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Heegaard genus of closed orientable Seifert 3-manifolds

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A Heegaard splitting of an orientable closed connected 3-manifold M is a closed connected orientable surface F imbedded in M and dividing M into two handlebodies. The genus of the Heegaard splitting is the genus of F and the *Heegaard genus of* M, $h(M)$, is the smallest integer h such that M has a Heegaard splitting of genus h.

By $r(M)$ we denote the *rank of the fundamental group* $\pi_1 M$, that is the minimum number of elements needed to generate $\pi_1 M$. Obviously one has the inequality $r(M) \leqq h(M)$. F. Waldhausen (see [Ha 1; p. 149] and [Wa 3; p. 320]) asked whether the equality $r(M) = h(M)$ is true for all M. A positive answer to this question is known for S^3 and the lens spaces and for a few other classes of manifolds (see [Ra], [TOc], [Wa 2]). We give here a negative answer to the question of F. Waldhausen by describing some Seifert-manifolds M with $2 = r(M) < h(M) = 3$ (see [BZ 1]).

The aim of this paper is to calculate the Heegaard genus of almost all closed orientable Seifert-manifolds and the rank of their fundamental group. In most of the cases these numbers coincide; however there are counterexamples, as mentioned above, and some cases still remain undecided.

1. Heegaard genus of Seifert 3-manifolds with orientable bases

In this section we consider closed Seifert 3-manifolds with orientable basis. These manifolds M admit a differentiable action of S^1 where the stabilizer of each point of M is finite (see for example [Se], [Or], [OVZ], [Si]).

To each such manifold there corresponds a system of invariants $(g; e_0; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$, $1 < \alpha_1 \leqq \ldots \leqq \alpha_m$, where g is the genus of the 2-dimensional orbit space (the basis of M), $e_0 \in \mathbb{Q}$ is the rational Euler number of the Seifert-fibration, and β_i/α_i are elements in $(\mathbb{Q}/\mathbb{Z})^*$ which characterize the holonomy of the exceptional fibres. The usual Euler class of the fibration is $e = e_0 + \sum_{i=1}^{m} \beta_i/\alpha_i \in \mathbb{Z}$. The principal purpose of this fractional presentation of the

Seifert invariants is to avoid the normalisation $0<\beta_i<\alpha_i$ as in [Se]. For more details on these fractional invariants we refer to [Si], [Th].

By H. Seifert [Se], any system $(g; e_0; \beta_1/\alpha_1, \ldots, \beta_m\alpha_m)$, such that $\left(e_0+\sum_{i=1}^{m}\beta_i/\alpha_i\right)\in\mathbb{Z}$, belongs to a closed Seifert 3-manifold with orientable basis, determined up to orientation preserving homeomorphisms. Such a Seifert-manifold will be denoted by $S(g; e_0; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$.

Conversely, in most cases the topological oriented type of M determines the system of fractional invariants of M (see [OVZ], and also [Si]).

We want to prove the following theorem:

1.1. Theorem. *Let* $M=S(g; e_0; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$, with $1<\alpha_1\leqq\ldots\leqq\alpha_m$ *and* $gcd(\alpha_i, \beta_i)=1$, $1\leqq i\leqq m$.

i) *If* $M=S(0; e_0; 1/2, \ldots, 1/2, \beta_m/2\lambda+1)$, *with* $\lambda>0$, m *even and* $m\geqq 4$, *then* $m-2=r(M)\leqq h(M)\leqq m-1$.

If, in addition, $e_0=\pm 1/2(2\lambda+1)$, *then* $r(M)=h(M)=m-2$.

ii) *If* $g>0$ *and* $m=2$, *or if* $m\geqq 3$ *and if* M *does not belong to the case* i), *then* $r(M)=h(M)=2g+m-1$.

iii) *If* $m=1$, *then* $r(M)=h(M)=2g$ *if* $e_0=\pm 1/\alpha$; *otherwise* $r(M)=h(M)=2g+1$.

If $m=0$, *then* $r(M)=h(M)=2g$ *if* $e_0=\pm 1$; *otherwise* $r(M)=h(M)=2g+1$.

1.2. Remarks. a) The manifolds with $m=2$ and $g=0$ are already among the manifolds with $m\leqq 1$ and $g=0$.

b) We will show in Sect. 3 that the only cases where $h(M)$ is still undetermined are the cases i) with $e\neq\pm 1/2(2\lambda+1)$ and $m\geqq 6$.

Before proving Theorem 1.1 we state some propositions.

1.3. Proposition. *Let* $M=S(g; e_0; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$ *with* $1<\alpha_1\leqq\ldots\leqq\alpha_m$, *and* $gcd(\alpha_i, \beta_i)=1$, $1\leqq i\leqq m$.

If $m\geqq 2$, M *admits a Heegaard splitting of genus* $2g+m-1$.

If $m\leqq 1$, M *admits a Heegaard splitting of genus* $2g+1$.

Proof. Let B be the basis of M and $x_1, \ldots, x_m$ be the images of the exceptional fibres of M in B. Let $D_1, \ldots, D_m$ be m small disjoint disks centered at $x_1, \ldots, x_m$. We choose a point x_0 on ∂D_1 and denote by $(u_1, v_1, u_2, v_2, \ldots, u_g v_g)$ a system of cuts for the surface B, that is, closed simple curves on B, passing through x_0 but through no x_i, $1\leqq i\leqq m$, and such that the surface obtained from B by cutting along these curves is a disk; see Fig. 1.

The restriction of the Seifert fibration to $B-\bigcup_{i=1}^{m}\mathring{D}_i$ is a trivial S^1-fibration. Therefore, we may assume that the point x_0 and the system of simple closed curves (u_j, v_j), $1\leqq j\leqq g$ belong to the manifold M.

