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Titel: Growth functions and Euler series.

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Jahr: 1984

PURL: https://resolver.sub.uni-goettingen.de/purl?356556735_0077|log49

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Invent. math. 77, 517-531 (1984)

Inventiones mathematicae
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Growth functions and Euler series

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§ 1. The growth series

As originally conceived (see for example Milnor [8], Bass [2]), a growth series was associated with a finitely generated group G and a set of generators $\{g_1, \ldots, g_d\}$: the n-th term a_n in the growth series is then the number of elements of the group G which can be written as a word of length n but no shorter in the given generators and their inverses. Most of the interest in the literature centres on the rate of growth of the a_n, rather than on the *growth function*, the function whose power series is $\sum a_n x^n$.

It is natural to consider non-symmetric generating sets (that is, sets which generate the group as a semigroup, but which do not necessarily contain inverses); and, particularly if subgroups are to be included in a general theory, to allow the generators to be considered of length other than 1 (see for example Cannon [5], Wagreich [11], Benson [4]). Thus the natural setting for the concept appears to be that of *filtered semigroups.*

Definition. A *filtered set* is a set X together with a collection of subsets X_s for non-negative integers s such that $X_s \subseteq X_{s+1}$ and $X = \bigcup X_s$; the elements of $X_s - X_{s-1}$ are said to be of *degree* s.

A *filtered semigroup* is a semigroup G whose underlying set is filtered in such a way that $G_0 = \{\text{identity}\}$ and $G_s \cdot G_t \subseteq G_{s+t}$. Throughout this work we shall further insist that G_s is a finite set for each s.

The *growth function* $\gamma_G(x)$ of a filtered semigroup is the power series $\sum a_s x^s$ where $a_s = \operatorname{card}(G_s - G_{s-1})$.

As has been noted by others (see Wagreich [11]), the growth function can be identified with the Hilbert-Poincare series of a filtered algebra, or more precisely, the associated graded algebra. Choose a co-efficient ring R: in this paper R will be either the ring of integers or a field. The algebra RG inherits a filtration from G, and a_s is the R-rank of $(RG)_s/(RG)_{s-1}$. (Note that our filtrations are always increasing, and for a filtered R-algebra Γ we assume $\Gamma_0 = R$.) The question of the rationality of this series in the setting of more general graded algebras is of current interest: see for example [1, 6].

Suppose that we are given a set of generators X for the semigroup G, and each generator is given a positive integer degree so that X is a filtered set with X_0 empty and each X_s is finite; then there is an induced filtration on G where the degree of each element g is the minimum of the sum of the degrees of the generators appearing in a word representing g. The filtered set X will be said to generate the filtered semigroup G. Note that not every filtration on G will be induced in this way from a *finite* filtered set of generators, even when the underlying semigroup is finitely generated.

§ 2. Filtered modules over filtered algebras

Our object is to connect the growth function of a filtered group with the Euler characteristic; this suggests that we consider resolutions of the ground ring R by filtered modules over the filtered algebra RG. There is however a difficulty here, in that the category of filtered modules over a filtered algebra (with filtration-preserving homomorphisms) is not an abelian category; the basic problem is that epimorphisms are simply surjective maps, not necessarily surjective in each degree, which for example has the consequence that RG is not even projective as an RG-module! The standard way out of this difficulty is to pass to the category of graded modules over the associated graded algebra. This method will no doubt suffice for much of what follows. However it is of interest to note that there is another abelian category available, which in fact contains both filtered modules and graded modules as objects. This category has some advantages for us; in particular it enables direct comparison with the standard unfiltered homological algebra of a group.

Definition. Let Γ be a filtered R-algebra. A *Γ-tower* M consists of

(i) a sequence of finitely generated (left) R-modules M_s, $s \geqq 0$,

(ii) R-morphisms $j_s: M_s \to M_{s+1}$,

(iii) R-morphisms $k_{st}: \Gamma_s \otimes_R M_t \to M_{s+t}$, defining an action of Γ on M, such that the following diagrams commute:

(a) (compatibility)

$$\begin{array}{ccc} \Gamma_s \otimes_R M_t & \longrightarrow & M_{s+t} \\ \downarrow & & \downarrow \\ \Gamma_s \otimes_R M_{t+1} & \longrightarrow & M_{s+t+1} \end{array}$$

and

$$\begin{array}{ccc} \Gamma_s \otimes_R M_t & \longrightarrow & M_{s+t} \\ \downarrow & & \downarrow \\ \Gamma_{s+1} \otimes_R M_t & \longrightarrow & M_{s+t+1}, \end{array}$$

(b) (associativity)

$$\begin{array}{ccc} \Gamma_s \otimes_R \Gamma_t \otimes_R M_u & \longrightarrow & \Gamma_{s+t} \otimes_R M_u \\ \downarrow & & \downarrow \\ \Gamma_s \otimes_R M_{t+u} & \longrightarrow & M_{s+t+u}, \end{array}$$

(c) (identity)

$$\begin{array}{ccc} \Gamma_0 \otimes_R M_t & \longrightarrow & M_t \\ \| & & \| \\ R \otimes_R M_t & = & M_t. \end{array}$$

A (degree 0) *morphism* of Γ-towers is a sequence of R-morphisms $f_s: M_s \to N_s$ for which all the appropriate diagrams commute.

