the notation of § 4. We assume that case (2) of Corollary (4.17) holds; that is $\Gamma = \Gamma(\mathfrak{C})$ is a locally finite, classical rank 3 Tits geometry over I with hyperbolic linear diagram Δ. Further, if 2 is the middle

node of Δ, then by (4.17) either $\bar{G}_2 \simeq \mathbb{Z}_2^3 L_2(7)$ and $|\Delta_1(2)| = |\Delta_3(2)| = 8$ or $A_6 \cong \bar{G}_2 \cong \mathrm{Aut}(A_6)$ and $|\Delta_1(2)| = |\Delta_3(2)| = 6$. We first show:

(5.1) **Proposition.** *Suppose $\bar{G}_2 \simeq \mathbb{Z}_2^3 L_2(7)$. Then the following holds:*

(1) $\Delta =$ 

(2) $G_1 \simeq G_3 \simeq PSp(4,7)$. *Further $M_2 = O_7(G_2)$ is extraspecial of order 7^3.*

Proof. As $\bar{G}_2 \simeq \mathbb{Z}_2^3 PSL_2(7)$ we have $X_1/Q_1 \simeq PSL_2(7) \simeq X_3/Q_3$. Hence by (3.8) $\bar{G}_i \simeq PSp(4,7)$ for $i = 1, 3$. Now X_2 acts irreducibly on $O_p(K_{1,3}/K_i)$, $i = 1, 3$ by the structure of these groups. (We have $O_p(K_{1,3}/K_1) \simeq \mathbb{Z}_7^3$ and $X_2/Q_2 \simeq PGL_2(7)$ acting irreducibly on $\mathbb{Z}_7^3$.) By (3.12) K_i is 7-closed with $[K_i, O^{7'}(G_i)] \leqq O_7(K_i)$ for $i = 1, 3$. Further $K_1 \leqq K_{1,2} = K_2$ and $K_3 \leqq K_2$ by the structure of $\bar{G}_2$.

Suppose now $O_7(K_1) \neq 1$ or $O_7(K_3) \neq 1$. Then by the irreducible action of X_2 we have

$$S \leqq O_7(K_1) O_7(K_3) \leqq K_2 \quad \text{for } S \in \mathrm{Syl}_7(K_{1,3}).$$

But by the structure of $\bar{G}_1$ we see that $\bar{S} \not\leqq \bar{K}_{1,2} = \bar{K}_2$. Hence $O_7(K_1) = 1 = O_7(K_3)$ and so $[K_i, O^{7'}(G_i)] = 1$ for $i = 1, 3$. Now, as K_i is centralized by $O^{7'}(X_2)$, the structure of $\bar{G}_j$ implies $K_i \leqq K_j$ for $\{i, j\} = \{1, 3\}$. Hence $K_1 = K_3 = 1$.

It is now obvious that $G_1 \simeq G_3 \simeq PSp(4,7)$. This proves (5.1).

We assume for the rest of this section that $A_6 \cong \bar{G}_2 \cong \mathrm{Aut}(A_6)$. We fix the following notation: $\tilde{G}_i = O^{5'}(G_i)$, $M_i = O_5(G_i) = O_5(\tilde{G}_i)$, $i \in I$.

(5.2) **Lemma.** *K_i is 5-closed with $[K_i, \tilde{G}_i] \leqq M_i$ for $i \in I$.*

Proof. By (3.12) we know that (5.2) holds for $i = 1$ and 3. Especially $S \lhd B$ for $S \in \mathrm{Syl}_5(B)$. Since $K_2 \leqq B$ this shows that also K_2 is 5-closed. Now we have

$$[S, K_2] \leqq S \cap K_2 \leqq O_5(K_2) = M_2.$$

Since $\tilde{G}_2 = \langle S^{G_2} \rangle$ this proves (5.2).

We next show:

(5.3) **Proposition.** *$\tilde{G}_i = M_i * C_{\tilde{G}_i}(M_i)$ for some $i \in I$.*

Proof. Since $\tilde{G}_i/M_i$ is for $i = 1, 2, 3$ a perfect central extension of a simple group by (5.2) it is easy to see that either (5.3) holds or $\tilde{G}_i$ is 5-constraint for each $i \in I$. By way of contradiction assume that the latter is the case. We will obtain a contradiction in several steps. Let for the whole proof $Z = \Omega_1(Z(S))$, $S \in \mathrm{Syl}_5(B)$ and $V_i = \langle Z^{\tilde{G}_i} \rangle$. Then $S \in \mathrm{Syl}_5(\tilde{G}_i)$ and $V_i \leqq \Omega_1(Z(M_i))$ for each $i \in I$.

First we show:

(1) $J(S) \not\leqq M_i$ and $V_i \neq Z$ for $i = 1, 3$.

