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# Analytic torsion and closed geodesics on hyperbolic manifolds

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For $X$ a closed, oriented hyperbolic manifold and $\rho$ an orthogonal representation of $\pi_1 X$, we define a Ruelle-type zeta function $R_\rho(s)$ for $\mathrm{Re}(s)$ large as the product over prime closed geodesics $\gamma$ of factors $\det(I-\rho(\gamma)\,e^{-sl(\gamma)})$, where $l(\gamma)$ is the length of $\gamma$. We analytically continue $R_\rho(s)$ and compute its leading term at $s=0$. In odd dimensions this gives a formula for the Ray-Singer analytic torsion $T_\rho(X)$ in terms of closed geodesics. This shows that the periodic orbits of a hyperbolic geodesic flow can be used to compute Reidemeister torsion via a zeta function in analogy to our earlier results for other classes of hyperbolic flows.

In an appendix we compute the functional determinant of the Laplacian on a hyperbolic surface.

Let $K$ be a finite complex and $\rho$ a representation of $\pi_1 K$ by real matrices of determinant $\pm 1$. Suppose $\rho$ is acyclic, i.e. the vector space $H^i(K;\rho)$ of twisted cohomology classes is zero for all $i$. Then there is an invariant $\tau_\rho(M)\in\mathbf{R}^+$ called Reidemeister torsion. It is computable from the twisted cochain complex of $K$ by taking a suitable alternating product of determinants [Mi1]. This invariant plays a role in simple homotopy theory [Co], where it is used to distinguish homotopy equivalent spaces, and in knot theory, where it arises as the Alexander polynomial [Mi2].

Reidemeister torsion also has a dynamical interpretation in many cases [F2]. For a flow on a closed manifold with circular chain recurrent set we showed in [F1] that $\tau$ can be computed from its closed orbits using a suitable zeta function. This includes flows with cross-section and Smale flows. The case of Smale flows on $S^3$ was first treated (with different terminology) by Franks, who showed how to compute the Alexander polynomial of a link from a suitable flow on its exterior [Fr]. These results depend on applying the Lefschetz fixed point formula to suitably constructed transversals to the flow.

Consider a closed Riemannian manifold $X$ of dimension $d$ and the geodesic flow $\phi$ on the unit tangent bundle $SX$. Trajectories of $\phi$ are the lifts to $SX$ of

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unit speed geodesic curves in $X$. Closed orbits of $\phi$ are thus closed geodesics on $X$. For $d>1$ the flow $\phi$ is not usually accessible to the topological techniques just mentioned. But these results do suggest that the Reidemeister torsion of $SX$ ought to be computable from the closed geodesics on $X$. We will show

**Theorem 1.** *Suppose $X$ is a closed, oriented hyperbolic manifold $\Gamma\backslash H^d$ and $\rho: \pi_1(SX)\to O(m)$ is acyclic. Then the function defined for* $\operatorname{Re} s>d-1$ *by*

$$R_\rho(s)=\prod_\gamma \det(I-\rho(\gamma)\,e^{-sl(\gamma)})$$

*extends meromorphically to* $\mathbf{C}$ *and, for* $\varepsilon=(-1)^{d-1}$,

$$|R_\rho(0)^\varepsilon|=\tau_\rho(SX).$$

*Here $\gamma$ runs over the prime closed geodesics of $X$ (*i.e. *those not expressible as multiples of some shorter closed geodesic) and $l(\gamma)$ denotes the length of $\gamma$.*

In [F2] we compare this to our earlier results and give analogous theorems for geodesic flows on some other Riemannian manifolds. In particular, the exponent $\varepsilon$ is explained as being the Poincaré index of the periodic orbits of the geodesic flow on a hyperbolic $d$-manifold: the right zeta function for computing torsion is $R^\varepsilon$.

The proof of this theorem begins by expressing $R_\rho$ in terms of certain Selberg-type zeta functions $S_0,\ldots,S_{d-1}$. Here each $S_i$ is characterized by the property that its logarithmic derivative $Y_i=S_i'/S_i$ is a certain sum over closed geodesics that arises when the Selberg Trace Formula is applied to $i$-dimensional twisted forms. One has $S_i=S_{d-1-i}$ and each $S_i$ satisfies a certain functional equation relating its values at $s$ and $d-1-s$. These properties alone allow one to prove Theorem 1 for $d=2$ [F5] but for $d>2$ more analysis is needed.

We recall that Ray and Singer defined an analytic torsion $T_\rho(M)$ for every closed Riemannian manifold $X$ and orthogonal representation $\rho$ of $\pi_1 X$ [RS1]. Their definition involves the spectrum of the Laplacian on twisted forms. They conjectured, and it was proved by Cheeger [C] and Müller [Mu], that when $\rho$ is acyclic and orthogonal $T_\rho(X)=\tau_\rho(X)$.

Our proof of Theorem 1 for $d>2$ will use this result. We have $\pi_1(SX)\cong\pi_1 X$ for $d>2$, so $\rho$ in Theorem 1 may be identified with a representation of $\pi_1 X$. Then topology gives $\tau_\rho(SX)=\tau_\rho(X)^2$ (see Section 1). We then need to relate $T_\rho(X)^2$ to $R_\rho(0)$. But $R_\rho$ is expressible by means of the geodesic terms in the Selberg Trace Formula and $T_\rho(X)^2$ is related to the eigenvalue terms. It will be seen that the third sort of terms, coming from the identity element in $\pi_1 X$, cancel out in the end thanks to the functional equations of the $S_i$'s.

When $\rho$ is orthogonal but not acyclic we can still identify $T_\rho(X)$ by means of the leading term of $R_\rho$ at $s=0$. This answers a problem posed by Singer at the Vancouver Congress in 1974 [Si]. The order of vanishing of $R_\rho$ at $s=0$ will be a certain linear combination of the twisted Betti numbers $\beta_i=\dim H^i(X;\rho)$ (see Theorem 3).

**Contents**



## Section 1. Analytic and Reidemeister torsion

For a closed, oriented Riemannian manifold $X$ and an orthogonal representation $\rho$: $\pi_1 X \to O(m)$ we will define the *analytic torsion* $T_\rho(X) \in (0, \infty)$.

First recall how one defines a *functional determinant* $\det(D)$ of a positive elliptic pseudodifferential operator $D$ on $X$. By a result of Seeley [S], $D^{-s}$ is a trace class operator for $\mathrm{Re}(s)$ large and the Dirichlet series $\zeta_D(s) = \mathrm{Tr}\, D^{-s}$ has a meromorphic continuation to $\mathbf{C}$ that is regular at $s=0$. The derivative $(\mathrm{Tr}\, D^{-s})'(0)$ is formally $\sum_\lambda (\lambda^{-s})'(0) = -\sum_\lambda \log \lambda = -\log(\prod \lambda)$, where $\lambda$ runs over $\mathrm{Spec}\, D$ (with multiplicity). Thus we define $\det D = \exp - \zeta'_D(0)$. This definition has proven useful in differential geometry and quantum field theory.

