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Continuous q-convex exhaustion functions

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Introduction

The investigation of cohomology groups with analytic coefficients plays an important part in complex analysis. There are a lot of theorems which state the vanishing of cohomology groups. One of the most important tools for proving this are the vanishing theorems and finiteness theorems of Andreotti and Grauert [1]. There the vanishing follows from the existence of a function with some geometric properties, the q-convexity.

Now there are many examples where the situation is not as good as in the above mentioned case. There we have to deal with functions which are locally the maximum of q-convex functions. In [5] some vanishing theorems are proved by using those functions. There one cannot take any $f=\max(f_1,\ldots,f_r)$ where $f_1,\ldots,f_r$ are q-convex. For instance, nothing can be proved, if there is no holomorphic tangent vector ξ such that all the Levi forms of $f_1,\ldots,f_r$ are positive-definite on the span of ξ. But in [3] Diederich and Fornaess prove that each continuous q-convex function, which means functions that are locally the maximum of q-convex functions, can be smoothed to a $\tilde{q}$-convex function (§ 1) which we need to state vanishing theorems for cohomology. Of course, the $\tilde{q}$ that depends on q gives a weaker theorem than q would do.

The construction of our functions is always as follows. If f is a q-convex function on $X\times X$ and $A\subset X$ is a submanifold, the function $g(x)=\sup\{f(x,y)\colon y\in A\}$ has some convexity properties if it is finite. Thus one has to ask if there is a q-convex exhaustion function of the complement of the diagonal in $X\times X$. So we define the conditions (E_r) and (E'_r) in § 2. The complement of a q-codimensional submanifold in $\mathbb{P}_n$ is q-convex [2] because (E_0) holds on $\mathbb{P}_n$. But this condition implies also that the complement is $\tilde{q}$-complete. If we take the Hopf manifold instead of $\mathbb{P}_n$, we have $q+1$-completeness of the complement. These are special cases of our result (§ 2) combined with the theorem of Diederich and Fornaess.

§ 1. Definitions

If X is a complex space, $C^\infty(X)$ is the set of all functions $f: X \to \mathbb{R}$ such that there are a neighbourhood U for each $x \in X$, a biholomorphic mapping Φ from U onto an analytic subset B of an open set V in $\mathbb{C}^n$ and a function $g \in C^\infty(V)$ with $g \circ \Phi = f|U$. If g can be chosen q-convex (the Leviform has $\dim_x X - q + 1$ positive eigenvalues) then f is called q-convex.

Definition 1. a) For a complex space X let $F_q(X)$ be the set of all functions $f: X \to \mathbb{R}$ such that for each $x \in X$ there are a neighbourhood U of x and q-convex functions $f_1, \ldots, f_s$ on U with $f|U = \max(f_1, \ldots, f_s)$.

b) For $x \in X$ the set $G_q(x)$ consists of all functions $f: X \to \mathbb{R}$ such that there are a neighbourhood U of x and a q-convex function $g: U \to \mathbb{R}$ with $g(x) = f(x)$ and $g \leqq f|U$. $G_q(X) := C^0(X) \cap \bigcap_{x \in X} G_q(x)$.

c) Be $H_q(X)$ the set of all $f \in C^0(X)$ with the following property: For each $x \in X$ and each neighbourhood U of x there is an open set V with $x \in V \Subset U$ and a continuous $g: \bar{V} \to \mathbb{R}$, q-convex on V such that $g(x) = f(x)$ and $g|\partial V < f|\partial V$.

The functions defined by 1a) are the "continuous q-convex functions" which are the subject of this paper. The other functions are auxiliary functions.

Definition 2. A function $f: X \to \mathbb{R}$ is called an exhaustion function, if f is continuous and $\{x \in X | f(x) \leqq c\}$ is a compact set for each $c \in \mathbb{R}$.

Definition 3. A complex space X is q-convex with corners, if there are a compact subset $K \subset X$ and an exhaustion function f of X such that $f|X - K \in F_q(X - K)$. If one can choose $K = \emptyset$, then X is called q-complete with corners. An open subset M of a complex space Y is called locally q-complete with corners, if and only if for each $y \in \partial M$ there is an open $U \ni y$ such that $U \cap M$ is q-complete with corners.

If U, V are open subsets of a complex space X with $U \cap V \neq \emptyset$, U, V q-convex with corners, then $U \cap V$ is q-convex with corners. If moreover U is q-complete with corners, then $U \cap V$ is q-complete with corners.

The following theorem shows the importance of the concept of "q-convexity with corners":

Theorem [4]. *If X is an irreducible n-dimensional complex space which is q-convex (q-complete) with corners, then X is $\tilde{q}$-convex (q-complete) where* $\tilde{q} = n - \left[\frac{n}{q}\right] + 1$.

There $\left[\frac{n}{q}\right]$ is the largest integral number, smaller or equal to $\frac{n}{q}$.

§ 2. The results

We introduce two conditions on complex spaces. Let $n = n(X) = \inf\{\dim_x X: x \in X\}$, $\Delta_X = \{(x, x): x \in X\}$, $X' := X \times X - \Delta_X$ for any complex space X. Let $r \in \mathbb{N} \cup \{0\}$. Then our conditions are as follows:

(E'_r) There is a neighbourhood U of Δ_X and an $f \in F_{n+r}(U - \Delta_X)$ such that $f(x_\nu, x) \to \infty$, if $(x_\nu, x) \in U - \Delta_X$ and $x_\nu \to x$.

(E_r) holds on X if and only if one can choose $U = X \times X$ in (E'_r).

In § 4 we give two examples: (E_0) holds on $\mathbb{P}_n$ and (E_1) holds on the Hopf manifold. Now we list our theorems.

Theorem 1. *If X is a homogeneous connected compact complex manifold and $M \subsetneqq X$ is an open set in X that is q-convex with corners then M is q-complete with corners.*

Theorem 2. *Let $M \subset X$ be locally q-complete with corners, $K \subset M$ compact, $f \in F_r(X - K)$ such that $\{x \in M - K \mid f(x) < c\} \Subset X$ for each $c \in \mathbb{R}$. Then*

(a) *M is $(q+r)$-convex with corners and $(q+r)$-complete with corners, if $K = \emptyset$.*

(b) *If X is a complex manifold, M is $(q+r-1)$-convex with corners and $(q+r-1)$-complete with corners, if $K = \emptyset$.*

Theorem 3. *Let $M \Subset X$ be locally q-complete with corners and (E'_r) valid on X. Then M is $(q+r)$-convex with corners.*

Theorem 4. *If X is compact, (E_r) holds on X and $M \subsetneqq X$ is q-convex with corners, then M is $(q+r)$-complete with corners.*

Now let $A \subset X$ be analytic and $\dim A_i \geqq n - q$ for each irreducible component A_i of A.

