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Total curvature for knotted surfaces

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1. Introduction

In 1949 J. Milnor [18, 19] and I. Fary [3] proved that a C^2-knot γ in three dimensional euclidean space $\mathbb{R}^3$ has total Gauss curvature $K(\gamma)=\int |K|\, ds$ greater or equal to 4π. Milnor even excluded equality as well:

$$K(\gamma)>4\pi. \tag{1.1}$$

This theorem presents one of the most beautiful relationships between the classical local differential geometry of a curve γ and its topological placements in $\mathbb{R}^3$. Before, in 1929, Fenchel [5] had proved that the total Gauss curvature for any closed curve γ obeys $K(\gamma)\geqq 2\pi$, and equality is realized if and only if γ is a plane convex curve. In this paper we restrict to maps of differentiable class C^∞, if not specified differently. (For the curvature of continuous maps see [11].)

The theorem of Milnor-Fary provides a starting point for later studies relating the local geometry of a compact surface in $\mathbb{R}^3$ and its topological placement [15–17, 22]. Since our theorems and proofs are motivated in part by techniques already employed by Fary and Milnor, we now briefly sketch the proof of their theorem.

First consider a C^1-boundary torus T_ε of a small ε-neighborhood $N_\varepsilon = \{q\in\mathbb{R}^3\colon \text{distance}(q,\gamma)\leqq\varepsilon\}$ of γ, and the Gauss map $G\colon T_\varepsilon\to S^2$ where $G(p)$ is the outward normal to N_ε at the point p, parallel translated to the origin. The area $K(T_\varepsilon)$ of the image of the Gauss map $G(T_\varepsilon)$, counting multiply covered points multiple, is equal to

$$K(T_\varepsilon)=\int_{T_\varepsilon} |K(p)\, dA|, \tag{1.2}$$

where $|K(p)|$ denotes the absolute value of the Gauss curvature of T_ε at p, and dA its area element. The quantity

$$\tau(T_\varepsilon)=K(T_\varepsilon)/2\pi \tag{1.3}$$

will be called the *total curvature* of T_ε. See [28]. It can be defined for an arbitrary imbedding or immersion, $f=M\to\mathbb{R}^3$ of a compact surface M without boundary in $\mathbb{R}^3$ by the same formulas (1.2) and (1.3). With this definition the total curvature $\tau(f)=\tau(M)$ can be interpreted as the *mean or expectation value* ($\mathscr{E}$) of the number of critical points (μ_L) of $\pi_L\circ f\colon M\to\mathbb{R}$, where $\pi_L\colon\mathbb{R}^3\to L$ is the orthogonal projection on a general oriented line L through the origin. Such oriented lines are parametrised by their forward intersection points with S^2, and the mean is with respect to the standard measure on S^2:

$$\tau(f)=\mathscr{E}_L\mu_L(\pi_L\circ f). \tag{1.4}$$

Having established this formula for surfaces we next observe that the formula for $\tau(f)$ gives a definition (of total curvature) for an immersion, or even for some more general maps, $f\colon M\to\mathbb{R}^N$ of any closed manifold M into $\mathbb{R}^N$. In particular Fary and Milnor had already for a closed curve γ in $\mathbb{R}^3$ the formula

$$\tau(\gamma)=\mathscr{E}_L\mu_L(\pi_L|\gamma)=K(\gamma)/\pi=\int|k|\,ds/\pi,$$

and it equals:

$$\frac{1}{2}\tau(T_\varepsilon)=\frac{1}{2}\mathscr{E}_L\mu_L(\pi_L|T_\varepsilon)=\frac{1}{4\pi}\int_{T_\varepsilon}|K(p)\,dA|.$$

Indeed, observe that if $\pi_L|\gamma$ is a Morse function (nondegenerate critical points), then $\pi_L|T_\varepsilon$ has for small $\varepsilon>0$, two critical points for any one critical point of $\pi_L|\gamma$, and all critical points of $\pi_L|T_\varepsilon$ arise in this manner. We next recall the main argument from Fary-Milnor. If $K(\gamma)<4\pi$, then $\tau(\gamma)<4$. As $\tau(\gamma)$ is a mean value, there is some oriented line L such that $\pi_L|\gamma$ has less than 4 and hence only two critical points, a maximum and a minimum. We then see that γ is the boundary of an imbedded disc D which is ruled by straight line segments in the planes orthogonal to L. Since γ is the boundary of the disc D, γ is unknotted.

Now Fary and Milnor observed that for any closed imbedded polygon γ' which is not in a plane, the total curvature $\tau(\gamma')$, properly defined, can be decreased by some small perturbation of the vertices. Milnor shows moreover that by adding one vertex to a polygon the curvature cannot decrease. (It suffices to look at the area swept out on S by planes through the origin orthogonal to the "tangents" of the (polygonal) curve.) It follows that for any C^2-curve γ the total curvature $\tau(\gamma)$ is an upper bound for the total curvature of all inscribed polygons. Then for any nonplanar closed imbedded curve γ we find a nearby isotopic curve γ'' with smaller total absolute curvature, and so $\tau(\gamma)=4$ is impossible for a knotted curve; Milnor's theorem (1.1) is established.

It should be noted that the trefoil knot can be represented by curves γ with $K(\gamma)=4\pi+\varepsilon$ with any positive ε, so that Milnor's theorem is a best possible result, for curves.

In our paper we will mainly investigate relationships between the total curvature of an imbedding $f\colon M\to\mathbb{R}^3$ of a compact surface and the topology of the imbedding. For example, the following is a generalization of Milnor's theorem (1.1) from curves γ to imbedded tori T in $\mathbb{R}^3$.

Theorem 3a. *If* f: $T \to \mathbb{R}^3$ *is an imbedded torus with* $\tau(T) \leqq 8$, *then* $T = f(T)$ *is unknotted.*

The proof for $\tau(T) < 8$, generalizes the proof of Fary-Milnor for curves γ with $\tau(\gamma) < 4$. Namely when $\tau < 8$ then, τ being a mean, there exists a line $L \subset \mathbb{R}^3$ such that $\pi_L | T$ is a nondegenate function on T with six critical points, if not four, and for the well chosen orientation of L it has at most one (local) maximum. Then it is shown that each component of $\mathbb{R}^3 \setminus T$ has one generator for its fundamental group, and one generator for its first homology group. Finally methods of geometric topology can be used to prove that a torus for which each component of its complement $T^c = \mathbb{R}^3 \setminus T$ has a fundamental group generated by one element is standardly imbedded up to ambient isotopy [27]. Actually, to prove that T is unknotted, we use a more general theorem for surfaces with an interesting generalization for all dimensions as follows. Let f: $M^n \to \mathbb{R}^N$ be a smooth imbedding of a closed manifold M^n in $\mathbb{R}^N$. Let $\beta_1^c = \text{rank } H_1(M^c, \mathbb{Z}_2)$ be the first Betti number with coefficients in $\mathbb{Z}_2$ of the *complement* $(M^c = \mathbb{R}^N \setminus f(M^n))$ and denote by $\beta_1^c + \sigma_1(f)$ the minimal number of generators of the fundamental group $\pi_1(M^c, *)$. The number $\sigma_1 = \sigma_1(f)$ is called the *first homotopy excess* of the imbedding f. *For codimension* $N - n \geqq 3$ *it is zero.*

Remark. Our choice of coefficients, $\mathbb{Z}_2$, is restrictive for dimensions $n \geqq 3$, but we keep the simplifying restriction as no interesting cases seem to arise without it. For a hypersurface, $N - n = 1$, σ_1 is the sum of a contribution σ_1^{int} concerning the bounded component M^{int} of M^c and a contribution σ_1^{ext} concerning the unbounded component M^{ext}:

$$\sigma_1 = \sigma_1^{\text{int}} + \sigma_1^{\text{ext}}, \qquad M^c = M^{\text{int}} \cup M^{\text{ext}}. \tag{1.5}$$

Finally, let β be the sum of the Betti numbers $\beta_i = \text{rank } H_i(M; \mathbb{Z}_2)$ of M. Then we prove the main inequality in

Theorem 1. *The total curvature of the imbedding of a closed manifold,* f: $M^n \to \mathbb{R}^N$, $1 \leqq n < N$,

$$\tau(f) = \mathscr{E}_L \mu_L(\pi_L \circ f),$$

obeys

$$\tau(f) \geqq \beta + 4\sigma_1, \tag{1.6}$$

except for curves, where one only has:

(Fox[6]) $$\tau(f) \geqq \beta + 2\sigma_1 \qquad n = 1 \tag{1.6F}$$

and for three dimensional hypersurfaces in $\mathbb{R}^4$, *where Wintgen's result* [30] *is the best we have*:

$$\tau(f) \geqq \beta + 4 \max(\sigma_1^{\text{int}}, \sigma_1^{\text{ext}}) \qquad \text{for } N = n + 1 = 4. \tag{1.6W}$$

For the connected sum $f = f_1 \# f_2$ *of two hypersurfaces* f_j: $M_j^n \to \mathbb{R}^{n+1}, j = 1, 2,$ *one has*

$$\sigma_1(f) = \sigma_1(f_1) + \sigma_1(f_2).$$

This theorem contains trivially the classical

Theorem 1b (Chern-Lashof [1]). *The total curvature of any immersion* f: $M^n \to \mathbb{R}^N$ *obeys*

$$\tau(f) \geqq \beta \geqq 2.$$

They also observed that $\tau(f)=2$ only occurs for convex hypersurfaces in $\mathbb{R}^{n+1} \subset \mathbb{R}^N$.

For *closed curves*, $n=1$ *in* $\mathbb{R}^3$, *we find a theorem of Fox* [6], generalizing the weak theorem of Milnor-Fary. By the τ-decreasing argument of Milnor for non-planar curves one concludes however the better

Theorem 1c (Fox-Milnor). *For knotted curves* $f: S^1 \to \mathbb{R}^3$ *the total curvature obeys*

$$\int \frac{|k\,ds|}{\pi} = \tau(f) > 2 + 2\sigma_1,$$

in which equality is excluded!

However this inequality is not sharp for all closed curves because the standard torus-knot $\gamma^{p,q}$ of type (p, q), $3 \leqq p \leqq q$ has $\sigma_1(f)=1$, but $\tau(f) > 2p > 4$, as the bridge number $\frac{1}{2} \min_L \mu_L \pi_L(\gamma^{p,q})$ is p by *Schubert* [26].

Our main interests are the imbeddings f: $M_g \to \mathbb{R}^3$ of compact surfaces of genus g. Theorem 1 states that if $\tau(f) < \beta + 4$, then $\sigma_1(f)=0$, and then both components (see (1.5)) of M^c have free fundamental groups. Each of them is obtained from a ball by attaching solid one-handles (they are one-handle bodies). A deep theorem of Waldhausen [27] states that if each component of M^c is a one-handle-body, then M is standard up to isotopy (:unknotted). Thus, we have the following corollary due to Meeks [17] and Morton [22] which was first proved by Langevin and Rosenberg [15], in the case of the torus T. These authors were the first to study the total curvature of knotted surfaces.

Corollary 1d. *If the total curvature of a compact imbedded surface* M_g *in* $\mathbb{R}^3$ *is less than* $\beta + 4 = 2g + 6$, *then* M_g *is unknotted.*

For $g=1$, this corollary concerns the torus. Surfaces f: $M \to \mathbb{R}^3$, of minimal total absolute curvature $\tau(f)=\beta$ are in particular unknotted. They are called *tight* surfaces and have the *characteristic property* that there exists a convex surface ∂B_0, boundary of the convex hull $\mathscr{H}M$ of $M = f(M)$, such that the set of all points of Gauss curvature $K>0$ in M, $M_{K>0}$, coincides with the analogous set for $\partial B_0 = \partial \mathscr{H} M$, $(\partial \mathscr{H} M)_{K>0}$.

Theorem 2a (Kuiper [10]). *The smooth immersion* f: $M^2 \to \mathbb{R}^3$ *is tight iff*

$$M_{K>0} = (\partial \mathscr{H} M)_{K>0}$$

Here ∂B_0 is in general only C^1 at the boundary of $\partial B_0 \setminus M$.[1]

Starting from ∂B_0 we obtain M by attaching and completing with surface pieces of non-positive Gauss curvature $K \leqq 0$. Such placements exist for all oriented closed surfaces M_g and for non-oriented surfaces M (here we must

[1] Notation for sets: $X \setminus Y = \{q \in X : q \notin Y\}$

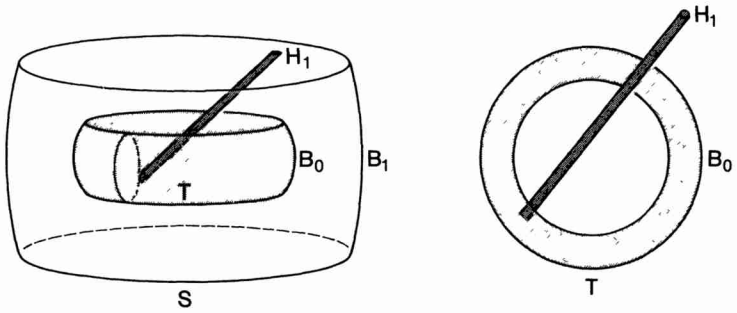


Fig. 1. A relatively isotopy tight torus $M=T\#S$

admit immersion) if they have Euler characteristic $\chi(M)\leqq -2$, but not for $\chi(M)\geqq 0$. The case $\chi(M)=-1$ is a longstanding open problem. For stable maps (in the sense of Thom) f: $M\to\mathbb{R}^3$ such tight placements exist for all closed surfaces [12]. Note the formula for the sum of the $\mathbb{Z}_2$-Betti numbers of a surface: $\beta=4-\chi(M)$.

By Milnor's τ-decreasing method it was shown that a knotted curve with total curvature $\tau(f)=4$ does not exist. There is no such method for surfaces as seen from the torus imbedding f: $M\to\mathbb{R}^3$ in Fig. 1. This surface $f(M)$ is obtained from a standard (tight) torus T inside a convex surface S, by attaching a connecting handle H of non-positive curvature $K\leqq 0$. Near the attachments, the surfaces S and T are made flat before.

Then $\pi_L\circ f$ has for almost all L exactly 8 critical points and for any nearby imbedding f': $M\to\mathbb{R}^3$ we see (!) that $\pi_L\circ f$ has at least as many. Hence

$$\tau(f')\geqq\tau(f)$$

and τ attains a relative minimum at f on the space of imbeddings $\mathrm{Imb}(M,\mathbb{R}^3)$. But as f is isotopic to the standard imbedding, $\tau(f)$ is not an absolute minimum in this example. As Ozawa observed, there even is an arc f_t in $\mathrm{Imb}(M,\mathbb{R}^3)$ from $f=f_0$ to a tight imbedding f_1, on which $\tau(f_t)\leqq 8$ for $0\leqq t\leqq 1$ and $\tau(f_1)=4$. So the expression "relative minimum" could be misleading.

This leads to the natural question of whether there exist global or local minima for τ on the various path components of $\mathrm{Imb}(M_g,\mathbb{R}^3)$. It suggests the following definition. An imbedding f is called *isotopy tight* if $\tau(h)\geqq\tau(f)$ for all imbeddings h in the ambient isotopy class $[f]$ of f, that is its path component in $\mathrm{Imb}(M,\mathbb{R}^3)$:

$$\tau(f)=\min_{h\in[f]}\tau(h).$$

We will also define a (more general) notion of *relative isotopy tightness*. The geometric properties are described in the following

Fundamental Lemma 2. *If f: $M_g\to\mathbb{R}^3$ is a (relatively) isotopy tight C^2-imbedding of a closed surface M, then there exists an integer k and $k+1$ convex*

surfaces $\partial B_0, \ldots, \partial B_k$ *in* $\mathbb{R}^3$ *such that the set* $M_{K>0}$ *of all points* p *of* M *with positive Gauss curvature* $K(p)>0$ *is exactly the analogous set for* $\bigcup_{i=0}^{k} \partial B_i$:

$$M_{K>0} = \bigcup_{i=0}^{k} (\partial B_i)_{K>0}.$$

The unique principal component M_i^+ *of* $\partial B_i \cap M$ *contains* $(\partial B_i)_{K>0}$, *and is obtained from* ∂B_i *by deleting disjoint open plane convex discs in* ∂B_i. *From now on we reserve the word top-circle of* M *for the boundary of such a disc. The plane* $\pi(\gamma)$ *of a top circle* γ *supports some neighborhood* $U \subset M$ *of* $\pi(\gamma) \cap M_i^+$, *that is* U *lies in one of the two closed half spaces bounded by* $\pi(\gamma)$. *The set* $\pi(\gamma) \cap M_i^+$ *is then called a (relative) top set. The total absolute curvature of* f *is*

$$\tau(f) = \beta + 4k = 2 + 2g + 4k.$$

All these properties together are sufficient for an imbedding to be relatively isotopy tight.

The obvious analogous conclusions for immersions also hold. We will not give the proof.

For $k=0$ we get back to the tight imbeddings of Theorem 2a. If $f \in \mathrm{Imb}(M_g, \mathbb{R}^3)$ is "knotted", then Corollary 1d states that $\tau(f) \geqq \beta + 4 = 2g + 6$ for a knotted imbedding. Thus if f is knotted and $\tau(f) = \beta + 4$, then f is isotopy tight. Then *the fundamental lemma describes how* $f(M_g)$ *must be made from two convex surfaces by connecting with pieces of surfaces of non-positive curvature.* We exploit this description to prove the following *main result*:

Theorem 3. *If* f: $M_g \to \mathbb{R}^3$ *is an imbedded knotted isotopy tight surface of genus* g, *with* $\tau(f) = \beta + 4 = 2g + 6$, *then* g *is greater than two. Furthermore, for all* $g > 2$ *there exists an isotopy tight surface* f *with* $\tau(f) = \beta + 4$.

This gives an answer to problem 5 in [13].