For $m\geqq 2$ we consider the graph Γ in M which is the union of the curves $(u_1, v_1, \ldots, u_g, v_g)$ and of the $m-1$ first exceptional fibres joined to x_0 in the following way (see Fig. 1): First, we join x_1 to x_0 by an arc in M which is projected homeomorphically on a radius of D_1 in the basis, and then we join the $m-2$ other exceptional fibres to x_1 by arcs which are projected homeomorphically on segments in the basis.

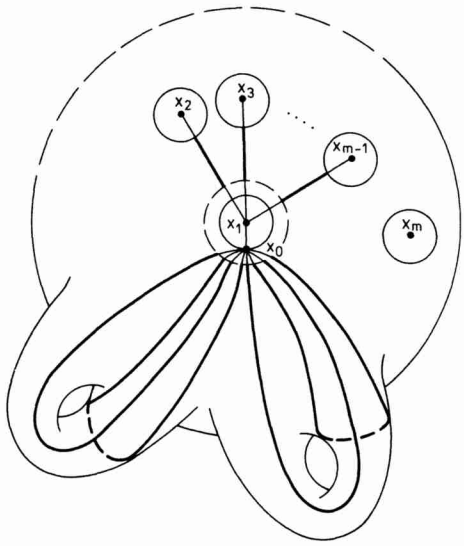


Fig. 1

A regular neighborhood of Γ, $N(\Gamma)$ is a handlebody of genus $2g+m-1$. To see that the closure of the complement $\overline{M-N(\Gamma)}$ is also a handlebody of genus $2g+m-1$, we consider the $2g+m-2$ segments indicated by the heavysolid lines in Fig. 1. There are $2g$ segments which are parts of the curves u_j, v_j, $1 \leqq j \leqq g$ and $m-2$ segments which are parts of the arcs joining x_1 to the exceptional fibres $x_2, \ldots, x_{m-1}$. The pre-image in M of each segment is an annulus in M; therefore, these give rise to $2g+m-2$ disks in $\overline{M-N(\Gamma)}$. Cutting along these $2g+m-2$ disks we obtain a solid torus which is a regular neighbourhood of the m-th exceptional fibre. This completes the proof for $m \geqq 2$.

If $m \leqq 1$, we consider the graph Γ which is the union of the curves $(u_1, v_1, \ldots, u_g, v_g)$ and of the non-singular fibre passing through x_0. Then $N(\Gamma)$, as well as $\overline{M-N(\Gamma)}$, are handlebodies of genus $2g+1$. (For the particular case $M = B \times S^1$, see also [Oc]). □

1.4. Proposition. *Let G be the group presented by:*

$$G = \left\langle s_1, \ldots, s_m, u_1, v_1, \ldots, u_g, v_g, f \mid [s_i, f],\ [u_j, f],\ [v_j, f],\ s_i^{\alpha_i} f^{\beta_i},\ s_1 \ldots s_m \prod_1^g [u_j, v_j]\, f^e \right\rangle$$

where $\alpha_i \geqq 2$ for $1 \leqq i \leqq m$.

i) *If $g=0$, m is even $\geqq 4$, one of the α_i is odd and all the others are equal to 2, then $r(G) \geqq m-2$.*

ii) *If $g>0$ and $m=2$, or if $m \geqq 3$ and G does not belong to the case* i), *then $r(G) \geqq 2g+m-1$.*

iii) *If* $m=1$ *and* $\beta_1-e\alpha_1=\pm 1$, *or if* $m=0$ *and* $e=\pm 1$, *then* $r(G)\geqq 2g$. *Otherwise* $r(G)\geqq 2g+1$.

Proof. The proof is an application of the rank formula for the Fuchsian groups, given in [PRZ; Satz 3]. The group $\bar{G}=G/\langle f\rangle$ is a Fuchsian group with presentation:

$$\bar{G}=\left\langle \bar{s}_1,\ldots,\bar{s}_m,\bar{u}_1,\bar{v}_1,\ldots,\bar{u}_g,\bar{v}_g \mid \bar{s}_i^{\alpha_i}(1\leqq i\leqq m),\bar{s}_1\ldots\bar{s}_m\prod_1^g[\bar{u}_j,\bar{v}_j]\right\rangle.$$

and, clearly, $r(G)\geqq r(\bar{G})$. By the result given in [PRZ; Satz 3], $r(\bar{G})=m-2$ in case i) and $r(\bar{G})=2g+m-1$ in case ii).

For the case iii), an easy calculation shows that

$$G/[G,G]=\begin{cases}\mathbb{Z}^{2g}\oplus\mathbb{Z}_{\beta_1-e\alpha_1} & \text{if } m=1,\\ \mathbb{Z}^{2g}\oplus\mathbb{Z}_e & \text{if } m=0.\end{cases}$$

Since $r(G)\geqq r(G/[G,G])$, this completes the proof. □

1.5. Proposition. *The group*

$$G=\langle s_1,\ldots,s_{2q},f\mid s_1^2 f,\ldots,s_{2q-1}^2 f,s_{2q}^{2\lambda+1}f^{\beta},s_1\ldots s_{2q}f^e\rangle,$$

with $q\geqq 2$, $\lambda>0$ *and* β *arbitrary, admits a presentation with* $2q-2$ *generators and* $2q-2$ *defining relators.*

1.6. Remarks. That the rank of G is $2q-2$ can easily been proved by repeating the argument of [BZ 1]; 2.2.A). The proof in [BZ 1] rests on the fact that f is in the centre of G. For the following proof we need balanced presentations at each stage and we have to ensure that in each reduction of the presentation or alteration of relators the unchanged relators themselves imply the commutativity with f needed in the calculations.

