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L^2 and intersection cohomologies for a polarizable variation of Hodge structure

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§ 1. Introduction

We consider a polarized variation of Hodge structure $(\mathbf{V}, \mathbf{V}_{\mathbf{Z}}, \mathbf{S}, \mathbf{F})$ of weight k, over a complex manifold X [16]. Here $\mathbf{V}$ denotes a locally constant sheaf of finite dimensional complex vector spaces, $\mathbf{V}_{\mathbf{Z}}$ a sheaf of lattices in $\mathbf{V}$, and $\mathbf{F}$ a decreasing filtration of $\mathbf{O}_X \otimes_{\mathbb{C}} \mathbf{V}$ by locally free sheaves of $\mathbf{O}_X$-modules $\mathbf{F}^p$. By assumption, the filtration $\mathbf{F}$ induces Hodge structure of weight k on the stalks of $\mathbf{V}$, and satisfies the Riemann bilinear relations, as well as the transversality relation, relative to the flat bilinear form $\mathbf{S}$. Typically variations of Hodge structure arise from the cohomology of the fibres in a family of smooth projective varieties.

Henceforth the base X will be assumed to lie as a Zariski open subset in a compact Kähler manifold $\bar{X}$, such that $\bar{X} - X$ is a divisor with normal crossings. In other words, every $x \in \bar{X}$ lies in some neighborhood U, with

$$U \cong \Delta^n, \qquad U \cap X \cong (\Delta^*)^m \times \Delta^{n-m} \tag{1.1}$$

(Δ = unit disc in $\mathbb{C}$, $\Delta^* = \Delta - \{0\}$), for some $m = m(x)$ between 0 and $n = \dim X$. Any smooth quasiprojective variety X has a projective completion $\bar{X}$ of this sort. We equip X with a complete Kähler metric g, whose restriction to any neighborhood U as in (1.1) is asymptotic to a product of metrics on the one dimensional factors: Euclidean on the $n-m$ factors Δ, and of constant negative curvature, complete at the punctures, on the m factors Δ^*; such a metric g always exists [9].

Whether a $\mathbf{V}$-valued form on X is square integrable over compact subsets of $\bar{X}$, relative to the metric g on X and the Hodge metrics [6, 18] on the stalks of $\mathbf{V}$, depends only on the asymptotic behavior of g, not on the particular choice of g, nor on the choice of polarization. The complex of sheaves $\mathbf{A}^{\bullet}_{(2)}(\mathbf{V})$ on $\bar{X}$, defined by the assignment

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(1.2) $U \mapsto$ complex of $\mathbf{V}$-valued forms ω on $U \cap X$, with measurable coefficients and measurable exterior derivative, such that ω and $d\omega$ have finite L^2 norm on $K \cap X$, for every compact $K \subset U$,

is therefore canonically attached to the completion $\bar{X}$. Because of the special nature of the metric g – on neighborhoods U as in (1.1) it splits into a product of metrics on the one dimensional factors, up to quasi-isometry – the sheaves $\mathbf{A}^p_{(2)}(\mathbf{V})$ are fine, so the complex of global sections computes the hypercohomology of $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$. By definition,

$$H^*_{(2)}(\bar{X}, \mathbf{V}) = H^*(\bar{X}, \mathbf{A}^\bullet_{(2)}(\mathbf{V})) = H^*(\Gamma \mathbf{A}^\bullet_{(2)}(\mathbf{V})) \tag{1.3}$$

is the L^2 cohomology of $\bar{X}$, with values in $\mathbf{V}$. The Poincaré lemma applies in the setting of L^2 forms, hence $H^*(Y, \mathbf{A}^\bullet_{(2)}(\mathbf{V})) = H^*(Y, \mathbf{V})$ if Y is an open subset of X.

For variations of Hodge structure over a compact Kähler manifold X, Deligne [10] (see also [22]) has constructed canonical Hodge structures on the cohomology groups $H^p(X, \mathbf{V})$, of weight $p+k$. In effect, Deligne extends the usual Kähler identities to $\mathbf{V}$-valued forms, graded according to the bidegree both in the fibre and the base. The Lefschetz decomposition and Riemann bilinear relations also carry over to this situation without essential change. When the base X is non-compact, Deligne's arguments still put Hodge structures on the L^2 cohomology groups of the completion $\bar{X}$, provided they are finite dimensional. This raises the question of the geometric significance of the L^2 cohomology.

In the case of a one dimensional base X, Zucker [22] has obtained a natural identification

$$H^*_{(2)}(\bar{X}, \mathbf{V}) \cong H^*(\bar{X}, i_* \mathbf{V}); \tag{1.4}$$

here $i_* \mathbf{V}$ denotes the direct image of $\mathbf{V}$ on $\bar{X}$. The L^2 cohomology groups of the completion $\bar{X}$ are then finite dimensional and come equipped with Hodge structures. Simple examples show that (1.4) breaks down without the restriction $\dim X = 1$. Instead, in analogy to similar conjectures in other contexts, Deligne [11] has predicted natural isomorphisms between the L^2 cohomology and the intersection cohomology of $\bar{X}$. This paper establishes Deligne's conjecture:

(1.5) **Theorem.** *The complex of sheaves* $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$ *satisfies the axioms of the (middle) intersection cohomology sheaf with values in the local system* $\mathbf{V}$. *In particular,* $H^*_{(2)}(\bar{X}, \mathbf{V}) \cong IH^*(\bar{X}, \mathbf{V})$.

(1.6) **Corollary.** *The intersection cohomology groups* $IH^p(\bar{X}, \mathbf{V})$ *carry canonical (pure) Hodge structures, of weight* $p+k$.

The proof of Theorem (1.5) is a local problem – it suffices to identify the L^2 cohomology and the intersection cohomology of any neighborhood $U \cong \Delta^n$ as in (1.1). Both cohomologies have weight filtrations, which are induced by the local monodromy. The nature of the metric g and of the Hodge metrics on $\mathbf{V}$ enables us to compare the L^2 cohomology groups of Δ^n to those of $\Delta^n - \{0\}$, via a Mayer-Vietoris spectral sequence: the L^2 cohomology of Δ^n agrees with that of $\Delta^n - \{0\}$ below a certain critical weight, and vanishes above it; the contribution of the critical weight itself is isomorphic to the tensor product of a

certain infinite dimensional space with the L^2 cohomology of $\Delta^n - \{0\}$, also at the critical level. On the other hand, the intersection cohomology groups of Δ^n and of $\Delta^n - \{0\}$ are directly related, for simple geometric reasons. We may assume (1.5) in the case of $\Delta^n - \{0\}$, by induction on the length of the stratification. To complete the argument, we need to know that the weights of $IH^*(\Delta^n, \mathbf{V})$ and $IH^*(\Delta^n - \{0\}, \mathbf{V})$ satisfy the appropriate restrictions; for example, there must be no cohomology at exactly the critical weight. These restrictions follow from a property of the local intersection cohomology – "purity" – which reflects the existence of a pure Hodge structure on $IH^*(\bar{X}, \mathbf{V})$ at the local level. We shall describe it presently.

There are striking formal similarities between the proof of Theorem (1.5) and proofs of similar statements in other situations [3, 8, 17, 23]. We should point out, however, that the geometry of the singularities plays a major role in these other cases, whereas it presents no difficulties for us. Indeed, the space $\bar{X}$ is smooth by hypothesis. Our proof depends crucially on purity and the estimates for the Hodge norms [6, 18]; the rest of the argument is relatively straightforward.

We begin the discussion of purity with some preliminary remarks. For arbitrary local systems $\mathbf{V}$, not necessarily attached to a variation of Hodge structure, the intersection cohomology groups over a neighborhood U of the type (1.1) can be expressed in terms of the local monodromy. The $n-m$ factors Δ of $U \cap X$ do not matter, so we assume $n=m$. Without essential loss of generality, we make the further assumption that the monodromy transformations γ_j, $1 \leqq j \leqq n$, around the punctures in the factors of $U \cap X \cong (\Delta^*)^n$, are unipotent. The γ_j and their logarithms N_j commute pairwise; they act on V, "the" stalk of the pullback of $\mathbf{V}$ to the universal covering of $U \cap X$. We define

$$B^p(N_1, N_2, \ldots, N_n; V) = \bigoplus_{1 \leqq j_1 < \ldots < j_p \leqq n} N_{j_1} N_{j_2} \ldots N_{j_p} V. \tag{1.7}$$

The transformations

$$(-1)^{s-1} N_{j_s} : N_{j_1} \ldots \hat{N}_{j_s} \ldots N_{j_p} V \to N_{j_1} \ldots N_{j_p} V$$

between the summands of $B^{p-1}(N_1, \ldots, N_n; V)$ and of $B^p(N_1, \ldots, N_n; V)$ turn $B^\bullet(N_1, \ldots, N_n; V)$ into a complex. This complex computes the local intersection cohomology:

$$H^*(B^\bullet(N_1, \ldots, N_n; V)) \cong IH^*(U, \mathbf{V}) \tag{1.8}$$

[11, 13].

We return to the original hypotheses, i.e. $\mathbf{V}$ underlies a polarized variation of Hodge structure. Since the N_j commute, their sum

$$N = N_1 + N_2 + \ldots + N_n \tag{1.9}$$

maps the complex $B^\bullet(N_1, \ldots, N_n; V)$ to $B^\bullet(N_1, \ldots, N_n; NV)$.

(1.10) **Theorem.** *The induced map*

$$N : H^*(B^\bullet(N_1, \ldots, N_n; V) \to H^*(B^\bullet(N_1, \ldots, N_n; NV))$$

reduces to zero.

Any linear combination of the N_j is nilpotent, and thus defines a weight filtration on V. As a general rule[1], we center the weight filtrations of nilpotent transformations at zero. According to [4], all elements of the cone

$$C = C(N_1, \ldots, N_n) = \{ \sum_{1 \leqq j \leqq n} \lambda_j N_j \mid \lambda_j > 0 \} \tag{1.11}$$

determine the same weight filtration $W = W(C; V)$. We filter the complex (1.7) by setting

$$W_l(N_{j_1} \ldots N_{j_p} V) = N_{j_1} \ldots N_{j_p} W_{l+p}(C; V). \tag{1.12}$$

From Theorem (1.10) we deduce:

(1.13) **Corollary ("Purity").** *The cohomology of* $B^\bullet(N_1, \ldots, N_n; V)$ *occurs entirely in weight zero or less.*

Deligne [11] had conjectured this result; it is an analogue of Gabber's purity statement in the l-adic case. Along the way towards (1.10) and (1.13), we prove a technical lemma, which has a surprising consequence quite apart from intersection cohomology. We recall that the local variation of Hodge structure over $(\Delta^*)^n$ has a "limit" at 0: a mixed Hodge structure $(W(C; V)[-k], F)$ on V, whose weight filtration is the weight filtration of the cone (1.11), with a shift of indices by $-k$ [6, 21]; the polarization form S, together with any $N \in C(N_1, \ldots, N_n)$, polarizes this limiting mixed Hodge structure. To state the lemma, we fix an element T in the closure of the cone (1.11) and a second element N in its interior. The polarization form S induces a non-degenerate bilinear form $\tilde{S}$ on TV,

$$\tilde{S}(Tu, Tv) = S(u, Tv), \quad u, v \in V. \tag{1.14}$$

We define filtrations $\tilde{F}$, $\tilde{W}$ on TV by the rules

$$\tilde{F}^p = TF^{p+1}, \quad \tilde{W}_l = TW_{l+1}(C; V), \tag{1.15}$$

and denote by $\tilde{N}$ the restriction of N to TV. Then

(1.16) **Lemma ("Descent Lemma").** $(\tilde{W}[1-k], \tilde{F})$ *is a mixed Hodge structure on* TV, *polarized by the pair* $(\tilde{S}, \tilde{N})$.

The passage from a local variation of Hodge structure to the limiting mixed Hodge structure can be reversed [6]. The full cohomology of a compact Kähler manifold, graded according to degree and equipped with the obvious Hodge filtration, provides a particularly simple example of a polarized mixed Hodge structure. In this setting the descent lemma implies:

(1.17) **Corollary.** *Let* Y *be a compact Kähler manifold,* c_1 *and* c_2 *two real cohomology classes of type* (1, 1), *such that* $\lambda_1 c_1 + \lambda_2 c_2$ *is positive – i.e., the cohomology class of a Kähler metric – whenever* $\lambda_1, \lambda_2 > 0$. *Cup product with* c_i *defines a "Kähler operator"* L_i,

$$L_i d = c_i \vee d, \quad d \in H^*(Y, \mathbb{C}).$$

[1] see § 3 for a discussion of the various different conventions

Then, for $l \leqq \dim Y$ *and* $l_1, l_2 \geqq 0$, *the kernel of*

$$L_1^{l_1} L_2^{l_2}: H^{l-l_1-l_2}(Y, \mathbb{C}) \to H^{l+l_1+l_2}(Y, \mathbb{C})$$

lies in the linear span of $\ker L_1^{l_1}$ *and* $\ker L_2^{l_2}$.

The corollary makes a non-obvious assertion even if c_1 and c_2 are both positive. For example, a cohomology class cannot simultaneously lie in the image of one Kähler operator and be primitive with respect to another, as can be seen by setting $l = \dim Y$ and $l_2 = 1$.

This paper is organized as follows. We prove the descent lemma in § 2, by arguments similar to those in [4]. Theorem (1.10) and the two Corollaries (1.13), (1.17) are derived in § 3. Section four prepares the ground for the proof of Theorem (1.5): we introduce a notion of weight on the intersection cohomology of Δ^n and $\Delta^n - \{0\}$, and relate the intersection cohomology groups of these two spaces. We complete the proof in § 5, by making the connection between the L^2 cohomology groups of Δ^n and $\Delta^n - \{0\}$.

Proofs of Theorem (1.5) have been announced by Kashiwara-Kawai [19], and by us[2] in [7], where we also state the descent lemma and indicate a proof of purity. Though the argument outlined in [19] differs from ours, it depends equally on the estimates for the Hodge norms and the purity of the local intersection cohomology. The norm estimates were deduced by us from our multivariable version of the SL_2-orbit theorem. We first described these results in lectures at Harvard in the spring of 1983. Our paper [6] contains the details. Kashiwara [18] has independently derived the norm estimates from the one-variable SL_2-orbit theorem [21] and the constancy of the monodromy weight filtration [4]. The purity statement (1.13) had been conjectured by Deligne [11]. Handwritten notes, in which we proved the descent lemma and purity for two variables, were circulated among several mathematicians during the summer of 1983. A formal announcement of purity for two variables, along with a very special case of Theorem (1.5), has appeared in [5].

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§ 2. The descent lemma

We begin this section by explaining some of our notational conventions. We shall work with weight filtrations on various vector spaces. To keep them apart, our notation for weight filtrations includes the underlying space as a variable; if the filtration is known to arise as the weight filtration of a nilpotent transformation N, we generally include N as well – e.g., $W(N; V)$ –, though we may deviate from this pattern when the context allows no ambiguity. Since we consider Hodge structures and mixed Hodge structures of more than one weight, we consistently center the weight filtration of a nilpotent transformation at zero:

[2] The proof in this paper agrees with our announcement [7], except for one small change: we now work directly with $IH^*(\Delta^n - \{0\}, \mathbf{V})$, instead of the double complex $B^{\bullet,\bullet}$; what remains unsaid in [7] is that the latter computes the intersection cohomology of $\Delta^n - \{0\}$

$W(N; V)$ is characterized by the properties

(2.1) a) $NW_l(N; V) \subset W_{l-2}(N; V)$,
b) N^l: $\mathrm{Gr}_l(W(N; V)) \to \mathrm{Gr}_{-l}(W(N; V))$ is an isomorphism.

We refer to

(2.2) $$\mathrm{PGr}_l(W(N; V)) = \ker \{N^{l+1}: \mathrm{Gr}_l(W(N; V)) \to \mathrm{Gr}_{-l-2}(W(N; V))\}$$

as the primitive suspace of $\mathrm{Gr}_l(N; V)$; one then has a Lefschetz decomposition

(2.3) $$\mathrm{Gr}_l(N; V) = \bigoplus_{s \geqq \max(0, -l/2)} N^s \mathrm{PGr}_{l+2s}(W(N; V))$$

of $\mathrm{Gr}_l(W(N; V))$ into a sum of isomorphic images of primitive pieces.

Usually N will be an infinitesimal isometry of a symmetric or skew-symmetric non-degenerate bilinear form S on V. In that case, $W(N; V)$ satisfies the self-duality condition

(2.4) $W_{-l-1}(N; V)$ is the annihilator of $W_l(N; V)$,

relative to S. The relation

(2.5) $$S_l(N)([u], [v]) = S(u, N^l v),$$

with $u, v \in W_l(N; V)$ representing classes $[u], [v] \in Gr_l(W(N; V))$, $l \geqq 0$, defines a non-degenerate bilinear form $S_l(N)$ on $\mathrm{Gr}_l(W(N; V))$, which makes the decomposition (2.3) orthogonal. In particular, $S_l(N)$ restricts non-degenerately to $\mathrm{PGr}_l(W(N; V))$.