The *key example is given in* Fig. 2. It is obtained from the surface $f(M)$ in Fig. 1 by suitably attaching two more handles H_2 and H_3 of non-positive

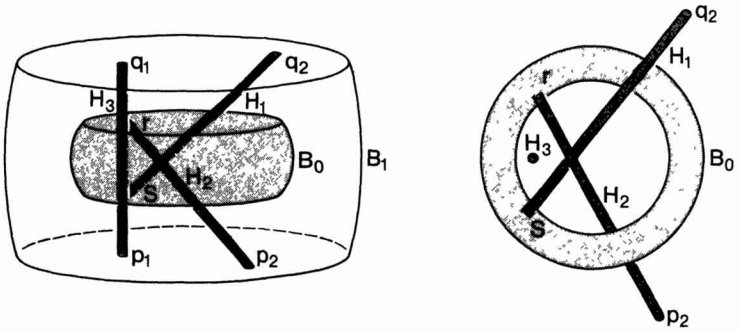


Fig. 2. An isotopy tight surface of genus $g=3$

curvature. The total curvature of the new surface f': $M_g \to \mathbb{R}^3$, $g=3$, is clearly

$$\tau(f')=\beta+4=12.$$

This curvature is a relative minimum of τ on $\mathrm{Imb}(M_3, \mathbb{R}^3)$, but as we will prove that $f'(M)$ is knotted, the imbedding is isotopy-tight.

We also give examples of isotopy tight imbeddings f: $M_g \to \mathbb{R}^3$ with $\tau(f)=\beta+4k$ for any integer $k \geqq 1$. For large values of k we seem to need large values of the genus g.

An outline of the paper is as follows:

In Sect. 2 we show how the fundamental group of the complement of a closed manifold M^n in $\mathbb{R}^N$, of codimension one or two, may influence the total absolute curvature τ. In Sect. 3 we give examples of isotopy tight knotted surfaces in $\mathbb{R}^3$ of genus $g \geqq 3$ (for which the infimum of τ in its isotopy class is reached!). We mention in 3b that such examples can be analytic and even given by a polynomial equation. In Sect. 3f we obtain as a by-product a hypersurface $W^n \subset \mathbb{R}^{n+1}$, $n \geqq 4$, with minimal absolute total curvature $\tau=\gamma=\beta+4$, but not tight, $\tau \neq \beta$. In Sect. 4 we prove the fundamental Lemma 2.

Sect. 5 gives the proof that for a knotted torus and a knotted surface of genus two, the infimum of the total curvature $\tau=\beta+4$ cannot be reached. We also introduce a notion of weak relative isotopy tightness (RIT-) and show that a RIT-torus with $\tau=8$ is unique as in Fig. 1 up to weak RIT-isotopy. (The readers is invited to isotope the surface in a standard unknotted situation.)

Finally we make plausible a complete classification of RIT-surfaces of genus two and curvature $\tau=10$, up to weak RIT-equivalence: There are four classes represented by E, F, G, H in Fig. 8.

Many problems are left to attack. We mention some in the paper. We are happy to acknowledge profitable conversations with many mathematicians and in particular with Hugh Morton.

2. The influence of the first homotopy excess on curvature

In this section we prove Theorem 1 concerning the embedding f: $M^n \to \mathbb{R}^N$ of a closed manifold $M=M^n=f(M^n)$. We first mention with the notation μ_L for $\mu_L(\pi_L \circ f)$ the obvious inequality

$$\tau(f)=\int_L \mu_L \geqq \min \mu_L, \tag{2.1}$$

where the minimal value is taken over all oriented lines L for which $\pi_L \circ f$: $M \to L$ is non-degenerate. We concentrate our attention on any such line L, call it *vertical* and assume a scaling $L=\mathbb{R}$ such that π_L: $\mathbb{R}^N \to \mathbb{R}$ is one of a system of N Euclidean coordinates, which we call height and denote by z. We first treat

a) Hypersurfaces: *For example surfaces in* $\mathbb{R}^3$. Each component of the complement $M^c=\mathbb{R}^{n+1} \setminus M=M^{\mathrm{int}} \cup M^{\mathrm{ext}}$ of a hypersurface $n+1=N$, can be closed in the one point compactification $S^N=\mathbb{R}^N \cup \infty$ of $\mathbb{R}^N$, to a compact manifold with

M as smooth boundary. From now on we use *the notations* M^{int} and M^{ext} for these compact $n+1$-manifolds with boundary. A good model is obtained as follows. Let

$$B_R = \{x \in \mathbb{R}^{n+1} : \|x\| \leqq R\},$$

be a large ball in $\mathbb{R}^{n+1}$, which contains $f(M) = M$ in its interior. Take two copies of B_R, one of them containing M, and attach them to each other along their boundaries. The union is an $n+1$-sphere S^{n+1}, in which M, M^{int} and M^{ext} have an obvious embedding. Also the function $z\colon S^{n+1} \to \mathbb{R}$ is well defined. Next we define (see Fig. 3):

$$\begin{aligned} M_t &= \{p \in M : z(p) \leqq t\}, \\ M_t^{\text{int}} &= \{p \in M^{\text{int}} : z(p) \leqq t\} \end{aligned} \tag{2.2}$$

and

$$M_t^{\text{ext}} = \{p \in M^{\text{ext}} : z(p) \leqq t\}$$

A point $q \in M^{\text{int}}$ is called a *critical point* of index i of the function $z \mid M^{\text{int}}$, in case

$$H_i(M_t^{\text{int}}, M_t^{\text{int}} \setminus q; \mathbb{Z}_2) = \mathbb{Z}_2 \qquad \text{for } t = z(q), \tag{2.3}$$

and analogous for $q \in M^{\text{ext}}$. For $z\colon M \to \mathbb{R}$ we get in this way the usual definition. See [20].

Any critical point of index i for z on M, is a critical point of index i for z, either on M^{int} or on M^{ext}, except the point of maximal height on M, which is neither critical on M^{int} nor on M^{ext}, as seen from the geometry in S^{n+1}.

Observe also that every critical point of index i for z on M is at the same time a critical point of index $n-i$ for the "depth-function" $-z$ on M. Those of these points which are critical of index $i<n$ for z on M, are also critical of index i for z on M^{int}, as well as critical of index $n-i$ for $-z$ on M^{ext}.

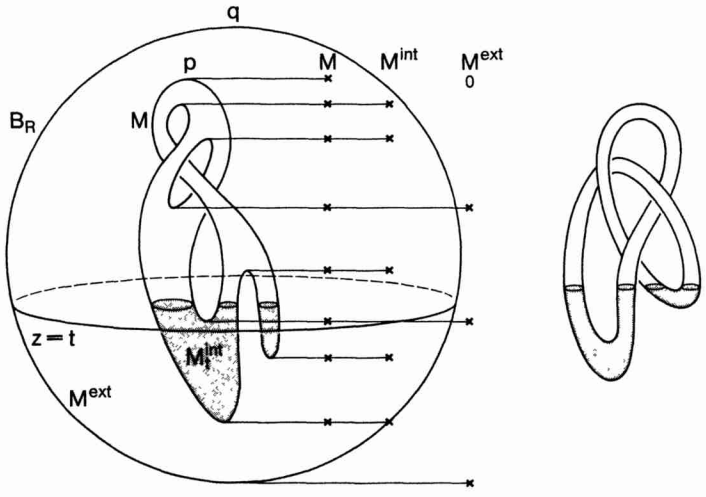


Fig. 3. a Critical points for z on M, M^{int}, M^{ext}. **b** trefoil torus knot $\tau > 8$

Let μ_i, μ_i^{int}, μ_i^{ext} be the number of critical points of index i, for z on M, M^{int} and M^{ext} respectively. Note that all critical points for z on M^{int} and M^{ext} are critical points for z on M except for the extreme z-values on M^{ext}. We can summarize our geometrical observations in the following formulas. For $1 \leqq i < n$,

$$\begin{aligned} \mu_0 &= \mu_0^{\text{int}} + \mu_0^{\text{ext}} - 1, \\ \mu_i &= \mu_i^{\text{int}} + \mu_i^{\text{ext}}, \\ \mu_n &= \mu_n^{\text{int}} + \mu_n^{\text{ext}}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \mu_i &= \mu_i(z) = \mu_{n-i}(-z) \\ \mu_i^{\text{int}} &= \mu_i^{\text{int}}(z) = \mu_{n-1}^{\text{ext}}(-z) \end{aligned} \tag{2.5}$$

$$\mu_i(z) = \mu_i^{\text{int}}(z) + \mu_{n-1}^{\text{int}}(-z). \tag{2.6}$$

We now apply an ambiant isotopy[1] to the embedding $f\colon M \to B_R$, which preserves the nondegenerency of the function zf on M, hence the numbers of critical points of index i, but yields a new embedding, *again called* f, for which the critical points are partly ordered by their height z as follows:

$$i-1 < z(p) < i, \quad i = 0, 1, n-1, \text{ or } n$$

for any critical point p of index $i = 0$, 1, $n-1$, or n, and

$$1 < z(q) < n-2$$

for any other critical point q.

In fact, it is clear from the geometry that there is no obstruction to isotoping the critical points of index 0 and n, (minima and maxima) in the desired place with respect to height. To get their z-values between -1 and 0 and between $n-1$ and n, is a matter of rescaling z. This suffices already for surfaces ($n=2$) in $\mathbb{R}^3$. In case $n \geqq 3$ we can now as well assume that all critical points of index $i > 1$ are already at height $z > 1$. For a critical point of index one we consider its 1-manifold (arc) of steepest z-descent and can assume by transversality that both sides of this arc end under the level $z = 0$. We can then suitably z-descent this arc in $\mathbb{R}^N$, and isotope at the same time some tubular neighborhood in M, keeping fixed its boundary, *under* the height $z = 1$, without creating any self intersections of M. We can do this such that the function z on M remains nondegenerate during the process with a constant number of critical points. This follows a classical procedure already found in Morse's work [21]. Finally we can bring the critical points of index $n-1$ in place, by repeating the analogous procedure for the function $-z$ and its critical points of index 1.

The spaces M_t, M_t^{int} and M_t^{ext}, are manifolds with boundary for $t = i = 0$, 1, $n-2$, and $n-1$. For $t=0$ these manifolds are topologically disjoint unions of μ_0, μ_0^{int} and μ_0^{ext} discs respectively. For $t=1$ they are connected one-handle bodies, homotopy-equivalent to a wedge of $1+\mu_1-\mu_0$, $1+\mu_1^{\text{int}}-\mu_0^{\text{int}}$, $1+\mu_1^{\text{ext}}-\mu_0^{\text{ext}}$ circles respectively. These circles represent a complete set of generators for the

[1] This isotopy is helpful but not necessary for the calculations and proof, as Ozawa observed

fundamental groups $\pi_1(M)$, $\pi_1(M^{\text{int}})$ and $\pi_1(M^{\text{ext}})$ respectively (!). Thus by the definition of the first homotopy excesses σ_1^{int} and σ_1^{ext}, we now find, in terms of the $\mathbb{Z}_2$-Betti numbers β_i, β_i^{int} and β_i^{ext} of M, M^{int} and M^{ext} respectively

$$\begin{aligned} \mu_1^{\text{int}} - \mu_0^{\text{int}} &\geqq \beta_1^{\text{int}} - \beta_0^{\text{int}} + \sigma_1^{\text{int}}, \qquad \beta_0^{\text{int}} = 1, \\ \mu_1^{\text{ext}} - \mu_0^{\text{ext}} &\geqq \beta_1^{\text{ext}} - \beta_0^{\text{ext}} + \sigma_1^{\text{ext}}, \qquad \beta_0^{\text{ext}} = 1. \end{aligned} \tag{2.8}$$

By addition we get with (2.4, 5, 6)

$$\mu_1 - \mu_0 \geqq \beta_1 - \beta_0 + \sigma_1, \qquad \sigma_1 = \sigma_1^{\text{int}} + \sigma_1^{\text{ext}},$$

and, as $\mu_0 \geqq 1 = \beta_0$,

$$\mu_1 \geqq \beta_1 + \sigma_1. \tag{2.9}$$

Next by one of the Morse inequalities for z on M:

$$\begin{aligned} \mu_2 - \mu_1 + \mu_0 &\geqq \beta_2 - \beta_1 + \beta_0, \\ \mu_2 &\geqq \beta_2 + [(\mu_1 - \mu_0) - (\beta_1 - \beta_0)] \geqq \beta_2 + \sigma_1. \end{aligned} \tag{2.10}$$

The same arguments for the function $-z$ give with (2.5):

$$\mu_{n-1} \geqq \beta_{n-1} + \sigma_1, \tag{2.9'}$$

and

$$\mu_{n-2} \geqq \beta_{n-2} + \sigma_1. \tag{2.10'}$$

Then for $\mu = \sum \mu_i$, $\beta = \sum \beta_i$ we find from (2.9, 9', 10, 10') for $n \geqq 5$:

$$\mu \geqq \beta + 4\sigma_1. \tag{2.11}$$

With the Morse equality $\sum(-1)^i \mu_i = \sum(-1)^i \beta_i$ we obtain for n even:

$$\mu \geqq \beta + 4\sigma_1. \tag{2.11}$$

Equality in (2.11) yields after examination:

$$\begin{aligned} &\text{For } n=2: \quad \mu_0 = \mu_2 = 1 + \sigma_1, \quad \mu_1 = \beta_1 + 2\sigma_1 \\ &\text{For } n \geqq 4: \quad \mu_0 = \mu_1 = 1 (!), \qquad \mu_i = \beta_i + \sigma_1 \quad \text{for } i = 1, 2, n-2, n-1 \\ &\qquad \text{and} \quad \mu_j = \beta_j \text{ for the remaining values of } j. \end{aligned} \tag{2.12}$$

But for $n = 3$ we can only conclude to

$$\mu \geqq \beta + 2\sigma_1.$$

In this dimension Wintgen's theorem [30] is better and we reproduce his argument for completeness. We concentrate on M^{int}, but can do the same for M^{ext}. As before we have for the manifold with boundary M^{int}:

$$\mu_1^{\text{int}} \geqq \beta_1^{\text{int}} + \sigma_1^{\text{int}}. \tag{2.8}$$

With the Morse inequality, true for our space M^{int}, we obtain also

$$\mu_2^{\text{int}} \geqq \beta_2^{\text{int}} + \sigma_1^{\text{int}}. \qquad \text{see (2.10)}$$

The sum gives:

$$\mu^{\text{int}} = \mu^{\text{int}}(z) \geqq \beta^{\text{int}} + 2\sigma_1^{\text{int}}.$$

The same holds for the function $-z$:

$$\mu^{\text{int}}(-z) \geqq \beta^{\text{int}} + 2\sigma_1^{\text{int}}.$$

But the critical points of z on M^{int} and those of $-z$ on M^{int} are *different critical points on* M. Therefore the sum of their numbers is

$$\mu \geqq 2\beta^{\text{int}} + 4\sigma_1^{\text{int}}. \tag{2.13}$$

For the field $\mathbb{Z}_2$ we apply duality of Poincaré (P) and Alexander (A):

$$\begin{aligned} H_i(M) &\overset{(P)}{=} H^i(M) \overset{(A)}{=} H_{n-i}(\mathbb{R}^{n+1} \setminus M) \\ &= H_{n-i}(M^{\text{int}}) \oplus H_{n-i}(M^{\text{ext}}) \\ &= H_{n-i}(M^{\text{int}}) \oplus H_{n-i}(\mathbb{R}^{n+1} \setminus M^{\text{int}}) \\ &\overset{(A)}{=} H_{n-i}(M^{\text{int}}) \oplus H^i(M^{\text{int}}) = H_{n-1}(M^{\text{int}}) \oplus H_i(M^{\text{int}}). \end{aligned}$$

Hence $\beta_i(M) = \beta_i(M^{\text{int}}) + \beta_{n-i}(M^{\text{int}})$ and by summation over i

$$\beta(M) = 2\beta(M^{\text{int}}),$$

$$\beta = 2\beta^{\text{int}}.$$

Substitution in (2.13) yields

$$\mu \geqq \beta + 4\sigma_1^{\text{int}},$$

and

$$\mu \geqq \beta + 4\sigma_1^{\text{ext}}.$$

We obtain Wintgen's theorem [30] of which also Suzanne Fornari [8] had the case $n=2$:

$$\mu \geqq \beta + 4\max(\sigma_1^{\text{int}}, \sigma_1^{\text{ext}}) \qquad \text{for } n+1 = N \geqq 3. \tag{2.14}$$

As the conclusions (2.11, 12, 14) hold for almost any oriented line L, Eq. (2.1) gives the principal part of Theorem 1 for hypersurfaces. The last part concerning connected sums of embeddings of hypersurfaces is proved in Sect. 3d.

Apart from Corollary 1d we also mention an observation of H. Morton [22]:

Corollary 1e. *If* $f\colon M \to \mathbb{R}^3$ *is an embedding of a closed surface and* $\tau(f) < \beta + 8$, *then at least one among* M^{int} *and* M^{ext} *is a one-handle body.*

Indeed, if not, then $\sigma_1^{\text{int}} \geqq 1$ and $\sigma_1^{\text{ext}} \geqq 1$, and $\tau(f) \geqq \beta + 8$. Compare Figs. 3b and 4a.

Problem 1. Find a closed three dimensional hypersurface $f\colon M^3 \to \mathbb{R}^4$, for which

$$\tau < \beta + 4(\sigma_1^{\text{int}} + \sigma_1^{\text{ext}}),$$

or improve Wintgen's result.

Here is an *algebraic variant* for which Wintgen's result can *not* be improved. Let V be an algebraic Veronese surface ($\mathbb{R}P^2$) imbedded in $\mathbb{R}^4$ and V_r the 3-manifold bounding the ε-tube around V. Then, we get in homology with coefficients Q (instead of $\mathbb{Z}_2$): $\tau=6$, $\beta=2$, $\sigma_1^{\text{int}}=\sigma_1^{\text{ext}}=1$, and so $\tau=\beta+4\max(\sigma_1^{\text{int}}, \sigma_1^{\text{ext}})<\beta+4(\sigma_1^{\text{int}}+\sigma_1^{\text{ext}})$.