Theorem 5. *If X is a complex manifold then $X - A$ is locally q-complete with corners.*

Theorem 6. *If X is compact and (E'_r) holds, then $X - A$ is $(q+r)$-convex with corners. If even (E_r) holds, then $X - A$ is $(q+r)$-complete with corners.*

§ 3. Density of H_q in F_q

In § 1 we introduced classes of functions $H_q(X)$, $G_q(X)$ and $F_q(X)$. We have $F_q(X) \subset G_q(X) \subset H_q(X)$. For the following constructions it is important that $H_q(X)$ is dense in $F_q(X)$: If $f \in H_q(X)$ and $\eta \in C^0(X)$ with $\eta > 0$ then there is a $g \in F_q(X)$ with $|g - f| < \eta$. But it is not necessary to prove this here. So we are merely stating

Lemma 1. *If $f \in H_q(X)$, $Y \Subset X$ open and $\varepsilon > 0$, there is a continuous function h on Y with $|h - f| < \varepsilon$ and $h \mid Y \in F_q(Y)$.*

Proof. First we prove that for each $x \in X$ there are a $\delta = \delta(x) > 0$, open sets $U = U(x)$, $V = V(x)$ with $x \in U \subset V \Subset X$ and functions $f_z = f_{z,x} \in C^0(\bar{V})$ for each $z \in U$ such that

(i) $f_z(z) = f(z)$

(ii) $f - f_z > \delta$ on ∂V

(iii) $f_z - f < \varepsilon$

(iv) f_z is q-convex on V (so $f_z \in C^\infty(V)$).

Let $x\in X$ and Z be a neighbourhood of x such that $f(z)-f(u)<\varepsilon$ for $z,u\in Z$. There is an open set V with $x\in V\Subset Z$ and some $g\in C^0(\bar{V})$ that is q-convex on V such that $f(x)=g(x)$ and $f-g>0$ on ∂V.

Let $\delta>0$ with $f-g|\partial V>2\delta$. There is an open set U with $x\in U\subset V$ and $|g-f|<\delta$ on U. Define $f_z:=g+f(z)-g(z)$ for $z\in U$.

Because g is q-convex and $\bar{V}$ is compact, there is a $u\in\partial V$ such that $g\leqq g(u)$. Thus $f_z(v)-f(v)=g(v)-f(v)+f(z)-g(z)<g(u)-f(v)+\delta=g(u)-f(u)+f(u)-f(v)+\delta<-\delta+\varepsilon+\delta=\varepsilon$ for each $v\in V$, because $g(u)-f(u)<-\delta$ for $u\in\partial V$ and $f(u)-f(v)<\varepsilon$.

There are $x_1,\ldots,x_s\in\bar{Y}$ such that $\bigcup_{i=1}^{s} U(x_i)\supset\bar{Y}$. Let $\delta:=\min(\delta(x_1),\ldots,\delta(x_s),\varepsilon)>0$. For each $z\in\bar{Y}$ there is an $i=i(z)$ with $1=i\leqq s$ and $z\in U(x_i)$. Let $W(z)\ni z$ be open with $W(z)\subset U(x_i)$ and $|f_{z,x_i}-f|<\delta$ on $W(z)$ where $i=i(z)$. Such a neighbourhood exists due to (*i*). There are $z_1,\ldots,z_k\in\bar{Y}$ with $\bigcup_{i=1}^{k} W(z_i)\supset Y$. Let $Z_j:=V(x_{i(z_j)})$ for $j=1,\ldots,k$. Now we define $h_j=f_{z_j,x_i}$ with $i=i(z_j)$ for $j=1,\ldots,k$ on Z_j and $h(y):=\max\{h_j(y)|\,y\in Z_j\}$, $y\in Y$.

Now $h_j<f-\delta$ on ∂Z_j. If $u\in\partial Z_j\cap Y$, there is an m with $u\in W(z_m)$. So $h_m(u)$ is defined and $|h_m(u)-f(u)|<\delta$. Therefore $h_m(u)>h_j(u)$. Thus $h\in F_q(Y)$ because h_j is q-convex on Z_j and $h_j<h$ on ∂Z_j.

Because $|h_j-f|<\varepsilon$ on $W(z_j)$ we have $f-h<\varepsilon$. By (iii) we conclude $h-f<\varepsilon$. So $|h-f|<\varepsilon$.

By lemma 1 one obtains only functions on a subset Y of X. To improve this one has to apply.

Lemma 2. *Assume that f is an exhaustion function and K a compact subset of X. If for each open $U\Subset X$ there is a $g\in F_q(U-K)$ with $|g-f|<1$ on $U-K$, then X is q-convex with corners. If $K=\emptyset$, then X is q-complete with corners.*

Proof. Obviously one may assume that $f>0$ on X. Let $U_n:=\{x\in X:\ f(x)<8n\}$ for each positive integer n. Then there are functions $g_n\in F_q(U_n-K)$ with $|g_n-f|<1$ on U_n-K. We define $f_n:=ng_n-2n(n-1)$.

For $x\in X-K$, $f(x)<n$, both f_n and f_{n+1} are defined and $f_{n+1}(x)<f_n(x)$, for $(f_{n+1}-f_n)(x)<(n+1)\quad(f(x)+1)-2n(n+1)-n(f(x)-1)+2n(n-1)=f(x)+1-2n<1-n\leqq 0$.

On the other hand, $f_{n+1}(x)>f_n(x)$, if $x\in X-K$ and $7n\leqq f(x)<8n$, for $f_{n+1}(x)-f_n(x)>(n+1)(f(x)-1)-2n(n+1)-n(f(x)+1)+2n(n-1)=f(x)-6n-1\geqq n-1\geqq 0$.

We now define $h(x):=\sup\{f_n(x):\ U_n\ni x\}$ for $x\in X-K$. Fix $x\in X-K$ and define k as the smallest integer with $f(x)<8k$. Now assume $7(k-1)\geqq f(x)$. Then $k-1>0$, because $f(x)>0$, and $f(x)<8(k-1)$ which is a contradiction to the definition of k. So $f(x)>7(k-1)$.

Thus $U:=\{y\in X-K:\ 7(k-1)<f(y)<8k\}$ is an open neighbourhood of X. For all $j\geqq k$ the functions f_j are defined on U. But for $y\in U$, $j>8k$ we have $f(y)<j-1$ and so $h(y)\geqq f_{j+1}(y)>f_j(y)$. For $y\in U_j\cap U$ and $j<k$ we have $f(y)>7j$, so $h(y)\geqq f_{j+1}(y)>f_j(y)$.

So $h|U=\max(f_k|U,\ldots,f_{8k}|U)$, each function being locally the maximum of q-convex functions. So $h\in F_q(U-K)$.

On $X-K$ we have $|f-h|<1$. If $K=\emptyset$, the function h is an exhaustion function of X and $f\in F_q(X)$. So X is q-complete with corners. If $K\neq\emptyset$, take a compact set K' whose interior contains K. If $\tilde{h}\in C^0(X)$ is a function with $\tilde{h}|X-K'=h|X-K'$, then $\tilde{h}|X-K'\in F_q(X-K')$ is an exhaustion function of X. So X is q-convex with corners.

From the above lemmas we obtain immediately.