Such operators arise in our situation in the following way. Consider those $\mathbf{R}^n$-valued differential forms $\omega$ on the universal cover $\tilde{X}$ of $X$ that transform according to $\rho$. That is if $g \in \pi_1 X$ acts on $\tilde{X}$ as a deck transformation then the components $\omega_1, \ldots, \omega_n$ of $\omega$ satisfy

$$\begin{pmatrix} g^* \omega_1 \\ \vdots \\ g^* \omega_n \end{pmatrix} = \rho(g) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}.$$

Since the exterior derivative $d$ preserves this equivariance property, one obtains a complex of such *twisted forms*. As $\rho$ is orthogonal, Hodge theory applies. Using the Riemannian measure on $X$ and the Hodge star operator one defines an inner product on twisted forms. Taking the adjoint $d^*$ of $d$ we let $\Delta_j = dd^* + d^* d$ acting on twisted $j$-forms. This Hodge Laplacian $\Delta_j$ is nonnegative and elliptic.

The twisted $j$-forms in $\ker \Delta_j$ are called *harmonic*. They are a set of representatives for the twisted cohomology $H^j(X; \rho)$. When these cohomology groups vanish for all $j$, we say $\rho$ is *acyclic*. In this case the $\Delta_j$ are all positive and we may define $T_\rho$ by

$$(*) \qquad T_\rho^2 = \frac{(\det \Delta_1)(\det \Delta_3)^3 \ldots}{(\det \Delta_2)^2 (\det \Delta_4)^4 \ldots}.$$

For an acyclic representation $\rho$ there is also a combinatorially defined invariant $\tau_\rho(X)$ called *Reidemeister torsion*. Taking a triangulation of $X$ one forms the (finite-dimensional) complex of twisted cochains $C^*(X; \rho)$ with coboundary operator $\delta$. There is a natural way to view this complex as a sum of copies of $\mathbf{R}^m$, one for each simplex in $M$, and this makes $C^*(X; \rho)$ a Hilbert space. Define the combinatorial Laplacian $\Delta_j^c = \delta\delta^* + \delta^*\delta$ and choose $\tau_\rho \in (0, \infty)$ by

$$\tau_\rho^2 = \frac{(\det \Delta_1^c)(\det \Delta_3^c)^3 \dots}{(\det \Delta_2^c)^2 (\det \Delta_4^c)^4 \dots}.$$

Ray and Singer proved this is equivalent to the earlier combinatorial definition of $\tau_\rho$, due to Reidemeister, De Rham and Milnor [RS1]. Their conjecture that $T_\rho = \tau_\rho$ was proved by Cheeger [C] and Müller [Mu].

When $\rho$ is not acyclic, the definition of Reidemeister torsion depends on choosing a density on the vector space $H^j(X; \rho)$. By using the inner product on harmonic $j$-forms, one obtains an inner product on $H^j$ and so a density. With this choice one still has a $\tau_\rho$, but now $\tau_\rho$ is no longer computable from a triangulation of $X$ as it involves the metric in an essential way.

The definition of $T_\rho$ extends to the case of nonacyclic $\rho$ as well: one alters the definition of functional determinant by deleting from the Dirichlet series the terms corresponding to eigenvalues $\lambda = 0$ and uses the same formula (*) as above. With these conventions, one still has $\tau_\rho = T_\rho$.

Recall that Hodge theory gives a $\Delta_j$-invariant decomposition of twisted $j$-forms as

$$\text{harmonic} \oplus \text{exact} \oplus \text{coexact}$$

where $\omega$ is coexact if $*\omega$ is exact. Then $d$ sets up an isomorphism from coexact $j$-forms to exact $j+1$-forms that commutes with $\Delta$. Setting

$$D_j = \Delta_j | \text{coexact}$$

we find $\det \Delta_j = (\det D_j)(\det D_{j-1})$. Substituting into (*) gives

$$(**) \qquad T_\rho^2 = \frac{(\det D_0)(\det D_2) \dots}{(\det D_1) \dots}.$$

The Hodge $*$ operator matches coexact $j$-forms with exact $(d-j)$ forms. We see

$$D_j \cong D_{d-j-1}, \qquad d = \dim X.$$

In particular if $d$ is even, (**) shows $T_\rho = 1$.

A representation $\rho$ of $\pi_1 X$ determines a representation $\rho_1$ of $\pi_1(SX)$. By the Gysin exact sequence for twisted cohomology

$$\dots \to H^k(X; \rho) \to H^k(SX; \rho_1) \to H^{k+1-d}(X; \rho) \to H^{k+1}(X; \rho) \to \dots$$

$\rho$ is acyclic if and only if $\rho_1$ is acyclic. By [A] we have in this case $\tau_{\rho_1}(SX) = \tau_\rho(X)\tau_\rho^*(X)$ where $\rho^*$ is the contragradient representation of $X$. So if $\rho$ is orthogonal we have $\rho = \rho^*$ and so $\tau_{\rho_1}(SX) = \tau_\rho(X)^2$. In particular we have $\tau_{\rho_1}(SX) = 1$ if $X$ has even dimension and $T_{\rho_1}(SX) = T_\rho(X)^2$.

If $d > 2$ then $\pi_1(SX) \cong \pi_1(X)$ and so all acyclic orthogonal representations of $\pi_1(SX)$ come from acyclic orthogonal representations of $\pi_1 X$.

If $\pi_1 X$ has an acyclic representation then $\chi(X) = 0$. This is because one can compute the Euler characteristic of the twisted cohomology from the chain level, getting $n \cdot \chi(X)$, or from the cohomology level, getting 0. For a closed hyperbolic manifold, Hirzebruch proportionality implies $\chi(X)=0 \leftrightarrow d$ odd. Thus we will mainly be interested in odd dimensional manifolds henceforth.

In differential geometry the operators $\Delta_j$ are usually studied via the trace of the heat kernel $\operatorname{Tr} e^{-t\Delta_j}$, where $e^{-t\Delta_j}$ is the semigroup of smooth integral operators that solves the heat equation for twisted $j$-forms at time $t>0$

$$\left(\frac{\partial}{\partial t}+\Delta_j\right)(e^{-t\Delta_j}\omega)=0$$
$$e^{-t\Delta_j}\omega \to \omega \quad \text{as } t\to 0.$$

This can be related to the functional determinant $\det \Delta_j$ as follows. From the definition of the $\Gamma$-function, the Mellin transform of the function $e^{-\lambda t}(\lambda>0$ fixed) is

$$M e^{-\lambda t}=\int_0^\infty t^{s-1} e^{-\lambda t}\,dt=\lambda^{-s}\int_0^\infty x^{s-1} e^{-x}\,dx=\lambda^{-s}\Gamma(s)$$

Summing over $\lambda \in \operatorname{Spec} \Delta_j$, $\lambda>0$, gives

$$M\left[\operatorname{Tr}(e^{-t\Delta_j^*})\right]=\Gamma(s)\operatorname{Tr}(\Delta_j^*)^{-s}$$

where $\Delta_j^*$ is the nonsingular part of $\Delta_j$ obtained by restricting $\Delta_j$ to the orthogonal complement of the harmonic forms. Here $\operatorname{Tr}(\Delta_j^*)^{-s}$ is the zeta function of Minakshisundaram and Pleijel and is denoted $\zeta_j(s)$.