Proof of Proposition 1.5. Define $v_j=s_1s_j$ for $2\leqq j\leqq 2q-1$. We will show that the elements v_j, $2\leqq j\leqq 2q-1$, generate the group G.

First, we obtain a new presentation (P_1) of G:

$$(P_1)\qquad G=\langle s_1,\ldots,s_{2q},f,v_2,\ldots,v_{2q-1}\mid s_i^2 f(1\leqq i\leqq 2q-1),\ s_{2q}^{2\lambda+1}f^{\beta},s_1\ldots s_{2q}f^e,\ v_js_j^{-1}s_1^{-1}(2\leqq j\leqq 2q-1)\rangle.$$

We denote the defining relations of the presentation (P_1) as follows:

(A) $s_i^2 f=1\quad(1\leqq i\leqq 2q-1)$

(B) $s_1\ldots s_{2q}f^e=1$

(C) $v_js_j^{-1}s_1^{-1}=1\quad(2\leqq j\leqq 2q-1)$

(D) $s_{2q}^{2\lambda+1}f^{\beta}=1.$

The relations (A), (B) and (C) imply that f commutes with $s_1,\ldots,s_{2q}$, $v_2,\ldots,v_{2q-1}$ and, therefore, f belongs to the centre of G.

We next define $w=v_2v_3^{-1}v_4\ldots v_{2q-1}^{-1}v_2^{-1}v_3v_4^{-1}\ldots v_{2q-1}$. Using relations (A), (B) and (C) we then obtain:

$$\begin{aligned} w&=(s_1s_2)(s_3^{-1}s_1^{-1})(s_1s_4)\dots(s_{2q-1}^{-1}s_1^{-1})(s_1s_3)\dots(s_1s_{2q-1})\\ &=s_1s_2s_3fs_4\dots s_{2q-1}fs_1fs_2fs_3\dots s_{2q-1}\\ &=(s_1\dots s_{2q-1})^2f^{2q-1}\\ &=s_{2q}^{-2}f^{-2e+2q-1}. \end{aligned}$$

Therefore, the relation (D) can be replaced by the relation,

(D^{I}) $\qquad s_{2q}f^{-\gamma}w^{-\lambda}=1 \quad$ with $\gamma=-\beta+\lambda(2e-2q+1)$.

The systems of defining relations (A), (B), (C), (D) and (A), (B), (C), (D^{I}) are equivalent. Using (D^{I}), we replace s_{2q} by $w^{\lambda}f^{\gamma}$ in (B) to get the relation

(B^{I}) $\qquad s_1\dots s_{2q-1}w^{\lambda}f^{\gamma+e}=1.$

Now we may drop s_{2q} and the relation (D^{I}) to obtain the following presentation (P$_2$) of G:

(P$_2$) $\qquad G=\langle s_1,\dots,s_{2q-1},f,v_2,\dots,v_{2q-1}\,|\,s_i^2f(1\leqq i\leqq 2q-1),$
$s_1\dots s_{2q-1}w^{\lambda}f^{\gamma+e},\ v_js_j^{-1}s_1(2\leqq j\leqq 2_{q-1})\rangle$

with $w=v_2v_3^{-1}v_4\dots v_{2q-1}^{-1}v_2^{-1}v_3v_4^{-1}\dots v_{2q-1}$.

By the relation (A),

$$\begin{aligned} s_1s_2\dots s_{2q-2}&=(s_1s_2)(s_3s_1^{-1}s_1s_4)\dots(s_{2q-3}s_1^{-1}s_1s_{2q-2})\\ &=(s_1s_2)(f^{-1}s_3^{-1}s_1^{-1}s_1s_4)\dots(f^{-1}s_{2q-3}^{-1}s_1^{-1}s_1s_{2q-2})\\ &=(s_1s_2)(s_3^{-1}s_1^{-1})(s_1s_4)\dots(s_{2q-3}^{-1}s_1^{-1})(s_1s_{2q-2})f^{-(q-2)}. \end{aligned}$$

(We have used that f commutes with $s_1,\dots,s_{2q-1}$ which follows from (A).)

Hence, if we define $x=v_2v_3^{-1}v_4\dots v_{2q-3}^{-1}v_{2q-2}$, then the relations (A) and (C) imply that

$$s_1\dots s_{2q-2}=xf^{-(q-2)}.$$

Therefore, using (A) and (C) the relation (B^{I}) $s_1\dots s_{2q-1}w^{\lambda}f^{\gamma+e}=1$ can be replaced by the relation

(B^{II}) $\qquad xs_{2q-1}f^{\delta}w^{\lambda}=1 \quad$ with $\delta=\gamma+e+2-q$.

(Here we use that the relations (A) and (C) imply that f commutes with $s_1,\dots,s_{2q-1},v_2,\dots,v_{2q-1}$.)

The system of defining relations (A), (B^{II}), (C) is equivalent to the system (A), (B^{I}), (C). Now, using (B^{II}), we may replace s_{2q-1} by $x^{-1}w^{-\lambda}f^{-\delta}$ in the relations (A) and (C) to get the relations:

(A^{I}) $\qquad s_i^2f=1 \quad (1\leqq i\leqq 2q-2),$

(A$^{\mathrm{I}}_{2q-1}$) $\qquad (x^{-1}w^{-\lambda}f^{-\delta})^2f=1,$

(C^{I}) $\qquad v_js_j^{-1}s_1^{-1}=1 \quad (2\leqq j\leqq 2q-2),$

(C$^{\mathrm{I}}_{2q-1}$) $\qquad v_{2q-1}f^{\delta}w^{\lambda}xs_1^{-1}=1.$

Now we drop s_{2q-1} and the relation (B^{II}) to obtain the following presentation (P_3) of G:

$$(P_3) \qquad G=\langle s_1,\dots,s_{2q-2},f,v_2,\dots,v_{2q-1} \mid s_i^2 f\ (1\leqq i\leqq 2q-2),\ (x^{-1}w^{-\lambda}f^{-\delta})^2 f,\ v_j s_j^{-1}s_1^{-1}\ (2\leqq j\leqq 2q-2),\ v_{2q-1}f^{\delta}w^{\lambda}xs_1^{-1}\rangle$$

with $x=v_2v_3^{-1}v_4\dots v_{2q-3}^{-1}v_{2q-2}$ and $w=v_2v_3^{-1}v_4\dots v_{2q-1}^{-1}v_2^{-1}v_3v_4^{-1}\dots v_{2q-1}$.