In this section it will be convenient to slightly weaken the hypotheses for a variation of Hodge structure: we replace the sheaf of lattices $\mathbf{V}_{\mathbf{Z}}$ by a sheaf of real forms $\mathbf{V}_{\mathbf{R}}$; the bilinear form $\mathbf{S}$ is assumed to take real values on $\mathbf{V}_{\mathbf{R}}$. We consider a variation of weight k, over a product of punctured discs $(\Delta^*)^n$, with unipotent monodromy transformations γ_j, $1 \leqq j \leqq n$. The two orbit theorems of [21] apply in this more general setting. The monodromy logarithms $N_j = \log \gamma_j$ act on V, "the" stalk of the pullback of $\mathbf{V}$ to the universal covering of $(\Delta^*)^n$. They are nilpotent, commute pairwise, and are infinitesimal automorphisms of the real structure $V_{\mathbf{R}}$ and the bilinear form S. We recall that all elements of the cone

(2.6) $$C = C(N_1, \ldots, N_n) = \{ \sum_{1 \leqq j \leqq n} \lambda_j N_j \mid \lambda_j > 0 \}$$

define the same weight filtration $W(C; V)$ [4]. The primitive subspaces (2.2) do depend on N, however. In fact, the descent lemma provides information on how the primitive spaces can vary with N.

Every $N \in C$ generates a one-variable nilpotent orbit, in the sense of [6]. Because of the constancy of $W(N; V)$ for $N \in C$, the arguments of [21] produce a single limiting mixed Hodge structure $(W(C; V)[-k], F)$, whose weight filtration is that of the cone C, with a shift of indices:

(2.7) $$(W(C; V)[-k])_l = W_{-k+l}(C; V).$$

Each $N \in C$ polarizes the limiting mixed Hodge structure. Since N is already known to be an infinitesimal automorphism of $(V_{\mathbf{R}}, S)$, with weight filtration $W(C; V)$, the polarization conditions come down to

(2.8) a) $S(F^p, F^{k-p+1}) = 0$,
b) N is a morphism of the mixed Hodge structure $(W(C; V)[-k], F)$ of type $(-1, -1)$, i.e. $NF^p \subset F^{p-1}$,
c) the induced pure Hodge structure of weight $l+k$ on $\mathrm{PGr}_l(W(N; V))$ is polarized by the form $S_l(N)$.

We note that b) gives meaning to c): the Hodge structure of $\mathrm{Gr}_l(W(N))$ restricts to a Hodge structure on the primitive subspace. The polarization condition c) for $\mathrm{PGr}_l(W(N))$ consists of the two Riemann bilinear relations; the first of these is already implied by a). To make explicit the second bilinear relation, we let $\mathscr{C}_l$ denote the Weil operator of the Hodge structure on $\mathrm{Gr}_l(W)$,

(2.9) $\mathscr{C}_l v = i^{p-q} v$ for
$v \in H^{p,q}(\mathrm{Gr}_l(W)) = F^p(\mathrm{Gr}_l(W)) \cap \overline{F^q(\mathrm{Gr}_l(W))}, \quad p+q = k+l.$

The hermitian form $H_l(N)$ on $\mathrm{Gr}_l(W)$, defined by

(2.10) $$H_l(N)(u, v) = S_l(N)(\mathscr{C}_l u, \bar{v}), \quad u, v \in \mathrm{Gr}_l(W),$$

is then positive definite on $\mathrm{PGr}_l(W(N))$. Since N commutes with the various Weil operators, this also fixes the sign of $H_l(N)$ on each of the summands of the Lefschetz decomposition (2.3).

All polarized mixed Hodge structures arise as limits of a variation of pure Hodge structure. More precisely:

(2.11) **Proposition** [6, (4.66)]. *Let $C = C(N_1, \ldots, N_n)$ be a cone of commuting nilpotent infinitesimal automorphisms of $(V_{\mathbf{R}}, S)$ and F a filtration of V. The following two statements are equivalent:*

a) Every $N \in C$ determines the same weight filtration $W(C, V)$, $N_j F^p \subset F^{p-1}$ for all j, p, and $(W(C, V)[-k], F)$ is a mixed Hodge structure, polarized by at least one $N_0 \in C$;

b) The correspondence $z \to \exp\left(\sum_{1 \leqq j \leqq n} z_j N_j\right) F$ defines a polarized variation of Hodge structure, of weight k, over the set $H^n(\alpha) = \{z \in \mathbf{C}^n \mid \mathrm{Im}\, z_j > \alpha;\ 1 \leqq j \leqq n\}$, for some real number α.

We note that the variation described in b) drops to a variation of polarized Hodge structure over a product of punctured discs[3], with the N_j as monodromy logarithms.

(2.12) *Remark.* In (2.11 a), the phrase "polarized by at least one $N_0 \in C$", can be replaced by "polarized by every $N_0 \in C$", without affecting the meaning. To see this, we assume inductively that $H_j(N)$ is positive definite on $\mathrm{PGr}_j(W(N))$ whenever $N \in C$ and $j > l$. We then know the sign of $H_l(N)$ on all summands of the decomposition (2.3), except for $\mathrm{PGr}_l(W(N))$ itself. One the other hand,

[3] In the sense of this section, without underlying rational structure

$H_l(N)$ is non-degenerate on $\mathrm{Gr}_l(W)$ and makes the decomposition (2.3) orthogonal, for every $N \in C$, so the sign of $H_l(N)$ must remain constant.

A mixed Hodge structure (W, F) is said to split over $\mathbb{R}$ if there exist a real splitting $U = \{U_j\}$ of W, compatible with the Hodge filtration – in other words, subspaces U_j of the underlying vector space, defined over $\mathbb{R}$, such that

$$W_l = \bigoplus_{j \leq l} U_j, \qquad U_j = \bigoplus_{p+q=j} U_j \cap \mathrm{F}^p \cap \overline{F^q}. \tag{2.13}$$

If such a splitting exists at all, it is unique, since

$$U_j = \bigoplus_{p+q=j} W_j \cap F^p \cap \overline{F^q}. \tag{2.14}$$

This latter identity makes the splitting compatible also with morphisms of the mixed Hodge structure, e.g.

$$N U_j \subset U_{j-2} \quad \text{if } N \text{ is a morphism of type } (-1, -1). \tag{2.15}$$

By a canonical procedure one can attach to each mixed Hodge structure (W, F) an $\mathbb{R}$-split mixed Hodge structure (W, F'), having the same weight filtration, with the following property: the Hodge filtration $F' = gF$ is related to F by a linear transformation g that preserves W and acts as the identity on W_l/W_{l-2} [6]. If a bilinear form S and nilpotent transformation N polarize the original mixed Hodge structure, then g is a (complex) isometry of S and commutes with N; in particular, the pair (S, N) polarizes also (W, F').

In the setting of the descent Lemma (1.16), the passage from the limiting mixed Hodge structure $(W(C)[-k], F)$ to the associated $\mathbb{R}$-split mixed Hodge structure does not affect the hypotheses – this follows from (2.11), since we have dispensed with underlying $\mathbb{Q}$-structures. Neither does it affect the conclusion: the transition map g commutes with the entire cone C, preserves S, and acts trivially on $\mathrm{Gr}(W(C))$. Without loss of generality, we now make the simplifying assumption

$$(W(C)[-k], F) \text{ splits over } \mathbb{R}. \tag{2.16}$$

We shift the indices of the grading U, so as to keep the labeling of weights compatible:

$$W_l(C) = \bigoplus_{j \leq l} U_j, \qquad U_j = \bigoplus_{p+q=k+j} U_j \cap \mathrm{F}^p \cap \overline{F^q}, \tag{2.17}$$

with the corresponding adjustment in (2.14); (2.15) remains correct as stated. With slight abuse of terminology, we call U "the splitting of $(W(C)[-k], F)$ over $\mathbb{R}$". The same convention applies to any $\mathbb{R}$-split mixed Hodge structure whose weight filtration is induced by a $(-1, -1)$ morphism. The splitting determines, and is determined by, the endomorphism $\Lambda = \Lambda(W(C)[-k], F)$, with

$$\Lambda v = l v \quad \text{for } v \in U_l; \tag{2.18}$$

Λ acts as an infinitesimal automorphism of $(V_{\mathbb{R}}, S)$ and stabilizes F. When $\mathrm{Gr}_l(W)$ is identified with U_l, the Weil operators $\mathscr{C}_l$ define a single Weil operator $\mathscr{C}$ on V. Because of (2.15), $\mathscr{C}$ commutes with every $N \in C$. The hermitian forms

$H_l(N)$ on $\mathrm{Gr}_l(W(C))$ are related to a hermitian form H on V in the same way in which the $S_l(N)$ are related to S:

$$H_l(N)([u],[v]) = H(u, N^l v) \quad \text{for } u, v \in W_l(C), \tag{2.19}$$
$$\text{where } H(u,v) = S(\mathscr{C}u, \bar{v});$$

cf. (2.5), (2.10).

We recall the hypotheses and the notation of the descent lemma (1.16), and the additional hypothesis (2.16). Thus T: $V \to V$ is an infinitesimal automorphism of $(V_{\mathbb{R}}, S)$ and commutes with every $N \in C$. As $(-1,-1)$-morphism of the mixed Hodge structure $(W(C;V)[-k], F)$, T preserves both filtrations strictly, with the appropriate shift of indices [16]. The Hodge structure on $U_{l-1} \cong \mathrm{Gr}_{l-1}(W(C;V))$ therefore restricts to a Hodge structure on $U_{l-1} \cap TV = TU_{l+1} \cong \mathrm{Gr}_l(\tilde{W})$, of weight $l+k-1$. The same reasoning also shows that the filtration $\tilde{F}$ induces this Hodge structure. We conclude:

(2.20) $(\tilde{W}[1-k], \tilde{F})$ is a mixed Hodge structure on TV split over $\mathbb{R}$ by $\tilde{U} = TU[1]$. The filtration $\tilde{W}$ is self-dual with respect to $\tilde{S}$, and $\tilde{N}$ acts on TV as a $(-1,-1)$-morphism of $(\tilde{W}[1-k], \tilde{F})$;

here $N \in C$ is arbitrary, as in the statement of the descent lemma.

(2.21) **Observation.** *If $(W(\tilde{N}; TV)[1-k], \tilde{F})$ is a mixed Hodge structure polarized by $(\tilde{N}, \tilde{S})$, for any particular $N \in C$, then $W(\tilde{N}; TV)$ coincides with $\tilde{W}$, so the conclusion of the descent lemma holds with this choice of N.*

Indeed, Lemma (2.17) of [4] contains the implication. We reproduce the argument. For $p+q = k-1+l$, $\tilde{N}^l$ is injective on

$$J^{p,q} = \text{intersection of } \tilde{F}^p \text{ and the complex conjugate of } \tilde{F}^q,$$

because $W_{l-1}(\tilde{N})$ contains $\ker \tilde{N}^l$ and $J^{p,q} \cap W_{l-1}(\tilde{N}) = 0$ by hypothesis. We now apply (2.14) to the $\mathbb{R}$-split mixed Hodge structure (2.20):

$$\tilde{U}_l = \bigoplus_{p+q=k-1+l} J^{p,q} \cap \tilde{U}_l, \qquad \tilde{U}_{-l} = \bigoplus_{p+q=k-1+l} J^{p-l,q-l} \cap \tilde{U}_{-l}.$$

The summands of the first decomposition are mapped by $\tilde{N}^l$ to those of the second, hence $\tilde{N}^l \colon \tilde{U}_l \to \tilde{U}_{-l}$ is injective, hence an isomorphism because of the self-duality of $\tilde{W}$. But then $\tilde{W} = W(\tilde{N}; TV)$, as had to be shown.

We shall use (2.21) to verify the descent lemma for any $N \in C$ sufficiently close to T. For this purpose we fix a specific N in C. Every element of $C(T, N) \subset C$ polarizes the mixed Hodge structure $(W(C)[-k], F)$. Hence, according to (2.11),

$$(z_1, z_2) \mapsto \exp(z_1 T + z_2 N) F \tag{2.22}$$

defines a polarized variation of Hodge structure over $H^2(\alpha)$, for some $\alpha \in \mathbb{R}$. Our next lemma gives information on the images of the Hodge structures (2.22) in TV; it is extracted from the multi-variable SL_2-orbit Theorem [6].

(2.23) **Lemma.** *There exists a unipotent element g of the complex orthogonal group of S with the following properties:*

a) $(W(T;V)[-k], F_1)$, *with* $F_1 = g^{-1}\exp(iN)F$, *is an* $\mathbb{R}$*-split mixed Hodge structure, polarized by* T;

b) The transformations $Y = \Lambda(W(C)[-k], F)$ *and* $Y_1 = \Lambda(W(T)[-k], F_1)$ *commute*;

c) $\log(g)$ *lies in the linear span of the eigenspaces of* $\operatorname{ad} Y_1$ *corresponding to strictly negative eigenvalues*;

d) g *commutes with* T.

Proof. We refer the reader to (4.20, i–iii) of [6], where the present quantities T, N, F, F_1, Y, Y_1, g are denoted by $N_1, N_2, \tilde{F}_2, \tilde{F}_1, \hat{Y}_2, \hat{Y}_1, \exp(iN_2)\exp(-i\hat{N}_2^-)$.

The assertions of the lemma remain valid when we replace (V, S) by $(TV, \tilde{S})$; k by $k-1$; N, T, g by their restrictions $\tilde{N}, \tilde{T}, \tilde{g}$ to TV; F and F_1 by the filtrations $\tilde{F} = TF[1]$ and $\tilde{F}_1 = TF_1[1]$; $W(C)$ by $\tilde{W}$; and Y, Y_1 by the infinitesimal automorphisms $\tilde{Y}, \tilde{Y}_1$ of $(TV_{\mathbb{R}}, \tilde{S})$ which are defined by

$$\tilde{Y}v = T(Y-1)u, \quad \tilde{Y}_1 v = T(Y_1-1)u \quad \text{for } v = Tu \in TV. \tag{2.24}$$

In the case of b)–d) this is immediate, while the analogue of a) follows from the fact that T is a $(-1,-1)$-morphism of the $\mathbb{R}$-split, polarized mixed Hodge structure $(W(T;V)[-k], F_1)$.

(2.25) **Lemma.** *For* $r \in \mathbb{R}^+$ *sufficiently large,* $(W(r\tilde{T}+\tilde{N}; TV)[1-k], \tilde{F})$ *is a mixed Hodge structure, polarized by* $(r\tilde{T}+\tilde{N}, \tilde{S})$.

Proof. We shall repeatedly use the analogues of (2.23 a–d) in the space TV. For simplicity, we refer to these statements as ã)–d̃). Both $\tilde{T}$ and $\tilde{N}$ are horizontal at $\tilde{F}$. Hence, by (2.11), it suffices to show:

(2.26) for $\operatorname{Im} z \gg 0$, $\exp(z(r\tilde{T}+\tilde{N}))\,\tilde{F}$ is a pure Hodge structure on TV, of weight $k-1$, polarized by $\tilde{S}$.

As $(-1,-1)$-morphisms of $(W(C)[-k], F)$, T and N lie in the (-2)-eigenspace of $\operatorname{ad} Y$, so

$$[\tilde{Y}, \tilde{T}] = -2\tilde{T}, \quad [\tilde{Y}, \tilde{N}] = -2\tilde{N}. \tag{2.27}$$

Because of the definition of Y and Y_1,

$$\tilde{Y} \text{ stabilizes the filtration } \tilde{F}, \text{ and } \tilde{Y}_1 \text{ stabilizes } \tilde{F}_1. \tag{2.28}$$

Coupled with ã)–d̃), this allows us to write

$$\begin{aligned}\exp(iy(r\tilde{T}+\tilde{N}))\,\tilde{F} &= \exp(-\tfrac{1}{2}\log y\,\tilde{Y})\exp(i(r\tilde{T}+\tilde{N}))\,\tilde{F}\\ &= \exp(-\tfrac{1}{2}\log y\,\tilde{Y})\exp(ir\tilde{T})\exp(i\tilde{N})\,\tilde{F}\\ &= \exp(-\tfrac{1}{2}\log y\,\tilde{Y})\exp(ir\tilde{T})\,\tilde{g}\tilde{F}_1\\ &= \exp(-\tfrac{1}{2}\log \tilde{Y})\,\tilde{g}\exp(ir\tilde{T})\,\tilde{F}_1,\end{aligned}$$

for $r, y \in \mathbb{R}^+$. Also because of (2.27–28),

$$\exp(ir\tilde{T})\,\tilde{F}_1 = \exp(-\tfrac{1}{2}\log r\tilde{Y}_1)\exp(i\tilde{T})\,\tilde{F}_1\,.$$

We set $\tilde{g}(r)=\exp(\frac{1}{2}\log r\tilde{Y}_1)\,\tilde{g}\exp(-\frac{1}{2}\log r\tilde{Y}_1)$ and combine the previous two identities:

(2.29) $\exp(iy(r\tilde{T}+\tilde{N}))\,\tilde{F}=\exp(-\frac{1}{2}\log y\tilde{Y})\exp(-\frac{1}{2}\log r\tilde{Y}_1)\,\tilde{g}(r)\exp(i\tilde{T})\,\tilde{F}_1$.