For an odd dimensional $X$, Seeley has shown that $\beta_j+\zeta_j(s)$ vanishes at $s=0$ [S]. Since $\Gamma(s)$ has a pole of residue $+1$ at $s=0$, this shows

$$(***) \qquad -\log\det\Delta_j=\lim_{s\to 0}\Gamma(s)\,\beta_j+M\left[\operatorname{Tr} e^{-t\Delta_j^*}\right].$$

This will be used below to evaluate $\det\Delta_j$.

## Section 2. The trace formula for the heat kernel

For a closed oriented hyperbolic manifold $X$ we will use the Selberg Trace Formula (STF) to compute the determinants that occur in the formula (**) for analytic torsion. For this we will view $X$ as a locally homogeneous space and describe the twisted $j$-forms and the corresponding Laplacians $\Delta_j$ in group theoretic terms, as in [M].

The universal cover $\tilde{X}$ of $X$ can be identified with hyperbolic $d$-space $H^d$, $d=\dim X$. We will view $H^d$ as one sheet of the hyperboloid of two sheets in $\mathbf{R}^{d+1}$ given by $q(x)=-x_0^2+x_1^2+\ldots+x_d^2=-1, x_0>0$ with the metric induced by the quadratic form $q(x)$. The orientation preserving isometries of $H^d$ form the group $G$ which is the identity component of $\mathrm{SO}(1,d)$. The isotropy group $K$ of the basepoint $(1,0,\ldots,0)$ is $\mathrm{SO}(d)$. The deck transformations $\pi_1 X$ form a subgroup $\Gamma$ of $G$. We have $X=\Gamma\backslash G/K$.

The Lie algebra $\mathscr{G}$ of $G$ carries the Killing form defined by

$$C(x,y)=\operatorname{Tr}(\operatorname{ad}(x)\cdot\operatorname{ad}(y))$$

where $x, y \in \mathcal{G}$ act on $\mathcal{G}$ by the adjoint representation ad. We have that $C$ is nondegenerate, i.e. $G$ is semisimple. The restriction of $C$ to the Lie algebra $\mathcal{K} \subset \mathcal{G}$ of $K$ is negative definite. The orthogonal complement $\mathcal{P}$ of $\mathcal{G}$ can be identified with the tangent space to $H^d$ at our basepoint. The restriction of $C$ to $\mathcal{P}$ is positive definite and invariant under the adjoint action of $K$. Thus it determines a $G$-invariant Riemannian metric on $H^d$. The metric of curvature $-1$ corresponds to the normalized Killing form $\langle x, y \rangle = \frac{1}{2d-2} C(x, y)$.

Choose an orthonormal basis for the subspace of $\mathcal{G}^*$ that annihilates $\mathcal{K}$. This basis determines left-invariant 1-forms $\omega_1, \ldots, \omega_d$ on $G$. A complex valued $j$-form $\omega$ on $G/K = H^d$ pulls back to a $j$-form $\omega'$ on $G$ expressible as

$$\omega' = \Sigma f_{i_1 \ldots i_j} \omega_{i_1} \wedge \ldots \wedge \omega_{i_j}.$$

The component functions $(f_{i_1 \ldots i_j})$ give a map $f: G \to \Lambda^j \mathbf{C}^d$ that satisfies

$$f(gk) = \xi_j(k^{-1}) f(g)$$

where $\xi_j$ is the usual representation of $\mathrm{SO}(d)$ on $\Lambda^j \mathbf{C}^d$.

Such functions on $f$ are naturally identified with the sections of a certain homogeneous vector bundle over $H^d$ with fiber $\Lambda^j \mathbf{C}^d$ called the bundle of $j$-forms. $\mathcal{G}$ acts by first order operators on sections of any homogeneous vector bundle, and this extends to an action of the universal enveloping algebra $\mathcal{U}$.

The center of $\mathcal{U}$ gives invariant differential operators. The Hodge Laplacian $\Delta_j$ on $j$-forms for the curvature $-1$ metric arises from the center in this way. Take $E_i$ a basis for $\mathcal{K}$ that is orthonormal for the form $-C$ and $E_k$ a basis for $\mathcal{P}$ orthonormal for $C$. Then $-\Sigma E_i^2 + \Sigma E_k^2$ is the Casimir element in the center of $\mathcal{U}$. The corresponding invariant operator $\Omega'$ is called the Casimir operator. Then $\Delta_j$ is a normalized version of this Casimir operator, namely $\Delta_j = (2-2d)\,\Omega'$.

Finally, let us consider the orthogonal representation $\rho: \Gamma \to O(m)$. The twisted $j$-forms on $\Gamma \backslash H^d$ correspond to equivariant $j$-forms on $H^d$ with values in $\mathbf{C}^m$. These are invariant under the action of $\Delta_j$, acting component by component.

Having thus algebraized the situation, we can bring to bear the representation theory of $G$. We first recall the Iwasawa decomposition $G = \mathrm{KAN}$. In the upper $\frac{1}{2}$-space model for $H^d$, $N$ is the group of all horizontal translations, $A$ is the group of all homotheties that fix the origin and $K$ is the isotropy group of $(0, \ldots, 0, 1)$. Thus $\dim A = 1$: one says that $G$ has split rank one and that $H^d$ is a rank one symmetric space. We let $M$ be the centralizer of $A$ in $K$: i.e. the rotations that fix the vertical axis over the origin, so $M \cong \mathrm{SO}(d-1)$. We let $H$ be an element in the Lie algebra $\mathcal{A} \subset \mathcal{G}$ of $A$ with $\langle H, H \rangle = 1$. The open ray containing $H$ is denoted $A^+$.

The principal series representations of $G$ are parametrized by $\hat{M} \times \mathbf{R}$. A $p$-dimensional representation $\sigma \in \hat{M}$ and a $\nu \in \mathbf{R}$ determine a $p$-dimensional unitary representation of the subgroup $B = \mathrm{MAN}$ by the rule (here $m \in M$, $H \in \mathcal{A}$, $n \in N$)

$$m e^{tH} n \to \sigma(m)\, e^{it\nu}.$$

This induces an infinite-dimensional unitary representation $\pi_{\sigma,\nu}$ of $G$ called a principal series representation. One considers those $B$-equivariant maps $G\to\mathbf{C}^p$ whose restriction to $K$ lie in $L^2(K)$. The natural action of $G$ on this Hilbert space is $\pi_{\sigma,\nu}$.

For our purposes the $\sigma$'s of interest are those that arise by restricting $\xi_j$ to $M$. Writing $\mathbf{C}^d=\mathbf{C}^{d-1}\oplus\mathbf{C}$ we see that every $\omega\in\Lambda^j\mathbf{C}^d$ is uniquely expressible in the form $\omega'+\omega''\wedge dx_d$, $\omega'\in\Lambda^j\mathbf{C}^{d-1}$, $\omega''\in\Lambda^{j-1}\mathbf{C}^{d-1}$. Thus $\xi_j|M=\sigma_j\oplus\sigma_{j-1}$, where $\sigma_j$ is the action of $M=\mathrm{SO}(d-1)$ on $\Lambda^j\mathbf{C}^{d-1}$. Here $\sigma_j$ is irreducible except when $d$ is odd and $j=\dfrac{d-1}{2}$ in which case it decomposes as the sum of the two spin representations $\sigma_j^+,\sigma_j^-$. In all cases, however, the irreducible summands of $\xi_j|M$ are nonisomorphic, i.e. $\xi_j$ is $M$-multiplicity free.

