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Bombieri's theorem in short intervals. II

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1. In our previous paper [10] we proved the following extension of the mean-value theorem of Bombieri and A.I. Vinogradov (see e.g. [1], p. 57, and [13]):

$$\sum_{q\leqq Q}\max_{(a,q)=1}\max_{h\leqq y}\text{and}\int_{\frac{1}{2}x<z\leqq x}\left|\psi(z+h,q;a)-\psi(z,q;a)-\frac{h}{\phi(q)}\right|\ll y\log^{-A}x \qquad (1.1)$$

where A is an arbitrary positive constant,

$$y=x^{\theta},\quad Q=x^{\psi}\log^{-B}x,\quad \psi=\theta-1/2, \qquad (1.2)$$

$B=B(A)$ is a positive constant depending on A, θ is arbitrary with $3/5<\theta\leqq 1$ and, as usual,

$$\psi(x,q;a)=\sum_{\substack{n\leqq x\\ n\equiv a(\mathrm{mod}\,q)}}\Lambda(n),$$

Λ is von Mangoldt's function.

We recall (see e.g. [4]) that the generalized density hypothesis, and even the generalized Riemann hypothesis, implies (1.1)-(1.2) for all $\theta>1/2$.

A result of type (1.1)-(1.2) was proved for $\theta>3/4$ by Huxley and Iwaniec [4] using density theorems for the Dirichlet L-functions and later for $\theta>2/3$ by Ricci [11], by means of Vaughan's method [12].

In [10] we used Heath-Brown's identity [2] and the approximate functional equation for the L-functions [7]. The method given there did not furnish any positive value of ψ for $\theta\leqq 3/5$, hence for $7/12<\theta\leqq 3/5$ only the result

$$\psi=3(\theta-7/12)-\varepsilon,\qquad \varepsilon>0 \text{ arbitrary}, \qquad (1.3)$$

proved in [4] using density theorems, seems to be available.

Our present aim is to show that a suitable modification of the method in [10] can yield (1.3), even with a slightly better value of ψ for every θ between 7/12 and 3/5. Hence it appears that in this particular problem Heath-Brown's identity is more powerful than either Vaughan's identity or density theorems.

It is interesting to note that the improvement over (1.3) tends to 0 when

$\theta \to 3/5$ but it tends to a definite positive number when $\theta \to 7/12$. Since the proof is relatively complicated we shall restrict ourselves to the case in which θ is near 7/12, thereby proving the following - somewhat surprising - extension of Huxley's prime number theorem [3]:

Theorem. *The estimate* (1.1) *holds if* $y = x^{\theta}$, $7/12 < \theta \leqq 1$ *and* $Q = x^{1/40}$.

For a more detailed history of the problem and for the earlier results of Jutila [5] and Motohashi [9] we refer the reader to [10] (or [4, 11]).

Our present method is a combination of the techniques in our previous paper [10] and of Heath-Brown's method in [2], where he proved Huxley's prime number theorem using his identity, without discussing hypothetical zeros of $\zeta(s)$ off the critical line.

2. Since in some places there is a similarity with the proof in [10], we shall be brief at these places, referring to [10] for more details.

Let us fix a sufficiently small positive $\Delta = \Delta(\theta) < (\theta - 7/12)/8$. In the sequel c will denote a suitable positive constant, whose value is not necessarily the same at each occurrence and it might depend on A, θ and Δ. Throughout the paper we will use the notation $L = \log x$.

Using the standard arguments of [10] our task reduces to proving that

$$\sum_{Q'/2 < q \leqq Q'} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\, q)} \max_{h \leqq y} \max_{\frac{1}{2}x < z \leqq x} \Big| \sum_{z < n \leqq z+h} \Lambda(n)\chi(n) \Big| \ll yQ'L^{-A-c}$$

where Σ^* denotes summation over all primitive characters $\chi (\mathrm{mod}\, q)$.

