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Mod 2 exterior H-spaces

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§ 1. Introduction

In this note we study H-spaces which have no two torsion in their integral cohomology. Throughout the paper, the symbol X will be reserved for a one connected H-space with integral homology of finite type and whose mod two cohomology is an exterior algebra on generators of odd degree. We add the additional assumption that the module of generators of degree not a power of two minus one be finite dimensional.

Note that these hypotheses are satisfied for all finite simply connected H-spaces whose mod two cohomology is two torsion free.

We prove the following theorems. Unless denoted otherwise, *all coefficients will be* $\mathbb{Z}_2$.

Theorem 1.1. (a) $QH^{4k+1}(X)=Sq^2QH^{4k-1}(X)+Sq^{2k}Sq^2QH^{2k-1}(X)$ *for* $k>0$.

(b) For $r>1$ *and* $k>0$

$$QH^{2^r+2^{r+1}k-1}(X)=Sq^{2^r}QH^{2^{r+1}k-1}(X).$$

(c) *For all* $r>0$, $k>0$

$$Sq^{2^r}QH^{2^r+2^{r+1}k-1}(X)=0.$$

Corollary 1.2. $H^*(X)$ *is primitively generated if and only if for* $r>1$, $QH^{2^r-1}(X)$ *is isomorphic to* $PH^{2^r-1}(X)$ *via the projection map.*

Theorem 1.3. *The first nonvanishing homotopy group of* X *occurs in degree three, or seven.*

These results are related to work done on finite H-spaces in the sixties and seventies. In the sixties, Thomas studied finite H-spaces whose mod two cohomology is primitively generated. For instance, he proved the following result which is analogous to Theorem 1.1 and Theorem 1.3.

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Theorem 1.4 (Thomas [12, 13]). *Let Z be a finite H-space with $H^*(Z)$ primitively generated. Then for $k>0$*

$$PH^{2^{r+2^{r+1}k-1}}(Z)=Sq^{2^r}PH^{2^{r+1}k-1}(Z)$$

and

$$Sq^{2^r}PH^{2^{r+2^{r+1}k-1}}(Z)=0.$$

Further if Z has no cohomology two torsion, the first non-vanishing homotopy group occurs in degree one, three, or seven.

Thomas' work showed that not all finite H-spaces had primitively generated mod two cohomology. In fact, the Lie group E_8 has a nine dimensional primitive not in the image of Sq^2; hence, Theorem 1.4 implies E_8 cannot admit a primitively generated mod 2 cohomology ring.

In the special case that the mod 2 cohomology is an exterior algebra, however, there is no known example of a finite H-space whose mod 2 cohomology is not primitively generated. A Hopf algebra theorem due to Samelson and Leray [10] shows that if an H-space has mod 2 cohomology exterior, then it is primitively generated if and only if its mod 2 homology is associative. Corollary 1.2 provides a further reduction to the generators of the cohomology in degrees a power of two minus one. At present, it is still a question of interest if a finite H-space always admits an H-structure such that its homology with any field coefficients is an associative ring.

I would like to thank the mathematics departments of M.I.T. and the University of Neuchatel for their hospitality while this work was done.

2. In this chapter, we collect some results about the c-invariant, the suspension map, and several Hopf algebra theorems. The chapter concludes with a proof of $QH^{4k+1}(X)=Sq^2QH^{4k-1}(X)+Sq^{2k}Sq^2QH^{2k-1}(X)$. We begin with an idea due to Kane [4].

Lemma 2.1. *Let $t,u\in PH_*(X)$. Then $t^2=0$, $u^2=0$ and $[t,u]=0$.*

Proof. Because $H^*(X)$ is exterior on odd degree generators, t,u must have odd degree. Then t^2,u^2, and $[t,u]$ are even degree primitive. Hence, they must be trivial. Q.E.D.