A filtered module can be identified with a tower in which all the maps j_s are injections. At the other extreme, towers in which all the maps j_s are zero can be identified with graded modules!

[It is also possible to view towers as special cases of graded modules! For Γ can be identified with the graded sub-algebra $\Gamma^* = \sum_{s \geq 0} \Gamma_s t^s$ of the polynomial algebra $\Gamma[t]$; then a Γ-tower M can be regarded as a graded Γ^*-module $M^* = \sum M_s$, where the action of t is provided by the map j_s. The compatibility condition above corresponds to the fact that t commutes with the elements of Γ.]

It is straight-forward to verify that the category of Γ-towers is an abelian category. From the point of view of this enlarged category, the difficulty with filtered modules arises from the fact that a quotient of filtered modules is a tower but may not be a module.

The unfiltered R-algebra underlying Γ will be denoted by $|\Gamma|$. Comparisons with the standard theory of $|\Gamma|$-modules can be made via the functor sending the filtered tower M to the unfiltered module $\lim M$, which we shall denote by $|M|$. For a filtered module this direct limit is just $\bigcup M_s$; towers for which $|M| = 0$, of which graded modules are special cases, will be called *null-towers.*

For a Γ-tower M, the *translate* of M, denoted by xM, is defined to be the tower with

$$(xM)_0 = 0,$$
$$(xM)_s = M_{s-1}, \qquad s > 0$$

with the obvious definitions of the maps j_s and k_{st}. Observe that $|xM| = |M|$. Note that there is a map of xM into M (the map in degree s is just j_{s-1}); for a filtered module M, this is an inclusion map, and the quotient tower M/xM is precisely the associated graded module, which we shall denote by aM (more generally we set $a_n M = M/x^n M$).

We can next define $(x+1)M$ to be the direct sum $xM \oplus M$, and indeed for any polynomial (or even a power series) $p(x)$ with non-negative integer coefficients we define a tower $p(x)M$. Observe that $|p(x)M| = p(1)|M|$, the direct sum of $p(1)$ copies of $|M|$.

Let X be a filtered set with each X_s finite. We can associate with X a power series $p(x)$ (a polynomial if X is finite) by

$$p(x) = \sum (\operatorname{card} X_s - X_{s-1})\, x^s.$$

There is also associated with X a free Γ-tower generated by X. It is then easy to see that this free tower is just $p(x)\Gamma$; it is in fact a filtered module. The

series $p(x)$ clearly plays the role of the rank of the free module, and so we shall call it the rank series. Observe that in the case of Γ being the algebra RG of a filtered semigroup G, we can consider Γ to be a free filtered module over R; its rank series over R is simply the growth function $\gamma_G(x)$.

We shall not need to consider projectives *per se* in this paper, but in fact projective Γ-towers turn out to be free modules for the co-efficient rings R under consideration. This can be seen by following the argument in, for example, Bass [3] Proposition 2.12, using a version of Nakayama's Lemma for towers (or graded modules). I am grateful to the referee for pointing this out, and also for the interpretation of towers as graded modules mentioned above.

§ 3. The Euler series of a tower

Let M be a Γ-tower. Since each M_s is finitely generated over R we may choose a power series $p_0(x)$ and a Γ-morphism ϕ mapping $p_0(x)\Gamma$ onto M; we may suppose that the first non-zero co-efficient in $p_0(x)$ occurs in the degree of the first non-zero term in the filtration of M, and moreover if this term happens to be a free R-module then we may choose the map to be an isomorphism in this degree. Let K be the kernel of this map. Now K_s is the kernel of ϕ_s; by our choice of R, K_s is finitely generated and free over R. Hence we may find a Γ-morphism mapping a free module $p_1(x)\Gamma$ onto K, and so on.

Thus we see that there is a Γ-resolution of M by free filtered Γ-modules

$$\ldots \to p_n(x)\Gamma \to \ldots \to p_1(x)\Gamma \to p_0(x)\Gamma \to M \to 0$$

where the $p_n(x)$ are power series with non-negative integer co-efficients and of increasing lowest degree.

