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Semiregular maximal abelian $*$-subalgebras and the solution to the factor state Stone-Weierstrass problem

S. Popa

Department of Mathematics, National Institute for Scientific and Technical Creation, Bdul Pacii 220, 79622 Bucharest, Romania

Let M be a factor von Neumann algebra and $A \subset M$ a maximal abelian $*$-subalgebra (abbreviated in the sequel as MASA) in M. A is called *semiregular* if the normalizing group of A in M, $\mathcal{N}(A) = \{u$ unitary element in $M | uAu^* = A\}$, acts ergodically on A ([6]).

In [10] we gave a constructive proof of the existence of semiregular MASA's in weakly separable factors of type II_1, II_∞. By Connes' discrete decomposition this automatically yields the existence of semiregular MASA's in factors of type III_λ, $0<\lambda<1$. Moreover, using the discrete decomposition of type III_0 factors (see 5.3.6 in [4]) and going along the line of the proofs in [10] (see also [1]), one can easily show the existence of semiregular MASA's in factors of type III_0. Further developments of these results may be found in [11]. In this paper we prove that type III_1 factors also have semiregular MASA's, thus obtaining the following general result:

Theorem 1. *Any separable factor has a semiregular MASA.*

Actually, as in the cases II_1, II_∞, III_λ, $0 \leqq \lambda < 1$, we prove that given a III_1 factor M and M_0 a uniformly hyperfinite C^*-algebra with diagonal A_0 ([8]), there exists a representation π_0 of M_0 in M (i.e. $\pi_0(M_0) \subset M$) such that $N = \pi_0(M_0)''$ is a factor and $\pi_0(A_0)''$ is maximal abelian in M. This clearly implies Theorem 1 (see also 4.1 and Remark 3.5 in [10]), and also the following:

Theorem 2. *Any separable factor M has an approximately finite dimensional subfactor N with trivial relative commutant in M,* $N' \cap M = \mathbb{C}$.

An important feature of these results is related to the Stone-Weierstrass problem for (norm) separable C^*-algebras. In [2] Anderson and Bunce, based on preceding work by Sakai [12], proved that if any separable factor has a semiregular MASA then the factorial Stone-Weierstrass conjecture holds true. Thus, our Theorem 1 constitutes the last "brick" in the proof of the following:

Theorem 3. *If $B_0 \subset B$ are separable C^*-algebras and B_0 separates the factor states of B then $B_0 = B$.*

Another consequence of our results is related to the possibility of extending factor states from a C^*-algebra B_0 to a larger one $B \supset B_0$. Indeed, by a result of Sakai (cf. Theorem 7 in [3]), the preceding Theorem 2 entails:

Theorem 4. *If $B_0 \subset B$ are separable C^*-algebras then every factor state of B_0 extends to a factor state of B.*

We mention that for properly infinite factors a different proof of Theorem 2 was independently obtained by R. Longo [9], using the machinery developed in [7]. We shall end the paper by giving a global and elementary proof for the existence of semiregular MASA's in all factors of type III, assuming the apriori existence of approximately finite dimensional subfactors with trivial relative commutant.

In the proof of the partial results obtained in [10] we considered a criterion for an abelian *-subalgebra of M, range of a normal conditional expectation and generated recursively by finite partitions, to be maximal abelian in M. We now use another criterion, that is also suitable to inductive constructions, but does not involve existence of normal conditional expectations. This criteria is an easy consequence of a result by Skau ([13], Corollary 1) and of the Kaplansky density theorem. Let a von Neumann algebra M be represented on a separable Hilbert space $\mathscr{H}$ with cyclic and separating vector ξ and choose a norm dense sequence of vectors $\{\xi_n\}_n$ in $\mathscr{H}$. Let $A \subset M$ be an abelian von Neumann subalgebra of M which is the weak limit of an increasing sequence of finite dimensional subalgebras $\{A_n\}_n$, $\overline{\bigcup_n A_n}^w = A$.

Lemma 1. *The following two conditions are equivalent:*

(i) *A is maximal abelian in M.*

(ii) *There exist $z_n \in (A_n \cup M')''$, $\|z_n\| \leqq 1$, such that $\|[A_n \xi]\xi_j - z_n \xi_j\| \underset{n}{\rightarrow} 0$, for all j.*

(For $B \subset \mathscr{B}(\mathscr{H})$, $[B\xi]$ denotes the orthogonal projection onto the closed linear span of $B\xi$).