Let $Q_2H^*(X)=IH^*(X)/IH^*(X)^3$ where $IH^*(X)$ is the augmentation ideal of $H^*(X)$. The reduced coproduct map induces a map of $\mathscr{A}(2)$ modules [see 4]:

$$d\colon\ Q_2H^*(X)\to QH^*(X)\otimes QH^*(X).$$

Definition 2.2. Let $\pi\colon H^*(X)\to Q_2H^*(X)$ be the projection and denote $\pi(x)$ by $\{x\}$.

Lemma 2.3. (a) *Given $\bar{x}\in QH^*(X)$, $\bar{x}$ has a representative x with $d\{x\}=0$.*

(b) *If z is odd degree decomposable with $d\{z\}=0$, then z is 3 fold decomposable.*

(c) $Sq^1H^*(X)=0$.

Proof. Since $QH^*(X)$ is concentrated in odd degrees, (a) follows. If z is odd degree decomposable and is not 3 fold decomposable $d\{z\}$ will have nontrivial

terms in the image of $1+T^*$. Finally, $H^*(X)$ is exterior on odd degree generators. Since Sq^1 is a derivation, the lowest degree element of image Sq^1 is Sq^1 of a generator. Hence, it will be even degree primitive. The exact sequence

$$0 \to P(\xi H^*(X)) \to P(H^*(X)) \to QH^*(X) \to 0 \tag{2.1}$$

[10], shows there are no even degree primitives. Q.E.D.

Theorem 2.4. *$H^*(\Omega X)$ is even dimensional. Further, the suspension map*

$$\sigma^*\colon\ QH^*(X) \to PH^*(\Omega X)$$

is an isomorphism of $\mathscr{A}(2)$ modules.

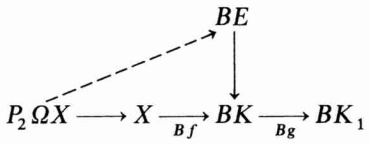
Proof. Since $H^*(X)$ is an exterior algebra on odd degree generators, $\mathrm{Tor}_{H^*(X)}(\mathbb{Z}_2, \mathbb{Z}_2)$ is a divided polynomial algebra on even degree primitives in the minus one line. The Eilenberg Moore spectral sequence collapses and E_∞ is isomorphic as coalgebras to $H^*(\Omega X)$.

The suspension map has been identified by

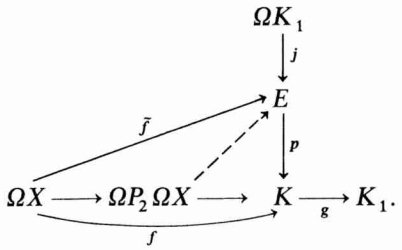
$$\sigma^*\colon\ QH^*(X) \to \mathrm{Tor}^{-1}_{H^*(X)}(\mathbb{Z}_2, \mathbb{Z}_2) \xrightarrow{\cong} E_\infty \cong g\,r\,H^*(\Omega X).$$

See [3]. It follows that σ^* is an isomorphism. Q.E.D.

We will often encounter the following diagram of H-spaces and maps:



Here BK, BK_1 are generalized Eilenberg MacLane spaces, BE is the fibre of Bg, $P_2\Omega X$ is the projective plane of ΩX, which includes into X. Note that all 3-fold cup products vanish in $H^*(P_2\Omega X)$. Therefore, if $Bg\,Bf$ has image in the module of 3-fold decomposables, there is a lifting of $P_2\Omega X$ to BE. Now ΩX is an H-retract of $\Omega P_2\Omega X$ [11], therefore, looping in the above diagram, we get



Now g and f are loop maps, $\tilde{f}$ is an H-map. Thinking of f as a cohomology class we have

$$c(f)=0, \quad \text{because } H^*(\Omega X) \text{ is even dimensional.} \tag{2.1}$$

This follows because whatever dimension f lies in, $c(f)$ shifts dimension by one.