Definition. The *Euler series* of such a resolution is the power series

$$\sum (-1)^n p_n(x).$$

Since there are only finitely many non-zero co-efficients in each degree, this sum is well-defined. Moreover the usual comparison theorem argument (see for example Stallings [10]) shows that this Euler series depends only on the Γ-tower M, not on the resolution chosen. We shall therefore denote it by $\chi_\Gamma(M)$.

Note that $\chi_\Gamma(p(x)\Gamma) = p(x)$.

The Euler characteristic of a group $|G|$ is classically defined to be $\chi_{R|G|}(R)$, assuming the trivial $R|G|$-module R has a finite length resolution by finitely generated $R|G|$-projectives. We therefore define the *Euler series of a filtered semigroup* G to be $\chi_{RG}(R)$; here R is considered a trivial filtered module by putting $R_s = R$ for all s.

Theorem. *The Euler series of the filtered semigroup G is the reciprocal of its growth function.*

Proof. Consider an RG-resolution of R

$$\ldots \to p_n(x)\,RG \to \ldots \to p_1(x)\,RG \to p_0(x)\,RG \to R \to 0.$$

Now this can also be considered to be a free R-resolution of R; since $\chi_R(R)=1$, we have

$$\begin{aligned} 1 &= \sum (-1)^n p_n(x)\,\gamma_G(x) \\ &= \chi_{RG}(R)\,\gamma_G(x). \end{aligned}$$

Remarks. 1) Suppose the filtered semigroup G happens to have a resolution as above which is of finite length and all the $p_n(x)$ are polynomials. Now the direct limit functor $|\ |$ preserves exactness, so we have a finite free $R|G|$-resolution of R

$$0 \to p_n(1)\,R|G| \to \ldots \to p_1(1)\,R|G| \to p_0(1)\,R|G| \to R \to 0.$$

Hence we would have the desired result that the classical Euler characteristic of $|G|$

$$\begin{aligned} \chi_{R|G|}(R) &= \text{the value of the Euler series of } G \text{ at } 1 \\ &= 1/\gamma_G(1). \end{aligned}$$

Unfortunately, while this situation can arise for semigroups, it is very unlikely to occur for groups; the growth function of the group would have to be the reciprocal of a polynomial, a property enjoyed as far as I know only by the trivial group.

2) In contrast with the classical Euler characteristic, which is defined only for the very restricted class of modules having a finite length resolution by finitely generated projectives, the Euler series is defined for all Γ-towers.

3) The Euler series is also a considerably stronger invariant than the Euler characteristic (and no doubt therefore far less tractable). For example if R is a field, we see from the construction of the resolution that the first non-zero coefficient in $\chi_{RG}(M)$ is the R-rank of the first non-zero term in the filtration of M. In particular, if R is a field and $\chi_\Gamma(M)=0$, we can conclude that $M=0$!

4) There is a standard way of constructing a free RG-resolution for a filtered semigroup G given a filtered set of generators for G, in fact the Gruenberg resolution (see [7]) can be given a filtration. We sketch the development here; the full details can be found in [9].

Suppose X is a filtered set of generators for G. Let S be the free semigroup on X, with the induced filtration. Then there is an epimorphism from RS onto RG (indeed this is an advantage of the category of towers - a filtered subset X generates G if and only if the induced map $RS \to RG$ is an epimorphism; this is not true in the category of filtered RS-modules). Let $\mathscr{R}$ denote the relation ideal, the kernel of this map:

$$0 \to \mathscr{R} \to RS \to RG \to 0.$$

It can be shown that $\mathscr{R}$ is a free filtered left RS-module; let us denote its RS-rank series by $r(x)$. The augmentation ideal $\mathscr{S}$ is also free, generated by

$\{g-1: g\in X\}$; we denote its rank series by $d(x)$. The construction of Gruenberg can then be followed to produce a free resolution of R by filtered RG-modules

$$\ldots \to \mathscr{R}^2/\mathscr{R}^3 \to \mathscr{R}\mathscr{S}/\mathscr{R}^2\mathscr{S} \to \mathscr{R}/\mathscr{R}^2 \to \mathscr{S}/\mathscr{R}\mathscr{S} \to RG \to R \to 0.$$

The RG-rank series of $\mathscr{R}^n/\mathscr{R}^{n+1}$ is easily seen to be $(r(x))^n$, and that of $\mathscr{R}^n\mathscr{S}/\mathscr{R}^{n+1}\mathscr{S}$ is $(r(x))^n d(x)$; it follows that

$$\chi_{RG}(R)=(1-d(x))/(1-r(x))$$

and

$$\gamma_G(x)=(1-r(x))/(1-d(x)).$$

This result can be regarded as a version of Govorov's theorem [6].