According to ã), $(W(\tilde{T})[1-k],\tilde{F}_1)$ is an $\mathbb{R}$-split mixed Hodge structure. In the setting of (2.11), if the mixed Hodge structure splits over $\mathbb{R}$, then the constant α in part b) can be chosen to be 0; cf. [6, (3.12)]. Consequently $\exp(i\tilde{T})\,\tilde{F}_1$ is a pure Hodge structure of weight $k-1$, polarized by $\tilde{S}$. Because of c̃) and d̃), the transformations $\tilde{g}(r)$ preserve the bilinear form $\tilde{S}$ and approach the identity as $r\to\infty$, hence map $\exp(i\tilde{T})\,\tilde{F}_1$ to another polarized Hodge structure for all sufficiently large $r\in\mathbb{R}^+$. As automorphisms of $(TV_{\mathbb{R}},\tilde{S})$, the first two factors on the right hand side of (2.29) act on the set of polarized Hodge structures, as does the real one parameter group generated by $r\tilde{T}+\tilde{N}$. This implies (2.26) and concludes the proof.

Taken together, (2.21) and (2.25) have the following implication: for each $N'\in C$, there exists $N''\in C(T,N')$ such that the assertion of the descent lemma is satisfied by every $N\in C(T,N'')$. In particular the equality $W(\tilde{N};\mathrm{TV})=\tilde{W}$ holds generically for $N\in C(T,N')$. Because of (2.12), the descent lemma (1.16) now reduces to:

(2.30) **Lemma.** *If $N''\in C$ has the property that every $N'\in C(T,N'')$ satisfies the conclusion of* (1.16), *then $W(\tilde{N}'';TV)=\tilde{W}$.*

Proof. To simplify the notation, we write N instead of N''. Since $\tilde{N}\tilde{W}_l\subset\tilde{W}_{l-2}$, and since $\tilde{W}$ is self-dual, the lemma can fail only if $\tilde{N}^l\colon \mathrm{Gr}_l(\tilde{W})\to\mathrm{Gr}_{-l}(\tilde{W})$ has a nontrivial kernel for some $l>0$. In that case

$$\begin{aligned}TW_l(C)&=\tilde{W}_{l-1}\supsetneq\tilde{N}^{-l}(\tilde{W}_{-l-1})=\tilde{N}^{-l}(TW_{-l}(C))=\tilde{N}^{-l}(TN^lW_l(C))\\&=\tilde{N}^{-l}(N^lTW_l(C))=TW_l(C)+T(\ker TN^l);\end{aligned}$$

at the final step we have used the strictness of the $(-1,-1)$-morphisms T, N (or see (2.15)). It therefore suffices to derive a contradiction from the non-containment

(2.31) $$\ker TN^l\not\subset W_l(C)+\ker T,\quad\text{with}\quad l>0.$$

On the other hand, $T(\ker TN^l)\subset\ker N^l\subset W_{l-1}(C)$, and T acts strictly in $W(C)$, hence

(2.32) $$\ker TN^l\subset W_{l+1}(C)+\ker T.$$

Since N and T preserve the splitting U of $(W(C)[-k],F)$ with a shift of indices, (2.31–32) imply

(2.33) $$K\underset{\text{def}}{=}\ker TN^l\cap U_{l+1}\quad\text{is not contained in }\ker T.$$

We claim:

(2.34) the hermitian form $H(\cdot,T^2N^{l-1}\cdot)$ is negative semi-definite on K, but not identically zero.

Indeed, $TK \subset U_{l-1} \cap \ker N^l$, so TK maps into the primitive subspace $\mathrm{PGr}_{l-1}(W(N; V))$, on which $H_{l-1}(N)$ is positive definite; (2.34) follows because T is skew-hermitian and $TK \neq 0$ by (2.33).

To contradict (2.31), we shall show that $H(\cdot, T^2 N^{l-1} \cdot)$ is also positive semi-definite on K. We recall (2.20). As a transformation which commutes with $\tilde{N}$, $\tilde{T}$ preserves $W(\tilde{N}; TV)$ and induces a map

$$[\tilde{T}]: \mathrm{Gr}_{l-1}(W(\tilde{N}; TV)) \to \mathrm{Gr}_{l-1}(W(\tilde{N}; TV)). \tag{2.35}$$

Quite generally we let $[L]$ denote the image of $L \cap W_{l-1}(\tilde{N}; TV)$ in $\mathrm{Gr}_{l-1}(\mathrm{W}(\tilde{N}; TV))$, for any subspace or filtration L of TV. Then, because of the hypotheses of the lemma,

a) $W([\tilde{T}]) = [\tilde{W}[l-1]]$;
b) $(W([\tilde{T}])[2-k-l], [\tilde{F}])$ is a mixed Hodge structure, split over $\mathbb{R}$ by the image $[\tilde{U}[l-1]]$ of the splitting $\tilde{U} = TU[1]$ of $(\tilde{W}[1-k], \tilde{F})$, with a shift of indices by $l-1$;
c) $(W([\tilde{T}])[2-k-l], [\tilde{F}])$ restricts to a mixed Hodge structure on $\mathrm{PGr}_{l-1}(W(\tilde{N}; TV))$, polarized by $[\tilde{T}]$ and $S_{l-1}(\tilde{N})$;

in view of (2.11), these statements follow from [4, (3.3)], where a) is explicitly stated and b), c) are implicit in the proof. Alternatively a)–c) can be inferred from the results of [6], specifically the existence of commuting SL_2-triples.

From (2.32) we deduce $\ker \tilde{N}^l = T(\ker TN^l) \subset TW_{l+1}(C) = \tilde{W}_l$; coupled with a), this shows

d) $\mathrm{PGr}_{l-1}(W(\tilde{N})) = [\ker \tilde{N}^l] \subset [\tilde{W}_l] = W_1([\tilde{T}]; \mathrm{Gr}_{l-1}(W(\tilde{N})))$.

In particular, the weight filtration of $[\tilde{T}]$ on $\mathrm{PGr}_{l-1}(\mathrm{W}(\tilde{N}))$ only has weights $0, \pm 1$, hence

e) $\mathrm{Gr}_1(W([\tilde{T}]; \mathrm{PGr}_{l-1}(\mathrm{W}(\tilde{N})))) = \mathrm{PGr}_1(W([\tilde{T}]; \mathrm{PGr}_{l-1}(W(\tilde{N}))))$.

Because of b), e), the hermitian form $H_1([\tilde{T}])$ is positive definite on $\mathrm{Gr}_1(W([\tilde{T}]; \mathrm{PGr}_{l-1}(W(\tilde{N}))))$. But then $\tilde{H}(\cdot, TN^{l-1} \cdot)$ is positive semi-definite on the T-image of

$$\{v \in V \mid Tv \in W_{l-1}(\tilde{N}), [Tv] \in \mathrm{PGr}_{l-1}(W(\tilde{N}))\}; \tag{2.36}$$

equivalently, $H(\cdot, T^2 N^{l-1} \cdot)$ restricts to a positive semi-definite hermitian form on the space (2.36). Since $TK \subset \ker \tilde{N}^l \subset \mathrm{W}_{l-1}(\tilde{N})$ and $[TK] \subset [\ker \tilde{N}^l] = \mathrm{PGr}_{l-1}(W(\tilde{N}))$, the space (2.36) contains K. This completes the proof of (2.30). As has been pointed out, (2.30) implies the descent lemma.

§ 3. Purity

In this section we adopt the conventions and notation that were introduced at the beginning of § 2. In particular the simplifying assumption (2.16) remains in force, i.e., the existence of a real splitting U for the limiting mixed Hodge structure – without loss of generality, as has been explained.

The capital letters J, K, ... will be used as generic labels for ordered multi-indices $1 \leqq j_1 < j_2 < \ldots < j_p \leqq n$; $p = |J|$ is the length of J. Symbolically we write $J \cup \{j\}$ for the multi-index obtained by adjoining j to J and then reordering. We define $\varepsilon(j, J) = (-1)^s$ if j occurs at the $(s+1)st$ position in $J \cup \{j\}$, but not in J itself, and we assign the value zero to $\varepsilon(j, J)$ if $j \in J$. In terms of this notation, the complex (1.7) has the following description:

$$(3.1)\quad B^p(N_1, N_2, \ldots, N_n; V) = \bigoplus_{J;\,|J|=p} B_J(N_1, N_2, \ldots, N_n; V), \text{ with}$$

$$B_J(N_1, N_2, \ldots, N_n; V) = N_{j_1} N_{j_2} \ldots N_{j_p} V \quad \text{if } J = \{j_1, \ldots, j_p\}.$$

The differential d is given by the formula

$$(3.2)\qquad (d\omega)_K = \sum_{j, J;\, J \cup \{j\} = K} \varepsilon(j, J)\, N_j\, \omega_J\,;$$

here ω_J stands for the $B_J(\ldots)$-component of the chain $\omega \in B^\bullet(\ldots)$.

The N_j commute, hence $N = N_1 + N_2 + \ldots + N_n$ induces a surjective morphism of complexes

$$N\colon B^\bullet(N_1, N_2, \ldots, N_n; V) \to B^\bullet(N_1, N_2, \ldots, N_n; NV).$$

Its kernel consists of those chains ω which are annihilated by N, component-by-component. The assertion of Theorem (1.10) is therefore equivalent to:

(3.3) every cohomology class of the complex $B^\bullet(N_1, \ldots, N_n; V)$ can be represented by a cocycle ω such that $N\omega = 0$.

This is the statement we shall actually prove.

The bilinear form S, the weight filtration $W = W(C; V)$, the Hodge filtration F and the splitting U induce corresponding objects $\tilde{S}$, $\tilde{W}$, $\tilde{F}$, $\tilde{U}$ in the summands of the complex (3.1). We write N_J for the composition $N_{j_1} N_{j_2} \ldots N_{j_p}$; then

$$(3.4)\qquad \begin{aligned} \tilde{S}(N_J u, N_J v) &= S(u, N_J v) \quad \text{for } N_J u, N_J v \in B_J(\ldots), \\ \tilde{W}_l(B_J(\ldots)) &= N_J W_{l+|J|}, \quad \tilde{F}^p(B_J(\ldots)) = N_J F^{p+|J|}, \\ \tilde{U}_l(B_J(\ldots)) &= N_J U_{l+|J|}. \end{aligned}$$

The correspondence (2.11) between mixed Hodge structures and local variations of (pure) Hodge structure allows us to apply the descent Lemma (1.16) repeatedly, to conclude:

(3.5) **Proposition.** *The pair $(\tilde{W}[p-k], \tilde{F})$ defines a mixed Hodge structure on $B^p(N_1, \ldots, N_n; V)$, polarized with respect to $\tilde{S}$ by every element of $C = C(N_1, \ldots, N_n)$, and split over $\mathbb{R}$ by $\tilde{U}$. The coboundary operator d acts as a morphism of type $(-1, -1)$; d is homogeneous of degree -1, relative to the grading $\tilde{U}$.*

We digress briefly, to comment on the normalization of weights. Our labelings of the weight filtration of the limiting mixed Hodge structure and of the monodromy weight filtration $W(C; V)$ differ by k, the weight of the pure Hodge structures on the stalks of $\mathbf{V}$. The shift is often built into the labeling – we are deviating from this practice because the central weight k drops when one

descends from the mixed Hodge structure on V to that on $N_J V$. The drop by $p=|J|$ also accounts for the homogeneity of d of degree -1, even though d is a $(-1,-1)$ morphism. A different convention is more natural for the local intersection cohomology, which inherits a monodromy weight filtration; as will become apparent in § 4, the induced weight filtration on the complex (3.1), without any shift, has advantages in that context.

According to (3.5) the weight filtration of N, acting on the complex (3.1), coincides with $\tilde{W}$. Hence $\tilde{W}_0$ contains the kernel of N, and the purity statement (1.13) immediately follows from (3.3), once we have proved the latter.

We turn to the proof of (3.3). A small calculation shows that the natural inclusion

$$B_{J\cup\{j\}}(N_1, \ldots, N_n; V) \cong N_{J\cup\{j\}} V \hookrightarrow N_J V \cong B_J(N_1, \ldots, N_n; V)$$

is the right adjoint of the map

$$N_j\colon B_J(N_1, \ldots, N_n; V) \cong N_J V \to N_{J\cup\{j\}} V \cong B_{J\cup\{j\}}(N_1, \ldots, N_n; V),$$

relative to the bilinear forms $\tilde{S}$ on the domain and target. This leads to the expression

$$(d^*\omega)_J = \sum_j \varepsilon(j, J)\, \omega_{J\cup\{j\}} \tag{3.6}$$

for the right adjoint d^* of the coboundary operator d, again relative to $\tilde{S}$.

(3.7) **Lemma.** *$(dd^* + d^*d)\,\omega = N\omega$, for any $\omega \in B^\bullet(N_1, \ldots, N_n; V)$.*

Proof. We combine the identities (3.2) and (3.6):

$$\{(dd^* + d^*d)\,\omega\}_J = \sum_{j, K;\, K\cup\{j\}=J} \sum_i \varepsilon(j, K)\,\varepsilon(i, K)\, N_j\, \omega_{K\cup\{i\}}$$
$$+ \sum_i \sum_{j, I;\, I\cup\{j\}=J\cup\{i\}} \varepsilon(i, J)\,\varepsilon(j, I)\, N_j\, \omega_I.$$

The terms in the two sums corresponding to $i \neq j$ cancel, those corresponding to $i=j$ add up to $\sum_{j\in J} N_j \omega_J$ in the case of the first sum, and to $\sum_{j\notin J} N_j \omega_J$ in the case of the second. The lemma follows.

(3.8) **Corollary.** *Every cocycle $\omega \in \tilde{W}_{-1}(B^\bullet(N_1, \ldots, N_n; V))$ is cohomologous to a cocycle in the kernel of N.*

Proof. The image of N contains $\tilde{W}_{-1}$, hence $\omega = N\phi = (dd^* + d^*d)\,\phi$ for some cochain ϕ. We set $\psi = d^*d\phi$; then $\omega = \psi + d(d^*\phi)$, and $N\psi = Nd^*d\phi = d^*dN\phi = d^*d\omega = 0$.

The corollary establishes (3.3) for all cohomology classes of strictly negative weight. Since d is homogeneous of degree -1, we only need to consider classes represented by cocycles $\omega \in \tilde{U}_l$, $l \geqq 0$. Any cocycle or cochain in $\tilde{U}_l$ can be expressed uniquely as the sum of a primitive part ω' and an "imprimitive" part ω'':

$$\omega = \omega' + \omega'', \quad \omega' \in \ker\{N^{l+1}\colon \tilde{U}_l \to \tilde{U}_{-l-2}\},\ \omega'' \in N\tilde{U}_{l+2}. \tag{3.9}$$

As in § 2, we define hermitian forms

$$\tilde{H}_l(\omega, \phi) = \tilde{S}(\mathscr{C}\omega, N^l\bar{\phi}), \quad l \geqq 0, \tag{3.10}$$

in terms of the graded Weil operator $\mathscr{C}$. Then $\tilde{H}_l$ restricts non-degenerately to $\tilde{U}_l$, makes the decomposition (3.9) orthogonal, and is positive definite on the primitive subspace $\tilde{U}'_l \subset \tilde{U}_l$; cf. (3.5). Since N maps $\tilde{U}_1$ isomorphically onto $\tilde{U}_{-1}$, there exists a unique non-degenerate hermitian form $\tilde{H}_{-1}$ on $\tilde{U}_{-1}$, such that

$$\tilde{H}_{-1}(N\omega, N\phi) = -\tilde{H}_1(\omega, \phi), \quad \omega, \phi \in \tilde{U}_1. \tag{3.11}$$

We let δ denote the adjoint of d: $\tilde{U}_{l+1} \to \tilde{U}_l$, $l \geqq -1$, relative to the forms $\tilde{H}_{l+1}$ on the domain and $\tilde{H}_l$ on the target.

(3.12) **Lemma.** *a) For $l \geqq 0$, the map $(d\delta + \delta d)$: $\tilde{U}_l \to \tilde{U}_l$ leaves invariant the decomposition $\tilde{U}_l = \tilde{U}'_l \oplus \tilde{U}''_l$, and acts as the identity on the subspace of imprimitive cocycles $\tilde{U}''_l$. b) $\tilde{H}_l((d\delta + \delta d)\,\omega, \phi) = \tilde{H}_l(\omega, \phi) + \tilde{H}_{l-1}((d^*\omega)', (d^*\phi)')$ if $\omega, \phi \in \tilde{U}_l$, $l \geqq 1$.*

Proof. Some preliminary remarks: d^* and δ are homogeneous of degree -1 and $+1$, respectively, relative to the grading $\tilde{U}$; δ is both left and right adjoint to d, since the forms $\tilde{H}_l$ are hermitian; $\tilde{S}$ changes parity, depending on the degree of cocycles, so d^* is right adjoint to d but left adjoint to $-d$, in both cases with respect to $\tilde{S}$.

We fix $\omega \in \tilde{U}_l$, $l \geqq 0$. Since $d^*\omega$ has weight $l-1$, N^l annihilates its primitive part $(d^*\omega)'$ (for $l=0$, $d^*\omega$ is imprimitive by definition). We express $(d^*\omega)''$ as $N\psi$, with $\psi \in \tilde{U}_{l+1}$ uniquely determined by this relation. Then, for any $\phi \in \tilde{U}_{l+1}$,

$$\begin{aligned}
\tilde{H}_{l+1}(\phi, \delta\omega) &= \tilde{H}_l(d\phi, \omega) = \tilde{S}(\mathscr{C}\,d\phi, N^l\bar{\omega}) \\
&= \tilde{S}(d\mathscr{C}\phi, N^l\bar{\omega}) = \tilde{S}(\mathscr{C}\phi, d^*\,N^l\bar{\omega}) = \tilde{S}(\mathscr{C}\phi, \overline{N^l\,d^*\,\omega}) \\
&= \tilde{S}(\mathscr{C}\phi, \overline{N^l(d^*\omega)''}) = \tilde{S}(\mathscr{C}\phi, \overline{N^{l+1}\psi}) = \tilde{H}_{l+1}(\phi, \psi),
\end{aligned}$$

hence $\delta\omega = \psi$.