The following theorem is proved in [9]:

Theorem 2.5. *$c(\tilde{f})$: $\Omega X \wedge \Omega X \to \Omega E$ factors as $c(\tilde{f})$: $\Omega X \wedge \Omega X \xrightarrow{\gamma} \Omega^2 K_1 \xrightarrow{\Omega j} \Omega E$, and the cohomology class γ lies in*

$$\operatorname{im}\sigma^* \otimes F_2 H^*(\Omega X) + F_2 H^*(\Omega X) \otimes \operatorname{im}\sigma^* + PH^*(\Omega X) \otimes PH^*(\Omega X)$$

where $F_2 H^*(\Omega X)$ is the kernel of $\bar{\Delta}^2$.

Lemma 2.6. (a) $\gamma \in \operatorname{im}\sigma^* \otimes F_2 H^*(\Omega X) + F_2 H^*(\Omega X) \otimes \operatorname{im}\sigma^*$.

(b) *$F_2 H^*(\Omega X)$ is spanned by primitives, 2-fold products of primitives, and elements of the form $\gamma_2(u)$ where u is primitive and $\gamma_2(u)$ appears in a Borel coalgebra decomposition of $H^*(\Omega X)$.*

Proof. By Theorem 2.4 $PH^*(\Omega X) = \operatorname{im}\sigma^*$, and since $PH^*(\Omega X) \subseteq F_2 H^*(\Omega X)$ (a) follows. Part (b) follows by examining a coalgebra decomposition of $H^*(\Omega X)$ and [1.13, 5]. Q.E.D.

Proposition 2.7. *Let BBE be a space with an element $s \in H^*(BBE)$ such that $s^2 = 0$ and $\sigma^*[\sigma^*(s)] \neq 0$. Then if $\Omega^2 BBE = E$, there is an element $v \in H^*(E)$ with $c(v) = \sigma^*[\sigma^*(s)] \otimes \sigma^*[\sigma^*(s)]$.*

Proof. This follows from [3.1, 8] and [15]. Q.E.D.

We conclude this chapter with a proof of Theorem 1.1(a) and part of Theorem 1.1(c).

Theorem 2.8. *For $k > 0$, $QH^{4k+1}(X) = Sq^2 QH^{4k-1}(X) + Sq^{2k} Sq^2 QH^{2k-1}(X)$. Also $Sq^2 QH^{4k+1}(X) = 0$.*

Proof. The original hypothesis on X implies

$$\sum_{k>0} QH^{4k+1}(X).$$

is a finite dimensional vector space. Therefore, we may prove this theorem by downward induction. Assume that for $k' > k$,

$$QH^{4k'+1}(X) = Sq^2 QH^{4k'-1}(X) + Sq^{2k'} Sq^2 QH^{2k'-1}(S)$$

and $Sq^2 QH^{4k'+1}(X) = 0$. There is a relation $Sq^{4k+2} = Sq^2\theta$ where $\theta = Sq^{4k} + Sq^{4k-1} Sq^1$. Let $\bar{x} \in QH^{4k+1}(X)$. Then by downward induction and Lemma 2.3(c),

$$\begin{aligned}\theta\bar{x} &= Sq^{4k} Sq^2 \bar{y}_1 + Sq^2 \bar{y}_2 \\ &= Sq^{4k}\bar{x}.\end{aligned}$$

Hence $\bar{x}_1 = \bar{x} - Sq^2 \bar{y}_1$ has

$$\theta\bar{x}_1 = Sq^2 \bar{y}_2.$$

It clearly suffices to show that

$$\bar{x}_1 \in \operatorname{im} Sq^2 + \operatorname{im} Sq^{2k} Sq^2.$$

By Lemma 2.3(b), if x_1 and y_2 are representatives of $\bar{x}_1, \bar{y}_2$ with $d\{x_1\}=0$, $d\{y_2\}=0$ then $\theta x_1 - Sq^2 y_2$ is 3-fold decomposable and $Sq^1 y_2 = 0$. Let

