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SK_1 of finite abelian groups, II

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Introduction

This paper is a continuation of [A-D-S] in which we laid the theoretical foundation for computing the torsion, $SK_1(\mathbb{Z}[G])$, in the Whitehead group of a finite abelian group G. In that paper we showed how $SK_1(\mathbb{Z}[G])$ can be computed from a knowledge of $SK_1(\mathbb{Z}[G_p])$ for the Sylow p-subgroups of G, with some complications for $p=2$. In this paper we eliminate those complications, showing that when H is an abelian 2-group and $\mathcal{O}$ is a totally imaginary ring of integers in which 2 is unramified, then

$$SK_1(\mathcal{O}[H]) \approx SK_1(\mathbb{Z}[H] \oplus SK_1(\mathcal{O}[H/H^2]),$$

and we calculate the second summand.

A second foundational result in [A-D-S] described $SK_1(\mathbb{Z}[G])$ as the cokernel of a map between the K_2's of certain finite rings. More specifically, for a finite abelian p-group G, $SK_1(\mathbb{Z}[G])$ is the cokernel of the homomorphism

$$\varphi_S\colon K_2(\mathbb{Z}[G]/(p^k)) \to \Pi\, K_2(\mathbb{Z}[\chi]/(p^k))$$

where the product is taken over certain equivalence classes of rational characters χ of G, $\mathbb{Z}[\chi]$ is the ring of cyclotomic integers generated by $\mathrm{Im}(\chi)$, and k is large enough (e.g., $p^k \geqq |G|$). (This description requires some minor modification when $p=2$.) Now it follows from [D-S] that $K_2(\mathbb{Z}[\chi]/(p^k))$ is isomorphic to $\mathrm{Im}(\chi)$ under a homomorphism induced by the norm residue symbol, and this isomorphism is given explicitly by a formula of Artin and Hasse [A-H] involving the p-adic logarithm. The key idea of this paper, which is developed in §1, is to use the integral logarithm of Taylor and Oliver to give a simple combinatorial description of $\mathrm{Im}(\varphi_s)$ as a subgroup of $\Pi\,\mathrm{Im}(\chi)$ (Propositions 1.5 and 1.7).

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In §2 we use this new description to compute $SK_1(\mathbb{Z}[G])$ for elementary abelian p-groups (the rank 3 case for $p \geqq 3$ gave the first examples of non-trivial SK_1's [A-D-S_0]). We also compute $SK_1(\mathcal{O}[G])$ for elementary abelian 2-groups G and rings of algebraic integers $\mathcal{O}$ in which 2 is unramified.

We had previously shown [A-D-S, Theorem 3.4] that $SK_1(\mathcal{O}[G])$ vanishes if G is cyclic and the primes dividing its order do not ramify in $\mathcal{O}$. In §3 we generalize this result by removing the restriction on ramification. It then follows from induction theorems of Dress [Dr] and Oliver [O III] that $SK_1(\mathcal{O}[G])=1$ when G is metacyclic and $\mathcal{O}$ is any ring of algebraic integers; we prove, conversely, that metacyclic groups are the only finite groups with this property.

In §4 we compute explicitly the exponent of $SK_1(\mathbb{Z}[G])$ for a finite abelian group G (Theorem 4.8) and we determine that the only such groups with $SK_1(\mathbb{Z}[G])=1$ are those known prior to 1971: elementary abelian 2-groups, and groups all of whose Sylow p-subgroups are either cyclic or of type (p, p^n).

Finally, in §5 we present a selection of further computations. The aim here is to illustrate the patterns which occur for some other classes of p-groups in order to inspire conjectures about "general" patterns.

The character group $\mathrm{Hom}(G, C^*)$ of a finite abelian group G is denoted $\hat{G}$, and ζ_n denotes a primitive n-th root of unity. A set of elements L in a group or ring generates a subgroup or ideal denoted $\langle L \rangle$. All other notation is standard or defined when first introduced.

Section 1

In this section we will simplify the algorithm for computing $SK_1(\mathbb{Z}[G])$ obtained in [A-D-S] by using the p-adic logarithm. The first step is to find a small set of symbols which generate $K_2(\mathbb{Z}[G]/(p^k))$ when G is a finite abelian p-group.

The Jacobson radical of a ring R will be denoted $J(R)$. For any ideal $I \subset R$, we will write

$$K_2(R, I) = \ker(K_2(R) \to K_2(R/I)).$$

Note that this is *not* the relative group of the long exact sequence of algebraic K-theory.

Proposition 1.1. *Let R be a finite commutative local ring and let J be its Jacobson radical. Suppose $a_0, \ldots, a_n$ generate J as an ideal. Then*

(a) *$K_2(R)$ is generated by*

(i) all Dennis-Stein symbols $\langle a_i, \lambda a_{j_1} \ldots a_{j_s} \rangle$ where $s \geqq 1$, λ ranges over a fixed set of representatives for a set of generators of the additive group R/J, and, when $i=0$, $j_1=0$ (i.e. whenever a_0 appears on the left, it also appears on the right); or, equivalently, by

(ii) all Steinberg symbols $\{1+a_i, 1+\lambda a_{j_1} \ldots a_{j_s}\}$ with the same notations and restrictions as (i).

(b) *Let I be an ideal contained in J. Then $K_2(R,I)$ is generated by*

(iii) all Dennis-Stein symbols $\langle a_i, \lambda a_{j_1} \dots a_{j_{s-1}} b\rangle$, where b ranges over a set of generators of I as an ideal, $s \geqq 1$ and λ is as in (i), or equivalently, by

(iv) all Steinberg symbols $\{1+a_i, 1+\lambda a_{j_1} \dots a_{j_{s-1}} b\}$ with s, λ as in (iii).

Proof. Since R/J is a finite field, $K_2(R/J)=1$ and $K_2(R)=K_2(R,J)$. Since J is nilpotent, the filtration

$$K_2(R,J) \supset K_2(R,J^2) \supset \dots \supset \{1\}$$

is finite. It follows from [St1, Theorem 2.5] that $K_2(R,J^s)$ is generated by Steinberg symbols of the form $\{u, 1+b\}$, $u \in R^*$, $b \in J^s$, and we may write $u=u_0(1+a)$, $a \in J$, with u_0 chosen in a multiplicative set of representatives for $(R/J)^*$. Thus $\{u, 1+b\}=\{u_0, 1+b\}\{1+a, 1+b\}$, and since the orders of u_0 and $1+b$ are relatively prime, we see that $K_2(R,J^s)$ is generated by Steinberg symbols $\{1+a, 1+b\}$, $a \in J$, $b \in J^s$.

Computing in $K_2(R,J^s)/K_2(R,J^{s+1}) \approx K_2(J^s/J^{s+1})$, we see that $\{1+a,1+b\} = \langle a,b\rangle$, and these symbols are additive in a and b. Writing a and b as sums of $a_0, \dots, a_n$, this shows that $K_2(R,J^s)$ is generated by elements of the form

$$\begin{aligned}\langle x a_i, y a_{j_1} \dots a_{j_s}\rangle &= \langle a_i, xy a_{j_1} \dots a_{j_s}\rangle \\ &= \{1+a_i, 1+xy a_{j_1} \dots a_{j_s}\} \text{ modulo } K_2(R,J^{s+1}),\end{aligned}$$

where $x, y \in R$. Write $xy = \Sigma \lambda_k \bmod J$; our set of generators is then

$$\langle a_i, \lambda a_{j_1} \dots a_{j_s}\rangle = \{1+a_i, 1+\lambda a_{j_1} \dots a_{j_s}\}.$$

Now if $i \neq 0$, or if $i=0$ and $j_r=0$ for some r, no further reduction is necessary. Otherwise, we may rewrite $\langle a_0, \lambda a_{j_1} \dots a_{j_r}\rangle$, $j_r \geqq 1$ in the required form using the

Scholium. *Let $\alpha_0, \dots, \alpha_l$ be elements of a ring such that $1-\alpha_0 \dots \alpha_l$ is invertible. Let $\hat{\alpha}_i = \alpha_0 \dots \alpha_{i-1} \alpha_{i+1} \dots \alpha_l$. Then $1=\langle 1, \alpha_0 \dots \alpha_l\rangle = \prod_{i=0}^{l} \langle \alpha_i, \hat{\alpha}_i\rangle$.*

The scholium follows easily from relation (D3), and this completes the proof of (a).

The proof of (b) is similar, using [G-L, Theorem 3].

Corollary 1.2. *Let A be the of ring integers in a finite extension of $\hat{\mathbb{Q}}_p$, with maximal ideal $\mathfrak{p}$ generated by π. Let G be a finite abelian p-group with generators $g_1, \dots, g_m$.*

(a) *For any $n \geqq 1$, $K_2(A[G]/\mathfrak{p}^n A[G])$ is generated by the Steinberg symbols*

$$\{1+\pi, 1+\pi x\}, \quad \{g_i, 1+y\}, \quad x \in A[G]/\mathfrak{p}^n A[G], \quad y \in J(A[G]/\mathfrak{p}^n A[G]), \quad 1 \leqq i \leqq m,$$

(b) *For any $n > k \geqq 1$, $K_2(A[G]/\mathfrak{p}^n A[G], \mathfrak{p}^k)$ is generated by the Steinberg symbols*

$$\{1+\pi, 1+\pi^k x\}, \quad \{g_i, 1+\pi^k x\}, \quad i, x \text{ as in (a).}$$

Proof. Since $A[G]/\mathfrak{p}^n A[G]$ is a local ring, its unique maximal ideal is $J(A[G]/\mathfrak{p}^n A[G])$ which is the kernel of the composition

$$A[G]/\mathfrak{p}^n A[G] \xrightarrow{\text{aug}} A/\mathfrak{p}^n \xrightarrow{\text{mod } p} A/\mathfrak{p}.$$

Hence $J(A[G]/\mathfrak{p}^n A[G])$ is generated by π and $\{g_i - 1, 1 \leqq i \leqq m\}$. Thus (b) and (a) follow directly from Proposition 1.1.

We will now extend the modified logarithm map which was constructed in [OII] to study $K_1(\hat{\mathbb{Z}}_p[G])$ for a finite p-group G. To simplify the discussion, we limit ourselves here to the case of a finite *abelian* p-group G. In that case, $J(\hat{\mathbb{Z}}_p[G])$ is generated by p and $\{g-1 \mid g \in G\}$, and the series

$$\log(1+y) = \sum_{n>0} (-1)^{n+1} \frac{y^n}{n}$$

converges for all $y \in J(\hat{\mathbb{Z}}_p[G])$ (since $p \mid y^n$ for large enough n).

For $\Sigma \lambda_g g \in \hat{\mathbb{Z}}_p[G]$, set

$$\Phi(\Sigma \lambda_g g) = \Sigma \lambda_g g^p,$$

and define, for $u \in 1 + J(\hat{\mathbb{Z}}_p[G])$,

$$\Gamma = \Gamma_G \colon 1 + J(\hat{\mathbb{Z}}_p[G]) \to \hat{\mathbb{Q}}_p[G]$$

by

$$\Gamma(u) = \log(u) - \frac{1}{p}\Phi(\log u) = \frac{1}{p}\log\left(\frac{u^p}{\Phi(u)}\right).$$

Clearly Γ_G is natural with respect to group homomorphisms.

If $\lambda \in \hat{\mathbb{Z}}_p$ and $g \in G$, it makes sense to write g^λ: we may write λ as a convergent series $\lambda = \sum_{i \geqq 0} a_i p^i$ with $0 \leqq a_i \leqq p-1$; since G is a finite p-group, the product $g^\lambda = g^{\Sigma a_i p^i} = \Pi (g^{a_i})^{p^i}$ is finite. Define

$$\omega = \omega_G \colon \hat{\mathbb{Z}}_p[G] \to G$$

by $\omega_G(\Sigma \lambda_g g) = \Pi g^{\lambda_g}$.

Since we are using the same notation here as [OII, §§1, 2] for slightly different maps, a word of explanation is in order. Let $I(\hat{\mathbb{Z}}_p[G])$ be the augmentation ideal. The map Γ of [OII, pp. 208–9] is defined on $1 + I(\hat{\mathbb{Z}}_p[G])$; there it is shown that Γ in fact factors through $Wh'(\hat{\mathbb{Z}}_p[G])$ [OII, Prop. 3]. In this paper we have extended Γ to all of $1 + J(\hat{\mathbb{Z}}_p[G])$. Also note that when G is abelian, $\overline{I(\hat{\mathbb{Z}}_p[G])} = I(\hat{\mathbb{Z}}_p[G])$ [OII, p. 199]. So [OII, Prop. 10(i)] tells us that $\Gamma(1 + I(\hat{\mathbb{Z}}_p[G])) \subset I(\hat{\mathbb{Z}}_p[G])$ when G is abelian. Since it is clear that $\Gamma(1+px) \in \hat{\mathbb{Z}}_p[G]$ when $x \in \hat{\mathbb{Z}}_p[G]$, we see that the map Γ_G defined here takes values in $\hat{\mathbb{Z}}_p[G]$.

Lemma 1.3.

$$\Gamma_G(1 + J(\hat{\mathbb{Z}}_p[G])) = \begin{cases} \ker(\omega_G) & \text{if } p \text{ is odd,} \\ J(\hat{\mathbb{Z}}_2[G]) \cap \ker(\omega_G) & \text{if } p = 2. \end{cases}$$

Proof. We will apply [OII, Theorem 2, p. 211], keeping in mind the preceding discussion. According to that theorem,

$$\Gamma_G(1+I(\hat{\mathbf{Z}}_p[G]))=\ker(\omega_G|I(\hat{\mathbf{Z}}_p[G]))=I(\hat{\mathbf{Z}}_p[G])\cap\ker(\omega_G).$$

Now $1+J(\hat{\mathbf{Z}}_p[G])=(1+p\hat{\mathbf{Z}}_p)(1+I(\hat{\mathbf{Z}}_p[G]))$, and clearly

$$\Gamma_G(1+p\hat{\mathbf{Z}}_p)=\frac{1}{p}\log(1+p\hat{\mathbf{Z}}_p)=\begin{cases}\hat{\mathbf{Z}}_p & \text{if } p\neq 2,\\ 2\hat{\mathbf{Z}}_2 & \text{if } p=2.\end{cases}$$

Since Γ_G is a homomorphism, our lemma follows.

The homomorphism Γ will now be used to obtain a formula for the norm residue symbols which arise in computing $SK_1(\mathbf{Z}[G])$. Let G be a finite abelian p-group, and $\chi\in\hat{G}$ a character of G. Define a homomorphism

$$\psi_\chi\colon G\otimes_Z\hat{\mathbf{Z}}_p[G]\to\mathrm{Im}(\chi)\subset\hat{\mathbf{Q}}_p(\chi)$$

by

$$\psi_\chi(g\otimes h)=\begin{cases}\chi(g) & \text{if } \chi(h)=1,\\ 1 & \text{if } \chi(h)\neq 1,\end{cases}\tag{1}$$

$(g,h\in G)$ and extend linearly (so $\psi_\chi(g\otimes\sum_h\lambda_h h)=\sum_h\lambda_h\psi_\chi(g\otimes h)$).

Lemma 1.4. *Let G be a finite abelian p-group and $\chi\in\hat{G}$ a character of G with $|\mathrm{Im}(\chi)|=p^r>2$. For any $g\in G$ and $u\in 1+J(\hat{\mathbf{Z}}_p[G])$,*

$$(\chi(g),\chi(u))_v=\begin{cases}\psi_\chi(g\otimes\Gamma_G(u)) & \text{if } p \text{ is odd},\\ \psi_\chi(g\otimes\Gamma_G(u))^{2^{r-1}+1} & \text{if } p=2,\end{cases}$$

where $(\ ,\ )_v$ is the norm residue symbol on $\hat{\mathbf{Q}}_p(\chi)=\hat{\mathbf{Q}}_p(\zeta_{p^r})$ and χ denotes the homomorphism $\hat{\mathbf{Q}}_p[G]\to\hat{\mathbf{Q}}_p(\chi)$ induced by the character χ.

Proof. We will need to distinguish the ring $\hat{\mathbf{Q}}_p(\chi)=\hat{\mathbf{Q}}_p(\zeta_{p^r})$ from the group ring of $\mathrm{Im}(\chi)$ over $\hat{\mathbf{Q}}_p$, which we will denote $\hat{\mathbf{Q}}_p[\mathrm{Im}(\chi)]$. The induced homomorphism $\hat{\mathbf{Q}}_p[G]\to\hat{\mathbf{Q}}_p[\mathrm{Im}(\chi)]$ will be denoted $\tilde{\chi}$, and the quotient map $\hat{\mathbf{Q}}_p[\mathrm{Im}(\chi)]\to\hat{\mathbf{Q}}_p(\chi)$ will be called v, so that $\chi=v\tilde{\chi}$. We will abbreviate $\mathrm{Tr}_{\hat{\mathbf{Q}}_p(\chi)/\hat{\mathbf{Q}}_p}$ by Tr, and let $\tau\colon\hat{\mathbf{Q}}_p[\mathrm{Im}(\chi)]\to\hat{\mathbf{Q}}_p$ be the trace-like map sending an element of $\hat{\mathbf{Q}}_p[\mathrm{Im}(\chi)]$ to the coefficient of $1\in\mathrm{Im}(\chi)$.

With this notation, it is clear that $\psi_\chi(g\otimes x)=\chi(g)^{\tau\tilde{\chi}(x)}$ for $g\in G$, $x\in\hat{\mathbf{Z}}_p[G]$. By [A-H, p. 147], [Sen],

$$(\chi(g),\chi(u))_v=\chi(g)^t,$$

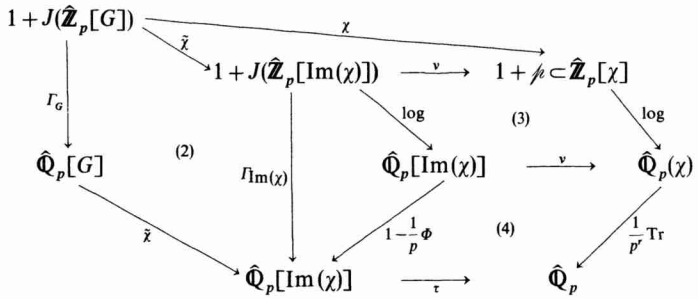
where

$$t=\begin{cases}\dfrac{1}{p^r}\mathrm{Tr}(\log(\chi(u))) & \text{if } p \text{ is odd},\\ \dfrac{2^{r-1}+1}{2^r}\mathrm{Tr}(\log(\chi(u))) & \text{if } p=2.\end{cases}$$

Thus to prove the lemma we must show, in either case, that

$$\frac{1}{p^r}\mathrm{Tr}(\log(\chi(u)))=\tau\tilde{\chi}(\Gamma_G(u)).$$

Let $\mathfrak{p}$ be the maximal ideal of $\hat{\mathbb{Z}}_p[\chi]$ $(\subset \hat{\mathbb{Q}}_p(\chi))$, and consider the following diagram:



The two triangles in this diagram commute by definition, and (2) commutes by the naturality of Γ, since $\tilde{\chi}$ is induced by the group homomorphism $\chi: G \to \mathrm{Im}(\chi)$. Because ν is a ring homomorphism, (3) also commutes. To prove the lemma we must show, therefore, that (4) is commutative, and it suffices to check this when $h \in \mathrm{Im}(\chi)$. If $h^p \neq 1$, then $\nu(h) \in \hat{\mathbb{Q}}_p(\chi)$ is *not* a primitive p-th root of unity, so $\mathrm{Tr}(\nu(h))=0$ (consider its minimal polynomial) and $\Phi(h)=h^p \neq 1$. Thus

$$\tau\left(h-\frac{1}{p}\Phi(h)\right)=\tau\left(h-\frac{1}{p}h^p\right)=0$$

and (4) commutes. If $h=1$, then

$$\frac{1}{p^r}\mathrm{Tr}(\nu(h))=\frac{1}{p^r}\mathrm{Tr}(1)=\frac{p^{r-1}(p-1)}{p^r}=1-\frac{1}{p}=\tau\left(1-\frac{1}{p}\right).$$

If $h \neq 1$, but $h^p=1$, then $\mathrm{Tr}(\nu(h))=-p^{r-1}$, so $\frac{1}{p^r}\mathrm{Tr}(\nu(h))=-\frac{1}{p}$. But then

$$\tau\left(h=\frac{1}{p}\Phi(h)\right)=\tau\left(h-\frac{1}{p}h^p\right)=\tau\left(h-\frac{1}{p}\right)=-\frac{1}{p}$$

and (4) again commutes.

Fix a character $\chi \in \hat{G}$ with $\chi \neq 1$ if $p=2$, and let

$$\lambda_\chi: K_2(\mathbb{Z}[\chi]/(p^k)) \to \mathrm{Im}(\chi) \subset \hat{\mathbb{Q}}_p(\chi)$$

be the homomorphism induced by the norm residue symbol: $\lambda_\chi(\{a,b\})=(a,b)_v$ (cf. [Ser, Section XIV.2, p. 215]). By [D-S, Theorem 4.3] λ_χ is well-defined and is an isomorphism if $k>r$, where $|\mathrm{Im}(\chi)|=p^r$. For such a character χ, we let

$$\varphi_\chi: K_2(\mathbb{Z}[G]/(p^k)) \to K_2(\mathbb{Z}[\chi]/(p^k))$$

be the homomorphism on K_2 induced by χ. Finally for any set of characters $T \subset \hat{G}$ and any k such that $\exp(G) | p^{k-1}$, we have the homomorphisms of direct products

$$\varphi_T: K_2(\mathbb{Z}[G]/(p^k)) \to \prod_{\chi \in T} K_2(\mathbb{Z}[\chi]/(p^k))$$

defined by $(\varphi_T(x))_\chi = \varphi_\chi(x)$, and when $p \neq 2$ or $1 \notin T$,

$$\lambda_T = \prod_{\chi \in T} \lambda_\chi \colon \prod_{\chi \in T} K_2(\mathbb{Z}[\chi]/(p^k)) \xrightarrow{\approx} \prod_{\chi \in T} \mathrm{Im}(\chi).$$

These maps clearly depend on k, but we do not indicate this to avoid making the notation too cumbersome. Besides, we shall mainly be interested in $\mathrm{Im}(\lambda_T \varphi_T)$ and this is independent of k because $K_2(\mathbb{Z}[G]/(p^{k+1})) \to K_2(\mathbb{Z}[G]/(p^k))$ is surjective. Note that $\varphi_{\hat{G}}$ and, for $p \neq 2$, $\lambda_{\hat{G}}$, are natural with respect to group homomorphisms.

Similarly, if $T \subset \hat{G}$, the homomorphisms ψ_χ defined in (1) induce a homomorphism of direct products

$$\psi_T \colon G \otimes_Z \hat{\mathbb{Z}}_p[G] \to \prod_{\chi \in T} \mathrm{Im}(\chi),$$

given by $(\psi_T(x))_\chi = \psi_\chi(x)$ for $x \in G \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p[G]$. When $T = \hat{G}$, $\psi_{\hat{G}}$ is natural with respect to group homomorphisms in the following sense: Given a group homomorphism $\alpha \colon H \to G$, let $\tilde{\alpha}$ be the induced homomorphism $\hat{\mathbb{Z}}_p[H] \to \hat{\mathbb{Z}}_p[G]$, and define

$$\alpha_\# \colon \prod_{\chi \in \hat{H}} \mathrm{Im}(\chi) \to \prod_{\chi \in \hat{G}} \mathrm{Im}(\chi) \quad \text{by } (\alpha_\#(x))_{\chi_0} = x_{\chi_0 \circ \alpha},$$

where $\chi_0 \in \hat{G}$ and $x = (x_\chi) \in \prod_{\chi \in \hat{H}} \mathrm{Im}(\chi)$. Then the diagram

$$\begin{array}{ccc} H \otimes_Z \hat{\mathbb{Z}}_p[H] & \xrightarrow{\psi_{\hat{H}}} & \prod_{\chi \in \hat{H}} \mathrm{Im}(\chi) \\ \downarrow{\scriptstyle \alpha \otimes \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha_\#} \\ G \otimes_Z \hat{\mathbb{Z}}_p[G] & \xrightarrow{\psi_{\hat{G}}} & \prod_{\chi \in \hat{G}} \mathrm{Im}(\chi) \end{array}$$

commutes.