We now suppose $l \geqq 1$. Combining the preceeding identity with lemma (3.7), we find

$$\begin{aligned}
\tilde{H}_l((d\delta + \delta d)\,\omega, \omega) &= \tilde{H}_{l+1}(\delta\omega, \delta\omega) + \tilde{H}_{l-1}(d\omega, d\omega) \\
&= \tilde{H}_{l+1}(\psi, \psi) + \tilde{S}(\mathscr{C}\,d\omega, N^{l-1}\,d\bar{\omega}) \\
&= \tilde{S}(\mathscr{C}\psi, N^{l+1}\bar{\psi}) + \tilde{S}(\mathscr{C}\,d\omega, d\,N^{l-1}\,\bar{\omega}) \\
&= -\tilde{S}(\mathscr{C}\,N\psi, N^l\bar{\omega}) - \tilde{S}(d^*\,\mathscr{C}\,d\omega, N^{l-1}\,\bar{\omega}) \\
&= -\tilde{S}(\mathscr{C}(d^*\omega)'', \overline{N^{l-1}(d^*\omega)''}) - \tilde{S}(d^*\,d\,\mathscr{C}\,\omega, N^{l-1}\,\bar{\omega}) \\
&= \tilde{S}(\mathscr{C}(d^*\omega)', \overline{N^{l-1}(d^*\omega)'}) \\
&\quad - \tilde{S}(\mathscr{C}\,d^*\,\omega, N^{l-1}\,d^*\,\bar{\omega}) - \tilde{S}(d^*\,d\,\mathscr{C}\,\omega, N^{l-1}\,\bar{\omega}) \\
&= \tilde{H}_{l-1}((d^*\omega)', (d^*\omega)') - \tilde{S}((d\,d^* + d^*\,d)\,\mathscr{C}\,\omega, N^{l-1}\,\bar{\omega}) \\
&= \tilde{H}_{l-1}((d^*\omega)', (d^*\omega)') - \tilde{S}(N\mathscr{C}\,\omega, N^{l-1}\,\bar{\omega}) \\
&= \tilde{H}_{l-1}((d^*\omega)', (d^*\omega)') + \tilde{H}_l(\omega, \omega).
\end{aligned}$$

Since $d\delta+\delta d$ is self-adjoint, the usual anti-symmetrization procedure now implies b). If $\omega=N\tau\in N\tilde{U}_{l+2}$ is imprimitive, then so is $d^*\omega=Nd^*\tau$, and the relation b) reduces to

$$\tilde{H}_l((d\delta+\delta d)\omega,\phi)=\tilde{H}_l(\omega,\phi)+\tilde{H}_{l-1}((d^*\omega)',(d^*\phi)')=\tilde{H}_l(\omega,\phi),$$

for any $\phi\in\tilde{U}_l$. We conclude that $d\delta+\delta d$ acts as the identity on $\tilde{U}_l''$, $l\geqq 1$. But then $d\delta+\delta d$ preserves also the orthogonal complement $\tilde{U}_l'$ of $\tilde{U}_l''$. The assertion a) follows, for $l\geqq 1$.

To verify a) in the remaining case $l=0$, we consider an imprimitive cocycle $\omega=N\tau\in N\tilde{U}_2$, and an arbitrary $\phi\in\tilde{U}_0$. In this situation, $(d^*\omega)''=(Nd^*\tau)''=Nd^*\tau$, hence $\delta\omega=d^*\tau$; similarly $\delta\phi=\sigma\in\tilde{U}_1$ is uniquely characterized by the equality $N\sigma=(d^*\phi)''=d^*\phi$ (the primitive part of $d^*\phi$ vanishes for weight reasons!). We compute:

$$\begin{aligned}\tilde{H}_0((d\delta+\delta d)\omega,\phi)&=\tilde{H}_1(\delta\omega,\delta\phi)+\tilde{H}_{-1}(d\omega,d\phi)\\&=\tilde{H}_1(d^*\tau,\sigma)+\tilde{H}_{-1}(Nd\tau,d\phi)=\tilde{S}(\mathscr{C}d^*\tau,N\bar{\sigma})-\tilde{S}(\mathscr{C}d\tau,d\bar{\phi})\\&=\tilde{S}(d^*\mathscr{C}\tau,d^*\bar{\phi})-\tilde{S}(d\mathscr{C}\tau,d\bar{\phi})=\tilde{S}((dd^*+d^*d)\mathscr{C}\tau,\bar{\phi})\\&=\tilde{S}(N\mathscr{C}\tau,\bar{\phi})=\tilde{H}_0(\omega,\phi);\end{aligned}$$

at the penultimate step we have used Lemma (3.7). From here on the argument proceeds exactly as in the case of strictly positive l.

(3.13) **Corollary.** *The kernel of* $(d\delta+\delta d)$: $\tilde{U}_l\to\tilde{U}_l$ *reduces to zero if* $l>0$, *and is contained in* $\ker N$ *if* $l=0$.

Proof. According to (3.12a), any ω in the kernel is necessarily primitive. For $l=0$ this simply means $N\omega=0$. For $l\geqq 1$ we appeal to (3.12b): since $\tilde{H}_l$ and $\tilde{H}_{l-1}$ are positive definite on the primitive subspaces of $\tilde{U}_l$ and $\tilde{U}_{l-1}$, the vanishing of

$$\tilde{H}_l((d\delta+\delta d)\omega,\omega)=\tilde{H}_l(\omega,\omega)+\tilde{H}_{l-1}((d^*\omega)',(d^*\omega)')$$

forces ω itself to vanish.

For the statement of the next Lemma, we consider a complex of finite dimensional vector spaces $(C^\bullet, d)$, whose terms C^p are equiped with non-degenerate bilinear or hermitian forms. The left adjoint of d: $C^{p-1}\to C^p$ will be denoted by δ.

(3.14) **Lemma.** *If* $d\delta+\delta d$ *is injective on* C^{p-1}, *the p-th cohomology group is isomorphic to* $\ker d\cap\ker\delta$. *In particular* $H^p(C^\bullet)$ *vanishes whenever* $d\delta+\delta d$ *is injective on both* C^{p-1} *and* C^p.

Proof. The subspaces dC^{p-1} and $\ker\delta$ of C^p have complementary dimensions. Any element v of the intersection can be expressed as $v=du$, with $u\in C^{p-1}$. By assumption $d\delta+\delta d$ has an inverse on C^{p-1}, hence $u=(d\delta+\delta d)w$, for some $w\in C^{p-1}$. We use the injectivity of $d\delta+\delta d$ on C^{p-1} once more: from the chain of equalities

$$0=\delta v=\delta du=\delta d(d\delta+\delta d)w=\delta d\delta d w=(d\delta+\delta d)\delta dw$$

we deduce $\delta dw=0$, hence $v=du=d^2\delta w=0$. We thus obtain the direct sum decomposition $C^p=dC^{p-1}\oplus\ker\delta$. The assertion of the Lemma is a formal consequence.

We apply the Lemma to the complex $C^\bullet = \tilde{U}_{-\bullet}(B^*(N_1, \dots, N_n; V)$; cf. (3.5). According to (3.13), $d\delta + \delta d$ is injective below degree zero, and $\ker N$ contains the intersection of the kernels of d and δ, acting on C^0. We conclude: any cocycle $\omega \in \tilde{U}_l$, with $l \geqq 0$, is cohomologous to a cocycle in the kernel of N. As was remarked earlier, this fact completes the verification of theorem (1.10) and of the purity statement (1.13).

Corollary (1.17) depends on a particularly simple case of (1.13). For $n=2$, the first cohomology group of the complex (3.1) equals

$$\{(N_1 u_1, N_2 u_2) | u_j \in V; N_1 N_2 (u_1 - u_2) = 0\} / \{(N_1 u, N_2 u) | u \in V\},$$

and the correspondence $(N_1 u_1, N_2 u_2) \to u_1 - u_2$ induces an isomorphism

$$H^1(B^\bullet(N_1, N_2; V)) \cong \ker(N_1 N_2)/(\ker N_1 + \ker N_2).$$

If $u_j \in V$ has weight l relative to the filtration $W(N_1 + N_2; V)$, the convention (3.4) assigns weight $l-1$ to $N_j u_j$, relative to $W(N_1 + N_2; N_j V)$. Hence purity gives the containment

$$\ker N_1 N_2 \subset W_1(N_1 + N_2; V) + \ker N_1 + \ker N_2. \tag{3.15}$$

This result can be generalized as follows:

(3.16) **Lemma.** $\ker(N_1^r N_2^s) \subset W_{r+s-1}(N_1 + N_2; V) + \ker N_1^r + \ker N_2^s$.

Proof. By induction on $r+s$. The Descent Lemma and Proposition (2.11) permit us to apply (3.15) to $\tilde{V} = V/\ker N_1^{r-1} N_2^{s-1} \cong N_1^{r-1} N_2^{s-1} V$ and the restrictions $\tilde{N}_1, \tilde{N}_2$ of the maps N_1, N_2:

$$\ker \tilde{N}_1 \tilde{N}_2 \subset W_1(\tilde{N}_1 + \tilde{N}_2; \tilde{V}) + \ker \tilde{N}_1 + \ker \tilde{N}_2.$$

Lifted back to V, the containment takes the form

$$\ker N_1^r N_2^s \subset W_{r+s-1}(N_1 + N_2; V) + \ker N_1^r N_2^{s-1} + \ker N_1^{r-1} N_2^s;$$

cf. (3.4). This immediately implies the Lemma.

We recall the hypotheses of corollary (1.17). Every linear combination $L = \lambda_1 L_1 + \lambda_2 L_2$ with strictly positive coefficients λ_1, λ_2 arises the Kähler operator of a Kähler metric on Y. The "Hard Lefschetz Theorem",

$$L^l: H^{k-l}(Y, \mathbb{C}) \xrightarrow{\sim} H^{k+l}(Y, \mathbb{C}), \quad \text{with} \quad k = \dim Y,$$

is equivalent to the assertion that any such L induces the weight filtration

$$W_l = \bigoplus_{j \geqq k-l} H^j(Y, \mathbb{C}).$$

We define a decreasing "Hodge filtration" F on the cohomology ring of Y,

$$F^p = \bigoplus_s \bigoplus_{r \leqq k-p} H^{r,s}(Y).$$

Then $(W[-k], F)$ is an $\mathbb{R}$-split mixed Hodge structure. Every L in the cone $C(L_1, L_2)$ of strictly positive linear combinations polarizes this mixed Hodge structure, relative to the Hodge-Riemann bilinear form – this, in effect, is the content of the two Riemann bilinear relations [16].

The correspondence (2.11) associates a local variation of Hodge structure to the polarized mixed Hodge structure $(W[-k], F)$. We can therefore apply the descent Lemma and its consequences, in particular Lemma (3.16):

$$\ker L_1^{l_1} L_2^{l_2} \subset W_{l_1+l_2-1} + \ker L_1^{l_1} + \ker L_2^{l_2}$$
$$= \bigoplus_{j \geqq k+1-l_1-l_2} H^j(Y, \mathbb{C}) + \ker L_1^{l_1} + \ker L_2^{l_2}.$$

Thus $\ker L_1^{l_1} + \ker L_2^{l_2}$ contains the kernel of $L_1^{l_1} L_2^{l_2}$ on $H^j(Y, \mathbb{C})$, as soon as $k+1-l_1-l_2$ exceeds j. Corollary (1.17) follows.

§4. Cohomological preliminaries

To prove Theorem (1.5), we only need to construct local isomorphisms between the L^2 and intersection cohomology,

$$H^*_{(2)}(U, \mathbf{V}) \cong IH^*(U, \mathbf{V}), \tag{4.1}$$

for any neighborhood U as in (1.1). Indeed, (4.1) implies the conditions on the support of the local cohomology [2, 15]. The dual conditions for compactly supported cohomology then follow from general properties of L^2 cohomology with respect to complete metrics[4] [1, 12]; alternatively, our proof of (4.1) can be modified, so as to apply to cohomology with compact support. The isomorphisms (4.1) also identify $\mathbf{V}$ with the cohomology sheaf of $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$ over the open stratum X. The remaining axioms – i.e., the local constancy of the cohomology along the various strata and the vanishing of $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$ below degree zero – are visibly satisfied.

We shall argue by induction on the integer m, in the notation of (1.1). When $m=0$, (4.1) reduces to the Poincaré Lemma. For the induction step, we can make two simplifying assumptions. If $m<n$, the Kähler metric on $U \cap X \cong (\Delta^*)^m \times \Delta^{n-m}$ and the Hodge metric on $\mathbf{V}$ [6, 18] are quasi-isometric to products of metrics of the same type on $(\Delta^*)^m$ with constant metrics on Δ^{n-m}. The L^2 cohomology is therefore constant along the factor Δ^{n-m}, as is the intersection cohomology. In other words, we are free to assume $m=n$; from now on,

$$U \cap X = (\Delta^*)^n, \qquad U = \Delta^n. \tag{4.2}$$

The monodromy transformations $\gamma_1, \ldots, \gamma_n$ around the punctures in the n copies of Δ^* become unipotent in some finite covering of $U \cap X$ [16]. Both the L^2 cohomology and the intersection cohomology of U coincide with the invariant parts of the cohomologies in the covering, extended to a branched covering of U. Thus we suppose

$$\gamma_j = e^{N_j}, \quad 1 \leqq j \leqq n, \quad \text{with } N_j \text{ nilpotent.} \tag{4.3}$$

The N_j act on the generic stalk V of the local system $\mathbf{V}$, they are infinitesimal isometries of the polarization form S and commute pairwise.

In the situation (4.2-3) we shall produce an explicit isomorphism (4.1), which

[4] Details can be found in §5, where the duality is established for a different purpose

will be functorial in both U and $\mathbf{V}$. The idea is to relate both cohomologies over Δ^n to the corresponding cohomologies over $\Delta^n - \{0\}$. By induction on the length of the stratification of $\bar{X}$ – i.e., the integer m – we already know

$$H^*_{(2)}(\Delta^n - \{0\}, \mathbf{V}) \cong IH^*(\Delta^n - \{0\}, \mathbf{V}), \tag{4.4}$$

functorially in both variables.

The intersection cohomology remains unchanged when we replace Δ^n by a small ball B centered at the origin. Similarly one can replace $\Delta^n - \{0\}$ by $B - \{0\}$, or even by ∂B, since $B - \{0\}$ is topologically a product of ∂B with an open interval. With these identifications the exact cohomology sequence of the pair $(B, \partial B)$, stratified by intersections of coordinate hyperplanes, takes the form

$$\to IH^p_c(\Delta^n, \mathbf{V}) \to IH^p(\Delta^n, \mathbf{V}) \to IH^p(\Delta^n - \{0\}, \mathbf{V}) \to IH^{p+1}_c(\Delta^n, \mathbf{V}) \to. \tag{4.5}$$

The support conditions for the local intersection cohomology around the codimension n stratum [2, 15],

$$IH^p(\Delta^n, \mathbf{V}) = 0 \text{ if } p \geqq n, \qquad IH^p_c(\Delta^n, \mathbf{V}) = 0 \text{ if } p \leqq n, \tag{4.6}$$

force the exact sequence (4.5) to collapse:

$$IH^p(\Delta^n, \mathbf{V}) \xrightarrow{\sim} IH^p(\Delta^n - \{0\}, \mathbf{V}) \quad \text{for } p < n, \tag{4.7a}$$

and $IH^p(\Delta^n - \{0\}, \mathbf{V}) \cong IH^{p+1}_c(\Delta^n, \mathbf{V})$ for $p \geqq n$. We use Poincaré duality [2, 15] to re-interpret this latter isomorphism:

$$IH^p(\Delta^n - \{0\}, \mathbf{V}) \xrightarrow{\sim} IH^{2n-p-1}(\Delta^n, \mathbf{V})^* \quad \text{for } p \geqq n; \tag{4.7b}$$

indeed, the local system $\mathbf{V}$ is self-dual via the polarization form, so Poincaré duality pairs $IH^*_c(\Delta^n, \mathbf{V})$ and $IH^*(\Delta^n, \mathbf{V})$ in complementary dimensions.

To construct the isomorphism (4.1), we shall establish partial analogues of (4.6-7) in the setting of L^2 cohomology:

(4.8) **Proposition.** *The inclusion* $\Delta^n - \{0\} \hookrightarrow \Delta^n$ *induces isomorphisms* $H^p_{(2)}(\Delta^n - \{0\}, \mathbf{V}) \cong H^p_{(2)}(\Delta^n, \mathbf{V})$ *if* $p < n$; $H^p_{(2)}(\Delta^n, \mathbf{V})$ *vanishes if* $p \geqq n$.