$$BK = K(\mathbb{Z}_2, 4k+1, 8k-1)$$
$$BK_1 = K(\mathbb{Z}_2, 8k+1, 8k).$$

Here the notation $K(\mathbb{Z}_2, n_1, n_2, \ldots, n_l)$ means $\prod_{i=1}^{l} K(\mathbb{Z}_2, n_i)$. Let $Bg\colon BK \to BK_1$, $Bf\colon X \to BK$ be defined by

$$Bg^*(i_{8k+1}) = \theta i_{4k+1} - Sq^2 i_{8k-1}, \qquad Bg^*(i_{8k}) = Sq^1 i_{8k-1}$$
$$Bf^*(i_{4k+1}) = x_1, \qquad Bf^*(i_{8k-1}) = y_2.$$

Then Bg is an infinite loop map. One can easily check using Proposition 2.7 that if BE is the fibre of Bg, there is a $v \in H^*(E)$ with $c(v) = u \otimes u$, $j^*(v) = Sq^2 i_{8k-1} + Sq^2 Sq^1 i_{8k-2}$, $u = p^*(i_{4k})$. We have the diagram

$$\begin{array}{ccccccc} & & & & BE & & \\ & & & & \downarrow & & \\ P_2\Omega X & \longrightarrow & X & \xrightarrow[Bf]{} & BK & \xrightarrow[Bg]{} & BK_1 \end{array}$$

which yields

$$\begin{array}{ccccc} \Omega K_1 & & & & \\ \downarrow & & & & \\ E & & & & \\ \downarrow & \nearrow \tilde{f} & & & \\ \Omega X & \xrightarrow[f]{} & K & \xrightarrow[g]{} & K_1. \end{array}$$

It follows that $\tilde{f}^*(v) = 0$, because $H^*(\Omega X)$ is even dimensional. Also by Lemma 2.6(a)

$$c(\tilde{f}^*(v)) \in \sigma^* x_1 \otimes \sigma^* x_1 + (Sq^2 + Sq^2 Sq^1)(\operatorname{im}\sigma^* \otimes F_2 + F_2 \otimes \operatorname{im}\sigma^*).$$

Here $F_2 = F_2 H^*(\Omega X)$. We conclude $\sigma^* x_1 \in Sq^2 F_2$.

Now there are no elements of the form $\gamma_2(u_{2k-1})$, because $H^*(\Omega X)$ is even dimensional. Therefore, by Lemma 2.6(b)

$$\sigma^* x_1 \in Sq^2(\operatorname{im}\sigma^*) + Sq^2(\text{2-fold products of primitives}).$$

By [5.3, 5] and Theorem 2.4 we conclude

$$\sigma^* x_1 \in Sq^2(\operatorname{im}\sigma^*) + Sq^{2k} Sq^2(\operatorname{im}\sigma^*).$$

Therefore, by induction

$$QH^{4k+1}(X) = Sq^2 QH^{4k-1}(X) + Sq^{2k} Sq^2 QH^{2k-1}(X).$$

Using the Adem relation $Sq^2 Sq^2 = Sq^3 Sq^1$ and

$$Sq^2 Sq^{2k} = a\, Sq^{2k+2} + b\, Sq^{2k+1} Sq^1, \qquad a, b \text{ in } \mathbb{Z}_2,$$

we conclude

$$Sq^2 QH^{4k+1}(X)=0. \quad \text{Q.E.D.}$$

Corollary 2.9. $QH^{8k+5}(X)=Sq^2 QH^{8k+3}(X)$.

Proof. This follows because $8k+5\equiv 1 \bmod 4$. Therefore, Theorem 2.8 implies

$$QH^{8k+5}(X)=Sq^2 QH^{8k+3}(X)+Sq^{4k+2} Sq^2 QH^{4k+1}(X),$$

and the last term is zero. Q.E.D.