Proposition 1. *Let $K\subset X$ be compact and f an exhaustion function of the complex space X. Assume that for each open set $U\Subset X$ there is a $g\in H_q(U-K)$ with $|g-f|<1$. Then X is q-convex with corners. If $K=\emptyset$, X is q-complete with corners.*

Proof. Let $K'\subset X$ be compact with $K'\supset \operatorname{int} K$: Chose $K'=\emptyset$ if $K=\emptyset$.

Take an open set $V\Subset X$. Then there is another open set U such that $V\Subset U\Subset X$ and an $h\in H_q(U-K)$ with $|h-f|<1$. Because $V-K'\Subset U-K$, by lemma 1 there is a $g\in F_q(V-K')$ with $|g-h|<1$ on $V-K'$. Now $g/2\in F_q(V-K')$ and $|g/2-f/s|<1$. So the proposition follows from lemma 2.

§ 4. Examples for (E_r)

Definition 4. A complex manifold X is called homogeneous if and only if for each $x, y\in X$ there is a biholomorphic map $f: X\to X$ with $f(x)=y$.

Theorem 1. *Let X be a homogeneous connected compact complex manifold and M an open subset of X, $M\neq X$. Then M is q-complete with corners, if it is q-convex with corners.*

Proof. Let h be an exhaustion function of M and $K\subset M$ a compact set such that $h|M-K\in F_q(M-K)$. One may assume that $K=\{x\in M|f(x)\leqq 0\}$. Let $V=\{x\in M|f(x)<1\}$. Then $K\subset V\Subset M$.

Aut X is a Lie group and the connected component of id_X is operating transitively on X. Let U be a neighbourhood of the identity in Aut X such that $\sigma^{-1}V\Subset M$ for each $\sigma\in U$. Then there are elements $\sigma_1,\ldots,\sigma_r\in U$ with the following property: Setting $U_0:=M-K$, $U_i=U_{i-1}\cup\sigma_i(U_{i-1})$ for $i=1,\ldots,r$, we have $U_r=M$.

Now we are inductively defining functions $f_0,\ldots,f_r: M\to\mathbb{R}$ with the following properties:

(1) $f_i|\partial V=1$
(2) $U_i=\{x|f_i(x)>0\}$
(3) $f_i|U_i\in F_q(U_i)$
(4) $f_i|M-V=f_{i-1}|M-V$

for $i=1,\ldots,r$.

Let $f_0=f$ then the condition (4) says nothing and (1), (2), (3) are valid, for $U_0=M-K$ and $V=\{x|f(x)<1\}$.

Now we assume that $i\geqq 1$ and (1)-(4) are valid for $i-1$. There is a $\delta>0$ such that for $x\in\partial V$ one has $\delta f_{i-1}\circ\sigma_{i-1}^{-1}(x)<f_{i-1}(x)$ because $f_{i-1}(x)=1$ and $\sigma_{i-1}^{-1}(V)$ is compact in M.

Now let $f_i := \max(\delta f_{i-1} \circ \sigma_{i-1}^{-1}, f_{i-1})$ on V and $f_i = f_{i-1}$ on $M - V$.

(1) and (4) are trivial.

On $M - V$ we have $f_i = f_0 = f \geqq 1$, so $f_i > 0$ on $M - V$. On V we have $f_i(x) > 0$, if and only if $f_{i-1} \circ \sigma_{i-1}^{-1} > 0$ or $f_{i-1} > 0$ which is equivalent to $x \in \sigma_{i-1}(U_{i-1}) \cup U_{i-1} = U_i$. So $\{x \in M \mid f_i(x) > 0\} = U_i \cup (M - V)$. But $U_i \supset U_0 = M - K \supset M - V$, so we have proved (2). Because $\delta f_{i-1} \circ \sigma_{i-1}^{-1} < f_{i-1}$ on ∂V we have $f_i = f_{i-1}$ on an open neighbourhood W of $M - V$. By $M - V \subset U_{i-1}$ and induction $f_i | W \cap U_{i-1} \in F_q(W \cap U_{i-1})$. So it is enough to prove $f_i | V \cap U_i \in F_q(V \cap U_i)$.

Let $x \in U_i \cap V$. We distinguish three cases.

First case: $\delta f_{i-1} \circ \sigma_{i-1}(x) < f_{i-1}(x)$. Then $f_{i-1}(x) > 0$, $x \in U_{i-1}$, and there is a neighbourhood S of x such that $S \subset U_{i-1}$ and $f_{i-1} = f_i$ on S. So by induction $f_i | S \in F_q(S)$. Second case: $\delta f_{i-1} \circ \sigma_{i-1}(x) > f_{i-1}(x)$. Then $x \in \sigma_i U_{i-1}$ and on a neighbourhood $S \subset \sigma_i U_{i-1}$ of x we have $f_i = \delta f_{i-1} \circ \sigma_{i-1}^{-1}$. Because σ_{i-1}^{-1} is biholomorphic and $\delta > 0$ we can conclude $f_i | S \in F_q(S)$.

Third case: $\delta f_{i-1} \circ \sigma_{i-1}(x) = f_{i-1}(x)$. Then $x \in \sigma_{i-1}(U_{i-1}) \cap U_{i-1}$. But

$$f_i \in F_q(\sigma_{i-1}(U_{i-1}) \cap U_{i-1})$$

because $\delta f_{i-1} \circ \sigma_{i-1}^{-1}$ and f_{i-1} are and f_i is the maximum of them.

Now $f_r \in F_q(M)$ and $f_r | M - V = f | M - V$ which proves that M is q-complete with corners.

We now prove that (E_0) holds on $\mathbb{P}_n$ and (E_1) on the Hopf manifold.

Let π: $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}_n$ be the canonical map $\langle x, y \rangle$ the euclidean scalar product and $\|x\|$ the euclidean norm in in $\mathbb{C}^{n+1}$. Define $g(x, y) := -\log \left(1 - \frac{|\langle x, y \rangle|^2}{\|x\|^2 \|y\|^2}\right)$. Then $g(x, y) = +\infty$, if and only if $\pi(x) = \pi(y)$. Furthermore $g(x_1, y_1) = g(x_2, y_2)$ if $\pi(x_1) = \pi(x_2)$ and $\pi(y_1) = \pi(y_2)$. So there is an exhaustion function f of $\mathbb{P}_n \times \mathbb{P}_n - \Delta \mathbb{P}_n$ with $f(\pi(x), \pi(y)) = g(x, y)$. This function is n-convex on $\{(\pi(x), \pi(y)) : \pi(x) \neq \pi(y), \langle x, y \rangle \neq 0\}$. So $\mathbb{P}_n \times \mathbb{P}_n - \Delta \mathbb{P}_n$ is n-convex. From the preceding lemma it follows that $\mathbb{P}_n \times \mathbb{P}_n - \Delta \mathbb{P}_n$ is n-complete with corners, so (E_0) holds.