Observe that the inequalities (2.11) can be sharpened sometimes since they remain valid with homology in coefficients in any field, not only Z_2.

b) Submanifolds of codimension $N-n=2$. Let f: $M^n\to\mathbb{R}^N$, $N=n+2$ be a smooth closed submanifold of codimension two. For small $\varepsilon>0$ we have the smooth hypersurface f_ε: $M_\varepsilon\to\mathbb{R}^N$, boundary of the ε-neighborhood

$$N_\varepsilon=\{q\in\mathbb{R}^N\colon \text{distance}(q, M)\leqq\varepsilon\}.$$

The complement $M_\varepsilon^c=M_\varepsilon^{\text{int}}\cup M_\varepsilon^{\text{ext}}$ consists of the interior, homotopy equivalent to M, and the exterior, homotopy-equivalent to the exterior M^c. Take any oriented line L for which $\pi_L\circ f$ is nondegenerate, call it vertical, and define the coordinate z as before. Every critical point p of index i for z on M gives rise to exactly one critical point of index $i+1$ for z on $M_\varepsilon^{\text{ext}}$, just at a distance ε above p. All critical points for z on $M_\varepsilon^{\text{ext}}$ are so obtained, exept for the unique minimum and maximum on ∂B_R. Recall that by Alexander duality

$$\beta_1(M_\varepsilon^{\text{ext}})=\beta_1(M^{\text{ext}})=\beta_1(\mathbb{R}^{n+2}\setminus M^n)\overset{(A)}{=}\beta_n(M)=1.$$

The definition of $\sigma_1=\sigma_1(f)$ now gives:

$$\mu_0=\mu_0(z|M)=\mu_1(z|M_\varepsilon^{\text{ext}})\geqq\beta_1(M_\varepsilon^{\text{ext}})+\sigma_1=1+\sigma_1;$$
$$\mu_0\geqq 1+\sigma_1.$$

By the Morse inequality for z on M we have:

$$\mu_1-\mu_0>\beta_1-\beta_0, \quad \text{hence} \quad \mu_1\geqq\beta_1+\sigma_1.$$

Applied to the function $-z$ we find

$$\mu_n\geqq 1+\sigma_1$$
$$\mu_{n-1}\geqq\beta_{n-1}+\sigma_1.$$

Therefore with $\mu=\sum\mu_i$, $\beta=\sum\beta_i$ we obtain

$$\mu\geqq\beta+4\sigma_1 \quad \text{for } n\geqq 2, \quad \text{and}$$
$$\mu\geqq\beta+2\sigma_1 \quad \text{for } n=1$$

as required for Theorem 1. But see the Corollaries 1b, c in the introduction! For codimension $N-n>2$, $\sigma_1(f)=0$ and there is nothing to prove.

c) On the infimum of τ *in an isotopy class.* If μ_i is the number of critical points of index i of a nondegenerate function ϕ: $M\to\mathbb{R}$, then $\sum(-1)^i\mu_i=\sum(-1)^i\beta_i$

$=\chi(M)$, so that $\mu \geqq \beta$ is of the form

$$\mu = \beta + 2m, \quad m \geqq 0.$$

We now formulate

Theorem 5a. *If* $f\colon M \to \mathbb{R}^N$ *is an embedding and*

$$\min_L \mu_L = \beta + 2m, \quad \varepsilon > 0,$$

then there exists an embedding g isotopic to f, with

$$\tau(g) \leqq \beta + 2m + \varepsilon.$$

Idea of the proof (see Wilson [29] for details). Let z be a Euclidean coordinate corresponding to a line L for which the minimum $\beta+2m$ is obtained. The height function z has $\beta+2m$ nondegenerate critical points on M. Multiply the z-coordinate by $\lambda \geqq 1$ and keep the other orthogonal coordinates fixed. The curvature $\tau(f_\lambda)$ of the new embedding f_λ converges to $\beta+2m$ for $\lambda \to \infty$.

As a consequence we find for the infinum of $\tau(g)$ for g in the isotopy class $[f]$ of f, the

Infimum-formula 5b.

$$\tau([f]) \overset{\text{def}}{=} \inf_{g \in [f]} \tau(g) = \min_{g \in [f]} \min_L \mu_L(\pi_L \circ g) = \beta + 2m$$

for some integer $m=0$ *or* $m \geqq 2$, $m=1$ being excluded by Theorem 1. Following Milnor [19], $\tau([f])$ is called the *crookedness* of f.

In Sect. 4 we will see for surfaces in $\mathbb{R}^3$ that if the infimum is obtained by an (isotopy tight) embedding g, then $\tau(g)=\beta+4k$ for some integer $k \geqq 0$. Hence in that case $m=2k$ is even.

Problem 2. Find a knotted surface $f\colon M \to \mathbb{R}^3$ for which the crookedness is:

$$\tau([f]) = \inf_{g \in [f]} \tau(g) = \beta + 6.$$

Recall that for no $g \in [f]$ one can have equality $\tau(g)=\beta+6$.

d) Examples. For the trefoil knot $f\colon S^1 \to \mathbb{R}^3$, $\tau([f])=4$, $\sigma_1=1$. For its tubular neighborhood boundary (Fig. 3b)

$$f_\varepsilon\colon T_\varepsilon \to \mathbb{R}^3, \quad \tau([f]) = \beta + 4 = 8.$$

In Fig. 4b we suggest a trefoil arc in $\mathbb{R}^3 = \{(x_1, x_2, x_3)\}$ inside the half space $x_3 \geqq 0$ and ending orthogonally at the plane $x_3=0$. We embed $\mathbb{R}^3$ as $\{(x_1, x_2, x_3, 0, \ldots, 0)\}$ in $\mathbb{R}^{n+2}$. By rotation of the trefoil arc about $x_3 = x_4 = - - = x_{n+2} = 0$ we find the *spun trefoil n-knot* f_n with $\sigma_1 = 1$, and $\tau(f_n) \geqq \beta + 4 = 6$. Observe that the function x_3 has 6 critical points on $f_n(S^n)$, so that the *crookedness* is $\tau([f]) = 6$.

Fig. 4. **a** $\tau > \beta + 8 = 16$ both sides non standard. **b** spun trefoil knot in $\mathbb{R}^{n+2}$

For completeness we now give a quick survey on another result of Wintgen (†1981) [30]:

e) Wintgen's curvature formula for hypersurfaces with boundary. Let f: $(M^n, \partial M) \to \mathbb{R}^{n+1}$, $n \geqq 2$, be a smooth hypersurface M with smooth boundary ∂M. The boundary is a submanifold of codimension 2 so that by Theorem 1

$$\tau(\partial M) \geqq \beta(\partial M) + \sigma_1(\partial M)^c. \tag{2.15}$$

For sufficiently small $\varepsilon > 0$, the *boundary* M_ε of the ε-neighborhood

$$N_\varepsilon = \{p \in \mathbb{R}^{n+1} : \text{distance}(p, M) \leqq \varepsilon\}$$

is a C^1-manifold, consisting of two parts with smooth interior, one parallel at distance ε to M, *and* another part with smooth interior which is "half" of the ε-neighborhood boundary of ∂M. Therefore we find by examining π_L for almost all L,

$$\tau(M_\varepsilon) = 2\tau(M \setminus \partial M) + \tau(\partial M).$$

By Theorem 1, we get with $\pi_1(M_\varepsilon^{\text{ext}}) = \pi_1(M^c)$, for the hypersurface M:

$$\tau(M_\varepsilon) \geqq \beta(M_\varepsilon) + 4\sigma_1(M^c).$$

Therefore we obtain Wintgen's formula

$$2\tau(M \setminus \partial M) + \tau(\partial M) \geqq \beta(M_\varepsilon) + 4\sigma_1(M^c) \tag{2.16}$$

where M_ε is homeomorphic to the double of M. In particular for a surface in $\mathbb{R}^3$ bounded by circles

$$2\int_M \frac{|K d\sigma|}{2\pi} + \int_{\partial M} \frac{|\rho ds|}{\pi} \geqq 2\beta(M) + 4\sigma_1(M^c). \tag{2.16a}$$

Here $\beta(M_\varepsilon) = 2\beta(M)$ is the sum of the $\mathbb{Z}_2$-Betti numbers of M_ε, the double of M. (2.15) and (2.16) can be combined to give other inequalities, like those of Langevin and Rosenberg [15] for surfaces.

Exercise. Suppose f: $M^n \to \mathbb{R}^{n+1}$ is for $n \geqq 4$ an isotopy tight hypersurface with total absolute curvature $\tau(f) = \beta + 4\sigma_1$. Then the set of the points of M where all principal curvatures k_i are different from zero and have the same sign, equals the same set for the boundary of the convex hull $\partial \mathscr{H} M \subset \mathbb{R}^{n+1}$. M has

Banchoff's two piece property (=0-tight) that is every hyperplane divides M in at most two pieces. See 2.12 for the proof.

Problem 3. Is there a *knotted* hypersurface in $\mathbb{R}^{n+1}$ for $n \geqq 3$ with $\tau(f)=\beta+4$?

3. Examples of isotopy tight surfaces in $\mathbb{R}^3$

a) The key example (with non-standard interior) is knotted. In Fig. 2 of the introduction we gave an example of a relatively isotopy tight embedding f of a surface $M=M_g$ of genus $g=3$ with total curvature $\tau(f)=\beta+4=12$. In order to prove that f is isotopy tight it suffices to show that $\sigma_1(M^{\text{int}}) \geqq 1$ so that M is knotted in $\mathbb{R}^3$. General references for knots are [9, 23–25].

Proof. We first construct a presentation of the fundamental group $\pi_1(M^{\text{int}})$ with four generators and one relation.

We isotope the key surface M in the position of Fig. 5a. While doing so we do preserve the two convex bodies B_0, convex hull of the inside torus and B_1, convex hull of M. We mainly isotope the connecting pieces of surface of non-positive curvature. The new body M^{int} is obtained from B_1 by boring the suggested holes. Next, we consider the compact set M^{int} as the union of two rather simple parts E and F as far as homotopy type is concerned:

$$F=[(B_1 \setminus \text{Interior } B_0) \cap M^{\text{int}}] \cup [r, s]$$

$$E=\text{closure}(\text{Int } B_0 \cap M^{\text{int}})$$

$E \cap F$ is a union of two horizontal annuli connected by the straight segment $[r, s]$.

As base point for fundamental groups we take a point p on the segment $[r, s]$. The fundamental group $\pi_1(E \cap F)$ is freely generated by the suggested arcs a and g beginning and ending at p. The fundamental group $\pi_1(F)$ is generated by a, g and t with the relation, read from right to left for composition,

$$g=tat^{-1}. \tag{3.1}$$

To study $\pi_1(E)$ we isotope E in the placement of Fig. 5b where we immediately see that it is freely generated by k, l, m and n in the horizontal plane $z=z(p)$ of p. Then we "unknot" E while isotoping the circles in that plane, and we introduce two more convenient generators N and L for $\pi_1(E)$. By *van Kampen's* theorem, as $E \cap F$ is connected, we get for $\pi_1(E \cup F)$ a complete set of generators $a, t, g, k, l, m, n, L, N$ and a complete set of relations, (3.1) together with the following relations:

$$\begin{aligned} &g=nm, \quad l=m^{-1}Lm, \quad n=LNL^{-1}, \\ &kN=1, \quad a=LNmk. \end{aligned} \tag{3.2}$$

Elimination of g yields $ta^{-1}t^{-1}nm=1$. Elimination of a, k, n, l yields *one relation*

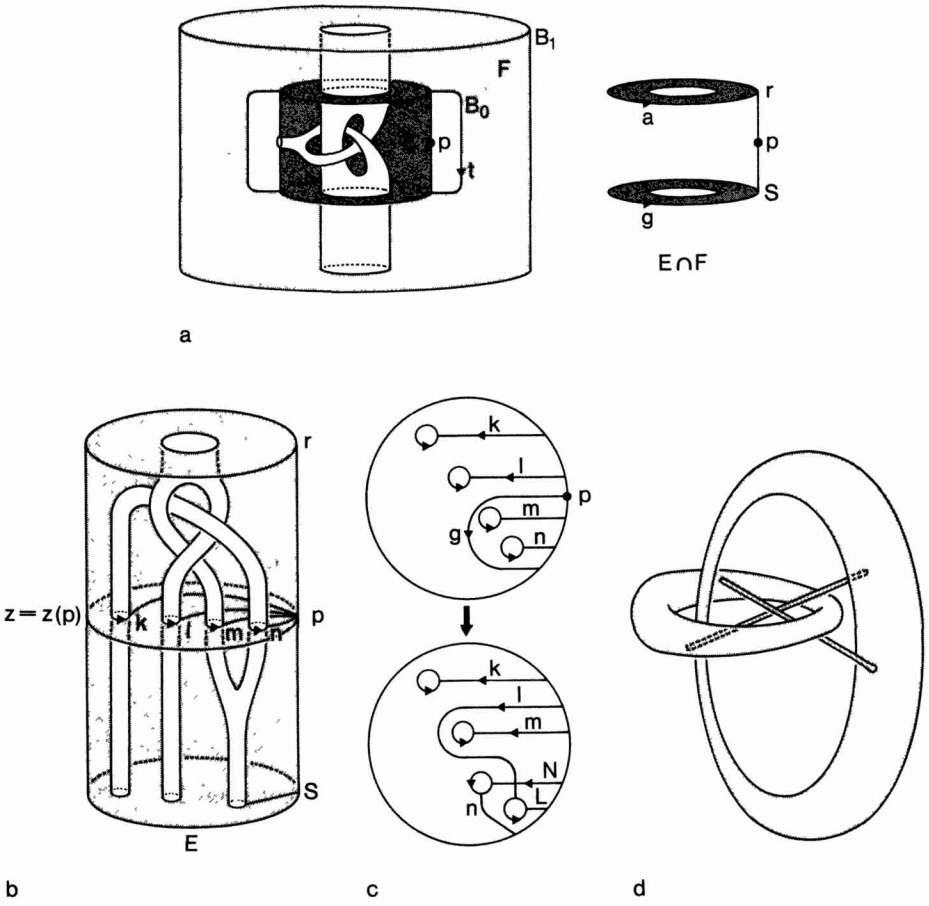


Fig. 5a–d. New placement for M and $M^{\text{int}} = E \cup F$

$$\varphi \equiv tNm^{-1}N^{-1}L^{-1}t^{-1}LNL^{-1}m = 1, \tag{3.3}$$

for four generators t, N, m, L of $\pi_1(M^{\text{int}})$.

Next we apply the *free calculus* of Fox [7, 2] to φ: We calculate the derivatives $\dfrac{\partial\varphi}{\partial t}$ etc., and apply the abeleanization homomorphism to the resulting elements of the group ring $\mathbb{Z}(t, t^{-1}, N, N^{-1}, m, m^{-1}, L, L^{-1})$, multiplying the final results by suitable units (like m) in order to simplify.

$$\frac{\partial\varphi}{\partial t} = 1 - tNm^{-1}N^{-1}L^{-1}t^{-1} \overset{\text{Abel.}}{=} 1 - m^{-1}L^{-1}; \quad m - L^{-1}.$$

$$\frac{\partial\varphi}{\partial m} = -tNm^{-1} + tNm^{-1}N^{-1}L^{-1}t^{-1}LNL^{-1} \overset{\text{Abel.}}{=} (-t + L^{-1})Nm^{-1}; \quad t - L^{-1}.$$

$$\frac{\partial\varphi}{\partial N}=t-tNm^{-1}N^{-1}+tNm^{-1}N^{-1}L^{-1}t^{-1}L \overset{\text{Abel.}}{=} t-tm^{-1}+m^{-1}; \quad mt-t+1.$$

$$\frac{\partial\varphi}{\partial L}=-tNm^{-1}N^{-1}L^{-1}+tNm^{-1}N^{-1}L^{-1}t^{-1}-tNm^{-1}N^{-1}L^{-1}t^{-1}LNL^{-1}$$

$$\overset{\text{Abel.}}{=} m^{-1}L^{-1}[-t+1-N]; \quad N+t-1.$$

The "Alexander" ideal generated by

$$m-L^{-1}, \quad t-L^{-1}, \quad mt-t+1 \quad \text{and} \quad N+t-1$$

(which contains t^2-t-1 and $N-t^2$), is an invariant of the group $\pi_1(M^{\text{int}})$ and so is the quotient ring

$$\begin{aligned} Q&=\mathbb{Z}[t,t^{-1},N,N^{-1},m,m^{-1},L,L^{-1}]/\text{ideal}\left(\frac{\partial\varphi}{\partial t},\frac{\partial\varphi}{\partial m},\frac{\partial\varphi}{\partial N},\frac{\partial\varphi}{L}\right)\\ &=\mathbb{Z}[t,t^{-1}]/\text{ideal}(t^2-t+1)\\ Q&=\mathbb{Z}[t]/\text{ideal}(t^2-t+1). \end{aligned} \tag{3.4}$$

This proves that $\pi_1(M^{\text{int}})$ is not a free group and M^{int} is not a one-handle body, as for a free group the Alexander ideal is zero.

b) A polynomial equation for the key examples. With the theory in [14] we can obtain a polynomial $\mathscr{E}(x_1, x_2, x_3)$ such that one component of the set $\mathscr{E}^{-1}=0$ is an isotopy tight surface $f'(M)\subset\mathbb{R}^3$ of genus $g=3$, isotopic and near to the example $f(M)$ just discussed, and with total curvature $\tau(f')=12$.

c) Another example with non standard EXTERIOR is obtained from the key-surface by isotoping it inside $S^3=\mathbb{R}^3\cup\infty$, while passing once over the point ∞. An isotopy tight smooth surface so obtained is suggested in Fig. 5d. Observe that the convex bodies B'_0 and B'_1 are convex hulls of two linking tight tori.

d) Connected sums of (hyper)surfaces. Suppose M_1 and M_2 are two closed n-manifolds in $\mathbb{R}^{n+1}$. After an isotopy, we can assume the following special properties, expressed with the help of euclidean coordinates $x_1,\ldots,x_n$, z for $\mathbb{R}^{n+1}$: the manifold M_1 (resp. M_2) meets the half space $z\leqq 0$ (resp. $z\geqq 0$) as well as the unit ball

$$\sum^{n} x_i^2\leqq 1,$$

exactly in the lower (resp. higher) hemisphere

$$\sum_1^n x_i^2+z^2=1, \quad z\leqq 0 \text{ (resp. } z\geqq 0).$$

We now define the connected sum $M=M_1\# M_2$ up to isotopy as the hyper-surface obtained by deleting the two hemisphere interiors from $M_1\cup M_2$. (The

fact that M is only C^2 at the connection $M_1 \cap M_2$ is not essential. We just make M smooth again by a small isotopy. It is clear that M^{int} has the homotopy type of $M_1^{\text{int}} \vee M_2^{\text{int}}$ the space obtained from the union M_1^{int} and M_2^{int} by identification of one point in M_1^{int} with one point in M_2^{int}. Then $\pi_1(M^{\text{int}})$ is the free product of $\pi_1(M_1^{\text{int}})$ and $\pi_1(M_2^{\text{int}})$ and by Grusko's theorem we have for the homotopy excesses:

$$\sigma_1(M^{\text{int}}) = \sigma_1(M_1^{\text{int}}) + \sigma_1(M_2^{\text{int}}).$$

Analogously,

$$\sigma_1(M^{\text{ext}}) = \sigma_1(M_1^{\text{ext}}) + \sigma_1(M_2^{\text{ext}}).$$

By addition we obtain the

Last part of Theorem 1 *for hypersurfaces*: *If* M *is a connected sum of hypersurfaces* M_1 *and* M_2 *then*

$$\sigma_1(M^c) = \sigma_1(M_1^c) + \sigma_1(M_2^c).$$

Let us return to the key example $f\colon M_3 \to \mathbb{R}^3$ of Sect. 3a. For the connected sum $f^{\#m}$ of m copies of this surface we conclude to a first homotopy excess

$$\sigma_1(f^{\#m}) = m.$$

The surface has genus $g = 3m$ so its total curvature is

$$\tau(f^{\#m}) \geqq \beta + 4m = (2+2g) + 4m = 2 + 10m.$$

Replace in the key example Fig. 2 the ball B_1 by a cube. Place m such cubes, with their borings between the two horizontal faces, in a row. Their union is a convex body with borings. Then after some smoothing we obtain an embedding (up to isotopy)

$$f^{\#m}\colon\ M_{3m} \to \mathbb{R}^3,$$

for which

$$(M_{3m})_{K>0} = \left(\bigcup_{i=0}^{m} \partial B_i\right)_{K>0}$$

for $m+1$-convex bodies $B_0, \ldots, B_m$, and with total curvature

$$\tau(f^{\#m}) = \beta + 4m = 2 + 10m. \tag{3.5}$$

Therefore this embedding is isotopy tight with a high curvature excess namely $4m$.