§ 3. We proceed with proving the rest of Theorem 1.1

Note that if $H^*(X)$ is an exterior algebra on generators $x_1,\ldots,x_l$, and $\sigma^*(x_i)=y_i$, then $H^*(\Omega Z)$ as a coalgebra, is a divided power coalgebra on $y_1,\ldots,y_l$. This follows from the Eilenberg Moore spectral sequence.

Lemma 3.1. *Let $u\in PH^*(\Omega X)$. Then $\gamma_2(u)\in H^*(\Omega X)$ and $Sq^{2n}\gamma_2(u)=\gamma_2(Sq^n u)+z+d$, where z is a primitive, d is decomposable, and $\gamma_2(Sq^n u)$ is primitive if $Sq^n u=0$.*

Proof. $Sq^{2n}\gamma_2(u)$ and $\gamma_2(Sq^n u)+\sum_{i=0}^{n-1} Sq^i u Sq^{2n-i} u$ have the same reduced coproduct. Therefore, they differ by a primitive z. Q.E.D.

Theorem 3.2. *Let $\bar{x}\in QH^{8k+3}(X)$ with $\sigma^* x\in Sq^2 F_2 H^*(\Omega X)$. Then $\bar{x}\in \operatorname{im} Sq^4$.*

Proof. By Eq. (2.1) $\sigma^* x$ is a primitive generator. If $\sigma^* x\in Sq^2 PH^{8k}(\Omega X)=Sq^2\sigma^*(QH^{8k+1}(X))$, then Theorem 2.8 implies $\sigma^* x=0$. But $\sigma^*: QH^*(X)\to PH^*(\Omega X)$ is an isomorphism, so this is not possible. Therefore by Lemma 2.6(b), $\sigma^* x$ is a linear combination of terms of the form $Sq^2\gamma_2(y)$ where degree $y=4k$. By Theorem 2.8,

$$y=Sq^2 y_1+y_2^2$$

where y_1, y_2 primitive suspensions.

By Lemma 3.1 and $Sq^2 PH^{8k}(\Omega X)=0$, we get

$$Sq^2\gamma_2(y)=Sq^2 Sq^4\gamma_2(y_1)+Sq^2(d) \tag{3.1}$$

where d is decomposable.

Claim. $Sq^{8k}\sigma^* x=0$.

It suffices to prove $Sq^{8k}\sigma^* x$ is decomposable. Then Eq. (2.1) implies $Sq^{8k}\sigma^* x=0$. By Eq. (3.1), it suffices to show $Sq^{8k} Sq^2\gamma_2(y)$ is decomposable. We have

$$Sq^{8k} Sq^2\gamma_2(y)=Sq^{8k} Sq^2 Sq^4\gamma_2(y_1)+Sq^{8k} Sq^2 d.$$

Using the Adem relations and $Sq^1 H^*(\Omega X)=0$,

$$Sq^{8k} Sq^2=Sq^4 Sq^{8k-2}+Sq^{8k+2}$$
$$Sq^{8k-2}=Sq^2 Sq^{8k-4}+Sq^{8k-3} Sq^1$$

we get

$$\begin{aligned} Sq^{8k}Sq^2\gamma_2(y) &= Sq^4Sq^{8k-2}Sq^4\gamma_2(y_1)+Sq^{8k}Sq^2 d \\ &= Sq^4Sq^2Sq^{8k-4}Sq^4\gamma_2(y_1)+Sq^{8k}Sq^2 d. \end{aligned} \tag{3.2}$$

By Lemma 3.1,

$$Sq^{8k-4}Sq^4\gamma_2(y_1)=\gamma_2(Sq^{4k-2}Sq^2 y_1)+z_2+d_2.$$

By Theorem 2.8, $Sq^{4k-2}Sq^2 y_1=0$. Therefore,

$$Sq^{8k}Sq^2\gamma_2(y)\in Sq^4Sq^2PH^{16k-4}(\Omega X)+d_3 \tag{3.3}$$

where d_3 is decomposable. But $Sq^2PH^{16k-4}(\Omega X)=0$ by Theorem 2.8. This proves the claim.