Let $\rho \in \mathbb{C}$ with $|\rho| > 1$. For $k \in \mathbb{Z}$ the map Φ_k: $\mathbb{C}^n - \{0\} \to \mathbb{C}^n - \{0\}$, defined by $\Phi_k(z) = \rho^k z$ is an automorphism of $\mathbb{C}^n - \{0\}$. In this way the action of $\mathbb{Z}$ on $\mathbb{C}^n - \{0\}$ is properly discontinous, and $X := \mathbb{C}^n - \{0\}/\mathbb{Z}$ is a compact complex manifold, called Hopf manifold. We claim that (E_1) holds on X. To prove this we need the following

Lemma 3. *Let Y be a complex manifold and $f \in C^0(Y)$. Assume that for each $y \in Y$ there is an open neighbourhood U and a q-codimensional complex submanifold $H \ni y$ of U such that $f | H$ is k-convex. Then $f \in H_{q+k}(Y)$.*

Proof. One may assume that Y is an open subset of $\mathbb{C}^n$. Let $y \in Y$, and U and H as in our assumption. By a coordinate transformation one can obtain that $y = 0$ and $H = \{z_1 = \ldots = z_q = 0\}$. Let W be an open neighbourhood of 0 in U. We have to prove that there is an open neighbourhood V of x with $V \Subset W$ and a continuous function g: $\bar{V} \to \mathbb{R}$ such that $g | V$ is $(q+k)$-convex on V, $g(x) = f(x)$ and $g | \partial V < f | \partial V$.

There are $\varepsilon>0$ and $\delta>0$ such that

$$V:=\{z=(z_1,\ldots,z_n)\colon \max_i |z_i|<\varepsilon\}\Subset W$$

and $\tilde{g}\colon \bar{V}\cap H\to\mathbb{R}$, defined by

$$g(z_{q+1},\ldots,z_n)=f|\bar{V}\cap H-\delta(|z_{q+1}|^2+\ldots+|z_n|^2)$$

is k-convex on $V\cap H$. There we consider $V\cap H$ as an open subset of $\mathbb{C}^{n-q}$.

Now define $h(z_1,\ldots,z_n)=\tilde{g}(z_{q+1},\ldots,z_n)-R\left(\sum_{i=1}^{q}|z_i|^2\right)$ on V. Then h is $(q+k)$-convex on V. If R is large enough, $h|\partial V<f|\partial V$.

We now prove that (E_1) holds on the Hopf manifold X. Let $a:=\log|\rho|^2/|\rho|^2>0$ and

$$h(u,v):=-\log\|u-v\|^2+\log(\|u\|^2+\|v\|^2)+a \quad \text{if } 0<\|u\|^2+\|v\|^2\leqq 1$$

and

$$h(u,v):=-\log\|u-v\|^2+a(\|u\|^2+\|v\|^2) \quad \text{if } 1\leqq\|u\|^2+\|v\|^2\leqq|\rho|^2.$$

So h is defined if $0<\|u\|^2+\|v\|^2\leqq|\rho|^2$ and $h(u,v)=\infty$ if and only if $u=v$.

Let $M:=\{(u,v)\in\mathbb{C}^n\times\mathbb{C}^n\colon 0<\|u\|^2+\|v\|^2<|\rho|^2,\ u\neq v\}$. Applying the preceding lemma we show that $h\in H_{n+1}(M)$. The function h is continous. If $(u,v)\in M$ and $\|u\|^2+\|v\|^2\neq 1$ then h is n-convex in (u,v). So we have to look upon the points (u,v) with $\|u\|^2+\|v\|^2=1$. But the ball $\{\|u\|^2+\|v\|^2<1\}$ is strongly pseudoconvex and there is a smooth hypersurface H through (u,v) such that $\|u\|^2+\|v\|^2\geqq 1$ on H. So $h|H=-\log\|u-v\|^2+a(\|u\|^2+\|v\|^2)|H$ which is n-convex. Now $h\in H_{n+1}(M)$ by lemma 3.

Let now $p\colon\mathbb{C}^n-\{0\}\to X$ be the quotient map. For $(x,y)\in X\times X$ we define $Q(x,y):=\{(u,v)\colon u=0$ or $p(u)=x;\ v=0$ or $p(v)=y;\ |\rho|^2>\|u\|^2+\|v\|^2>0\}$ and $f(x,y):=\sup\{h(u,v)\colon (u,v)\in Q(x,y)\}$.

The function $f\colon X\times X\to\mathbb{R}\cup\{\infty\}$ is continuous and $f(x,y)=\infty$, if and only if $x=y$, which is verified easily.

For each $(x,y)\in X'$ there is a $(u,v)\in M$ such that $p(u)=x$, $p(v)=y$ and $f(x,y)=h(u,v)$. This follows from the definition of h, because $h(\rho^{-\nu}u,\ \rho^{-\nu}v)=h(\rho^{-(\nu+1)}u,\ \rho^{-(\nu+1)}v)$ for large ν, $u\neq 0$, $v\neq 0$. Also, $h(u,v)<f(x,y)$, if $p(u)=x$, $p(v)=y$ and $\|u\|^2+\|v\|^2=|\rho|^2$, because

$$f(x,y)\geqq h(\rho^{-1}u,\rho^{-1}v)>h(u,v)+\log|\rho|^2-a|\rho|^2=h(u,v)$$

in this case.

Assume first that $u\neq 0$ and $v\neq 0$. Then take neighbourhoods U of u and V of v in $\mathbb{C}^n-\{0\}$ such that $U\times V\subset M$ and $p|U\to p(U)$, $p|V\to p(V)$ are biholomorphic. Let $S=p(U)\times p(V)$, $t\colon S\to U\times V$ the inverse mapping of $(p|U)\times(p|V)$ and consider the function $h\colon S\to\mathbb{R}$, defined by $h\colon S\to\mathbb{R}$, defined by $\tilde{h}=h\circ t$. Obviously $\tilde{h}\in H_q(S)$, because $h\in H_q(U\times V)$. Moreover $\tilde{h}(x,y)=f(x,y)$ and $\tilde{h}\leqq f|S$. From this it follows immediately from the definition of H_q that $f\in H_q(S)$.

Let now $u=0$ or $v=0$, one of them must be $\neq 0$. Taking the definition of h one obtains $h(u,v)\leqq a|\rho|^2$. So $f\in H_q(Z-\Delta_x)$, where $Z=\{(x,y)\colon f(x,y)>a|\rho|^2\}$ is an open set containing Δ_x. Because X is compact, X' is q-convex with corners. By theorem 1 the manifold X' is q-complete with corners which implies (E_1).