Exercises. Find an isotopy tight embedding $f\colon M_g \to \mathbb{R}^3$ for $g = 6$, with $\tau(f) = \beta + 8$, where M_g^{int} and M_g^{ext} are both non-standard. Find an isotopy tight embedding

$$f\colon M_g \to \mathbb{R}^3 \quad \text{for any } g > 3 \quad \text{with} \quad \tau(f) = \beta + 4.$$

e) A curious manifold with boundary in $\mathbb{R}^3$. Let $\gamma = \gamma(W)$ be the minimal number of critical points a nondegenerate function can have on a manifold

with (or without) boundary. Then for any embedding f: $W \to \mathbb{R}^N$ we find

$$\tau(f) \geqq \min_L \mu_L(\pi_L \circ f) \geqq \gamma \geqq \beta.$$

Observe that $\gamma = \beta$ for all compact surfaces with or without boundary. We have the

Theorem. *There exists an embedding f of a 3-manifold W with boundary in $\mathbb{R}^3$ with minimal total curvature (but not tight)*:

$$\tau(f) = \gamma(W) > \beta(W).$$

Proof. Take the closure $W = M^{\text{int}}$ of the interior of the key example surface of Sect. 3a. It has curvature

$$\tau(M^{\text{int}}) = \gamma(M^{\text{int}}) = 6 = \beta(M^{\text{int}}) + 2 = 4 + 2.$$

f) A curious hypersurface of minimal τ but not tight. Let M be the smooth (key) isotopy tight surface of genus $g = 3$ in $\mathbb{R}^3$ of Sect. 3a. Consider the composition of imbeddings $M^{\text{int}} \subset \mathbb{R}^3 \subset \mathbb{R}^{n+1}$.

Denote by $M_\varepsilon^{\text{int}} \subset \mathbb{R}^{n+1}$ the ε-neighborhood of M^{int} in $\mathbb{R}^{n+1}$ for $\varepsilon > 0$ small. It has as boundary the closed n-manifold $W^n = \partial M_\varepsilon^{\text{int}}$ which is however only C^1 at some points. Looking at the function μ_L for $W^n \subset \mathbb{R}^{n+1}$ and for $M \subset \mathbb{R}^{n+1}$, we see (!) that for almost all L, $\mu_L(W^n) = \mu_L(M)$. Therefore $\tau(W^n) = \tau(M) = 12$. Note that

$$\beta(W^n) = 2\beta(W^{n\,\text{int}}) = 2\beta(M^{\text{int}}) = \beta(M) = 2 + 2g = 8.$$

Note also that *W has the two piece property.* Next assume $n \geqq 4$.

As $\sigma_1(W^{n\,\text{int}}) = 1$, any imbedding f: $W^n \to \mathbb{R}^{n+1}$ isotopic to $W^n \subset \mathbb{R}^{n+1}$ has total curvature

$$\tau|f| \geqq \beta(W^n) + 4 \cdot \sigma_1 \geqq 12.$$

For $n \equiv 3$ Wintgen's theorem says the same $\tau(f) \geqq \beta(W^n) + 4$. $\sigma_1^{\text{int}} \geqq 12$. So we conclude

Theorem 4a. *The imbedding $W^n \subset \mathbb{R}^{n+1}$ is isotopy tight for $n \geqq 3$.*

For $n \geqq 4$ we can say more because in that case any non degenerate function φ: $W^n \to \mathbb{R}$ has in μ_1, μ_2, and μ_{n-1} and μ_{n-2} one extra contribution at least, due to $\pi_1(W^n)$. This homotopy group is not free (and has more generators than $H_1(W^n)$), because it contains the non free subgroup

$$\pi_1(M^{\text{int}}) \xrightarrow{\text{inject}} \pi_1(W^n) \xrightarrow[\text{back}]{\text{project}} \pi_1(M^{\text{int}}).$$

identity

Here the *injection* $M^{\text{int}} \to W^n$ is into some suitable topset of W^n namely with respect to a coordinate x_i for $\mathbb{R}^{n+1}$, $i \geqq 4$; the *projection back* is orthogonal projection into $\mathbb{R}^3$:

$$x_4 = \ldots = x_{n+1} = 0.$$

Therefore $\mu(\varphi) \geqq \beta(W^n)+4 \geqq 12$, hence $\tau(f) \geqq 12$ for any imbedding f: $W^n \to \mathbb{R}^{n+1}$ and we have

Theorem 4b. *The imbedding $W^n \subset \mathbb{R}^{n+1}$ has minimal total curvature $\tau = \beta(W^n)+4 = 12 > \beta(W^n) = 8$ but is not tight for $n \geqq 4$. It has the two piece property.*

This gives an answer to problem 1 in [13], but not to problem 3 of our Sect. 2, as W^n can not be considered knotted.

4. Relatively isotopy tight surfaces in $\mathbb{R}^3$

Proof of the fundamental Lemma 2

An attractive definition would seem to be the following:

Definition B. The imbedding (immersion) f: $M \to \mathbb{R}^3$ of the closed surface into $\mathbb{R}^3$ is called relatively isotopy tight in case

$$\tau(f) = \mathscr{E}_L \mu_L = \max_L \mu_L = \min_L \mu_L, \qquad \mu_L = \mu_L(\pi_L \circ f) \tag{4.1}$$

and

$$\tau(g) \geqq \tau(f) \quad \text{for all } g \in \mathscr{N}, \tag{4.2}$$

where $\mathscr{N}$ is some neighborhood of f in the space of imbeddings $\mathrm{Imb}(M, \mathbb{R}^3)$ with C^1-topology (or immersions $\mathrm{Imm}(M, \mathbb{R}^3)$ respectively). For $\mathscr{N}$ the connected component of f we get the definition of *isotopy tight* mentioned before.

For *practical reasons* (namely the proof of the fundamental lemma) we prefer however another

Definition A. We call f *relatively isotopy tight* in case (4.1) holds and

$$\min_L \mu_L(\pi_L \circ g) \geqq \tau(f) \quad \text{for all } g \in N \tag{4.3}$$

as before $\mathscr{N}$ being some neighborhood of f.

Clearly, given (4.1), then condition (4.3) of Definition A implies (4.2) of Definition B, but the opposite implication is not obvious. Observe however that if f fulfills (4.1) but not (4.3) for any $\mathscr{N}$, then there is an imbedding g, as near to f as we wish, and a line L for which

$$\mu_L(\pi_L \circ g) < \tau(f)$$

and with an affine transformation as in Sect. 2c, we get an isotopic imbedding g_λ for which $\tau(g_\lambda) < \tau(f)$.

If we take for $\mathscr{N}$ the whole component of f then the Definition A reduces to the same definition of isotopy tight as Definition B.

Let f: $M \to \mathbb{R}^3$ be a smooth *relatively isotopy tight imbedding* (by Definition A). For any oriented line L, $0 \in L \subset \mathbb{R}^3$, we denote by z^* the unit vector represented by the forward intersection of L with the unit sphere $S \subset \mathbb{R}^3$, and (as before) by z the (euclidean coordiante and) linear function whose gradient is z^*. The relation $\{(q, z^*)$: z^* is normal to M at $q \in M\} \subset M \times S$ is the two-valued

normal Gauss-map G^+: $M \to S$, whose *regular values* z^*, (for which $z \circ f$ is a nondegenerate function on M) form an open dense set $\Omega \subset S$. We will denote by G the usual oriented one valued Gauss map for oriented M and oriented $\mathbb{R}^3$.

Let $1+k_z$ be the *number of relative maxima* of the nondegenerate function $z \circ f$, $z^* \in \Omega$. The integer k_z is a constant on every component of Ω. Let $z_j^* \in \Omega$, $j=1, 2, 3, \ldots$ be a sequence of unit normal vectors converging to $z^* \in S$, for which the functions $z_j \circ f$ have relative maxima at points p_j converging to $p \in M$.

If the compact set $C_p^a = \{q \in M\colon z(q)=z(p)\}$ has at least k connected isolated compact components, then so has the set $\{q \in M\colon z'(q)=z'(p)\}$ for any linear function z' sufficiently near to z with non-degenerate restriction $z'|M$. Then $z'|M$ has at least k critical points and therefore k is finite. So we can *define* $C_p \subset C_p^a$ as the isolated connected component of p. We claim the following properties for (C_p, M):

Lemma 4.1. a) The compact set C_p is a *relative top set* for z: There exists a neighborhood $U(C_p) \subset M$ of C_p, such that

$$z(q) \leqq z(p) \quad \text{for } q \in U(C_p) \quad \text{and} \quad z(q) < z(p) \quad \text{for } q \in U(C_p) \setminus C_p.$$

b) The plane set C_p contains the boundary (∂) of its convex hull ($\mathscr{H}$):

$$\partial \mathscr{H} C_p \subset C_p.$$

c) C_p is obtained from its plane convex hull $\mathscr{H} C_p$ by deleting a finite number of disjoint interiors of plane convex sets in C_p.

Proof of a). We call our specific vector z^* *vertical*, we call z the height function, and we assume $p=0 \in \mathbb{R}^3$, $z(p)=0 \in \mathbb{R}$. The level planes $z=$constant are horizontal planes.

For any point $q \in M$ let the *slope* $\rho(q)$ of M at q be defined as the square of the length of the orthogonal projection of a unit normal vector at q, into a horizontal plane. The function ρ: $M \to [0, 1] \subset \mathbb{R}$ is a smooth function.

The set C_p is the union of a compact subset $C_p^0 = \{q \in C_p\colon \rho(q)=0\}$ and its complement $C_p \setminus C_p^0$, a *smooth* 1-manifold in the horizontal plane $z=0$, as some disc in M around any point $q \in C_p \setminus C_p^0$ meets the horizontal plane (and C_p) in one arc by transversality.

Take a small regular value $\delta > 0$ of the function ρ restricted to $C_p \setminus C_p^0$, and *define* C_p^δ as the component of p of the compact set $\{q \in C_p\colon \rho(q) \leqq \delta\}$. Clearly $C_p^\delta \subset C_p$, and $C_p^\delta \setminus C_p^0$ consists of a countable number of arcs each with at most one end q for which $\rho(q)=\delta$, and all other ends in C_p^0. There is only a finite number of ends with $\rho(q)=\delta$. If not, they would accumulate at some point q^0 with $z(q^0)=0$ and $\rho(q^0)=\delta$, near which C_p is of course just one arc. This excludes such accumulations.

Let $\varepsilon > 0$ be a small regular value of $|z \circ f|$: $M \to \mathbb{R}$. Then we have the surfaces with boundary

$$W_\varepsilon = \{q \in M\colon -\varepsilon \leqq z(q) \leqq \varepsilon\},$$

and

$$W^\delta = \{q \in M\colon \rho(q) \leqq \delta\}.$$

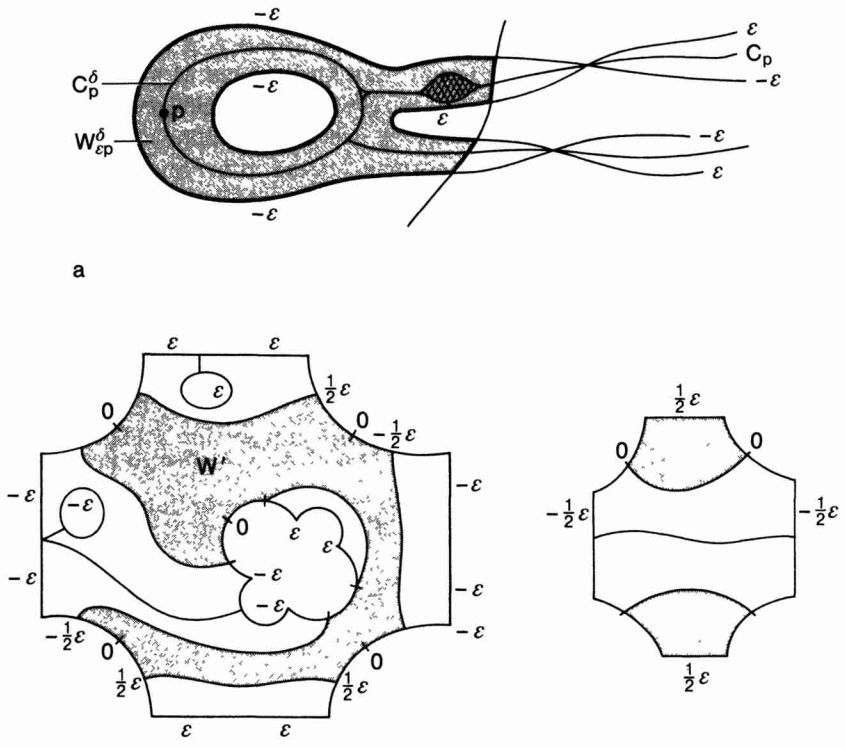


Fig. 6.

Let $W_{\varepsilon p}^\delta$ be the component of p of the intersection (see Fig. 6a)

$$W_\varepsilon^\delta = W_\varepsilon \cap W^\delta.$$

Observe that $C_p^\delta \subset W_{\varepsilon p}^\delta$, and $W_{\varepsilon p}^\delta$ converges to C_p^δ for $\varepsilon \to 0$. As δ was chosen to be a regular value of $\rho|(C_p \setminus C_p^0)$, the boundary of W^δ meets C_p transversally at the end points, finite in number, of C_p^δ and is a smooth curve near these points. A closed piecewise smooth imbedded curve in a surface will be called a *polygon.* It is called an n-gon if it has n vertices, the points where it is not smooth. For $n=0$ it is a smooth closed curve.

From our construction of $W = W_{\varepsilon p}^\delta$ it follows that the boundary ∂W for small ε is a union of polygons, each with 0, 4, 8 or etc. vertices: The sides of each polygon for $n>0$ are alternating, *either* smooth arcs at height $z=-\varepsilon$ or ε *or* arcs on which the slope $\rho=\delta$ is constant and the height z is monotone between $z=-\varepsilon$ and $z=\varepsilon$.

As C_p^0 is embedded in the horizontal plane $z=0$ we also have for small $\varepsilon>0$ that $W=W_{\varepsilon p}^{\delta}$ *is the graph of a function*

$$z=\varphi(x, y), \quad x, y, z \text{ euclidean coordinates,}$$

defined *over its orthogonal projection* $\pi(W)$ into the horizontal plane.

We now *assume* that C_p *is not a relative topset for* z, so that $W_{\varepsilon p}^{\delta}$ for small $\varepsilon>0$ does have points at level $z=\varepsilon$. We want to deduce a contradiction. By transversality we find for some $p_j \in W$ defined earlier, very near to p, hence z_j^* very near to z^*, and z_j very near to z, an analogously defined set $W_j=W_{\varepsilon p j}^{\delta}$ for the same values of ε and δ, very near to and isotopic to W in M. The surface W_j is the graph of a function $z_j=\varphi_j(x_j, y_j)$ over its orthogonal projection $\pi_j(W_j)$ into the plane $z_j=z_j(p_j)$.

The important difference between z and z_j is that $z_j \circ f$ is a non-degenerate function with $\tau(f)$ critical points, and in particular *one relative maximum at* $p_j \subset W_j$. We next proceed to modify the function φ_j in $\pi_j(W_j)$ but not near the boundary $\pi_j(\partial W_j)$, and obtain a new nondegenerate function ψ_j which has inside $\pi_j(W_j)$ only critical points of index 1 (saddle points). Then with the new isotopic imbedding $f'\colon M \to \mathbb{R}^3$ so obtained, which can be made near to f in the C^1-topology, we find that $z_j \circ f_j'\colon M \to \mathbb{R}$ has less critical points than $z_j \circ f$, which had $\tau(f)$. This contradicts our definition of relative isotopy tightness.