Using the results of Huxley and Iwaniec [4] we may restrict ourselves to the case

$$Q' = x^{\alpha + 3\Delta}, \qquad 0 \leqq \alpha \leqq 1/40. \tag{2.1}$$

Applying Heath-Brown's identity

$$\frac{\zeta'}{\zeta}(s) = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} \zeta^{j-1}(s)\, \zeta'(s)\, M^j(s) + \frac{\zeta'}{\zeta}(s)\,(1 - \zeta(s)\, M(s))^k$$

with $k = 4$ and $X = (2x)^{1/4}$, where $M(s) = M_X(s) = \sum_{n \leqq X} \mu(n) n^{-s}$, we see that, similarly to [10], it is sufficient to show that

$$\sum_{Q'/2 < q \leqq Q'} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\, q)} \max_{h \leqq y} \max_{\frac{1}{2}x < z \leqq x} |S^*| \ll yQ'L^{-A-c}$$

for any S^* of the form

$$S^* = \sum_{N_i \leqq n_i < 2N_i} \chi(n_1) \log n_1\, \chi(n_2) \dots \chi(n_4)\, \chi(n_5)\, \mu(n_5) \dots \chi(n_8)\, \mu(n_8), \tag{2.2}$$

in which

$$x \ll \prod_{i=1}^{8} N_i \ll x \quad \text{and} \quad N_i \leqq x^{1/4} \quad \text{if } i = 5, \dots, 8$$

(the case $j < 4$ corresponds to $N_v = 1$ for some v).

Using the Pólya-Vinogradov inequality and the Perron formula, the arguments of [10] show that in order to prove (2.2) it is sufficient to show

$$\sum_{Q'/2<q\leqq Q'} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \int_{1/2+iT_1/2}^{1/2+iT_1} |f(s,\chi)|\,|ds| \ll x^{1/2-\Delta} Q' Y \tag{2.3}$$

where

$$10\leqq T_1\leqq Q' x^{2\Delta} T_0, \quad T_0=\frac{x}{y}, \quad Y=\max\left(\frac{T_1}{T_0}, 1\right), \tag{2.4}$$

$$f(s,\chi)=\prod_{i=1}^{8}\Big(\sum_{N_i\leqq n_i<2N_i}\chi(n_i)a_i(n_i)n_i^{-s}\Big)=\prod_{i=1}^{8} f_i(s,\chi) \tag{2.5}$$

with $a_1(n)=\log n$, $a_i(n)=1\,(2\leqq i\leqq 4)$, $a_i(n)=\mu(n)\,(5\leqq i\leqq 8)$ and

$$N_i<yL^{-3\Delta}(1\leqq i\leqq 4), \qquad N_i\leqq x^{1/4}(5\leqq i\leqq 8).$$

Now, using the approximate functional equation for the L-functions, in the form given by Lavrik [7], in the same way as in [10], we see that we may assume, for $1\leqq i\leqq 4$,

$$N_i<(Q'T_1)^{1/2}<Q'x^{\Delta}T_0^{1/2}<x^{1/40+5/24}<x^{1/4},$$

even dropping the condition $x\ll\prod_{i=1}^{8}N_i$ and allowing more generally any sequence $a_i(n)$ such that

$$|a_i(n)|\leqq L \tag{2.6}$$

in the definition of $f_i(s,\chi)$ in (2.5).

Therefore we shall assume only (2.6) for the $a_i(n)$, and we will suppose

$$N_i=x^{\alpha_i},\ 0\leqq\alpha_i\leqq 1/4 \quad \text{for } i=1,\dots,8. \tag{2.7}$$

Since none of the indices $i=1,\dots,8$ plays now any special role, we may further assume

$$\alpha_i\geqq\Delta/8 \text{ for } i\leqq t, \qquad \alpha_i<\Delta/8 \text{ for } t<i\leqq 8,$$

with some integer t, $0\leqq t\leqq 8$.