Let M be a separable type III_1 factor. The next Lemma is a consequence of Connes-Störmer transitivity theorem [5]:

Lemma 2. *Let $A_0 \subset M$ be an abelian finite dimensional von Neumann subalgebra of M, φ a normal faithful state on $A'_0 \cap M$ and $\varepsilon > 0$. There exist a matrix algebra $M_0 \subset M$ with diagonal subalgebra A_0 and a normal faithful state φ' on $A'_0 \cap M$, such that $\|\varphi' - \varphi\| < \varepsilon$ and $\varphi'(xy) = \varphi'(x)\varphi'(y)$, for $x \in M'_0 \cap M$ and $y \in A_0$.*

Proof. Let $e_1, \ldots, e_n$ be the minimal projections in A_0 and choose partial isometries $v_1, v_2, \ldots, v_n$ in M with $v_i v_i^* = e_1$, $v_i^* v_i = e_i$ (this is possible because M is of type III). Consider the normal faithful positive forms $\varphi(v_i \cdot v_i^*)$ and $\varphi(e_i \cdot e_i)$ on $e_i M e_i$. By [5] there exist unitary elements u_i in $e_i M e_i$ such that $v_1 u_1 = e_1$ and

$$\|c_i\, \varphi(v_i u_i \cdot u_i^* v_i^*) - \varphi(e_i \cdot e_i)\| < \varepsilon/n$$

where $c_i = \varphi(e_i)\varphi(e_1)^{-1}$. For $x \in A'_0 \cap M = \sum_i e_i M e_i$ let $\varphi'(x) = \sum_i c_i \varphi(v_i u_i x u_i^* v_i^*)$.

Clearly $\|\varphi'-\varphi\|<\varepsilon$ and if M_0 is the algebra generated by the partial isometries $v_i u_i$, then for any $x\in M_0'\cap M$ we have $x=\sum_i u_i^* v_i^* x v_i u_i$ so that $\varphi'(xe_i)=\varphi'(u_i^* v_i^* x v_i u_i)=c_i\varphi(e_1 x e_1)=\varphi'(x)\,\varphi'(e_i)$.

To prove Theorem 1 in the remaining case when M is of type III_1, suppose M acts on the separable Hilbert space $\mathscr{H}$ with cyclic and separating vector ξ and fix $\{\xi_n\}_n$ a norm dense sequence in $\mathscr{H}$. Denote by φ_0 the state on M given by ξ. For each $n\geqq 1$ we construct by induction the objects:

a) A finite dimensional subfactor $M_n\subset M$ with matrix units $\{e_{ij}^n\}_{i,j}$ and diagonal subalgebra $A_n=\mathrm{span}\{e_{ii}^n\}_i$;
b) A normal faithful state φ_n on $A_n'\cap M$;
c) An element $z_n\in(A_n\cup M')''$, $\|z_n\|\leqq 1$;

such that the following conditions hold:

(1) $M_n\supset M_{n-1}$, $A_n\supset A_{n-1}$ and each e_{rs}^{n-1} is the sum of some e_{ij}^n;
(2) $\|\varphi_n-\varphi_{n-1}|_{A_n'\cap M}\|<2^{-n}$ and $\varphi_n(xy)=\varphi_n(x)\varphi_n(y)$ for all $x\in A_n$, $y\in M_n'\cap M$;
(3) $\|[A_n\xi]\xi_j-z_n\xi_j\|<2^{-n}$, $1\leqq j\leqq n$.