It remains to define ψ_j. *While doing so we suppress all indices* j for convencience. We must define ψ over the horizontal connected surface $\pi(W)$ whose boundary consists of polygons (n-gons, $n=0, 4, 8 \ldots$). The function ψ will be equal to φ near to the boundary. In other words we must redefine the surface M above $\pi(W)$ without changing it above some neighborhood of the boundary ∂W. (See Fig. 6b.) Choose on each n-gon, $n>0$, of $\pi(\partial W)$ one edge at which $\varphi=\psi=-\varepsilon$. Choose one point in the interior of this edge as well as one point on each 0-gon at which $\varphi=-\varepsilon$. Connect one of these points to each of the others by disjoint smooth arcs inside $\pi(W)$. Define the new surface between heights $-\varepsilon \leqq \psi(w) \leqq -\frac{1}{2}\varepsilon$ in such a way that the arcs lift to "bridges" each giving rise to one critical point of index 1 for the function ψ, such that no other critical points are created, and such that the surface meets the horizontal plane $z=-\frac{1}{2}\varepsilon$ transversely.

The boundary of the new surface so far reconstructed does contain points with value $z=\varepsilon$ by our assumption. These points constitute 0-gons (circles) or edges of at most one 4-m-gon for some $m \geqq 1$.

Next we define a further extension of the surface between heights $\frac{1}{2}\varepsilon \leqq \psi(w) \leqq \varepsilon$, with the same method, but with arcs and upside down bridges. The function ψ then obtained is again non-degenerate with critical points of index one only. Denote by $\pi(W')$ the closure of the open set in $\pi(W)$ where ψ is not yet defined. The boundary of $\pi(W')$ can be a pair of 0-gons where ψ has values $-\frac{1}{2}\varepsilon$ and $\frac{1}{2}\varepsilon$ respectively. Then $\pi(W')$ is a horizontal annulus and the function ψ can be completed smoothly without any new critical point. The only other possibility is that $\pi(W')$ is a topological disc bounded by a $4m$-gon on whose even sides ψ has values $-\frac{1}{2}\varepsilon$ and $\frac{1}{2}\varepsilon$ respectively, whereas ψ is monotone on the odd sides. In that case, we extend the surface for the values $-\frac{1}{2}\varepsilon \leqq \psi \leqq 0$ with "bridges" between edges where $\psi=-\frac{1}{2}\varepsilon$ leading to critical

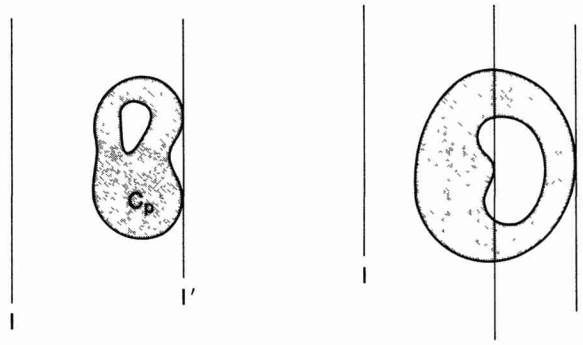


Fig. 7

points of index 1 only, and complete without any further critical points (last stage).

While taking ε and δ small enough, we can arrange that the new surface is isotopic to the original one, and as near in the C^1-topology as we please. This completes the construction of $\psi = \psi_j$ and f', as well as the proof of a).

Proof of b). The proof of b) and c) is by standard arguments for top-sets of tight surfaces in $\mathbb{R}^3$ as follows.

We can now assume a neighborhood $U(C_p)$, $C_p \subset U(C_p) \subset M$, with $z(q) < z(p)$ for $q \in U(C_p) \setminus C_p$. Suppose $\partial \mathscr{H} C_p \not\subset C_p$. Then two parallel lines l and l' exist in the plane $z=0$ of C_p such that l' is tangent in at least two components and C_p lies between l and l' (see Fig. 7). If we turn the horizontal plane over a small angle about l, we can get a plane with equation (say) $z' = c$ such that $U(C_p) \cap (z' \geqq c)$ is a connected set with its smooth boundary in $z' = 0$, and the function z' has at least two (relative) maxima. The connected set is the graph of that function z' over its orthogonal projection in the plane $z' = 0$. The function can be assumed non-degenerate. We can replace the height function z' (and the graph) by a nearby non-degenerate function with one maximum only. This again contradicts isotopy tightness.

Proof of c). As $\partial \mathscr{H} C_p \subset C_p$ then C_p is obtained from $\mathscr{H} C_p$ by deleting a finite number of simply connected open sets. If one of these V is not convex, then we can find again a line l in the plane $z=0$ disjoint from C_p such that the distance of points of C_p to l has at least two relative maxima. As before, we turn the horizontal plane slightly about l, and observe that we can decrease the number of relative maxima of a new linear function z'.

By Lemma 4.1, we can define for every $z^* \subset S$ the number $1 + k_z$ of connected relative topsets, either nondegenerate critical points for z or not. By the proof of b) and c) we see that if $z_1^* \in \Omega \subset S$ is a regular value, sufficiently near to $z^* \notin \Omega$ not regular, then $z_1 f$ has *at least and at most one* relative nondegenerate maximal value critical point near each degenerate or nondegenerate relative

topset of zf. Therefore

$$k_{z_1} = k_z \quad \text{is constant for } z \in S.$$

For each relative maximum of $z_1 f$ there is a relative minimum for $-z_1 f$. The function $z_1 f$ has therefore, for any regular value z_1^*, $1+k$ relative maxima, $1+k$ relative minima, and $(\beta-2)+2k$ critical points of index 1. Consequently

$$\tau(f) = \beta + 4k.$$

End of the proof of the fundamental lemma. Next we construct an abstract surface Y as follows. It is the union of

a) the closure $\overline{M}_{K>0}$ in M of the set

$$\{q \in M : K(q) > 0\} \subset M = f(M) \subset \mathbb{R}^3.$$

b) *together with the plane convex topset hulls* $\mathscr{H} C_p$ for all limit points $p \in \overline{M}_{K>0}$ (as studied in this section). The normal map ν: $\overline{M}_{K>0} \to S$ which sends every point q of $M_{K>0}$ into that unit normal vector z^*, for which zf attains a relative maximum at q, is constant on $\partial \mathscr{H} C_p$ for $p \in \overline{M}_{K>0}$. The map ν^+ can be extended to ν^+: $Y \to S$ by constant maps along each convex set $\mathscr{H} C_p$. It induces a locally one-to-one map between the set of equivalence classes (convex sets) $(Y/\sim) = Y/\{q \sim p$ if $q \in \mathscr{H} C_p\}$ and S. Each equivalence class is a contractible plane set and so we get a cellular map $Y \to S$ and a local homeomorphism $(Y/\sim) \to S$. Then Y is a covering space of S, and it consists of $1+k$ copies of S. Each of these copies is represented in $\mathbb{R}^3$ by a locally convex closed surface ∂B_i, which then is convex, $i = 0, \ldots, k$. *Then the fundamental lemma follows, and clearly*

$$M_{K>0} = \bigcup_{i=0}^{k} (\partial B_i)_{K>0}.$$

5. Relatively isotopy tight surfaces with $\tau = \beta + 4$

In this section f: $M \to \mathbb{R}^3$ will denote a relatively isotopy tight (RIT)-imbedding of a closed surface with total absolute curvature $\tau(f) = \beta + 4$. We will derive some elementary topological properties of f that will be essential in detecting all classes of such imbeddings for surfaces of genus $g \leqq 2$, and in proving that they must be unknotted. As before we frequently identify M with $f(M)$, and M itself will denote the RIT-surface in question. The fundamental lemma of Sect. 4 implies that M has two associated convex balls B_0 and B_1.

We now recall and fix notations for this section:

(1) M_i^+ is that (compact) component of $M \cap \partial B_i$, which contains the points of positive curvature on ∂B_i.

(2) The points of $M^+ = M_0^+ \cup M_1^+$ are called *top points.*

(3) The set of (open) components of $M^- = M \setminus M^+$ is $\{M_1^-, \ldots, M_l^-\}$.

(4) The component M_j^- is called *connecting* if it has ends bounded by top circles in ∂B_0 as well as in ∂B_1.

(5) $\Gamma_0=\{\alpha_1,\ldots,\alpha_{n_0}\}$ is the set of top circles of M in $M_0^+\subset\partial B_0$.
(6) $\Gamma_1=\{\beta_1,\ldots,\beta_{n_1}\}$ is the set of top circles of M in $M_1^+\subset\partial B_1$.
(7) $\Gamma=\Gamma_0\cup\Gamma_1$ is the collection of all top circles.

a) Decomposition of M into pieces. A preliminary classification for $g\leqq 2$. We start from the more general case of a closed orientable RIT-surface M of genus g with total curvature

$$\tau=\beta+4k=2+2g+4k,$$

and decomposition:

$$M=M^+\cup M^-=\left[\bigcup_{i=0}^{k} M_i^+\right]\cup\left[\bigcup_{j=1}^{l} M_j^-\right]. \tag{5.1}$$

M_j^- can be compactified to a piecewise differentiable immersed closed surface by filling each end with the convex hull of the corresponding top circle. Thus vice versa, the abstract open manifold M_j^- can be obtained from a closed orientable surface of (some) genus $g_j\geqq 0$, by deleting discs, say a_j discs. Then

$$\chi(M_j^-)=(2-2g_j)-a_j. \tag{5.2}$$

In view of the curvature condition ($K\leqq 0$) on M_j^- we have

$$a_j\geqq 2, \quad \text{and} \quad a=\sum_{j=1}^{l} a_j\geqq 2l. \tag{5.3}$$

We calculate the Euler characteristic of M (making for convenience the non-essential assumption that all top circles are mutually disjoint).

$$\begin{aligned}2-2g=\chi(M)&=\sum_{i=0}^{k}\chi(M_i^+)+\sum_{j=1}^{l}\chi(M_j^-)=[(k+1)2-a]+\left[\sum_{j=1}^{l}(2-2g_j-a_j)\right]\\&=2-2(a-l-k)-2\sum g_j.\end{aligned}$$

Hence

$$g=(a-l-k)+\sum g_j. \tag{5.4}$$

The surface M_j^- can be imbedded in a plane iff $g_j=0$. It is then called *planar*. If we make a new *abstract* surface M' by replacing each component M_j^- by a planar surface with the same ends then $g_j'=0$, and we get the inequality

$$g'=g(M')=(a-l-k)\geqq 0. \tag{5.5}$$

Exercise. For $k=0$ we now have formulas for tight surfaces and we read:

$$g=0\Rightarrow l=0; \quad g=1\Rightarrow l=1, \quad a=a_1=2, \quad g_1=0.$$

Study $g=2$.

We now return to our main interest $\tau=\beta+4$, $k=1$ and find with (5.3), (5.4) and (5.5)

$$g=(a-l-1)+\sum g_j\geqq a-l-1\geqq l-1. \tag{5.6}$$

We have

$$n_0+n_1=a. \tag{5.7}$$

Then we determine all solutions $\{g, l, a, \sum g_j, a_j, n_0, n_1\}$ with $n_0 \geqq n_1 \geqq 1$ of (5.3), (5.6) and (5.7) for $g \leqq 2$ and give them in Table 1.

Table 1. RIT-surfaces. A classification for $g \leqq 2$

g	l	$a+\sum g_j \sum g_j$	$a=a_1+a_2+a_3$	$\{n_0, n_1\}$	Realisations	
					Imm.	Imb.
0	1	2 0	2=2	1 1	A_1	
1	1	3 1	2=2	1 1	A_2	
		0	3=3	2 1		D
	2	4 0	4=2+2	3 1	A_3	
		0		2 2	B_1	
2	1	4 2	2=2	1 1	A_4	
		1	3=3	2 1		$E_1 F_1$
		0	4=4	3 1		E_2
				2 2		E_3, F_3, G, H
	2	5 1	4=2+2	3 1	A_5	
		1		2 2	B_2	
		0	5=3+2	4 1		E_4
				3 2	B_3	$E_5 F_5$
	3	6 0	6=2+2+2	5 1	A_6	
				4 2	B_4	
				3 3	A_7, C	

On the other subsections of 5

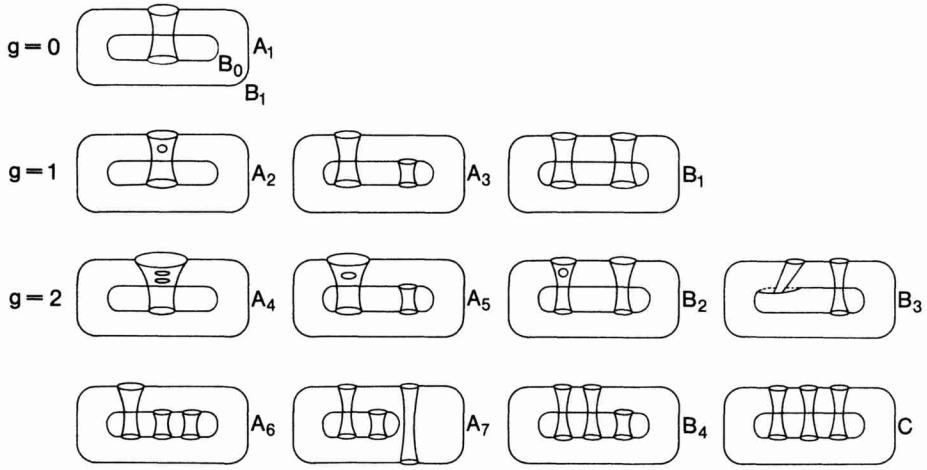
Many of the cases in Table 1 cannot be realized by RIT-imbeddings. We prove this for the classes A, B and C in Sect. 5b, as a consequence of general lemmas. For example, in all cases A there is exactly one connecting component M_1^-. It has exactly two boundary top circles, one on M_0^+ and one on M_1^+. Each of them disconnects M and then Lemma 5.2 gives the contradictions $B_0 \subset B_1$ and $B_1 \subset \mathring{B}_0$. There do exist however relatively isotopy tight immersions for A, B and C as seen in Fig. 8a.

We conclude in the remark after Lemma 5.3 that the RIT-torus D (Fig. 1) must be unknotted.

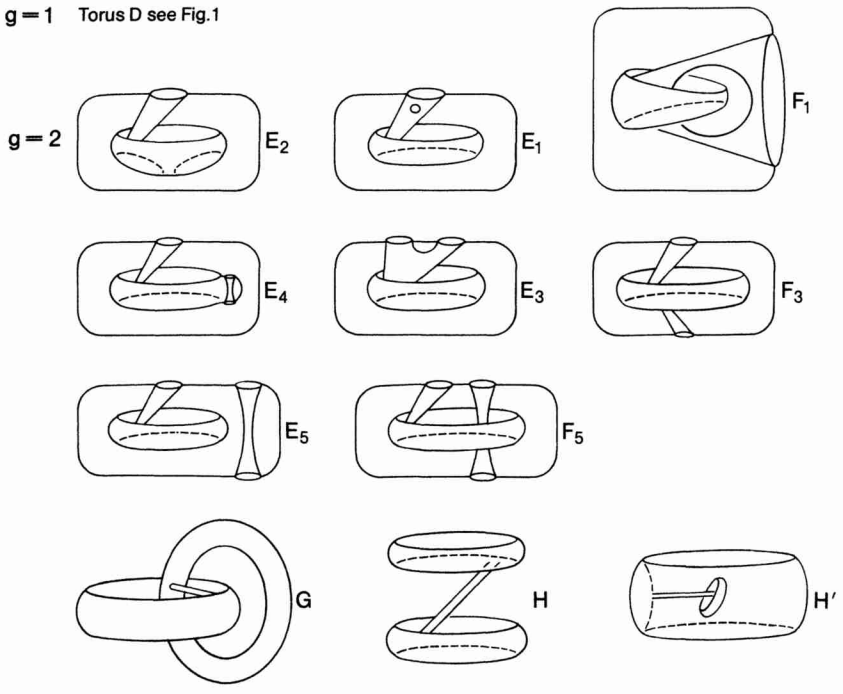
In Fig. 8b we present some RIT-imbeddings realizing the remaining cases of Table 1: E_i, F_i, G, H, and again by Lemma 5.3, E_1, F_1, E_2, E_4 are seen to be unknotted.

If $\{f_t: M \to \mathbb{R}^3, 0 \leqq t \leqq 1\}$ is a continuous family of RIT-imbeddings, then f_0 and f_1 are called RIT-*isotopic*.

In Sect. 5c, we define a weaker (?) equivalence relation, *weak* RIT-*isotopy*, and prove that the RIT-torus of Fig. 1 with $\tau = 8$ is unique up to weak RIT-isotopy. We use Meeks' uniqueness of multipants theorem in a convenient formulation.



a RIT-immersions



b RIT-imbeddings, classes E, F, G, H

Fig. 8. a RIT-immersions. **b** RIT-imbeddings, closses *E*, *F*, *G*, *H*. Torus *D* see Fig. 1

In Sect. 5d we deduce further general lemmas and conclude that all RIT-imbeddings for $g=2$ with $\tau=10$ are unknotted. This finishes the proof of Theorem 3.

Finally in Sect. 5e, we discuss weak RIT-isotopy for $g=2$. We indicate already in 5c why E, F, G, H are not equivalent and we make plausible that it is a complete classification (for classes) for $g=2$ up to weak RIT-isotopy.

b) General lemmas for imbeddings. In this subsection, where as before M is an imbedded RIT-surface with $\tau=\beta+4$, we can and will sometimes tacitly assume that $M\cap \mathrm{Int}\,\mathscr{H}(\gamma)$ is a transversal intersection for every top circle γ. It is then a union of imbedded circles.

Lemma 5.1. If M_j^- has exactly one boundary curve α in Γ_0 and all others in Γ_1, then $\mathscr{H}(\alpha)\cap M_1^+=\emptyset$. (And analogous for Γ_0 and Γ_1 interchanged.)