Given a geodesic $\gamma$ on $X$ we let $\gamma_0$ be the unique prime geodesic such that $\gamma$ is a multiple of $\gamma_0$, i.e. $\gamma=\gamma_0^{\mu(\gamma)}$ where $\mu(\gamma)\in\mathbf{Z}^+$ is the multiplicity of $\gamma$. Identifying $\gamma$ with the conjugacy class in $\Gamma$ that it represents, we can conjugate $\gamma$ in $G$ to an element of $MA^+$, say $m_\gamma\exp Hl(\gamma)$. Here $l(\gamma)$ is indeed the length of $\gamma$ for the curvature $-1$ metric [M]. The rotation $m_\gamma$ describes the holonomy when one parallel transports normal vectors around $\gamma$. If one views $\gamma$ as a closed orbit for the geodesic flow then $\gamma$ is hyperbolic i.e. the linear Poincaré map around $\gamma$ is the direct sum of an expansion $A^u(\gamma)$ and a contraction $A^s(\gamma)$. Indeed $A^u(\gamma)=e^{l(\gamma)}m_\gamma$, $A^s(\gamma)=e^{-l(\gamma)}m_\gamma$. We let $\Delta(\gamma)=\det I-A^s(\gamma)$.

Recall that $\hat{M}\times\mathbf{R}$ carries a smooth Plancherel density. On each line $\sigma\times\mathbf{R}$ this can be written $P_\sigma(\nu)\,d\nu$, where $P_\sigma$ is even and of polynomial growth. For $d$ odd, $P_\sigma$ is a polynomial of degree $d-1$.

We now state the Selberg Trace Formula for the trace of the heat kernel on twisted $j$-forms.

**Theorem 2.** *For each $j=0,\dots,d-1$ let*

$$G_t(\sigma_j)=\sum_\gamma a_\gamma\frac{1}{\sqrt{4\pi t}}e^{-l(\gamma)^2/4t}e^{-tc^2}e^{-(d-1)l(\gamma)/2}$$

$$I_t(\sigma_j)=a_1\int_{-\infty}^{+\infty}e^{-t(\nu^2+c^2)}P_{\sigma_j}(\nu)\,d\nu$$

*where $\gamma$ runs over closed geodesics in $X$, and where*

$$c=\left|\frac{d-1}{2}-j\right|,\quad a_1=m\binom{d-1}{j}\mathrm{vol}(X),\quad a_\gamma=\mathrm{Tr}\,\rho(\gamma)\cdot\mathrm{Tr}\,\sigma_j(m_\gamma)\cdot l(\gamma_0)/\Delta(\gamma).$$

*Then*

$$\mathrm{Tr}\,e^{-t\Delta_j}=[I_t(\sigma_j)+I_t(\sigma_{j-1})]+[G_t(\sigma_j)+G_t(\sigma_{j-1})].$$

The notation $G_t$, $I_t$ is to suggest the geodesic and identity terms in the trace formula. This decomposition of the heat trace arises naturally by regarding it as the average amount of heat that returns to its source after time $t$ and dividing these loops up according to their free homotopy class. One compares the heat kernels in $X$ and $\tilde{X}$, say $K_t(x,y)$ and $\tilde{K}_t(\tilde{x},\tilde{y})$, by the rule

$$K_t(x,y)=\sum_{g\in\Gamma}\tilde{K}_t(\tilde{x},g\tilde{y})$$

where $\tilde{x}, \tilde{y} \in \hat{X}$ lie over $x, y \in X$. Setting $x=y$, $\tilde{x}=\tilde{y}$ gives

$$K_t(x,x)=\hat{K}_t(x,x)+\sum_{\gamma}\Big(\sum_{g\in\gamma}\hat{K}_t(\tilde{x}, g\tilde{x})\Big)$$

where $\gamma$ runs over the nontrivial conjugacy classes in $\Gamma$. The left hand side represents the heat that flows from $x$ to $x$ in time $t$. The first term on the right corresponds to heat flow around contractible loops and the inner sum over $g\in\gamma$ gives the heat flow in the free homotopy class of $\gamma$. Now integrate over $X$. Thanks to the highly symmetric situation, one can evaluate the terms arising on the right explicitly. This is done for $d=2, j=0$, and $\rho$ trivial by McKean [Mc] in a paper that's the best introduction to STF.

In Selberg's original description, STF evaluates the trace of an invariant integral operator on functions. In the more modern and general approach through representation theory one considers instead a function $f: G\to C$ which one uses to average the unitary action $\pi$ of $G$ on the sections of a locally homogeneous vector bundle so as to obtain trace class operators $\pi(f) = \int_G \pi(g) f(g)\, dg$. The resulting traces are then expressed as sums over conjugacy classes or, via harmonic analysis, as sums over the irreducible representations of $G$ that occur in this unitary action. The equality of these two sums is STF. This approach is developed in [W] for $C^\infty$ functions $f$ of compact support. However less stringent growth conditions suffice: it is enough that $f$ be in Harish-Chandra's $L^1$ Schwarz space $\mathscr{C}_1(G)$.

For $m=1$ and $\rho$ trivial, our Theorem 2 is just a case considered by Miatello [Mia], formula (2), p. 26, where an $f_t$ in $\mathscr{C}_1(G)$ is exhibited for which $\operatorname{Trace}\pi(f_t) = \operatorname{Trace} e^{-t\Delta_j}$. His proof uses that $\xi_j$ is $M$-multiplicity free (although Wallach informs us that this condition isn't necessary).

Knowing only that $f_t$ exists, one can compute the other side of STF to prove Theorem 2. One needs to know what scalar corresponds to the normalized Casimir operator $\Delta_j$ on the discrete series representation $\pi_{\sigma,\nu}$ for $\sigma\in\hat{M}$ occurring in the restriction $\xi_j|M=\sigma_j\oplus\sigma_{j-1}$.

**Lemma 1.** *If* $j=\frac{d-1}{2}$ *then for a spin representation* $\sigma=\sigma_j^{\pm}$, $\Delta_j=\nu^2$ *on* $\pi_{\sigma,\nu}$. *If* $j\neq\frac{d-1}{2}$ *then for* $\sigma=\sigma_j$, $\Delta_j=\nu^2+c^2$ *on* $\pi_{\sigma,\nu}$.

The first statement of Lemma 1 is Millson's Lemma 1.4 in [M]. The second statement follows by the same proof, which uses the formula for the normalized Casimir in terms of roots (see [F4], [F6]):

$$\Delta_j=\nu^2+\langle\rho_K,\rho_K\rangle-\langle\mu+\rho_M,\mu+\rho_M\rangle$$

where $\mu$ is the highest weight of $\sigma$, $\rho_M$ is half the sum of the positive roots of $M$ and $\rho_K$ is the same for $K$. Here, in the notation of [M], for an odd $d=2n+1$, and $j<n$ we have $\mu=\theta_1+\ldots+\theta_j$, $\rho_M=(n-1)\theta_1+(n-2)\theta_2+\ldots+\theta_{n-1}$ and $\rho_K=n r_0+\rho_M$. Thus

$$\langle\rho_K,\rho_K\rangle=\sum_{i=1}^{n} i^2,\quad \langle\mu+\rho_M,\mu+\rho_M\rangle=\sum_{i=1}^{j}(n+1-i)^2+\sum_{i=j}^{n-1}(n-i)^2$$

and the difference is $(n-j)^2=c^2$, as desired (c.f. [B], pp. 254–5). The case of even $d$ is similar (and not needed in later sections).