The main tool in our previous paper in proving (1.1)–(1.2) was the Lemma of [10], which showed that (2.3) holds if we can group the indices $i=1,\dots,8$ into two blocks, say I and J, such that

$$\max\Big(\prod_{i\in I}N_i, \prod_{i\in J}N_i\Big)\ll yL^{-3\Delta} \tag{2.8}$$

provided only $\theta>1/2$ and $x^{\Delta}\leqq Q'\leqq x^{\theta-1/2-\Delta}$ with any constant $\Delta>0$.

Such a grouping is clearly possible by (2.7) if $t\leqq 2$, therefore we may also assume that

$$t\geqq 3. \tag{2.9}$$

The inequality (2.3) will trivially follow if we can show

$$\sum_{Q'/2<q\leqq Q'} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \int_{1/2+iT_1/2}^{1/2+iT_1} \prod_{i=1}^{t} |f_i(s,\chi)|\,|ds| \ll x^{1/2-2\Delta} Q' Y, \tag{2.10}$$

under the restrictions (2.4), (2.9) and

$$|a_i(n)| \leqq 1/100,\quad N_i = x^{\alpha_i},\quad \Delta/8 \leqq \alpha_i \leqq 1/4 \qquad i=1,\dots,t. \tag{2.11}$$

Let $\sigma_i = b(v_i)$ $(i=1,\dots,t)$ be one of the numbers

$$b(0) = -100/\Delta + 1/2,\quad b(v) = 1/2 + v/L,\quad v=1,\dots,[L]-1. \tag{2.12}$$

For all possible choices of $\sigma_1,\dots,\sigma_t$ we define $\Omega(\sigma_1,\dots,\sigma_t)=\Omega$ to be the set of the pairs (r,χ) for which $r\in[\frac{1}{2}T_1, T_1]$, χ is primitive (mod q), $Q'/2<q\leqq Q'$, and

$$N_i^{b(v_i+1)-1/2} > |f_i(1/2+ir,\chi)| \geqq N_i^{b(v_i)-1/2} \qquad i=1,\dots,t. \tag{2.13}$$

Let Ω_0 be the remaining set, i.e.

$$\Omega_0 = \{(r,\chi):\ \exists i,\ 1\leqq i\leqq t,\ |f_i(1/2+ir,\chi)| < N_i^{-100/\Delta}\}. \tag{2.14}$$

Owing to (2.11) the total integral over Ω_0 in (2.10) is clearly $\ll x^{-10}$.

Using large sieve mean-value estimates (see [8], Theorem 7.3) and the Halász-Montgomery-Huxley inequality (see Jutila [6]) we can bound the weighted measure $|\Omega|$ (where every point r is calculated with multiplicity m if there are exactly m characters χ with $(r,\chi)\in\Omega$) as follows.

Let us rise the Dirichlet polynomial $f_i(s,\chi)$ to a suitable integral power g_i with

$$1 \leqq g_i \leqq 100/\Delta \tag{2.15}$$

and take into account the inequality

$$\sum_{n=N}^{2N} d_g^2(n) \ll N(\log N)^{c(g)},\qquad d_g(n) = \sum_{n=n_1\dots n_g} 1.$$

Then we have

$$|\Omega| \leqq \min_{1\leqq i\leqq t} R_i L^c \tag{2.16}$$

where

$$R_i \ll (Q')^2 T_1 N_i^{g_i(1-2\sigma_i)} + N_i^{g_i(2-2\sigma_i)} \qquad (\sigma_i \leqq 3/4), \tag{2.17}$$

$$R_i \ll (Q')^2 T_1 N_i^{g_i(4-6\sigma_i)} + N_i^{g_i(2-2\sigma_i)} \qquad (3/4 \leqq \sigma_i < 1). \tag{2.18}$$