Remark. In particular, if each component of M^- is connecting and has exactly two boundary curves, then

$$\partial B_0\cap\partial B_1=\emptyset.$$

Proof. Suppose that $\mathscr{H}(\alpha)\cap M_1^+\neq\emptyset$. Choose α horizontal with equation $z=0$ and let p and q be points in the *interior* $\mathring{M}_1^+$ of $M_1^+\subset\partial B_1$, which have values of z that are maximal on M_1^+ and minimal respectively, or if not possible very nearly. Then, make a smooth arc σ_1 on $\mathring{M}_1^+$ connecting p to q and another arc σ_2 starting at p going vertically up, and ending by coming vertically up to connect with q. We can choose σ_2 so that $\sigma=\sigma_1\cup\sigma_2$ is a closed curve meeting $\mathscr{H}(\alpha)$ transversally in a single point. Fill in the boundary curves of M_j^- by plane discs to obtain a piecewise immersed closed surface N. By the convex hull property

$$N\subset\mathscr{H}(\partial M_j^-)\subset\mathscr{H}(\alpha\cup B_1).$$

We note that $N\cap\sigma$ is a transversal intersection consisting of *one point* in $\mathrm{Int}\,\mathscr{H}(\alpha)$ which is impossible by intersection theory. This contradiction implies $\mathscr{H}(\alpha)\cap M_1^+=\emptyset$.

Lemma 5.2. *If* $\beta\in\Gamma_1=\{\beta_1,\ldots,\beta_m\}$ *disconnects* M, *then*

(1) $$B_0\subset\mathring{B}_1,$$

(2) β *is a unique element in* Γ *which disconnects* M,

(3) $$m\neq 2.$$

Proof. Choose the coordinate z, called height, so that β is in the plane $z=0$ and B_1 below it. The curve β disconnects M into components X_0 and X_1 where we index so that $M_1^+\subset X_1$. Since $z|X_0$ has a minimum, X_0 contains a top point and then $M_0^+\subset X_0$. The surface $W=X_1\setminus M_1^+$ contains no top points and so by the convex hull property

$$W\subset\mathscr{H}(\partial W)\subset\mathscr{H}(\partial B_1)=B_1.$$

As all top points on X_1 are in $M_1^+\setminus\beta\subset\partial B_1$, the surface $N=X_1\cup\mathscr{H}(\beta)$ is a closed imbedded C^1 surface with no points above $z=0$.

Let R be the closure of that component of $B_1 \setminus N$ which contains $\mathscr{H}(\beta)$. Near β the surface X_0 *enters* $\mathring{R}$ and as $W \subset B_1$, it can only *escape* B_1 through Int $\mathscr{H}(\beta)$. So let $X_{01} \subset X_0$ be the component of $X_0 \cap R$ which has β at one end. The whole boundary of X_{01} is now contained in $\mathscr{H}(\beta)$. Since $z|X_{01}$ has a minimum at an interior point, that point is a top point, and $M_0^+ \cap \mathring{R}$ is non-empty. If M_0^+ is not contained in R, then $M_0^+ \cap \partial R = M_0^+ \cap \mathscr{H}(\beta) \neq \emptyset$ which is impossible by Lemma 5.1. Hence, $M_0^+ \subset \mathring{R} \subset B_1$ and

$$B_0 = \mathscr{H}(M_0^+) \subset \text{Int } \mathscr{H}(B_1) \subset \mathring{B}_1$$

and the first part of the lemma is proved.

Now suppose that there exists another element $\gamma \in \Gamma$ which also disconnects M. If $\gamma \in \Gamma_0$ then not only $B_0 \subset \mathring{B}_1$, but also $B_1 \subset \mathring{B}_0$ by part (1) with respect to γ, a contradiction.

If $\gamma \in \Gamma_1$, then define $Y_1 \supset M_1^+$ and $Y_0 \supset M_0^+$ for γ, as we defined X_1 and X_0 for β. Then clearly apart from

$$M_0^+ \subset Y_0,$$

also, as we observe from the geometrical situation:

$$M_0^+ \subset X_0 \subset X_0 \cup M_1^+ \subset Y_1,$$

and Y_0 and Y_1 cannot be separated by γ.

Finally part (3) holds, because if $m=2$, then both curves β_1 and β_2 in Γ_1 will disconnect M in contradiction with part (2).

Remark. Since the classes of Table 1, represented by examples A_i (see Fig. 8a), each have two disconnecting top circles, Lemma 5.2 implies that none of them can be realized by imbedded surfaces.

Lemma 5.3. *If $\beta \in \Gamma_1 = \{\beta_1, \ldots, \beta_m\}$ bounds a disk D on M, then*

(1) $$B_0 \subset \mathring{B}_1,$$

(2) *β is the unique curve in Γ which disconnects M,*

(3) $$m=1 \quad \text{and} \quad M_1^+ = D,$$

(4) *M is unknotted.*

Proof. As β bounds a disc $D \subset M$, it disconnects M, and by Lemma 5.2 we have (1) $B_0 \subset \mathring{B}_1$ and (2) β is the unique disconnecting top circle in Γ. The disc D is one of the parts, X_0 or X_1, of the proof of Lemma 5.2. So D contains either M_0^+ or M_1^+. If $M_0^+ \subset D$, then each (at least one!) top circle in $M_0^+ \subset D$ is also disconnecting and β is not unique, a contradiction to (2). If $M_1^+ \subset D$, then every top circle on M_1^+ is disconnecting, by uniqueness there cannot be others, and $m=1$. This takes care of parts (1), (2), (3).

Next, choose the height z, so that it takes its maximal value $z=0$ in all points of the top circle β. Compactify $\mathbb{R}^3$ by one point to obtain $S^3 = \mathbb{R}^3 \cup \infty$. Then isotope the disc $D = M_1^+ \subset \partial B_1$, outside $\mathring{B}_1$, over ∞, to its image by reflection in the plane $z=0$. With a small suitable perturbation of the resulting

surface M' (which will be C^1 along β), we obtain a smooth surface $M'' \subset \mathbb{R}^3 \subset S^3$, for which $z|M''$ is nondegenerate with only *one* (relative) minimum: Then M'' is unknotted in $\mathbb{R}^3$ by Meeks, Morton (see Corollary 1d), hence unknotted in S^3. By isotopy, M is also unknotted in S^3, hence in $\mathbb{R}^3$.

Remark 5.4.R. This applies to all cases in our table for which $n_1 = 1$. The imbeddings D, E_1, F_1, E_2, E_4 are therefore all unknotted.

Lemma 5.4. *Suppose $B_0 \subset B_1$. Let M_1^- be a connecting component of M^-, and X_0 a component of $M_1^- \cap B_0$ ($\neq M_1^-$ and $\neq \emptyset$) with a boundary curve $\alpha_1 \in \Gamma_0$, and a boundary curve $\gamma \subset \operatorname{Int} \mathscr{H}(\alpha_2)$ for some $\alpha_2 \in \Gamma_0$, where α_2 is a boundary curve of a component M_2^- of M^- different from M_1^-. Then M_2^- has (1) at least two boundary curves in Γ_1, and (2) at least two boundary curves in Γ_0 (see Fig. 9).*

Proof. Choose an arc $\{p(t): 0 < t < 2\}$ in M_1^- starting at a point $p(0)$ in a boundary circle $\beta_1 \subset M_1^+ \subset \partial B_1$, ending at a point $p(2) \in \alpha_1 \subset M_0^+$, and intersecting ∂B_0 transversally and in exactly one point $p(1)$ of $\gamma \subset \operatorname{Int} \mathscr{H}(\alpha_2)$. Take any vector v, denote the inner product $q \to \langle q, v \rangle$ by z, and call that coordinate

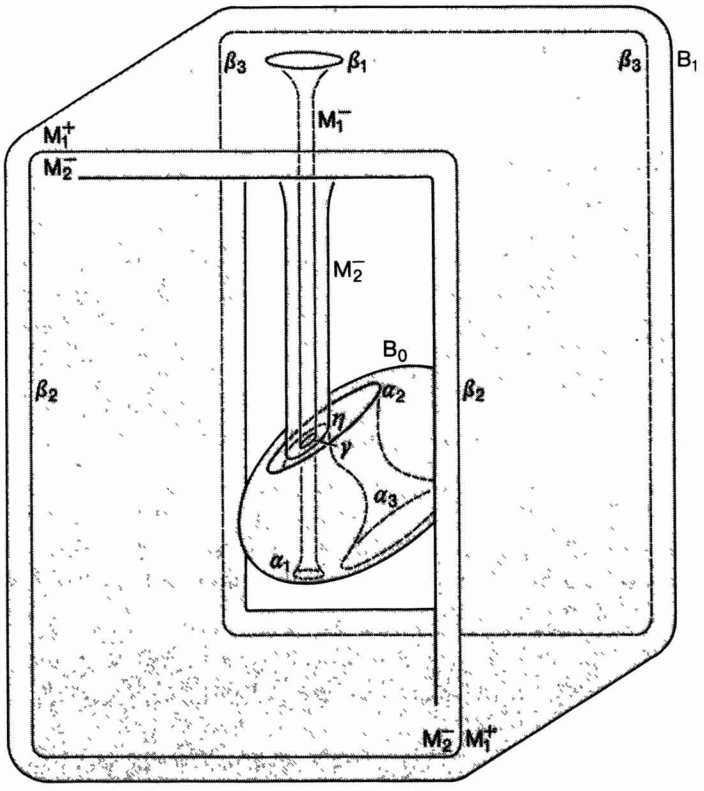


Fig. 9. $\partial M_1^- = \alpha_1 \cup \beta_1$, $\partial M_2^- = \alpha_2 \cup \alpha_3 \cup \beta_2 \cup \beta_3$

height. Let $p(3)$ be a highest point on M_0^+, and connect $p(2)$ to $p(3)$ by an arc $\{p(t)\colon 2\leqq t\leqq 3\}$ in M_0^+.

To get out of B_1 we continue with an ascending curve $\{p(t)\colon 3\leqq t\leqq 5\}$, which avoids M_2^- and meets ∂B_1 transversally in exactly one point $p(4)$. Finally, connect $p(5)$ to $p(6)=p(0)$ by an arc outside B_1 to get a closed piecewise smooth curve $\sigma_z=\{p(t)\colon 0\leqq t\leqq 6,\ p(6)=p(0)\}$.

Let ΔM_2^- be the collection of boundary curves of M_2^-. Then

$$N=M_2^-\bigcup_{\delta\in\Delta M_2^-}\mathscr{H}(\delta)$$

is a piecewise differentiable closed immersed surface and σ_z meets N transversally in $p(1)$ and at most in one more point namely $p(4)$. The even intersection number compels $p(4)$ to be such an intersection point and therefore $p(4)$ is contained in $\mathscr{H}(\beta_2)$ for some boundary curve $\beta_2\in\Gamma_1$ of M_2^-. To get a second one, we reconsider the construction, but for that linear function z which takes its minimal value on B_1 in all points of $\beta_2\subset\partial B_1$. Then we find a boundary curve $\beta_3\in\Gamma_1$ for M_2^- which by the construction cannot be β_2. So, M_2^- has at least the two boundary curves β_2 and β_3 in Γ_1.

We now prove that M_2^- also has at least two boundary curves in Γ_0 (one more, apart from α_2). We again use the closed curve σ_z constructed above. Let Y_0 be the closure of that component of $M_2^-\cap B_0$ with boundary curve α_2. For every $\alpha\in\Gamma_0$ denote by Δ_α the unique planar domain on $\mathscr{H}(\alpha)$ with boundary $Y_0\cap\mathscr{H}(\alpha)$. Then

$$N_2=Y_0\bigcup_{\alpha\in\Gamma_0}\Delta_\alpha$$

is a piecewise smooth closed imbedded surface. Since $\sigma_z\cap N_2\subset\sigma_z\cap\mathscr{H}(\alpha_2)=\{p(1)\}$ (*one* point), and it consists of an even number of points, $\sigma_z\cap N_2$ is empty. Hence $\Delta_{\alpha_2}\neq\mathscr{H}(\alpha_2)$ and so Y_0 must have a boundary curve η on $\mathscr{H}(\alpha_2)$ different from α_2. The convex hull property implies that Y_0 has another boundary curve $\delta\subset\mathscr{H}(\alpha')$ for some $\alpha'\neq\alpha_2$. Suppose α' is a boundary curve of some component C of M^- different from M_2^-. It is straightforward to construct a closed curve ρ on $M_2^-\cup M_1^+$ such that ρ is transverse to ∂B_0 and such that ρ intersects ∂B_0 in exactly two points $p_1\in\delta$ and $p_2\in\eta$. Let

$$\Delta C=\{\alpha\in\Gamma\,|\,\alpha\text{ is a boundary curve of }C\}$$

$$N_3=C\bigcup_{\alpha\in\Delta C}\mathscr{H}(\alpha).$$

Then ρ intersects N_3 transversely in the single point p_2 which is a contradiction. This contradiction implies that α' is a boundary curve of M_2^- which completes the proof of the lemma.

Corollary. *If $B_0\subset B_1$ and M_1^- is a connecting component with one boundary curve in Γ_0, then there exists a connecting component M_2^- with at least 2 boundary curves in Γ_0 and also in Γ_1.*

Proof. If $\alpha_1\in\Gamma_0$ is a boundary curve of M_1^-, then let X_0 be the component of $M_1^-\cap B_0$ with boundary curve α_1. Let $\gamma\subset\mathscr{H}(\alpha_2)$ be another boundary curve of

X_0. Then M_2^- being defined as in Lemma 5.4 gives the required component which proves the corollary.

Remark. We now analyse the last case C illustrated in Fig. 8a where each component of M^- is a connecting annulus and has exactly two boundary curves. Then by Lemma 5.1, we conclude

$$\partial B_0 \cap \partial B_1 = \emptyset.$$

Then either $B_0 \cap B_1 = \emptyset$ or $B_0 \subset B_1$ or $B_1 \subset B_0$. If $B_0 \subset B_1$ or $B_1 \subset B_0$, then the corollary to Lemma 5.4 prevents a realization by an imbedding. If $B_0 \cap B_1 = \emptyset$, then a simple application of the convex hull property proves that for any component M_1^- of M^-, the part $M_1^- \cap B_1$ is an annulus A_1. Fix the component M_1^- and let γ be the boundary curve of A_1 which is not a top circle of M. In this case $\gamma \subset \mathscr{H}(\alpha)$ for some top circle α of M. Let M_2^- be the component of M^- with boundary curve α. Then a linking argument as before proves that $A_2 = M_2^- \cap B_1$ must intersect A_1 and hence M cannot be realized as an imbedding.

Lemma 5.5. *If $\Gamma_1 = \{\beta_1, \beta_2\}$, then no connecting component of M^- is an annulus.*

Proof. Suppose $\Gamma_1 = \{\beta_1, \beta_2\}$, M_1^- is a connecting annulus with β_1 at one end, and M_2^- is the other connecting component with β_2 at one end. Lemma 5.1 implies that $M_0^+ \cap \partial B_1 = \emptyset$. If $M_0^+ \subset B_1$ then $B_0 \subset \mathring{B}_1$ and Lemma 5.4 obliges M_2^- to have at least two boundary curves in Γ_1, which is false.

Therefore we must have $M_0^+ \cap B_1 = \emptyset$. Then the annulus M_1^- must escape the ball B_1, and so must M_2^-. Escape is possible through $\mathrm{Int}\,\mathscr{H}(\beta_1)$ and $\mathrm{Int}\,\mathscr{H}(\beta_2)$.

If $M_1^- \cap \mathscr{H}(\beta_1)$ is non-empty, then there exists a curve γ (general position assumption) $\gamma \subset M_1^- \cap \mathscr{H}(\beta_1)$. The curve γ bounds a disc D on M_1^- or γ together with β_1 bounds a subannulus $A \subset M_1^-$. As in either case D or A is not contained in $\mathscr{H}(\beta_1)$, it cannot have non-positive curvature, a contradiction. Thus, $M_1^- \cap \mathscr{H}(\beta_1) = \emptyset$ and M_1^- leaves B_1 through $\mathscr{H}(\beta_2)$ and in fact $M_1^- \cap \mathscr{H}(\beta_2)$ consists of a single Jordan curve and no more. Elementary separation properties (see for example the proof of Lemma 5.2) show that M_2^- must also escape B_1 through $\mathscr{H}(\beta_2)$ and hence $M_2^- \cap B_1$ has its boundary in the plane disk $\mathscr{H}(\beta_2)$ which again shows $M_2^- \cap B_1$ would contain a top point in M_0^+ which is impossible. The lemma is proved.

Remark. Lemma 5.5 shows that none of the cases B can occur as imbedded surfaces. This completes the proof that the cases A, B, C in Table 1 cannot be realized by imbedded surfaces. For an imbedded RIT-torus with $\tau = 8$, only case D of Table 1 remains and it is unknotted by Remark 5.4. This takes care of Theorem 3a.

c) Uniqueness of multipants, and weak RIT-uniqueness of the RIT-torus with $\tau = 8$. An imbedded compact surface M with boundary $\partial M = \partial_- M \cup \partial_+ M$,

$$p\colon (M, \partial_- M, \partial_+ M) \to (V, \partial_- V, \partial_+ V), \quad a < b$$

for $V = \mathbb{R}^2 \times [a, b] \subset \mathbb{R}^3 = \{(x, y, z)\}$,

$$\partial_- V = \mathbb{R}^2 \times a, \quad \partial_+ V = \mathbb{R}^2 \times b, \quad a < b,$$

is called a *multipants, in case* $z|M$ is a nondegenerate (=Morse) function with no (resp. one) relative maxima, and $\partial_+ M$ is *one* imbedded circle (resp. empty).

Theorem 4 (Meeks [17]). *A multipants p is unique up to ambient isotopy relative to* ∂V.

In other words, for pants p and $\tilde{p}$, if M is homeomorphic to $\tilde{M}$ and $(\partial_- M, \partial_+ M, \partial V) = (\partial_- \tilde{M}, \partial_+ \tilde{M}, \partial V)$, then p is ambiently isotopic to p relative to ∂V.