Proposition 1.5. *Let p be an odd prime, G a finite abelian p-group, and k an integer such that $\exp(G) | p^{k-1}$. Then for any set of characters $T \subset \hat{G}$, $\lambda_T \circ \varphi_T$ and ψ_T have the same image in $\prod_{\chi \in T} \mathrm{Im}(\chi)$.*

Proof. Let $pr_T \colon \prod_{\chi \in \hat{G}} \mathrm{Im}(\chi) \to \prod_{\chi \in T} \mathrm{Im}(\chi)$ be the projection map. Then clearly $\psi_T = pr_T \circ \psi_{\hat{G}}$, $\lambda_T \circ \varphi_T = pr_T \circ \lambda_{\hat{G}} \circ \varphi_{\hat{G}}$, so it suffices to prove this proposition when $T = \hat{G}$. To simplify notation write ψ, φ, λ for $\psi_{\hat{G}}, \varphi_{\hat{G}}, \lambda_{\hat{G}}$ respectively.

For each fixed $g \in G$, define homomorphisms

$$\psi_g \colon \hat{\mathbb{Z}}_p[G] \to \prod_{\chi \in \hat{G}} \mathrm{Im}(\chi)$$

and

$$\varphi_g \colon (\hat{\mathbb{Z}}_p[G])^*_{(p)} = 1 + J(\hat{\mathbb{Z}}_p[G]) \to \prod_{\chi \in \hat{G}} \mathrm{Im}(\chi)$$

by

$$\psi_g(x) = \psi(g \otimes x), \quad x \in \hat{\mathbb{Z}}_p[G],$$
$$\varphi_g(u) = \lambda \varphi(\{g, u\}), \quad u \in 1 + J(\hat{\mathbb{Z}}_p[G]).$$

It follows from Lemma 1.4 that

$$\begin{aligned}(\varphi_g(u))_\chi &= (\lambda\varphi(\{g,u\}))_\chi = \lambda(\{\chi(g),\chi(u)\})\\ &= (\chi(g),\chi(u))_v = \psi_\chi(g\otimes\Gamma_G(u))\\ &= (\psi(g\otimes\Gamma_G(u)))_\chi\\ &= (\psi_g(\Gamma_G(u)))_\chi.\end{aligned}$$

This shows that $\mathrm{Im}(\psi_g(\Gamma_G(1+J(\hat{\mathbb{Z}}_p[G]))\subset\mathrm{Im}(\varphi_g)$, and also that $\mathrm{Im}(\varphi_g)\subset\mathrm{Im}(\psi_g)$.

Conversely, since p is odd, ir follows from (1) that

$$\psi_g(h)=\psi(g\otimes h)=\psi(g\otimes h^2)=\psi_g(h^2)$$

for any $h\in G$, and so, by Lemma 1.3,

$$\psi_g(h)=\psi_g(2h-h^2)\in\ker(\omega_g)=\psi_g(\Gamma_G(1+\mathrm{J}(\hat{\mathbb{Z}}_p[G])))\subset\mathrm{Im}(\varphi_g).$$

Thus $\mathrm{Im}(\varphi_g)=\mathrm{Im}(\psi_g)$.

By Corollary 1.2, $K_2(\mathbb{Z}[G]/(p^k))$ is generated by Steinberg symbols of the form $\{1+p,\ 1+px\}$, $\{g,u\}$, where $g\in G$, $u\in 1+J(\mathbb{Z}[G]/(p^k))$, and $x\in\mathbb{Z}[G]/(p^k)$. But the subgroup of $(\mathbb{Z}[G]/(p^k))^*$ generated by all $1+px$, $x\in\mathbb{Z}[G]/(p^k)$ is also generated by all $1+p^ig$, $1\leqq i<k$, $g\in G$, so $K_2(\mathbb{Z}[G]/(p^k))$ is also generated by the Steinberg symbols

$$\{1+p,\ 1+p^ig\},\{g,u\}\qquad g\in G,\ u\in 1+J(\mathbb{Z}[G]/(p^k)).$$

It follows that $\mathrm{Im}(\lambda\circ\varphi)$ is generated by all

$$\mathrm{Im}(\varphi_g),\qquad \lambda\varphi(\{1+p,1+p^ig\}),\qquad g\in G,\ 1\leqq i<k.$$

Since we have already shown that $\mathrm{Im}(\varphi_g)=\mathrm{Im}(\psi_g)\subset\mathrm{Im}\,\psi$, it remains to prove that

$$\lambda\varphi(\{1+p,1+p^ig\})\in\mathrm{Im}(\psi),\tag{5}$$

and it will suffice, by naturality of φ, λ, ψ, to prove (5) when G is the cyclic group generated by g.

Fix $a\in\mathbb{N}$ prime to p and let $\sigma\in\mathrm{Gal}(\hat{\mathbb{Q}}_p(\chi)/\hat{\mathbb{Q}}_p)$ be the automorphism $\sigma(\zeta)=\zeta^a$. For any $\chi\in\hat{G}$, we have

$$\begin{aligned}(\lambda\varphi(\{1+p,1+p^ig\}))_{\chi^a} &= \lambda(\{1+p,1+p^i\chi^a(g)\})\\ &= (1+p,1+p^i\chi^a(g))_v = (1+p,1+p^i(\chi(g))^a)_v\\ &= (\sigma(1+p),\sigma(1+p^i\chi(g)))_v = \sigma((1+p,1+p^i\chi(g))_v)\\ &= ((1+p,1+p^i\chi(g))_v)^a = \lambda(\{1+p,1+p^i\chi(g)\})^a\\ &= ((\lambda\varphi(\{1+p,1+p^ig\}))_\chi)^a.\end{aligned}$$

Thus $\lambda\varphi(\{1+p,\ 1+p^ig\})\in H$, where H is the subgroup of elements $x\in\prod_{\chi\in\hat{G}}\mathrm{Im}(\chi)$ satisfying $(x)_{\chi^a}=((x)_\chi)^a$ for all $a\in\mathbb{Z}$ relatively prime to p and all $\chi\in\hat{G}$. But it is straightforward to verify that if g has order p^n, H is generated by the elements $\psi(g\otimes g^{p^i})$, $1\leqq i\leqq n$. This completes the proof of (5) and the lemma.

When $p=2$, the corresponding description of $\mathrm{Im}(\lambda_T\circ\varphi_T)$ is more complicated. It will be given in Proposition 1.7 after the following preparatory lemma.

Lemma 1.6. *Let G be a finite abelian 2-group and let $T=\{\chi\in\hat{G}\,|\,|\mathrm{Im}\,\chi|>2\}$. Fix $k\geqq 2$. Let x be any element of $\mathbb{Z}[G]/(2^k)$. Then*

(a) $\{-1,1+4x\}=1$ *in* $K_2(\mathbb{Z}[G]/(2^k))$.

(b) $\varphi_T(\{-1,1+2x\})$ *is contained in the subgroup generated by all* $\varphi_T(\{g,1+4y\})$, *where* $g\in G$ *and* $y\in\mathbb{Z}[G]/(2^k)$.

Proof. Note first that the subgroup $1+4(\mathbb{Z}[G]/(2^k))$ of $(\mathbb{Z}[G]/(2^k))^*$ is generated by 5 and all $1+4g+4g^2$, $g\in G$. Since $\{-1,5\}=1$ in $K_2(\mathbb{Z}/2^k\mathbb{Z})$ [St1, p. 179], and $\{-1,1+4g+4g^2\}=\{-1,1+2g\}^2=1$, (a) is proved.

Next, note that $1+2(\mathbb{Z}[G]/(2^k))$ is generated by -1 and all $1+2g+2g^2$, $g\in G$. Since $\{-1,-1\}=\{-1,i\}^2=1$ in $K_2(\mathbb{Z}[i])$, it follows that $\varphi_T(\{-1,-1\})=1$, since it is an element of $K_2(\mathbb{Z}[\zeta_{2^r}]/(2)^k$ for some $r\geqq 2$. Thus (b) is equivalent to the assertion that for every $g\in G$,

$$\begin{aligned}\varphi_T(\{-1,1+2g+2g^2\})&=\varphi_T(\{g,1+4g+4g^2\})\\&=\varphi_T(\{g,1+2g\}^2).\end{aligned}\tag{6}$$

That is, we must prove that if ζ is a primitive 2^r-th root of unity, and $n\geqq\max\{r,2\}$, then

$$\{-1,1+2\zeta+2\zeta^2\}=\{\zeta,1+2\zeta\}^2 \quad \text{in } K_2(\mathbb{Z}[\zeta_{2^n}]/(2)^k.\tag{7}$$

Moreover, since

$$1+2\zeta+2\zeta^2=(1+2\zeta)(1+2\zeta^2)(1+4x),$$

with $x=-\zeta^3/(1+2\zeta)(1+2\zeta^2)$, it follows from (a) that (7) is equivalent to showing that

$$\{-1,1+2\zeta^2\}=\{-\zeta^2,1+2\zeta\}.\tag{8}$$

This is clear if $\zeta=\pm 1$ or $\zeta=i(=\zeta_4)$ (recall that $\{-1,-1\}=1$ in $K_2(\mathbb{Z}[i])$ and its quotients). So we may assume that $n=r\geqq 3$. Using the formulas of [A-H, p. 147] we obtain, for $\zeta=\zeta_{2^n}$, $n\geqq 3$, and $N=N_{\mathbb{Q}(\zeta)/\mathbb{Q}}$:

$$\begin{aligned}(-1,1+2\zeta^2)_v&=(-1)^{(N(1+2\zeta^2)-1)/2^n}\\(-\zeta^2,1+2\zeta)_v&=(-\zeta^2)^{(N(1+2\zeta)-1)/2^n}.\end{aligned}$$

Also, $N(1+2\zeta)=1+N(2\zeta)=1+2^{2^{n-1}}$, since all other symmetric polynomials in $\zeta,\zeta^3,\ldots,\zeta^{2^n-1}$ vanish. Similarly,

$$N(1+2\zeta^2)=(1+2^{2^{n-2}})^2\equiv 1+2^{1+2^{n-2}}(\mathrm{mod}\,2^{2^{n-1}}).$$

It follows that $(-1,1+2\zeta^2)_v$ and $(-\zeta^2,1+2\zeta)_v$ both equal -1 when $n=3$, and both are trivial if $n\geqq 4$. Thus (8) follows from [D-S, Theorem 4.3].

When G is a finite abelian 2-group and $T\subset\hat{G}$, define

$$\mu\colon\prod_{\chi\in T}\mathrm{Im}(\chi)\to\prod_{\chi\in T}(\mathrm{Im}(\chi)\quad\text{by }(\mu(x))_\chi=((x)_\chi)^{2^{r-1}+1}$$

when $|\mathrm{Im}\,\chi|=2^r$. It is clear that μ is an automorphism. We define

$$\psi'_T=\mu\circ\psi_T\colon G\otimes\hat{\mathbb{Z}}_2[G]\to\prod_{\chi\in T}\mathrm{Im}(\chi).$$

Proposition 1.7. *Let G be a finite abelian 2-group and k an integer such that $\exp(G)|2^{k-1}$. Let $T\subset\hat{G}$ be a set of characters such that $|\mathrm{Im}\,\chi|>2$ for all $\chi\in T$. Then*

(a) $\lambda_\chi\varphi_\chi(\{g,\bar{v}\})=(\psi'_T(g\otimes\Gamma_G(v)))_\chi$, *for* $g\in G$, $v\in 1+J(\hat{\mathbb{Z}}_2[G])$ *for all* $\chi\in T$, *where* $\bar{v}$ *is the image of* v *in* $\mathbb{Z}[G]/(2^k)$.

(b) $\mathrm{Im}(\lambda_T\circ\varphi_T)=\psi'_T(G\otimes\langle 2,J(\hat{\mathbb{Z}}_2[G])^2\rangle)$
$=\psi'_T(G\otimes\langle 2g,(1-g)(1-h)|g,h\in G\rangle)$.

Proof. Corollary 1.2 shows that $K_2(\mathbb{Z}[G]/(2^k))$ is generated by all $\{-1,u\}$ and $\{g,v\}$, $g\in G$, $u\in 1+2(\mathbb{Z}[G]/(2^k))$, $v\in 1+J(\mathbb{Z}[G]/(2^k))$. But Lemma 1.6 shows that $\varphi_T(\{-1,u\})$ is already contained in the subgroup generated by all $\varphi_T(\{g,v\})$. Moreover, by Lemma 1.4, if $|\mathrm{Im}(\chi)|=2^r$ and $v\in 1+J(\hat{\mathbb{Z}}_2[G])$,

$$\lambda_\chi\varphi_\chi(\{g,\bar{v}\})=(\chi(g),\chi(v))_v=(\psi'_T(g\otimes\Gamma_G(v)))_\chi$$

for all $\chi\in T$, proving (a). Hence

$$\mathrm{Im}(\lambda_T\circ\varphi_T)=\psi'_T(G\otimes\Gamma_G(1+J(\hat{\mathbb{Z}}_2[G]))).$$

Now if $x=\Sigma\lambda_g g\in\Gamma_G(1+J(\hat{\mathbb{Z}}_2[G]))$, Lemma 1.3 implies

$$\Sigma\lambda_g\in 2\hat{\mathbb{Z}}_2$$
$$\Pi g^{\lambda_g}=1\in G.$$

Since $(g-1)+(h-1)=(gh-1)-(g-1)(h-1)\equiv(gh-1)$ modulo $J(\hat{\mathbb{Z}}_2[G])^2$, for all $g,h\in G$, it follows that

$$x=\Sigma\lambda_g g\equiv\Sigma\lambda_g(g-1)\equiv\Pi g^{\lambda_g}-1\equiv 0$$

modulo $\langle 2,J(\hat{\mathbb{Z}}_2[G])^2\rangle$, so

$$\Gamma_G(1+J(\hat{\mathbb{Z}}_2[G]))\subset\langle 2,\hat{J}(\mathbb{Z}_2[G])^2\rangle.$$

Conversely, Lemma 1.3 implies $(1-g)(1-h)\in\Gamma_G(1+J(\hat{\mathbb{Z}}_2[G]))$ for any $g,h\in G$. Also, since $g+g^{-1}\in\ker(\omega_G)$, and $\chi(g)=1$ if and only if $\chi(g^{-1})=1$, it follows that

$$\psi_T(h\otimes 2g)=\psi_T(h\otimes(g+g^{-1}))\in\psi_T(G\otimes\Gamma_G(1+J(\hat{\mathbb{Z}}_2[G]))),$$

and the same holds for $\psi'_T=\mu\circ\psi_T$. Thus

$$\psi'_T(G\otimes\langle 2,J(\hat{\mathbb{Z}}_2[G])^2\rangle=\psi'_T(G\otimes\langle 2g,(1-g)(1-h)\rangle\subset\psi'_T(G\otimes\Gamma(1+J(\hat{\mathbb{Z}}_2[G]))),$$

which completes the proof of this Proposition.

Definition. Let G be a finite abelian p-group and let $\mathcal{O}$ be the ring of integers in an algebraic number field F. A subset $S\subset\hat{G}$ will be called an *F-cluster* of characters for G id S contains exactly one character for each simple

$F[G]$-module. When $F=\mathbf{Q}$, $\mathcal{O}=\mathbf{Z}$, we will drop F from the notation and call S a *cluster* of characters for G (cf. [A-D-S, Definition 2.1].

If S is a cluster of characters for G, define

$$S_0=\{\chi\in S\,|\,\mathbf{Z}[\chi]\neq\mathbf{Z}\}.$$

S_0 will be called an *imaginary cluster* of characters for G. Note that

$$S_0=\begin{cases}S-\{\text{trivial character}\} & \text{if } p \text{ is odd,}\\ \{\chi\in S\,|\,|\mathrm{Im}(\chi)|>2\} & \text{if } p=2.\end{cases}$$

We can now state the major theorem which will be used for computations in the succeeding sections.

Theorem 1.8. *Let G be a finite abelian p-group and let S_0 be an imaginary cluster of characters for G. Then*

$$SK_1(\mathbf{Z}[G])\approx\begin{cases}[\prod_{\chi\in S_0}\mathrm{Im}(\chi)]/\psi_{S_0}(G\otimes\hat{\mathbf{Z}}_p[G]) & \text{if } p \text{ is odd,}\\ [\prod_{\chi\in S_0}\mathrm{Im}(\chi)]/\psi'_{S_0}(G\otimes\langle 2, J(\hat{\mathbf{Z}}_2[G])^2\rangle) & \text{if } p=2.\end{cases}$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} K_2(\mathbf{Z}[G]/(p^k)) & \xrightarrow{\varphi_{S_0}} & \prod_{\chi\in S_0} K_2(\mathbf{Z}[\chi]/(p^k)) & \longrightarrow & SK_1(\mathbf{Z}[G]) & \longrightarrow & 1\\ & \searrow^{\lambda_{S_0}\varphi_{S_0}} & \downarrow{\scriptstyle \lambda_{S_0}} & & & & \\ & & \prod_{\chi\in S_0}\mathrm{Im}(\chi) & & & & \end{array}$$

where k is chosen so that $\exp(G)|p^{k-1}$. The top row is exact by [A-D-S, Theorem 2.10], and λ_{S_0} is an isomorphism. Hence

$$SK_1(\mathbf{Z}[G])\approx\mathrm{cok}(\lambda_{S_0}\circ\varphi_{S_0})\approx[\prod_{\chi\in S_0}\mathrm{Im}(\chi)]/\mathrm{Im}(\lambda_{S_0}\circ\varphi_{S_0}).$$

The theorem now follows from Propositions 1.5 and 1.7.

To complete the section we will derive similar formulas for computing relative SK_1. We first generalize [A-D-S, Theorem 2.9].

Suppose $\mathcal{O}$ is the ring of integers in an algebraic number field which is unramified over p, G is a finite abelian p-group, and $S\subset\hat{G}$ is a cluster of characters for G. For any integer $n\geqq 1$ and any ideal $a\subset\mathcal{O}[G]/(p^n)$, denote by $\varphi_{S,a}$ the composite map

$$K_2(\mathcal{O}[G]/(p^n),(a))\xrightarrow{i}K_2(\mathcal{O}[G]/(p^n))\xrightarrow{\varphi_S}\prod_{\chi\in S}K_2(\mathcal{O}[\chi]/(p^n))$$

$$\xrightarrow{(\partial_\chi)}\prod_{\chi\in S}SK_1(\mathcal{O}[\chi],(p^n)),$$

where i and ∂_χ are the maps arising in the K-theory exact sequences for $\mathcal{a}$, $p^n\mathcal{O}[\chi]$ respectively. We drop the dependence on $n, \mathcal{a}$ from the notation.

Proposition 1.9. *Let $\mathcal{O}$ be the ring of integers in an algebraic number field F which is unramified over p, let G be a finite abelian p-group, and let $S \subset \hat{G}$ be an F-cluster of characters for G. Let $I \subset \mathcal{O}[G]$ be an ideal of finite index and choose n so that $\exp(G) | p^{n-1}$ and $p^n\mathcal{O}[G] \subset I$.*

Then $SK_1(Z[G], I) \approx \operatorname{cok}(\varphi_{S,I})$.

Proof. Under the isomorphism $F[G] \xrightarrow{\approx} \prod_{\chi \in S} F(\chi)$, we may identify $\mathcal{M} = \prod_{\chi \in S} \mathcal{O}[\chi]$ with the maximal order in $F[G]$ [A-D-S, Proposition 2.2]. Choose $k \geqq n$ such that $p^k\mathcal{M} \subset p^n\mathcal{O}[G]$, and let $c = p^k\mathcal{M}$. Consider the following diagram:

$$\begin{array}{ccccccc} K_2(\mathcal{O}[G]/c, I/c) & \xrightarrow{\partial} & SK_1(\mathcal{O}[G], c) & \longrightarrow & SK_1(\mathcal{O}[G], I) & \longrightarrow & 1 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \\ & & SK_1(\mathcal{M}, c) & & & & \\ & & \downarrow{\scriptstyle f_3} & & & & \\ K_2(\mathcal{O}[G]/(p^n), I/p^n\mathcal{O}[G]) & \xrightarrow{\varphi_{S,I}} & \prod_{\chi \in S} SK_1(\mathcal{O}[\chi], p^n\mathcal{O}[\chi]) & & & & \end{array}$$

The top row is exact [Mi, Remark 6.6]. The map f_2 is an isomorphism by [A-D-S, Theorem 1.3]; f_3 is an isomorphism by [B-M-S, Corollary 4.3]; and f_1 is onto (since Dennis-Stein symbols can be lifted). Since the diagram commutes, we have

$$SK_1(\mathcal{O}[G], I) \approx \operatorname{cok}(\varphi_{S,I}).$$

Recall that if $\Sigma \lambda_g g \in \hat{\mathbb{Z}}_2[G]$, the map Φ is defined by

$$\Phi(\Sigma \lambda_g g) = \Sigma \lambda_g g^2.$$

Theorem 1.10. *Let G be a finite abelian 2-group, and S_0 an imaginary cluster of characters for G. Then:*

(a) *If $\exp(G) = 2^{n-1}$, $SK_1(\mathbb{Z}[G], (2^k)) \approx [\prod_{\chi \in S_0} \operatorname{Im}(\chi)]/\lambda_{S_0}(V)$,*

where V is the subgroup of $K_2(\mathbb{Z}[G]/(2^n), (2^k))$ generated by all $\{g, 1+2^k x\}$, $g \in G$, $x \in \mathbb{Z}[G]/(2^n)$ for all $k \geqq 1$.

(b) $SK_1(\mathbb{Z}[G], 2\mathbb{Z}[G]) \approx [\prod_{\chi \in S_0} \operatorname{Im}(\chi)]/\psi_{S_0}(G \otimes (2-\Phi)(J(\hat{\mathbb{Z}}_2[G])))$.

(c) *For $k \geqq 2$,*

$SK_1(\mathbb{Z}[G], 2^k\mathbb{Z}[G]) \approx [\prod_{\chi \in S_0} \operatorname{Im}(\chi)]/\psi_{S_0}(G \otimes 2^{k-1}(2-\Phi)(\hat{\mathbb{Z}}_2[G]))$.