§ 5. Construction of continuous q-convex functions

The construction of continuous q-convex functions on a subset of a complex space X using the property (E_r) is is done by the following

Lemma 4. *Let U be a complex space, V a complex manifold with $\dim V=m$, $f\in F_r(U\times V)$, $\sup f<\infty$, $s(u):=\sup\{f(u,v)|v\in V\}$ for $u\in U$, and $x\in U$, $y\in V$ with $s(x)=f(x,y)$. Then $s\in G_{r-m}(x)$.*

Proof. First we assume that U is an open set in $\mathbb{C}^n$, $x=0$ and V is an open set in $\mathbb{C}^m$ with $y=0$. Then there are open neighbourhoods $U_1\subset U$ and $V_1\subset V$ of x resp. y and strongly q-convex functions $f_1,\ldots,f_r\in C^\infty(U_1\times V_1)$ such that $f|U_1\times V_1=\max(f_1,\ldots,f_r)$ and $f_1(0,0)=f(0,0)$. There is a subspace H of the vectorspace $\mathbb{C}^{n+m}$ with $\dim H=n+m-r+1$ such that the Levi form of $f_1|U_1\times V_1\cap H$ is positive-definite in $(0,0)$. Now $f_1|\{0\}\times V_1$ has a local Maximum in $(0,0)$, so the Leviform of $f_1|\{0\}\times V_1$ in $(0,0)$ is negative-semidefinite. Therefore $H\cap\{0\}\times\mathbb{C}^m=\{0\}$.

If $p_1\colon\mathbb{C}^n\times\mathbb{C}^m\to\mathbb{C}^n$ is the projection on the first component, $p_1|H\to p_1(H)$ is an isomorphism of vectorspaces. Let $p_2\colon\mathbb{C}^n\times\mathbb{C}^m\to\mathbb{C}^m$ be the projection on the second component and $q_1=p_2\circ(p_1|H)^{-1}\colon p_1(H)\to\mathbb{C}^m$. There is a linear map $q\colon\mathbb{C}^n\to\mathbb{C}^m$ with $q|p_1(H)=q_1$.

The set $U_2:=U_1\cap q^{-1}(V_1)$ is an open neighbourhood of 0 in U_1. For $z\in U_2$ one can define $g(z):=f_1(z,q(z))$, By definition of s and $f_1\leqq f$ we get $g\leqq s|U_2$. Furthermore $g(0)=f_1(0,0)=f(0,0)=s(0)$ and g is differentiable; because $g=f_1\circ(\mathrm{id}\times\mathrm{q})U_2$. Because $\mathrm{id}\times\mathrm{q}$ is injective and the Leviform of $f_1|H\cap U_1\times V_1$ in 0 is positive-definite, $g|U_2\cap(\mathrm{id}\times\mathrm{q})^{-1}(H)$ is strongly 1-convex in a neighbourhood of 0. Because the dimension of this submanifold is the same as the dimension of H which is $n+m-r+1$, the function g is in a neighbourhood of 0 strongly $(r-m)$-convex and $s\in G_{r-m}(0)$.

New let U be a complex space and V a complex manifold. Then there are

(1) open neighbourhoods U_1 of x and V_1 of y,

(2) open neighbourhoods U_2 of $0\in\mathbb{C}^n$ and V_2 of $0\in\mathbb{C}^m$,

(3) an analytic subset A of U_2 containing 0

(4) biholomorphic mappings $F_1\colon U_1\to A$ and $F_2\colon V_1\to V_2$ with $F_1(x)=0$ and $F_2(y)=0$ and

(5) r-convex functions $\tilde{f}_1,\ldots,\tilde{f}_r$ on $U_2\times V_2$ such that $\sup\tilde{f}_i<\infty$ and $f=\max(\tilde{f}_1,\ldots,\tilde{f}_r)\circ(F_1\times F_2)$.

Now define $\tilde{f}=\max(\tilde{f}_1,\ldots,\tilde{f}_r)$ and $\tilde{s}(u)=\sup\{\tilde{f}(u,v)|v\in V_2\}$. Then $\tilde{s}\in G_{r-m}(0)$ due to the first part of our proof and $s=\tilde{s}\circ F_1\in G_{r-m}(x)$.

§ 6. Avoiding analytic subsets

Let X be a complex space, A an analytic subset of codimension q in X and $f\in F_{n+r}(X')$ an exhaustion function of X where $n=\dim X$.

If X is compact the function $s(x)=\sup\{f(x,y),\ y\in A\}$ is an exhaustion function of $X-A$. If A is smooth one obtains by lemma 4 and §3 that $X-A$ is $(q+r)$-complete with corners.

If A has singularities one has to change the function f such that it remains $(n+r)$-convex with corners and such that the sup is not taken on $\operatorname{Sg} A$. This is done by

Lemma 5. *Let X be a complex space and $B\subset X$ an analytic subset. Then for each $Y\Subset X$ there are functions $g_1,\dots,g_r\in C^\infty(X)$ and a continous function $g\colon Y\to\mathbb{R}\cup\{-\infty\}$ such that $B=\{x\in X\mid g(x)=-\infty\}$ and for each $y\in Y-B$ there is an open neighbourhood $U\subset Y-B$ and a $1\leqq i\leqq r$ with $g|U+g_i|U\in F_1(U)$.*

Proof. There are open sets $W_i\Subset V_i\Subset U_i$ in X; $i=1,\dots,k$, such that $\bigcup_{i=1}^{k} W_i\supset Y$, $h_i\in C^\infty(X)$, $f_\rho^{(i)}\in\mathcal{O}(U_i)$, $i=1,\dots,k$, $\rho=1,\dots,r$ such that the $f_\rho^{(i)}$ generate the ideal of B in U_i, $\operatorname{supp} h_i\subset U_i$, h_i is 1-convex in a neighbourhood of $\bar{V}_i$.

Let $s_i:=\log\sum_{\rho=1}^{r}|f_\rho^{(i)}|^2$, $s_i\colon U_i-B\to\mathbb{R}$. Because the $f_\rho^{(i)}$ and $f_\rho^{(j)}$ generate the ideal of B in $U_i\cap U_j$ there is a constant C such that $|s_i-s_j|<C$ on $V_i\cap V_j-B$ for $i,j=1,\dots,k$. Let $p_i\in C^\infty(X)$ such that $p_i|W_i\equiv C$, $\operatorname{supp} p_1\subset V$. There is an $a>0$ such that ah_i+p_j is 1-convex on V_i for $i,j=1,\dots,k$. Set $g_i:=ah_i$ and $g(x):=\max\{s_i(x)+p_i(x)\mid x\in V_i\}$ for $x\in Y-B$ and $g(x)=-\infty$ for $x\in B$. The function $g\colon Y\to\mathbb{R}\cup\{-\infty\}$ is continuous. Let $y\in Y-b$.

There is a neighbourhood U of y in $Y-B$ with the following properties:

(1) $U\subset V_j$ if $y\in V_j$

(2) $U\cap\operatorname{supp} p_j=\emptyset$ if $p_j(y)=0$.

Then $g|U=\max\{s_j+p_j|y\in V_j\}$.

For let $y\notin V_j$, $z\in U\cap V_j$. Then $p_j(z)=0$, so $s_j(z)+p_j(z)=s_j(z)$. Let $z\in W_i$. Then $g(z)\geqq s_i(z)+p_i(z)=s_i(z)+C>s_j(z)=s_j(z)+p_j(z)$.

Now $g|U+g_j|U\in F_1(U)$ for each j with $y\in V_j$.