Because σ^* is an isomorphism, $Sq^{8k}x$ must be decomposable.

Consider the relation

$$Sq^{8k+4}=Sq^4(Sq^{8k}+Sq^{8k-2}Sq^2)+(Sq^{8k}Sq^3+Sq^{8k+3})Sq^1.$$

We may assume x has $d\{x\}=0$. Then $Sq^1x=0$, and $Sq^{8k}x$ is 3-fold decomposable, by Lemma 2.3. $Sq^{8k-2}=Sq^2Sq^{8k-4}+Sq^{8k-3}Sq^1$. Therefore, Theorem 2.8 together with Lemma 2.3 imply

$$Sq^{8k-2}Sq^2x \quad \text{is 3-fold decomposable.}$$

Let

$$\begin{gathered} BK=K(\mathbb{Z}_2,8k+3) \\ BK_1=K(\mathbb{Z}_2,16k+3,8k+4) \\ Bg\colon BK\to BK_1, \\ Bg^*(i_{16k+3})=(Sq^{8k}+Sq^{8k-2}Sq^2)i_{8k+3} \\ Bg^*(i_{8k+4})=Sq^1 i_{8k+3}. \end{gathered}$$

As in the proof of Theorem 2.8 we get

$$\sigma^*x\otimes\sigma^*x\in(Sq^4+Sq^{8k}Sq^3+Sq^{8k+3})(F_2\otimes\operatorname{im}\sigma^*+\operatorname{im}\sigma^*\otimes F_2).$$

Therefore $\sigma^*x\otimes\sigma^*x\in Sq^4(F_2\otimes\operatorname{im}\sigma^*+\operatorname{im}\sigma^*\otimes F_2)$. If Sq^4 splits in the Cartan formula, that would imply $\sigma^*x\in(Sq^1+Sq^2)(\operatorname{im}\sigma^*)$. This is impossible, because $Sq^2PH^{8k}(\Omega X)=0$ and $Sq^1H^*(\Omega X)=0$. We conclude

$$\sigma^*x\in Sq^4F_2H^*(\Omega X).$$

Since σ^*x is indecomposable,

$$\overline{\sigma^*x}\in Sq^4QH^{8k-2}(\Omega X)\cong Sq^4PH^{8k-2}(\Omega X)\cong Sq^4\sigma^*QH^{8k-1}(X).$$

Therefore $\bar{x}\in\operatorname{im}Sq^4$. Q.E.D.

Theorem 3.3. *Suppose that for* $2\leqq i<r$, $k>0$

$$QH^{2^i+2^{i+1}k-1}(X)=Sq^{2^i}QH^{2^{i+1}k-1}(X)$$

and

$$Sq^{2^i} QH^{2^i+2^{i+1}k-1}(X)=0.$$

Then if $r>2$, $k>0$ *and* $\bar{x}\in QH^{2^r+2^{r+1}k-1}(X)$ *with* $\sigma^* x\in Sq^2 F_2 H^*(\Omega X)$, *then* $\bar{x}\in \operatorname{im} Sq^{2^r}$.

Proof. $\sigma^* x$ is a primitive generator, and by Theorem 2.8 $\sigma^* x\notin Sq^2 PH^*(\Omega X)$. Therefore, $\sigma^* x$ is a linear combination of terms of the form $Sq^2\gamma_2(y)+d_0$ where y is primitive of degree $2^{r-1}+2^r k-2$ and d_0 decomposable. By assumption and Theorem 2.4, $y=Sq^{2^{r-1}} y_1$ where degree $y_1=2^r k-2$, and y_1 is primitive. By Lemma 3.1,