Proof. Let $k \geqq 1$. We will apply Proposition 1.9 with $\mathcal{O} = \mathbb{Z}$, $p = 2$, $I = 2^k\mathbb{Z}[G]$, $1 \leqq k < n$. In this case, if $|\operatorname{Im}\chi| \leqq 2$, $\mathbb{Z}[\chi] = \mathbb{Z}$ and $SK_1(\mathbb{Z}[\chi], (2^k)) = 1$, so we have $SK_1(\mathbb{Z}[G], 2^k\mathbb{Z}[G]) \approx \operatorname{cok}(\alpha_{S_0})$, where S_0 is an imaginary cluster of characters for G. Moreover, since there is only one prime ideal in $\mathbb{Z}[\chi]$ lying over (2), $\partial_\chi : K_2(\mathbb{Z}[\chi]/(2^n)) \to SK_1(\mathcal{O}[\chi], (2^n))$ is an isomorphism for each $\chi \in S_0$ (cf. [D-S, Remarks following Theorem 5.1, p. 225]). Hence

$$SK_1(\mathbb{Z}[G], 2^k\mathbb{Z}[G]) \approx \operatorname{cok}(\varphi_{S_0} \circ i),$$

where i: $K_2(\mathbb{Z}[G]/(2^n),(2^k))\to K_2(\mathbb{Z}[G]/(2^n))$. Finally, since under our hypotheses

$$\lambda_{S_0}\colon \prod_{\chi\in S_0} K_2(\mathbb{Z}[\chi]/(2^n))\to \prod_{\chi\in S_0}\mathrm{Im}(\chi)$$

is an isomorphism, we have

$$SK_1(\mathbb{Z}[G],2^k\mathbb{Z}[G])\approx \mathrm{cok}(\lambda_{S_0}\circ\varphi_{S_0}\circ i).$$

Now by Corollary 1.2, $K_2(\mathbb{Z}[G]/(2^n),(2^k))$ is generated by the Steinberg symbols $\{-1,1+2^k x\}$, $\{g,1+2^k x\}$, $g\in G$, $x\in\mathbb{Z}[G]/(2^n)$. Hence $\mathrm{Im}(\lambda_{S_0}\circ\varphi_{S_0}\circ i)$ is generated by all $\lambda_{S_0}\circ\varphi_{S_0}(\{g,1+2^k x\})$, since Lemma 1.6 says we can ignore the contribution of the symbols $\{-1,1+2^k x\}$ to this image. This proves (a).

Now Proposition 1.7a says that

$$\lambda_\chi\varphi_\chi(\{g,1+2^k x\})=(\psi'_{S_0}(g\otimes\Gamma_G(1+2^k x)))_\chi$$

(where the x on the right lifts x to $\hat{\mathbb{Z}}_2[G]$) and $\Gamma_G(1+2^k x)=(1-\frac{1}{2}\Phi)\log(1+2^k x)$, so the Theorem will be established if we can prove

$$\log(1+2\hat{\mathbb{Z}}_2[G])=2J(\hat{\mathbb{Z}}_2[G]) \tag{9}$$

$$\log(1+2^k\hat{\mathbb{Z}}_2[G])=2^k\hat{\mathbb{Z}}_2[G]\quad\text{for } k\geqq 2. \tag{10}$$

To prove (9), note that $\log(1+2x)=2(x-x^2)+\frac{8}{3}x^3-4x^4+\ldots\in 2J(\hat{\mathbb{Z}}_2[G])$. Conversely, if $v\in J(\hat{\mathbb{Z}}_2[G])$, $\lim_{m\to\infty} v^m=0$, and the series $e^{2v}=1+2v+2v^2+\ldots$ converges in $1+2\hat{\mathbb{Z}}_2[G]$. So

$$2v=\log(e^{2v})\in\log(1+2\hat{\mathbb{Z}}_2[G]).$$

The proof of (10) is similar and will be left to the reader.

We now apply this to obtain a "binomial theorem" for SK_1 of abelian 2-groups.

Theorem 1.11. *Let G be a finite abelian 2-group, and C_2 a cyclic group of order 2 generated by σ. Then*

$$SK_1(\mathbb{Z}[G\times C_2^m])\approx\prod_{k=0}^{m}\binom{m}{k}SK_1(\mathbb{Z}[G],2^k\mathbb{Z}[G])$$

for all $m\geqq 1$.

Proof. The square

$$\begin{array}{ccc}\mathbb{Z}[G\times C_2] & \xrightarrow{\sigma\to 1} & \mathbb{Z}[G]\\ {\scriptstyle\sigma\to -1}\downarrow & & \downarrow\\ \mathbb{Z}[G] & \longrightarrow & \mathbb{F}_2[G]\end{array}$$

is cartesian, and the top horizontal arrow is a split surjection. Thus

$$SK_1(\mathbb{Z}[G\times C_2])\approx SK_1(\mathbb{Z}[G])\oplus SK_1(\mathbb{Z}[G],(\sigma-1)).$$

Since the left vertical arrow induces an excision isomorphism [G-L, p. 182]

$$SK_1(\mathbb{Z}[G\times C_2],(\sigma-1))\xrightarrow{\approx} SK_1(\mathbb{Z}[G],2\mathbb{Z}[G]),$$

we see that

$$SK_1(\mathbb{Z}[G\times C_2])\approx SK_1(\mathbb{Z}[G])\oplus SK_1(\mathbb{Z}[G],2\mathbb{Z}[G]), \tag{11}$$

which is the case $m=1$ of the Theorem.

Suppose we can prove

$$SK_1(\mathbb{Z}[G\times C_2],(2^k))\approx SK_1(\mathbb{Z}[G],2^k))\oplus SK_1(\mathbb{Z}[G],(2^{k+1})) \tag{12}$$

for any finite abelian 2-group G, and any $k\geqq 0$. Then taking $H=G\times C_2$, we have, by induction on m,

$$\begin{aligned} SK_1(\mathbb{Z}[G\times C_2^m])&\approx SK_1(\mathbb{Z}[H\times C_2^{m-1}])\\ &\approx \prod_{k=0}^{m-1}\binom{m-1}{k} SK_1(\mathbb{Z}[H],(2^k))\\ &\approx \prod_{k=0}^{m-1}\binom{m-1}{k}(SK_1(\mathbb{Z}[G],(2^k))\oplus SK_1(\mathbb{Z}[G],(2^{k+1})))\\ &\approx \prod_{k=0}^{m}\binom{m}{k} SK_1(\mathbb{Z}[G],(2^k)) \end{aligned}$$

as desired. So it remains to prove (12) by induction on k. The case $k=0$ is (11) above.

Assume $k\geqq 1$, and let S_0 be an imaginary cluster of characters for G. Extend each $\chi\in S_0$ to $G\times C_2$ in two ways by

$$\chi_\pm(\sigma)=\pm 1,$$

and let $T_0=\{\chi_+,\chi_-\mid\chi\in S_0\}$. Then T_0 is an imaginary cluster of characters for $G\times C_2$. Define $f_i\colon \prod_{\chi\in T_0}\mathrm{Im}(\chi)\to\prod_{\chi\in S_0}\mathrm{Im}(\chi)$, $i=1,2$, for $\chi\in S_0$, $x\in\prod_{\chi\in T_0}\mathrm{Im}(\chi)$, by

$$\begin{aligned} (f_1(x))_\chi&=(x)_{\chi_+}\\ (f_2(x))_\chi&=[(x)_{\chi_-}][(x)_{\chi_+}^{-1}] \end{aligned}$$

Then $f_1\oplus f_2\colon \prod_{\chi\in T_0}\mathrm{Im}(\chi)\to\prod_{\chi\in S_0}\mathrm{Im}(\chi)$ is an isomorphism.

Let $\exp(G)=2^{n-1}$, so that by Theorem 1.10(a),

$$SK_1(\mathbb{Z}[G\times C_2],(2^k))\approx[\prod_{\chi\in T_0}\mathrm{Im}(\chi)]/\lambda_{T_0}\varphi_{T_0}(V),$$

where V is the subgroup of $K_2(\mathbb{Z}[G\times C_2]/(2^n),(2^k))$ generated by all $\{h,1+2^k y\}$, $h\in G\times C_2$, $y\in\mathbb{Z}[G\times C_2]/(2^n)$. Since

$$\langle 1+2^k y\mid y\in\mathbb{Z}[G\times C_2]/(2^n)\rangle$$

is also generated by all $1+2^k x$, $1+2^k(\sigma-1)x$, $x\in\mathbb{Z}[G]/(2^n)$, it follows that

$$V=V_1+V_2,$$

where $V_1=\langle\{g,1+2^k x\}\mid g\in G,\ x\in\mathbb{Z}[G]/(2^n)\rangle$ (these generators all come from $K_2(\mathbb{Z}[G]/(2^n),(2^k))$) and

$$V_2=\langle\{\sigma,1+2^k x\},\{g,1+2^k(\sigma-1)x\},\{\sigma,1+2^k(\sigma-1)x\}\rangle$$

with g, x as for V_1. Since $\chi_+(\sigma)=1$, it follows that $f_1\lambda_{T_0}\varphi_{T_0}(V_2)=1$; since $\chi_+(g)=\chi_-(g)=\chi(g)$ for $g\in G$, it follows that $f_2\lambda_{T_0}\varphi_{T_0}(V_1)=1$. Thus

$$\begin{aligned} SK_1(\mathbb{Z}[G\times C_2],(2^k)) &\approx [\prod_{\chi\in T_0}\mathrm{Im}(\chi)]/\lambda_{T_0}\varphi_{T_0}(V) \\ &\approx [\prod_{\chi\in T_0}\mathrm{Im}(\chi)]/f\lambda_{T_0}\varphi_{T_0}(V) \\ &\approx \prod_{\chi\in S_0}\mathrm{Im}(\chi)/f_1\lambda_{T_0}\varphi_{T_0}(V)\oplus\prod_{\chi\in S_0}\mathrm{Im}(\chi)/f_2\lambda_{T_0}\varphi_{T_0}(V). \end{aligned}$$

Now the above calculation shows that $f_1\lambda_{T_0}\varphi_{T_0}(V)=\lambda_{S_0}\varphi_{S_0}(V_1)$, so that

$$[\prod_{\chi\in S_0}\mathrm{Im}(\chi)]/f_1\lambda_{T_0}\varphi_{T_0}(V)\approx SK_1(\mathbb{Z}[G],2^k))$$

by Theorem 1.10(a). Moreover,

$$f_2\lambda_{T_0}\varphi_{T_0}(V)=f_2\lambda_{S_0}\varphi_{S_0}(V_2),$$

and, for $\chi\in S_0$, we have

$$\begin{aligned} f_2\lambda_\chi\varphi_\chi(\{\sigma,1+2^kx\}) &= (-1,\chi(1+2^kx))_v; \\ f_2\lambda_\chi\varphi_\chi(\{g,1+2^k(\sigma-1)x\}) &= (\chi(g),\chi(1-2^{k+1}x))_v; \\ f_2\lambda_\chi\varphi_\chi(\{\sigma,1+2^k(\sigma-1)x\}) &= (-1,\chi(1-2^{k+1}x))_v. \end{aligned}$$

So $f_2\lambda_{S_0}\varphi_{S_0}(V_2)=\lambda_{S_0}\varphi_{S_0}(\langle\{g,1+2^{k+1}x\}\rangle)$ (we can ignore $(-1,\chi(1-2^{k+1}x))_v$ by Lemma 1.6), and

$$[\prod_{\chi\in S_0}\mathrm{Im}(\chi)]/f_2\lambda_{T_0}\varphi_{T_0}(V)\approx SK_1(\mathbb{Z}[G],(2^{k+1}))$$

by Theorem 1.10(a). This completes the proof of (12) and the Theorem.

Section 2

The first known examples of groups G with $SK_1(\mathbb{Z}[G])\neq 1$ were the elementary abelian p-groups of rank $\geqq 3$ ($p\neq 2$); this result was announced in [A-D-S_0] and [St 2] and the proofs were given in [De] and [A]. In this section we will re-derive that result using the description of $SK_1(\mathbb{Z}[G])$ given in § 1.

Fix a prime p, not necessarily odd. It will be convenient when dealing with elementary abelian p-groups in this section to write the groups and their characters additively. Let $\mathbb{F}_p$ be a finite field with p elements, and let $A\approx\mathbb{F}_p^n$ be an elementary abelian p-group of rank n. Characters $\xi\in\hat{A}=\mathrm{Hom}_{\mathbb{F}_p}(A,\mathbb{F}_p)$ take their values in $\mathbb{F}_p$. Let $S^p(A)$ be the p-fold symmetric power of A [La, Chapter XVI, § 7, p. 427], i.e., the homogeneous part of degree p in the symmetric algebra $S(A)$. Define a homomorphism

$$\eta\colon S^p(A)\to\prod_{\xi\in\hat{A}}\mathbb{F}_p$$

by

$$(\eta(g_1 \otimes \ldots \otimes g_p))_\xi = \xi(g_1) \ldots \xi(g_p)$$

for any $g_1, \ldots, g_p \in A$ and $\xi \in \hat{A}$.

Lemma 2.1. *The homomorphism* $\eta: S^p(A) \to \prod_{\xi \in \hat{A}} \mathbb{F}_p$ *is injective.*

Proof. Fix a basis $a_1, \ldots, a_n$ of A over $\mathbb{F}_p$. Define $\tilde{\eta}: \mathbb{F}_p[x_1, \ldots, x_n] \to \prod_{\xi \in \hat{A}} \mathbb{F}_p$ by $(\tilde{\eta}(f))_\xi = f(\xi(a_1), \ldots, \xi(a_n))$. Clearly for $1 \leqq i_1, \ldots, i_p \leqq n$,

$$\eta(a_{i_1} \otimes \ldots \otimes a_{i_p}) = \tilde{\eta}(x_{i_1} \ldots x_{i_p}).$$

Since the elements $a_{i_1} \otimes \ldots \otimes a_{i_p}$ form a basis for $S^p(A)$, it follows that $\eta(S^p(A))$ is generated by

$$U = \{\tilde{\eta}(x_1^{m_1} \ldots x_n^{m_n}) \mid m_i \geqq 0, m_1 + \ldots + m_n = p\};$$

the injectivity of η will be established is we can show that U is linearly independent in $\prod_{\xi \in \hat{A}} \mathbb{F}_p$. For $\xi_0 \in \hat{A}$, define

$$f(x_1, \ldots x_n) = \prod_{i=1}^{n} \prod_{\substack{r=0 \\ r \neq \xi_0(a_i)}}^{p-1} (x_i - r) \in \mathbb{F}_p[x_1, \ldots, x_n].$$

Clearly

$$(\tilde{\eta}(f))_\xi = \begin{cases} 0 & \text{if } \xi \neq \xi_0, \\ (-1)^n & \text{if } \xi = \xi_0. \end{cases}$$

This shows that $\tilde{\eta}$ is surjective, and, in fact, that

$$B = \{\tilde{\eta}(x_1^{t_1} \ldots x_n^{t_n}) \mid 0 \leqq t_i \leqq p-1 \text{ for all } i\}$$

spans $\prod_{\xi \in \hat{A}} \mathbb{F}_p$. But then B must be a basis, by counting. Since $\tilde{\eta}(x_i^p) = \tilde{\eta}(x_i)$, it follows that the elements of U are linearly independent.

Now assume p is *odd*. Recall that in §1 we defined and proved certain properties of the map $\psi_T: G \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}_p[G] \to \prod_{\chi \in T} \mathrm{Im}(\chi)$. With our change to additive notation, we have, for any subset $T \subset \hat{A}$,

$$\psi_T: A \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}_p[A] \to \prod_{\xi \in T} \mathbb{F}_p$$

given by

$$\psi_T(a \otimes b))_\xi = \begin{cases} \xi(a) & \text{if } \xi(b) = 0, \\ 0 & \text{if } \xi(b) \neq 0. \end{cases}$$

Lemma 2.2. $\psi_{\hat{A}}(A \otimes \hat{\mathbf{Z}}_p[A]) = \eta(S^p(A))$.

Proof. Let $a, b \in A$, $\xi \in \hat{A}$. Then

$$(\eta(a \otimes b \otimes \ldots \otimes b))_\xi = \xi(a)\, \xi(b)^{p-1} = \begin{cases} \xi(a) & \text{if } \xi(b) \neq 0, \\ 0 & \text{if } \xi(b) = 0. \end{cases}$$

In particular, $(\eta(a\otimes\ldots\otimes a)_\xi=\xi(a)$. Hence

$$(\psi_{\hat{A}}(a\otimes b))_\xi=(\eta(a\otimes\ldots\otimes a))_\xi-(\eta(a\otimes b\otimes\ldots\otimes b))_\xi$$

which shows that $\psi_{\hat{A}}(A\otimes\hat{\mathbb{Z}}_p[A])\subset\eta(H)$, where

$$H=\langle a\otimes b\otimes\ldots\otimes b\in S^p(A)\,|\,a,b\in A\rangle.$$

Conversely, $\eta(a\otimes b\otimes\ldots\otimes b)=\psi_{\hat{A}}(a\otimes 1)-\psi_{\hat{A}}(a\otimes b)$, so $\eta(H)\subset\psi_{\hat{A}}(A\otimes\hat{\mathbb{Z}}_p[A])$. But according to [Bou, Chapitre III, §6, n°1, Remarque 3, p. 68], $S^{p-1}(A)$ is generated by all $b\otimes\ldots\otimes b$, $b\in A$, which shows that $H=S^p(A)$ and completes the proof of this lemma.

The proof of the next lemma follows easily from the definition of a cluster.

Lemma 2.3. *Let A be an elementary abelian p-group and let S be a cluster of characters for A. Then for each $\xi\in\hat{A}$, there is an r, $1\leqq r\leqq p-1$ such that $(r\xi)\in S$, the map*

$$\theta\colon\prod_{\xi\in S}\mathbb{F}_p\to\prod_{\xi\in\hat{A}}\mathbb{F}_p$$

defined by $(\theta(y))_\xi=\frac{1}{r}((y)_{(r\xi)})$ is an injective homomorphism, and

$$\theta\circ\psi_S=\psi_{\hat{A}}.$$

Theorem 2.4. *Let p be an odd prime and A an elementary abelian p-group of rank $n\geqq 1$. Then $SK_1(\mathbb{Z}[A])$ is itself an elementary abelian p-group, of rank $\frac{p^n-1}{p-1}-\binom{p+n-1}{p}$. In particular, $SK_1(\mathbb{Z}[A])\neq 1$ if and only if $n\geqq 3$.*

Proof. Let $S_0\subset\hat{A}$ be an imaginary cluster of characters for A. It is easy to see that $|S_0|=\frac{p^n-1}{p-1}$, so $\prod_{\xi\in S_0}\mathbb{F}_p$ is an elementary abelian p-group of rank $\frac{p^n-1}{p-1}$. By Theorem 1.8, $SK_1(\mathbb{Z}[A])\approx\operatorname{cok}(\psi_{S_0})$. Since the map ψ_0 corresponding to the trivial character is 0, we also have $\operatorname{cok}(\psi_{S_0})=\operatorname{cok}(\psi_S)$. Now

$$\begin{aligned}\psi_S(A\otimes\hat{\mathbb{Z}}_p[A])&=\theta(\psi_S(A\otimes\hat{\mathbb{Z}}_p[A])) && \text{(Lemma 2.3)}\\ &\approx\psi_{\hat{A}}(A\otimes\hat{\mathbb{Z}}_p[A]) && \text{(Lemma 2.3)}\\ &=\eta(S^p(A)) && \text{(Lemma 2.2)}\\ &\approx S^p(A). && \text{(Lemma 2.1)}\end{aligned}$$

But $S^p(A)$, when identified with its image in $\mathbb{F}_p[x_1,\ldots,x_n]$, has as a basis the monomials of degree exactly p, and there are $\binom{p+n-1}{n-1}=\binom{p+n-1}{p}$ of these. Thus $\operatorname{Im}(\psi_S)$ has rank $\binom{p+n-1}{p}$, and the Theorem is proved.

We isolate the following consequence of the proof of Theorem 2.2 which is useful in calculations for non-abelian groups (see [O IV], Lemma 5.4, Theorems 5.5 and 5.6).

Proposition 2.5. *Let p be an odd prime and $A=\mathbb{F}_p^n$ an elementary abelian p-group of rank $n\geqq 1$. Let $\mathcal{M}$ be the maximal order in $\mathbb{Q}[A]$. Then for any $k\geqq 2$, there is a natural isomorphism*

$$H=\mathrm{Im}(K_2(\mathbb{Z}[A]/(p^k))\to K_2(\mathcal{M}/(p^k)))\approx S^p(A).$$

Proof. Let S be a cluster of (additive) characters of A. Under the isomorphism $\mathbb{Q}[A]\xrightarrow{\approx}\prod_{\xi\in S}\mathbb{Q}[\xi]$, we may identify $\mathcal{M}$ with $\prod_{\xi\in S}\mathbb{Z}[\xi]$ (cf. the proof of Proposition 1.9), and $K_2(\mathcal{M}/(p^k))$ with $\prod_{\xi\in S}K_2(\mathbb{Z}[\xi]/(p^k))$. Then

$$\begin{aligned} H&\approx\varphi_S(K_2(\mathbb{Z}[A]/(p^k))\approx\lambda_S\varphi_S(K_2(\mathbb{Z}[A]/(p^k)))\\ &\approx\psi_S(A\otimes\hat{\mathbb{Z}}_p[A]) \qquad \text{(by Proposition 1.5)}\\ &=S^p(A)\end{aligned}$$

as in the proof of Theorem 2.4.

We now consider the case $p=2$, so that A is an elementary abelian 2-group of rank n. It has long been known that $SK_1(\mathbb{Z}[A])=1$ [B-M-S, Corollary 4.13] (this is, of course, an easy corollary of Theorem 1.8 as well). We will consider, therefore, the group rings $\mathcal{O}[A]$, where $\mathcal{O}$ is the ring of integers in a totally imaginary number field unramified at 2. We begin by considering more general 2-groups.

Theorem 2.6. *Let $\mathcal{O}$ be the ring of integers in a totally imaginary number field which is unramified at 2, and let G be a finite abelian 2-group (written multiplicatively). Then the quotient map $G\to G/G^2$ and the transfer $SK_1(\mathcal{O}[G])\to SK_1(\mathbb{Z}[G])$ induce an isomorphism*

$$SK_1(\mathcal{O}[G])\xrightarrow{\approx}SK_1(\mathbb{Z}[G])\oplus SK_1(\mathcal{O}[G/G^2]).$$

Proof. Choose k so that $\exp(G)\mid 2^{k-1}$, and let S be a cluster of characters for G. The diagram

$$\begin{array}{ccccc} K_2(\mathcal{O}[G]/(2^k)) & \xrightarrow{\varphi_{S,e}} & \prod_{\chi\in S}K_2(\mathcal{O}[\chi]/(2^k)) & \xrightarrow{(\partial)} & \prod_{\chi\in S}SK_1(\mathcal{O}[\chi],(2^k)) \\ \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle (t_2)} & & \\ K_2(\mathbb{Z}[G]/(2^k)) & \xrightarrow{\varphi_S} & \prod_{\chi\in S}K_2(\mathbb{Z}[\chi]/(2^k)) & & \end{array}$$

is commutative (cf. the proof of [A-D-S, Theorem 3.9]) and the transfer maps t_1 and t_2 are surjective [A-D-S, Proposition 3.8]. We claim that $\ker(\partial)=\ker(t_2)$. It clearly suffices to prove this for the restrictions of (∂) and (t_2) to a factor of the direct products for an arbitrary $\chi\in S$. That is, we must show that the kernels of the horizontal and vertical arrows coincide in the diagram

$$\begin{array}{ccc} K_2(\mathcal{O}[\chi]/(2^k)) & \xrightarrow{\partial} & SK_1(\mathcal{O}[\chi],(2^k)) \\ \downarrow{\scriptstyle t_2} & & \\ K_2(\mathbb{Z}[\chi]/(2^k)). & & \end{array} \tag{1}$$

If $|\mathrm{Im}\,\chi|>2$, we may complete (1) to a commutative square

$$\begin{array}{ccc} K_2(\mathcal{O}[\chi]/(2^k)) & \xrightarrow{\partial_{\mathcal{O}}} & SK_1(\mathcal{O}[\chi],(2^k)) \\ \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t} \\ K_2(\mathbb{Z}[\chi]/(2^k)) & \xrightarrow{\partial_{\mathbb{Z}}} & SK_1(\mathbb{Z}[\chi],(2^k)) \end{array}$$

in which $\partial_{\mathbb{Z}}$ and t are isomorphisms (since they are clearly surjections, and their orders coincide by [B-M-S, Corollary 4.3c] and [D-S, Theorem 4.3]).