Since $M_1 M_3 \leqq M_2$ and $G = \langle G_i, G_2 \rangle$ the first statement is obvious. By symmetry assume $V_1 = Z$. Then $C_{V_j}(\tilde{G}_j) = 1$ for $j = 2, 3$, since $\tilde{G} = \langle \tilde{G}_1, \tilde{G}_j \rangle \trianglelefteqq G$ and $G = \tilde{G}H$ for a 5′-complement H to S in B. Now, as $J(S) \not\leqq M_3$, V_3 is an FF-module for $\tilde{G}_3/M_3$. Let W_3 be an irreducible submodule of V_3 and $Z_0 = Z \cap W_3$. Then W_3 is also an FF-module and by (3.15) $\tilde{G}_3/M_3 \simeq SL_3(5)$, $Sp(4,5)$ or $SU(4,5)$ and W_3 is either the natural $\tilde{G}_3/M_3$-module or $\tilde{G}_3/M_3 \simeq \Omega^-(6,5)$ and W_3 an orthogonal module.

If now W_3 is the natural module for $SU(4,5)$, then, as $\tilde{X}_2 \leqq N(Z_0)$, $X_1/Q_1 \cong PSL_2(5^2)$ which is impossible since $X_1 \leqq G_2$. Hence we get in any case that $V = \langle Z_0^{\tilde{X}_1} \rangle$ is the natural module for $SL_2(5)$.

Let $g_0=1,\ldots,g_5$ be coset representatives of $\tilde{X}_3\cap\tilde{X}_1$ in $\tilde{X}_1$. As $\tilde{X}_i=X_i\cap\tilde{G}_2$, $i=1,3$, $g_0,\ldots,g_5$ are also coset representatives of $\tilde{X}_3$ in $\tilde{G}_2$. But then

$$V=\sum_{i=0} Z_0^{g_i}\leqq V_2$$

is $\tilde{G}_2$-invariant, since $\tilde{G}_2=\tilde{X}_1\tilde{X}_3$. This is obviously impossible, since $\hat{A}_6$ cannot act non-trivially on $\mathbb{Z}_5^2$.

Next we show:

(2) $\tilde{G}_i/M_i\simeq Sp(4,5)$ for $i=1,3$ and $\tilde{X}_2/M_i$ is the normalizer of a root group of transvections in $\tilde{G}_i/M_i$.

By (1) V_i is an FF-module for $\tilde{G}_i/M_i$, $i=1,3$. Hence by (3.16) $\tilde{G}_i/M_i\in\{Sp(4,5), SU_4(5)\}$ for $i=1,3$, since Δ is hyperbolic. If now $\tilde{G}_1/M_1\simeq SU_4(5)$, then $PSL_2(25)\cong X_2/Q_2$ since $X_3\leqq G_2$. Hence $\tilde{G}_3/M_3\simeq SU_4(5)$ and either $M_1=M_3=1$ or M_1M_3 is a p-Sylow subgroup of $K_{1,3}$, since X_2 acts irreducibly on $O_p(Q_2/K_i)$, $i=1,3$ by the structure of $SU_4(5)$. Now the first possibility contradicts our assumption, while the second contradicts $M_1M_3\leqq M_2$ and the structure of $U_4(5)$.

So $\tilde{G}_i/M_i\simeq Sp(4,5)$ for $i=1,3$. If now $\tilde{X}_2/M_i$ is not the normalizer of a root group in $\tilde{G}_i/M_i$, then $\tilde{X}_2$ acts irreducibly on $O_p(Q_2/K_i)$, a contradiction as before. This proves (2).

Now $V_2\leqq\Omega_1(Z(M_1))$ as $\tilde{G}_1$ is 5-constraint. Since $\tilde{G}_1=\langle M_2, M_2^g\rangle$ for some $g\in\tilde{G}_1$ and since $V_1/C_{V_1}(\tilde{G}_1)$ is by (3.16) the natural $\tilde{G}_1/M_1$-module, we obtain $V_2=C_{V_1}(M_2)$.

But then $V_2/C_{V_1}(\tilde{G}_1)$ is the natural $SL_2(5)$-module. Since $C_{V_1}(\tilde{G}_1)\cap C_{V_3}(\tilde{G}_3)=1$, as $\langle\tilde{G}_1,\tilde{G}_3\rangle\leqq G$, we have $|C_{V_1}(\tilde{G}_1)|\leqq 5$. Hence $|V_2|\leqq 5^3$ which is impossible as $3^2\nmid|SL_3(5)|$. This proves (5.3).

(5.4) **Corollary.** *$M_i=1$ for $i=1$ or 3.*

Proof. Suppose false. Since $M_1M_3\leqq M_2$, (5.3) implies that $\tilde{G}_i=M_i*C_{\tilde{G}_i}(M_i)$ for $i=1$ or 3. By symmetry suppose that $\tilde{G}_1=M_1*C_{\tilde{G}_1}(M_1)$. Let $R_1=O^5(\tilde{G}_1)$. Then $\tilde{G}_1=M_1*R_1$ and $M_1\cap R_1\leqq Z(R_1)\cap R_1'$. Hence [2, p. 20 Table 3 and 4] implies $M_1\cap R_1=1$, since $R_1/Z(R_1)$ is a simple rank 2 group of Lie-type in char. 5. This shows $\tilde{G}_1=M_1\times R_1$, R_1 a quasisimple rank 2 group of Lie-type.