Since there is no more distinction among the indices $i=1,\dots,t$ we may obviously assume that

$$\sigma_1 \geqq \sigma_2 \geqq \dots \geqq \sigma_t \quad (\text{i.e. } v_1 \geqq v_2 \geqq \dots \geqq v_t) \tag{2.19}$$

and let t' be defined as

$$t'=0 \quad \text{if } v_1=0 \quad \text{and} \quad t' = \max\{j:\ 1\leqq j\leqq t,\ \sigma_j > 1/2\ (\text{i.e. } v_j>0)\} \text{ otherwise.}$$

Owing to the definitions in (2.12) we have clearly that

$$|f_i(1/2+ir,\chi)|<e \quad \text{if } t'<i\leqq t \quad \text{and} \quad (r,\chi)\in\Omega.$$

Hence in case of $t'\leqq 2$ we obtain, for the integral over Ω, an estimate of the form (2.3), which follows from the Lemma of [10]. Thus we may suppose

$$t'\geqq 3. \tag{2.20}$$

Let us introduce the notation

$$\sigma_1-\sigma_2=\delta_1, \quad \sigma_2-\sigma_3=\delta_2. \tag{2.21}$$

In view of (2.10)–(2.16) the inequality (2.3) for the integral over Ω will follow if we can show that

$$R_i\prod_{j=1}^{t'} N_j^{\sigma_j-1/2}\ll x^{1/2-3\Delta}Q'Y \tag{2.22}$$

for some i with $1\leqq i\leqq t'$. Let us suppose that

$$\prod_{j=1}^{t'} N_j^{\sigma_j-1/2}\leqq x^{\sigma_i-1/2+a_i}. \tag{2.23}$$

For $i=1$ we can clearly take $q_1=0$ by (2.19). Taking into account (2.1), (2.4), (2.17), (2.18) and (2.23), the inequality (2.22) follows if we can prove the existence of a natural number $g_i=g_i(\sigma_i)$ such that the inequalities

$$\alpha_i g_i(1-2\sigma_i)\leqq 1-\sigma_i-a_i-3\Delta-(1-\theta)-\alpha-3\Delta \quad (1/2<\sigma_i\leqq 3/4), \tag{2.24}$$

$$\alpha_i g_i(4-6\sigma_i)\leqq 1-\sigma_i-a_i-3\Delta-(1-\theta)-\alpha-3\Delta \quad (3/4\leqq\sigma_i<1), \tag{2.25}$$

$$\alpha_i g_i(2-2\sigma_i)\leqq 1-\sigma_i-a_i+\alpha \quad (1/2<\sigma_i<1) \tag{2.26}$$

simultaneously hold if σ_i satisfies the conditions indicated in the brackets.

3. In the course of the proof we need to investigate (5 times) systems of inequalities of type (2.24)–(2.26) where

$$-1/12\leqq a\leqq\alpha/4\leqq 1/160 \quad (0\leqq\alpha\leqq 1/40). \tag{3.1}$$

Using the relations $\alpha_i\leqq 1/4$, $\Delta\leqq(\theta-7/12)/8$ it is easy to see that (2.24)–(2.26) are satisfied if for every $\sigma\in(1/2,1)$ we can find a natural number $g=g(\sigma)\leqq 100/\Delta$ such that the inequalities

$$\beta g\geqq 1/2+\frac{\alpha+a-1/12}{2\sigma-1}=A_1(\sigma) \quad (1/2<\sigma\leqq 3/4), \tag{3.2}$$

$$\beta g\geqq 1/6+\frac{\alpha+a+1/12}{6\sigma-4}=A_2(\sigma) \quad (3/4\leqq\sigma\leqq 1), \tag{3.3}$$

$$\beta g\leqq 1/2+\frac{\alpha-a}{2-2\sigma}=A_3(\sigma) \quad (1/2<\sigma<1) \tag{3.4}$$

simultaneously hold if σ satisfies the conditions indicated in the brackets and $\beta \leqq 1/4$.