Using the relation (C^{I}_{2q-1}) we replace s_1 by $v_{2q-1}w^{\lambda}f^{\delta}x$ in the relations (A^{I}) and (C^{I}) to obtain the relations:

$$(A_1^{II}) \qquad (v_{2q-1}f^{\delta}w^{\lambda}x)^2 f=1$$

$$(A^{II}) \qquad s_i^2 f=1 \qquad (2\leqq i\leqq 2q-2)$$

$$(C^{II}) \qquad v_j s_j^{-1}x^{-1}w^{-\lambda}f^{-\delta}v_{2q-1}^{-1}=1 \qquad (2\leqq j\leqq 2q-2).$$

Next we drop s_1 and the relation (C^{I}_{2q-1}) and obtain the following presentation (P_4) for G:

$$(P_4) \quad G=\langle s_2,\dots,s_{2q-2},f,v_2,\dots,v_{2q-1} \mid (v_{2q-1}f^{\delta}w^{\lambda}x)^2 f,\ s_i^2 f\ (2\leqq i\leqq 2q-2),\ (x^{-1}w^{-\lambda}f^{-\delta})^2 f,\ v_j s_j^{-1}x^{-1}w^{-\lambda}f^{-\delta}v_{2q-1}^{-1}\ (2\leqq j\leqq 2q-2)\rangle.$$

Using the relations (C^{II}), we replace s_j $(2\leqq j\leqq 2q-2)$ by $x^{-1}w^{-\lambda}f^{-\delta}v_{2q-1}^{-1}v_j$ in the relations (A^{II}) and obtain the relations:

$$(A^{III}) \qquad (x^{-1}w^{-\lambda}f^{-\delta}v_{2q-1}^{-1}v_i)^2 f=1 \qquad (2\leqq i\leqq 2q-2).$$

Next we drop s_j $(2\leqq j\leqq 2q-2)$ and the relations (C^{II}) to obtain the following presentation (P_5) of G:

$$(P_5) \quad G=\langle f,v_2,\dots,v_{2q-1} \mid (v_{2q-1}f^{\delta}w^{\lambda}x)^2 f,\ (x^{-1}w^{-\lambda}f^{-\delta}v_{2q-1}^{-1}v_i)^2 f\ (2\leqq i\leqq 2q-2),\ (x^{-1}w^{-\lambda}f^{-\delta})^2 f\rangle.$$

The relation $(A^{I}_{2q-1})(x^{-1}w^{-\lambda}f^{-\delta})^2 f=1$ implies that f commutes with $x^{-1}w^{-\lambda}$. Further, the relations (A^{I}_{2q-1}) and (A_1^{II}) imply that f commutes with v_{2q-1}. Therefore, the relations (A^{I}_{2q-1}) and (A_1^{II}) imply that $(x^{-1}w^{-\lambda})^2 f^{-2\delta+1}=1$ and $(v_{2q-1}w^{\lambda}x)^2 f^{2\delta+1}=1$.

Hence, the relation

$$(1) \qquad (x^{-1}w^{-\lambda})^2(v_{2q-1}w^{\lambda}x)^2 f^2=1$$

is a consequence of (A^{I}_{2q-1}) and (A_1^{II}). The relation (1) implies that f^2 belongs to the subgroup $\langle v_2,\dots,v_{2q-1}\rangle$ of G.

Using the fact that f commutes with $x^{-1}w^{-\lambda}$ (a consequence of (A^{I}_{2q-1})) the relation

$$(A_2^{III}) \qquad (x^{-1}w^{-\lambda}f^{-\delta}v_{2q-1}^{-1}v_2)^2 f=1.$$

can be replaced by the relation

$$(A_2^{IV}) \qquad f^{-\delta+1}yf^{-\delta}y=1, \quad \text{where } y=x^{-1}w^{-\lambda}v_{2q-1}^{-1}v_2.$$

By definition, w, x and y belong to $\langle v_2, \ldots, v_{2q-1}\rangle$.

The relation (1), which is a consequence of the relations (A^{I}_{2q-1}) and (A^{II}_{1}), implies that either $f^{-\delta+1}$ or $f^{-\delta}$ belongs to the subgroup $\langle v_2, \ldots, v_{2q-1}\rangle$ and that the relation (A^{IV}_{2}) is equivalent to a relation (A^{V}_{2}) $fz=1$ where z belongs to $\langle v_2, \ldots, v_{2q-1}\rangle$.