If $\mathrm{Im}\,\chi\subset\{\pm 1\}$, $\mathbb{Z}[\chi]=\mathbb{Z}$ and $\mathcal{O}[\chi]=\mathcal{O}$; since 2 is unramified in $\mathcal{O}$, both $SK_1(\mathcal{O}[\chi],(2^k))$ and $K_2(\mathbb{Z}[\chi]/(2^k))$ have order 2. Moreover, the map

$$\prod_{\mathfrak{p}|2} K_2(\mathcal{O}[\chi]/\mathfrak{p}^k)\to K_2(\mathcal{O}[\chi]/(2^k))$$

is an isomorphism, and each composite map

$$K_2(\mathcal{O}[\chi]/\mathfrak{p}^k)\to K_2(\mathcal{O}[\chi]/(2^k))\to SK_1(\mathcal{O}[\chi],(2^k))$$
$$K_2(\mathcal{O}[\chi]/\mathfrak{p}^k)\to K_2(\mathcal{O}[\chi]/(2^k))\to K_2(\mathbb{Z}[\chi]/(2^k))$$

is an isomorphism [B-M-S, loc. cit.], [D-S, loc. cit.]. Since $\mathbb{Z}/2\mathbb{Z}$ has no non-trivial automorphisms, we may again complete (1) to a commutative square

$$\begin{array}{ccc} K_2(\mathcal{O}[\chi]/(2^k)) & \xrightarrow{\partial_{\mathcal{O}}} & SK_1(\mathcal{O}[\chi],(2^k)) \\ \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle \beta} \\ K_2(\mathbb{Z}[\chi]/(2^k)) & \xrightarrow[\alpha]{} & \mathbb{Z}/2\mathbb{Z} \end{array}$$

in which α and β are isomorphisms.

Thus in either case, an easy diagram chase shows that $\ker(\partial_{\mathcal{O}})=\ker(t_2)$, proving our claim. Hence it follows from [A-D-S, Theorem 2.9] that

$$\begin{aligned} SK_1(\mathcal{O}[G]) &\approx \mathrm{cok}(\partial\varphi_{S,\mathcal{O}}) \\ &\approx \mathrm{cok}(\varphi_S). \end{aligned}$$

Now let us write S as the disjoint union $S_0\cup S_1$, where S_0 is an imaginary cluster of characters for G and $S_1=\{\chi\in S\,|\,\mathbb{Z}[\chi]=\mathbb{Z}\}$, and let φ_i, $i=0,1$ be the composite

$$K_2(\mathbb{Z}[G]/(2^k))\xrightarrow{\varphi_S}\prod_{\chi\in S}K_2(\mathbb{Z}[\chi]/(2^k))\longrightarrow\prod_{\chi\in S_i}K_2(\mathbb{Z}[\chi]/(2^k)).$$

The Jacobson radical $J(\mathbb{Z}[G]/(2^k))$ is generated by $-2, g_1-1,\ldots,g_n-1$, where the g_i's generate G, and so

$$Y=\{1+(-2)^t(g_1-1)^{t_1}\ldots(g_n-1)^{t_n}\,|\,t,t_i\geqq 0\}$$

generates $(\mathbb{Z}[G]/(2^k))^*$. Let

$$\begin{aligned} Y_0&=\{-1,g_1,\ldots,g_n\},\\ Y_1&=\{1+(-2)^t(g_1-1)^{t_1}\ldots(g_n-1)^{t_n}\,|\,t+t_1+\ldots+t_n\geqq 2\}, \end{aligned}$$

so that Y is the disjoint union $Y_0 \cup Y_1$. If $\chi \in S_0$ and $u, v \in Y_0$, $\{\chi(u), \chi(v)\} = 1$ in $K_2(\mathbb{Z}[\chi]/(2^k))$, since $\chi(u)$, $\chi(v)$ are powers of some 2^r-th root of unity for $r \geqq 2$ (cf. [Ca, Lemma 1]).

Hence all Steinberg symbols of the form $\{u, v\}$, $u, v \in Y_0$, are contained in $\ker(\varphi_0)$.

If $u \in Y_0$, $w \in Y_1$, and $\chi \in S_1$, then $\chi(u) = \pm 1$ and $\chi(w) \equiv 1 \pmod 4$ in $\mathbb{Z}[\chi]/(2^k) = \mathbb{Z}/2^k\mathbb{Z}$. Since $K_2(\mathbb{Z}/2^k\mathbb{Z}) \xrightarrow{\approx} K_2(\mathbb{Z}/4\mathbb{Z})$ for $k \geqq 2$ [St 1, Prop. 3.3(b)], it follows that all Steinberg symbols of the form $\{u, w\}$, $u \in Y_0$, $w \in Y_1$, lie in $\ker(\varphi_1)$.

But Corollary 1.2 tells us that $K_2(\mathbb{Z}[G]/(2^k))$ is generated by all $\{u, x\}$, $u \in Y_0$, $x \in Y$, so

$$SK_1(\mathcal{O}[G]) \approx \operatorname{cok}(\varphi) \approx \operatorname{cok}(\varphi_0) \oplus \operatorname{cok}(\varphi_1) \approx SK_1(\mathbb{Z}[G]) \oplus \operatorname{cok}(\varphi_1).$$

Finally, S_1 contains precisely those characters of G which factor through, and thus form a cluster of characters for, G/G^2, and

$$K_2(\mathbb{Z}[G]/(2^k)) \to K_2(\mathbb{Z}[G/G^2]/(2^k))$$

is onto by [A-D-S, Proposition 1.1]. Hence $\operatorname{cok}(\varphi_1) \approx SK_1(\mathcal{O}[G/G^2])$, which completes the proof of this Theorem.

Theorem 2.7. *Let $\mathcal{O}$ be the ring of integers in a totally imaginary number field which is unramified at 2 and let A be an elementary abelian 2-group of rank $n \geqq 1$. Then $SK_1(\mathcal{O}[A])$ is an elementary abelian 2-group of rank $2^n - 1 - n - \binom{n}{2}$.*

Proof. We will apply Theorem 2.6. Let S be a cluster of characters for A; in the notation of Theorem 2.6, $S = S_1 = \hat{A}$, and

$$\prod_{\xi \in \hat{A}} K_2(\mathbb{Z}[\xi]/(2^k)) \xrightarrow{\approx} (\mathbb{Z}/2)^{2^n}.$$

We know that

$$SK_1(\mathbb{Z}[A]) \approx \operatorname{cok}(\varphi_S \colon K_2(\mathbb{Z}[A]/(2^k)) \to \prod_{\xi \in \hat{A}} K_2(\mathbb{Z}[\xi]/(2^k)))$$

and that $\operatorname{Im}(\varphi_S) = \operatorname{Im}(\varphi_{S_1})$ is generated by all $\varphi_S(\{u, v\})$, $u, v \in Y_0 = \{-1, a_1, \ldots, a_n\}$, where the a_i are a minimal generating set (= basis over $\mathbb{F}_2$) for A. Since $\{-1, a_i\} = \{a_i, a_i\}$, $\operatorname{Im}\varphi_S$ is generated by all $\varphi_S(\{-1, -1\})$, $\varphi_S(\{a_i, a_j\})$, $1 \leqq i \leqq j \leqq n$. The composite

$$S^2(A) \xrightarrow{i} K_2(\mathbb{Z}[A]/(2^k)) \longrightarrow \prod_{\xi \in \hat{A}} K_2(\mathbb{Z}[\xi]/(2^k)) \longrightarrow \prod_{\xi \in \hat{A}} \mathbb{F}_2,$$

where $i(g \otimes h) = \{g, h\}$ is just the map η of Lemma 2.1, and is therefore injective. This shows that the elements $\varphi(\{a_i, a_j\})$ are linearly independent over $\mathbb{F}_2$. Clearly $\varphi(\{-1, -1\})$ is independent of them, since it is the only symbol not killed when $\chi = 1$. Thus $\dim(\operatorname{Im}(\varphi_S)) = 1 + n + \binom{n}{2}$, which completes the proof of this Theorem.

If we combine Theorem 2.6 with [A-D-S, Theorem 3.11], we can completely reduce the calculation of $SK_1(\mathbb{Z}[G])$ for finite abelian groups to the case of p-groups.

Corollary 2.8. *Let G be a finite abelian group and let $\{G_p\}$ be its set of Sylow p-subgroups. Let $q(G/G_p)$ denote the number of irreducible rational representations of G/G_p, and let n be the 2-rank of G_2. Then*

$$SK_1(\mathbb{Z}[G]) \xrightarrow{\approx} (\mathbb{Z}/2\mathbb{Z})^{[q(G/G_2)-1]\left[2^n-1-n-\binom{n}{2}\right]} \times \prod_{p\mid|G|} SK_1(\mathbb{Z}[G_p])^{q(G/G_p)}.$$

Section 3

In this section, we prove that $SK_1(\mathcal{O}[G])$ vanishes when G is any finite cyclic group and $\mathcal{O}$ is the ring of integers in *any* algebraic number field (thus removing the hypotheses on ramification of [A-D-S, Theorem 3.4]). This, combined with results in [O III], allows us to characterize metacyclic groups as precisely those finite groups for which SK_1 vanishes over all rings of algebraic integers.

The following is the main technical result needed to handle the necessary symbol calculations.

Lemma 3.1. *Fix a prime p and an integer $k\geqq 1$. Let K be any finite ramified extension of $\hat{\mathbb{Q}}_p[\zeta_{p^k}]$ which contains $p^{(1/p^{k-1})}$. Let $A\subseteq K$ be its valuation ring, let $v\colon K^*\to\mathbb{Z}$ be the valuation, and let $e=v(p)$ be the ramification index of p. Then, for any $x\in A$ satisfying*

$$v(x)=\frac{e}{p^{k-1}(p-1)}-1,$$

$1-x^{p^k}$ is not a p-th power in $A/p(1-\zeta_p)A$, where $\zeta_p=(\zeta_{p^k})^{p^{k-1}}$.

Proof. Set $\zeta=\zeta_p\in A$, and let $\mathfrak{p}\subseteq A$ be the maximal ideal. If $k>1$, assume inductively that the lemma holds for $k-1$. Note that

$$e\geqq 2\cdot e(\hat{\mathbb{Q}}_p[\zeta_{p^n}])=2p^{k-1}(p-1)\geqq p^k$$

by assumption. In particular, $x\in\mathfrak{p}$. Furthermore,

$$v(p\,x^{p^k})=e+p\,e/(p-1)-p^k\geqq p\,e/(p-1),$$
$$v(x^{2p^k})=2p\,e/(p-1)-2p^k\geqq p\,e/(p-1)+2p^k-2p^k=p\,e/(p-1);$$

and hence

$$p\,x^{p^k}, x^{2p^k}\in\mathfrak{p}^{pe/(p-1)}=p(1-\zeta)A.$$

Thus, if $1-x^{p^k}$ is a p-th power $\bmod p(1-\zeta)A$, then so is

$$(1+x^{p^{k-1}})^p(1-x^{p^k})\equiv 1+p\left[x+\frac{p-1}{2}x^2+\ldots+x^{p-1}\right]^{p^{k+1}}$$
$$\equiv 1-y^{p^{k-1}} \bmod p(1-\zeta)$$

where

$$y=(-1)^p\,p^{(1/p^{k-1})}\left[x+\frac{p-1}{2}x^2+\ldots+x^{p-1}\right].$$

In particular,

$$v(y)=e/p^{k-1}+v(x)=[e/(p^{k-2}(p-1))]-1.$$

If $k>1$, this contradicts our induction hypothesis. If $k=1$, then there exists z such that

$$1-y\equiv(1+z)^p=1+p\,z+\ldots+z^p \bmod p(1-\zeta)\,A=\mathfrak{p}^{pe/(p-1)}.$$

Since $p\nmid v(y)$, we must have

$$v(y)=v(p\,z)<v(z^p).$$

But this is impossible, since it would imply

$$v(z^p)=p\,v(z)=p(v(y)-e)=p\,e/(p-1)-p<v(y).$$

The next lemma will be needed to show that $SK_1(\mathcal{O}[C_n])$ vanishes for arbitrary $\mathcal{O}$, if it vanishes when $\mathcal{O}$ contains the n-th roots of unity.

Lemma 3.2. *If G is any abelian group, and if $K_2\supseteq K_1$ are algebraic number fields with rings of integers $\mathcal{O}_2\supseteq\mathcal{O}_1$, then the transfer map*

$$t: SK_1(\mathcal{O}_2[G])\to SK_1(\mathcal{O}_1[G])$$

is surjective.

Proof. This can be proved directly using the results in Sect. 2 of [A-D-S], but such a proof is fairly complicated. It seems simpler therefore to deduce this lemma as a case of [O I, Proposition 11].

As in [O I], for any finite group π and any algebraic number field K with ring of integers $\mathcal{O}\subseteq K$, we set

$$\mathrm{Cl}_1(\mathcal{O}[\pi])=\ker[l: SK_1(\mathcal{O}[\pi])\to\prod_p SK_1(\hat{\mathcal{O}}_p[\pi])].$$

Here, the product is taken over all rational primes p, $\hat{\mathcal{O}}_p=\hat{\mathbf{Z}}_p\otimes_{\mathbf{Z}}\mathcal{O}$, and l is the localization homomorphism (which is a surjection via localization sequences). The product is, in fact, finite: $SK_1(\hat{\mathcal{O}}_p[\pi])=0$ whenever $p\nmid|\pi|$.

Since matrices over $\hat{\mathcal{O}}_p[\pi]$ can be diagonalized ($\mathcal{O}_p[\pi]$ is semilocal), $SK_1(\hat{\mathcal{O}}_p[\pi])=0$ whenever π is abelian ([W, Prop. 1.3]; see also [O II, Theorem 3]). In particular, $SK_1(\mathbf{Z}[\pi])=\mathrm{Cl}_1(\mathbf{Z}[\pi])$ for any abelian π. The lemma now follows from [O I, Proposition 11], which says that the transfer homomorphism

$$t: \mathrm{Cl}_1(\mathcal{O}_2[\pi])\to\mathrm{Cl}_1(\mathcal{O}_1[\pi])$$

is onto for any finite π and any pair $\mathcal{O}_2\supseteq\mathcal{O}_1$ of rings of integers in algebraic number fields.

The main result of this section can now be proved.

Theorem 3.3. *Let $\mathcal{O}$ be the ring of integers in an algebraic number field, and let C_n ($n\geqq 1$) be a cyclic group of order n. Then $SK_1(\mathcal{O}[C_n])=1$.*

Proof. By Lemma 3.2, it suffices to prove this when $\mathcal{O}$ contains the n-th roots of unity.

Fix a prime p, and write $n=p^k m$, where $p\nmid m$. Then $C_n\approx C_m\times C_{p^k}$. By [A-D-S, Proposition 3.1],

$$SK_1(\mathcal{O}[C_n])_p\cong SK_1(\mathcal{M}[C_{p^k}]),$$

where $\mathcal{M}=\prod_{\chi\in\hat{C}_m}\mathcal{O}[\chi]=\prod_{\chi}\mathcal{O}$ (since $\mathrm{Im}(\chi)\subseteq\mathcal{O}$).

We are thus reduced to proving that $SK_1(\mathcal{O}[C_{p^k}])=1$ when $\mathcal{O}$ is the ring of integers in a number field K containing the p^k-th roots of unity. Finally, using Lemma 3.2 again, it suffices to prove this under the assumption that

$$\zeta_{p^{k+1}}, p^{(1/p^{k-1})}\in K. \tag{1}$$

To simplify our notation, we write $G=C_{p^k}$.

Step 1. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_r\subseteq\mathcal{O}$ be the primes dividing p, and set $A_i=\hat{\mathcal{O}}_{\mathfrak{p}_i}$ for each i. By [A-D-S, Theorem 2.9],

$$SK_1[\mathcal{O}[G])\cong\mathrm{cok}\Big(\prod_i K_2(A_i[G]/(p^k))\xrightarrow{\prod\varphi_{i,\chi}}\prod_{i,\chi}K_2(A_i/(p^k))\xrightarrow{\partial_{\mathcal{O}}}\prod_{\chi}SK_1(\mathcal{O},p^k\mathcal{O})\Big). \tag{2}$$

Here, for each $1\leqq i\leqq r$ and each $\chi\in\hat{G}$, $\varphi_{i,\chi}$ is induced by $\chi\colon A_i[G]\to A_i$. The boundary map $\partial_{\mathcal{O}}$ is the product (over $\chi\in\hat{G}$) of the boundary maps

$$\partial\colon\prod_i K_2(A_i/(p^k))=K_2(\mathcal{O}/(p^k))\to SK_1(\mathcal{O},p^k\mathcal{O}).$$

In particular, it will suffice to show that for each i, $\prod_{\chi}\varphi_{i,\chi}$ is onto.

Step 2. Fix i, and set $A=A_i$ for simplicity. Let F be the field of fractions of A ($F=\hat{K}_{\mathfrak{p}_i}\supseteq\hat{\mathbf{Q}}_p$). By (1),

$$\zeta_{p^{k+1}}, p^{(1/p^{k-1})}\in F.$$

In particular, $F/\hat{\mathbf{Q}}_p[\zeta_{p^k}]$ is a ramified extension.

Using Lemma 3.1, we can now fix $x\in A$ such that $w=1-x^{p^k}$ is not a p-th power in $A/p(1-\zeta_p)A$. The extension $F(w^{1/p})/F$ is ramified [He, Satz 119]; and so there is some $u\in A^*$ which is not a norm from $F(w^{1/p})$ [La, Lemma 4, p. 188]. Thus, by [Ser, Chapitre XIV, Proposition 7(iii)],

$$(u,w)_r \text{ generates } (\mu_F)_p.$$

(Here $(\mu_F)_p$ denotes the group of p-power roots of unity.) It now follows from [D-S, Theorem 4.3] that

$$\{u,w\}=\prod_{i=0}^{p^k-1}\{u,1-(\zeta_{p^k})^i x\} \text{ generates } K_2(A/(p^k))\cong\langle\zeta_{p^{k-1}}\rangle. \tag{3}$$

Step 3. Fix a generator $g\in G=C_{p^k}$, and set $\zeta=\zeta_{p^k}$. For any $i\in\mathbf{Z}$, set

$$w_i=1-\zeta^i x g\in(A[G])^*$$

(x has positive valuation by construction). For each $j \in \mathbb{Z}$, let $\chi_j \in \hat{G}$ denote the character given by $\chi_j(g) = \zeta^j$. Then for any i and j,

$$\varphi_{\chi_j}(\{u, w_i\}) = \{u, 1 - \zeta^{i+j} x\} \in K_2(A/(p^k)).$$

So if we set

$$\varphi = \prod_j \varphi_{\chi_j} : K_2(A[G]/(p^k)) \to \prod_{\chi} K_2(A/(p^k)) = \prod_{j=0}^{p^k-1} K_2(A/(p^k)),$$

then for any i

$$\varphi(\{u, w_i\}) = (\{u, 1 - \zeta^{i+j} x\})_{j=0}^{p^k-1}. \tag{4}$$

Step 4. We now claim that $\prod_{j=0}^{p^k-1} K_2(A/(p^k))$ is generated by the elements $\varphi(\{u, w_i\})$. The key idea here is to use the fact that $\mathbb{Z}[G]/(p^{k-1})$ is a local ring (recall that $G = C_{p^k}$). By [D-S, Theorem 4.3], $K_2(A/(p^k)) \approx \mathbb{Z}/p^{k-1}\mathbb{Z}$. The group

$$M = \prod_{\chi \in \hat{G}} K_2(A/(p^k)) \approx \prod_{j=0}^{p^k-1} K_2(A/(p^k))$$

is thus a free $\mathbb{Z}[G]/(p^{k-1})$-module of rank 1; and by (4) the elements $\varphi(\{u, w_i\})$ generate a submodule of M. The maximal ideal, J, of $\mathbb{Z}[G]/(p^{k-1})$, is generated by p and $1-g$. In particular, JM has index p in M, and so any proper submodule of M lies in JM. But by (3) and (4), $\varphi(\{u, w_i\}) \notin JM$ (for any i), and so by Nakayama's Lemma,

$$M = \prod_{\chi \in \hat{G}} K_2(A/(p^{k-1})) = \langle \varphi(\{u, w_i\}) \mid i \in \mathbb{Z} \rangle.$$

In particular, it follows that

$$\varphi : K_2(A[G]/(p^{k-1})) \to \prod_{\chi \in \hat{G}} K_2(A/(p^{k-1}))$$

is surjective. Since we took $A = A_i$ for arbitrary i, and $\hat{\mathcal{O}}_p = \prod A_i$, this implies that (in the notation of (2))

$$\prod \varphi_{i,\chi} : \prod_i K_2(A_i[G]/(p^k)) \to \prod_{i,\chi} K_2(A_i/(p^k))$$

is surjective. Thus, (2) implies that $SK_1(\mathcal{O}[C_{p^k}]) = 1$; and this proves the theorem.

If G is a metacyclic group (i.e., if all Sylow subgroups of G are cyclic), it follows from Theorem 3.3 and [O III, Theorem 3.3] that $SK_1(\mathcal{O}[G]) = 1$ whenever $\mathcal{O}$ is the ring of integers in an algebraic number field. We will next show that the metacyclic groups are, in fact, the only finite groups with this property. Two preliminary lemmas will be needed.

Lemma 3.4. *Let p be a prime, and let C_p be a cyclic group of order p. Let*

$$\mathcal{O} = \begin{cases} \mathbb{Z}[i] & \text{if } p = 2, \\ \mathbb{Z}[\zeta_p] & \text{otherwise.} \end{cases}$$

Then for any $k \geqq 2$, there is a surjection

$$\rho\colon SK_1(\mathcal{O}[(C_p)^k]) \to \mathbf{Z}/p\mathbf{Z}$$

such that

(i) *for any $\alpha \in \mathrm{Aut}((C_p)^k)$, $\rho\,\alpha_* = \rho$;*
(ii) *for any subgroup $K \subsetneqq (C_p)^k$, the composite $SK_1(\mathcal{O}[K]) \to SK_1\,\mathcal{O}[(C_p)^k] \xrightarrow{\rho} \mathbf{Z}/p\mathbf{Z}$ is zero.*

Proof. Fix $k \geqq 2$, write $H = (C_p)^k$, and set

$$\zeta = \begin{cases} i & \text{if } p = 2 \\ \zeta_p & \text{otherwise.} \end{cases}$$

Let $G = \langle\sigma\rangle \times H$, where $|\sigma| = |\zeta|$ (i.e., $|\sigma| = 4$ or p). Then

$$\alpha\colon \mathbf{Z}[G] \to \mathcal{O}[H] \qquad (\alpha(\sigma) = \zeta)$$

induces a homomorphism

$$\alpha_*\colon K_2(\mathbf{Z}[G]/(p^3)) \to K_2(\mathcal{O}[H]/(p^3));$$

which is surjective by [A-D-S, Proposition 1.1].

There are homomorphisms

$$\begin{array}{ccccc} K_2(\mathcal{O}[H]/(p^3)) & \xrightarrow{\varphi_H} & \prod_{\chi \in \hat{H}} K_2(\mathcal{O}/(p^3)) & \xrightarrow{\partial_H} & \prod_{\chi \in \hat{H}} SK_1(\mathcal{O}, p^3\mathcal{O}) \\ & & & \searrow^{\varepsilon} \quad \swarrow_{s} & \\ \mathbf{Z}/p\mathbf{Z} \approx \langle\zeta\rangle/\langle\zeta^p\rangle \approx \mathbf{Z}/p\mathbf{Z} \otimes SK_1(\mathcal{O}, p^3\mathcal{O}) & & \xleftarrow{\rho} & & SK_1(\mathcal{O}[H]) \end{array} \tag{5}$$

Here, φ_H is induced by the map $\mathcal{O}[H] \to \prod_{\chi} \mathcal{O}$, ∂_H is the product of the boundary maps, ε is the sum of the projections, and s is induced by the isomorphism $SK_1(\mathcal{O}[H]) \approx \mathrm{cok}(\partial_H \circ \varphi_H)$ of [A-D-S, Theorem 2.9]. In particular, s is onto and $\ker(s) = \mathrm{im}(\partial_H \circ \varphi_H)$.