Now $M_2=M_1\times(M_2\cap R_1)$ and $M_2\cap R_1=O_5(X_3\cap R_1)$. Let $S\in\mathrm{Syl}_5(B)$ and $T=S\cap R_1$. Suppose first $\Phi(M_2\cap R_1)\neq 1$. Then the structure of the parabolic subgroups of the rank 2 groups of Lie-type shows that either $Z(M_2\cap R_1)=Z(T)=Z$ or $|Z(M_2\cap R_1):Z|=5$ and $|Z(M_2\cap R_1)|=5^2$, since $\tilde{G}_1/M_1\not\simeq U_5(5)$. In the first case $Z(M_2)=Z(M_1)\times Z\leqq Z(\tilde{X}_3)$. Hence $Z(M_2)=Z(\tilde{G}_2)$ and so $Z(M_1)=1$ as $Z(M_1)\lhd\langle\tilde{G}_1,G_2\rangle\leqq G$, a contradiction. In the second case also $|Z(M_2):Z(S)|=5$. Since $\tilde{G}_2=\langle S,S^g\rangle$ and $Z(\tilde{G}_2)\leqq M_1$ (in case $|Z(M_2\cap R_1)|=5^2$!) this shows $/Z(S)|=5$. But then again $Z(M_1)=1$ as $Z\leqq Z(S)$, a contradiction.

This shows $\Phi(M_2\cap R_1)=1$. But then $\Phi(M_2)=\Phi(M_1)=1$. Now by the same argument as above $|M_2:M_1|>5^2$. As $X_3/Q_3\simeq PSL_2(5)$ this shows that $\tilde{G}_1=M_1\times PSp(4,5)$. But then again by the same argument $M_1\simeq\mathbb{Z}_5$, since otherwise $Z(S)\cap Z(S^g)\neq 1$ for each $g\in\tilde{G}_2$. Hence $\tilde{X}_2\simeq(\mathbb{Z}_5\times 5^{1+2})SL_2(5)\times\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting on $Z(S)$. But then easily also $\tilde{G}_3\simeq\mathbb{Z}_5\times PSp(4,5)$ and $Z(\tilde{G}_1)=Z(\tilde{G}_3)\simeq\mathbb{Z}_5$, a contradiction as before. This proves (5.4).

From (5.4) it is easy to obtain

(5.5) **Proposition.** *Up to symmetry one of the following holds:*

(1) $\Delta = \overset{1}{\circ}\!\!=\!\!=\overset{2}{\circ}\!\!=\!\!=\overset{3}{\circ}$, $G_1 \simeq G_3 \simeq G_2(5)$ *and* $G_2 \simeq 5^{1+4}(\hat{\Sigma}_6 * \mathbb{Z}_4)$; *where* $\hat{\Sigma}_6$ *is the perfect central extension of* Σ_6 *by* $\mathbb{Z}_2$.

(2) $\Delta = \overset{1}{\circ}\!\!-\!\!-\overset{2}{\circ}\!\!=\!\!=\overset{3}{\circ}$, $G_1 \simeq G_2(5)$, $G_3 \simeq SL_3(5)/\mathbb{Z}_5^3$ *(non-split) and* G_2 *as in* (1).

Proof. By symmetry and (5.4) we may assume that $M_1 = 1$. We will show that this implies $\tilde{G}_1 \simeq G_2(5)$. By (5.2), (5.4) and (4.3) $\tilde{G}_1$ is a perfect central extension of a rank 2 simple Chevalley-group defined over $GF(5)$ by a 5′-group. We show that all other possibilities than $\tilde{G}_1 \simeq G_2(5)$ are impossible.

Since $(X_3/Q_3)' \simeq PSL_2(5)$, $\tilde{G}_1/Z(\tilde{G}_1) \neq PSU_5(5)$. Assume $\tilde{G}_1/Z(\tilde{G}_1) \simeq {}^3D_4(5)$. Then $(X_1/Q_1)' \simeq PSL_2(5)$ and $(X_2/Q_2)' \simeq PSL_2(5^3)$. Hence $|S| = 5^{12}$ and $\tilde{G}_3/Z(\tilde{G}_3) \simeq {}^3D_4(5)$. Now $|M_2| = 5^{11}$ and $M_2 = O_5(\tilde{X}_3) = O_5(\tilde{X}_1)$. But the structure of the parabolic subgroups of ${}^3D_4(5)$ implies $|Z(O_5(\tilde{X}_3))| = 5^2$ and $(X_3/Q_3)'$ acts non-trivially on $Z(O_5(X_3))$, which is obviously impossible since $\hat{A}$ cannot act non-trivially on $\mathbb{Z}_5^2$.

Next assume $\tilde{G}_1/Z(\tilde{G}_1) \simeq PSL_3(5)$ or $PSp(4,5)$. Then $|S| \leqq 5^4$ for $S \in \mathrm{Syl}_5(B)$. Since Δ is non-spherical this implies $\tilde{G}_1/Z(\tilde{G}_1) \simeq PSp(4,5) \simeq \tilde{G}_3/Z(\tilde{G}_3)$. But this is impossible, as $\tilde{G}_2$ acts non-trivially on M_2 and $3^2 \nmid |SL_3(5)|$.