Definition. *If* $\{f_t: M \to \mathbb{R}^3 : 0 \leqq t \leqq 1\}$ *is a continuous family of* C^∞*-RIT-imbeddings, then* f_0 *and* f_1 *are called RIT-isotopic.*

It seems hard to assert this equivalence-relation in many cases where we would immediately conjecture it. For example, we have:

Problem 4: Are any two smooth tight embedded tori in $\mathbb{R}^3$ RIT-isotopic?

We therefore introduce as follows an interesting weaker (that is not stronger) equivalence relation.

First, we call the C^∞-RIT-imbeddings f_j: $M \to \mathbb{R}^3$, $j=0,1$, *restricted* C^1*-isotopic* in case

(1) there is a continuous ambient isotopy of C^1-imbeddings f_t: $M \to \mathbb{R}^3$, $0 \leqq t \leqq 1$, connecting f_0 and f_1, such that

(2) there is also a continuous isotopy of convex C^1-imbedded surfaces

$$g_{it}: \partial B_i \to \mathbb{R}^3 \qquad i=0,1,\ldots,k,\ 0 \leqq t \leqq 1,$$

with closed subsets $N_i^+ \subset \partial B_i$, for which $g_{it}(\partial B_i \setminus N_i^+)$ is a finite union of open convex plane discs with $g_{ij}(N_i^+) = [f_j M]_i^+$ $j=0,1$ and

$$g_{it}(N_i^+) \subset f_t(M), \qquad 0 \leqq t \leqq 1.$$

Definition. *The equivalence relation generated by RIT-isotopy and restricted* C^1*-isotopy, between RIT-imbedded surfaces, is called weak RIT-isotopy. Clearly RIT-isotopy implies weak RIT-isotopy.*

Here are some weak RIT-isotopic perturbations which suggest the freedom this equivalence gives.

a) A given RIT-surface can be made flat at the points of a (convex) top circle $\gamma \subset M_i^+$. That is all derivatives of the distance to the topset plane $T_\gamma(M)$ vanish at γ.

b) Some neighborhood of γ *in* M_i^+ can be made to lie in the tangent plane $T_\gamma(M)$.

c) The convex part M_i^+ can be perturbed keeping its boundary with tangent planes fixed as long as it remains part of a convex surface.

d) If M_i^- is an annulus connecting top circles α and β, and the interior body is free of M, then we can shrink α and β to very small circles, and M_i^- to a thin tube around a straight line segment. A top circle γ can also be moved over ∂B_i, while perturbing ∂B_i coherently, as long as other parts of M are not in the way.

e) If M_j^- has three ends, then often we can arrange that two of its boundary circles are in one plane, surrounded by a plane part of M_i^+ with convex boundary.

f) Having reached the situation in e), we can obtain that the two circles are two triangles forming a square, with a diagonal as common edge, in a horizontal plane supporting B_i from below. We then can move the common edge up, and obtain a situation in which the square is one top circle. M_j^- has lost one end but its genus is increased by one. We do not elaborate this delicate construction here.

With these observations we can see the weak RIT-isotopy equivalences for several imbeddings suggested in Fig. 8:

$$E_3 \to E_1$$
$$E_2 \to E_1$$
$$E_4 \to E_2$$
$$E_5 \to E_3$$
$$F_3 \to F_1$$
$$F_5 \to F_3.$$

On the other hand, we conclude that no two among E_5, F_5, G and H are weak RIT-equivalent, as they are distinguished as follows.

$(E, F \neq G, H)$: In E_5 and F_5 an orientation on M will induce opposite orientations in the convex surfaces ∂B_0 and ∂B_1; in G and H they will induce the same orientations.

$E \neq F$: In E_5 only one top circle disc for a top circle in M_0^+ meets M^-. This is not so in F_5.

$G \neq H$: In G the parts M_0^+ and M_1^+ are linked, in H they are not linked.

So we have the

Conclusion 6A. *There are at least four different RIT-imbeddings of the surface of genus $g=2$ in $\mathbb{R}^3$ with $\tau=10$, different with respect to weak RIT-isotopy.*

We now formulate the main result of this section.

Theorem 5. *A RIT-imbedded torus with $\tau=8$ is unique up to weak RIT-isotopy.*

See Fig. 1 for the unique example. As we can unknot the torus of Fig. 1, this implies (again) unknottedness.

A useful tool is the following

Annulus Lemma 5.6. *Let $A \subset \mathbb{R}^3$ be a compact smooth imbedded annulus of non-positive Gauss curvature, whose boundary consists of two plane imbedded circles α and β. Then any plane P that separates α from β meets A transversally in a circle γ homotopic to γ so that γ divides A in two annuli.*

Proof. Suppose P is the horizontal plane $z=0$ and α is below and β above this plane. Suppose P is not transversal. Then, it is tangent to A, say at p: $(x, y, z) = (0, 0, 0)$.

If the Gauss curvature at p is $K \neq 0$, hence $K<0$, then the same is true in some small neighborhood U_p of p. Consider for small $\varepsilon \geqq 0$ an RIT-isotopy with support in U_p (that is points remain fixed outside U_p), sending p into $\tilde{p}=(0,0,\varepsilon)$, preserving $K<0$ in U_p, and with $T_{\tilde{p}}(\tilde{M})$ horizontal with equation

$$z=\varepsilon.$$

Choose ε moreover such that $z|A$ has ε as a regular value. Then the component of $\tilde{p}$ of the level curve $\{q \in \tilde{A}: z(q)=\varepsilon\}$ is a smooth curve C with one singularity, the double point $\tilde{p}$. But then C divides $\tilde{A}$ in three parts one of which has boundary $C \subset T_{\tilde{p}}(\tilde{A})$. In this part the surface $\tilde{A}$ must have a top point, hence it has a point of positive curvature a contradiction. We *conclude that* $K=0$ in the *open* set $U \subset A$:

$$U=\{q \in A: T_q(A) \text{ separates } \alpha \text{ and } \beta\}.$$

The Gauss normal map $G: U \to G(U) \subset S^2$ has derivative dG of rank $dG \leqq 1$. Let $U_0 \subset U$ be the set of points where this rank is 0 and $U_1 = U \setminus U_0$ the set of points where this rank is 1. As G is not constant, U_1 is not empty and $G(U_1)$ is an immersed curve of Hausdorff dimension one. By a refinement of the theorem of Sard and A.P. Morse [4] the Hausdorff dimension of $G(U_0)$ is zero. Therefore $G(U_0)$ can be neglected with respect to $G(U)$.

By elementary differential geometry we know that some neighborhood U_p of a point $p \in U_1$ is part of a developable surface. Each tangent plane $T_q(U_p)$ meets U_p in a straight segment. Let $q \in U_1$ be such that $G(q)$ is a regular value (as map into an immersed curve) of $G=U_1 \to G(U_1)$, that is $G(q) \notin G(U_0)$. The component of $q \in U$ of the set $G^{-1}G(q)$ is then a closed curve in the tangent plane $T_q(A)$, which is locally straight. This contradiction proves the annulus lemma.

Proof of Theorem 5. From Table 1 and Sect. 5b we know already that an RIT-imbedded torus with $\tau=8$ has the following properties (in the earlier notations):

$B_0 \subset \mathring{B}_1$, M^- is a connected planar domain with three ends in boundary circles α_1, $\alpha_2 \subset M_0^+$ and $\beta_1 \subset M_1^+$. We also know that M_1^+ is a compact 2-disc. We must still prove however that M_0^+ is a compact annulus with α_1 and α_2 disjoint from each other. This will be shown by proving that α_1 and α_2 are contained in different planes. In fact if α_1 and α_2 are in the same plane with equation $z=0$, and B_0 is under it, then the function $z|(M^- \cap \mathring{B}_0)$ has a minimum $z=c<0$, at some top point inside $\mathring{B}_0 \subset B_1$, which cannot lie in M_0^+ nor in M_1^+, a contradiction. (In short: there is an impossible top point.)

As M^- is attached from the inside of B_0 along α_1 and α_2 to M_0^+, it must have a nonempty intersection with $\mathscr{H}(\alpha_1) \cup \mathscr{H}(\alpha_2) \subset \partial B_0$ in order to connect with $M_1^+ \subset \partial B_1$. Suppose $M^- \cap \mathscr{H}(\alpha_2) \neq 0$, and let $z=0$ be the equation of the horizontal tangent plane $T_{\alpha_2}(M)$ and B_0 under this plane. We can assume for our arguments that $z|M$ is a nondegenerate function except for the points of α_2. Observe that a nondegenerate $z'|M$ must have $\tau(M)=8$ critical points, of which four are of index 1 (saddle points). The plane $z=0$ meets M in the (tangent) circle α_2, (and) in a circle outside α_2 that contains $M \cap M_1^+ \subset \partial B_1$, and

in r circles $\gamma_1, \ldots, \gamma_r$ inside α_2. But if $r>1$, we need too many saddle points above the plane $z=0$ and below the plane $z=0$ in order to connect all these components to build up M. So $r=1$:

$$M^- \cap \mathscr{H}(\alpha_2) = \gamma.$$

The curve γ does not bound a disc in M^-, because such a disc would contain impossible top points. As M^- is homeomorphic to a sphere from which three points are deleted, corresponding to α_1, α_2 and β, then there is an annulus A between γ and either α_2, α_1 or β. If $\partial A = \gamma \cup \alpha_2$, then ∂A lies in the plane $z=0$ and A must contain an impossible top point. If $\partial A = \gamma \cup \alpha_1$, then A must be inside B_0, as any part of A outside B_0 gives rise to impossible top points. But $M^- \setminus A$ is attached from the interior of B_0 along α_2 to M_0^+, and then it cannot get outside B_0, as it has to pass through $\mathscr{H}(\alpha_2) \setminus$(Interior of γ). Consequently, $\partial A = \gamma \cup \beta$. The annulus A cannot have any points inside B_0 and the remaining part $M^- \setminus A$ cannot have points outside B_0, as this would again lead to impossible top points.

After a projective transformation involving B_0 (and not necessarily all of M) we can assume that α_1 and α_2 are in horizontal planes $z=-1$ and $z=0$ respectively. Then the closure of $M^- \cap B_0$ is imbedded as a multipants (upside down) with horizontal boundaries α_1 in $z=-1$, and α_2, γ in $z=0$. So this part is unique by multipants uniqueness (in its very simplist form).

Observe now that γ in fact *must* be a transversal intersection of M^- and $z=0$, because we can assume this for intersections with $z=\varepsilon$ and $z=-\varepsilon$ near to γ, for regular values ε and $-\varepsilon$ of $z|M$, and then transversality along γ follows from the annulus lemma.

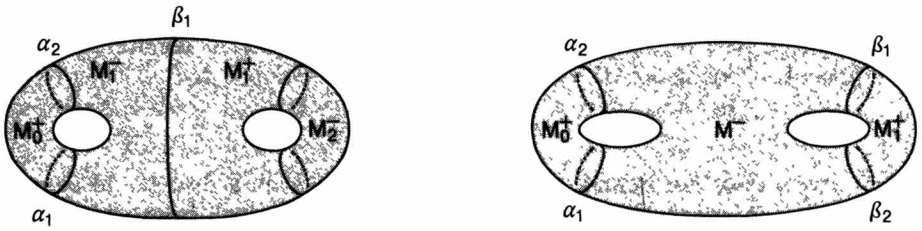
To study the part $A = M^- \setminus B_0$ we examine its placement inside $B_1 \setminus B_0$. Choose a straight segment $I \subset B_1 \setminus B_0$ which connects ∂B_0 to ∂B_1 but does not meet

$$A \subset \mathscr{H}(\gamma \cup \beta) \subset \mathscr{H}(\alpha_2 \cup \beta).$$

Choose a convex function $d\colon B_1 \setminus \mathring{B}_0 \to \mathbb{R}$ with $d|B_0 = 0$, $d|B_1 = 1$, and monotone on I. Then A is a one-pant with nonrelative d-maxima inside $V = B_1 \setminus (\mathring{B}_0 \cup I)$ with respect to the levels of d and the structure of an appropriate one-slab $V = \mathbb{R}^2 \times I$. Again Meeks' multipants uniqueness theorem applies.

To go from any RIT-torus M with $\tau=8$ to another one $\tilde{M}$, we can first treat the given RIT-torus M with $\tau=8$. We apply coherently weak RIT-isotopies to $M \cap B_0$ and $M \setminus B_0$ to obtain an RIT-torus for which M_0^+ is a standard round torus T, and M^- is then obtained from $T \setminus M_0^+$ by suitably attaching a thin straight tube that sticks out of B_0 and ends in a small top circle of ∂M_1^+. Finally we can give M_1^+ and M a suitable standard form. We can do the same for the other RIT-torus $\tilde{M}$ with $\tau=8$ and therefore obtain a weak RIT-isotopy from M to $\tilde{M}$. This completes the proof of Theorem 5.

d) The unknottedness theorem for genus two. Throughout this section $f\colon M \to \mathbb{R}^3$ will denote an isotopy tight surface of genus two with $\tau(f)=10$. This case is a special case of the situation considered in Sect. 5b where f was an RIT-surface of genus g and $\tau(f)=\beta+4$. We continue to use the notation of Sect. 5b.



a $\lambda=2$ b $\lambda=1$

Fig. 10

From Table 1 and the remarks given in Sect. 5b referring to the possibilities for an RIT-surface of genus 2 with $\tau=10$, we note that there are few possible configurations for the top circles of M. By Lemma 5.3 we know that if a top circle of M bounds a disc on M, then M is unknotted. Thus, if M is knotted, then the Table 1 shows that the top circles, if pairwise disjoint, appear as in Fig. 10a (for $l=2$) or as in Fig. 10b (for $l=1$) up to diffeomorphism.

Our analysis of the topology of $M\subset\mathbb{R}^3$ will depend on the possible configurations of the top curves Γ on M and possible positions of the associated convex balls B_0 and B_1. The next lemma is actually true for any RIT-surface $M_g\subset\mathbb{R}^3$ with $\tau(M_g)=\beta+4$.

Lemma 5.7. *If $B_0\cap B_1=\emptyset$, then M is unknotted.*

Proof. If $B_0\cap B_1=\emptyset$, then we may assume that

(1) B_0 lies below the plane P with equation $z=0$.

(2) B_1 lies above P.

(3) P is transverse to M.

(4) the height function $z|M$ is a Morse function.

In this case, consider $X^+=z^{-1}[0,\infty)$ and $X^-=z^{-1}(-\infty,0]$. The multipants uniqueness theorem in Sect. 5c implies that X^+ and X^- are standardly embedded up to ambient isotopy in their respective half spaces relative to the plane P. A simple induction argument on the number of curves in $P\cap M$ proves that whenever a plane cuts a compact surface N in $\mathbb{R}^3$ into two components each of which is "unknotted" in its half space, then the surface N is itself unknotted. This proves Lemma 5.7. (For the definition of a standardly embedded surface M see the related discussion given in the proof of Theorem 1 in [17].)

In our topological analysis we need a uniqueness theorem similar to the multipants uniqueness theorem. We shall call a surface $(M,\partial M_1,\partial M_2)\subset(V,\partial V_1,\partial V_2)$ a *special double pair of pants* if M is a planar domain with 4 boundary curves, $\partial M=\partial M_1\cup\partial M_2$ where ∂M_i contains two boundary curves of M, $V=\mathbb{R}\times[0,3]$, $\partial V_1=\mathbb{R}\times\{0\}$, $\partial V_2=\mathbb{R}\times\{3\}$, ∂M_2 is the boundary of an annulus in ∂V_2 and $z|\mathring{M}$ is a Morse function with critical points of index 1 only.

Lemma 5.8. *If* $(M, \partial M_1, \partial M_2) \subset (V, \partial V_1, \partial V_2)$ *is a special double pair of pants, then* M *is unique up to ambient isotopy relative to* ∂V.

Proof. The proof of this proposition is a rather simple application of the method of the proof of multipants uniqueness and we refer the reader to [17] for a more precise description of the constructions we make here. Without loss of generality in the proof we may assume ∂M_2 to be the union of two concentric circles γ_1, γ_2 in $\partial V_2 = \mathbb{R} \times \{3\}$ and ∂M_1 to be the union of circles δ_1, δ_2 in the plane $\partial V_1 = \mathbb{R} \times \{0\}$.

Note that there are exactly two critical points p and q of $z|\mathring{M}$ of index 1 and we assume that $z(q)=1$ and $z(p)=2$. If we denote $\xi_t = z^{-1}(t)$, then $\xi_t \cap M$ are plane 1-manifolds, for $1 \leqq t \leqq 3$, $t \neq 1, 2$, a pair of Jordan curves $\delta_1(t)$, $\delta_2(t)$ for $0 \leqq t < 1$ and another pair $\gamma_1(t)$, $\gamma_2(t)$ for $2 < t \leqq 3$.

There are basically three cases to consider.

Case A. In this case the critical point p occurs as points $p_1(t) \in \gamma_1(t)$ and $p_2(t) \in \gamma_2(t)$ approach each other as t decreases to $t=2$. Passing $t=2$ we get a single curve $\eta(t)$ defined for $1 < t < 2$. For $0 \leqq t < 1$ the curves $\delta_1(t)$ and $\delta_2(t)$ are forced to come together to form $\eta(t)$ near $t=1$. From this simple description we deduce that M is isotopic to a surface as described in Fig. 11 A.

Case B. In this case points $p_1(t)$, $p_2(t) \in \gamma_2(t)$ approach each other as t decreases to $t=2$ and we arrive at a surface isotopic to Fig. 11B.

Case C. In this remaining case points $p_1(t)$, $p_2(t) \in \gamma_1(t)$ approach each other as t decreases to $t=2$ and we arrive at a surface isotopic to Fig. 11C.

It is finally straightforward to check that the isotopy classes of the surfaces described in Fig. 11 depend only on the configuration of the curves in $\partial M_1 \subset \partial V_1$, namely whether they bound pairwise disjoint disc in ∂V_1 or not. This observation proves the lemma.

Remark. None of the hypotheses of Lemma 5.8 can be left out. For example, if γ_1 and γ_2 are not "concentric" there is an infinite number of possibilities for isotopy classes of M. Also if $z|M$ is allowed to have either a local maximum or minimum there are again an infinite number of possible isotopy classes for M. If M is not a planar domain, the lemma fails similarly.