Before turning to the proof, we note that this statement formally implies (4.1). Combining the proposition with (4.4) and (4.6-7), one finds

$$H^p_{(2)}(\Delta^n, \mathbf{V}) \cong H^p_{(2)}(\Delta^n - \{0\}, \mathbf{V}) \cong IH^p(\Delta^n - \{0\}, \mathbf{V}) \cong IH^p(\Delta^n, \mathbf{V})$$

if $p < n$, and $H^p_{(2)}(\Delta^n, \mathbf{V}) = IH^p(\Delta^n, \mathbf{V}) = 0$ if $p \geqq n$. In either case $H^p_{(2)}(\Delta^n, \mathbf{V})$ and $IH^p(\Delta^n, \mathbf{V})$ are isomorphic, as asserted by (4.1). The isomorphism comes about as a composition of functorial maps, and is therefore functorial itself.

The proof of (4.8) depends on a certain notion of weights for the L^2 and intersection cohomologies of Δ^n and $\Delta^n - \{0\}$. The next few paragraphs introduce this notion. We recall the description (1.8) of the local intersection cohomology. It will be convenient to rephrase the definition of the complex (1.7) in slightly different but equivalent terms. We regard V as a module for the abelian Lie algebra

$$\mathfrak{n} = \mathbb{C}N_1 \oplus \mathbb{C}N_2 \oplus \ldots \oplus \mathbb{C}N_n. \tag{4.9}$$

The coboundary operator of Lie algebra cohomology turns

$$A^\bullet(N_1, \ldots, N_n; V) = V \otimes \bigwedge^\bullet \mathfrak{n}^* \tag{4.10}$$

into a complex. It contains $B^\bullet(N_1, \ldots, N_n ; V)$ with

$$B^p(N_1, \ldots, N_n ; V) = \text{linear span of } \{v \otimes N_{j_1}^* \wedge \ldots \wedge N_{j_p}^* \mid v \in N_{j_1} N_{j_2} \ldots N_{j_p} V\}, \tag{4.11}$$

as subcomplex. This new definition of $B^\bullet(N_1, \ldots, N_n ; V)$ is consistent with (1.7).

It was mentioned already in §1 that all elements of the cone (1.11) determine the same weight filtration [4]. We let Y denote a particular "splitting endomorphism" of $W(C; V)$ [6]: an infinitesimal automorphism of $(V_{\mathbf{R}}, S)$ with integral eigenvalues, such that

$$W_l(C; V) = \oplus_{r \leqq l} V_r, \quad \text{where } V_r = \{v \in V \mid Yv = rv\}, \tag{4.12}$$

and $N_j V_r \subset V_{r-2}$ for all r, j; the latter condition can be rephrased as follows:

$$[Y, N_j] = -2N_j, \quad 1 \leqq j \leqq n. \tag{4.13}$$

In accordance with these commutation relations we make Y act on $\mathfrak{n}^*$ as multiplication by 2; cf. (4.9). The complexes (4.10-11) then inherit actions of Y which commute with the differentials. Our present convention does not agree with the labeling of weights (1.12), but is compatible with the natural weight filtration on the local intersection cohomology. Via the isomorphisms (1.8) Y operates on the cohomology groups $IH^p(\Delta^n, \mathbf{V})$ – semisimply, with integral eigenvalues, as in the case of V and $\mathfrak{n}^*$. We use (4.7a) and a twisted version of (4.7b) to transfer the action of Y to $IH^*(\Delta^n - \{0\}, \mathbf{V})$: since $\bigwedge^n \mathfrak{n}^*$ has dimension one, (4.7b) is equivalent to

$$IH^p(\Delta^n - \{0\}, \mathbf{V}) \cong IH^{2n-p-1}(\Delta^n, \mathbf{V})^* \otimes \bigwedge^n \mathfrak{n}^* \text{ for } p \geqq n. \tag{4.14}$$

The explicit form of Poincaré duality dictates this twist, as will become apparent later. By definition, Y acts on $IH^*(\Delta^n - \{0\}, \mathbf{V})$ so as to make the isomorphisms (4.7a) and (4.14) Y-equivariant.

As a general convention, we write $\{\ldots\}_l$ for the l-eigenspace in any vector space $\{\ldots\}$ on which Y acts. When the purity statement (1.13) is adjusted to our new convention for assigning weights, we find

$$IH^p(\Delta^n, \mathbf{V})_l = 0 \text{ unless } l \leqq p; \tag{4.15}$$

cf. (1.12), (4.12-13). For $IH^*(\Delta^n - \{0\}, \mathbf{V})$, (4.15) implies:

$$IH^p(\Delta^n - \{0\}, \mathbf{V})_l \text{ vanishes if either } p < n \text{ and } l \geqq n, \text{ or } p \geqq n \text{ and } l \leqq n. \tag{4.16}$$

Purity will enter the Proof of (4.8) via this statement.

Our next task is to define an action of Y on the L^2 cohomology of Δ^n and $\Delta^n - \{0\}$. For the moment, D shall be an arbitrary open subset of Δ^n. We recall the definition of $H^*_{(2)}(D, \mathbf{V})$ as the cohomology of the complex of global sections of $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$ over D. Any $\omega \in \Gamma_D \mathbf{A}^\bullet_{(2)}(\mathbf{V})$ is a $\mathbf{V}$-valued differential form on $D \cap (\Delta^*)^n$, with locally L^2 coefficients and locally L^2 exterior derivative, such that

$$\int_{K\cap(\Delta^*)^n} \|\omega\|^2\, dm<\infty, \qquad \int_{K\cap(\Delta^*)^n} \|d\omega\|^2\, dm<\infty, \tag{4.17}$$

whenever $K\subset D$ is compact; the measure dm and the pointwise norms are taken with respect to the Poincaré metric on the n factors of $(\Delta^*)^n$ (i.e., the complete metric on Δ^* of constant negative curvature -1) and the Hodge metrics on the stalks of $\mathbf{V}$ [6, 18].

We let H denote the upper half plane in $\mathbb{C}$, with coordinate $z=x+iy$. Then H^n covers $(\Delta^*)^n$,

$$p: H^n\to(\Delta^*)^n, \qquad p(z_1,\ldots,z_n)=(e^{2\pi i z_1},\ldots,e^{2\pi i z_n}). \tag{4.18}$$

By translation, $\mathbb{R}^n$ acts on the x-coordinates. This action drops to the rotation action of the n-torus $T^n=\mathbb{R}^n/\mathbb{Z}^n$ on $(\Delta^*)^n$. We extend the T^n-action to $\mathbf{V}$-valued forms, as follows. Each $v\in V$ may be regarded as a flat section of $p^*\mathbf{V}$, so

$$\tilde{v}(z)=\exp\Big(\sum_{1\le j\le n} x_j N_j\Big)\, v \tag{4.19}$$

defines a section of $\mathbf{C}^\infty\otimes p^*\mathbf{V}$. This section is invariant under all covering translations, and consequently drops to a section of $\mathbf{C}^\infty\otimes\mathbf{V}$ over $(\Delta^*)^n$. In view of the definition, $\tilde{v}$ has differential

$$d\tilde{v}=\sum_{1\le j\le n} \widetilde{N_j v}\otimes dx_j. \tag{4.20}$$

We now suppose that $D\subset\Delta^n$ is a circular domain – i.e., open and T^n-invariant –, and consider a $\mathbf{V}$-valued differential form ω on $D\cap(\Delta^*)^n$. When the pullback $p^*\omega$ to $p^{-1}(D)\subset H^n$ is expressed in terms of a frame $\{\tilde{v}_k\}$ corresponding to some basis $\{v_k\}$ of V, the coefficients become invariant under the covering transformations $x_j\to x_j+1$. Thus T^n operates on such forms ω by translation of the coefficients. The operation commutes with the exterior derivative; cf. (4.20). It also preserves the L^2 norm, because the Poincaré metric on Δ^* is invariant under rotations. In short, we have constructed an operation of T^n on the complex $\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathbf{V})$. The space of invariants,

$$\{\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathbf{V})\}^{T^n}=\{\omega\in\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathbf{V})\mid\omega\text{ is }T^n\text{-invariant}\}, \tag{4.21}$$

forms a subcomplex.

(4.22) **Lemma.** *If $D\subset\Delta^n$ is a circular domain, the inclusion*

$$\{\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathbf{V})\}^{T^n}\hookrightarrow\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathbf{V})$$

induces an isomorphism of cohomology $H^*(\{\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathbf{V})\}^{T^n})\cong H^*_{(2)}(D,\mathbf{V})$.

Proof. The weight filtration $W(C;V)$ determines a filtration $\mathbf{W}$ of the sheaf $\mathbf{V}$. We use the induced Hodge metrics on $\mathbf{W}_l$ and $\mathrm{Gr}_l(\mathbf{W})$ to define complexes of sheaves $\mathbf{A}^\bullet_{(2)}(\mathbf{W}_l)$, $\mathbf{A}^\bullet_{(2)}(\mathrm{Gr}_l(\mathbf{W}))$, in complete analogy to $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$. For any circular domain D, the torus T^n operates on the spaces of global sections $\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathbf{W}_l)$, $\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathrm{Gr}_l(\mathbf{W}))$, as it does on $\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathbf{V})$. Since $\mathrm{Gr}_l(\mathbf{W})$ has trivial monodromy, a standard averaging argument gives isomorphisms

$$H^*(\{\Gamma_D\mathbf{A}^\bullet_{(2)}(\mathrm{Gr}_l(\mathbf{W}))\}^{T^n})\cong H^*_{(2)}(D,\mathrm{Gr}_l(\mathbf{W})).$$

The assertion of the lemma will follow by the usual spectral sequence argument, once $0 \to \mathbf{W}_{l-1} \to \mathbf{W}_l \to \mathrm{Gr}_l(\mathbf{W}) \to 0$ is shown to induce an exact sequence of complexes

$$0 \to \Gamma_D \mathbf{A}^\bullet_{(2)}(\mathbf{W}_{l-1}) \to \Gamma_D \mathbf{A}^\bullet_{(2)}(\mathbf{W}_l) \to \Gamma_D \mathbf{A}^\bullet_{(2)}(\mathrm{Gr}_l(\mathbf{W})) \to 0. \tag{4.23}$$

This fact depends on the behavior of the Hodge metric near the singular locus as described in Theorem (5.21) of [6] (see also [18]), which we now recall.

If σ is a permutation of $(1, 2, \ldots, n)$ and $1 \leqq r \leqq n$, we define ${}^\sigma W^r = W(N_{\sigma 1} + N_{\sigma 2} + \ldots + N_{\sigma r}; V)$. The filtrations ${}^\sigma W^1, \ldots, {}^\sigma W^n$ admit simultaneous $\mathbb{Z}^n$-gradings over $\mathbb{R}$, which are compatible with the grading (4.12):

$${}^\sigma W^r_l = \bigoplus_{l_r \leqq l} {}^\sigma V_{l_1, l_2, \ldots, l_n}, \qquad \bigoplus_{l_n = l} {}^\sigma V_{l_1, l_2, \ldots, l_n} = V_l.$$

We note that ${}^\sigma W^n$ coincides with W for each σ. As σ ranges over all permutations of $(1, 2, \ldots, n)$, the images $p({}^\sigma R(\varepsilon))$ of the sets

$${}^\sigma R(\varepsilon) = \{z \in H^n \mid y_{\sigma 1}/y_{\sigma 2} > \varepsilon,\ y_{\sigma 2}/y_{\sigma 3} > \varepsilon,\ \ldots,\ y_{\sigma n} > \varepsilon\},$$

$\varepsilon > 0$, cover $(\Delta^*)^n$. Theorem (5.21) of [6] asserts

$$\| \tilde{v} \|^2 \sim (y_{\sigma 1}/y_{\sigma 2})^{l_1} (y_{\sigma 2}/y_{\sigma 3})^{l_2} \ldots (y_{\sigma n})^{l_n} \text{ on } {}^\sigma R(\varepsilon) \quad \text{if } v \in {}^\sigma V_{l_1, \ldots, l_n} \text{ and } \varepsilon > 0, \tag{4.24}$$

in the sense that the ratio of the two quantities remains bounded and bounded away from zero. The transformation Y (cf. (4.12)) stabilizes all of the filtrations ${}^\sigma W^r$, so (4.24) implies uniform bounds for the action of Y on sections $\tilde{v}$,

$$\| \widetilde{Yv} \|_t \leqq c(K) \| \tilde{v} \|_t, \qquad t \in K \cap (\Delta^*)^n, \tag{4.25}$$

for any compact $K \subset \Delta^n$. In particular, the map on forms induced by the projection $W_l \to \mathrm{Gr}_l(W) \cong V_l$ is bounded in the L^2 norm. Since this map commutes with exterior differentiation, the sequence (4.23) is well defined.

To verify exactness, we lift sections of $\mathbf{A}^\bullet_{(2)}(\mathrm{Gr}_l(\mathbf{W}))$ to $\mathbf{W}_l$-valued forms via $\mathrm{Gr}_l(\mathbf{W}) \cong \mathbf{V}_l \hookrightarrow \mathbf{W}_l$. Any such lifting can be expressed a sum of terms $\phi \otimes \tilde{v}$, with a locally L^2 scalar form ϕ as coefficient and $v \in V_l$, such that both $\phi \otimes \tilde{v}$ and $d\phi \otimes \tilde{v}$ are square-integrable over intersections $K \cap (\Delta^*)^n$, for any compact $K \subset D$. Now $d(\phi \otimes \tilde{v}) = d\phi \otimes \tilde{v} + \sum_j \phi \wedge dx_j \otimes \widetilde{N_j v}$; the differential dx_j has length y_j, N_j stabilizes the filtrations ${}^\sigma W^r$, and N_j lowers the ${}^\sigma W^r$-weight by two whenever $\sigma^{-1} j \leqq r$. In view of (4.24), this shows that $\tilde{v} \to dx_j \otimes \widetilde{N_j v}$ acts as a bounded operator over each of the regions ${}^\sigma R(\varepsilon)$, $\varepsilon > 0$, and therefore over $K \cap (\Delta^*)^n$. The exactness of (4.23) follows.

We identify the cotangent space of $\mathbb{R}^n$ with the dual of the Lie algebra (4.9) via the correspondence $dx_j \leftrightarrow N_j^*$. Because of (4.20), there is a natural isomorphism between the complex of T^n-invariant $\mathbf{V}$-valued differential forms on any particular T^n-orbit in $(\Delta^*)^n$ and the complex $A^\bullet(N_1, \ldots, N_n; V)$, which was introduced in (4.10). The T^n-orbits are parametrized by the "positive imaginary octant" in H^n,

$$Q = \{y \in \mathbb{R}^n \mid y_j > 0,\ 1 \leqq j \leqq n\}.$$

For any open subset $Q' \subset Q$, we define

$$M^\bullet(Q') = \text{complex of } \mathbb{C}\text{-valued differential forms on } Q', \text{ with}$$
$$\text{locally } L^2 \text{ coefficients}^5 \text{ and locally } L^2 \text{ exterior derivative.}$$

Then $M^\bullet(Q') \otimes A^\bullet(N_1, \ldots, N_n; V)$ can be thought of as the complex of T^n-invariant $\mathbf{V}$-valued forms, with locally L^2 coefficients and locally L^2 exterior derivative, on the union of the T^n-orbits through $p(iQ')$. If the circular domain $D \subset \Delta^n$ intersects $(\Delta^*)^n$ in this union of orbits, we obtain an inclusion of complexes

$$\{\Gamma_D \mathbf{A}^\bullet_{(2)}(\mathbf{V})\}^{T^n} \hookrightarrow M^\bullet(Q') \otimes A^\bullet(N_1, \ldots, N_n; V). \tag{4.26}$$

The L^2 boundedness conditions (4.17) characterize the image.

We make Y act on $M^\bullet(Q') \otimes A^\bullet(N_1, \ldots, N_n; V)$: trivially on the first factor, and in the natural manner on the second factor; cf. the discussion below (4.13). This action is locally finite, semisimple, has integral eigenvalues, and commutes with the differential. Because of (4.25), Y leaves invariant the image of the inclusion (4.26). The cohomology of the image, i.e. $H^*_{(2)}(D, \mathbf{V})$, thus inherits an action of Y; it is functorial in D by construction. We have shown:

(4.27) **Lemma.** *The transformation Y acts naturally on $H^*_{(2)}(D, \mathbf{V})$, for all circular domains $D \subset \Delta^n$. The action is locally finite, semi-simple, with integral eigenvalues, ands commutes with the restriction map induced by any inclusion of circular domains $D_1 \hookrightarrow D_2$.*

To clarify the structure of the argument, we reduce proposition (4.8) to two lemmas; their proofs occupy the next section. We recall that $\{\ldots\}_l$ refers to the l-eigenspace of Y in $\{\ldots\}$.

(4.28) **Lemma.** *The isomorphism* (4.4) *preserves the actions of Y.*

(4.29) **Lemma.** *a) The inclusion $\Delta^n - \{0\} \hookrightarrow \Delta^n$ induces isomorphisms $H^*_{(2)}(\Delta^n, \mathbf{V})_l \cong H^*_{(2)}(\Delta^n - \{0\}, \mathbf{V})_l$ for $l < n$; b) $H^*_{(2)}(\Delta^n, \mathbf{V})_l$ vanishes if $l > n$; c) there exists a vector space M, such that $H^*_{(2)}(\Delta^n, \mathbf{V})_n \cong M \otimes H^{*-1}_{(2)}(\Delta^n - \{0\}, \mathbf{V})_n$.*

Because of (4.28), the weight restrictions (4.16) apply also to $H^*_{(2)}(\Delta^n - \{0\}, \mathbf{V})$. Proposition (4.8) and with it the local isomorphism (4.1) and Theorem (1.5), are then direct consequences of (4.29).