This yields the following presentation (P_6) of G:

$$(P_6)\qquad G=\langle f, v_2, \ldots, v_{2q-1} \mid (v_{2q-1}f^{\delta}w^{\lambda}x)^2 f, fz, (x^{-1}w^{-\lambda}f^{-\delta}v_{2q-1}^{-1}v_i)^2 f \quad (3\leqq i\leqq 2q-1), (x^{-1}w^{-1}f^{-\delta})^2 f\rangle.$$

Now we replace f by z^{-1} in the other defining relations of the presentation (P_6) of G, and then drop f and the relation $fz=1$ to obtain the desired presentation (P_7) of G with $2q-2$ generators and $2q-2$ defining relations:

$$(P_7)\qquad G=\langle v_2, \ldots, v_{2q-1} \mid (v_{2q-1}z^{-\delta}w^{\lambda}x)^2 z^{-1}, (x^{-1}w^{-\lambda}z^{\delta}v_{2q-1}^{-1}v_i)^2 z^{-1} \quad (3\leqq i\leqq 2q-2), (x^{-1}w^{-\lambda}z^{\delta})^2 z^{-1}\rangle.$$

(For the definitions of λ, w, x, δ, z see 1.5, (P_2), (B^{II}) and (A^{V}_{2}).) □

Proof of Theorem 1.1.

1.7. Case i): By Proposition 1.3, $r(M)\leqq h(M)\leqq m-1$ and by Proposition 1.5, $r(M)=m-2$. Therefore, $m-2=r(M)\leqq h(M)\leqq m-1$.

If $e_0=\varepsilon/2(2\lambda+1)$, $\varepsilon=\pm 1$, then $\beta_m/2\lambda+1\equiv\varepsilon\lambda/2\lambda+1 \bmod 1$, since

$$e=e_0+\frac{m-1}{2}+\beta_m/2\lambda+1\in\mathbb{Z}.$$

The Seifert manifold $S(0;\ \varepsilon/2(2\lambda+1);\ 1/2, \ldots, 1/2,\ \varepsilon\lambda/2\lambda+1)$ with $\lambda>0$ and $m\geqq 4$ exceptional fibres, m even, is a 2-fold covering of the Seifert manifold $S(0;\ \varepsilon/2\lambda+1;\ 2\varepsilon\lambda/2\lambda+1)$ branched along $m-1$ ordinary fibres (see [Se; §14]). The manifold $S(0;\ \varepsilon/2\lambda+1;\ 2\varepsilon\lambda/2\lambda+1)$ is a lens space with fundamental group $\langle s_1, f \mid s_1^{2\lambda+1}f^{2\varepsilon\lambda}, s_1 f^{\varepsilon}\rangle$, $\varepsilon=\pm 1$, which is trivial. Therefore $S(0;\ \varepsilon/2\lambda+1;\ 2\varepsilon\lambda/2\lambda+1)$ is homeomorphic to S^3 and the branching set is the torus link $T(m-1;\ (m-1)(2\lambda+1))$ (cf. [BuM]). This link can be presented with $m-1$ bridges; hence according to Birman-Hilden [BiH], the 2-fold branched covering $S(0;\ \varepsilon/2(2\lambda+1);\ 1/2. \ldots, 1/2,\ 2\varepsilon\lambda/2\lambda+1)$ admits a Heegaard splitting of genus $m-2$.

1.8. Case ii): from Proposition 1.3. $r(M)\leqq h(M)\leqq 2g+m-1$, and from Proposition 1.4 ii), $r(M)\geqq 2g+m-1$. Therefore, $r(M)=h(M)=2g+m-1$.

1.9. Case iii): if $m=1$ and $e_0\neq\pm 1/\alpha_1$ or if $m=0$ and $e_0\neq\pm 1$, then, by Proposition 1.3, $r(M)\leqq h(M)\leqq 2g+1$ and by Proposition 1.4 iii), $r(M)\geqq 2g+1$. Therefore, in these cases $r(M)=h(M)=2g+1$.

If $m=1$ and $e_0=\pm 1/\alpha_1$ or if $m=0$ and $e_0=\pm 1$, by Proposition 1.4 iii) $r(M)\geqq 2g$. By [Mo 1; 5.11, b] the manifold $S(g;\ -\varepsilon/\alpha_1)$, $\varepsilon=\pm 1$ is a 2-fold covering of S^3 branched over the torus link $T(2g+1,\ 2\alpha_1(2g+1))$, which has a $2g+1$ bridge presentation. As in Case i), its 2-fold branched covering has a Heegaard splitting of genus $2g$. □

For the last argument we give another proof which uses a different geometric construction due to K. Reidemeister [Re].

Remark. If $m=1$ and $e_0=\varepsilon/\alpha_1$, $\varepsilon=\pm 1$, then $\beta_1/\alpha_1 \equiv -\varepsilon/\alpha_1 \bmod 1$ since $e=\varepsilon/\alpha_1 + \beta_1/\alpha_1 \in \mathbb{Z}$.

1.10. Reidemeister construction of a Heegaard splitting of genus $2g$ for $M=S(g;\ \varepsilon/\alpha_1;\ -\varepsilon/\alpha_1)$, $\varepsilon=\pm 1$

For the manifolds $M=S(g;\ \varepsilon/\alpha_1;\ -\varepsilon/\alpha_1)$, $\varepsilon=\pm 1$, we perform the same construction as used by Reidemeister in [Re] for the case $g=1$. In the following proof α_1 may take the value 1; that allows us to extend the construction to the case $m=0$ and $e_0=\pm 1$.

Following Seifert [Se] (see also [Si]) the Seifert manifold $M=S(g;\ \varepsilon/\alpha_1;\ -\varepsilon/\alpha_1)$ can be obtained in the following way: Let F be a closed connected orientable surface of genus g. We choose one disk D on F and denote $F-\mathring{D}$ by $\check{F}$. The Seifert manifold $M=S(g;\ \varepsilon/\alpha_1;\ -\varepsilon/\alpha_1)$ will be obtained by gluing two Seifert manifolds $\check{M}$ and T, where $\check{M}$ is the trivial S^1 bundle over $\check{F}$ and T is a solid Seifert fibred torus associated with the pair $(\alpha_1, -\varepsilon)$. Next we describe the gluing map.