We will use the following

Lemma. *Under the conditions* (3.1) *and* $\beta \leqq 1/4$ *the inequalities* (3.2)–(3.4) *simultaneously hold with some* $g \leqq 100/\Delta$ *unless*

$$1/6+\frac{\alpha-a}{2}<\beta<1/6+\alpha+a.$$

Proof. If for $\sigma \leqq 3/4$

$$\alpha \leqq A_3(\sigma)-A_1(\sigma) \tag{3.5}$$

holds, then $g=\max\left(1,\left[\frac{A_1(\sigma)}{\alpha}\right]+1\right)$ obviously satisfies (3.2), (3.4) and (2.15). So we may suppose (3.5) to be false. Let us consider first the case $1/2<\sigma \leqq 2/3$.

Taking $g=2$, from $\beta \leqq 1/4$ and $a \leqq \alpha$ we clearly obtain (3.4). On the other hand an easy calculation shows that from (3.1) it follows that

$$2A_3'(\sigma)-3A_1'(\sigma)=\frac{\alpha-a}{(1-\sigma)^2}-\frac{3/2(1/12-\alpha-a)}{(\sigma-1/2)^2} \leqq 0 \tag{3.6}$$

for $1/2<\sigma \leqq 3/4$. So from (3.1) we obtain for $\sigma \leqq 2/3$

$$\begin{aligned} 2A_3(\sigma)-3A_1(\sigma) &\geqq 2A_3(2/3)-3A_1(2/3)=-1/2+3(\alpha-a)-9(\alpha+a-1/12) \\ &=1/4-6\alpha-12a \geqq 0. \end{aligned} \tag{3.7}$$

In view of (3.7) the falsity of (3.5) implies (3.2) with $g=2$, since

$$2\beta \geqq 2A_3(\sigma)-2A_1(\sigma) \geqq A_1(\sigma).$$

Let us suppose now

$$2/3<\sigma \leqq 3/4. \tag{3.8}$$

Similarly to (3.6) we have in this case

$$3A_3'(\sigma)-4A_1'(\sigma)=\frac{3}{2}\frac{(\alpha-a)}{(1-\sigma)^2}-\frac{2(1/12-\alpha-a)}{(\sigma-1/2)^2} \leqq 0, \tag{3.9}$$

hence from (3.1) we get

$$\begin{aligned} 3A_3(\sigma)-4A_1(\sigma) &\geqq 3A_3(3/4)-4A_1(3/4) \\ &=-1/2+6(\alpha-a)-8(\alpha+a-1/12)=1/6-2\alpha-14a \geqq 0. \end{aligned} \tag{3.10}$$

Thus the falsity of (3.5) implies (3.2) with $g=3$. If we have additionally

$$\beta \leqq 1/6+\frac{\alpha-a}{2} \tag{3.11}$$

then from $\sigma>2/3$ we get

$$3\beta \leqq 1/2+\frac{3(\alpha-a)}{2}<1/2+\frac{\alpha-a}{2(1-\sigma)}, \tag{3.12}$$

hence (3.4) is also true with $g=3$. On the other hand if

$$\beta \geqq 1/6+\alpha+a \tag{3.13}$$

then taking $g=2$ in (3.2) we have from $\sigma \leqq 3/4$ and $\alpha+a<1/12$

$$2\beta \geqq 1/2+2(\alpha+a-1/12) \geqq 1/2+\frac{\alpha+a-1/12}{2\sigma-1}, \tag{3.14}$$

whilst for $g=2$ (3.4) is satisfied as well with $\beta \leqq 1/4$ and $a \leqq \alpha$. This settles the case (3.8).

Since for $3/4 \leqq \sigma<1$ we have $A_2(\sigma) \leqq A_2(3/4)$ and $A_3(\sigma) \geqq A_3(3/4)$, it is clearly sufficient to consider the case $\sigma=3/4$. But in view of $A_1(3/4)=A_2(3/4)$ this case is settled by (3.8)–(3.14), and the Lemma is proved.