In practice we usually deal with finitely generated groups with all generators of degree 1; here $d(x)$ is simply dx, where d is the number of generators. However even in the simplest cases, $r(x)$ will not be a polynomial. For example, if G is the group of integers, with degree 1 generating set $\{1, -1\}$, we have

$$\gamma_G(x)=(1+x)/(1-x)$$

and so

$$r(x)=2x^2/(1-x).$$

Nevertheless, perhaps for some classes of groups it may be possible to exhibit explicit free generators for $\mathscr{R}$ and so compute $r(x)$ directly.

It will be observed that the above construction appears to be slightly different from Gruenberg's, in that he deals with groups G and so uses the free group F in place of our free semigroup S. The group ring RF can be given a suitable filtration, in which of course the inverse of a generator in X is assigned its degree in G; but the augmentation ideal $\mathscr{F}$ is *not* free as a filtered RF-module! However, if we denote the relation ideal in RF by $\mathscr{U}$ and consider RS as a subset of RF, it can be shown that $\mathscr{U}\cap\mathscr{S}=\mathscr{R}$; it follows that the RG-resolution given above really is the same as the Gruenberg resolution.

We close this section with an additivity theorem for the Euler series.

Theorem. *If*

$$\ldots \to M_n \to \ldots \to M_1 \to M_0 \to 0$$

is an exact sequence of Γ-towers, such that for each degree s there is an integer $v(s)$ with $(M_n)_s=0$ for $n\geqq v(s)$, then

$$\sum(-1)^n \chi_\Gamma(M_n)=0.$$

Proof. If the sequence is of finite length, then the usual proof will work. Thus we first consider the case of a short exact sequence of Γ-towers

$$0 \to A \to B \to C \to 0.$$

Choose a free Γ-resolution of A:

$$\ldots \to X_1 \to X_0 \to A \to 0.$$

Then we can extend this to a free resolution of B:

$$\ldots \to Y_1 \to Y_0 \to B \to 0$$

in such a way that $X_n \subseteq Y_n$. Then

$$\ldots \to X_1 \oplus Y_2 \to X_0 \oplus Y_1 \to Y_0 \to C \to 0$$

is exact, and is a free Γ-resolution of C. Hence

$$\chi_\Gamma(C) = \chi_\Gamma(B) - \chi_\Gamma(A).$$

By induction, the theorem follows for finite length sequences.

Finally consider an infinite exact sequence. Let K be the kernel in dimension $v(s)$. Thus we have a finite exact sequence

$$0 \to K \to M_{v(s)} \to \ldots \to M_0 \to 0.$$

By assumption, the first non-zero term in the filtration of $M_{v(s)}$ and therefore also of K occurs in degree $> s$. Hence $\chi_\Gamma(K) = x^s k(x)$. By the result for finite sequences,

$$[\textstyle\sum (-1)^n \chi_\Gamma(M_n) \text{ truncated at degree } s] = 0.$$

Since this holds for every degree s, we are finished.

In particular, for a filtered module M, the Euler series of the associated graded module aM can be calculated from the sequence

$$0 \to xM \to M \to aM \to 0$$

to be

$$\chi_\Gamma(aM) = (1-x)\,\chi_\Gamma(M)$$

and indeed

$$\chi_\Gamma(a_n M) = (1-x^n)\,\chi_\Gamma(M).$$

§ 4. Homology

In order to discuss homology of towers, we need a tensor product. This is provided by the following definition.

Definition. Let M be a right Γ-tower and N be a left Γ-tower. We define the R-tower $M \otimes_\Gamma N$ to have degree s term $\sum_{u+v=s} M_u \otimes_R N_v$ modulo the submodule generated by elements

$$k_{uv}(m \otimes \gamma) \otimes n - m \otimes k_{vw}(\gamma \otimes n) \quad \text{for } m \in M_u,\ \gamma \in \Gamma_v,\ n \in N_w,\ u+v+w=s$$

and

$$j_u(m) \otimes n - m \otimes j_w(n) \quad \text{for } m \in M_u,\ n \in N_w,\ u+w=s-1.$$

The j and k maps for the tensor product are defined in the obvious way.

We note that this tensor product commutes with direct limits so that

$$|M \otimes_\Gamma N| = |M| \otimes_{|\Gamma|} |N|.$$

As usual, the homology of a filtered semigroup G with co-efficients in a right RG-tower M is defined to be the homology of $M \otimes_{RG} Y$, where Y is a free left RG-resolution of R. We shall only be interested in the case of homology with co-efficients in the trivial module R. Again since homology commutes with direct limits, we have

$$|H_q(G:R)| = H_q(|G|:R).$$

Moreover it is straightforward to prove that

$$\chi_\Gamma(R) = \sum (-1)^q \chi_R(H_q(G:R)).$$

We are led therefore to a consideration of R-towers and their Euler series.