§7. Q-pseudoconvexity

Definition 5. Let X be a complex space. An open subset M of X is called q-pseudoconvex if there is an analytic subset B of X such that $\overline{X-(M\cup B)}=X-M$ and for each $x\in X-(M\cup B)$ there is a locally analytic subset $x\in H\subset X-M$ which is a complex manifold of dimension $\dim X-q$.

If A is an analytic subset of X with $\dim_x A\geqq n-q$ for each $x\in A$, then $X-A$ is q-pseudoconvex. To see this you have to take $B:=\operatorname{Sg} A$ in the definition and $H=A-\operatorname{Sg} A$ for $x\in A$. Of course, the condition need only be verified for the points of $\mathcal{O}M-B=A-\operatorname{Sg} A$.

A second example arises from continuous q-convex exhaustion functions.

Lemma 6. *Let X be a complex space, $K\subset X$ be compact, g an exhaustion function of X such $g|X-K\in F_q(X-K)$ and $X_c:=\{x\in X\colon g(x)<c\}$. Then there are arbitrary large $c\in\mathbb{R}$ such that X_c is q-pseudoconvex.*

Proof. Applying the theorem of Sard we prove that there is a zero set $N \subset \mathbb{R}$ and a $c_0 \in \mathbb{R}$ such that X_c is q-pseudoconvex if $c \notin N$ and $c > c_0$.

To do this we introduce the notion of critical points. A point $x \in X$ is not a critical point, if $x \in \operatorname{Sg} X$ or there is an open neighbourhood U of x and a q-convex function $h: U \to \mathbb{R}$ such that $h(x) = g(x)$, $h \leqq g | U$ and $dh(x) \neq 0$.

Let C be the set of critical points of $g | X - K$. Obviously M_c is q-pseudoconvex if $\sup g | K < c$ and $c \notin g(C)$. So we get our assertion from the fact that $g(C)$ is a zero set in $\mathbb{R}$ with respect to the Lebesgue measure. To prove this take $x \in X - K$, a neighbourhood V of x and q-convex functions $g_1, \ldots, g_r$ in V such that $g | V = \max(g_1, \ldots, g_r)$.

Let $c \in g(C \cap V)$ and $y \in V \cap C$ such that $g(y) = c$. There is a $1 \leqq \rho \leqq r$ with $g_\rho(y) = g(y)$. From our definition we get $dg_\rho(y) = 0$. If C_ρ is the set of critical points of g_ρ, i.e. $C_\rho := \{z \in V | dg_\rho(z) = 0\}$, we conclude that $c \in g_\rho(C_\rho)$. So $g(C \cap V) \subset \bigcup_{N=1}^{r} g_N(C_N)$. By the theorem of Sard $g(C \cap V)$ is a zero set because the $g_\rho(C_\rho)$ are zero sets. Thus $g(C)$ is also a zero set as a countable union of zero sets.

Lemma 7. *Let X be a complex space, $n = n(X) = \min_{x \in X} \dim_x X$, $S \subset X \times X$ be open and $f \in F_{n+r}(S - \Delta_X)$. Let $M \Subset N \Subset X$ be open such that $N - M \neq \emptyset$ and $M \times N \Subset S$. Define $s(x) = \sup\{f(x, y) | y \in \bar{N} - M$ for $x \in M$ and assume $s(x) > f(x, y)$ if $y \in \partial N$.*

If M is q-pseudoconvex or q-convex with corners, then for each $D \Subset M$ and each $\varepsilon > 0$, there is a $g \in F_{q+r}(D)$ such that $|g - s| < \varepsilon$ on D.

Proof. By Lemma 1 it is sufficient to find such a $g \in G_{q+r}(D)$. We first prove the statement in the case that M is q-pseudoconvex. Let B be the analytic set in the definition. Because $D \times \partial N$ is a compact subset of $S - \Delta_X$ there is an $a \in \mathbb{R}$, $a > 0$ such that $s(x) - f(x, y) > a$ for $x \in D$, $y \in \partial N$.

Due to lemma 5 there are functions $t_1, \ldots, t_r \in C^\infty(X)$ and a continous t: $X \to \mathbb{R} \cup \{-\infty\}$ such that $B = \{x \in X : t(x) = -\infty\}$ and for each $x \in N - B$ there is an open neighbourhood $U \subset N - B$ and a $1 \leqq i \leqq r$ with $t | U + t_i | U \in F_1(U)$.

Let $f_\delta^{(i)}(x, y) = f(x, y) - \delta t_i(y)$ and $f_\delta(x, y) = f(x, y) + \delta t(y)$.
If $\delta > 0$ is small enough, $f_\delta^{(i)} \in F_{n+r}(D \times (N - M))$ for $1 \leqq i \leqq r$ which implies $f_\delta | D \times (N - M - B) \in F_{n+r}$. Let $s_\delta(x) = \sup\{f_\delta(x, y) | y \in \bar{N} - M\}$.

Because of $\overline{X - (M \cup B)} = X - M$, $|s_\delta - s| < \min(a, \varepsilon)$ on D for small enough $\delta > 0$. Thus $s_\delta(x) > f(x, y)$ if $x \in D$, $y \in \partial N$ and for each $x \in D$ there is a $y \in N - M$ such that $s_\delta(x) = f(x, y)$. Now the assertion follows from lemma 4 and the q-pseudoconvexity of M.

Let now M be q-convex with corners. There is a compact set $K \subset M$ and an exhaustion function g of M such that $g | M - K \in F_q(M - K)$. Let $M_c := \{x \in M | g(x) < c\}$. By lemma 6 the set M_c is q-pseudoconvex for arbitrary large c. If c is large enough such that $D \Subset M_c$, $M \times (N - M_c) \Subset S$ and $|t - s| < \varepsilon/2$ on D where $t(x) = \sup\{f(x, y) : y \in \bar{N} - M_c\}$, there is an $h \in G_{r+q}(D)$ such that $|h - t| < \varepsilon/2$ on D. This follows from the first part. But then $h \in G_{1+r}(D)$ and $|h - s| < \varepsilon$.

Lemma 8. *Let Y be a complex space, $S_i \subset W_i \subset V_i$ open sets of Y with $\bigcup_{i=1}^{k} S_i = Y$ and $\partial W_i \subset V_i$. Let $h_i \in F_s(V_i)$, $g \in C^0(Y)$, $\varepsilon > 0$, $\delta \geqq 0$, $|h_i - g| < \varepsilon$ on V_i, $h_i - g > \delta$ on W_i and $h_i - g < \delta$ on $V_i - W_i$. Then there is an $h \in F_s(V)$ such that $|h - g| < \varepsilon$.*

Proof. Let $h(x) = \max\{h_i(x): x \in W_i\}$. Then $|h - g| < \varepsilon$. Let $x \in X$ and $I = \{i | x \in W_i\}$. Let $j \in I$ such that $x \in S_j$. If $x \in \partial V_m$ the inequation $h_j(x) > h_m(x)$ holds and is thus valid in a neighbourhood of x. Therefore we have an open neighbourhood U of x in Y such that $U \subset \bigcap_{i \in I} V_i$, $U \cap V_m = \emptyset$ if $x \notin \bar{V}_m$ and $h_j > h_m$ on U if $x \in \partial V_m$. Now $h|U = \max\{h_i|U: i \in I\}$ which proves that $h \in F_s(Y)$.