The idea now is to prove that $\varepsilon \circ \partial_H \circ \varphi_H = 0$. Once this is shown, then ε factors through the required epimorphism ρ on $SK_1(\mathcal{O}[H])$, and the desired properties of ρ can be checked.

Step 1. For each $\chi \in \hat{H}$, define $\bar{\chi} \in \hat{G}$ by setting

$$\bar{\chi}(h) = \chi(h) \quad \text{if } h \in H, \qquad \bar{\chi}(\sigma) = \zeta.$$

Set $T = \{\bar{\chi} \mid \chi \in \hat{H}\} \subseteq \hat{G}$. The composite

$$K_2(\mathbf{Z}[G]/(p^3)) \xrightarrow{\alpha_*} K_2(\mathcal{O}[H]/(p^3)) \xrightarrow{\partial_H \circ \varphi_H} \prod_{\chi \in \hat{H}} SK_1(\mathcal{O}, p^3\mathcal{O}) \approx \prod_{\chi} \langle\zeta\rangle$$

is just the homomorphism $\lambda_T \circ \varphi_T$ of Proposition 1.5 (p odd) or 1.7 ($p = 2$). Thus, by 1.5 or 1.7,

$$\mathrm{Im}(\partial_H \circ \varphi_H) = \mathrm{Im}(\partial_H \circ \varphi_H \circ \alpha_*) \subseteq \mathrm{Im}(\psi_T) \qquad (\mathrm{mod}\,\{\pm 1\}) \tag{6}$$

where

$$\psi_T = \prod \psi_{\bar{\chi}} : G \otimes \mathbf{Z}[G] \to \prod_{\chi \in \hat{H}} \langle \zeta \rangle,$$

and for any $\chi \in \hat{H}$,

$$\psi_{\bar{\chi}}(g \otimes h) = \begin{cases} \bar{\chi}(g) & \text{if } \bar{\chi}(h) = 1, \\ 1 & \text{if } \bar{\chi}(h) \neq 1. \end{cases}$$

For each $h \in G$, set

$$X(h) = \{\chi \in \hat{H} \mid \bar{\chi}(h) = 1\} \subseteq \hat{H}.$$

Then for any h, $X(h) = \hat{H}$, or is a coset of an index p subgroup of $\hat{H}$. Since $H \approx (C_p)^k$ with $k \geqq 2$, it follows that for any $g, h \in G$

$$\prod_{\chi \in \hat{H}} \psi_{\bar{\chi}}(g \otimes h) = \prod_{\chi \in X(h)} \bar{\chi}(g) \in \{\pm 1\}.$$

So by (6) and diagram (5),

$$\varepsilon \circ \partial_H \circ \varphi_H = 0. \tag{7}$$

Step 2. We have now shown that, in the notation of (5),

$$\mathrm{Ker}(s) = \mathrm{Im}(\partial_H \circ \varphi_H) \subseteq \mathrm{Ker}(\varepsilon).$$

In other words, ε factors through an epimorphism

$$\rho : SK_1(\mathcal{O}[H]) \to SK_1(\mathcal{O}, p^3\mathcal{O}) \otimes \mathbf{Z}/p\mathbf{Z} \approx \mathbf{Z}/p\mathbf{Z}$$

Furthermore, any automorphism of H acts on

$$\prod_{\chi \in \hat{H}} SK_1(\mathcal{O}, p^3\mathcal{O})$$

only by permuting the factors, and so the induced map on $SK_1(\mathcal{O}[H])$ commutes with ρ by construction.

Finally, if $K \subsetneq H$ is any proper subgroup, then there is a commutative diagram

$$\begin{array}{ccccc} \prod_{\chi \in \hat{K}} \langle \zeta \rangle & \xrightarrow{\text{inc}} & \prod_{\chi \in \hat{H}} \langle \zeta \rangle & \xrightarrow{\text{sum}} & \langle \zeta \rangle / \langle \zeta^p \rangle \\ \downarrow{\scriptstyle \partial_K} & & \downarrow{\scriptstyle \partial_H} & & \downarrow{\scriptstyle \cong} \\ SK_1(\mathcal{O}[K]) & \xrightarrow{\text{ind}} & SK_1(\mathcal{O}[H]) & \xrightarrow{\rho} & \mathbf{Z}/p\mathbf{Z} \end{array} \tag{8}$$

(recall that $\zeta = i$ or ζ_p). Here, for any $y \in \prod_{\chi \in \hat{K}} \langle \zeta \rangle$,

$$(\mathrm{inc}(y))_\chi = (y)_{\chi|K}$$

for all $\chi \in \hat{K}$. The composite of the top row in (8) is thus zero; and so $\rho \circ \mathrm{ind} = 0$.

By [Mag, Theorem 1], a surjection $G \to G'$ of finite groups, where G' is abelian, induces a surjection $SK_1(\mathcal{O}[G]) \to SK_1(\mathcal{O}[G'])$. It thus follows easily

from Lemma 3.4 that $SK_1(\mathcal{O}[G]) \neq 1$ when $\mathcal{O} = \mathbf{Z}[\zeta_p]$ (resp., $\mathbf{Z}[i]$), and G is any non-cyclic p-group for p odd (resp. $p=2$). We will now extend this result to arbitrary finite groups.

Theorem 3.5. *Let p be a prime, and let $\mathcal{O}$ be the ring of integers in an algebraic number field which contains a primitive p-th (resp., 4-th) root of unity if p is odd (resp., $p=2$). Let G be a finite group. Then $SK(\mathcal{O}[G])_p = 1$ if and only if the Sylow p-subgroups of G are cyclic.*

Proof. Let G_p be a Sylow p-subgroup of G. If G_p is cyclic, so is any p-elementary subgroup of G. But $SK_1(\mathcal{O}[G])_p$ is generated by induction from such subgroups [O III, Theorem 3, p. 327]; hence $SK_1(\mathcal{O}[G])_p = 1$ by Theorem 3.3.

Now assume G_p is not cyclic. By [O I, Proposition 11], it will suffice to show that $\mathrm{Cl}_1(\mathcal{O}[G])_p \neq 1$ when $\mathcal{O} = \mathbf{Z}[\zeta_p]$ (resp. $\mathcal{O} = \mathbf{Z}[i]$ if $p=2$). Let $F \subset G_p$ be the Frattini subgroup [G, Chapter 5, § 1]; G_p/F is an elementary abelian p-group and the image of any proper subgroup of G_p in G_p/F is a *proper* subgroup. The induced map $\mathrm{Cl}_1(\mathcal{O}[G_p]) \to \mathrm{Cl}_1(\mathcal{O}[G_p/F])$ is surjective [O I, Lemma 1], and $\mathrm{Cl}_1(\mathcal{O}[G_p/F]) = SK_1(\mathcal{O}[G_p/F])$ since G_p/F is abelian (see the proof of Lemma 3.2). Hence the composite map $\tilde{\rho}$: $\mathrm{Cl}_1(\mathcal{O}[G_p]) \to \mathbf{Z}/p\mathbf{Z}$ defined by

$$\mathrm{Cl}_1(\mathcal{O}[G_p]) \to \mathrm{Cl}_1(\mathcal{O}[G_p/F]) = SK_1(\mathcal{O}[G_p/F]) \xrightarrow{\rho} \mathbf{Z}/p\mathbf{Z},$$

where ρ is the homomorphism constructed in Lemma 3.4, is also surjective.

If H is a p-hyperelementary subgroup of G, define ρ_H: $\mathrm{Cl}(\mathcal{O}[H]) \to \mathbf{Z}/p\mathbf{Z}$ as follows. Choose a Sylow p-subgroup H_p of H. If H_p is conjugate to a *proper* subgroup of G_p, let $\rho_H = 0$. If H_p is conjugate to G_p itself, let ρ_H be the composite

$$\mathrm{Cl}_1(\mathcal{O}[H]) \to \mathrm{Cl}_1(\mathcal{O}[H_p]) \xrightarrow{\approx} \mathrm{Cl}_1(\mathcal{O}[G_p]) \xrightarrow{\tilde{\rho}} \mathbf{Z}/p\mathbf{Z}$$

where the first map is induced by the projection $H \to H_p$, and the second is induced by conjugation. Since $K \subsetneqq G_p$ implies $K/F \subsetneqq G_p/F$, it follows from (i) and (ii) of Lemma 3.4 that the maps ρ_H are consistent with inclusion and conjugation on the set $\mathscr{H}$ of p-hyperelementary subgroups of G. Thus they induce a surjection

$$\varinjlim_{H \in \mathscr{H}} \mathrm{Cl}_1(\mathcal{O}[H]) \to \mathbf{Z}/p\mathbf{Z}.$$

But it follows from [Dr, Propositions 1.1′ and 1.2] and [O III, Proposition 1(ii), p. 302] that

$$\mathrm{Cl}_1(\mathcal{O}[G])_p \approx \varinjlim_{H \in \mathscr{H}} \mathrm{Cl}_1(\mathcal{O}[H]),$$

which shows that $\mathrm{Cl}_1(\mathcal{O}[G])_p \neq 1$ and completes the proof of the Theorem.

Theorem 3.6. *Let G be a finite group. The following statements are equivalent:*

(a) *For every algebraic number field F with ring of integers $\mathcal{O}$, $SK_1(\mathcal{O}[G]) = 1$.*

(b) *G is metacyclic.*

Section 4

The exponent of $SK_1(\mathbb{Z}[G])$ for an arbitrary finite abelian group G will now be computed, determining in particular exactly when $SK_1(\mathbb{Z}[G])=1$. By Corollary 2.8 it suffices to make this computation when G is a p-group. An upper bound for $\exp(SK_1(\mathbb{Z}[G]))$ in this case is obtained from Theorem 3.3 and [A-D-S, Theorem 1.3]. Explicit calculation gives a lower bound for this exponent. These bounds agree except for a small class of 2-groups for which some extra explicit calculation is necessary.

To obtain upper bounds for $\exp(SK_1(\mathbb{Z}[G]))$, it suffices to compute the exponents of certain K_2's of which $SK_1(\mathbb{Z}[G])$ is a quotient.

Lemma 4.1. *Fix a (rational) prime p and integers $m,n\geqq 1$ and $1\leqq k\leqq m$. Let ζ_{p^k} be a primitive p^k-th root of unity and let $\zeta_p=(\zeta_{p^k})^{p^{k-1}}$. Then*

(a) $\exp(K_2(\mathbb{Z}[C_{p^n}]/(p^m)))$ *divides* p^{m-1}.

(b) $\exp\left(K_2\left(\dfrac{\mathbb{Z}[\zeta_{p^k}]}{p^{m-k}(1-\zeta_p)}[C_{p^n}]\right)\right)$ *divides* p^{m-1}.

Proof. Let $\pi=1-\zeta_{p^k}$; π generates the unique maximal ideal of $\mathbb{Z}[\zeta_{p^k}]$ lying over (p). Then

$$\frac{\mathbb{Z}[\zeta_{p^k}]}{p^{m-k}(1-\zeta_p)}[C_{p^n}]/(\pi)\approx\mathbb{F}_p[C_{p^n}],$$

and $K_2(\mathbb{F}_p[C_{p^n}])=1$ by [D-S, Theorem 3.4]. Thus it suffices in (b) to prove

$$\exp\left(K_2\left(\frac{\mathbb{Z}[\zeta_{p^k}]}{p^{m-k}(1-\zeta_p)}[C_{p^n}]\right),(\pi)\right)\leqq p^{m-1}. \tag{1}$$

Similarly in (a) it suffices to prove

$$\exp(K_2(\mathbb{Z}[C_{p^n}]/(p^m),(p))\leqq p^{m-1}. \tag{2}$$

Each of these relative groups is generated by Steinberg symbols of the form $\{u,v\}$, where $v\equiv 1$ modulo the ideal in question [St 1, Theorem 2.5], so it suffices to prove

$$(1+px)^{p^{m-1}}\equiv 1\,(\mathrm{mod}\,p^m)\quad\text{for } x\in\mathbb{Z}[C_{p^n}] \tag{3}$$

and

$$(1+\pi x)^{p^{m-1}}\equiv 1\,(\mathrm{mod}\,p^{m-k}(1-\zeta_p))\quad\text{for } x\in\mathbb{Z}[\zeta_{p^k}][C_{p^n}]. \tag{4}$$

If we keep in mind that $(p)=(\pi)^{p^{k-1}(p-1)}$ and $(1-\zeta_p)=(\pi)^{p^{k-1}}$, these statements follow easily by induction on m (in (4) the induction begins at $m=k$).

Proposition 4.2. *Let G be a noncyclic abelian p-group. Then*

$$\exp(SK_1(\mathbb{Z}[G]))\quad\text{divides}\quad\frac{|G|}{p\cdot\exp(G)}.$$

Proof. Set $\exp(G)=p^n$ and write $G=C_{p^n}\times H$, where $|H|=p^m$ for some $m\geqq 1$. With this notation we must show that

$$\exp(SK_1(\mathbf{Z}[G])) \quad \text{divides } p^{m-1}.$$

Let $\mathcal{M} = \prod_{i=0}^{q} \mathbf{Z}[\chi_i]$ be the maximal order of $\mathbf{Q}[H]$ containing $\mathbf{Z}[H]$, where $\{\chi_1, \ldots, \chi_q\}$ is a cluster of characters for H [B-M, Proposition 8.6], and let c be the conductor from $\mathcal{M}$ to $\mathbf{Z}[H]$. As an ideal of $\mathcal{M}$, we can write $c = \prod_{i=0}^{q} c_i$ as a direct product of ideals $c_i \subset \mathbf{Z}[\chi_i]$, where

$$\begin{aligned} c_i &= (p^m) && \text{if } \chi_i = 1 \\ c_i &= (p^{m-k}(1-\zeta_{p^k})^{p^{k-1}}) = (p^{m-k}(1-\zeta_p)) && \text{if } |\operatorname{Im}\chi_i| = p^k \end{aligned}$$

Hence $\mathcal{M}/c$ is the product of $\mathbf{Z}/p^m\mathbf{Z}$ and rings of the form $\mathbf{Z}[\zeta_{p^k}]/p^{m-k}(1-\zeta_p)$ for $1 \leqq k \leqq m$. Hence, by Lemma 4.1,

$$\exp(K_2(\mathcal{M}[C_{p^n}]/c\mathcal{M}[C_{p^n}])) \quad \text{divides } p^{m-1}.$$

Now the square

$$\begin{array}{ccc} \mathbf{Z}[G] = \mathbf{Z}[H][C_{p^n}] & \longrightarrow & \mathcal{M}[C_{p^n}] \\ \downarrow & & \downarrow \\ \mathbf{Z}[H][C_{p^n}]/c\mathbf{Z}[H][C_{p^n}] & \longrightarrow & \mathcal{M}[C_{p^n}]/c\mathcal{M}[C_{p^n}] \end{array}$$

is cartesian, and gives rise to the exact sequence [A-D-S, Theorem 1.3]

$$\begin{aligned} &K_2(\mathcal{M}[C_{p^n}]/c\mathcal{M}[C_{p^n}]) \\ &\quad \to SK_1(\mathbf{Z}[G]) \to SK_1(\mathbf{Z}[H][C_{p^n}]/c\mathbf{Z}[H][C_{p^n}]) \oplus SK_1(\mathcal{M}[C_{p^n}]). \end{aligned}$$

But $SK_1(\mathcal{M}[C_{p^n}]) = 1$ by Theorem 3.3, and $SK_1(\mathbf{Z}[H][C_{p^n}]/c\mathbf{Z}[H][C_{p^n}]) = 1$ because $\mathbf{Z}[H][C_{p^n}]/c\mathbf{Z}[H][C_{p^n}]$ is a finite commutative ring [B, Chapter V, Corollary 9.2]. Hence $SK_1(\mathbf{Z}[G])$ is a quotient of $K_2(\mathcal{M}[C_{p^n}]/c\mathcal{M}[C_{p^n}])$, whose exponent we have seen divides p^{m-1}.

The following technical proposition will be used to obtain a lower bound for $\exp(SK_1(\mathbf{Z}[G]))$.

Proposition 4.3. *Let G be an abelian p-group which is not cyclic, and let*

$$p^r = \gcd\left(\exp(G), \frac{|G|}{p \cdot \exp(G)}\right).$$

Let $l = 1$ if

(i) $G \approx C_{2^n} \times C_{2^n}$, $n \geqq 2$, *or*

(ii) $G \approx C_{2^n} \times C_{2^n} \times C_2$, $n \geqq 2$;

otherwise let $l = 0$.

Then there is a surjection

$$\hat{\Delta}\colon SK_1(\mathbf{Z}[G]) \to \mathbf{Z}/p^{r-l}\mathbf{Z}.$$

The proof of this proposition will be given below, following the proof of Lemma 4.4. The following discussion and Lemma 4.4 will also be used in § 5.

Let us keep the notation of Proposition 4.3. Set $\exp(G)=p^n$. Choose $g\in G$ of order p^n and write

$$G\approx\langle g\rangle\times H$$

where $H\subset G$ is a subgroup of order p^k; then $\min\{n,k-1\}=r$ (as defined in Proposition 4.3). Let $m=\operatorname{rank}(H)$, and let ζ be a primitive p^n-th root of unity. For each character $\chi\in\hat{H}$, we consider two extensions χ', $\chi''\in\hat{G}$ of χ obtained by setting

$$\chi'(g)=\zeta;\quad \chi''(g)=\zeta^p.$$

Choose a cluster of characters $S\subset\hat{G}$ such that $\{\chi'|\chi\in\hat{H}\}\cup\{\chi''|\chi\in(\hat{H})^p\}\subset S$ and define

$$\Delta\colon\prod_{\chi\in S}\operatorname{Im}(\chi)\to\langle\zeta\rangle$$

by setting

$$\Delta((y))=(\prod_{\chi\in\hat{H}}y_{\chi'})^{p^{n-r}}\cdot(\prod_{\chi\in(\hat{H})^p}y_{\chi''})^{-p^{n-r+m-2}}. \tag{5}$$

Note that this definition makes sense, since $n-r+m-2\geqq 0$. This is clear if $r<n$ (since $m\geqq 1$). If $r=n$, then $n<k$, and so $|H|=p^k>p^n$. But $\exp(H)\leqq p^n$, and therefore $m=r\,k(H)\geqq 2$.

Lemma 4.4. *With the above notation,* $\Delta\psi_S=1$ *except possibly in cases* (i) *and* (ii) *of* Proposition 4.3; *in those cases* $\operatorname{Im}(\Delta\psi_S)\subset\{\pm 1\}$.

Proof. We must compute $\Delta\psi_S$ on the two kinds of generators of $G\otimes\hat{\mathbb{Z}}_p[G]$, viz.

$$g\otimes g^c h,\quad k\otimes g^c h,\quad h,k\in H,\quad c\in\mathbb{Z}.$$

Step 1. We begin with elements of the form $g\otimes g^c h$. We have

$$\begin{aligned}\Delta\psi_S(g\otimes g^c h)&=[\prod_{\chi\in\hat{H}}\psi_{\chi'}(g\otimes g^c h)]^{p^{n-r}}[\prod_{\chi\in\hat{H}^p}\psi_{\chi''}(g\otimes g^c h)]^{-p^{n-r+m-2}}\\&=[\prod\zeta]^{p^{n-r}}[\prod\zeta^p]^{-p^{n-r+m-2}},\end{aligned} \tag{6}$$

where the first product is taken over all $\chi\in\hat{H}$ with $\zeta^c\chi(h)=1$, and the second over all $\chi\in\hat{H}^p$ with $\zeta^{pc}\chi(h)=1$. Clearly $\zeta^c\chi_0(h)=1$ for some character $\chi_0\in\hat{H}$ if and only if $\operatorname{ord}(\zeta^c)\leqq\operatorname{ord}(h)$. Set $K=H/\langle h\rangle$. Then $\hat{K}\approx\{\chi\in\hat{H}|\chi(h)=1\}$, and $\{\chi\in\hat{H}|\zeta^c\chi(h)=1\}\approx\chi_0\hat{K}$ for any such χ_0. Thus the first product in (6) is

$$(\prod\zeta)^{p^{n-r}}=\zeta^t$$

where

$$\begin{aligned}&t=0 && \text{if } \operatorname{ord}(\zeta^c)>\operatorname{ord}(h)\\&t=p^{n-r}|K| && \text{if } \operatorname{ord}(\zeta^c)\leqq\operatorname{ord}(h).\end{aligned} \tag{7}$$

Let ${}_pH$ be the p-torsion subgroup of H (so $|{}_pH|=p^m$) and let $H_0=H/{}_pH$, $K_0=H/\langle{}_pH,h\rangle$. Then $\zeta^{pc}\chi_0^p(h)=1$ for some character $\chi_0\in\hat{H}$ if and only if $\operatorname{ord}(\zeta^c)>\operatorname{ord}(h)$ and $\operatorname{ord}(\zeta^c)>p$. Identifying $\hat{H}^p$ with $\hat{H}_0$, we have

$$\begin{aligned}\{\chi\in\hat{H}^p|\zeta^{pc}\chi(h)=1\}&\approx\{\tilde{\chi}\in\hat{H}_0|\tilde{\chi}(\bar{h})=1\}\\&\approx\chi_0^p\hat{K}_0,\end{aligned}$$

when $\bar{h}$ is the image of h in H_0. Thus the second product in (6) is

$$[\prod \zeta]^{-p^{n-r+m-2}} = \zeta^{t_0}$$

where

$$\begin{aligned} t_0 &= 0 && \text{if } \operatorname{ord}(\zeta^c) > \operatorname{ord}(h) \quad \text{and} \quad \operatorname{ord}(\zeta^c) > p \\ t_0 &= -p^{n-r+m-1}|K_0| && \text{otherwise.} \end{aligned}$$

Now if $h=1$,

$$p^{n-r}|K| = p^{n-r}|H| = p^{n-r+k} \equiv 0 \pmod{p^n} \tag{8}$$

since $r = \min\{n, k-1\}$. Also,

$$p^{n-r+m-1}|K_0| = p^{n-r+m-1}|H_0| = p^{n-r+k-1} \equiv 0 \pmod{p^n}. \tag{9}$$

If $h \neq 1$, then $|_pH \cap \langle h\rangle| = p$, and thus

$$p^{n-r}|K| = p^{n-r+m}\frac{|H_0|}{|\langle h\rangle|} = p^{n-r+m-1}|K_0|. \tag{10}$$

Thus we have shown that $\Delta\psi_S(g \otimes g^c h) = 1$.