Theorem. *A finitely generated R-tower M is an extension*

$$0 \to x^u M^* \to M \to Q \to 0$$

where Q is a finitely generated R-null-tower and M^* is the trivially filtered R-module on $|M|$, that is, $M^*_s = |M|$ for all s.

Proof. Since M is finitely generated there is a polynomial $p(x)$ and an epimorphism from $p(x)R$ onto M. Let t be the degree of $p(x)$. Then for $s \geqq t$, M_s is a quotient of $p(1)R$ and M_{s+1} is a quotient of M_s. Since R is Noetherian, for some degree u we must have

$$M_u = M_{u+1} = M_{u+2} = \ldots = |M|.$$

Then $x^u M^*$ is clearly a subtower of M and the quotient is a null-tower.

Now $\chi_R(x^u M^*)$ is clearly just $m x^u$, where m is the R-rank of $|M|$. For the null-tower part of M we need

Theorem. *If Q is a finitely generated R-null-tower, then*

$$\chi_R(Q) = (1-x)\, q(x)$$

where $q(x)$ is a polynomial with non-negative integer co-efficients.

Proof. By the argument of the previous theorem, $Q_s = 0$ for s greater than some degree u; we may suppose $Q_u \neq 0$. Let Q' be the null-tower with $Q'_0 = Q_u$ and zero in all other degrees. Then $x^u Q'$ is a subtower of Q; the quotient tower Q'' is zero in all degrees greater than $u-1$. We have $\chi_R(Q) = \chi_R(Q'') + x^u \chi_R(Q')$.

Let T be the trivially filtered R-module on Q_u, that is, $T_s = Q_u$ for all s. Then $Q' = aT$. It follows that

$$\chi_R(Q') = (1-x)\, \chi_R(T) = (1-x)\, \mathrm{rank}_R(Q_u).$$

By induction then $\chi_R(Q) = (1-x)\, q(x)$, where $q(x)$ is a polynomial whose co-efficient of x^s is the R-rank of Q_s.

Corollary. *If M is a finitely generated R-tower, then $\chi_R(M)$ is a polynomial whose value at 1 is the rank of $|M|$.*

Let vM denote the subtower of M consisting of elements which eventually vanish in higher degrees of the filtration; that is,

$$(vM)_s = \{m \in M_s : j_t \dots j_{s+1} j_s(m) = 0 \text{ for large } t\}$$

vM is the largest null-tower contained in M. The quotient M/vM is then a module which we shall denote by μM (if R is a field, $M = vM \oplus \mu M$). Thus we have a canonical decomposition of $\chi_R(M)$ into a sum of polynomials

$$\mu(x) + (1-x)\, v(x)$$

where $\mu(x) = \chi_R(\mu M)$ and $(1-x)\, v(x) = \chi_R(vM)$. It follows that $\mu(1) =$ the rank of $|M|$.

Let us suppose now that we have a filtered group G whose underlying group $|G|$ has a finite projective $|RG|$-resolution. Although we cannot expect that R will have a finite RG-resolution by free RG-modules, it does seem to be not unreasonable to hope that there is a free resolution all of whose terms are finitely generated over RG; G is then said to be of type FP_∞. Then the homology of G with co-efficients in R is finitely generated in each dimension, and since it agrees in the direct limit with the homology of $|G|$, all but a finite number of the homology towers are null-towers. Indeed we see that

$$\chi_{RG}(R) = \mu_G(x) + (1-x)\, v_G(x)$$

where $\mu_G(x)$ is a polynomial whose value at 1 is the Euler characteristic of $|G|$, and $v_G(x)$ is a power series determined by the homology of G. Thus we have

Theorem. *If G is a filtered (semi-)group which is of type FP_∞, $|G|$ has a finite resolution, and the Euler series of G is the power series of a function which can be extended to 1, then the value of this function at 1 is the Euler characteristic of $|G|$ if and only if the power series $v_G(x)$ does not have a pole at 1.*

We are then left with two questions: which filtered groups are of type FP_∞, and what can be said about the series $v_G(x)$? The examples of Sect. 6 seem to point to the class of type FP_∞ groups being reasonably large. As to the second question, the theorem is quite useless unless some restrictions can be placed on $v_G(x)$, but it does not seem possible to have enough control over the properties of the $v(H_n(G:R))$ for arbitrary filtered groups to exclude the possibility of a pole at 1. There seems to be no reason, for example, why we could not have

$$\begin{aligned}\sum (-1)^n v_n(x) &= (1+2x) - x + (x^2 + 2x^3) - x^3 + \dots \\ &= 1/(1-x).\end{aligned}$$

§ 5. Periodic semigroups and rational growth functions

While the notion of "finite cohomological dimension" seems to be inappropriate for the study of filtered semigroups, I suggest that the following concept may play a useful role.