The complex $\Gamma_{\Delta^n} \mathbf{A}^\bullet_{(2)}(\mathbf{V})$ consists of $\mathbf{V}$-valued differential forms on $(\Delta^*)^n$, with locally L^2 coefficients and locally L^2 exterior derivative, which satisfy the L^2 boundedness conditions (4.17) on compact subsets $K \subset \Delta^n$. Those forms ω that are globally square integrable and have a globally square integrable exterior derivative, constitute a subcomplex $\bar{A}^\bullet(\Delta^n) \subset \Gamma_{\Delta^n} \mathbf{A}^\bullet_{(2)}(\mathbf{V})$; here we use the metric of the ambient manifold X, i.e., one which is asymptotic to the Poincaré metric of $(\Delta^*)^n$, but extends across $\partial \Delta^n$. For technical reasons we need to know that the subcomplex $\bar{A}^\bullet(\Delta^n)$ also computes the L^2 cohomology of Δ^n. This will be proved in § 5, along with Lemmas (4.28–29):

(4.30) **Lemma.** *The inclusion $\bar{A}^\bullet(\Delta^n) \hookrightarrow \Gamma_{\Delta^n} \mathbf{A}^\bullet_{(2)}(\mathbf{V})$ induces an isomorphism $H^*(\bar{A}^\bullet(\Delta^n)) \cong H^*_{(2)}(\Delta^n, \mathbf{V})$.*

[5] Unlike in the definition of $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$, we only impose a regularity condition on the coefficients, not a bound near the punctures of $(\Delta^*)^n$

§ 5. Proof of Theorem (1.5)

We continue with the assumptions and the notation of the previous section; in particular, the hypotheses (4.2–3) remain in force. To complete the argument which was begun in § 4, we shall establish Lemmas (4.28–30). By induction we may assume the conclusion of (4.30) in dimension less than n.

As first step towards (4.28), we interpret the action of Y on the intersection cohomology in topological terms. We recall the explicit form (4.18) of the covering $p: H^n \to (\Delta^*)^n$. The local system $\mathbf{V}$ over $(\Delta^*)^n$ can be identified with the quotient of $H^n \times V$ by the action of $\pi_1((\Delta^*)^n)$. For $r \in \mathbb{N}$, the map

$$\tilde{\tau}_r : H^n \times V \to H^n \times V, \quad \tilde{\tau}_r(z, v) = (rx + iy, \exp(-\tfrac{1}{2} \log r\, Y)v) \tag{5.1}$$

sends orbits to orbits: if e_j denotes the j-th standard unit vector, $(z + e_j, (\exp N_j)v)$ and (z, v) lie in the same orbit, as do

$$\begin{aligned} \tilde{\tau}_r(z + e_j, (\exp N_j)v) &= (rx + re_j + iy, \exp(-\tfrac{1}{2} \log r\, Y) \exp N_j v) \\ &= (rx + re_j + iy, \exp(rN_j) \exp(-\tfrac{1}{2} \log r\, Y)v) \end{aligned}$$

and $\tilde{\tau}_r(z, v)$; cf. (4.13). Thus $\tilde{\tau}_r$ drops to a map τ_r of the local system $\mathbf{V}$. Its restriction to the base $(\Delta^*)^n$ extends continuously to Δ^n and preserves the stratification of Δ^n by intersections of coordinate hyperplanes.

Because of the functorial nature of intersection cohomology, τ_r induces endomorphisms of $IH^*(\Delta^n, \mathbf{V})$ and $IH^*(\Delta^n - \{0\}, \mathbf{V})$; we refer to these by the same symbol τ_r. The action of τ_r on $\mathbf{V}$-valued differential forms preserves the L^2 boundedness conditions (4.17), as can can be seen from the bound (4.25). Consequently τ_r acts on the sheaf $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$, which at least over $\Delta^n - \{0\}$ computes the intersection cohomology; cf. (4.4). We recall the definition (4.19) of the local sections $\tilde{v}$. A small calculation shows $\tau_r \tilde{v} = r^{-l/2} \tilde{v}$ if $v \in V_l$ ($= l$-eigenspace of Y). Also, τ_r acts as the identity on T^n-invariant functions, and as multiplication by r on the differentials dx_j. We conclude

$$\tau_r c = r^{l/2} c \quad \text{if } c \in H^*_{(2)}(\Delta^n - \{0\}, \mathbf{V})_l. \tag{5.2}$$

Caution: τ_r acts contravariantly on forms, whereas the above formula for $\tau_r \tilde{v}$ involves τ_r covariantly – this reconciles the apparent discrepancy in the exponents.

Our next task is to describe the action of τ_r on $IH^*(\Delta^n, \mathbf{V})$ in terms of the complex (4.11). The proof of the isomorphism (1.8) via $\mathbf{D}$-modules [11, 13] constructs the intersection cohomology sheaf as a complex of sheaves $\Omega^\bullet(\mathbf{V})$ of holomorphic differential forms, with certain prescribed singularities along the divisor $\bar{X} - X$. Concretely, $\Omega^\bullet(\mathbf{V})$ is the smallest subsheaf of $i_*(\Omega^\bullet_X \otimes \mathbf{V})$ which is invariant under the flat connection and contains the canonical extension of $\Omega^\bullet_X \otimes \mathbf{V}$. The complex of global sections $\Gamma_{\Delta^n} \Omega^\bullet(\mathbf{V})$ is quasi-isomorphic to the stalk of $\Omega^\bullet(\mathbf{V})$ at the origin, and thus computes the local intersection cohomology. Up to isomorphism, the complex (4.11) can be embedded into $\Gamma_{\Delta^n} \Omega^\bullet(\mathbf{V})$, and the inclusion

$$B^\bullet(N_1, \ldots, N_n; V) \hookrightarrow \Gamma_{\Delta^n} \Omega^\bullet(\mathbf{V}) \tag{5.3}$$

induces an isomorphism of cohomology – in essence, this is the proof of (1.8).

As a non-holomorphic map, τ_r does not operate in the complex of sheaves $\Omega^\bullet(\mathbf{V})$. However, we can modify the definition (5.1), by scaling both the real and imaginary part of $z \in H^n$ with the integer r, to obtain a holomorphic map σ_r of $\mathbf{V}$ over $(\Delta^*)^n$. In § 4, we described the complex (4.11) as a complex of $\mathbf{V}$-valued differential forms. We now follow the same procedure, but first make the modification in (4.19) analogous to that of (5.1) – i.e., we replace x_j by z_j – and correspondingly identify N_j^* with dz_j instead of dx_j. The resulting description of $B^\bullet(N_1, \ldots, N_n; V)$ as a complex of differential forms explicitly realizes the inclusion (5.3). Arguing exactly as in the case of τ_r and $H^*_{(2)}(\Delta^n - \{0\}, \mathbf{V})$, we find

$$\sigma_r c = r^{l/2} c \quad \text{if } c \in IH^*(\Delta^n, \mathbf{V})_l. \tag{5.4}$$

In this identity Y acts on $IH^*(\Delta^n, \mathbf{V})$ via the isomorphism (1.8), in accordance with our previous definition.

The maps τ_r and σ_r are related by a homeomorphism of Δ^n which preserves the stratification, and therefore have the same effect on the intersection cohomology of Δ^n and $\Delta^n - \{0\}$. Because of the functorial definition, the action of τ_r commutes with the isomorphism (4.4) and with restriction from Δ^n to $\Delta^n - \{0\}$. At this point (4.7a), (5.2), (5.4) imply the assertion of Lemma (4.28) up to degree $n-1$.

We use Poincaré duality to extend the validity of (4.28) to all degrees. The action of Y on $IH^*(\Delta^n - \{0\}, \mathbf{V})$ was defined so as to make it compatible with the isomorphism (4.14), or equivalently with

$$IH^p(\Delta^n - \{0\}, \mathbf{V}) \cong IH^{2n-p-1}(\Delta^n - \{0\}, \mathbf{V})^* \otimes \bigwedge^n \mathfrak{n}^*. \tag{5.5}$$

If the same isomorphism for the L^2 cohomology groups also commutes with Y, Lemma (4.28) will follow. We recall that (5.5) comes about by Poincaré duality for Δ^n and the collapsing of the exact sequence (4.5). The identical isomorphism results from Poincaré duality for $\Delta^n - \{0\}$ and the natural isomorphism

$$IH^{2n-p-1}(\Delta^n - \{0\}, \mathbf{V}) \cong IH_c^{2n-p}(\Delta^n - \{0\}, \mathbf{V}), \tag{5.6}$$

which reflects the homeomorphism $\Delta^n - \{0\} \cong S^{2n-1} \times I$ (I = open unit interval in $\mathbb{R}$).

According to the inductive hypothesis, $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$ satisfies the axioms of the intersection cohomology sheaf on $\Delta^n - \{0\}$. The isomorphism (4.4) therefore has a counterpart for cohomology with compact support,

$$H_c^*(\Delta^n - \{0\}, \mathbf{A}^\bullet_{(2)}(\mathbf{V})) \cong IH_c^*(\Delta^n - \{0\}, \mathbf{V}). \tag{5.7}$$

An action of Y on $H_c^*(\Delta^n - \{0\}, \mathbf{A}^\bullet_{(2)}(\mathbf{V}))$ can be defined in exactly the same fashion as in the case of $H^*(\Delta^n - \{0\}, \mathbf{A}^\bullet_{(2)}(\mathbf{V}))$. Via (4.4) and (5.7), the isomorphism (5.6) is given by wedge product with a compactly supported 1-form, which can be made T^n-invariant and normal to the T^n-orbits. We conclude: the isomorphism (5.6), expressed in terms of L^2 cohomology and compactly supported L^2 cohomology, commutes with the action of Y. Contraction by means of the polarization form and integration over $\Delta^n - \{0\}$ results in a natural pairing

$$H^p(\Delta^n - \{0\}, \mathbf{A}^\bullet_{(2)}(\mathbf{V})) \times H_c^{2n-p}(\Delta^n - \{0\}, \mathbf{A}^\bullet_{(2)}(\mathbf{V})) \to \bigwedge^n \mathfrak{n}^*; \tag{5.8}$$

we have replaced the obvious target $\mathbb{C}$ by the one dimensional space $\bigwedge^n \mathfrak{n}^*$ to make the pairing compatible with action of Y: when two T^n-invariant, $\mathbf{V}$-valued forms of complementary degrees are contracted, one obtains a T^n-invariant scalar form of top degree, on which Y acts as it does on $\bigwedge^n \mathfrak{n}^*$. Comparing (5.5) to (5.6–8), we see that Lemma (4.28) is now a consequence of:

(5.9) **Lemma.** *The pairing* (5.8) *induces Poincaré duality.*

Although experts know that the L^2 cohomology groups satisfy Poincaré duality, we have not found this fact stated in a form which applies to the present situation. Our proof is quite general; we leave it to the reader to extract a general statement of the lemma from the argument.

Proof. The sheaves $\mathbf{A}^p_{(2)}(\mathbf{V})$ are fine, so $U \mapsto \mathrm{Hom}(H^0_c(U, \mathbf{A}^\bullet_{(2)}(\mathbf{V})), \mathbb{C})$ defines a complex of sheaves, the Borel-Moore dual of the complex $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$ [2, 15]. It suffices to show that the integration pairing sets up a quasi-isomorphism between $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$ and its own dual, with a shift of indices by the real dimension $2n$ – equivalently

$$H^p(D, \mathbf{A}^\bullet_{(2)}(\mathbf{V})) \cong H^{2n-p}_c(D, \mathbf{A}^\bullet_{(2)}(\mathbf{V}))^* \tag{5.10}$$

for any open polydisc $D \subset \Delta^n - \{0\}$, or more generally for open subsets $D \subset \Delta^n - \{0\}$ such that $\dim H^*(D, \mathbf{A}^\bullet_{(2)}(\mathbf{V})) < \infty$. To verify (5.10) in this situation, we exhaust D by an increasing sequence of relatively compact open subsets $\tilde{U}_k$, and set $U_k = \tilde{U}_k \cap (\Delta^*)^n$. The $\mathbf{V}$-valued p-forms on U_k, with L^2 coefficients, constitute a Hilbert space $L^p(U_k)$. We shall work with both the minimal and the maximal closed extension to $L^p(U_k)$ of the exterior derivative, defined on the subspace of smooth forms with compact support in U_k; these will be referred to as the strong closure and the weak closure of d. We topologize $\Gamma_D \mathbf{A}^\bullet_{(2)}(\mathbf{V})$ as inverse limit of Hilbert spaces,

$$\Gamma_D \mathbf{A}^p_{(2)}(\mathbf{V}) = \varprojlim C^p(U_k), \quad \text{with} \tag{5.11}$$

$$C^p(U_k) = \{(\omega, \phi) \in L^p(U_k) \oplus L^{p+1}(U_k) \mid d\omega = \phi \text{ weakly}\}.$$

As a general rule, we let W' denote the continuous dual of a locally convex topological vector space W. Contraction via $\mathbf{S}$, followed by integration over U_k, gives a natural identification

$$C^p(U_k)' = L^{2n-p}(U_k) \oplus L^{2n-p-1}(U_k) / \{(\eta, \psi) \mid \eta = d\psi \text{ strongly}\}; \tag{5.12}$$

here we are using the fact that the polarization form induces an isometry between the local system $\mathbf{V}$ and its dual, relative to the Hodge norms. Since $\Gamma_D \mathbf{A}^\bullet_{(2)}(\mathbf{V})$ is a Fréchet complex with finite dimensional cohomology groups, these groups are algebraically dual to the homology groups of the topologically dual complex:

$$H^p(D, \mathbf{A}^\bullet_{(2)}(\mathbf{V}))^* = H_p(\varinjlim C^\bullet(U_k)'). \tag{5.13}$$

Unravelling the definitions, one sees that homology classes of the dual complex are represented by L^2 forms ψ with support in some U_k, which satisfy the equation $d\psi = 0$ in $L^\bullet(U_k)$ in the strong sense. Any such ψ represents zero if it can be expressed as $d\eta$, for some L^2 form η with support in one of the sets U_l, again

in the strong sense. This description remains unchanged when we replace "strong" by "weak": on a complete Riemannian manifold the equation $d\omega=\phi$ holds strongly if it holds weakly; more generally this is true for an incomplete metric which can made complete by a modification outside the support of ω [1, 12], as is the case in our situation. We conclude:

$$H_p(\varinjlim C^\bullet(U_k)') = H_c^{2n-p}(D, \mathbf{A}^\bullet_{(2)}(\mathbf{V})). \tag{5.14}$$

The duality (5.10), and with it the statement of the lemma, now follows from (5.13–14).

The crucial ingredient of the proof of Lemma (4.29) is a Mayer-Vietoris spectral sequence for the L^2 cohomology of Δ^n and $\Delta^n-\{0\}$. In preparation for the statement, we recall the definition (4.10) of the complex $A^\bullet(N_1, \ldots, N_n; V)$, and the discussion following (4.22). For $r>1$, the open cones

$$Q_j(r)=\{y\in\mathbb{R}^n \mid y_i>0,\ r\,y_i>y_j \text{ for } i\neq j\}, \tag{5.15}$$

$1\leqq j\leqq n$, cover the positive octant Q. The ordered multi-indices $J=\{1\leqq j_1<\ldots<j_p\leqq n\}$ correspond to intersections of the $Q_j(r)$,

$$Q_J(r)=Q_{j_1}(r)\cap Q_{j_2}(r)\cap\ldots\cap Q_{j_p}(r). \tag{5.16}$$

Each $Q_J(r)$ generates a T^n-invariant domain $D_J(r)$ in $(\Delta^*)^n$, the union of all T^n-orbits through $p(iQ_J(r))$. In analogy to the situation in § 4, we identify $M^\bullet(Q_J(r))\otimes A^\bullet(N_1, \ldots, N_n; V)$ with the complex of T^n-invariant, V-valued differential forms on $D_J(r)$, with locally L^2 coefficients and locally L^2 exterior derivative. We let $A^\bullet(Q_J(r))$ denote the subcomplex of all those forms ω which satisfy the L^2 finiteness conditions

$$\int\limits_{K\cap D_J(r)} \|\omega\|^2\,dm<\infty, \qquad \int\limits_{K\cap D_J(r)} \|d\omega\|^2\,dm<\infty, \tag{5.17}$$

relative to the Poincaré metric on $(\Delta^*)^n$, for every compact subset $K\subset\Delta^n$. Similarly $\tilde{A}^\bullet(Q_J(r))$ denotes the complex of forms $\omega\in M^\bullet(Q_J(r))\otimes A^\bullet(N_1, \ldots, N_n; V)$ which satisfy (5.17) whenever K is a compact subset of $\Delta^n-\{0\}$.