Assume finally $\tilde{G}_1/Z(\tilde{G}_1) \simeq PSU(4,5)$. Then $|S| = 5^6$ and $M_2 = O_5(X_3)$ is extraspecial of order 5^5. As $(X_2/Q_2)' \simeq PSL_2(25)$, $3 \| B|$. Hence $3^3 \| |G_2|$. Since $3^3 \nmid |GL_4(5)|$ there exists a $\mathbb{Z}_3 \simeq Z \leqq C_{G_2}(M_5)$. Obviously $Z \leqq C_B(\tilde{G}_2)$. But then, as $[Z, \tilde{X}_3] = 1$, $Z \leqq C_B(\tilde{G}_1)$ and so $Z \leqq C_B(\tilde{G})$, $\tilde{G} = \langle \tilde{G}_1, \tilde{G}_2 \rangle \trianglelefteqq G$. This is obviously impossible.

So the case $\tilde{G}_1/Z(\tilde{G}_1) \simeq G_2(5)$ remains. Then, as $Z(\tilde{G}_1) \leqq \tilde{G}_1'$, Table 2 of [2] implies $Z(\tilde{G}_1) = 1$. Suppose $M_3 \neq 1$. Then the structure of the parabolic subgroups of $G_2(5)$ implies either $M_3 \simeq \mathbb{Z}_5$ or $\tilde{G}_3$ acts non-trivially on M_3 and thus $|M_3| \geqq 5^3$. In the first case $M_3 \leqq O_5(X_2)'$ so that $M_3 \leqq Z(\tilde{G}_3) \cap \tilde{G}_3'$. But then $\tilde{G}_3/Z(\tilde{G}_3) \simeq PSp(4,5)$ since $|S/M_3| = 5^4$ and Table 2 of [2] provides a contradiction.

In case $|M_3| \geqq 5^3$ and $[M_3, \tilde{G}_3] \neq 1$ order arguments imply $M_3 \simeq \mathbb{Z}_5^3$ and $\tilde{G}_3/M_3 \simeq SL_3(5)$. (Again by Table 2 of [2]!) Now $G_2(5)$ does not admit an outer diagonal automorphism. Hence $G_1 = \tilde{G}_1 \times K_1$. But then easily $K_1 \leqq C_B(\tilde{G}_3)$ and so $K_1 = 1$ by the above argument. This shows $G_1 = \tilde{G}_1$, $G_3 = \tilde{G}_3$. Now $X_3 \simeq 5^{1+4}GL_2(5)$. Since $GL_2(5) \simeq \hat{\Sigma}_5 * \mathbb{Z}_4$ it is now easy to see that case (2) of (5.5) holds.

So we may finally assume $M_1 = 1 = M_3$. But then also $\tilde{G}_3 \simeq G_2(5)$ and arguing as above $G_1 = \tilde{G}_1$, $G_3 = \tilde{G}_3$. It is now obvious that case (1) of (5.5) holds.

(5.6) **Lemma.** *The case* $\Delta = \overset{1}{\circ}\!\!=\!\!=\overset{2}{\circ}\!\!=\!\!=\overset{3}{\circ}$ *with* $|\Delta_1(2)| = 6 = |\Delta_3(2)|$ *is impossible.*

Proof. Suppose we are in this case. Then by (5.5) $G_1 \simeq G_3 \simeq G_2(5)$ and $G_2 \simeq 5^{1+4}(\hat{\Sigma}_6 * \mathbb{Z}_4)$. By the structure of $G_2(5)$ we have $X_1' = Q_1 L_1$, $X_3' = Q_3 L_3$ with $L_i \simeq SL_2(5)$ acting irreducibly on $Q_i/\Phi(Q_i)$. Now $Q_1 = M_2 = Q_3$. But it is easy to see that in an embedding $\hat{A}_6 \cong Sp(4,5)$ one of the two conjugacy classes of $SL_2(5)$'s in $\hat{A}_6$ cannot act irreducibly.

Now (4.17), (5.1), (5.5) and (5.6) together imply Theorem 2.

§ 6. Proof of Theorem 1

We now assume for the rest of this section that $\mathfrak{C}$ is a chamber system of minimal rank, satisfying the hypothesis but not the conclusion of Theorem 1. Let G be the transitive automorphism group of $\mathfrak{C}$ with finite chamber stabilizer B and Δ the diagram of $\mathfrak{C}$. By minimality of rank $\mathfrak{C}$ and (2.3) we obtain:

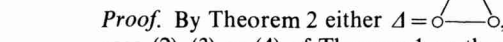
(6.1) One of the following holds:

(1) Δ is hyperbolic or

(2) Δ is a one-node extension of one of the diagrams in case (2), (3) or (4) of Theorem 1 such that all proper connected subdiagrams of Δ are either spherical or again of type (2), (3) or (4).

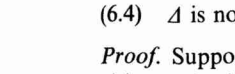
Next we show:

(6.2) If Δ is hyperbolic, then Theorem 1 holds.