Step 2. Next we check elements of the form $k \otimes g^c h$, where $h, k \in H$ and $c \in \mathbb{Z}$. By definition,

$$\begin{aligned} \Delta\psi_S(k \otimes g^c h) &= [\prod_{\chi \in \hat{H}} \psi_{\chi'}(k \otimes g^c h)]^{p^{n-r}} [\prod_{\chi \in \hat{H}^p} \psi_{\chi''}(k \otimes g^c h)]^{-p^{n-r+m-2}} \\ &= [\prod \chi(k)]^{p^{n-r}} [\prod \chi(k)]^{-p^{n-r+m-2}}, \end{aligned} \tag{11}$$

where the first product is taken over all $\chi \in \hat{H}$ with $\zeta^c \chi(h) = 1$, and the second over all $\chi \in \hat{H}^p$ with $\zeta^{pc}\chi(h) = 1$. For each $\chi \in \hat{H}$, choose $s_\chi \in \mathbb{Z}$ such that $\chi(k) = \zeta^s \chi$. If $\operatorname{ord}(\zeta^c) \leqq \operatorname{ord}(h)$ choose $\chi_0 \in \hat{H}$ as above so that $\zeta^c \chi_0(h) = 1$ (and, therefore, $\zeta^{pc}\chi_0^p(h) = 1$; otherwise set $\chi = 1$ and $s_{\chi_0} = 0$. As in Step 1 we set $K = H/\langle h\rangle$; then the first product in (11) is

$$[\prod \chi(k)]^{p^{n-r}} = \zeta^t,$$

where

$$\begin{aligned} t &= 0 \quad \text{if } \operatorname{ord}(\zeta^c) > \operatorname{ord}(h) \\ t &= p^{n-r}\sum_{\chi \in \hat{K}} (s_{\chi_0} + s_\chi) = p^{n-r} s_{\chi_0}|K| + p^{n-r}\sum_{\chi \in \hat{K}} s_\chi \quad \text{if } \operatorname{ord}(\zeta^c) \leqq \operatorname{ord}(h). \end{aligned} \tag{12}$$

Similarly, the second product in (11) is

$$[\prod \chi(k)]^{-p^{n-r+m-2}} = \zeta^{t_0}$$

where

$$\begin{aligned} t_0 &= 0 \quad \text{if } \operatorname{ord}(\zeta^c) > \operatorname{ord}(h) \quad \text{and} \quad \operatorname{ord}(\zeta^c) > p \\ t_0 &= -p^{n-r+m-1} s_{\chi_0}|K_0| + p^{n-r+m-2}\sum_{\chi \in \hat{K}_0} s_\chi \quad \text{otherwise.} \end{aligned} \tag{13}$$

Now according to (9) and (10), $p^{n-r}|K| \equiv p^{n-r+m-1}|K_0| \pmod{p^n}$; moreover if $h=1$, both are $0 \pmod{p^n}$. Hence

$$\Delta\psi_S(k \otimes g^c h) = \zeta^x, \quad \text{where } x = t + t_0 \equiv p^{n-r}\sum_{\chi \in \hat{K}} s_\chi + p^{n-r+m-2}\sum_{\chi \in \hat{K}_0} s_\chi.$$

Case 1. $\mathrm{ord}(\zeta^c) > \mathrm{ord}(h)$ and $\mathrm{ord}(\zeta^c) > p$.

In this case $\Delta\psi_S(k \otimes g^c h) = \zeta^0 = 1$.

Case 2. $\mathrm{ord}(\zeta^c) > \mathrm{ord}(h)$ and $\mathrm{ord}(\zeta^c) \leqq p$.

In this case $p > \mathrm{ord}(h)$ implies $h = 1$, and we must evaluate

$$\Delta\psi_S(k \otimes g^c h) = \zeta^x = [\prod_{\chi \in \hat{K}_0} \chi(k)]^{p^{n-r+m-2}}, \quad \text{where } x = t + t_0 = p^{n-r+m-2} \sum_{\chi \in \hat{K}_0} s_\chi.$$

Case 3. $\mathrm{ord}(\zeta^c) \leqq \mathrm{ord}(h)$ and $\mathrm{ord}(\zeta^c) > p$.

We have $\Delta\psi_S(k \otimes g^c h) = \zeta^x$, where

$$x = t_0 + t = p^{n-r} \sum_{\chi \in \hat{K}} s_\chi - p^{n-r+m-2} \sum_{\chi \in \hat{K}_0} s_\chi.$$

$$\Delta\psi_S(k \otimes g^c h) = [\prod_{\chi \in \hat{K}_0} \chi(k)]^{p^{n-r}} [\prod_{\chi \in \hat{K}_0} \chi(k)]^{p^{n-r+m-2}}.$$

In order to study the terms involving the s_χ, we need some general information. The only self-inverse elements of a group Q are the elements of order dividing 2; in particular, if Q is a finite p-group and p is odd, the product of the elements of Q is 1. If Q is a finite *cyclic* 2-group, there is a unique element of order 2 which equals the product of all the group elements. If Q is a finite non-cyclic 2-group, it is easy to see that the product of all its elements is again 1.

Let $H_1 \subset H$ be any subgroup, and let us apply these remarks to $\widehat{(H/H_1)}$. We see that

$$\xi = \prod_{\chi \in \widehat{(H/H_1)}} \chi = \begin{cases} -1 & \text{if } p = 2,\ H/H_1 \text{ is cyclic of order } \geqq 2 \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\xi(\bar{k}) = 1$ unless $p = 2$, H/H_1 is cyclic of order $\geqq 2$ and $\bar{k}$ generates H/H_1; in the latter case $\xi(\bar{k}) = -1$. Here $\bar{k}$ denotes the image of k in H/H_1. Since $\xi(k) = \prod_{\chi \in \widehat{(H/H_1)}} \zeta^s \chi = \chi^x$, where $x = \sum_{\chi \in \widehat{(H/H_1)}} s_\chi$, it follows that

$$\sum_{\chi \in \hat{K}} s_\chi \equiv \begin{cases} 2^{n-1} \pmod{2^n} & \text{if } p = 2 \text{ and } \langle h \rangle \neq \langle h, k \rangle = H \\ 0 \pmod{2^m} & \text{otherwise.} \end{cases}$$

Hence $\bmod p^n$ we have

$$p^{n-r} \sum_{\chi \in \hat{K}} s_\chi \equiv \begin{cases} 2^{n-1} & \text{if } p = 2,\ n = r,\ \langle h \rangle \neq \langle h, k \rangle = H \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

and

$$p^{n-r+m-2} \sum_{\chi \in \hat{K}_0} s_\chi \equiv \begin{cases} 2^{n-1} & \text{if } p = 2,\ n-r+m-2 = 0, \text{ and } \langle h, {}_pH \rangle \neq \langle h, k, {}_pH \rangle = H \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Thus $\Delta\psi_S(k \otimes g^c h) = 1$ when p is odd, and $\Delta\psi_S(k \otimes g^c h) \subset \{\pm 1\}$ when $p = 2$. Thus we are done unless $p = 2$ and G is *not* isomorphic to $C_{2^n} \times C_{2^n}$ or

$C_2 \times C_{2^n} \times C_2$ for $n \geqq 2$, i.e., when $H \not\approx C_{2^n}$ and $H \not\approx C_{2^n} \times C_2$. In these cases assume (14) holds. Then $n = r = \min\{k-1, n\} \leqq k-1$, so $|H| > 2^n$ and $m = \operatorname{rank}(H) = 2$. Thus $n - m + r = 2$. Conversely, if (15) holds and $m = 1$, then $n - 1 = r = \min\{k-1, n\}$ and $k = n$, which would imply the contradiction $H \approx C_{2^n}$. So if (15) holds, $m = 2$ and $n = r$. Thus if either (14) or (15) holds, $\operatorname{rank}(H) = 2$, $|H| > 2^n$, $\exp(H) \leqq 2^n$. Because $H \not\approx C_{2^n} \times C_2$, we must have $H \approx C_{2^i} \times C_{2^j}$, $i, j \geqq 2$, $i + j > n$. But then $\langle h, k \rangle = H$ if and only if $\langle h, k, {}_pH \rangle = H$. Thus (14) holds if and only if (15) holds, which shows $x \equiv 0 \pmod{p^n}$ in Case 3. Since neither (14) nor (15) can hold if $h = 1$, $x \equiv 0 \pmod{p^n}$ in Case 2 as well, which completes the proof of this lemma.

Proof of Proposition 4.3. Define

$$\mathscr{U} = \{\chi' \mid \chi \in \hat{H}\} \cup \{\chi'' \mid \chi \in \hat{H}^p\} \subset S$$

with χ', χ'' as in Lemma 4.4. If $\mathscr{U} \subset S_0$, then there are induced epimorphisms

$$\hat{\Delta}\colon SK_1(\mathbf{Z}[G]) \to \operatorname{cok}(\psi_{S_0}) \to \operatorname{Im}(\Delta) \approx \mathbf{Z}/p^{r-1}\mathbf{Z}.$$

The only problem arises when $p = 2$ and $n = 2$; in this case the characters χ'' take values in $\{\pm 1\}$ and hence do not lie in S_0.

In this case

$$SK_1(\mathbf{Z}[G]) \approx \prod_{\chi \in S_0} \operatorname{Im}(\chi)/\psi_{S_0}(G \otimes \langle 2, J(\hat{\mathbf{Z}}_2[G])^2 \rangle).$$

But if $\chi(G) \subset \{\pm 1\}$ and $g_1, g_2, g_3 \in G$, we have

$$\begin{aligned} &\psi_\chi(g_1 \otimes (1-g_2)(1-g_3)) = \psi_\chi(g_1 \otimes (1 - g_2 - g_3 + g_2 g_3)) \\ &= \psi_\chi(g_1 \otimes 1)\psi_\chi(g_1 \otimes g_2)^{-1}\psi_\chi(g_1 \otimes g_3)^{-1}\psi_\chi(g_1 \otimes g_2 g_3) \\ &= 1 \end{aligned}$$

and

$$\psi_\chi(g_1 \otimes 2g_2) = \psi_\chi(g_1 \otimes g_2)^2 = 1.$$

Thus when $p = 2$, $n = 2$, we may omit the characters χ'' in Lemma 4.4 and obtain the necessary surjection.

In order to completely determine $\exp(SK_1(\mathbf{Z}[G]))$ for finite abelian G, we will need certain explicit calculations for the exceptional 2-groups $C_{2^n} \times C_{2^n}$ and $C_{2^n} \times C_{2^n} \times C_2$, $n \geqq 2$. These are given in the next two propositions.

Proposition 4.6. *$SK_1(\mathbf{Z}[C_4 \times C_4])$ has exponent 2.*

Proof Let $G = C_4 \times C_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^4 = 1, \sigma\tau = \tau\sigma \rangle$.

According to Proposition 4.2, $\exp(SK_1(\mathbf{Z}[G]))|2$, so it will suffice to show that $SK_1(\mathbf{Z}[G]) \neq 1$. Let $\zeta = \zeta_4$ be a primitive fourth root of unity, and consider the imaginary cluster of characters $S_0 = \{\chi_{1j},\ 0 \leqq j \leqq 3;\ \chi_{i1},\ i = 0, 2\}$, where for any i, j, χ_{ij} is defined by setting

$$\chi_{ij}(\sigma) = \zeta^i, \quad \chi_{ij}(\tau) = \zeta^j.$$

For any $r, s \in \mathbf{Z}$,

$$\psi_{\chi_{ij}}(\sigma \otimes \sigma^r \tau^s) = \begin{cases} \zeta^i & \text{if } 4 \mid ri + sj \\ 1 & \text{otherwise} \end{cases}.$$

So $\chi_{i1}(\sigma \otimes \sigma^r \tau^s) = \pm 1$ for $i = 0,2$, and $\prod_{\chi \in S_0} \psi_\chi(\sigma \otimes \sigma^r \tau^s) = \pm \zeta^s = \zeta^t$, where $t = |\{j | 0 \leqq j \leqq 3, sj \equiv -r \bmod 4\}|$. Hence for any $x \in \langle 2, J(\hat{\mathbb{Z}}_2[G])^2 \rangle$, $\prod_{\chi \in S_0} \psi_\chi(\sigma \otimes x) = \pm 1$, and similarly $\prod_{\chi \in S_0} \psi_\chi(\tau \otimes x) = \pm 1$. Thus the composite map

$$G \otimes \langle 2, J(\hat{\mathbb{Z}}_2[G])^2 \rangle \xrightarrow{\psi_{S_0}} \prod_{\chi \in S_0} \mathrm{Im}(\chi) \xrightarrow{\pi} \langle \zeta \rangle / \{\pm 1\} \approx \{\pm 1\}$$

is trivial, where π is the product of coordinates map. By Theorem 1.8, this induces a surjection $SK_1(\mathbb{Z}[G]) \to \{\pm 1\}$, as desired

Remark. In Section 5 we will see that

$$SK_1(\mathbb{Z}[C_4 \times C_4]) \approx \mathbb{Z}/2\mathbb{Z}.$$

Proposition 4.7. (a) *For* $n \geqq 3$, $\exp(SK_1(\mathbb{Z}[C_{2^n} \times C_{2^n}])) = 2^{n-2}$.
(b) *For* $n \geqq 2$, $\exp(SK_1(\mathbb{Z}[C_{2^n} \times C_{2^n} \times C_2])) = 2^{n-1}$.

Proof. Fix $n \geqq 2$, and let

$$G = C_{2^n} \times C_{2^n} = \langle \sigma, \tau | \sigma^{2^n} = \tau^{2^n} = 1, \sigma\tau = \tau\sigma \rangle.$$

According to Theorem 1.11,

$$SK_1(\mathbb{Z}[G \times C_2]) \approx SK_1(\mathbb{Z}[G]) \times SK_1(\mathbb{Z}[G], 2\mathbb{Z}[G]).$$

Thus it will suffice to show that

$$\begin{aligned} &\exp(SK_1(\mathbb{Z}[G])) \leqq 2^{n-2} && \text{for } n \geqq 3 \\ &\exp(SK_1(\mathbb{Z}[G], 2\mathbb{Z}[G])) \leqq 2^{n-1} && \text{for } n \geqq 2. \end{aligned}$$

(Note that Proposition 4.3 yields the opposite inequalities.)

Let $\zeta = \zeta_{2^n}$ be a primitive 2^n-th root of unity, and define characters $\chi_{ij} \in \hat{G}$ by

$$\chi_{ij}(\sigma) = \zeta^i, \quad \chi_{ij}(\tau) = \zeta^j.$$

Then the set

$$S_0 = \{\chi_{2m, 2mj}, \chi_{2m+1\, i, 2m} | 0 \leqq m \leqq n-2\}$$

is an imaginary cluster of characters for G. By Theorems 1.8 and 1.10,

$$SK_1(\mathbb{Z}[G]) \approx [\prod_{\chi \in S_0} \mathrm{Im}(\chi)]/A \quad \text{and} \quad SK_1(\mathbb{Z}[G], 2\mathbb{Z}[G]) \approx [\prod_{\chi \in S_0} \mathrm{Im}(\chi)]/A';$$

where

$$A = \psi(G \otimes \langle 2\mathbb{Z}[G], J(\hat{\mathbb{Z}}_2[G])^2 \rangle)$$

($\psi = \psi_{S_0}$); and

$$A' = \psi(G \otimes (2 - \Phi) J(\hat{\mathbb{Z}}_2[G])).$$

For any $\chi_{ij} \in S_0$, $\sigma(i,j) \in \prod_{\chi \in S_0} \mathrm{Im}(\chi)$ will denote the element

$$\sigma(i,j)_\chi = \begin{cases} \zeta^{2^m} & \text{if } \chi = \chi_{ij} \text{ and } |\mathrm{Im}(\chi_{ij})| = 2^{n-m} \\ 1 & \text{if } \chi \neq \chi_{ij}. \end{cases}$$

The actual calculation will be carried out in three steps. For convenience, elements in $\prod_{\chi\in S_0} \mathrm{Im}(\chi)$ will be written additively.

Step 1. We first show that for all j, and $0\leqq m\leqq n-1$,

$$2^{n-m-1}\sigma(1,j)\equiv 2^{n-m-1}\sigma(1,j+2^m) \pmod{A} \tag{16}$$

$$2^{n-m}\sigma(2,2j)\equiv 2^{n-m}\sigma(2,2j+2^m) \pmod{A} \tag{17}$$

and

$$2^{n-m}\sigma(1,j)\equiv 2^{n-m}\sigma(1,j+2^m) \pmod{A'} \tag{18}$$

To see (16), note that

$$\psi(g\otimes(1-\sigma^{2^{n-1}})\sigma^{-j}\tau)=\sigma(1,j)-\sigma(1,j+2^{n-1})\in A$$

and

$$\begin{aligned}\psi(\sigma\otimes(1-\sigma^{2^{n-1}})\sigma^{-2j}\tau^2)&=\sigma(1,j)-\sigma(1,j+2^{n-2})+\sigma(1,j+2^{n-1})-\sigma(1,j+3\cdot 2^{n-2})\\&\equiv 2\sigma(1,j)-2\sigma(1,j+2^{n-2})\in A.\end{aligned}$$

So by induction,

$$\begin{aligned}&\psi(g\otimes(1-\sigma^{2^{n-1}})\sigma^{-2^k j}\tau^{2^k})=\sum_{r=0}^{2^{k+1}-1}(-1)^r\sigma(1,j+r\cdot 2^{n-k-1})\\&\equiv 2^k\sigma(1,j)-2^k\sigma(1,j+2^{n-k-1})\in A.\end{aligned}$$

Similarly, (18) follows since the elements

$$\psi(\sigma\otimes 2(1-\sigma^{2^{n-1}})\sigma^{-2^k j}\tau^{2^k})$$

lie in A'. (Note that $\Phi((1-\sigma^{2^{n-1}})x)=0$ for any $x\in\hat{\mathbb{Z}}_2[G]$.) Finally, (17) follows from the relations

$$\begin{aligned}&\psi(g\otimes(1-\sigma^{2^{n-2}})(1+\sigma^{2^{n-1}})\sigma^{-j}\tau-\psi(\sigma\otimes(1-\sigma^{2^{n-1}})\sigma^{-2j}\tau^2)\\&=2\sigma(2,2j)-2\sigma(2,2j+2^{n-1})\in A;\\&\psi(\sigma\otimes(1-\sigma^{2^{n-2}})(1+\sigma^{2^{n-1}})\sigma^{-2j}\tau^2)-\psi(\sigma\otimes(1-\sigma^{2^{n-1}})\sigma^{-4j}\tau^4)\\&=2\sigma(2,2j)-2\sigma(2,2j+2^{n-2})+2\sigma(2,2j+2^{n-1})-2\sigma(2,2j+3\cdot 2^{n-2})\\&\equiv 4\sigma(2,2j)-4\sigma(2,2j+2^{n-2})\in A;\end{aligned}$$

and so forth.

Step 2. We have shown that $\exp SK_1(\mathbb{Z}[G])\leqq 2^{n-1}$ (Proposition 4.2). Hence calculations of A can be done modulo 2^{n-1}. Consider the elements

$$\begin{aligned}&\psi(\sigma\otimes(\tau^{2^{n-1}}-\tau^{2^{n-2}}))\\&=\sum_{i=0}^{2^{n-2}-1}\sigma(1,4i+2)+\sum_{i=0}^{2^{n-2}-1}\sigma(2,4i+2)+\sum_{i=1}^{2^{n-2}-1}2i\,\sigma(4i,2)\end{aligned}$$

and

$$\begin{aligned}&\psi(\tau\otimes(\sigma^{2^{n-1}}-\sigma^{2^{n-2}}))\\&=\sum_{i=0}^{2^{n-2}-1}\sigma(4i+2,1)+\sum_{i=0}^{2^{n-2}-1}(2i+1)\sigma(2,4i+2)+\sum_{i=1}^{2^{n-2}-1}2i\,\sigma(2,4i).\end{aligned}$$

Apply (16) and (17) (and similar formulas obtained by switching coordinates) to get:

$$\sum_{i=0}^{2^{n-2}-1} \sigma(1,4i+2) \equiv 2^{n-2}\sigma(1,0) \pmod{A}$$

$$\sum_{i=0}^{2^{n-2}-1} \sigma(4i+2,1) \equiv 2^{n-2}\sigma(0,1) \pmod{A}$$

$$\sum_{i=1}^{2^{n-1}-1} 2i\,\sigma(4i,2) \equiv 2^{n-2}\sigma(0,2) \pmod{A}$$

$$\sum_{i=1}^{2^{n-2}-1} 2i\,\sigma(2,4i) \equiv 2^{n-2}\sigma(2,0) \pmod{A};$$

and finally

$$\sum_{i=0}^{2^{n-2}-1} (2i+1)\,\sigma(2,4i+2) - \sum_{i=0}^{2^{n-2}-1} \sigma(2,4i+2)$$
$$= \sum_{i=0}^{2^{n-2}-1} 2i\,\sigma(2,4i+2) \equiv 2^{n-2}\sigma(2,2) \pmod{A}.$$

In other words,

$$\psi(\tau \otimes (\sigma^{2^{n-1}} - \sigma^{2^{n-2}})) - \psi(\sigma \otimes (\tau^{2^{n-1}} - \tau^{2^{n-2}}))$$
$$\equiv 2^{n-2}[\sigma(1,0)+\sigma(0,1)+\sigma(2,0)+\sigma(0,2)+\sigma(2,2)] \in A. \tag{19}$$

Furthermore, if $n \geqq 3$,

$$\psi(\sigma \otimes 2^{n-2}\tau) = 2^{n-2}\sigma(1,0) + 2^{n-2}\sigma(2,0) \in A; \tag{20}$$

and similarly

$$2^{n-2}[\sigma(0,1)+\sigma(0,2)], \quad 2^{n-2}[\sigma(1,1)+\sigma(2,2)] \in A. \tag{21}$$

Combining (19), (20), and (21) shows that

$$2^{n-2}\sigma(1,1), \quad 2^{n-2}\sigma(2,2) \in A;$$

and by symmetry $2^{n-2}\sigma(i,j) \in A$ whenever $4 \nmid i$ or $4 \nmid j$. Since any $\sigma(4i,4j)$ has order at most 2^{n-2} in $\prod_{\chi \in S_0} \mathrm{Im}(\chi)$, $\exp(SK_1(\mathbb{Z}[G])) = 2^{n-2}$.

Step 3. Now consider the elements

$$2^{n-1}a + a^{2^{n-1}} - 1 = (2-\Phi)(2^{n-2}a + 2^{n-3}a^2 + \ldots + a^{2^{n-2}} + a^{2^{n-1}})$$

for $a = \sigma$ or τ. We have

$$\psi(\sigma \otimes (2^{n-1}\tau + \tau^{2^{n-1}} - 1))$$
$$= 2^{n-1}\sigma(1,0) - \sum_{i=0}^{2^{n-1}-1} \sigma(1,2i+1) - \sum_{i=1}^{2^{n-1}-1} 2i\,\sigma(2i,1) \in A'$$

and

$$\psi(\tau\otimes(2^{n-1}\sigma+\sigma^{2^{n-1}}-1))$$
$$=2^{n-1}\sigma(0,1)-\sum_{i=0}^{2^{n-1}-1}(2i+1)\sigma(1,2i+1)-\sum_{i=1}^{2^{n-1}-1}2i\sigma(1,2i)\in A'.$$

Applying (18), we get

$$\sum_{i=1}^{2^{n-1}-1}2i\sigma(2i,1)\equiv 2^{n-1}\sigma(0,1) \quad (\mathrm{mod}\ A')$$
$$\sum_{i=1}^{2^{n-1}-1}2i\sigma(1,2i)\equiv 2^{n-1}\sigma(1,0) \quad (\mathrm{mod}\ A')$$

and

$$\sum_{i=0}^{2^{n-1}-1}(2i+1)\sigma(1,2i+1)-\sum_{i=0}^{2^{n-1}-1}\sigma(1,2i+1)$$
$$=\sum_{i=0}^{2^{n-1}-1}2i\sigma(1,2i+1)\equiv 2^{n-1}\sigma(1,1) \quad (\mathrm{mod}\ A').$$

Combining these gives

$$\psi(\tau\otimes(2^{n-1}\sigma+\sigma^{2^{n-1}}-1))-\psi(\sigma\otimes(2^{n-1}\tau+\tau^{2^{n-1}}-1))\equiv 2^{n-1}\sigma(1,1)\in A'.$$

By symmetry, $2^{n-1}\sigma(i,j)\in A'$ for any i,j not both even; and so $\exp(SK_1(\mathbb{Z}[G],2\mathbb{Z}[G]))=2^{n-1}$.