Recursive Definition. A Γ-tower M will be said to be *periodic* if

(i) it is finitely generated and free
or (ii) it is a finite direct sum of periodic towers
or (iii) there is an exact sequence

$$0 \to q(x)\, M \to X_n \to \ldots \to X_1 \to X_0 \to M \to 0$$

in which $X_0, \ldots, X_n$ are periodic Γ-towers, $n \geqq 0$, and $q(x)$ is a polynomial with non-negative integer co-efficients.

A filtered semigroup G will be said to be periodic if the trivial filtered module R is periodic over RG.

If the X_i in (iii) are free, a resolution for M which has a kind of periodicity can be constructed by repeating translates of the sequence, hence the name. Periodicity may be an algebraic way of explaining the appearance of rational growth functions:

Theorem. *If M is periodic over Γ, then $\chi_\Gamma(M)$ is the power series of a rational function.*

Moreover if M appears in an exact sequence as given in the above definition, and

(i) *each $|X_i|$ has an Euler characteristic over $|\Gamma|$*

(ii) *the rational function $\chi_\Gamma(X_i)$ evaluated at 1 is equal to this Euler characteristic for each i*

(iii) *$|M|$ has an Euler characteristic over $|\Gamma|$*

(iv) *n is even, or $q(x) \neq x^u$*

then $\chi_\Gamma(M)$ evaluated at 1 is the Euler characteristic of $|M|$.

(In most of the examples of the following section, the X_i are free so that (i) and (ii) hold.)

Proof. We have seen that the Euler series of a finitely generated free module is a polynomial; and the Euler series of a direct sum is just the sum of the series of the summands. Thus we may assume M appears in a sequence as in part (iii) of the definition; further we may assume by induction that each $\chi_\Gamma(X_i)$ is the power series of a rational function. Then by additivity we have

$$\chi_\Gamma(M) + (-1)^n q(x)\, \chi_\Gamma(M) = \sum (-1)^i \chi_\Gamma(X_i)$$

and so

$$\chi_\Gamma(M) = (\sum (-1)^i \chi_\Gamma(X_i))/(1 + (-1)^n q(x)).$$

We also have an exact sequence of unfiltered $|\Gamma|$-modules

$$0 \to q(1)\,|M| \to |X_n| \to \ldots \to |X_1| \to |X_0| \to |M| \to 0.$$

If conditions (i) and (iii) hold, we can conclude

$$\chi_{|\Gamma|}(|M|) + (-1)^n q(1)\, \chi_{|\Gamma|}(|M|) = \sum (-1)^i \chi_{|\Gamma|}(|X_i|).$$

Finally condition (iv) ensures that $1 + (-1)^n q(1)$ is not zero, and condition (ii) gives the result.

In the following section we give some examples to show that periodicity does actually occur in nature.

§ 6. Examples

1. The free semigroup

Let X be a finite filtered set with associated polynomial

$$d(x)=\sum(\text{card } X_s - X_{s-1})\, x^s$$

and let S be the free semigroup on X with induced filtration. Then

$$0 \to \mathscr{S} \to RS \to R \to 0$$

is exact, and the augmentation ideal $\mathscr{S}$ is free as a filtered RS-module with rank series $d(x)$.

Hence

$$\chi_{RS}(R)=1-d(x)$$

and

$$\gamma_S(x)=1/(1-d(x)).$$

2. The free group

(a) Let F be the free group generated by $\{y_1,\ldots,y_d\}$, with the filtration induced by assigning degree 1 to these generators and their inverses. We have an exact sequence

$$0 \to \mathscr{F} \to RF \to R \to 0.$$

The unfiltered augmentation ideal $|\mathscr{F}|$ is free, but $\mathscr{F}$ is not free in the filtered sense (basically because y_1-1 would have to be a generator, but $y_1^{-1}(y_1-1)$ is not of degree 2). However it is not hard to show that $\mathscr{F}$ is generated by the elements y_i-1, $y_i^{-1}-1$ and so we have an epimorphism $2\,dx\,RF \to \mathscr{F}$; it can then be shown that the kernel of this map is isomorphic to $x\mathscr{F}$:

$$0 \to x\mathscr{F} \to 2\,dx\,RF \to \mathscr{F} \to 0.$$

Thus $\mathscr{F}$ and hence R are periodic over RF. So F is a periodic group, with Euler series

$$\begin{aligned}\chi_{RF}(R) &= 1-\chi_{RF}(\mathscr{F}) \\ &= 1-2\,dx/(1+x).\end{aligned}$$

The theorem of Sect. 5 applies so the Euler characteristic of $|F|$ is $1-d$, the value of this function at 1.