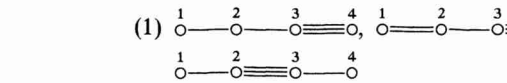
Proof. By Theorem 2 either $\Delta =$  and satisfies case (2), (3) or (4) of Theorem 1 or there exists a $G^* \trianglelefteq\trianglelefteq G$, $X_i^* \trianglelefteq\trianglelefteq X_i$, $B^* \trianglelefteq\trianglelefteq B$ such that $\mathfrak{p} = \{X_i^* | i \in I\}$ is a parabolic system of type Δ with common Sylow p-normalizer B^*. Assume the latter holds. Since we assume that $|\Delta_i(c)| \geqq 6$ for each $i \in I$, $O^{p'}(X_i^*/O_p(X_i^*))$ is non-solvable for each $i \in I$. Hence $\mathfrak{p}$ is a strong parabolic system. Since $O^{p'}(X_i^*/O_p(X_i))$ is defined over a field with at least 5 elements for each $i \in I$, (2.4) implies that $G_0^* = \langle O^{p'}(X_i^*) | i \in I \rangle$ has a BN-pair of type Δ with finite Borel-subgroup B_0. So if $\Gamma = \Gamma(G_0^*, B_0)$ is the building corresponding to this BN-pair, then Γ has type Δ and B_0 acts transitively on the apartments containing the chamber B_0. Now, since Δ is hyperbolic, Γ is infinite. But then, by (2.7) B_0 must be infinite, which is not the case.

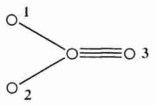
(6.3) Δ is not an extension of a complete graph.

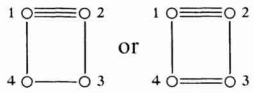
Proof. Suppose $\Delta(i)$ is a complete graph. Then, as Theorem 1 holds for $\Delta_{I-i}(c)$, $\Delta_{\{j,k\}}(c)$ is a projective plane over $GF(8)$ and $X_{j,k}$ is a Frobenius-group of order $73 \cdot 9$ for each pair $j, k \in I - i$. Choose $j \in I - i$ such that i and j are connected in Δ. Then by assumption either $\Delta|_{\{i,j,k\}}$ is spherical or it is a triangle. Since $X_{j,k}$ is a Frobenius-group, the first case is by (2.3) impossible. In the second case Theorem 2 implies that also $X_{i,j}$ and $X_{i,k}$ are Frobenius-groups. Since this holds for each $k \in I - \{i,j\}$ case (2) of Theorem 1 holds, a contradiction.

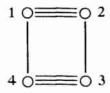
(6.4) Δ is not an extension of  .

Proof. Suppose (6.4) is false. Since all proper connected subdiagrams of Δ are either spherical or of type , it is easy to see that one of the following holds for Δ.

(1)  or

(2) 

(3) 

(4) 

In case (1) we have by Theorem 2, $(X_{2,4}/Q_{2,4})' \simeq \hat{A}_6$. On the other hand by (4.10) $\bar{G}_3 = (\bar{R}_{3,1} \times \bar{R}_{3,4})\bar{B}$. (The hypothesis that Δ is hyperbolic is not used in (4.10). One only needs that $\Delta(3)$ consists of a single node and a spherical linear part!) Hence

$$[X_4^\infty, X_{1,2}^\infty] \leqq K_3 \leqq Q_{2,4},$$

a contradiction since $(X_4/Q_{2,4})^\infty \simeq SL_2(5) \simeq (X_2/Q_{2,4})^\infty$ and both groups do not commute elementwise.

Next suppose we are in case (2). Then by Theorem 2, $K_1 = K_{1,2} = K_2$ and so $K_1 = K_2 = K_{1,2} = 1$. Hence $G_{1,2} \simeq G_2(5)$ by Theorem 2 and $|S| = 5^6$ for $S \in \mathrm{Syl}_5(B)$. On the other hand $(G_3/K_3)' \simeq PSL_4(5)$, so that $|K_3|_5 = 1$. As $K_{1,3} \simeq (\mathbb{Z}_5)^3$ this implies $K_3 = 1$ and

$$PSL_4(5) \cong G_3 \cong PGL_4(5).$$

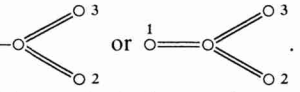
But it is obvious that 5-Sylow subgroups of $PSL_4(5)$ and $G_2(5)$ are non-isomorphic. (For example the 5-rank of $PSL_4(5)$ is 4 while the 5-rank of $G_2(5)$ is only 3!)

Now, in case (3) or (4), by the same argument as above $K_3 = K_{3,4} = K_4 = 1$. Hence $|S| = 5^6$ for $S \in \mathrm{Syl}_5(B)$. But then case (3) is impossible, since either $(G_1/K_1)' \simeq PSL_4(5)$ or $PSp(4,5)$. Now in case (4) also $K_1 = K_{1,2} = K_2 = 1$. Let $Z = Z(S)$, $S \in \mathrm{Syl}_5(B)$. Then by Theorem 2

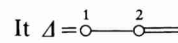
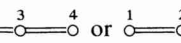
$$G = \langle G_{1,3}, G_{2,4} \rangle \leqq N(Z)$$

which is obviously impossible since G is an automorphism group of $\mathfrak{C}$. This proves (6.4).