Corollary. *If* $\beta \leqq 1/4$, (3.1) *holds and* $a \leqq -\alpha/3$, *then* (3.2)–(3.4) *hold for a suitable choice of* $g \leqq 100/\Delta$.

4. Taking $i=1$ we see from the Lemma that (2.22) is true unless

$$\alpha_1 \in (1/6+\alpha/2, 1/6+\alpha), \tag{4.1}$$

since in this case we may take $a_1=0$, as observed after (2.23). Thus we may assume the truth of (4.1).

Owing to (2.21) and (4.1) we have in this case

$$\prod_{j=1}^{t'} N_j^{\sigma_j-1/2} \leqq x^{\sigma_1-1/2-\delta_1} N_1^{\delta_1} < x^{\sigma_1-1/2-\delta_1(5/6-\alpha)}.$$

In view of the Corollary, (2.22) is true unless

$$\delta_1(5/6)-\alpha)<\alpha/3. \tag{4.2}$$

So we may suppose that (4.2) holds further on.

We take now $i=2$. By (2.21) we have

$$\prod_{j=1}^{t'} N_j^{\sigma_j-1/2} \leqq x^{\sigma_2-1/2} N_1^{\delta_1} \leqq x^{\sigma_2-1/2+a_2}$$

where, by (4.1), (4.2) and $0 \leqq \alpha \leqq 1/40$,

$$a_2<(1/6+\alpha)\delta_1<(1/6+\alpha)(5/6-\alpha)^{-1}\alpha/3<\alpha/12.$$

Thus in view of the Lemma we obtain (2.22) unless

$$\alpha_2 \in (1/6+\alpha 11/24, 1/6+\alpha 13/12). \tag{4.3}$$

Hence we will assume (4.3) further on.

Let us consider again $i=1$. From (2.21), (4.1) and (4.3) we get

$$\prod_{j=1}^{t'} N_j^{\sigma_j-1/2} \leqq x^{\sigma_1-1/2-\delta_1-\delta_2} N_2^{\delta_2} N_1^{\delta_1+\delta_2} < x^{\sigma_1-1/2-\delta_1(5/6-\alpha)-\delta_2(2/3-\alpha 25/12)}.$$

The Corollary implies then the truth of (2.22) unless

$$\delta_1(5/6-\alpha)+\delta_2(2/3-\alpha 25/12)<\alpha/3, \tag{4.4}$$

and we may assume (4.4) too.

We take now $i=3$. Taking into account (2.21) we obtain

$$\prod_{j=1}^{t'} N_j^{\sigma_j-1/2} \leqq x^{\sigma_3-1/2} N_2^{\delta_2} N_1^{\delta_1+\delta_2} \leqq x^{\delta_3-1/2+a_3}$$

where from (4.1), (4.3), (4.4) and $0 \leqq \alpha \leqq 1/40$ we get

$$a_3<(1/6+\alpha)\delta_1+(1/3+\alpha 25/12)\delta_2 \leqq \alpha/3\frac{1/3+\alpha 25/12}{2/3-\alpha 25/12}<\alpha/4.$$

The Lemma implies then that (2.22) holds unless

$$a_3 \in (1/6+\alpha 3/8, 1/6+\alpha 5/4), \tag{4.5}$$

and again we may assume (4.5) too.

But if (4.1), (4.3) and (4.5) all hold, then we have from $\prod_{i=1}^{8} N_i \ll x$

$$\prod_{j=1}^{3} N_j<x^{1/2+\alpha 40/12} \leqq x^{7/12}, \qquad \prod_{j=4}^{8} N_j \ll x^{1/2},$$

hence (2.8) is satisfied. Therefore we obtain, from the Lemma of [10], the crucial estimate (2.3) for the integral over Ω in the case (2.20), and the Theorem follows.

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