We denote by c the curve $\partial\check{F}$ in $\partial\check{M}$ and let $s: \partial D \to \partial T$ (D is the base space (orbifold) of the Seifert fibration of T) be the oriented section of the S^1-fibration of ∂T such that a meridian m of T is given by $m=s^{\alpha_1}f^{-\varepsilon}$, where f is an oriented fibre of the S^1-fibration of ∂T verifying $s\cdot f=+1$ on ∂T.

We glue $\check{M}$ and T by a diffeomorphism θ: $\partial\check{M}\to\partial T$, which reverses the orientation and sends the curve c to the section s on ∂T. Then. according to [Se], we have $M=\check{M}\cup_\theta T$. If we let $H=\overline{\check{M}-N(\check{F})}$, where $N(\check{F})$ is a regular neighbourhood of $\check{F}$ in $\check{M}$, then H is a handlebody of genus $2g$.

We consider $H'=N(\check{F})\cup_{\theta'} T$, where θ' is the restriction of the gluing map θ to the annulus $A=N(\check{F})\cap\check{M}$. Then H' is obtained from the handlebody $N(\check{F})$ by sewing a solid torus on $\partial N(\check{F})$ along an annulus A with core the curve c. To show that H' is a handlebody of genus $2g$ we have to verify that $\pi_1(H')$ is a free group of rang $2g$.

By the Seifert-van Kampen theorem, $\pi_1 H'=\pi_1 N(\check{F})*_{\pi_1 A}\pi_1 T$, where the image of the group $\pi_1 A$ in $\pi_1 N(\check{F})$ is generated by the curve c, and in the group $\pi_1 T$ by the section s. In $\pi_1 T$, $s^{\alpha_1}=f^{\varepsilon}$ and $f=l^{\alpha_1}$, where l is a generator of $\pi_1 T$; therefore $s=l^{\varepsilon}$ in $\pi_1 T$.

Hence, $\pi_1 H'=\pi_1 N(\check{F})_{c=l^{\varepsilon}}*\langle l|\rangle$, with $\varepsilon=\pm 1$ and $\pi_1 H'$ is isomorphic to $\pi_1 N(\check{F})$, so H' is a handlebody of genus $2g$. Since $M=H\cup H'$ with $\partial H=\partial H'$, M admits a Heegaard splitting of genus $2g$. □

1.11. Remark. If $e_0=\mu/\alpha_1$ with $\mu\neq\pm 1$, then the above construction does not give a Heegaard splitting for $M=S(g;\ \mu/\alpha_1;\ -\mu/\alpha_1)$ because H' is no longer a handlebody. Further, since the curve c is null-homologous in $H(\check{F})$, the group $\pi_1(H')$ contains an element with torsion $\mu\neq\pm 1$ and $\pi_1 H'$ is no longer a free group.

2. Heegaard genus of Seifert manifolds with non orientable basis

An orientable Seifert manifold supports an S^1-action of the circle only if the basis is orientable. In the general case a Seifert manifold is defined as a manifold foliated by circles (the fibres) such that each fibre admits a regular fibred neighbourhood (see [Se], [Or], [OVZ], [Si]). The exceptional fibres can be described in the same way as for the case of an orientable basis (see [Se]) using the number $\beta/\alpha \in (\mathbb{Q}/\mathbb{Z})^*$ characterizing the holomomy of these exceptional fibres. Therefore, a closed orientable Seifert manifolds with non-orientable basis can be associated with a system of invariants $(-g;\ e_0;\ \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$ where $e_0 \in \mathbb{Q}$, $\beta_i/\alpha_i \in (\mathbb{Q}/\mathbb{Z})^*$ and g is the genus of the basis (see [Se], [Or], [OVZ], [Si]). The usual Euler number of the Seifert fibration is still $e = e_0 + \sum_{i=1}^{m} \beta_i/\alpha_i \in \mathbb{Z}$.

In the same way as for Seifert manifolds with orientable basis, see [Se], any system of invariants $(-g;\ e_0;\ \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$ such that $e = e_0 + \sum_{i=1}^{m} \beta_i/\alpha_i \in \mathbb{Z}$ belongs to a closed Seifert manifold with non-orientable basis, determined up to orientation preserving homeomorphism. Such a Seifert manifold will be denoted by $S(-g;\ e_0;\ \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$.

Then, using the rank formula for the factor group $\pi_1 M/\langle f \rangle$ given in [PRZ; Satz 3, Korollar], when M is a Seifert manifold with non-orientable basis, and extending Proposition 1.3 for the case $m > 1$, and for $m \leqq 1$ repeating the Reidemeister construction 1.10 or noting that by [Mo 1; 3.1] M is a 2-fold covering of S^3 branched over a link with $g+2$ bridges or $g+1$ bridges if $e_0 = \pm\frac{1}{\alpha_1}$ (see [BZ 2; Théorème 2.1]) the next theorem follows.

2.1. Theorem. *Let $M = (-g;\ \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$ be a Seifert manifold with non-orientable basis and with $1 < \alpha_1 \leqq \ldots \leqq \alpha_m$ and $gcd(\alpha_i, \beta_i) = 1$, $1 \leqq i \leqq m$.*

i) *If $m \geqq 2$, then $r(M) = h(M) = g + m - 1$.*

ii) *If $m = 1$, then* $r(M) = h(M) = \begin{cases} g, & e_0 = \pm 1/\alpha_1 \\ g+1, & \text{otherwise.} \end{cases}$

iii) *If $m = 0$, then* $r(M) = h(M) = \begin{cases} g, & e_0 = \pm 1 \\ g+1, & \text{otherwise.} \end{cases}$ □

2.2. Remark. In the case of a Seifert manifold with non-orientable basis, Heegaard genus and rank of the fundamental group coincide.