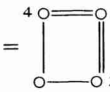
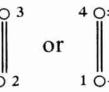
(6.5) Suppose rank $\mathfrak{C} = 4$ and Δ is an extension of $\circ\!=\!\circ\!=\!\circ$. Then case (4) of Theorem 1 holds.

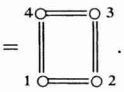
Proof. To prove (6.4) we first show that $\Delta =$  or .

Since all connected subdiagrams of Δ are either spherical or of type $\circ\!=\!\circ\!=\!\circ$ it obviously suffices to show that Δ is neither linear nor of quadrangle type.

It $\Delta =$  or  exactly the same arguments as in (6.5) using (4.10) shows that

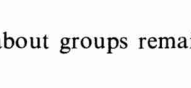
$$[X_2^\infty, X_4^\infty] \leqq Q_{2,4},$$

a contradiction since by (4.16) $X_{2,4}/Q_{2,4} \simeq (\mathbb{Z}_2^3)PSL_2(7)$. Next suppose $\Delta =$  or  . Then $X_{2,4}/Q_{2,4} \simeq (\mathbb{Z}_2^3)PSL_2(7)$ and so $X_2/Q_2 \simeq PSL_2(7)$. But on the other hand $X_2 < G_{3,4}$ and $(G_{3,4}/K_{3,4}) \simeq PSL_3(7)$. Now the structure of $PSL_3(7)$ implies already that X_2 induces $PGL_2(7)$ on $\Delta_2(c)$, a contradiction to $X_2/Q_2 \simeq PSL_2(7)$.

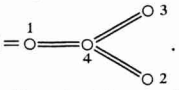
So we are left with the case $\Delta =$  . Now by (5.1) $K_1 = K_{1,2} = K_2 = 1$. Similarly $K_3 = K_4 = 1$. Hence $|S| = 7^4$ for $S \in \mathrm{Syl}_7(B)$. Especially if $Z = Z(S)$ then $|Z| = 7$ and by (5.1) (2)

$$G = \langle X_{2,4}, X_{1,3} \rangle \leqq N(Z)$$

which is obviously impossible.

We have shown that Δ is as in case (4) of Theorem 1. Now the statement about groups remains to be verified. So suppose first $\Delta =$  . Then as $X_2/Q_2 \simeq X_3/Q_3 \simeq PSL_2(7)$ it follows that $\bar{G}_2 \ncong P\Omega^-(8,7) \ncong \bar{G}_3$ and $\bar{G}_2 \ncong P\Omega(7,7) \ncong \bar{G}_3$. So we obtain $\bar{G}_2 \simeq \bar{G}_3 \simeq PSp(6,7)$. Now by (5.1) $K_2 \leqq K_{1,2} = K_1$ and $K_3 \leqq K_1$. Since by (3.9) $G_{2,3}$ acts irreducibly on $O_p(K_{2,3}/K_i)$, $i = 2, 3$ this implies either $K_2 = K_3 = 1$ or $S \cap K_{2,3} = S \cap K_2 K_3$ for $S \in \mathrm{Syl}_7(B)$.

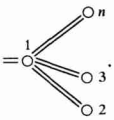
Suppose the latter case holds. Then $|S/S \cap K_{2,3}| = 7^3$. Since $S \cap K_{2,3} = S \cap K_2 K_3 \leqq S \cap K_1$ and $|S : S \cap K_1| = 7^4$ this is impossible.

Hence $K_2 = K_3 = 1$. By (4.17) this shows that the statement about the groups in case (4) of Theorem 1 is satisfied in this case. So assume finally $\Delta =$  . Then by (5.1) $K_1 = K_{1,2} = K_2 = 1$ and similarly $K_1 = K_{1,3} = K_3 = 1$. Hence $K_{1,2} = K_{2,3} = K_{1,3} = 1$ and so by (5.1) we are left with the proof of the statement about G_4. Now by (4.17) applied to G_1 we have $K_{1,4} = K_{1,2,4} = K_{1,3,4}$. Since $K_{1,4} \cap K_{2,4} \cap K_{3,4} = K_4$ this implies by symmetry

$$K_4 = K_{1,2,4} = K_{1,3,4} = K_{2,3,4}.$$

So we get $X_i/K_4 \simeq PSL_2(7)$ and $X_{i,j}/K_4 \simeq (\mathbb{Z}_2)^3 PSL_2(7)$ for $\{i,j\} \subseteq \{1,2,3\}$. Now $G_4 = X_1 X_2 X_3 = X_2 X_3 X_1 = X_3 X_1 X_2$, since $X_i X_j = X_j X_i$ for $i, j \in \{1,2,3\}$. Hence $|G_4/K_4| = 2^6 \cdot 168$ and $|G_4 : X_i X_j| = 8$. It is now obvious that $G_4/K_4 \simeq (\mathbb{Z}_2)^6 PSL_2(7)$. Further, since $|S| = 7^4$, $M_4 = O_7(K_4)$ is extraspecial of order 7^3.