Knowing Lemma 1, we see that $f_t$ corresponds to the function $e^{-t(v^2+c^2)}$ on each $\sigma_j$, $0 \leqq j \leqq d-1$. The Fourier transform of this in $v$ is

$$\frac{1}{\sqrt{4\pi t}}\, e^{-l^2/4t}\, e^{-tc^2}$$

which, together with the geometric interpretation of the Jacobian determinant $\Delta(\gamma)$ given above, proves that the geodesic terms in Theorem 2 correspond to those in STF as given in [W]. The identity terms in Theorem 2 follow immediately, since the Plancherel density is exactly the way of recovering the value $f(1)$ from the value of $\pi(f)$ in each $\pi_{\sigma,v}$.

For nontrivial $\rho$, the above methods work as well: one replaces $G$ by $G \times O(m)$, $\Gamma$ by graph $(\rho) \subset G \times O(m)$, $K$ by $K \times O(m)$ and $\xi_j$ by $\xi_j \otimes \mathbf{C}^m$. Alternatively, one may alter the proofs in [M], [Mia] appropriately, keeping $G$ the same.

## Section 3. Ruelle and Selberg functions

For a given orthogonal representation $\rho\colon \pi_1 X \to O(m)$ we will decompose the *Ruelle function* $R=R_\rho$ as an alternating product of more complicated factors, each of which is a Selberg-type zeta function $S_j$, or *Selberg function* for short. The advantage is that each factor is related by STF to differential forms in a single dimension. This is reminiscent of the way Atiyah and Bott decompose the Lefschetz index of a $C^1$ map at an isolated fixed point as an alternating sum, in which each term measures the local trace of the action on differential forms in a certain dimension [AB]. The global version of this decomposition is the well known formula for the Lefschetz number (defined as the intersection number of the graph and the diagonal) as a sum of homology traces in various dimensions: in turn this is analogous to formula (**) of Sect. 1 for analytic torsion. For our calculations we will need the functional equations of these Selberg functions which we also review in this section. Our chief references are [F3], [F4], where much related material is developed.

For fixed $\rho\colon \pi_1 X \to O(m)$ and $j \in \{0, \ldots, d-1\}$ we let $Y(z)=Y_j(z)=\sum a_\gamma e^{-zl(\gamma)}$ and $S(z)=S_j(z)=\exp - \sum a_\gamma e^{-zl(\gamma)}/l(\gamma)$, where $\operatorname{Re} z > d-1$ and we use the notation of Theorem 2. The sums in question converge uniformly on each half-plane $\operatorname{Re} z > d-1+\varepsilon$ and so $Y, S$ are holomorphic functions. Clearly $Y=S'/S$. Now in fact $S$ has a meromorphic continuation to $\mathbf{C}$ [F3] and this satisfies [F4]

$$S_j\left(\frac{d-1}{2}+w\right)=S_j\left(\frac{d-1}{2}-w\right)\exp \int_0^w 4\pi a_1 P_{\sigma_j}(iz)\,dz. \qquad (FE)$$

The corresponding functional equation for $Y$ is simpler:

$$Y_j\left(\frac{d-1}{2}+z\right)+Y_j\left(\frac{d-1}{2}-z\right)=4\pi a_1 P_{\sigma_j}(iz). \qquad (FE)'$$

The case $j=0$ of these results was evidently known to Selberg [Se] but for $d>2$ they were first presented by Gangolli [G]. For $d=2, 3$ and any $j$ one has the work of Hejhal [H] and Scott [Sc], respectively. The case $j=\frac{d-1}{2}$, $d\equiv 3 \pmod 4$, is close to Millson's work in [M], but he actually used the difference (not the sum!) of the two spin representations that decompose $\sigma_j$ in this case (see [F7]).

The relation of $R$ and $S_j$ is (c.f. [F3])

$$R(z)=\prod_{j=0}^{d-1} S_j(j+z)^{(-1)^j}. \tag{RS}$$

The proof of this is simple. For $\operatorname{Re} z$ large, one takes the log of both sides and compares the two contributions of a closed geodesic $\gamma$. The problem reduces to showing

$$(\operatorname{Tr}\rho(\gamma))\, e^{-zl(\gamma)}/\mu(\gamma)=\sum_j (-1)^j a_\gamma\, e^{-(z+j)l(\gamma)}/l(\gamma).$$

which, substituting the definition of $a_\gamma$ and simplifying, becomes

$$\det I - A^s(\gamma)=\sum_j (-1)^j e^{-jl(\gamma)} \operatorname{Tr}\sigma_j(m_\gamma).$$

But since $\sigma_j$ is $j$th exterior power, a linear algebra identity (used also in [AB])

$$\det I - A = \Sigma(-1)^j \operatorname{Tr} \Lambda^j A$$

applied to $A=A^s(\gamma)=e^{-l(\gamma)} m_\gamma$ finishes the proof.

From the functional equation for $S_0$, together with knowledge of its poles, zeroes and order of growth, one can recover STF for sections of the flat bundle. There is a close analogy with the way that the functional equation of the Riemann zeta function gives the explicit formulas of prime number theory: indeed this reasoning (in reverse, since he already had STF) led Selberg to define $S_0$. The geometric meaning of $(FE)$ is discussed in [F3] where we relate it to Poincaré duality on the "space of stable leaves" of the geodesic flow. An intuitive sketch of the proof of the simplest case of $(FE)$ from STF appears in [Mc].

## Section 4. The leading term of $R_\rho$ at 0

We take an odd $d=2n+1$, $n\geqq 1$, and compute $T_\rho$ using STF. This will lead to evaluating the functional determinant of $\Delta_j$ in terms of special values of Selberg functions (see (16) and (26) below). In the end we identify $T_\rho$ as a factor in the leading term of the Ruelle function at 0 (Theorem 3).

We begin with (***) from Sect. 1. Since $\Delta_j^* \cong D_j \oplus D_{j-1}$ we can take an alternating sum over all $j\leqq k$ to obtain

$$-\log\det D_k=\lim_{s\to 0}\Gamma(s)\,\alpha_k + M\,[\operatorname{Tr} e^{-tD_k}] \tag{1}$$

where we have set $\alpha_k=\beta_k-\beta_{k-1}+\ldots\pm\beta_0$. Similarly, using

$$\operatorname{Tr} e^{-t\Delta_j}=\beta_j+\operatorname{Tr} e^{-tD_j}+\operatorname{Tr} e^{-tD_{j-1}}$$

in Theorem 2 and forming an alternating sum gives

$$\alpha_k+\operatorname{Tr} e^{-tD_k}=I_t(\sigma_k)+G_t(\sigma_k) \tag{2}$$

for $t>0$.