3. Discrepancy between the Heegaard genus and the rank of the fundamental group

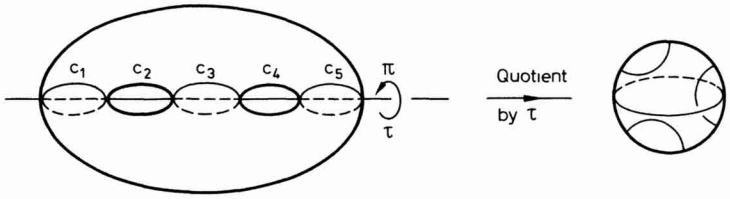
Theorems 1.1 and 2.1 show that $r(M) = h(M)$ for an orientable closed Seifert manifold M, except perhaps in the case 1.1 i) of Theorem 1.1. where we are only able to prove that $h(M) - r(M) \leqq 1$. In these cases, we have not been able

to calculate $h(M)$ if $e_0 \neq \pm 1/2(2\lambda+1)$ and $m \geqq 6$, but for $m=4$ we have the following result, which provides examples of prime 3-manifolds M such that $r(M)<h(M)$ (see also BZ 1]):

3.1. Theorem. *Let* $M=S(0; e_0; 1/2, 1/2, 1/2, \beta/2\lambda+1)$ *with* $\lambda>0$, $gcd(\beta, 2\lambda+1)=1$. *If* $e_0 \neq \pm 1/2(2\lambda+1)$, *then* $2=r(M)<h(M)=3$.

The proof of Theorem 3.1 follows from two lemmas the first of which is well-known.

We recall that a link L in S^3 admits an *n-bridge presentation* if there exists a 2-sphere S^2 embedded in S^3, meeting L transversally in $2n$ points and diving S^3 in two balls B_i, $i=1,2$, such that $L \cap B_i$, $i=1,2$, is formed by n arcs unknotted, unliked and trivially embedded in B_i as in Fig. 2.



Handlebody of genus 2 with the standard rotation of angle π about an axis meeting the boundary in 6 points.

Ball which contains 3 unknotted and unlinked arcs corresponding to the image of the image of the fixed point set of τ.

Fig. 2

3.2. Lemma (see [BiH], [Vi]). *A closed orientable connected 3-manifold M admits a Heegaard splitting of genus 2 if and only if M is a 2-fold covering of* S^3 *branched along a link with a 3-bridge presentation.*

The proof of Lemma 3.2 follows from the fact that any homeomorphism of a surface of genus 2 is isotopic to a product of Dehn-twists along the curves $c_1, \ldots, c_5$ of Fig. 2 (cf. [De], see also [Bi; 4.3], [Li]). Therefore any diffeomorphism of a surface of genus 2 is isotopic to a τ-equivariant one, where τ is the standard orientation preserving involution of the surface of genus 2 with 6 fixed points (see Fig. 2). The proof now follows easily using Fig. 2 which describes the quotient of a handlebody of genus 2 by the involution τ. □

3.3. Lemma. *The Seifert manifold* $M=S(0; e_0; 1/2, 1/2, 1/2, \beta/2\lambda+1)$, $\lambda>0$, $gcd(\beta, 2\lambda+1)=1$, *is a 2-fold covering of* S^3 *branched along a link L which admits a 3-bridge presentation if and only if* $e_0=\pm 1/2(2\lambda+1)$.

Proof. If $e_0=\pm 1/2(2\lambda+1)$, M has Heegaard genus 2 by Theorem 1.1 i) and the assertion follows from Lemma 3.2.

Thus, suppose $M=S(0; e_0; 1/2, 1/2, 1/2, \beta/2\lambda+1)$ is a 2-fold branched covering of S^3 and let τ denote the covering involution. By Bonahon [Bo] and Tollefson [To] every involution on M is conjugate to a fibre preserving one. Hence, we may assume that τ is fibre-preserving. So we have to consider two cases depending on the action of τ on the fibres:

3.4. τ reverses to orientation of the fibres. In this case the involution $\bar{\tau}$, induced by τ on the basis S^2 of M, reverses the orientation of S^2. As τ has a non-empty

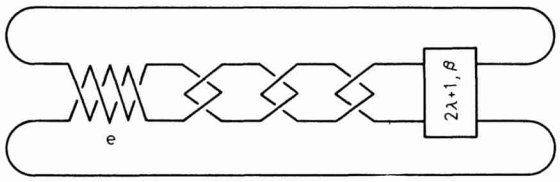
fixed point set, $\bar{\tau}$ has a non empty fixed point set. Therefore $\bar{\tau}$ is the reflection through a circle of S^2. Since τ respects the set of exceptional fibres, $\bar{\tau}$ respects the set of the images of the exceptional fibres in S^2. Then there are two possibilities, as shown in Fig. 3:

a) τ respects each exceptional fibre

b) τ exchanges two exceptional fibres of type $1/2$

Fig. 3

a) In this case, by Montesinos [Mo 1], [Mo 2] (see also [BuZ, §12], [Zi]) M/τ is S^3 and the branch set is the Montesinos link $m(0; e_0; 1/2, 1/2, 1/2, \beta/2\lambda+1)$ (see Fig. 4) with 3 components but with 4 bridges because two components are trivial and the third is a 2-bridge knot (see also [BZ 2]).