This shows that case (4) of Theorem 1 holds in this case as well.

(6.6) Suppose $\Delta =$  . Let $I=\{1,\ldots,n\}$ and $I'=\{2,\ldots,n\}$. Then the following holds:

(1) $K_1=K_J$ for each $\{1\}\subseteq J\subset I$.

(2) $K_L=1$ for each $L\subseteq I'$.

(3) $G_L\simeq PSp(4,7)$ for all $L\subset I'$ with $|I'-L|=1$.

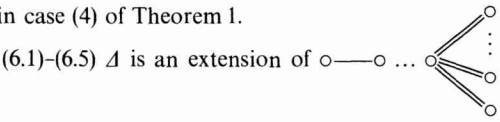
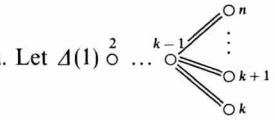
(4) $G_1/K_1\simeq(\mathbb{Z}_2)^{3(n-2)}PSL_2(7)$. Further $M_1=O_7(K_1)$ is extraspecial of order 7^3.

Proof. Let J_1, J_2 be two subsets of I satisfying $\{1\}\subseteq J_i\subset I$ and $|I-J_i|=1$ for $i=1,2$. Then, arguing as in (6.5), $K_{J_1}=K_{J_1\cap J_2}=K_{J_2}$. Since $G_1=\langle G_J|\{1\}\subseteq J\subset I$ with $|I-J|=1\rangle$ this implies $K_1=K_J$ for each such J. But then (1) holds.

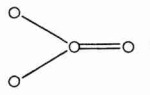
Let $L_i\subset I'$ with $|I'-L_i|=1, i=1,2$. Then by Theorem 2 $K_{L_1}=K_{L_1\cap L_2}=K_{L_2}$. This implies $K_L=1$ for each $L\subset I'$ with $|I'-L|=1$ and so (2) holds. (3) is an obvious consequence of (2) and (5.1), since $(X_1/Q_1)'\simeq PSL_2(7)$.

We prove (4) by induction on n. By (1) we have $X_i/K_1\simeq PSL_2(7)$ for each $i\in I'$. Since the X_i, $i\in I'$ commute pairwise $G_1=\prod_{i\in I'}X_i$. By induction assumption $G_{1,i}/K_1\simeq(\mathbb{Z}_2)^{3(n-2)-3}\cdot PSL_2(7)$. Further, since $G_1=G_{1,i}\cdot G_{1,j}$ with $G_{1,i}\cap G_{1,j}=G_{1,i,j}$, for $i\neq j\in I'$, it follows that $|G_1/K_1|=2^{3(n-2)}\cdot 168$. So $G_{1,i}$ is a subgroup of index 8 of G_1. Hence the kernel of the permutation representation of G_1/K_1 on the cosets of $G_{1,i}/K_1$ has index 8 in $O_2(G_{1,i}/K_1)$. Now it is obvious that (4) holds.

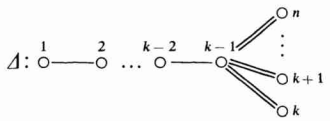
(6.7) Δ is one of the diagrams in case (4) of Theorem 1.

Proof. Suppose this is false. By (6.1)-(6.5) Δ is an extension of  such that all proper connected subdiagrams are of the same type or spherical. Further $|\Delta|\geqq 5$. Choose now the enumeration of the nodes of Δ so that $\Delta(1)$ is the above diagram. Let $\Delta(1)$  and suppose first $2=k-1$. Then as 1 is only connected with 2 in Δ (6.8) holds.

So we have $k-1>2$. Since $\Delta(1)$ is not spherical we have $n-k\geqq 2$. But then 1 can only be connected with 2 or $k-1$ in Δ, since otherwise Δ would contain a hyperbolic subdiagram of a different type.

If 1 is connected with 2, then 1 and 2 is connected by a single bond since we assume $k-1>2$. So 1 is connected only with $k-1$. Since we assume (6.7) is false, 1 and $k-1$ must be connected by a single bond. But then Δ contains a subdiagram of type  which is not possible by Theorem 2. This proves (6.7).

We now come to the final contradiction. By (6.6), (6.7)



with $k-1 \geqq 2$. Further $n \geqq 5$ by (6.5). Let $K=\{k, k+1, \ldots, n\}$. We need to show $K_J=1$ for each $J \subsetneqq K$. Let $J_i \subset K$ with $|K-J_i|=1$ for $i=1,2$. If $K_{J_2} \nleqq K_{J_1}$ then by (3.9) $S \cap K_K \leqq S \cap K_{J_1} K_{J_2}$ for $S \in \mathrm{Syl}_7(B)$. But since we assume that Theorem 1 holds $\Delta_{I-1}(c)$ we have $K_{J_i} \leqq K_1$ for $i=1,2$. This would imply

$$S \cap K_K \leqq K_{J_1 \cup \{1\}}$$

which is a contradiction to (3.1) applied to the Chevalley-group G_{J_1}/K_{J_1}.