The results of this section can now be summarized in the following theorem.

Theorem 4.8. *Let G be a finite abelian group, and let k_G be the product of the distinct primes p dividing $|G|$ for which the Sylow p-subgroup G_p is not cyclic. Then*

$$\exp(SK_1(\mathbb{Z}[G]))=\varepsilon\cdot\gcd\left(\exp(G),\frac{|G|}{k_G\cdot\exp(G)}\right),$$

where $\varepsilon=\frac{1}{2}$ if

(i) $G\approx(C_2)^n$ *for some* $n\geqq 3$, *or*
(ii) $G_2\approx C_{2^n}\times C_{2^n}$ *for some* $n\geqq 3$, *or*
(iii) $G_2\approx C_{2^n}\times C_{2^n}\times C_2$ *for some* $n\geqq 2$;

and $\varepsilon=1$ otherwise.

Proof. If G is a non-cyclic p-group, it follows from Theorem 1.8, Proposition 4.2 and Proposition 4.7 that $\exp(SK_1(\mathbb{Z}[G]))$ divides

$$\varepsilon\cdot\gcd\left(\exp(G),\frac{|G|}{p\cdot\exp(G)}\right).$$

On the other hand, Propositions 4.3, 4.4 and 4.5 show that

$$\varepsilon\cdot\gcd\left(\exp(G),\frac{|G|}{p\cdot\exp(G)}\right)\leqq\exp(SK_1(\mathbb{Z}[G])).$$

This completes the proof for p-groups. For arbitrary finite abelian groups, the Theorem now follows from the formula in Corollary 2.8.

The preceding theorem allows us to determine those abelian groups with trivial SK_1.

Theorem 4.9. *Let G be a finite abelian group. Then $SK_1(\mathbb{Z}[G])=1$ if and only if either*

(i) $G \approx (C_2)^n$ *for some* $n \geqq 1$, *or*
(ii) *every Sylow p-subgroup of G has the form C_{p^n} or $C_p \times C_{p^n}$ for some $n \geqq 1$.*

Theorem 4.10. *Let p be an odd prime, and let G be any (not necessarily abelian) finite p-group. Then $SK_1(\mathbb{Z}[G])=1$ if and only if $G \approx C_{p^n}$ or $G \approx C_p \times C_{p^n}$ for some $n \geqq 1$.*

Proof. Combine Theorem 4.9 with [O I, Theorem 3].

Remark 4.11. It is not really necessary to use the full force of Theorem 4.8 to obtain Theorems 4.9 and 4.10. These two Theorems follow from the following observations:

1. A surjection of abelian groups induces a surjection of the corresponding SK_1's [A-D-S, Theorem 2.6].
2. A p-group is not of the type in Theorem 4.10 if and only if it maps onto one of the following groups:
   (i) $C_p \times C_p \times C_p$ or $C_{p^2} \times C_{p^2}$, if p is odd,
   (ii) $C_4 \times C_2 \times C_2$ or $C_4 \times C_4$ for $p=2$.
3. An abelian group fails to satisfy the hypotheses of Theorem 4.9 if and only if it maps onto one of the following groups:
   (i) one of the groups in 2(*i*) or 2(*ii*),
   (ii) $C_2 \times C_2 \times C_2 \times C_p$, for p an odd prime.
4. Each of the groups given in the lists above has non-trivial SK_1. This can be derived from the proof of Theorem 4.8. The actual computation of the necessary SK_1's is very easy, except for that of the group $C_{p^2} \times C_{p^2}$, which is a bit more tedious. Explicit computations are given in Sect. 5.

Section 5

In this section we have collected a substantial number of further calculations. These illustrate the techniques of computation based on § 1 and provide many examples of the patterns which occur for various classes of p-groups. At the end of the section we formulate some general conjectures based on these calculations.

Theorem 5.1. *For any integers $n, k \geqq 1$,*

$$SK_1(\mathbb{Z}[C_{2^n} \times (C_2)^k]) \approx \prod_{r=2}^{k} \binom{k}{r} (\mathbb{Z}/2^{r-1}\mathbb{Z}) \bigotimes_{\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \ldots \times \mathbb{Z}/2^n\mathbb{Z}).$$

Proof. According to Theorem 1.11,

$$SK_1\left(\mathbb{Z}[C_{2^n}\times(C_2)^k]\approx\prod_{r=0}^{k}\binom{k}{r}(SK_1(\mathbb{Z}[C_{2^n}],(2^r))\right), \tag{1}$$

so it will suffice to compute $SK_1(\mathbb{Z}[C_{2^n}],(2^r))$, which has exponent less than or equal to 2^{r-1} (Proposition 4.2). In particular, we may assume $r\geqq 2$, since otherwise $SK_1(\mathbb{Z}[C_{2^n}],(2^r))=1$.

Let S_0 be an imaginary cluster of characters for C_{2^n}. By Theorem 1.10.,

$$SK_1([C_{2^n}],(2^r))\approx\prod_{\chi\in S_0}\mathrm{Im}(\chi)/\psi(C_{2^n}\otimes(2^{r-1})(2-\Phi)\hat{\mathbb{Z}}_2[C_{2^n}]).$$

Since $\psi(C_{2^n}\otimes(2^{r-1})(2-\Phi)\hat{\mathbb{Z}}_2[C_{2^n}])$ is clearly a subgroup of $[\prod_{\chi\in S_0}\mathrm{Im}(\chi)]^{2^{r-1}}$, it follows that

$$\begin{aligned}SK_1(\mathbb{Z}[C_{2^n}],(2^r))&\approx\mathbb{Z}/2^{r-1}\mathbb{Z}\otimes\prod_{\chi\in S_0}\mathrm{Im}(\chi)\\&\approx\mathbb{Z}/2^{r-1}\mathbb{Z}\otimes(\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}\times\ldots\times\mathbb{Z}/2^n\mathbb{Z}).\end{aligned}$$

We now deduce a result mentioned in [A-D-S] which shows that an exponent for G can fail to be an exponent for $SK_1(\mathcal{O}[G])$ if a prime dividing the order of G ramifies in $\mathcal{O}$ [A-D-S, Remark 1 following Proposition 3.2].

Proposition 5.2. *The group $SK_1(\mathbb{Z}[i][(C_2)^k])$ is isomorphic to $SK_1(\mathbb{Z}[C_4\times(C_2)^k])$. In particular, $SK_1(\mathbb{Z}[i][(C_2)^3])$ contains an element of order* 4.

Proof. Consider the cartesian square

$$\begin{array}{ccc}\mathbb{Z}[C_4\times(C_2)^k] & \longrightarrow & \mathbb{Z}[i][(C_2)^k]\\ \downarrow & & \downarrow\\ \mathbb{Z}[(C_2)^{k+1}] & \longrightarrow & \mathbb{F}_2[(C_2)^{k+1}],\end{array}$$

which gives rise to the Mayer-Vietoris sequence [A-D-S, Theorem 1.3]

$$\begin{aligned}&K_2(\mathbb{Z}[i][(C_2)^k])\oplus K_2(\mathbb{Z}[(C_2)^{k+1}])\to K_2(\mathbb{F}_2[(C_2)^{k+1}])\\&\to SK_1(\mathbb{Z}[C_4\times(C_2)^k])\to SK_1(\mathbb{Z}[i][(C_2)^k])\oplus SK_1(\mathbb{Z}[(C_2)^{k+1}])\\&\to SK_1(\mathbb{F}_2[(C_2)^{k+1}]).\end{aligned}$$

Now $\mathbb{F}_2[(C_2)^{k+1}]$ is a local ring and thus has trivial SK_1. The group $SK_1(\mathbb{Z}[(C_2)^{k+1}])$ is also trivial (as follows from Theorem 1.8). Finally, the map

$$K_2(\mathbb{Z}[(C_2)^{k+1}])\to K_2(\mathbb{F}_2[(C_2)^{k+1}])$$

is surjective, since its cokernel is $SK_1(\mathbb{Z}[(C_2)^{k+1}],(2))$, which vanishes according to (2) of Theorem 2.1. This proves the Proposition.

We now begin a series of Lemmas and Propositions which will culminate in the computation analogous to Theorem 5.1 for odd primes (Theorem 5.8). Along the way we will calculate SK_1 for several other classes of p-groups.

Lemma 5.3. *Let p be any prime, and let $\alpha: G \to G'$ be a surjection of abelian p-groups. Let S_0 be an imaginary cluster of characters for G. Let*

$$S_0^1 = \{\chi \in S_0 \mid \chi \neq 1 \text{ on } \ker(\alpha)\}; \quad S_0^2 = \{\chi \in S_0 \mid \chi = 1 \text{ on } \ker(\alpha)\},$$

so that S_0 is the disjoint union $S_0^1 \cup S_0^2$. If we regard $\prod_{\chi \in S_0^1} \mathrm{Im}(\chi)$ as a subgroup of $\prod_{\chi \in S_0} \mathrm{Im}(\chi)$, the kernel of the surjection $SK_1(\mathbb{Z}[G]) \to SK_1(\mathbb{Z}[G'])$ induced by α is isomorphic to

$$\prod_{\chi \in S_0^1} \mathrm{Im}(\chi)/(\psi_{S_0}(G \otimes X) \cap \prod_{\chi \in S_0^1} \mathrm{Im}(\chi)),$$

where $X = \begin{cases} \hat{\mathbb{Z}}_p[G] & \text{if } p \text{ is odd.} \\ \langle 2, J(\hat{\mathbb{Z}}_2[G])^2 \rangle & \text{if } p = 2. \end{cases}$

Proof. The set S_0^2 is an imaginary cluster of characters for G', and X is mapped onto

$$X' = \begin{cases} \hat{\mathbb{Z}}_p[G] & p \text{ odd} \\ \langle 2, J(\hat{\mathbb{Z}}_2[G])^2 \rangle & p = 2. \end{cases}$$

Thus $\psi_G(G \otimes X)$ maps onto $\psi_{G'}(G' \otimes X')$, and the lemma follows from Theorem 1.8.

Lemma 5.4. *Suppose $\alpha: \tilde{G} \to G$ is a surjection of abelian p-groups, and let H be a subgroup of G. Set $\tilde{H} = \alpha^{-1}(H)$. Then the transfer map $SK_1(\mathbb{Z}[\tilde{G}]) \to SK_1(\mathbb{Z}[\tilde{H}])$ maps $\ker(SK_1(\mathbb{Z}[\tilde{G}]) \xrightarrow{\alpha_*} SK_1(\mathbb{Z}[G]))$ onto*

$$\ker(SK_1(\mathbb{Z}[\tilde{H}]) \xrightarrow{\alpha_*^H} SK_1(\mathbb{Z}[H])).$$

In particular,

$$|SK_1(\mathbb{Z}[\tilde{G}])| \geqq \frac{|SK_1(\mathbb{Z}[G])| \cdot |SK_1(\mathbb{Z}[\tilde{H}])|}{|SK_1(\mathbb{Z}[H])|}.$$

Proof. Both α_* and α_*^H are surjective [A-D-S, Theorem 2.6].

Let $\mathcal{M}(\tilde{G}) \subset \mathbb{Q}[\tilde{G}]$ and $\mathcal{M}(\tilde{H}) \subset \mathbb{Q}[\tilde{H}]$ be the respective maximal orders. For any k, the square

$$\begin{array}{ccc} K_2(\mathbb{Z}[\tilde{G}]/(p^k)) & \xrightarrow{i_{\tilde{G}}} & K_2(\mathcal{M}(\tilde{G})/(p^k)) \\ \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle t_1} \\ K_2(\mathbb{Z}[\tilde{H}]/(p^k)) & \xrightarrow{i_{\tilde{H}}} & K_2(\mathcal{M}(\tilde{H})/(p^k)) \end{array} \tag{3}$$

is commutative, where $i_{\tilde{G}}$, $i_{\tilde{H}}$ are induced by inclusion, and t_0, t_1 are transfer maps. This is so because we may identify the groups on the right with subquotients of $K_2(\mathbb{Q}[\tilde{G}])$, $K_2(\mathbb{Q}[\tilde{H}])$ [D-S, Theorem 2.2] and inclusion of orders into $\mathbb{Q}[\tilde{G}]$, $\mathbb{Q}[\tilde{H}]$ commutes with transfer.

Choose clusters of characters $S_{\tilde{G}}$ for $\tilde{G}$, $S_{\tilde{H}}$ for $\tilde{H}$ so that for each $\chi \in S_{\tilde{G}}$, its restriction to $\tilde{H}$ lies in $S_{\tilde{H}}$. Then the transfer

$$\prod_{\chi \in S_{\tilde{G}}} K_2(\mathbb{Z}[\chi]/(p^k)) \approx K_2(\mathcal{M}(\tilde{G})/(p^k)) \xrightarrow{t_1} K_2(\mathcal{M}(\tilde{H})/(p^k)) \approx \prod_{\chi \in S_{\tilde{H}}} K_2(\mathbb{Z}[\chi]/(p^k))$$

is the sum, over all $\chi \in S_{\tilde{G}}$, of the transfers $K_2(\mathbb{Z}[\chi]/(p^k)) \to K_2(\mathbb{Z}[\chi_{|\tilde{H}}]/(p^k))$, and these are surjective ([Mi, Corollary A.15] and [D-S, Theorem 2.2]). Thus if we define $(S_{\tilde{G}})_0^1$, $(S_{\tilde{H}})_0^1$ as in Lemma 5.3, t_1 restricts to a surjection of $\prod_{\chi \in (S_{\tilde{G}})_0^1} \mathrm{Im}(\chi)$ onto $\prod_{\chi \in (S_{\tilde{H}})_0^1} \mathrm{Im}(\chi)$, since any character in $(S_{\tilde{H}})_0^1$ extends to a character in $(S_{\tilde{G}})_0^1$. Thus it follows from (3) and Lemma 5.1 that this induces a surjection

$$\ker(SK_1(\mathbb{Z}[\tilde{G}]) \to SK_1(\mathbb{Z}[G])) \to \ker(SK_1(\mathbb{Z}[\tilde{H}]) \to SK_1(\mathbb{Z}[H])),$$

and the construction shows that this map is induced by the transfer.

Theorem 5.5. *For any prime p and $n \geqq 1$,*

$$SK_1(\mathbb{Z}[C_{p^n} \times C_{p^2}]) \approx (\mathbb{Z}/p\mathbb{Z})^{(n-1)(p-1)}.$$

Proof. For $n=1$, this is a theorem of Lam [L, Chapter 5, Theorem 1.1] and also is a consequence of Proposition 4.2. So we will assume that $n \geqq 2$, and write $C_{p^n} \times C_{p^2} = G$. Since $SK_1(\mathbb{Z}[G])$ has exponent p (Theorem 4.8), it will suffice to compute its order. We will accomplish this in three steps.

Step 1. Assume first that $p \neq 2$ and $n=2$. Let $G_1 = G/G^p$ and view $\hat{G}_1$ as a subgroup of $\hat{G}$. Let S_1 be a cluster of characters for G_1 and S a cluster of characters for G such that if $\chi \in S \setminus S_1$, then $|\mathrm{Im}\,\chi| = 4$ and $\chi^p \in S_1$. Consider the diagram

$$\begin{array}{ccc} G \otimes \hat{\mathbb{Z}}_p[G] & \xrightarrow{\psi_S} & \prod_{\chi \in S} \mathrm{Im}(\chi) \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle \kappa} \\ G_1 \otimes \hat{\mathbb{Z}}_p[G_1] & \xrightarrow[\psi_1]{} & \prod_{\chi \in S_1} \mathrm{Im}(\chi) \approx (\mathbb{Z}/p\mathbb{Z})^{p+1} \end{array} \tag{4}$$

where pr is projection and κ, ψ_1 are defined as follows:

(i) If $x \in \prod_{\chi \in S} \mathrm{Im}(\chi)$ and $1 \neq \chi \in S_1$, let $\chi_1, \ldots, \chi_p$ be the characters in $S \setminus S_1$ with $\chi_i^p = \chi$. Then $(\kappa(x))_\chi = x_\chi \cdot \prod_{i=1}^{p} (x_{\chi_i})^{-p}$.

(ii) $(\psi_1(g \otimes h))_\chi = \begin{cases} 1 & \text{if } h \neq 1 \\ \chi(g) & \text{if } h = 1 \end{cases}$ for $g, h \in G_1$, $\chi \in S_1$.

We want to prove that (4) commutes. Fix $1 \neq \chi \in S_1$, and let $\chi_1, \ldots, \chi_p \in S \setminus S_1$ be the character with $\chi_i^p = \chi$. Let $g, h \in G$. To compute $y = (\kappa(\psi_S(g \otimes h)))_\chi$ we consider three cases:

(a) If $\chi(h) \neq 1$, then $\chi_i(h) \neq 1$ for all i, and so $y = 1$.

(b) If $\chi(h) = 1$ and $h \notin G^p$, there is a unique index t, $1 \leqq t \leqq p$, such that $\chi_t(h) = 1$, and therefore

$$y = (\psi_S(g \otimes h))_\chi \cdot (\psi_S(g \otimes h)_{\chi_t})^{-p} = \chi(g) \cdot \chi_t(g)^{-p} = 1.$$

(c) If $h=k^p\in G^p$, then $\chi(h)=1$ and for all i, $1\leqq i\leqq p$, $\chi_i(h)=\chi_i(k)^p=\chi(k)$. Thus it follows that

$$y=(\psi_S(g\otimes h))_\chi\cdot\prod_{i=1}^{p}(\psi_S(g\otimes h)_{\chi_i})^{-p}=\chi(g)\quad\text{if }\chi(k)\neq 1$$

$$y=\chi(g)\prod_{i=1}^{p}\chi_i(g)^{-p}=\chi(g)\quad\text{if }\chi(k)=1.$$

Thus diagram (4) commutes, and κ induces a surjection of $SK_1(\mathbb{Z}[G])\approx\operatorname{cok}(\psi_S)$ onto $\operatorname{cok}(\psi_1)$. Since, by definition, $\operatorname{im}(\psi_1)$ has rank at most two, it follows that

$$|SK_1(\mathbb{Z}[C_{p^2}\times C_{p^2}])|\geqq|\operatorname{cok}(\psi_1)|\geqq p^{p-1}. \tag{5}$$

Step 2. Now let p be any prime. If $p=2$, formula (5) is still valid by Proposition 4.6. By Lemma 5.4, for any $n\geqq 2$ we have

$$|SK_1(\mathbb{Z}[C_{p^n}\times C_{p^2}])|\geqq|SK_1(\mathbb{Z}[C_{p^{n-1}}\times C_{p^2}])|\cdot|SK_1(\mathbb{Z}[C_{p^2}\times C_{p^2}])|.$$

It follows from (5) by induction on n that

$$|SK_1(\mathbb{Z}[C_{p^n}\times C_{p^2}])|\geqq p^{(n-1)(p-1)}. \tag{6}$$

Step 3. Now fix $n\geqq 2$, write $\zeta=\zeta_{p^n}$, and define $\chi_j\in\hat{G}$ for $0\leqq j<p^2$ by

$$\chi_j(g)=\zeta;\quad \chi_j(h)=\zeta^{jp^{n-2}},$$

where g is a generator of C_{p^n} of order p^n and h is a generator of C_{p^2} of order p^2. Let S_0 be an imaginary cluster of characters for G which contains the χ_j. It follows from Lemma 5.3 that

$$\begin{aligned}&\ker(SK_1(\mathbb{Z}[G])\to SK_1(\mathbb{Z}[G/\langle g^{p^{n-1}}\rangle]))\\ &\quad\approx\prod_{j=0}^{p^2-1}\operatorname{Im}(\chi_j)/\Big[\psi(G\otimes X)\cap\prod_{j=0}^{p^2-1}\operatorname{Im}(\chi_j)\Big],\end{aligned} \tag{7}$$

where $X=\hat{\mathbb{Z}}_p[G]$ if $p\neq 2$ and $X=\langle 2,J(\hat{\mathbb{Z}}_2[G])^2\rangle$ if $p=2$. For each j as above define $\sigma_j\in\prod_{\chi\in S_0}\operatorname{Im}(\chi)$ by

$$(\sigma_j)_\chi=\begin{cases}\zeta & \text{if } \chi=\chi_j\\ 1 & \text{if } \chi\neq\chi_j.\end{cases}$$

Then if $p\leqq j<p^2$,

$$\psi(g\otimes g^{-jp^{n-2}}h(1-g^{p^{n-1}}))=\sigma_j-\sigma_{j-p}\in\psi(G\otimes X).$$

If $n\geqq 3$, then

$$\begin{aligned}&\psi(g\otimes(1+g^{p^{n-2}}+g^{2p^{n-2}}+\ldots+g^{(p-1)p^{n-2}})(1-h))\\ &-\psi(g\otimes(1+g^{p^{n-3}}+\ldots+g^{(p-1)p^{n-3}})(1-h^p))\\ &\equiv\sum_{j=0}^{p^2-p-1}\sigma_j-\sum_{\substack{j=1\\ p\nmid j}}^{p^2-1}\sigma_j=\sum_{j=0}^{p-2}\sigma_{pj}-\sum_{j=1}^{p-1}\sigma_{p^2-j}\quad(\operatorname{mod}p).\end{aligned}$$

This gives p^2-p+1 independent relations among the σ_j when $n\geqq 3$, and by (7)

$$rk(\ker(SK_1(\mathbb{Z}[G])\to SK_1(\mathbb{Z}[G/\langle g^{p^{n-1}}\rangle])))\leqq p-1. \tag{8}$$

By Lemma 5.4, (8) holds for all n, and by induction, $|SK_1(\mathbb{Z}[G])|\leqq p^{(n-1)(p-1)}$. This, combined with (6), completes the proof of the Theorem.

Definition. For any abelian p-group M and any $r\geqq 1$, we write $p^r-rk(M)$ for $\dim_{\mathbb{F}_p}(p^{r-1}M/p^rM)$.

Theorem 5.6. *For any prime p,*

$$SK_1(\mathbb{Z}[C_{p^3}\times C_{p^3}]\approx\begin{cases}(\mathbb{Z}/p^2\mathbb{Z})^{p-1}\times(\mathbb{Z}/p\mathbb{Z})^{p^2-1} & \text{if } p \text{ is odd}\\ (\mathbb{Z}/2\mathbb{Z})^4 & \text{if } p=2.\end{cases}$$

Proof. Fix p, and set $G=\langle a\rangle\times\langle b\rangle$, where $|\langle a\rangle|=|\langle b\rangle|=p^3$. Let $K=\langle a^{p^2}\rangle\subseteq G$. The proof will be carried out in three steps, as follows:

Step 1: $rk(SK_1(\mathbb{Z}[G]))\geqq p^2+p-2$.

Step 2: if p is odd, then $p^2-rk(SK_1(\mathbb{Z}[G]))\geqq p-1$

Step 3: $|\mathrm{Ker}[SK_1(\mathbb{Z}[G])\to SK_1(\mathbb{Z}[G/K])]|\leqq\begin{cases}p^{p^2-1} & \text{if } p \text{ is odd}\\ 2^2 & \text{if } p=2.\end{cases}$

The theorem will follow immediately from these three steps, since $SK_1(\mathbb{Z}[G/K])\approx(\mathbb{Z}/p\mathbb{Z})^{2(p-1)}$ by Theorem 5.5.