Now we fix $k$ and, to emphasize the dependence on $t$, we write $\alpha=\alpha_k$, $H(t)=\operatorname{Tr} e^{-tD_k}$ for the heat term, $I(t)=I_t(\sigma_k)$ for the identity term and $G(t)=G_t(\sigma_k)$ for the geodesic term. In order to analyze $MH$ near $s=0$, as required by (1), we study $MI$ and $MG$, as suggested by (2). First we consider $MI$.

**Lemma 2.** *Given an even polynomial $P$ of degree $2n$ let $F(t)=\int_{-\infty}^{\infty} e^{-tv^2}P(v)\,dv$, for $t>0$. Then $F(t)=t^{-\frac{1}{2}}Q(t^{-1})$ for some polynomial $Q$ of degree $n$.*

**Lemma 3.** *Given $c>0$, let $E(s)=M(e^{-tc^2}F(t))$ for $\operatorname{Re} s$ large. Then $E$ has a meromorphic continuation to $\mathbf{C}$ and is regular at 0 with*

$$E(0)=-2\pi\int_0^c P(iy)\,dy.$$

By linearity we may suppose $P(v)=v^{2a}$. Then

$$\begin{aligned}F(t)&=\left(-\frac{\partial}{\partial t}\right)^a\int_{-\infty}^{\infty} e^{-tv^2}\,dv\\&=\left(-\frac{\partial}{\partial t}\right)^a\sqrt{\pi}\,t^{-\frac{1}{2}}\\&=b_a t^{-a-\frac{1}{2}}\end{aligned}$$

where $b_a=\sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{2a-1}{2}$. This proves Lemma 2. Next

$$\begin{aligned}E(s)&=b_a\int_0^\infty t^{(s-a-\frac{1}{2})-1}e^{-tc^2}\,dt\\&=b_a\Gamma(s-a-\tfrac{1}{2})\,c^{-2(s-a-\frac{1}{2})}\end{aligned}$$

for $\operatorname{Re} s>a+\frac{1}{2}$. The $\Gamma$ function has a meromorphic continuation, so $E$ does too and we get

$$E(0)=b_a\Gamma(-a-\tfrac{1}{2})\,c^{2a+1}.$$

Since

$$\Gamma(-a-\tfrac{1}{2})\cdot-(a+\tfrac{1}{2})\cdot-(a-\tfrac{1}{2})\cdot\ldots\cdot-\tfrac{1}{2}=\Gamma(\tfrac{1}{2})=\sqrt{\pi},$$

this simplifies to $E(0)=(-1)^{a+1}\frac{2\pi}{2a+1}c^{2a+1}=-2\pi\int_0^c(iy)^{2a}\,dy$ so Lemma 3 holds.

Now in our application we take $c=|n-k|$ and $P=P_{\sigma_k}$ as in Theorem 2. This gives for $c>0$

(3) $$MI(0)=-2\pi a_1 \int_0^c P(iy)\,dy.$$

Taking $c>0$, we use characteristic functions to write (2) as

(4) $$H(t)-I(t)+\alpha\chi_{(0,1]}=-\alpha\chi_{(1,\infty)}+G(t).$$

We note that each term on the left is exponentially small at $+\infty$ whereas each term on the right is exponentially small at $0^+$. Thus we can define an entire function $J(s)$ by taking the Mellin transform of both sides of (4). We have that $M\chi_{(0,1]}=\frac{1}{s}$, for $\operatorname{Re} s>0$, and by Lemma 3 $MI$ has a meromorphic continuation to $\mathbf{C}$. Thus

(5) $$MH=J+MI-\frac{\alpha}{s}$$

defines the meromorphic continuation of $MH$ from large $\operatorname{Re} s$ to all of $\mathbf{C}$. On the right, we use $M\chi_{(1,\infty]}=-\frac{1}{s}$, $\operatorname{Re} s<0$, to see that

(6) $$MG=J-\frac{\alpha}{s}$$

defines a meromorphic continuation of $MG$ from $\operatorname{Re} s<0$ to all of $\mathbf{C}$. Also we have

(7) $$MH=MG+MI.$$

For $c=0$, $MI$ is nowhere defined so we must group $I$ with $\alpha$. Writing (2) as

(8) $$H(t)+(\alpha-I(t))\,\chi_{(0,1]}=(I(t)-\alpha)\,\chi_{(1,\infty)}+G(t)$$

we define, like before, an entire function $K(s)$ as the Mellin transform of either side. Using that $M(t^{-a-\frac{1}{2}}\chi_{(0,1]})=(s-a-\frac{1}{2})^{-1}$ for $\operatorname{Re} s$ large, we see that $M(\alpha-I(t))\,\chi_{(0,1]}$ has a meromorphic continuation to $\mathbf{C}$, with (at worst) simple poles at $s=a+\frac{1}{2}$, $a=0,\ldots,n$, and at $s=0$. Subtracting this from $K$ gives a meromorphic continuation of $MH$ to $\mathbf{C}$ with the same singularities. Likewise we use $M(t^{-a-\frac{1}{2}}\chi_{(1,\infty)})=-(s-a-\frac{1}{2})^{-1}$ for $\operatorname{Re} s<0$ to continue $MG$ to a meromorphic function on $\mathbf{C}$, only here we find

(9) $$MH=MG.$$

This discussion, and our application of STF to the heat kernel to compute $\det D_k$ can be summarized in the following way.

**Proposition 1.**

$$-\log\det D_k=(\alpha\Gamma+MG)(0)-2\pi a_1\int_0^c P(iy)\,dy.$$

For $c>0$ this is a consequence of (1), (7) and (3). For $c=0$, one uses (1) and (9).

Now we study $MG$ for $\operatorname{Re} s<0$ using the substitution

$$t^{s-1}=\frac{1}{\Gamma(1-s)}\int_0^\infty (x(x+2c))^{-s}\, e^{-x(x+2c)t}(2x+2c)\,dx \tag{10}$$

suggested by [RS2] and [M]. We switch the order of integration in the Mellin transform and integrate term by term using

$$\int_0^\infty e^{-x(x+2c)t}(2x+2c)\frac{1}{\sqrt{4\pi t}}\, e^{-l^2/4t}\, e^{-tc^2}\,dt=e^{-l(x+c)}. \tag{11}$$

This gives, for $Y_k$ as in Sect. 3,

$$MG(s)=\frac{1}{\Gamma(1-s)}\int_0^\infty (x(x+2c))^{-s}\, Y_k(n+c+x)\,dx. \tag{12}$$

Suppose at first that $\rho$ is acyclic, this being the simplest and most important case. Then $\alpha=0$, $MG$ is regular at 0, $S_k$ is regular at $s=n+c$ and we can let $s\uparrow 0$ to get

$$MG(0)=\int_0^\infty Y_k(n+c+x)\,dx=-\log S_k(n+c). \tag{13}$$

Then Proposition 1 implies

$$\det D_k=S_k(n+c)\exp 2\pi a_1\int_0^c P_{\sigma_k}(iy)\,dy. \tag{14}$$

Now by $(FE)$ of Sect. 3, we find

$$(\det D_k)^2=S_k(2n-k)\,S_k(k). \tag{15}$$

Using $\det D_k\cdot\det D_{k-1}=\det \Delta_k$, we find

$$\det \Delta_k=[S_{k-1}(k-1)\,S_k(k)\,S_k(2n-k)\,S_{k-1}(2n-k+1)]^{\frac{1}{2}}. \tag{16}$$