So we have $K_{J_2} \subseteq K_{J_1}$ and so by symmetry $K_{J_1}=K_{J_2}=K_{J'}$ for each subset $J' \subset K$ *with* $|K-J'|=1$. Since

$$G=\langle G_J | J \subset K \text{ with } |K-J|=1\rangle$$

this proves $K_J=1$ for each such J.

This of course implies $G_J \simeq PSp(2k, 7)$ for all $J \subset K$ with $|K-J|=1$. Now

$$K_{\{J,k-1\}} \leqq K_{\{1,\ldots,k-1,J\}} \leqq K_{1,\ldots,k-1}$$

by (6.6). Suppose again $K_{J,k-1} \neq K_{J',k-1}$ for $J, J' \subset K$ with $|K-J|=1=|K-J'|$ and $J \neq J'$. Then as before

$$S \cap K_{K,k-1} \leqq S \cap K_{J,k-1} K_{J',k-1} \leqq K_{1,\ldots,k-1}.$$

But this contradicts (3.1) applied to the spherical group G_J/K_J. So $K_{J,k-1} = K_{J',k-1}$ for each pair of subsets $J, J' \subset K$ of corank 1 in K. This implies

$$K_{k-1,J}=K_{k-1} \quad \text{for each } J \subsetneqq K.$$

Let $i \in K$ and $J=K-i$. Then by the structure of G_J/K_J we have

$$[X_i^\infty, X_{1,\ldots,k-2}^\infty] \leqq K_{k-1,J}=K_{k-1}.$$

Since $X_{1,\ldots,k-2}^\infty/K_{k-1} \simeq PSL_{k-1}(7)$ and since

$$\langle X_i | i \in K\rangle/K_{1,\ldots,k-1} \simeq (\mathbb{Z}_2)^{3(n-k)} PSL_2(7)$$

by (6.6) we obtain

$$O^{7'}(G_{k-1}/K_{k-1}) \simeq PSL_{k-1}(7) \times (\mathbb{Z}_2^{3(n-k)} PSL_2(7)).$$

This shows that all statements in case (4) of Theorem 1 are satisfied.

References

1. Carter, R.: Simple groups of Lie-Type. New York: John Wiley & Sons 1972
2. Collins, M.: A survey of the classification project. In finite simple groups II. New York: Academic Press 1980

3. Delgado, A., Stellmacher, B.: Weak (B, N)-pairs of rank 2. DMV-Seminar. Band 6. Boston: Birkhäuser 1985
4. Gorenstein, D.: Finite groups. New York: Harper and Row 1968
5. Huppert, B.: Endliche Gruppen I. Berlin-Heidelberg-New York: Springer 1967
6. Kantor, W.M.: Some geometries that are almost buildings. J. Comb. **2**, 239–247 (1981)
7. McLaughlin, J.: Some groups generated by transvections. Arch. Math. **18**, 364–368 (1967)
8. Meixner, Th.: Gruppen mit parabolischen Systemen. Habilitationsschrift, Universität Gießen Mitt. aus dem Math. Sem. Gießen 174 (1986)
9. Meixner, Th., Timmesfeld, F.: Chamber systems with string diagrams. Geom. Dedicata **15**, 115–123 (1983)
10. Mitchell, H.H.: Determination of the ordinary and modular ternary linear groups. Trans. Am. Math. Soc. **12**, 207–247 (1911)
11. Niles, R.: Finite groups with parabolic type subgroups must have a BN-pair. J. Algebra **75**, 484–494 (1982)
12. Seitz, G.: Flag-transitive subgroups of Chevalley-groups. Ann. Math. **2**, 27–56 (1973)
13. Suzuki, M.: On a class of doubly transitive groups I. Ann. Math. **75**, 105–145 (1962)
14. Timmesfeld, F.: Tits geometries and parabolic systems in finitely generated groups I, II. Math. Z. **184**, 377–396, 449–487 (1983)
15. Timmesfeld, F.: Tits geometries and revisionism of the classification of finite simple groups of char 2-type. Proc. of the Rutgers Group Theory Year 1983–84, Cambridge Univ. Press (1984)
16. Timmesfeld, F.: On the non-existence of flag-transitive triangle geometries. Mitt. aus dem Gießener Math. Sem., Coxeter Festschrift **163**, 19–44 (1984)
17. Timmesfeld, F.: A remark on irreducible modules for finite Lie-type groups. Arch. Math. **46**, 499–500 (1986)
18. Timmesfeld, F.: Tits geometries and finite group theory. Buildings and the Geometry of Diagrams. Lecture Notes 1181, Springer Verlag
19. Timmesfeld, F.: Locally finite classical Tits chamber systems of rank 3. (To appear)
20. Tits, J.: Buildings of spherical type and finite BN-pairs. Lecture Notes in Math., vol. 386. Berlin-Heidelberg-New York: Springer 1974
21. Tits, J.: A local approach to Buildings: I: The geometric vein. Coxeter Festschrift, pp. 519–547. Berlin-Heidelberg-New York: Springer 1981
22. Wong, W.: On finite groups whose Sylow 2-subgroups have cyclic subgroups of index 2. J. Austr. Math. Soc. **4**, 90–112 (1964)

Oblatum 11-IX-1985 & 17-III-1986