From now on we fix a specific $r>1$, and drop the dependence on r in the notation if there is no ambiguity. The order relation (by inclusion) among the cones Q_J and domains D_J is opposite to the order among the indices J. Thus

$$C^{\bullet,\bullet}=\bigoplus_{p,q\geqq 0} C^{p,q}, \qquad \tilde{C}^{\bullet,\bullet}=\bigoplus_{p,q\geqq 0} \tilde{C}^{p,q}, \quad \text{with} \tag{5.18}$$

$$C^{p,q}=\bigoplus_{|J|=p+1} A^q(Q_J), \qquad \tilde{C}^{p,q}=\bigoplus_{|J|=p+1} \tilde{A}^q(Q_J),$$

are double complexes in a natural manner: the cochain complexes of the standard $(n-1)$-simplex, with values in the directed systems of complexes $\{A^\bullet(Q_J)\}$ and $\{\tilde{A}^\bullet(Q_J)\}$. We observe that Y operates in the subcomplexes $A^\bullet(Q_J)$, $\tilde{A}^\bullet(Q_J)$ of $M^\bullet(Q_J)\otimes A^\bullet(N_1, \ldots, N_n; V)$ and in the double complexes $C^{\bullet,\bullet}$, $\tilde{C}^{\bullet,\bullet}$; c.f. (4.25).

(5.19) **Lemma.** *There exist natural isomorphisms* $H^*_{(2)}(\Delta^n, \mathbf{V})\cong H^*(C^{\bullet,\bullet})$, $H^*_{(2)}(\Delta^n-\{0\}, \mathbf{V})\cong H^*(\tilde{C}^{\bullet,\bullet})$, *which are compatible with the actions of* Y. *The inclusion* $C^{\bullet,\bullet}\hookrightarrow\tilde{C}^{\bullet,\bullet}$ *induces the restriction map from* $H^*_{(2)}(\Delta^n, \mathbf{V})$ *to* $H^*_{(2)}(\Delta^n-\{0\}, \mathbf{V})$.

Proof. We begin by constructing a partition of unity $\{f_j\}$ subordinate to the covering $\{D_j\}$ of $(\Delta^*)^n$, such that

(5.20) a) each f_j is T^n-invariant,

b) df_j is bounded, relative to the Poincaré metric.

For this purpose we fix constants s, t, with $r>s>t>1$. We then select a function $g\in C^\infty(\mathbb{R})$ which takes values between 0 and 1, is identically equal to zero below $x=s^{-1}$, and identically equal to one above $x=t^{-1}$. For $1\leqq j\leqq n$, the product

$$\tilde{h}_j(y)=\prod_{i\neq j} g(y_i/y_j)$$

defines a C^∞ function $\tilde{h}_j$ on the positive imaginary octant Q, with the following properties:

(5.21) $\qquad 0\leqq\tilde{h}_j\leqq 1, \quad \tilde{h}_j\equiv 1 \text{ on } Q_j(t), \quad \tilde{h}_j\equiv 0 \text{ on } Q-Q_j(s).$

The differential dy_i has norm y_i, relative to the Poincaré metric, hence

$$\|d(y_i/y_j)\|^2=y_j^{-2}\|dy_i\|^2+y_i^2y_j^{-4}\|dy_j\|^2=2(y_i/y_j)^2.$$

But y_i/y_j is bounded on the support of $g'(y_i/y_j)$, so $d\tilde{h}_j$ has bounded norm. The functions $h_j=(\sum_i \tilde{h}_i)^{-1}\tilde{h}_j$ inherit this latter property, since $\sum_i \tilde{h}_i\geqq 1$. By construction, $\{h_j\}$ is a partition of unity for Q, subordinate to $\{Q_j(r)\}$. Via $p: H^n\to(\Delta^*)^n$, the h_j correspond to T^n-invariant functions f_j on $(\Delta^*)^n$. These constitute the required partition of unity.

The arguments in § 4, in particular (4.22) and (4.26), provide isomorphisms

$$H^*_{(2)}(\Delta^n,\mathbf{V})\cong H^*(A^\bullet(Q)),\qquad H^*_{(2)}(\Delta^n-\{0\},\mathbf{V})\cong H^*(\tilde{A}^\bullet(Q));$$

here we think of Q as Q_J, corresponding to the empty multi-index J. These isomorphisms commute with the action of Y, and the inclusion $A^\bullet(Q)\hookrightarrow\tilde{A}^\bullet(Q)$ induces the restriction map. Because of (5.20), we may regard the product of any $\omega\in A^\bullet(Q_{J\cup(j)})$ with h_j as an element of $A^\bullet(Q_J)$. Similarly $h_j\omega$ belongs to $\tilde{A}^\bullet(Q_J)$ if $\omega\in\tilde{A}^\bullet(Q_{J\cup(j)})$.In view of these observations, the usual combinatorial proof of the acyclicity of the $(n-1)$-simplex can be adapted to the present situation, to show:

$$H^p(C^{\bullet,q})=H^p(\tilde{C}^{\bullet,q})=0 \quad \text{if } p>0,$$

$$H^0(C^{\bullet,q})\cong A^q(Q),\qquad H^0(\tilde{C}^{\bullet,q})\cong\tilde{A}^q(Q).$$

A standard spectral sequence argument now implies the assertion of the lemma.

The spectral sequences of $C^{\bullet,\bullet}$ and $\tilde{C}^{\bullet,\bullet}$, filtered according to the first index, have E_1-terms

(5.22) $$E_1^{p,q}=\bigoplus_{|J|=p+1} H^q(A^\bullet(Q_J)),\qquad \tilde{E}_1^{p,q}=\bigoplus_{|J|=p+1} H^q(\tilde{A}^\bullet(Q_J)).$$

According to the lemma, they abut to $H^*_{(2)}(\Delta^n,\mathbf{V})$ and $H^*_{(2)}(\Delta^n-\{0\},\mathbf{V})$, respectively. We shall establish Lemma (4.29) at the E_1-level.

To study the summands in (5.22), we fix a particular ordered multi-index J, of length $p+1$, $p\geqq 0$. For simplicity we renumber the coordinates y_j, so

that $J=\{1, 2, \ldots, p+1\}$. In terms of the new coordinate system

$$t=y_1;\quad u_i=\frac{y_i}{y_1},\quad 2\leqq i\leqq p+1;\quad v_j=\frac{y_j}{y_1},\quad p+1<j\leqq n \tag{5.23}$$

the region $Q_J=Q_J(r)$ has the following description:

$$\begin{gathered}0<t<\infty;\quad r^{-1}<u_i<r;\quad r^{-1}<v_j<\infty;\\ u_i<r\min\{u_k, v_j\}.\end{gathered} \tag{5.24}$$

In particular,

$$Q_J=\mathbb{R}^+\times S_J,\quad \text{with}\quad S_J=Q_J\cap\{t=1\}. \tag{5.25}$$

We recall the Definition (4.19) of the sections $\tilde{v}$ of $\mathbf{C}^\infty\otimes p^*\mathbf{V}$.

(5.26) **Lemma.** *The restriction of the Poincaré metric to $\mathbb{R}^n+iQ_J$ is quasi-isometric to*

$$t^{-2}(dt^2+dx_1^2)+\sum_i(du_i^2+t^{-2}\,dx_i^2)+\sum_j v_j^{-2}(dv_j^2+t^{-2}\,dx_j^2).$$

Up to quasi-isometry, the Hodge metric on $\mathbf{V}$ satisfies the identity $\|\tilde{v}\|_{(x,t,u,v)}=t^{l/2}\,\|\tilde{v}\|_{(x,1,u,v)}$ if $v\in V_l$ (= l-eigenspace of Y), and $\{\tilde{v}\,|\,v\in V_l\}$ is pointwise orthogonal to $\{\tilde{v}\,|\,v\in V_m\}$ if $l\neq m$. The family of subsets

$$p^{-1}(K)\cap(\mathbb{R}^n+iQ_J)\subset\mathbb{R}^+\times(\mathbb{R}^n+iS_J),$$

with K compact in $\Delta^n-\{0\}$ (respectively, compact in Δ^n) and the family $I\times(\mathbb{R}^n+iS_J)$, where $I\subset\mathbb{R}^+$ ranges over compact intervals (respectively, intervals (δ,∞), $\delta>0$), are cofinal.

Proof. A small calculation shows that the matrix of inner products of the covectors $t^{-1}dt$, $u_i^{-1}du_i$, $v_j^{-1}dv_j$ is constant. Also, u_i is bounded and bounded away from zero on Q_J. The dx_j have length y_j; they are perpendicular to each other and to the dy_j, hence perpendicular to dt, du_i, dv_j. This implies the first statement. The second is a consequence of the norm estimates [6, 18]: if we are willing to modify the metric of $\mathbf{V}$ by a quasi-isometry, we can replace the variation of Hodge structure over $(\Delta^*)^n$ by one that corresponds to an $\mathbb{R}$-split nilpotent orbit [6]; in this special case the norm of a section $\tilde{v}$, with $v\in V_l$, is homogeneous of degree $l/2$ along rays in the positive imaginary octant Q, and the various eigenspaces of Y are orthogonal. The last statement can be verified directly.

The T^n-orbits through $p(iS_J)$ make up a T^n-invariant, locally closed submanifold $T_J\subset(\Delta^*)^n$. We let $L^\bullet(S_J)$ denote the Hilbert space of T^n-invariant $\mathbf{V}$-valued differential forms on T_J, with locally L^2 coefficients, which are globally square integrable over T_J, relative to the restriction of the Poincaré metric on T_J and the Hodge metric on $\mathbf{V}$; we emphasize that we impose no condition on the exterior derivative of forms in $L^\bullet(S_J)$. In analogy to (4.25), we embed $L^\bullet(S_J)$ into the tensor product of $A^\bullet(N_1, \ldots, N_n; V)$ with the space of scalar forms on S_J, with locally L^2 coefficients. Thus Y operates on $L^\bullet(S_J)$. We write d_0 for the exterior derivative, acting on smooth, $\mathbf{V}$-valued differential forms on T_J or S_J. The restriction of d_0 to the space of compactly supported forms has

a maximal and a minimal closed extension to the Hilbert space $L^\bullet(S_J)$; we refer to these as the weak and the strong closure of d_0. Both commute with the action of Y.

Functions on Q_J may be viewed as functions on $\mathbb{R}^+$ with values in appropriate function spaces. We apply this principle to differential forms: because of Lemma (5.26), every ω in $A^\bullet(Q_J)$ or $\tilde{A}^\bullet(Q_J)$ has a unique representation as a sum

$$\omega = \phi(t) + dt \wedge \psi(t), \tag{5.27}$$

where $\phi(t)$, $\psi(t)$ are locally L^2 functions on $\mathbb{R}^+$ with values in the Hilbert space $L^\bullet(S_J)$. The existence of $d\omega$ means that also

$$\frac{\partial \phi}{\partial t} + d_0\psi \quad \text{and} \quad d_0\phi \quad \text{are } L^\bullet(S_J)\text{-valued, locally } L^2 \text{ functions;} \tag{5.28}$$

here d_0 and $\dfrac{\partial}{\partial t}$ are interpreted in the weakest possible sense: as differential operators on Q_J, which act on distributions. In the case of $\omega \in \tilde{A}^\bullet(Q_J)$, the L^2 conditions (5.17) are already implicit in the local L^2 property of ϕ, ψ, $d_0\phi$, $\dfrac{\partial \phi}{\partial t} + d_0\psi$. For $\omega \in A^\bullet(Q_J)$ the square-integrability of ω and $d\omega$ over compact subsets of Δ^n translates into the finiteness conditions

$$\begin{aligned} &\int_\delta^\infty t^{l-n-1} \|\phi_l\|^2 \, dt < \infty, \qquad \int_\delta^\infty t^{l-n+1} \|\psi_l\|^2 \, dt < \infty, \\ &\int_\delta^\infty t^{l-n-1} \|d_0\phi_l\|^2 \, dt < \infty, \qquad \int_\delta^\infty t^{l-n+1} \left\| \frac{\partial \phi_l}{\partial t} + d_0\psi_l \right\|^2 dt < \infty \end{aligned} \tag{5.29}$$

on the components $\phi_l(t)$, $\psi_l(t)$ of $\phi(t)$, $\psi(t)$ in the eigenspaces $L^\bullet(S_J)_l$ of Y, for some (or equivalently, any) $\delta > 0$: the factor t^{-n} measures the distortion of the volume form along the toroidal directions; $t^{-1}\,dt$ is the volume form on $\mathbb{R}^+$; the square of the norm of the values in $\mathbf{V}$ and the factors dx_j account for t^l; and $t^2 = \|dt\|^2$ is contributed by dt whenever it occurs.

(5.30) **Lemma.** *The image of $L^\bullet(S_J)$ under the weak closure of d_0 has finite codimension in the kernel of the weak closure of d_0. In particular, this image is closed.*

Proof. For simplicity we make no notational distinction between d_0 and its weak closure throughout this proof. The differential forms in $L^\bullet(S_J)$ which are constant in the u-directions and do not involve any factor du_i constitute a closed subspace $L^\bullet_v(S_J)$. We claim:

$$\operatorname{Ker} d_0 \cap L^\bullet_v(S_J)/d_0\, L^\bullet_v(S_J) \cong \operatorname{Ker} d_0/d_0\, L^\bullet(S_J). \tag{5.31}$$

To see this, we consider S_J as bundle over $\{u_2 = 1\}$ with uniformly bounded intervals as fibres. Up to quasi-isometry, the metric is the product of the Euclidean metric on the fibres with the restricted metric on the base; cf. (5.24–26). The usual proof of the Poincaré lemma (in the variable u_2, with the remaining

u_i and the v_j as parameters) produces a homotopy operator, bounded in L^2 norm by a constant which depends only on r. In this way we may pass to the subcomplex of forms that are independent of u_2 and do not involve du_2, without changing the cohomology. Continuing inductively we establish (5.31).

The complex $L^\bullet_v(S_J)\cap\{\text{domain of } d_0\}$ can be filtered according the number of dx_i, $1\leqq i\leqq p+1$. From the proof of Lemma (4.22) we recall that $\tilde{v}\mapsto dx_j\otimes\widetilde{N_j v}$ acts boundedly on sections $\tilde{v}$ over compact subsets of Δ^n; hence, on the graded level, the filtration results in a finite number of copies of the complex of T^{n-p-1}-invariant, globally square-integrable $\mathbf{V}$-valued forms, with globally L^2 exterior derivative, on the region

$$\{z\in\Delta^n \mid z_i=e^{-2\pi} \text{ for } 1\leqq i\leqq p+1, 0<|z_j|<e^{-2\pi/r} \text{ for } j>p+1\};$$

cf. (5.24–26). Because of Lemma (4.30) in dimension $n-p-1$, coupled with the analogue of Lemma (4.22) in the present setting, this latter complex computes the intersection cohomology of $\mathbf{V}$ over a polydisc, and therefore has finite dimensional cohomology. The lemma follows.

The description (5.27–29) of differential forms in $A^\bullet(Q_J)$ and $\tilde{A}^\bullet(Q_J)$ and the closed range property (5.30) of d_0 lead to an L^2 Künneth theorem. Our situation is very similar to that of a "warped product" [23], but not directly covered by it. Since we need certain intermediate statements, we shall sketch the argument.

We call a form $\phi\in L^\bullet(S_J)$ harmonic if it satisfies the equation $d_0\phi=0$ weakly and the adjoint equation $d_0^*=0$ strongly. In defining the adjoint d_0^*, we use the metrics described in Lemma (5.26). Then d_0^* commutes with the action of Y, and Y acts on the space of harmonic forms. We shall work only with the weak closure of d_0 and the strong closure of d_0^* from now on, so we simply write d_0 and d_0^* without further comment. These two operators are adjoints in the technical sense, hence Lemma (5.30) and the open mapping theorem[6] imply:

(5.32) **Corollary.** *Every d_0-closed $\phi\in L^\bullet(S_J)$ can be expressed uniquely as $\phi=\phi_1+d_0\phi_2$, where $\phi_1\in L^\bullet(S_J)$ is harmonic and ϕ_2 orthogonal to $\ker d_0$. The linear maps $\phi\mapsto\phi_1$, $\phi\mapsto\phi_2$ are bounded and commute with the action of Y.*

(5.33) **Lemma.** *Every cohomology class of $A^\bullet(Q_J)$ or $\tilde{A}^\bullet(Q_J)$ has a representative $\omega=\phi+dt\wedge\psi(t)$ as in (5.27–29), with ϕ harmonic and independent of t, and $\psi(t)$ pointwise harmonic. If a form of this type is exact, it can be expressed as the coboundary of a form $\Phi(t)$ whose values are harmonic.*

Proof. If a d-closed form ω in $A^\bullet(Q_J)$ or $\tilde{A}^\bullet(Q_J)$ is represented as in (5.27), the values $\phi(t)$ are d_0-closed for almost every t, and $(d_0^*\alpha,\psi(t))+\dfrac{\partial}{\partial t}(\alpha,\phi(t))=0$ in the sense of distributions on $\mathbb{R}^+$, for every smooth, compactly supported $\alpha\in L^\bullet(S_J)$. Integrating against a test function and recalling the closure properties of d_0^*, we see that the equation remains valid for any α in the domain of d_0^*, in particular for any harmonic α. We now use (5.32) pointwise, to write

[6] Note: d_0 may be viewed as a bounded operator defined on the graph

$\phi(t)=\phi_1(t)+d_0\,\phi_2(t)$. Since $\dfrac{\partial}{\partial t}(\alpha,\phi_1(t))=0$ for every harmonic α, $\phi_1(t)=\phi_1$ does not depend on t. If $\phi(t)$ satisfies the square-integrability condition (5.29), then so does the constant function ϕ_1; moreover, interpreted as element of $A^\bullet(Q_J)$ or $\tilde{A}^\bullet(Q_J)$, ϕ_1 is closed.