Using $S_{2n-k}=S_k$ and the isomorphism $D_k\cong D_{2n-k}$ of Sect. 1, (15) can be written

$$\det D_k\cdot\det D_{2n-k}=S_{2n-k}(2n-k)\,S_k(k). \tag{17}$$

Substituting (17) into (**) for $k<n$ and (14) for $k=n\,(c=0)$ gives, by $(RS)$ of Sect. 3,

$$T_\rho^2=\frac{S_0(0)\,S_2(2)\dots S_{2n}(2n)}{S_1(1)\dots S_{2n-1}(2n-1)}=R_\rho(0). \tag{18}$$

Recalling from Sect. 1 that $T_\rho^2=\tau_\rho^2=\tau_{\rho_1}(SX)$, this proves Theorem 1 for $d$ odd. The only even $d$ for which $\pi_1 SX$ has an acyclic representation is $d=2$ and this case is treated in [F5]. So Theorem 1 holds (and only for $d=2$ does one need the absolute value sign).

In general, when $\rho$ is not assumed acyclic, $S_k$ is regular for $|\operatorname{Re} z-n|>c$ but may have a pole or zero at $k$. Let $\delta x^r$ $(\delta=\delta_k, r=r_k)$ be the leading term in the

Laurent expansion of $S_k(k+x)$ at $x=0$. By $(FE)$ we find

$$r_k = r_{2n-k},\ \delta_{2n-k} = (-1)^{r_k}\,\delta_k \exp \int_0^{n-k} 4\pi a_1 P(iy)\,dy \tag{19}$$

where $P = P_{\sigma_k} = P_{\sigma_{2n-k}}$.

Take $k \geqq n$ and write $Y(k+x) = r x^{-1} + b(x)$ where $b$ is bounded near $x=0$. Then we break the integration interval of (12) into $(0, \varepsilon)$ and $(\varepsilon, \infty)$ for some $\varepsilon > 0$. Then, for $s<0$,

$$\int_0^\infty (x(x+2c))^{-s}\, Y(k+x)\,dx = O(s) + \int_\varepsilon^\infty Y(k+x)\,dx + \int_0^\varepsilon b(x)\,dx + r\int_0^\varepsilon x^{-s-1}(x+2c)^{-s}\,dx. \tag{20}$$

Using integration by parts in case $k>n$, this last integral can be computed as

$$\int_0^\varepsilon x^{-s-1}(x+2c)^{-s}\,dx = \begin{cases} O(s) + \log 2c\varepsilon - \dfrac{1}{s}, & c>0 \\[2ex] O(s) + \log \varepsilon - \dfrac{1/2}{s}, & c=0. \end{cases} \tag{21}$$

Since the limit in Proposition 1 is known to exist, and the residue of $\Gamma(s)$ at $s=0$ is 1, we obtain by (12), (20), and (21)

$$r_k = \begin{cases} \alpha_k, & k>n \\ 2\alpha_k, & k=n. \end{cases} \tag{22}$$

We write (20) as

$$O(s) + O(\varepsilon) - \log S(k+\varepsilon) + r\log\varepsilon + r\,\Phi(s), \tag{23}$$

$$\text{where } \Phi(s) = \begin{cases} (\log 2c) - \dfrac{1}{s}, & c>0 \\[2ex] -\dfrac{1/2}{s}, & c=0. \end{cases}$$

Letting $\varepsilon \to 0$, we find this is

$$r\,\Phi(s) - \log\delta + O(s). \tag{24}$$

Using the functional equation of $\Gamma$, we find

$$(\alpha\Gamma + MG)(0) = \lim_{s\uparrow 0} \frac{1}{\Gamma(1-s)}\left(\frac{\alpha\pi}{\sin\pi s} + r\,\Phi(s) - \log\delta\right) = \begin{cases} -\log\delta + r(\log 2c), & c>0 \\ -\log\delta, & c=0. \end{cases} \tag{25}$$

From this Proposition 1 gives (for $n \geqq k$)

$$\det D_k = \delta_k \exp\left(2\pi a_1 \int_0^c P_{\sigma_k}(iy)\,dy\right)(2c)^{-\alpha_k} \tag{26}$$

generalizing (14), where we introduce the convenient convention $0^{-\alpha_k}=1$ in case $c=0$. Now (19) gives (for all $k$)

$$(\det D_k)^2 = (-4c^2)^{-\alpha_k}\,\delta_k\,\delta_{2n-k} = \det D_k \cdot \det D_{2n-k} \tag{27}$$

$$\det \Delta_k = [(-4(n-k)^2)^{-\alpha_k}(-4(n-k+1)^2)^{-\alpha_{k-1}}\,\delta_k\,\delta_{k-1}\,\delta_{2n-k}\,\delta_{2n-k+1}]^{\frac{1}{2}} \tag{28}$$

generalizing (15), (16) and (17).

Using (27) for $k<n$ in (**) and (26) for $k=n$, we find

$$T_\rho^2 = \frac{\delta_0\,\delta_2 \ldots \delta_{2n}}{\delta_1 \ldots \delta_{2n-1}}\,C_\rho^{-1} \tag{29}$$

where $C_\rho \in \mathbf{Q}$ is defined by

$$C_\rho = \prod_{k=0}^{n-1} (-4(n-k)^2)^{\alpha_k(-1)^k}. \tag{30}$$

By (19), (22) and (29), the relation (*RS*) of Sect. 2 gives our main result.

**Theorem 3.** *The leading term in the Laurent expansion of $R_\rho(s)$ at $s=0$ is*

$$C_\rho\, T_\rho^2\, s^e$$

*where*

$$e = 2\sum_{k=0}^{n} (-1)^k \alpha_k = (2n+2)\,\beta_0 - 2n\,\beta_1 + (2n-2)\,\beta_2 - \ldots + (-1)^n\, 2\,\beta_n$$

Treating the $\beta_i$ as known (they are combinatorial invariants of $X$), we see that the leading term in $R_\rho$ at 0 determines $T_\rho$ and conversely.

There are some extensions of Theorems 1 and 3 that are worth investigating. One is to allow nonoriented closed manifolds. This would give interesting results for even dimensions, since then $T_\rho$ need not equal 1. Another is to allow elements of finite order. If $\Gamma$ is a discrete cocompact subgroup of $G$ which acts freely on $SH$ and $\rho\colon \Gamma \to O(m)$ a representation then $R_\rho(s)$ is still defined for the geodesic flow on the manifold $\Gamma\backslash SH = M$. One hopes that its behavior at $s=0$ again determines the torsion of $X$, even if the action of $\Gamma$ on $H$ is not free. For $d=2$ and $\rho$ acyclic, this was shown in [F5]. Finite volume cases would also be interesting, and perhaps of number theoretic value. All these extensions can presumably be made by using suitably general versions of STF and the procedure followed above.