From these exact sequences we can construct a resolution

$$\ldots \to 2\,dx^3\,RF \to 2\,dx^2\,RF \to 2\,dx\,RF \to RF \to R \to 0$$

and compute the homology of F to be

$$\begin{aligned} H_0(F:R) &= R, \\ H_1(F:R) &= dx R \oplus dx a R, \\ H_i(F:R) &= dx^i a R \quad \text{for } i>1, \\ v_G(x) &= -dx + dx^2 - \ldots \\ &= -dx/(1+x). \end{aligned}$$

(b) More generally, consider the free group F with filtration induced by assigning degree $\lambda(i)$ to the generator y_i and degree $\mu(i)$ to its inverse. Again F is periodic.

Here the augmentation ideal $\mathscr{F}$ is a direct sum of sub-modules $\mathscr{T}_i$ generated by $y_i - 1$ (in degree $\lambda(i)$) and $y_i^{-1} - 1$ (in degree $\mu(i)$). Each $\mathscr{T}_i$ is periodic because of an exact sequence

$$0 \to x^{\lambda(i)+\mu(i)} \mathscr{T}_i \to 2 x^{\lambda(i)+\mu(i)} RF \to (x^{\lambda(i)} + x^{\mu(i)}) RF \to \mathscr{T}_i \to 0.$$

Hence $\mathscr{F}$ and R are periodic, and

$$\chi_{RF}(R) = 1 - \sum (x^{\lambda(i)} + x^{\mu(i)} - 2 x^{\lambda(i)+\mu(i)})/(1 - x^{\lambda(i)+\mu(i)}).$$

(c) There are other filtrations on a free group $|F|$. We can choose any set of semigroup generators, and assign degrees. Whether the resulting filtered group F is always periodic seems to be a very difficult question. It is of course intimately connected with the problem of what happens to the growth function of a group under a change of generating set; absolutely no progress seems to have been made on this problem. It may even be true that there is a group which has a rational growth function with respect to one set of (degree 1) generators, but a transcendental growth function for some other set.

As an example, consider the free group of rank two generated by y and z with filtration induced by assigning degree 1 to y, z and $y^{-1} z^{-1}$. It is then a moderately interesting exercise to prove that the augmentation ideal $\mathscr{F}$ fits into an exact sequence

$$0 \to x^3 \mathscr{F} \to 3 x^3 RF \to 3 x RF \to \mathscr{F} \to 0.$$

Once again, F is periodic, and

$$\begin{aligned} \chi_{RF}(R) &= 1 - (3x - 3x^3)/(1 - x^3) \\ &= (1 - 2x - 2x^2)/(1 + x + x^2). \end{aligned}$$

This function happens to have the correct value at 1; but this does *not* follow from the theorem of Sect. 5! However we can compute the homology of this group from the resolution

$$\ldots \to 3 x^6 RF \to 3 x^4 RF \to 3 x^3 RF \to 3 x RF \to RF \to R \to 0$$

derived from the above sequences. We find

$$
\begin{aligned}
H_0(F:R) &= R,\\
H_1(F:R) &= 2xR \oplus x a_2 R,\\
H_2(F:R) &= 2x^3 aR,\\
H_3(F:R) &= x^4 a_2 R,\\
H_{i+2}(F:R) &= x^3 H_i(F:R) \quad \text{for } i \geqq 2,\\
v_G(x) &= -x(1+x)+2x^3-x^4(1+x)+2x^6-\ldots\\
&= -x(1+2x)/(1+x+x^2).
\end{aligned}
$$

$v_G(x)$ does not have a pole at 1 so the theorem of Sect. 4 applies.

3. *The free abelian group*

Let A be the free abelian group generated by $\{y_1, \ldots, y_d\}$, with the filtration induced by assigning degree $\lambda(i)$ to the generator y_i and degree $\mu(i)$ to its inverse.