Lemma 9. *Let $M \Subset X$ be q-pseudoconvex or locally q-complete with corners. If (E'_r) holds on X then M is $(q+r)$-convex with corners.*

Proof. We may assume $M \neq X$. There is an open set $T \supset \Delta_X$ and an $f \in F_{n+r}(T - \Delta_X)$ where $n = n(X)$ such that (E'_r) satisfied.

There are open sets $S_i \Subset W_i \Subset V_i \Subset U_i \Subset P_i$ for $i = 1, \ldots, k$ such that $\bigcup_{i=1}^{k} S_i \supset \partial M$, $P_i \times P_i \Subset T$, $U_i \subset P_j$ if $U_i \cap U_j \neq \emptyset$ and $U_i \cap M$ is either q-pseudoconvex for each i or q-complete with corners for each i.

Let $S := \bigcup_{i=1}^{k} S_i$, $V := \bigcup_{i=1}^{k} V_i$, $U := \bigcup_{i=1}^{k} U_i$.

We now define $s(x) := \sup\{f(x, y) | (x, y) \in \bigcup_{i=1}^{k} V_i \times \bar{P}_i, y \notin M\}$ for $x \in M \cap V$. $\bigcup_{i=1}^{k} \bar{V}_i \times (\bar{P}_i - U_i)$ is a compact subset of $T - \Delta_X$. Thus there is an $a \in \mathbb{R}$ such that $f(x, y) < a$ if $x \in V_i$ and $y \in \bar{P}_i - U_i$ for $i = 1, \ldots, k$. Also there is a compact set $K_1 \subset M$ such that $K_1 \supset M - S$ and $s > a + 1$ on $M - K_1$. Let $K \subset M$ be compact with $K_1 \subset \operatorname{int} K$. We prove that s is continuous on $M - K_1$ and that for each open N with $K \subset N \Subset M$ there is a $g \in F_{q+r}(N - K)$ with $|g - s| > 1$ on $N = K$. We then get our assertion from proposition 1.

Let $x \in V_i \cap M - K_1$. Then by construction $s(x) = \sup\{f(x, y) | x \in V_j, y \in U_j - M\}$. From $x \in V_i \cap V_j$ we obtain $U_j \subset P_i$, so $s(x) \leqq \sup\{f(x, y) | y \in P_i - M\} \leqq s(x)$. So we have $s(x) = \sup\{f(x, y) | y \in U_i - M\}$ on $V_i \cap M - K_1$. There are functions $t_i \in C^\infty(X \times X)$ such that $0 \leqq t_i \leqq 1$, $t_i | (N \cap S_i) \times (U_i - M) \equiv 1$ and $\operatorname{supp} t_i \Subset (T - \Delta_X) \cap (M \cap W_i) \times X$. Let $f_i^{(\delta)} = f + \delta \psi_i$ and $s_i^{(\delta)}(x) = \sup\{f_i^{(\delta)}(x, y) | y \in U_i - M\}$ for $x \in U_i \cap M - K_1$. If $0 < \delta < 1/2$ is small enouh, then $f_i^{(\delta)} \in F_{n+r}(T - \Delta_X)$.

By lemma 7 there is a $g_i \in F_{n+r}(V_i \cap N - K)$ such that $|g_i - s_i^{(\delta)}| < \delta/2$ on $V_i \cap N - K$. We apply lemma 8 to the complex space $N - K$. Note that in this topological space $\partial W_i \cap (N - K)$ is the boundary of W_i. On $V_i \cap (N - K)$ we have $0 \leqq s_i^{(\delta)} - s \leqq \delta$, so $|g_i - s| < 3/4 < 1$ on $V_i \cap (N - K)$. On $N \cap S_i - K$ we have $s_i^{(\delta)} - s = \delta$, so $g_i - s > \delta/2$ on $N \cap S_i - K$ and on $V_i - W_i - K$ we have $s_i^{(\delta)} = s$, so $g_i - s < \delta/2$ there. This proves our assertion.

§ 8. Proof of theorems 3–6

Theorem 3 is a part of lemma 9. So we have to prove theorem 4. Let $K \subset M$ be a compact set and s an exhaustion function of M with $s|M - K \in F_q(M - K)$. Let

$M_c := \{x \in M : s(x) < c\}$ for $c \in \mathbb{R}$ and $f \in F_r(X \times X - \Delta_X)$ coming from the condition (E_r). We define $p_c(x) = \sup\{f(x, y) | y \in X - M_c\}$ for $x \in M_c$ and $p(x) = \sup\{f(x, y) | y \in X - M\}$ for $x \in M$. Let $N \Subset M$ be open. By lemma 6 there is a large c such that M_c is q-pseudoconvex and $N \Subset M_c$, $|g - p_c| < 1/2$ on N. Take in lemma 7 $S = X \times X$ and $N = X$. Then there is an $h \in F_{q+r}(N)$ with $|h - p_c| < 1/2$. The assertion now follows from proposition 1 and $|h - g| < 1$ on N.

Proof of theorem 5. One may assume that X is an open subset of $\mathbb{C}^n$. Now (E_0) holds on $\mathbb{C}^n$. Let $x \in A \subset X$ and $V \Subset X$ a neighbourhood of x which is q-pseudoconvex. Then $V - A$ is q-convex with corners by Lemma 9. So $U - A$ is q-complete with corners for some smaller neighbourhood U of x.

Proof of theorem 6. The first part follows from lemma 9 because $X - A$ is q-pseudoconvex. To prove the second part let $f \in F_{n+r}(X \times X - \Delta_X)$, fullfilling (E_r).

Let $g(x) = \sup\{f(x, y) | y \in A\}$, $x \in X - A$. By lemma 7 for each $M \Subset X - A$ there is an $h \in F_{q+r}(M)$ such that $|h - g| < 1$ on M. Thus the lemma follows from proposition 1.

§ 9. Proof of theorem 2

Let X be a complex space, $M \subset X$ open, $K \subset M$ compact, $f \in C^0(X)$, $f | X - K \in F_r(X - K)$, such that $\{x \in X | f(x) < c\} \cap M \Subset X$ for each $c \in \mathbb{R}$. Now we must first prove

Lemma 10. *Assume that I is an index set, U_i, V_i, W_i are open sets in X, $f_i \in C^0(U_i \cap M)$ exhaustion functions for $i \in I$ such that*

(i) *$(U_i)_{i \in I}$ is a locally finite family, $W_i \Subset V_i \Subset U_i$ and $\bigcup_{i \in I} W_i \supset M$,*

(ii) *$f_i | V_i \cap V_j \cap M - f_j | V_i \cap V_j \cap M$ is bounded,*

(iii) *for each $U \Subset U_i \cap M$ there is a $g \in F_q(U)$ with $|g - f_i| < 1$ on U.*

Then M is $(q+r-1)$-convex with corners. If $K = \emptyset$ then M is $(q+r-1)$-complete with corners.