It is notable that our proof relies on Selberg's functional equation: we know of no earlier geometric application of it. In [M], for example, it is only proven after the eta invariant is expressed as a special value. Without it one cannot deal with the terms coming from the identity class in $\pi_1$: these terms were eliminated in [RS2, M] by taking the difference of two representations, i.e. the ratio of two zeta functions.

We believe Theorem 1 is the first *application* of the Cheeger/Müller theorem (as opposed to a mere calculation of the analytic torsion, such as Theorem 3). The $d=2$ case uses Reidemeister torsion $\tau$, the $d>2$ case uses analytic torsion $T$ and only through $T=\tau$ can these be seen as a single result.

Prompted by a question of S. Rosenberg concerning the conformal Laplacian, we show how the methods of this section allow one to compute $\det(D_k + x)$ in terms of $S_k$. Take $x>0$ and let

$$G_x(t)=e^{-tx}G(t)$$
$$H_x(t)=e^{-tx}H(t)$$
$$I_x(t)=e^{-tx}(I(t)-\alpha).$$

Then $G_x=H_x-I_x$, $G_x$ decays exponentially at $+\infty$, and $H_x$, $I_x$ decay exponentially at $0^+$. So the Mellin transform $MG_x$ is entire. Lemmas 2 and 3 give a meromorphic continuation of $MI_x$ and one obtains (c.f. Proposition 1)

$$-\log\det(D_k+x)=MG_x(0)-2\pi a_1\int_0^z P(iy)\,dy$$

where $z=((n-k)^2+x)^{\frac{1}{2}}>|n-k|$. Using (10) with $z$ for $c$ gives $MG_x(0)=-\log S_k(n+z)$ and (FE) gives (c.f. (15))

$$\det{}^2(D_k-(n-k)^2+z^2)=S_k(n+z)\,S_k(n-z).$$

Following Quillen [Q] we let $\det_\zeta D$ be $\exp -\zeta'_D(0)$ if $0\notin\operatorname{Spec} D$ and 0 if $0\in\operatorname{Spec} D$, $D=D_k-\lambda$. Then by analytic continuation we find for all $z\in\mathbf{C}$

**Theorem 4.** $\det_\zeta^2(D_k-(n-k)^2+z^2)=S_k(n+z)\,S_k(n-z)$.

By (FE) this can be written

$$\det_\zeta(D_k-\lambda)=S_k(w)\exp\int_0^{w-n} 2\pi a_1 P_{\sigma_k}(iz)\,dz \qquad (\det_\zeta)$$

where $\lambda=(n-k)^2-(w-n)^2$. Thus $S_k$ is the essential part (i.e. the part corresponding to heat flow around noncontractible loops) of the regularized characteric polynomial of $D_k$. The Selberg function $S_\sigma$ associated to a compact locally symmetric space of negative curvature $X=\Gamma\backslash G/K$ and $\sigma\in\hat{M}$ ([F4], [F6]) can be given a similar interpretation. See [D], Theorem 4.4, for a related result when $d=2$, $m=1$.

## Appendix. Functional determinants for Riemann surfaces

Certain calculations in superstring theory involve the functional determinant of the Laplacian on a closed hyperbolic surface [Al], i.e. $\det\Delta_k$ for $k=0$, $d=2$ and $\rho$ trivial. The methods of Sect. 4 allow one to calculate such determinants in terms of a Selberg zeta function. While we will here only deal with the case $d=2$, similar calculations could be made for any even $d$ and indeed for any closed locally symmetric space of negative curvature.

First note that $\det\Delta_1=(\det\Delta_0)^2$ and $\det\Delta_2=\det\Delta_0$. Thus we may suppose $k=0$, $\Delta=\Delta_0$. We write $\alpha=\alpha_0=\dim\ker\Delta$, $D=D_0$, $H(t)=\operatorname{Tr}(e^{-tD})$, $I(t)=I_t(\sigma_0)$, $G(t)=G_t(\sigma_0)$, $c=n=\frac{1}{2}$ and $\zeta(s)=\zeta_0(s)$.

Then (4) through (7) hold as before. One finds

$$\begin{aligned}\zeta'(0) &= \lim_{s\to 0} \Gamma(s)(\zeta(s)-\zeta(0)) \\ &= \lim_{s\to 0} MH - \Gamma(s)\zeta(0) \\ &= \lim_{s\to 0} (MG+\alpha\Gamma(s)) + \lim_{s\to 0} (MI-\alpha\Gamma(s))\end{aligned}$$

where $a$ is the residue of $MI$ at $s=0$. Note that unlike the case $d$ odd one cannot assume $\zeta(0)=0$.

The first limit can be evaluated just as in (25), giving $-\log\delta$ where $\delta$ is the leading coefficient of the Selberg zeta function $S_\rho$ at $x=1$. Since $I$ is proportional to $g-1$ and $m$, the second limit has the form $(g-1)mK$ where $K$ is its value for $g=2$ and the trivial representation. Setting $v=e^{-K}$ one obtains the determinental formula

$$(D) \qquad \det \Delta = \delta v^{m(g-1)}$$

where $v$ is some positive constant that we will explicitly evaluate below.

In the case of physical interest, when $m=1$ and $\rho$ is trivial, $S=S_\rho$ has a simple zero at $s=1$ and so $\delta=S'(1)$. Using the formula $R(z)=S(z)/S(z+1)$ we see that $\delta=R'(1)S(2)=R'(1)R(2)R(3)\ldots$. Since $R(z)=\prod_\gamma 1-e^{-zl(\gamma)}$ where $\gamma$ runs over the prime closed geodesics, we see that $\delta$ depends only on the (prime) length spectrum $\{l(\gamma)\}$ of $M$. The factor $v^{g-1}$ is independent of the moduli of the Riemann surface. The quantity $R'(1)$ is rather mysterious but is formally analogous to the reciprocal of the residue at $s=1$ of the Dedekind zeta function of an algebraic number field. This residue is evaluated in Dedekind's class number formula and involves a certain regulator, which is defined as a determinant. It would be interesting to better understand this analogy.

One can define the functional determinant and Selberg zeta function for a unitary representation $\rho\colon \pi_1 M \to U(m)$ by the samc formulas that we used for orthogonal representations, using complex traces and determinants instead of real ones. Taking $m=1$ and $\rho$ nontrivial, $S_\rho$ is regular at $x=1$ so $\delta=S_\rho(1)$ [Se]. Then $(D)$ improves the result of [RS2], which gave only the ratio of the determinants for a pair of such nontrivial unitary characters.

To compute $K$ we use the formula [R]

$$\zeta(s)=\frac{MH}{\Gamma}=\frac{A}{8(s-1)}\int_{-\infty}^{\infty}\left(\frac{1}{4}+r^2\right)^{1-s}\operatorname{sech}^2 \pi r\, dr + \text{geodesic terms}$$

for the Minakshisundaram-Pleijel zeta function of a closed hyperbolic surface of area $A$ (here one has $\rho$ trivial). Taking $g=2$ one has $A=4\pi$. From this one obtains

$$K=\pi\int_0^{\infty}\left(\frac{1}{4}+r^2\right)\operatorname{sech}^2 \pi r\left[-1+\log\left(\frac{1}{4}+r^2\right)\right]dr$$

Although the integral is not known precisely, one has approximately $K=-.6762$, $v=1.966$.

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