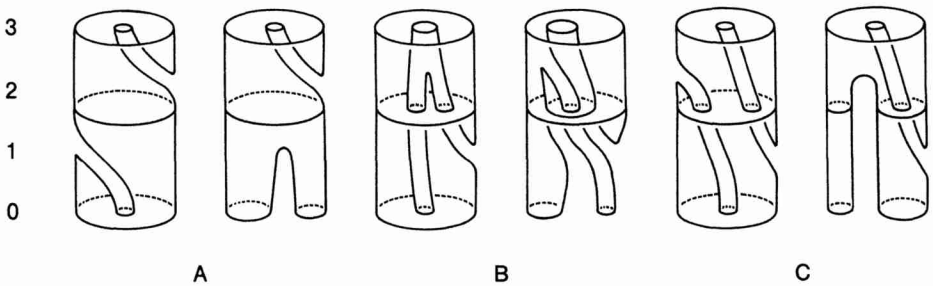


Fig. 11

Lemma 5.9. *If* $B_0 \subset \mathring{B}_1$ *(or* $B_1 \subset \mathring{B}_0$*), then* M *is unknotted.*

Proof. Suppose that M is knotted and $B_0 \subset B_1$. From the discussion at the beginning of this section either Fig. 10a or b applies and by Lemma 5.2 we conclude that $\Gamma_0 = \{\alpha_1, \alpha_2\}$. Define $E = \mathscr{H}(\alpha_1)$ and $F = \mathscr{H}(\alpha_2)$ and assume that M is transverse to $\mathring{E} \cup \mathring{F}$.

Assertion 1. *Let* $B_\infty = (B_0)^c \cup \{\infty\} \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. *Then* $W = M \cap B_\infty$ *is unknotted in* B_∞.

Proof of Assertion 1. Let $R = B_1 \setminus \mathring{B}_0$ and choose a convex function d: $R \to \mathbb{R}$ with $\partial B_0 = d^{-1}(0)$ and $\partial B_1 = d^{-1}(1)$ and such that $d|(W \setminus \partial B_1)$ is a Morse function. Since d is convex, $d|(W \setminus \partial B_1)$ has no relative maxima. After a small change of d near ∂B_1 we get a convex function d' defined on a neighborhood of R in B_0^c such that d' is a Morse function with a unique local maximum. At this point one can almost apply the multipants uniqueness theorem in Sect. 5c except that the region V that appears here is diffeomorphic to $S^2 \times [0,1]$. The multipants uniqueness theorem implies that there are two possibilities for W in B_0^c depending on which side of W contains the point at infinity. Thus in B_∞ there is no ambiguity for W which proves the assertion.

We now assume that E and F are in distinct planes which are parallel to the xy-plane (the reader can easily modify the following arguments in case E and F are not contained in parallel planes). Consider the stretching transformation L_b: $(x, y, z) \to (x, y, bz)$ applied to M for large values of b. After stretching, we may assume that M is almost a vertical cylinder over $X = M \cap (\mathring{E} \cup \mathring{F})$ near X (For the reader who is not familiar with the construction of a standardly embedded surface, it is perhaps a good time to refer to Fig. 2 in [17] and the corresponding discussion in the proof of Theorem 1 in [17]).

If $M \cap \mathring{E}$ consists of more than one curve, then since $M \cap B_\infty$ is unknotted, there exists a disc D in B_∞ such that

(1) $\partial D = D \cap (E \cup \dot{M})$

(2) ∂D intersects exactly two curves in X and intersects each of these curves transversely in exactly one point.

Once D is found, a small strip neighborhood S of $\partial D \cap M$ in M can be pushed down along D in a small neighborhood of D in $\mathbb{R}^3$ until S lies below E as in Fig. 12. (See also the proof of Theorem 1 in [17] for a more explicit

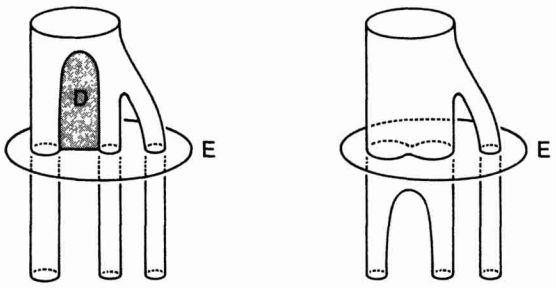


Fig. 12

description of this type of isotopy.) This isotopy of M has the property that the newly obtained surface M' is unknotted in B_∞ and furthermore $z|M' \cap B_0$ has no relative maxima or minima except along the curves α_1 and α_2. Thus, one by one we can carry out isotopies of M to reduce the number of intersection curves in $M \cap E$ and similarly in $M \cap F$ until the resulting surface $\overline{M}$ intersects each of E and F in at most one curve. Furthermore, the final surface $\overline{M}$ has the property that $\overline{M} \cap B_\infty$ is unknotted in B_∞ and $z|(\overline{M} \cap \mathring{B}_0)$ has no interior local maxima or minima.

Let K be the closure of the surface $\overline{M} \cap \mathring{B}_0$. If K has only one boundary curve in $\mathring{E} \cup \mathring{F}$, then multipants uniqueness implies that K is unknotted in B_0. If K has one boundary curve in $\mathring{E}$ and one boundary curve in $\mathring{F}$, then one checks that K is a planar domain and Lemma 5.8 shows that K is unknotted in B_0. Since K is unknotted, $\overline{M} \cap B_0$ is unknotted (i.e. standard up to diffeomorphism of B_0). Since $\overline{M} \cap B$ is also unknotted we conclude that $\overline{M}$ is unknotted in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ which implies that $\overline{M}$ is unknotted in $\mathbb{R}^3$. Since M is isotopic to $\overline{M}$, M is also unknotted, which proves the lemma.

Corollary. *If M is knotted, then Γ_0 and Γ_1 each contain exactly two curves (see Fig. 10).*

Proof. Lemma 5.2 states that if M contains a top circle that disconnects, then either $B_0 \subset \mathring{B}_1$ or $B_1 \subset \mathring{B}_0$. The corollary now follows from Lemma 5.9.

By the above corollary and by general position we can assume that when M is knotted that Fig. 10b applies. In this case M_- is a planar domain with boundary $\partial M_- = \bigcup \Gamma$. As usual we assume that M is transverse to the interior of the convex hull of each top circle.

Lemma 5.10. *If M_0^+ and M_1^+ are unlinked, then M is unknotted.*

Proof. Suppose that M_0^+ and M_1^+ are unlinked and M is knotted. In this case, Lemma 5.7 implies that $B_0 \cap B_1 \neq \emptyset$ and hence, after a possible change of index, that $M_0^+ \cap \mathring{B}_1 \neq \emptyset$ and $M_1^+ \cap B_0 = \emptyset$. Lemma 5.9 shows that $M_0^+ \cap \partial B_1 \neq \emptyset$ and so we will assume that $M_0^+ \cap \mathscr{H}(\beta_1) \neq \emptyset$.

Assertion 1. *The surface $W = M_- \cap B_0$ is a connected planar domain with boundary curves $\partial W = \{\gamma, \alpha_1, \alpha_2\}$. Furthermore, $B_0 \cap M$ is unknotted in B_0 (i.e. standard up to diffeomorphism).*

Proof of Assertion 1. First note that the convex hull property insures that no boundary curve of W bounds a disc on M_-. Let γ denote any boundary curve of W which is not a boundary curve of M_-. We will say that γ is parallel to a curve δ in M_- if γ and δ form the boundary of an annulus in M_-. If γ is parallel to α_1, then γ must be contained in $\mathscr{H}(\alpha_2)$. It follows that the component C of $M_- \cap B_0$ which has the boundary curve α_2 must contain a top point p which is the point of C furthest from the plane containing α_2. We then conclude that $p \in M_1^+ \cap B_0$ which contradicts the fact that $M_1^+ \cap B_0 = \emptyset$. Hence γ cannot be parallel to α_1 and similarly to α_2. Suppose γ is parallel to β_1 in M_- and let A denote the annulus with boundary consisting of γ and β_1. Our choice of β_1 was such that $M_0^+ \cap \mathscr{H}(\beta_1) \neq \emptyset$ but the proof of Lemma 5.1 applied to the annulus A shows that $M_0^+ \cap \mathscr{H}(\beta_1) = \emptyset$, and this contradiction proves that γ

cannot be parallel to β_1. If γ is parallel to β_2, then γ, α_1, α_2, and β_1 bound a connected surface $S \subset M_-$. The proof of Lemma 5.1 applied to S shows that $M_0^+ \cap \mathscr{H}(\beta_1) = \emptyset$ and this contradiction proves γ is not parallel to β_2. Thus, if γ does not disconnect M_- into two components N_1 and N_2 where α_1 and α_2 are part of the boundary of N_1 and β_1 and β_2 are part of the boundary of N_2, then one of the components, say N_1, of $M_- \setminus \gamma$ has β_1 as one boundary curve and its other boundary curves on ∂B_0. Again the proof of Lemma 5.1 shows M_- cannot exist. Hence we conclude that the curve γ disconnects M_- into components N_1 and N_2 where $\partial N_1 = \{\gamma, \alpha_1, \alpha_2\}$ and $\partial N_2 = \{\gamma, \beta_1, \beta_2\}$. By the convex hull property we note that $\mathring{N}_1$ is contained in $\mathring{B}_0$. Thus N_1 is the component of W with boundary containing α_1 and α_2. However since γ was arbitrary $N_1 = W$ which proves the first part of the Assertion 1. Multipants uniqueness applied to W completes the proof of the assertion.

Suppose that γ is contained in the plane P with equation $z=0$ and suppose that $z|M$ is a Morse function. Assertion 1 shows that after an ambient isotopy of M in a small neighborhood of B_0 which is the identity on $M \cap B_0^c$ we can move $M \cap B_0$ into a position so that for the new surface $\tilde{M}$, $z|\tilde{M}$ is a Morse function and $z|(\tilde{M} \setminus M \cap B_0^c)$ has only one critical point which is not of index 1. It follows that $z|\tilde{M}$ has at most 3 critical points of index different from 1 and therefore $\tilde{M}$ is unknotted. As M and $\tilde{M}$ are isotopic we conclude that M is also unknotted, which proves the lemma.

The only remaining case in our analysis is when M_0^+ and M_1^+ are linked. This we prove now, and conclude therefore by

Theorem 3 last part. *If $\tau(f) = 2g+6 = 10$ for a surface of genus 2, then f is unknotted.*

Proof. By Lemma 5.10 we need only to prove that M cannot exist when α_1 and β_1 are linked (see Fig. 13a). Let $D_1 = \mathscr{H}(\alpha_1)$, $D_2 = \mathscr{H}(\alpha_2)$ and let $D_3 = \mathscr{H}(\beta_1)$. As usual we assume M is transverse to D_1 and D_2. We will first prove Lemma 7.5 in the easier to visualize case where D_1 is contained in the xy-plane, D_2 contained in a different parallel plane and the points of B_0 lie below the xy-

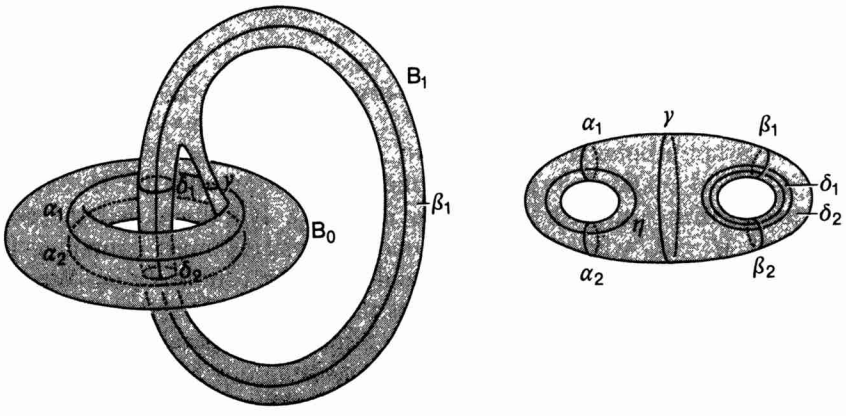


Fig. 13

plane. Let δ_1 be the curve in $M \cap D_1$ that contains $\beta_1 \cap D_1$, let δ_2 be the curve in $M \cap D_2$ that contains $\beta_1 \cap D_2$ and let η be the curve in $D_3 \cap M$ which contains $D_3 \cap \alpha_1$. From the simple topology of surfaces of genus 2 we conclude that the set of curves $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2, \eta\}$ appear as in Fig. 13b, up to diffeomorphism.

Suppose that $\Delta = \{\gamma_1, \ldots, \gamma_k\}$ are the curves in $M \cap \mathring{D}_1$ different from δ_1. Let $A = M \setminus (M^+ \cup \delta_1 \cup \eta)$ and note that A is an annulus. Since the curves in Δ do not bound disks in A, they are parallel. Let W be the subannulus of A with boundary $\gamma_1 \cup \gamma_2$. By the convex hull property $W \subset D_1$ which is impossible. Hence we have proved that $k \leqq 1$.

Now suppose that $\Delta = \{\gamma_1\}$ and suppose $\mathring{D}_2 \cap M$ contains a curve γ_2 other than δ_2. By the argument in the previous paragraph γ_1 and γ_2 bound a subannulus W of A. After a possible reindexing we may assume that γ_2 is contained in the component C of $M \setminus \gamma_1$ which contains M_0^+. The annulus W is contained in B_0 by the convex hull property and so the component C leaves B_0 at the disk D_2 and hence $\min(z|C) < \min z|M_0^+$ where min refers to the minimum value of the function. This inequality implies C has a top point p not in M_0^+ which contradicts the fact that M_1^+ is disjoint from C. This proves γ_2 cannot exist.

Suppose now that $M \cap (\mathring{D}_1 \cup \mathring{D}_2)$ consists of the curves δ_1 and δ_2, then $N = M \cap B_1$ is connected and Lemma 5.8 implies that N is unknotted. To prove that M is unknotted it remains to prove that the annulus $W \subset M$ with boundary $\delta_1 \cup \delta_2$ is unknotted in B_0^c. To prove that W is unknotted is simple because if we let P_1 be the plane containing D_1 and P_2 be the plane containing D_2, then multipants uniqueness implies that W is unknotted in each component of $\mathbb{R}^3 \setminus (P_1 \cup P_2)$. If $M \cap (\mathring{D}_1 \cup \mathring{D}_2)$ consists of the three curves γ, δ_1 and δ_2 where $\gamma \subset D_1$, we can quickly reduce to the case when γ does not appear as follows. Suppose γ is contained in the plane above P_1 as in Fig. 13a and let $K = (z|M)^{-1}([0, \infty))$ be the part of M above P_1. $z|K$ has a single local maximum and so is unknotted. Once K is known to be unknotted, the sliding technique used in the proof of Lemma 5.9 shows we may assume that $M \cap D_1 = \delta_1$ and $M \cap D_2 = \delta_2$.

Summarizing, we have proven above that Lemma 5.9 holds when α_1 and α_2 are contained in distinct parallel planes. The proof of the general case follows from a modification of the special case and we outline its modification. By any one of several general position arguments one may assume that D_1 and D_2 are contained in different planes and after a linear transformation of $\mathbb{R}^3$ we can make these planes almost parallel. The proof that we may assume that $M \cap (D_1 \cup D_2)$ consists of δ_1 and δ_2 is essentially the same as before. Since $N = M \cap B_0$ is unknotted, then after a diffeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ which is the identity outside of a small neighborhood of B_0 we produce a new example $f(M)$ such that $f(D_1) = D_1$, $f(D_2)$ is contained in a plane parallel to $f(D_1)$ and $z|(f(M) \setminus f(B_0))$ has a single local maximum and a single local minimum. After these modifications of the proof, the theorem follows from the special case.

e) Remarks for the classification of RIT-surface of genus two with $\tau = 10$. (Preliminary report.) We can push the analysis in Sect. 5d further. If M_0^+ and

M_1^+ are linked, then we can easily determine all possible cases of intersection between ∂B_0 and $M_0^- \cup M_1^+$ (assumed to be transverse) and find that all such surfaces are weak RIT-isotopic to each other and to G in Fig. 8.

If M_0^+ and M_1^+ are unlinked and if an orientation of M induces opposite orientations in ∂B_0 and ∂B_1, then we can assume $B_0 \subset \mathring{B}_1$. In that case the possible intersections of ∂B_0 and M^-, assumed to be transverse, can be classified given the information in Table 1. These placements of intersections lead to a first classification into cases, each of them being then individually seen to be weak RIT-isotopic to one of $E_1, \ldots, F_5$. In Sect. 5c we suggested already that this reduces to E_5, F_5 up to weak RIT-isotopy.

Next, suppose M_0^+ and M_1^+ are unlinked, an orientation of M induces the same orientations in ∂B_0 and ∂B_1, *and* $B_0 \subset B_1$. Then we can deduce that M^- meets ∂B_0 in one circle γ and $M^- \cap B_0$ is unique in B_0 by multipants-uniqueness. Also unique is $M^- \setminus B_0$ in $B_1 \setminus B_0$ by the arguments of Sect. 5d. We can then construct a weak RIT-isotopy to the case H' of Fig. 8, which is weak RIT-equivalent to H.

If $B_0 \cap B_1 = \emptyset$, a plane P that separates B_0 from B_1 will meet M in one circle γ and weak RIT-isotopy with H in Fig. 8 becomes obvious. All remaining cases also seem to be weak RIT-equivalent to H in Fig. 8. So the following seems true:

Pre-theorem 6. *If f: $M \to \mathbb{R}^3$ is a RIT-surface of genus two and curvature $\tau = 10$, then it is weak RIT-isotopic to exactly one of the four surfaces E, F, G, H in Fig. 8.*

Problem 5. It there a relatively isotopy tight imbedding of the two-sphere with $\tau = 2 + 4k$ for some $k > 0$?

Problem 6. Is there an isotopy tight torus imbedding with $\tau = 4 + 4k$ for some $k > 1$? [The authors recently showed that the answer is *no*.]

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