Proof. Let $K' \subset M$ be compact with $K \subset \operatorname{int} K'$, $K' = \emptyset$ if $K = \emptyset$. We construct an exhaustion function $h \in C^0(M)$ such that for each open $N \Subset M$ there is a $p \in F_{q+r-1}(N - K')$ with $|p - h| < 1$ on the domain of p. Our conclusion follows from proposition 1.

There are functions $p_i \in C^\infty(X)$ with $\operatorname{supp} p_i \subset V_i$ such that $f_i + p_i + 2 < f_j + p_j$ on $\partial V_i \cap W_j$. Furthermore there is an $r \in C^\infty(\mathbb{R})$ such that $\lim_{x \to \infty} r(x) = \infty$ and $r \circ f + p \in F_r(X - K)$ for $i \in I$ because (U_i) is locally finite.

Set $h(x) = r \circ f(x) + \max\{f_i(x) + p_j(x) | x \in V_i\}$ for $x \in M$. Obviously h is an exhaustion function of M.

Let $N \Subset M$. Then there are open sets Z_i with $V_i \Subset Z_i \Subset U_i$. By (iii) there are functions $g_i \in F_q(N \cap Z_i)$ such that $|g_i - f_i| < 1$ on $N \cap Z_i$.

Define $p(x) = r \circ f(x) + \max\{g_i(x) + p_i(x) | x \in V_i\}$ for $x \in N$.

On $N \cap \partial V_i \cap W_j$ we have $g_i + p_i < g_j + p_j$ because $f_i + p_i + 2 < f_j + p_j$.

For each $x\in N-K$ there is an open neighbourhood $U\subset N-K$ such that $p|U=r\circ f+\max\{g_i+p_i|x\in V_i\}$. But $g_i\in F_q(U)$ and $r\circ f+p_i\in F_r(U)$, so $p\in F_{q+r-1}(U)$. This proves $p\in F_{q+r-1}(N)$.

In order to prove (a) of theorem 2 we need

Lemma 11. *Let $f_n\colon \mathbb{R}\to\mathbb{R}$ be monotone functions, that is $f_n(x)\leqq f_n(y)$ if $x\leqq y$, for $n\in\mathbb{N}$ with $\inf\{f_n(x)|x\in\mathbb{R}\}>-\infty$ and $\sup\{f_n(x)|x\in\mathbb{R}\}=\infty$ for each $n\in\mathbb{N}$. Then there is an $r\in C^\infty(\mathbb{R})$, such that $r'>0$, $\lim_{x\to\infty} r(x)=\infty$ and $r\circ f_n-r\circ f_m$ is bounded for $n,m\in\mathbb{N}$.*

Proof. Let (a_i) be a sequence with $a_i<a_{i+1}$, $\sup a_i=\infty$ and

(1) $\min\{f_n(a_i)|n\leqq i\}<\max\{f_n(a_{i+1})|n\leqq i+1$,

(2) $\max\{f_n(a_i)|n\leqq i\}<\min\{f_n(a_{i+1})|n\leqq i+1\}$

for each fixed $i\in\mathbb{N}$.

Let $c_i:=\min\{f_n(a_i)|n\leqq i\}$ for odd i and $c_i:=\max\{f_n(a_i)|n\leqq i\}$ for even i.

Then $c_i<c_{i+1}$ and $\sup c_i=\infty$. So there is an $r\in C^\infty(\mathbb{R})$ such that $r'>0$ and $r(c_i)=i$. Obviously $\lim_{x\to\infty} r(x)=\infty$.

Let $n,m\in\mathbb{N}$. If x is large enough there is an $i\geqq n,m$ such that $a_{2i}\leqq x\leqq a_{2i+2}$. So $c_{2i+2}\geqq\max(f_n(x), f_m(x))$. Now $a_{2i-1}\leqq x$; $n,m\leqq 2i-1$, so $\min(f_n(x), f_m(x))\geqq c_{2i-1}$. Now by definition $r\circ f_n(x)-r\circ f_m(x)\leqq 3$.

Proof of theorem 2 (a). Let M be locally q-complete with corners. It is possible to construct the families U_i, V_i, W_i and f_i such that (i) hold and $f_i\in F_q(U_i\cap M)$, so also (iii) is valid.

For $i,j\in I$, $i\neq j$, $V_i\cap V_j\cap M\neq\emptyset$ we define $B_{ij}(x)=\inf\{f_i(z)|z\in V_i\cap V_j\cap M, f_j(z)\geqq x\}$. B_{ij} is a monotone function on $\mathbb{R}$ with $\sup B_{ij}=\infty$. Applying lemma 11 we get an $r\in C^\infty(\mathbb{R})$ with $r'>0$, $\lim_{x\to\infty} r(x)=\infty$ such that $r\circ B_{ij}(x)-x$ is bounded for all i,j with $V_i\cap V_j\cap M\neq\emptyset$.

Now $r\circ f_i\in F_{q+1}(U_i\cap M)$ are exhaustion functions.

Let $z\in V_i\cap V_j\cap M$. Then $B_{ij}(f_j(z))\leqq f_i(z)$, so $r\circ f_j(z)-r\circ f_i(z)\leqq r(f_j(Z))-r\circ B_{ij}(f_j(z))$ which is bounded.

So $r\circ f_j-r\circ f_i\leqq C$ with a constant C and lemma 10 proves our theorem.

Proof of (b). Let $h(z,w)=-\log\|z-w\|^2$ for $z,w\in\mathbb{C}^n$ and $z\neq w$. For $S\Subset\mathbb{C}^n$ let $h_S(z):=\sup\{h(z,w)|w\notin M\}$. Let S be q-complete with corners and $T\Subset S$. Let $S'\Subset\mathbb{C}^n$ be open such that $S\Subset S'$ and $h_S(z)>\sup\{h(z,w)|w\notin S'\}$. There is an $\varepsilon>0$ such that

$$h_S(z)>\sup\{h(z,w)+\varepsilon(\|z\|^2+\|w\|^2)\colon w\in\partial S'\}$$

and $\varepsilon(\|z\|^2+\|w\|^2)<1/4$ for $z,w\in S'$. So by lemma 7 there is a $g\in F_q(T)$ such that $|g-h_S|<1/2$ because $h(z,w)+\varepsilon(\|z\|^2+\|w\|^2)$ is n-convex.

There are families of open sets (P_i), (U_i), (V_i), (W_i), $i\in I$, biholomorphic maps $\Phi_i\colon P_i\to B=\{z\in\mathbb{C}^n\colon\|z\|<1\}$ such that $W_i\Subset V_i\Subset U_i\Subset P_i$, $\bigcup_{i\in I} W_i\supset M$, (U_i) is locally finite, $U_i\cap M$ is q-complete with corners.

Let $f_i(z)=h_{\Phi_i(U_i\cap M)}(\Phi_i(z))=\sup\{h(\Phi_i(z),\Phi_i(w))\colon w\in\partial(U_i\cap M)\}$. Obviously the conditions (ii) and (iii) of lemma 10 are satisfied, so (b) is proved.

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