Montesinos link $m(0, e_0, 1/2, 1/2, 1/2, \beta/2\lambda+1)$,
$e = e_0 + 3/2 + \beta/2\lambda+1 \in \mathbb{Z}$

Fig. 4

b) In the second case the quotient M/τ is not S^3. In fact, according to Montesinos' construction ([Mo 1], [Mo 2], see also [BuZ; §12], [Zi]), M/τ is obtained by Dehn-surgery with coefficient 1/2 along a trivial solid torus in S^3. (After the surgery, this solid torus corresponds to the image in M/τ of regular neighbourhoods of the two exceptional fibres of type 1/2 which are exchanged by τ.) Therefore $\pi_1(M/\tau) \cong \mathbb{Z}_2$ and M is a 2-fold branched covering of the lens space $L(2, 1)$ (see Fig. 5).

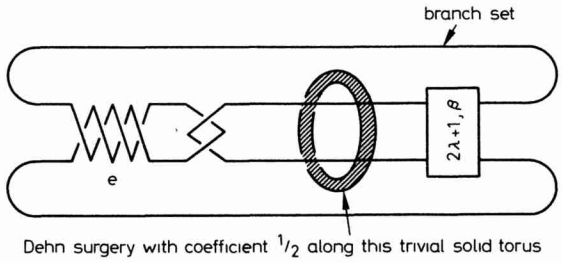


Fig. 5

Hence, if τ reverses the orientation of the fibres, M is never a 2-fold covering of S^3 branched along a link which admits a 3-bridge presentation.

3.5. τ preserves the orientation of the fibres. In this case the manifold M/τ is a Seifert manifold, and the basis of M/τ is $S^2/\bar{\tau}$, where $\bar{\tau}$ preserves the orientation of S^2. Since, by assumption, M/τ is homeomorphic to S^3, $S^2/\bar{\tau}$ is homeomorphic to S^2 and $\bar{\tau}$ admits 2 fixed points on S^2 (case c) or $\bar{\tau}$ is the identity on S^2 (case d).

c) If $\bar{\tau}$ admits 2 fixed points on S^2, then the fixed point set of τ in M admits at most 2 connected components. Since M is a 2-fold covering of S^3 branched along a link L, then it is well known (see for example [BuZ; E 9.5]) that the number of connected components of L is equal to $\operatorname{rank}(H_1(M;\mathbb{Z}_2))+1$. In this case $\operatorname{rank}(H_1(M;\mathbb{Z}_2))=2$; therefore, the fixed point set of τ in M will have 3 connected components, so this case cannot occur.

d) If $\bar{\tau}$ is the identity on S^2, then τ determines a $\mathbb{Z}_2$-action on M embedded in the S^1-action defining the Seifert fibration of M. Therefore, according to [Se; §14], $M/\tau=S(0;\ 2e_0;\ 2\beta/2\lambda+1)$. (This can be also seen on the fundamental group $\pi_1 M$ by sending f to f^2.) In particular M/τ is a lens space and $\pi_1(M/\tau)=\langle s_1, f \mid s_1^{2\lambda+1} f^{2\beta}, s_1 f^e\rangle$ with $e=2e_0+2\beta/2\lambda+1$. Therefore $\pi_1(M/\tau)\cong\mathbb{Z}_\alpha$ with $\alpha=|2e_0(2\lambda+1)|$ and M/τ is S^3 iff $\alpha=1$ or $e_0=\pm 1/2(2\lambda+1)$, which is the announced condition. □

3.6. Proof of Theorem 3.1. Lemma 3.2 and 3.3 imply that M has Heegaard genus $h(M)\geqq 3$ if $e_0 \neq \pm 1/2(2\lambda+1)$. Therefore, Propositions 1.3 and 1.5 show that $2=r(M)<h(M)$ in this case. □

3.7. Remarks

1) For $M=S(0;\ e_0;\ 1/2,\ 1/2,\ 1/2,\ \beta/2\lambda+1)$, $\lambda>0$, $gcd(\beta, 2\lambda+1)=1$, $\pi_1 M$ admits a presentation with 2 generators and 2 defining relations, by Proposition 1.5. Therefore the answer to the genus question of Waldhausen is again negative even if we consider a "balanced" presentation of $\pi_1 M$. Moreover by Waldhausen's results [Wa 1], the "balanced" presentation obtained in Proposition 1.5 for $\pi_1 M$ can never correspond to a Heegaard splitting of a 3-manifold since M is a closed Haken manifold.

2) The examples in Theorem 3.1 are prime manifolds. Since $h(M)$ and $r(M)$ behave additively with respect to the connected sum (see [Ha 2]) there are non-prime closed connected 3-manifold M with arbitrarily large discrepancy $h(M)-r(M)$. We do not known whether there are prime manifolds M with arbitrarily large discrepancy $h(M)-r(M)$.

3) The examples of Theorem 3.1 do not give any indication on the Poincaré Conjecture. However, according to F. Waldhausen [Wa 1], there may be some interest in looking for a negative answer to the genus question for the 3-manifolds with boundary embedded in S^3 and, in particular, for the knot spaces.

4) R. Osborne (Fort Collins, Colorado) informed us that the fundamental group of the manifold $M=S(0;\ e_0;\ 1/2,\ 1/2,\ 1/2,\ \beta/2\lambda+1)$ has the following presentation obtained from a Heegaard splitting of genus 3:

$$\pi_1 M=\langle a, b, c | a^{-2}b^2, a^{-2}c^2, W_{2\lambda+1,\beta+e(2\lambda+1)}(c^{-1}b^{-1}a^{-1}, a^{-2})\rangle,$$

where $e=e_0+3/2+\beta/(2\lambda+1)$ and $W_{m,n}(x, y)$ denotes the unique primitive in $\langle x, y\rangle$ whose abelianization is $x^n y^m$ (see [OZ; Theorem 1.2]).

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