Let A_k denote the subgroup generated by $\{y_1, \ldots, y_k\}$. Then RA_{d-1} is an RA-module with trivial action of y_d, and there is an exact sequence

$$0 \to \mathscr{T}_{d-1} \to RA \to RA_{d-1} \to 0$$

where $\mathscr{T}$ is the ideal generated by $(y_d - 1,\ y_d^{-1} - 1)$. It can then be shown that there is an exact sequence

$$0 \to x^{\lambda(d)+\mu(d)} \mathscr{T}_{d-1} \to 2x^{\lambda(d)+\mu(d)} RA \to (x^{\lambda(d)} + x^{\mu(d)}) RA \to \mathscr{T}_{d-1} \to 0.$$

So $\mathscr{T}_{d-1}$ and hence RA_{d-1} are periodic modules over RA; the Euler series of RA_{d-1} over RA is

$$\chi_{RA}(RA_{d-1}) = (1-x^{\lambda(d)})(1-x^{\mu(d)})/(1-x^{\lambda(d)+\mu(d)}).$$

Now strip off the next generator, producing exact sequences

$$
\begin{aligned}
&0 \to \mathscr{T}_{d-2} \to RA_{d-1} \to RA_{d-2} \to 0,\\
&0 \to x^{\lambda(d-1)+\mu(d-1)} \mathscr{T}_{d-2} \to 2x^{\lambda(d-1)+\mu(d-1)} RA_{d-1}\\
&\qquad\qquad \to (x^{\lambda(d-1)} + x^{\mu(d-1)}) RA_{d-1} \to \mathscr{T}_{d-2} \to 0.
\end{aligned}
$$

Thus eventually we have that R is periodic over RA, with Euler series

$$\chi_{RA}(R) = \prod [(1-x^{\lambda(i)})(1-x^{\mu(i)})/(1-x^{\lambda(i)+\mu(i)})].$$

Probably all finitely generated filtered abelian groups are periodic. This may also be true for nilpotent groups (it is known that their growth functions are rational).

4. *The cyclic group*

Let C be cyclic of order m, generated by t. With the filtration induced by assigning degree 1 to t, we find an exact sequence

$$0 \to x^m R \to xRC \to RC \to R \to 0$$

and so $\chi_{RC}(R) = (1-x)/(1-x^m) = 1/(1+x+\ldots+x^{m-1})$.

Another natural filtration would be to assign degree 1 to all elements of C other than the identity. In this case we have exact sequences

$$0 \to \mathscr{C} \to RC \to R \to 0,$$
$$0 \to (m-1)x\mathscr{C} \to (m-1)xRC \to \mathscr{C} \to 0$$

and so again C is periodic, with $\chi_{RC}(R) = 1/(1+(m-1)x)$.

5. *The trefoil group*

Finally we consider some filtrations on the trefoil group

$$G = \langle y, z : y^2 z^3 = 1 \rangle.$$

We choose semigroup generators y and z, both having degree 1. Then there are exact sequences

$$0 \to M \to 2xRG \to RG \to R \to 0,$$
$$0 \to x^5 M \to (2x^6 + x^5)RG \to (x^3 + x^4 + x^5)RG \to M \to 0.$$

The construction of the latter exact sequence is a long and detailed calculation relying on the canonical form for elements of G.

It follows that G is periodic, with

$$\chi_{RG}(R) = (1-x^2-x^3)(1-x)/(1+x+x^2+x^3+x^4).$$

We cannot deduce from the theorem of Sect. 5 that the value at 1 is the Euler characteristic. Again we compute the homology:

$H_0(G:R) = R$,

$H_1(G:R) = xR \oplus x a_4 R$,

$H_2(G:R)$ is a null-tower with $v_2(x) = x^3(1+x)^2$,

$H_3(G:R) = x^6 a_4 R$,

$H_{i+2}(G:R) = x^5 H_i(G:R)$ for $i \geqq 2$,

$v_G(x) = -x(1+x+x^3-2x^4)/(1+x+x^2+x^3+x^4)$ which does not have a pole at 1.

It may be tempting to conjecture that finitely generated filtrations on 1-relator groups are always periodic. However the reader interested in tackling this problem, or the problem of the connection between growth functions for different generating sets, may find that the following examples of (probable) Euler series for some other filtrations on the trefoil group pose a rather formidable mystery.

The following examples are guesses based on fitting a simple (?) rational function to computer calculations of the first 18 or so terms in the growth series. For the trefoil group with degree 1 generators y and z^{-1}, it appears that the Euler series is

$$(1-x)(1-x^2-x^3)/(1+x+x^2).$$

If we choose degree 1 generators y, z and y^{-1}, we appear to have Euler series

$$(1-x)(1-x^2-x^3)^2/(1+2x+2x^2-3x^4-4x^5-3x^6-x^7).$$

And if the degree 1 generators are y, z, y^{-1} and z^{-1}, we seem to get

$$\frac{(1-x)(1-x^2-x^3)(1-4x^2+4x^4)}{(1+x)(1+2x+x^2-7x^3-10x^4+6x^5+6x^6+4x^7)}.$$

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Oblatum 6-II-1984