$$\gamma_2(y)=Sq^{2^r}\gamma_2(y_1)+z+d$$

where z is primitive, d is decomposable. There is an Adem relation for $r>2$

$$Sq^2 Sq^{2^r}=Sq^{2^r} Sq^2+Sq^4 Sq^2 Sq^{2^r-4}+Sq^1 \text{ doubletons.}$$

By Theorem 2.8, $Sq^2 z=0$. Therefore,

$$Sq^2\gamma_2(y)=Sq^{2^r} Sq^2\gamma_2(y_1)+Sq^4 Sq^2 Sq^{2^r-4}\gamma_2(y_1)+Sq^2 d. \tag{3.4}$$

By Lemma 3.1 and Theorem 2.4,

$$Sq^2\gamma_2(y_1)=\sigma^*(w_1)+d_1$$
$$Sq^2 Sq^{2^r-4}\gamma_2(y_1)=\sigma^*(w_2)+d_2$$

where the w_i are generators, the d_i are decomposables.

Now since $Sq^4 QH^{8l+3}(X)=0$ by assumption, it follows that $Sq^4\sigma^*(w_2)=0$. Equation (3.4) becomes

$$Sq^2\gamma_2(y)=Sq^{2^r}\sigma^*(w_1)+d_3 \tag{3.5}$$

where d_3 is decomposable. Therefore, $\sigma^* x-Sq^{2^r}\sigma^*(w)$ is primitive decomposable for some $w\in H^{2^{r+1}k-1}(X)$. By Eq. (2.1) we conclude $\sigma^* x=\sigma^*(Sq^{2^r} w)$ or

$$\bar{x}\in \operatorname{im} Sq^{2^r}. \quad \text{Q.E.D.}$$

We are now ready to prove Theorem 1.1(b) and (c). The proof is by induction on r and downward induction on k. Consider the following inductive assumptions

$$\mathscr{P}(l):\ QH^{2^i+2^{i+1}m-1}(X)=Sq^{2^i} QH^{2^{i+1}m-1}(X)$$

and

$$Sq^{2^i} QH^{2^i+2^{i+1}m-1}(X)=0 \quad \text{for } 2\leqq i<l, \quad m>0.$$

Assume $\mathscr{P}(r-1)$ is true. Our task will be to prove $\mathscr{P}(r-1)$ implies $\mathscr{P}(r)$. Our original assumption about X implies

$$\sum_{k>0} QH^{2^r+2^{r+1}k-1}(X)$$

is a finite dimensional vector space. By downward induction on k, assume that for $k'>k$

$$QH^{2^{r+2^{r+1}k'-1}}(X)=Sq^{2^r}QH^{2^{r+1}k'-1}(X)$$

and

$$Sq^{2^r}QH^{2^{r+2^{r+1}k'-1}}(X)=0.$$

The induction will be completed if we prove

$$QH^{2^{r+2^{r+1}k-1}}(X)=Sq^{2^r}QH^{2^{r+1}k-1}(X)$$

and $Sq^{2^r}QH^{2^{r+2^{r+1}k-1}}(X)=0$. The following relations hold:

$$Sq^{2^{r+2^{r+1}k}}=\sum_{i=0}^{r} Sq^{2^i}\alpha_i \tag{3.6}$$

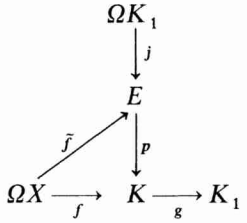
$$Sq^{2^j}Sq^{2^j}=\sum_{i=0}^{j-1} Sq^{2^i}\beta_i. \tag{3.7}$$

Let $\bar{x}\in QH^{2^{r+2^{r+1}k-1}}(X)$ have representative x with $d\{x\}=0$. Then the inductive assumptions together with Lemma 2.3, and Corollary 2.9 imply there are generators x_i with $\alpha_i x\equiv Sq^{2^i}x_i$ modulo 3-fold decomposables. Using (3.7) and introducing more variables, we get for example

$$\alpha_j x\equiv Sq^{2^j}x_j$$
$$\beta_i x_j\equiv Sq^{2^i}x_{ij}.$$

Successive relations will involve Eq. (3.7) with j decreasing until the final relation is $Sq^1Sq^1=0$. This is described in [6].