Step 1. Set $H=\langle a^{p^2}\rangle\times\langle b^{p^2}\rangle\subseteq G$, and let $H_0,\dots,H_p\subseteq H$ be the subgroups of order p. Let

$$f\colon G\to G/H \quad\text{and}\quad f_i\colon G/H_i\to G/H$$

be the projections. Fix a cluster S of characters for G, and set

$$T=\{\chi\in S\,|\,\chi|_H=1\},\qquad T_i=\{\chi\in S\,|\,\chi|_H\neq 1,\ \chi|_{H_i}=1\}\qquad 0\leqq i\leqq p.$$

Then S is the disjoint union of T and the T_i.

Set $X=\hat{\mathbb{Z}}_p[G]$ (p odd), or $X=\langle 2, J(\hat{\mathbb{Z}}_2[G])^2\rangle$ ($p=2$), and set

$$Y=\{x\in G\otimes X\subseteq G\otimes\hat{\mathbb{Z}}_p[G]\,|\,\psi_T(x)=1\}.$$

By Lemma 5.3,

$$\ker(SK_1(f))\approx\Big[\prod_{i=0}^{p}\prod_{\chi\in T_i}\mathrm{Im}(\chi)\Big]\Big/\psi_{\cup T_i}(Y);$$

and for $0\leqq i\leqq p$,

$$\ker(SK_1(f_i))\approx[\prod_{\chi\in T_i}\mathrm{Im}(\chi)]/\psi_{T_i}(Y).$$

The induced homomorphism

$$\mathrm{Ker}[SK_1(\mathbb{Z}[G])\to SK_1(\mathbb{Z}[G/H])]\to\prod_{i=0}^{p}\ker[SK_1(\mathbb{Z}[G/H_i])\to SK_1(\mathbb{Z}[G/H])]$$

is thus surjective. It follows that $SK_1(\mathbb{Z}[G])$ maps onto

$$\left\{(x_0,\ldots,x_p)\in\prod_{i=0}^{p} SK_1(\mathbb{Z}[G/H_i])\,\middle|\, SK_1(f_i)(x_i)=SK_1(f_j)(x_j),\ \text{for all } i,j\right\}$$
$$\approx(\mathbb{Z}/p\mathbb{Z})^{(p-1)(p+2)}$$

(recall that $SK_1(\mathbb{Z}[G/H_i])\approx(\mathbb{Z}/p\mathbb{Z})^{2(p-1)}$; $SK_1(\mathbb{Z}[G/H])\approx(\mathbb{Z}/p\mathbb{Z})^{p-1}$); and so

$$rk(SK_1(\mathbb{Z}[G]))\geqq(p-1)(p+2)=p^2+p-2.$$

Step 2. Now assume that p is odd. Write $S=S_3\cup S_2\cup S_1\cup\{1\}$, where $|\mathrm{Im}(\chi)|=p^i$ for all $\chi\in S_i$ $(1\leqq i\leqq 3)$. Assume that S is the "standard cluster": for $i=1,2,3$, and any $\chi\in S_i$, either $\chi(a)=\zeta_{p^i}$, or $\chi(b)=\zeta_{p^i}$ and $|\chi(a)|<p^i$.

Consider the homomorphisms

$$G\otimes\hat{\mathbb{Z}}_p[G]\xrightarrow{\psi_S}\prod_{\chi\in S}\mathrm{Im}(\chi)\xrightarrow{\kappa}\prod_{\chi\in S_1}\langle\zeta_{p^2}\rangle\approx(\mathbb{Z}/p^2\mathbb{Z})^{p+1}.$$

Here κ is defined by setting, for any $x\in\prod_{\chi\in S}\mathrm{Im}(\chi)$, and any $\chi_0\in S_1$,

$$(\kappa(x))_{\chi_0}=\prod\{(x_\chi)^p\,|\,\chi\in S_3,\chi^{p^2}=\chi_0\}\cdot\prod\{x_\chi)^{-1}\,|\,\chi\in S_2,\chi^p=\chi_0\}.$$

A straightforward check shows that for any $g,h\in G$,

$$\kappa\circ\psi_S(g\otimes h)=\begin{cases}1 & \text{if } h^p\neq 1\\ \kappa\circ\psi_S(g\otimes 1) & \text{if } h^p=1.\end{cases}$$

So $\mathrm{Im}(\kappa\circ\psi_S)$ has rank at most two.

There are thus induced epimorphisms

$$SK_1(\mathbb{Z}[G])\approx\mathrm{cok}(\psi_S)\to\mathrm{cok}(\kappa\circ\psi_S)\to(\mathbb{Z}/p^2\mathbb{Z})^{p-1}.$$

It follows that $p^2-rk(SK_1(\mathbb{Z}[G]))\geqq p-1$.

Step 3. Recall that $K=\langle a^{p^2}\rangle\subseteq G$. Let $S\subseteq\hat{G}$ be as in Step 2, and write $S=S'\cup S''$, where

$$S'=\{\chi\in S\,|\,\chi|_K\neq 1\}=\{\chi\in\hat{G}\,|\,\chi(a)=\zeta\}\qquad(\zeta=\zeta_{p^3}).$$

For $i\in\mathbb{Z}/p^3\mathbb{Z}$, let $\chi_i\in S'$ be the character $\chi_i(b)=\zeta^i$, so that $S'=\{\chi_0,\chi_1,\ldots,\chi_{p^3-1}\}$. Choose a basis $\{\sigma_i\}$ for $\prod_{\chi\in S'}\mathrm{Im}(\chi)$ by setting

$$(\sigma_i)_\chi=\begin{cases}\zeta & \text{if } \chi=\chi_i\\ i & \text{if } \chi=\chi_j,\quad \text{some } j\not\equiv i\ (\mathrm{mod}\,p^3).\end{cases}$$

Now set $X=\hat{\mathbb{Z}}_p[G]$ if p is odd, $X=\langle 2,J(\hat{\mathbb{Z}}_2[G])^2\rangle$ if $p=2$. Define

$$Y=\{x\in G\otimes X\,|\,\psi_{S''}(x)=1\},\quad\text{and}\quad R=\psi_S(Y)\subseteq\prod_{\chi\in S'}\mathrm{Im}(\chi).$$

Then by Lemma 5.3,

$$\ker[SK_1(\mathbb{Z}[G])\to SK_1(\mathbb{Z}[G/K])]\approx[\prod_{\chi\in S'}\mathrm{Im}(\chi)]/R.$$

For convenience in the calculations which follow, $\prod \mathrm{Im}(\chi)$ will be written additively.

Step 3A. Assume p is odd. Then $\exp(SK_1(\mathbb{Z}[G]))=p^2$ (Theorem 4.8), and so all calculations can be made mod p^2.

(a) For $i\in\mathbb{Z}$, $\psi_S(a\otimes[a^{-i}b-a^{-i-p^2}b]=\sigma_i-\sigma_{i+p^2}\in R$. Set

$R_0=\langle\sigma_i-\sigma_{i+p^2}|i\in\mathbb{Z}\rangle+\langle p^2\sigma_i|i\in\mathbb{Z}\rangle\subseteq R$.

(b) For $g,h\in G$, set

$\eta(g,h)=g^p-g(1+h^p+\ldots+h^{(p-1)p})-h^p+h(1+g^p+\ldots+g^{(p-1)p})$.

For $i,j\in\mathbb{Z}$ such that $i\not\equiv j \pmod p$,

$$\psi(a\otimes\eta(a^{-i}b,a^{-j}b))\equiv\left[p\sigma_i-\sum_{r=0}^{p-1}\sigma_{i+pr}\right]-\left[p\sigma_j-\sum_{r=0}^{p-1}\sigma_{j+pr}\right]\quad(\mathrm{mod}\,R_0).$$

Hence

$$R_1=R_0+\left\langle\left[p\sigma_i-\sum_{r=0}^{p-1}\sigma_{i+pr}\right]-\left[p\sigma_j-\sum_{r=0}^{p-1}\sigma_{j+pr}\right]\middle| i\not\equiv j\quad(\mathrm{mod}\,p)\right\rangle\subseteq R.$$

Note in particular that for any i,j:

$$p\left(\sum_{r=0}^{p-1}\sigma_{i+pr}-\sum_{r=0}^{p-1}\sigma_{j+pr}\right),\qquad\sum_{r=1}^{p-1}pr\,\sigma_{i+pr}\in R_1.$$

(c) For any $i\not\equiv j \pmod p$,

$$\psi(a^{-i}b\otimes\eta(a^{-i}b,a^{-j}b))$$
$$\equiv-\left[\sum_{r=1}^{p-1}pr\,\sigma_{i+pr}\right]-\left[p(j-i)\sigma_j-\sum_{r=0}^{p-1}(j-i+pr)\sigma_{j+pr}\right]\quad(\mathrm{mod}\,R_0).$$

Thus,

$$R_2=R_0+\left\langle p\sigma_i-\sum_{r=0}^{r-1}\sigma_{i+pr}\right\rangle\subseteq R.$$

Note that $[\prod_{\chi\in S'}\mathrm{Im}(\chi)]/R_2\approx[(\mathbb{Z}/p^2\mathbb{Z})\times(\mathbb{Z}/p\mathbb{Z})^{p-2}]^p$ has order p^{p^2}.

(d) $$\psi_S\left(a\otimes\left[1-b^{p^2}-\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}(b^{p^2s}-1)b^{pr}a\right]\right)\equiv(p-1)\sum_{i=0}^{p^2-1}\sigma_i\quad(\mathrm{mod}\,R_0).$$

Hence $R\supsetneq R_0$, and so

$$|\mathrm{Ker}[SK_1(\mathbb{Z}[G])\to SK_1(\mathbb{Z}[G/K])]|=|[\prod_{\chi\in S'}\mathrm{Im}(\chi)]/R|\leqq p^{p^2-1}.$$

Step 3B. Now assume $p=2$. Then $\exp(SK_1(\mathbb{Z}[G]))=2$ by Theorem 4.8, and we can work mod 2. For any i

$$\psi_S(a\otimes[(1+a^2)(1-a^{-i}b)-(1+a)(1-a^{-2i}b^2)]\equiv\sigma_{i+4}-\sigma_{i+2}\quad(\mathrm{mod}\,2),$$

so that $\sigma_{i+2}-\sigma_i \in R$ for all i, and

$$|\mathrm{Ker}[SK_1(\mathbb{Z}[G])\to SK_1(\mathbb{Z}[G/K])]| = |[\prod_{\chi\in S'} \mathrm{Im}(\chi)]/R| \leqq 2^2.$$

This completes the proof of the theorem.

Our final computations in this section are directed to the analog of Theorem 5.1 when p is odd. The answer is more complicated, however, and requires some preparation.

Lemma 5.7. *Fix $k\geqq 1$. For any subgroup $H\subset(C_p)^k$, let $\sigma_H=\sum_{h\in H} h$. Denote by I the ideal of $\mathbb{Z}[(C_p)^k]$ generated by all $g\sigma_H$, where $g\in(C_p)^k$ and $(C_p)^k/H$ is cyclic, and write $M_k=\mathbb{Z}[(C_p)^k]/I$. Then M_k is finite of exponent p^{k-1}, and for each r, $1\leqq r\leqq k-1$, $p^r-r\,k(M_k)=$ the number of k-tuples $(t_1,\ldots,t_k)\in\mathbb{Z}^k$ such that $0\leqq t_i<p$ for all i and $\sum t_i\leqq(p-1)(k-r)-1$.*

In particular, $p-r\,k(M_k)=p^k-\binom{p+k-1}{k}$ and $p^{k-1}-r\,k(M_k)=\binom{p+k-2}{k}$.

Proof. For convenience, we denote by $P(k,r)$ the cardinality of

$$\{(t_1,\ldots,t_k)\in\mathbb{Z}^k \mid 0\leqq t_i<p, \textstyle\sum t_i\leqq(p-1)(k-r)-1\}.$$

Set $G=(C_p)^k$, and let $\{g_1,\ldots,g_k\}$ be generators. Let $K\subseteq\mathbb{Z}[G]$ be the ideal

$$K=\langle\sigma_H \mid H\subseteq G,\ G/H \text{ cyclic}\rangle\subseteq\mathbb{Z}[G].$$

Step 1. Fix $r\geqq 1$, and $u_1,\ldots,u_k\geqq 0$ such that $\sum u_i\geqq r(p-1)+1$. For any $H\subseteq G$ of index p, and any $g\in G-H$, choose α_i such that $g_i\in g^{\alpha_i}H$ $(1\leqq i\leqq k)$; then

$$(1-g_1)^{u_1}\ldots(1-g_k)^{u_k}\sigma_H=(1-g^{\alpha_1})^{u_1}\ldots(1-g^{\alpha_k})^{u_k}\sigma_H\in(1-g)^{r(p-1)+1}\mathbb{Z}[G]\subseteq p^r\mathbb{Z}[G]$$

since $p(1-g)\mid(1-g)^p$. In other words, when $\sum u_i\geqq r(p-1)+1$, then

$$(1-g_1)^{u_1}\ldots(1-g_k)^{u_k}K\subseteq p^r\mathbb{Z}[G]. \tag{9}$$

Consider the set

$$T=\{t=(t_1,\ldots,t_k) \mid 0\leqq t_i\leqq p-1, \textstyle\sum t_i\leqq(p-1)(k-r)-1\}.$$

Assume there exist $\alpha_t\in\mathbb{Z}$ for $t\in T$, such that $p\nmid\alpha_t$ for some t, and

$$L=\sum_{t\in T}\alpha_t p^{r-1}(1-g_1^{t_1})\ldots(1-g_k^{t_k})\in(K\cap p^{r-1}\mathbb{Z}[G])+p^r\mathbb{Z}[G]. \tag{10}$$

Choose $t=(t_1,\ldots,t_k)$ which is minimal (under the partial ordering) among those with $p\nmid\alpha_t$. Then

$$\begin{aligned}&(1-g_1^{p-1-t_1})\ldots(1-g_k^{p-1-t_k})L\\ &\quad\equiv\alpha_t p^{r-1}(1-g_1^{p-1})\ldots(1-g_k^{p-1})\not\equiv 0 \pmod{p^r};\end{aligned}$$

and by (9) (applied to $u_i = p-1-t_i$), $L \notin K + p^r \mathbb{Z}[G]$. This contradicts (10). In other words, the elements $p^{r-1}(1-g_1^{t_1}) \dots (1-g_k^{t_k})$ are linearly independent, and

$$p^r - r\,k(M_k) = \dim_{\mathbb{F}_p}(p^{r-1}\mathbb{Z}[G]/[K \cap p^{r-1}\mathbb{Z}[G]) + p^r\mathbb{Z}[G]]) \geqq |T| = P(k,r). \tag{11}$$

Step 2. For any $1 \leqq r \leqq k$, let $I_r \subseteq \mathbb{F}_p[G]$ be the ideal

$$\begin{aligned} I_r &= \langle \sigma_H \in \mathbb{F}_p[G] \mid H \subseteq G,\, r\,k(G/H) = r \rangle \\ &= \langle (1-h_1)^{p-1} \dots (1-h_{k-r})^{p-1} \mid h_1, \dots, h_{k-r} \in G \rangle. \end{aligned}$$

Let $J = J(\mathbb{F}_p[G]) = \langle 1-g \mid g \in G \rangle$ be the Jacobson radical. By [Bou, Chapitre III, § 6, n° 1, Remarque 3] J^{p-1}/J^p is generated by $(p-1)$-st powers of elements in J/J^2: there is an obvious natural surjection of $S^{p-1}(J/J^2)$ onto J^{p-1}/J^p (cf. Lemma 2.2). Since

$$(1-g)+(1-h) \equiv (1-gh) \pmod{J^2}$$

for each $g, h \in G$; each element of J/J^2 is represented by $1-g$ for some $g \in G$. So as ideals,

$$J^{p-1} = \langle (1-g)^{p-1} \mid g \in G \rangle + J^p = \langle (1-g)^{p-1} \mid g \in G \rangle = I_{k-1}.$$

Since $I_r = (I_{k-1})^{k-r}$ for $1 \leqq r \leqq k$, it follows that $I_r = J^{(k-r)(p-1)}$. Thus, the elements $\prod (1-g_i)^{t_i}$ for $0 \leqq t_1 \leqq p-1$ and $\sum t_i < (k-r)(p-1)$ form a basis for $\mathbb{F}_p[G]/I_r$; and

$$r\,k(\mathbb{F}_p[G]/I_r) = P(k,r).$$

Step 3. We now work in $\mathbb{Z}[G]$ again. For any r, and $H \subseteq G$ of corank r,

$$\sum \{\sigma_K \mid H \subseteq K \subseteq G, [G:K] = p\} = (p^{r-1}-1)/(p-1)\,\sigma_G + p^{r-1}\,\sigma_H.$$

Hence, $p^{r-1}\,\sigma_H \in K$. It follows that $\exp(\mathbb{Z}[G]/I) \leqq p^{k-1}$, and that

$$p^r - r\,k(\mathbb{Z}[G]/K) \leqq r\,k(\mathbb{F}_p[G]/I_r) = P(k,r).$$

Together with (11), this proves the lemma.

Theorem 5.8. *Let p be a fixed odd prime, and let M_k $(k \geqq 1)$ be the group defined and described in Lemma 5.7. Then for any $n \geqq 1$ and $k \geqq 1$,*

$$SK_1(\mathbb{Z}[C_{p^n} \times (C_p)^k]) \approx SK_1(\mathbb{Z}[(C_p)^k]) \times (M_k \otimes (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times \dots \times \mathbb{Z}/p^n\mathbb{Z})).$$

Proof. Write $G = \langle g \rangle \times H$, where g has order p^n and $H \approx (C_p)^k$. For any $K \subseteq H$, set $\tilde{K} = \langle g \rangle \times K \subseteq G$ (so $G = \tilde{H}$); and identify

$$\mathbb{Q}[\tilde{K}] \approx \mathbb{Q}[K] \oplus \sum_{r=1}^{n} \mathbb{Q}[\zeta_{p^r}][K].$$

Let $\mathcal{M}_k^0 \subseteq \mathbb{Q}[K]$ and $\mathcal{M}_k^r \subseteq \mathbb{Q}[\zeta_{p^r}][K]$ $(1 \leqq r \leqq n)$ be the maximal orders. Since the projection $\tilde{K} \to K$ is split,

$$\begin{aligned} SK_1(\mathbb{Z}[\tilde{K}]) &\approx \operatorname{cok}\left[\hat{\lambda}^{-1}\psi_{\tilde{K}} \colon \tilde{K} \otimes \hat{\mathbb{Z}}_p[\tilde{K}] \to \sum_{r=0}^{n} K_2(\mathcal{M}_k^r/(p^{n+1}))\right] \\ &\approx SK_1(\mathbb{Z}[K]) \oplus \operatorname{cok}\left[\hat{\psi}_K = \operatorname{proj} \circ \hat{\lambda}^{-1}\psi_{\tilde{K}} \colon \tilde{K} \otimes \hat{\mathbb{Z}}_p[\tilde{K}] \to \sum_{r=1}^{n} K_2(\mathcal{M}_k^r/(p^{n+1}))\right]. \end{aligned} \tag{12}$$

Furthermore, this is natural with respect to inclusions $K \subseteq H$.

In particular, since $SK_1(\mathbb{Z}[\tilde{K}])=0$ if $|K|=p$ (Proposition 4.2),

$$\mathrm{Im}(\hat{\psi}_H)\supseteq \sum_{r=1}^{n} \mathrm{Im}[\sum_{|K|=p} K_2(\mathscr{M}_K^r/(p^{n+1}))\to K_2(\mathscr{M}_H^r/(p^{n+1}))]. \tag{13}$$

Since

$$\mathscr{M}_H^r/(p^{n+1})\approx \prod_{\chi\in\hat{H}} (\mathbb{Z}[\zeta_{p^r}]/(p^{n+1})),$$

we can identify $K_2(\mathscr{M}_H^r/(p^{n+1}))$ with $\mathbb{Z}[\hat{H}]/(p^r)$. Under this identification, (13) takes the form

$$\mathrm{Im}(\hat{\psi}_H)\supseteq \sum_{r=1}^{n} I^{(r)}: \tag{14}$$

where

$$I=\langle \sigma_K | K\subseteq \hat{H}, [\hat{H}:K]=p, \sigma_K=\sum_{k\in K} k\rangle$$

(the ideal generated by the σ_K); and $I^{(r)}$ is the image of I in $\mathbb{Z}[\hat{H}]/(p^r)$. If $H\approx(C_p)^2$, then by Lemma 5.2 and Theorem 2.4

$$\begin{aligned}&|SK_1(\mathbb{Z}[C_{p^n}\times H])|/|SK_1(\mathbb{Z}[C_{p^{n-1}}\times H])|\\ &\geqq |SK_1(\mathbb{Z}[C_p\times H])|/|SK_1(\mathbb{Z}[H])|=p^{p(p-1)/2}.\end{aligned}$$

So by induction on n and Lemma 5.7,

$$|SK_1(\mathbb{Z}[C_{p^n}\times H])|\geqq p^{np(p-1)/2}=\prod_{r=1}^{n} |\mathbb{Z}/p^r\mathbb{Z}\times M_2|=\prod_{r=1}^{n} |(\mathbb{Z}[\hat{H}]/(p^r))/I^{(r)}|.$$

It follows by (14) that $\mathrm{Im}(\hat{\psi}_H)=\sum_{r=1}^{n} I^{(r)}$ in this case.

Now assume $H\approx(C_p)^k$ for $k>2$. Then $G\otimes\hat{\mathbb{Z}}_p[G]$ is generated by elements of the form $g^i\otimes g^j h$ and $h'\otimes g^j h$ for $h,h'\in H$; i.e., by elements in $\tilde{K}\otimes\tilde{\mathbb{Z}}_p[\tilde{K}]$ for rank two subgroups $K\subseteq H$. So in this case we still get

$$\mathrm{Im}(\hat{\psi}_H)=\sum_{r=1}^{n} I^{(r)};$$

and by (12):

$$\begin{aligned}SK_1(\mathbb{Z}[G]&\approx SK_1(\mathbb{Z}[H])\oplus \sum_{r=1}^{n} (\mathbb{Z}[\hat{H}]/(p^r))/I^{(r)})\\ &\approx SK_1(\mathbb{Z}[H])\oplus \sum_{r=1}^{n} (\mathbb{Z}/p^r\mathbb{Z}\otimes M_k).\end{aligned}$$

Corollary 5.9. *For any odd prime p and any $n\geqq 1$,*

$$SK_1(\mathbb{Z}[C_{p^n}\times(C_p)^2]\approx(\mathbb{Z}/p\mathbb{Z})^{np(p-1)/2}$$

and

$$\begin{aligned}&SK_1(\mathbb{Z}[C_{p^n}\times(C_p)^3])\\ &\quad\approx(\mathbb{Z}/p\mathbb{Z})^{5p(p^2-1)/6+p(4p+1)(p-1)(n-1)/6}\times(\mathbb{Z}/p^2\mathbb{Z})^{p(p^2-1)(n-1)/6}.\end{aligned}$$

Given these results, the possibility of proving, or even imagining, a general formula for $SK_1(\mathbb{Z}[G])$ for an arbitrary finite abelian p-group seems rather remote. But the above calculations do suggest a number of conjectures which seem much more approachable. One of the most obvious ones is:

Conjecture 5.10. *Fix integers $n_1, n_2, \ldots, n_k \geqq 1$. Then, for each $r \geqq 1$, there is a polynomial $f_r \in \mathbb{Q}[x]$ such that for any odd prime p,*

$$p^r - r\,k(SK_1(\mathbb{Z}[C_{p^{n_1}} \times \ldots \times C_{p^{n_k}}])) = f_r(p).$$

Such formulas fail to extend to the corresponding 2-groups.

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