We argue similarly in the case of $\omega-\phi_1=d_0\,\phi_2(t)+dt\wedge\psi(t)$. The integrated form of the equation $(d_0^*\alpha,\psi(t))+\dfrac{\partial}{\partial t}(\alpha,d_0\,\phi_2(t))=0$ holds for any α in the domain of d_0^*. Since $\phi_2(t)$ takes values in the orthogonal complement of $\ker d_0$, and since d_0 and d_0^* are adjoints in the technical sense, we find $\dfrac{\partial}{\partial t}\phi_2(t)=\psi(t)-\tilde{\psi}(t)$ weakly, where $\tilde{\psi}(t)$ denotes the orthogonal projection of $\psi(t)$ into $\ker d_0$. Hence $d\phi_2(t)=d_0\,\phi_2(t)+dt\wedge\psi(t)-dt\wedge\tilde{\psi}(t)$; moreover $\phi_2(t)$ satisfies the finiteness condition (5.29) if $\phi(t)$ does. In effect, we have made ω cohomologous to $\phi_1+dt\wedge\tilde{\psi}(t)$ in $A^\bullet(Q_J)$ or $\tilde{A}^\bullet(Q_J)$, respectively. We now apply the decomposition (5.32) to $\tilde{\psi}(t)$ and observe that $dt\wedge d_0\,\tilde{\psi}_2(t)=-d(dt\wedge\tilde{\psi}_2(t))$ is exact, to make ω cohomologous even to $\phi_1+dt\wedge\tilde{\psi}_1(t)$.

If $\phi_1+dt\wedge\tilde{\psi}_1(t)$ is an exact form, say $\phi_1+dt\wedge\tilde{\psi}_1(t)=d(\Phi(t)+dt\wedge\Psi(t))$, $\Phi(t)$ must be d_0-closed pointwise since ϕ_1 is harmonic. Hence $\Phi(t)=\Phi_1(t)+d_0\,\Phi_2(t)$ as in (5.32). The harmonicity of $\tilde{\psi}_1(t)$ forces the vanishing of $\dfrac{\partial}{\partial t}d_0\,\Phi_2(t)+d_0\,\Psi(t)$, i.e., the exact part of $\dfrac{\partial}{\partial t}\Phi(t)+d_0\,\Psi(t)$. Thus $\Phi_1(t)$ has the same boundary in $A^\bullet(Q_J)$ or $\tilde{A}^\bullet(Q_J)$ as $\Phi(t)+dt\wedge\Psi(t)$. This completes the proof.

In the case of the complex $\tilde{A}^\bullet(Q_J)$, the functions $\psi(t)$, $\Phi(t)$ in Lemma (5.33) are only required to be locally L^2. The equation $\psi(t)=\dfrac{\partial}{\partial t}\Phi(t)$ can always be integrated within the class of locally L^2 functions, so $\phi+dt\wedge\psi(t)$ is cohomologous to ϕ. Corollary (5.32) identifies the space of harmonic forms $\phi\in L^\bullet(S_J)$ with the cohomology of the complex

$$\bar{A}^\bullet(S_J)=\text{domain of } d_0 \text{ in } L^\bullet(S_J). \tag{5.34}$$

Hence, when we embed $\bar{A}^\bullet(S_J)$ into $\tilde{A}^\bullet(Q_J)$ as the subcomplex of forms which are independent of t and do not involve dt, the inclusion induces an isomorphism of cohomology

$$H^*(\tilde{A}^\bullet(Q_J))\cong H^*(\bar{A}^\bullet(S_J)). \tag{5.35}$$

If the forms $\phi+dt\wedge\psi(t)$ and $\Phi(t)$ are to belong to $A^\bullet(Q_J)$, the three functions ϕ, $\psi(t)$, $\Phi(t)$ must satisfy the conditions (5.29). This is automatic for the components ϕ_l of weight less than n, but excludes components ϕ_l of weight greater or equal to n. For $k\in\mathbb{Z}$, we set

$$M_k=\frac{\left\{f\in L^2_{\mathrm{loc}}(\mathbb{R}^+)\,\middle|\,\int_1^\infty |f|^2\,t^k\,dt<\infty\right\}}{\left\{f'\,\middle|\,f,f'\in L^2_{\mathrm{loc}}(\mathbb{R}^+),\ \int_1^\infty |f|^2\,t^{k-2}\,dt<\infty,\ \int_1^\infty |f'|^2\,t^k\,dt<\infty\right\}}. \tag{5.36}$$

The obstruction to solving the equation $\psi_l(t)=\dfrac{\partial}{\partial t}\Phi_l(t)$ for $\Phi_l(t)$, subject to the finiteness conditions (5.29), is measured by the tensor product of M_{l-n+1} with the harmonic space in $L^\bullet(S_J)_l$. Hardy's inequality [20, 22] implies the vanishing of M_k for $k\neq 1$, hence

$$H^*(A^\bullet(Q_J)_l)\cong\begin{cases}H^*(\bar{A}^\bullet(S_J)_l) & \text{if } l<n,\\ M\otimes H^{*-1}(\bar{A}^\bullet(S_J)_n) & \text{if } l=n,\\ 0 & \text{if } l>n,\end{cases} \tag{5.37}$$

with $M=M_1$.

The first of the isomorphisms (5.37) is related to (5.35) by the inclusion $C^{\bullet,\bullet}\hookrightarrow\tilde{C}^{\bullet,\bullet}$. Thus (5.35), (5.37) imply parts a) and b) of Lemma (4.29).

As it stands, the second isomorphism in (5.37) does not arise from a morphism of double complexes. To dispose of this problem, we note that $\bar{A}^\bullet(S_J)\subset\tilde{A}^\bullet(Q_J)$ consists precisely of those forms that can be expressed in terms of homogeneous coordinates y_α/y_β and their differentials. In particular, restriction from Q_J to Q_I maps $\bar{A}^\bullet(S_J)$ to $\bar{A}^\bullet(S_I)$ if $J\subset I$. Replacing the summands $\tilde{A}^\bullet(Q_J)$ of $\tilde{C}^{\bullet,\bullet}$ by their subcomplexes $\bar{A}^\bullet(S_J)$, one obtains a double complex $D^{\bullet,\bullet}$ whose columns are quasi-isomorphic to those of $\tilde{C}^{\bullet,\bullet}$,

$$D^{\bullet,\bullet}\hookrightarrow\tilde{C}^{\bullet,\bullet},\qquad H^*(D^{p,\bullet})\cong H^*(\tilde{C}^{p,\bullet}); \tag{5.38}$$

cf. (5.35). A small calculation shows that the differential $d\tau$ of

$$\tau(y)=\min\{y_j\,|\,1\leqq j\leqq n\}, \tag{5.39}$$

interpreted in the sense of distributions, is piecewise continuous: Stokes' theorem applies to the integrand $d(\tau\alpha)$, for any smooth form α with compact support, because $\tau\alpha$ is then piecewise smooth. Unlike t, the function τ is canonically defined on all of Q.

(5.40) **Lemma.** *For $f\in C^\infty(\mathbb{R}^+)\cap L^2([1,\infty),t\,dt)$ and $\phi\in\bar{A}^\bullet(S_J)_n$, the product $f(\tau)\,d\tau\wedge\phi$ belongs to $A^\bullet(Q_J)_n$, and $d(f(\tau)\,d\tau\wedge\phi)$ equals $-f(\tau)\,d\tau\wedge d\phi$. If ϕ is closed, $f(\tau)\,d\tau\wedge\phi$ and $f(t)\,dt\wedge\phi$ are cohomologous in $A^\bullet(Q_J)_n$.*

Before proving the lemma, we indicate how it implies part c) of Lemma (4.29). A standard smoothing argument [7] permits us to pick a specific representative $f_m\in C^\infty(\mathbb{R}^+)\cap L^2([1,\infty),t\,dt)$ for each $m\in M$, linearly in the variable m. Because of (5.40), the map

$$T_m\colon\ \bar{A}^q(S_J)_n\to A^{q+1}(Q_J)_n,\qquad T_m(\phi)=(-1)^q f_m(\tau)\,d\tau\wedge\phi, \tag{5.41}$$

is well defined and extends to a morphism of double complexes

$$T_m\colon\ (D^{\bullet,\bullet})_n\to(C^{\bullet,\bullet})_n, \tag{5.42}$$

of bidegree (0, 1). Via the quasi-isomorphism (5.38), the assignment $m\otimes\phi\mapsto T_m(\phi)$ induces the second of the isomorphisms (5.37) at the E_1 level; this, in effect,

[7] Identify M with the cokernel of differentiation acting on locally L^2 functions on $\mathbb{R}$ which are globally L^2 on $\mathbb{R}^+$, via the change of coordinates $t=e^x$; convolution with any $\phi\in C_0^\infty(\mathbb{R})$ is a smoothing operator, homotopic to the identity if ϕ has total integral one

is the final assertion of (5.40). Since we have now implemented the second of the isomorphisms (5.37) by a morphism of double complexes, it also commutes with the higher differentials of the spectral sequences (5.22).

Proof of (5.40). The continuity of $f(\tau)\,d\tau$ away from the hyperplanes $y_i=y_j$ ensures that $\omega=f(\tau)\,d\tau\wedge\phi$ has locally L^2 coefficients, as does $d\omega$, provided $d\omega=-f(\tau)\,d\tau\wedge d\phi$ as claimed. To verify this latter identity, we note that $f(\tau)\,d\tau$ is (weakly) closed: Stokes' theorem applies to continuous, piecewise smooth integrands, hence

$$\int f(\tau)\,d\tau\wedge d\alpha=0 \tag{5.43}$$

for every smooth form α with compact support. Now let α be a form with locally L^2 coefficients, supported in a compact subset $K\subset Q_J$, whose exterior derivative exists as a locally L^2 form. It is then possible to construct a sequence $\{\alpha_n\}$ of smooth forms, with support in some fixed compact neighborhood of K in Q_J, such that $\alpha_n\to\alpha$ and $d\alpha_n\to d\alpha$ in L^2 norm [1, 12]. Substituting α_n for α and passing to the limit, we see that (5.43) remains valid for any such α. In particular we may set $\alpha=\phi\wedge\beta$, if β is smooth and has compact support. The resulting identity

$$\int f(\tau)\,d\tau\wedge\phi\wedge d\beta=(-1)^{\deg\phi+1}\int f(\tau)\,d\tau\wedge d\phi\wedge\beta \tag{5.44}$$

is equivalent to $d(f(\tau)\,d\tau\wedge\phi)=-f(\tau)\,d\tau\wedge d\phi$ (in the weak sense).

To show that $f(\tau)\,d\tau\wedge\phi$ and $f(\tau)\,d\tau\wedge d\phi$ have finite L^2 norms on $K\cap Q_J$, for any compact $K\subset\Delta^n$, we split up the region of integration into the disjoint subsets $K\cap Q_J\cap Q_i(1)$, $1\leqq i\leqq n$; cf. (5.15). On $Q_i(1)$ we use the coordinates $\tau=y_i$ (this is consistent with (5.39)!) and $v_j=\tau^{-1}y_j$, $j\neq i$, as in (5.23–26). Since ϕ and $d\phi$ are invariant under changes of scale $y\mapsto cy$, their restrictions to the sections $\tau=$const. of $Q_J\cap Q_i(1)$ do not depend on the value of τ. These restrictions are square integrable because of the L^2 conditions (5.17) on ϕ and $d\phi$, both of which belong to $\tilde{A}^\bullet(Q_J)_n$. At this point, an application of Lemma (5.26), with $J=\{i\}$, implies the L^2 finiteness of $f(\tau)\,d\tau\wedge\phi$ and $f(\tau)\,d\tau\wedge d\phi$ on $K\cap Q_J\cap Q_i(1)$.

We shall produce a continuous, piecewise smooth function F on Q_J, which is bounded on subsets $\{\tau\geqq\delta>0\}$, such that

$$dF=f(\tau)\,d\tau-f(t)\,dt. \tag{5.45}$$

Then $F\phi\in A^\bullet(Q_J)_n$ if $\phi\in\bar{A}^\bullet(S_J)_n$, as follows from (5.26). The identity $d(F\phi)=dF\wedge\phi+Fd\phi$ holds for the same reason as the analogous formula for $d(f(\tau)\,d\tau\wedge\phi)$. Hence $f(\tau)\,d\tau\wedge\phi$ and $f(t)\,dt\wedge\phi$ are indeed cohomologous whenever ϕ is closed. We note that (5.45) and the normalization $F=0$ on $\{\tau=t\}$ completely characterize F; we only need to check the boundedness on $\{\tau\geqq\delta>0\}$. The integral of f from $\tau=\tau(y)$ to $t=t(y)$ gives the value of F at any particular point y. On Q_J the ratio t/τ is bounded by r. Thus, if $\tau\geqq\delta$, Schwarz' inequality implies

$$|F(y)|^2\leqq\int_\delta^\infty|f(s)|^2s\,ds\int_\tau^t s^{-1}\,ds\leqq\log r\int_\delta^\infty|f(s)|^2s\,ds,$$

which is finite. This completes the proof.

Only Lemma (4.30) remains to be established. We recall that the complex $\bar{A}^\bullet(\Delta^n)$ was defined in terms of a metric g which continues across the boundary in the ambient manifold X. The closure $\overline{\Delta^n}$ of Δ^n can be covered by finitely many open (in $\overline{\Delta^n}$) subsets V_s of the following type:

$$\{z\in\overline{\Delta^n} \mid |z_j|<1-\delta \text{ for } j\in J;\ |z_j-\zeta_j|<2\delta \text{ for } j\notin J\}, \tag{5.46}$$

with $J=\{1\leqq j_1<j_2<\ldots<j_p\leqq n\}$, $0\leqq p\leqq n$, $\delta\ll 1$, and $\zeta\in\partial\Delta^n$. The coordinates z_j that are not bounded away from zero split off as factors, hence

$$\begin{aligned}&\text{there exists a partition of unity } \{f_s\} \text{ for } \overline{\Delta^n}, \text{ subordinate}\\ &\text{to } \{V_s\}, \text{ with uniformly bounded differentials } df_s\end{aligned} \tag{5.47}$$

(bounded relative to the metric g). For each multi-index S, we let V_S denote the intersection of the V_s, $s\in S$, and set $U_S=V_S\cap\Delta^n$. We define $A^\bullet(U_S)=$complex of sections of $\mathbf{A}^\bullet_{(2)}(\mathbf{V})$ over U_S. Those $\omega\in A^\bullet(U_S)$ whose L^2 norm on $K\cap(\Delta^*)^n$ is finite, and whose exterior derivatives $d\omega$ have finite L^2 norm on $K\cap(\Delta^*)^n$, for every compact $K\subset V_S$, constitute a subcomplex $\bar{A}^\bullet(U_S)\subset A^\bullet(U_S)$: the subcomplex of forms which are square-integrable up to the boundary of Δ^n, away from the boundary of U_S in Δ^n. Because of (5.47), but otherwise for formal reasons, there exist double complexes

$$A^{p,q}=\bigoplus_{|S|=p} A^q(U_S), \qquad \bar{A}^{p,q}=\bigoplus_{|S|=p}\bar{A}^q(U_S), \tag{5.48}$$

which compute the L^2 cohomology of Δ^n and the cohomology of the complex $\bar{A}^\bullet(\Delta^n)$, respectively; the inclusion $\bar{A}^{\bullet,\bullet}\hookrightarrow A^{\bullet,\bullet}$ corresponds to the natural map $H^*(\bar{A}^\bullet(\Delta^n))\to H^*_{(2)}(\Delta^n,\mathbf{V})$.

At this point it suffices to show that $\bar{A}^\bullet(U_S)\hookrightarrow A^\bullet(U_S)$ induces an isomorphism of cohomology, for each multi-index S. The sets U_S are either relatively compact in Δ^n – in which case $\bar{A}^\bullet(U_S)$ coincides with $A^\bullet(U_S)$ – or can be expressed as a product $D\times B$, as follows: D is a polydisc in a linear subspace $\mathbb{C}^k\subset\mathbb{C}^n$, centered at the origin, of radius less than 1, and B a convex domain in the complementary subspace $\mathbb{C}^{n-k}\subset\mathbb{C}^n$, such that the closure $\bar{B}$ does not meet the coordinate hyperplanes. In particular

$$U_S\cap(\Delta^*)^n=(D\cap(\Delta^*)^k)\times B, \qquad \partial U_S\cap\partial\Delta^n\subset D\times\partial B. \tag{5.49}$$

Restricted to $U_S\cap(\Delta^*)^n$, the Hodge metric of $\mathbf{V}$ and the metric g of the base are quasi-isometric to products of metrics of the same type on D with constant metrics on B. Thus we can argue as in the proof of Lemma (5.30): $A^\bullet(U_S)$ and $\bar{A}^\bullet(U_S)$ are quasi-isomorphic to their subcomplexes $A^\bullet_D(U_S)$, $\bar{A}^\bullet_D(U_S)$, consisting of all forms which are constant along the B-directions. In view of (5.49), the latter two complexes coincide. This proves Lemma (4.30).

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