If one lets BK be a generalized Eilenberg MacLane space with fundamental classes in the degrees of the variables and BK_1 a generalized Eilenberg MacLane space with classes in the degrees of the relations, then we can let Bg: $BK\to BK_1$ describe the relations. It follows that we will get a diagram of H-maps,



and an element $v\in H^*(E)$ with $c(v)=u\otimes u$, $\bar{f}^*u=\sigma^*x$ and $j^*(v)\in\sum_{i=0}^{r}\operatorname{im}Sq^{2^i}$. By Lemma 2.6,

$$\sigma^*x\otimes\sigma^*x\in\sum_{i=0}^{r}Sq^{2^i}(F_2\otimes\operatorname{im}\sigma^*+\operatorname{im}\sigma^*\otimes F_2).$$

This implies $\sigma^*x\in\sum_{i=0}^{r}Sq^{2^i}F_2$. If $i=1$, Theorems 3.2 and 3.3 imply $(Sq^2F_2)^{2^{r+2^{r+1}k-2}}\subseteq\operatorname{im}Sq^{2^r}$. For $i>1$, because σ^*x is a primitive generator and $PH^{4m+2}(\Omega X)\cong QH^{4m+2}(\Omega X)$ for $m\geqq 0$ it follows that $(Sq^{2^i}F_2)^{2^{r+2^{r+1}k-2}}\subseteq Sq^{2^i}PH^*(\Omega X)$. Therefore by Theorem 2.4,

$$\sigma^*x\in\sum_{i>1}Sq^{2^i}PH^*(\Omega X)=(\sum Sq^{2^i})\operatorname{im}\sigma^*.$$

But since $Sq^{2^l} QH^{2^i+2^{i+1}m-1}(X)=0$ it follows that $\sigma^* x \in Sq^{2^r}\sigma^* H^*(X)$. We conclude $\bar{x}\in \operatorname{im} Sq^{2^r}$. This proves

$$QH^{2^r+2^{r+1}k-1}(X)=Sq^{2^r}QH^{2^{r+1}k-1}(X).$$

The Adem relation $Sq^{2^r}Sq^{2^r}=\sum_{i=0}^{r-1} Sq^{2^i}\beta_i$ together with the inductive assumption imply $Sq^{2^r}QH^{2^r+2^{r+1}k-1}(X)=0$. This completes the proof of Theorem 1.1. Q.E.D.

Proof of Corollary 1.2. By Theorem 1.1 all generators of $H^*(X)$ are connected by Steenrod operations to generators in degrees of the form 2^l-1 for $l>1$. Because Steenrod operations preserve primitives, the corollary follows. Q.E.D.

Proof of Theorem 1.3. The Hurewicz theorem together with the fact that $H_*(X;\mathbb{Z})$ is two torsion free [1, 2] implies the first non-vanishing homotopy group occurs in the same degree as the first non-vanishing positive mod two homology group. Let this degree be $v_2(X)$. By Theorem 1.1, $v_2(X)=2^q-1$ for some $q>1$. Note that $H_*(X)$ is primitively generated because of the exact sequence [10].

$$PH_*(X)\to QH_*(X)\to Q[\lambda H_*(X)]\to 0.$$

Therefore the first decomposable primitive in $H_*(X)$ must be a 3-fold product by Lemma 2.1.

It follows that $H^*(X)$ is primitively generated in degrees less that $3v_2(X)$. The proof now proceeds exactly as in [12]. All that is needed is the structure of $H^*(P_2, X)$ in degrees less than $3v_2(X)$. This is the same as in the primitively generated case. The details are left to the reader. Q.E.D.

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