Suppose these objects have been constructed up to $n-1$. Let B^0 be an arbitrary maximal abelian subalgebra in $e_{11}^{n-1}Me_{11}^{n-1}$. Then $B=\sum_i e_{i1}^{n-1}B^0e_{1i}^{n-1}$ is maximal abelian in M. Since B^0 is separable there exists an increasing sequence of finite dimensional subalgebras $B_n^0\subset B^0$, $\overline{\bigcup_n B_n^{0}}^{w}=B^0$. Thus $\{\sum_i e_{i1}^{n-1}B_m^0e_{1i}^{n-1}\}_m$ increase to B. By "(i) implies (ii)" in Lemma 1, there exists an m such that if $A_n=\sum_i e_{i1}^{n-1}B_m^0e_{1i}^{n-1}$ then $\|[A_n\xi]\xi_j-z_n\xi_j\|<2^{-n}$, $1\leqq j\leqq n$, for some appropriate element $z_n\in(A_n'\cup M)''$, $\|z_n\|\leqq 1$. Denote by $P=e_{11}^{n-1}Me_{11}^{n-1}$ and $\varphi=\varphi_{n-1}(e_{11}^{n-1})^{-1}\varphi_{n-1}|_P$. We now apply the preceding lemma to $B_m^0=A_ne_{11}^{n-1}\subset e_{11}^{n-1}Me_{11}^{n-1}=P$ and to φ to get a state φ' on $B_m^{0\prime}\cap P$ and a matrix algebra $M_0\subset P$ with matrix unit $\{f_{ij}\}\subset P$ such that f_{ii} are the minimal projections of B_m^0, $\|\varphi'-\varphi|B_m^{0\prime}\cap P\|<2^{-n}$, $\varphi'(xy)=\varphi'(x)\,\varphi'(y)$ for all $x\in B_m^0$, $y\in M_0'\cap P$. Let M_n be the algebra generated by $f_{ij}e_{rs}^{n-1}$, φ_n the state on $A_n'\cap M$ defined by $\varphi_n(x)=\sum_i c_i\varphi'(e_{1i}^{n-1}xe_{i1}^{n-1})$, where $c_i=\varphi_{n-1}(e_i)\,\varphi_{n-1}(e_1)^{-1}$. It is easy then to verify that M_n, φ_n, z_n defined like this verify conditions (1), (2), (3).

Let $A=\overline{\bigcup_n A_n}^{w}$. By condition (3) and "(ii) implies (i)" in Lemma 1, A is maximal abelian in M. Since φ_n are all defined on $A\subset A_n'\cap M$, by (2) the sequence $\{\varphi_n|A\}_n$ converges to a normal state ψ on A. Moreover if $N=\overline{\bigcup_n M_n}^{w}$ then $\psi(xy)=\psi(x)\psi(y)$ for all $x\in\overline{\bigcup A_n}^{w}=A$, $y\in(\bigcup M_n)'\cap M=N'\cap M$. But $A\subset N$ so that $N'\cap M\subset A'\cap M=A$. Thus $\psi(xy)=\psi(x)\psi(y)$ for all $x,y\in N'\cap M$. This means that $N'\cap M$ has a nonzero atom, say e. Hence $Ae\subset eMe$ is a semiregular maximal abelian subalgebra in eMe, $(Ne)'\cap eMe=\mathbb{C}e$ and because eMe is isomorphic to M the proof of Theorem 1 is completed.

Finally let us consider a type III factor M acting on the separable Hilbert space $\mathscr{H}$ with cyclic and separating vector ξ and having an approximately finite dimensional subfactor $N\subset M$ such that $N'\cap M=\mathbb{C}$. We fix $\{\xi_j\}_j$ a norm dense sequence of vectors in $\mathscr{H}$ and $\{y_j\}_j \subset \mathscr{B}(\mathscr{H})$ a sequence of operators dense in the unit ball in the strong operator topology. For each $n \geqq 1$ we construct by induction:

(α) A matrix algebra $M_n \subset M$ with matrix unit $\{e^n_{ij}\}_{i,j}$ and diagonal subalgebra $A_n = \text{span}\{e^n_{ii}\}_i$;

(β) Elements z_n in the unit ball of $(A_n \cup M')''$ and $x^n_1, \ldots, x^n_n$ in the unit ball of $(M_n \cup M')''$;

such that the following conditions are satisfied:

(1) $M_n \supset M_{n-1}$, $A_n \supset A_{n-1}$ and each e^{n-1}_{rs} is the sum of some e^n_{ij};

(2) $\|[A_n \xi] \xi_j - z_n \xi_j\| < 2^{-n}$, $1 \leqq j \leqq n$;

(3) $\|y_k \xi_j - x^n_k \xi_j\| < 2^{-n}$, $1 \leqq j$, $k \leqq n$.

Assume the construction for $1, 2, \ldots, n-1$. Since $e^{n-1}_{11} M e^{n-1}_{11}$ is isomorphic to$M$ there exists an increasing sequence of finite dimensional subfactors N^0_k in $e^{n-1}_{11} M e^{n-1}_{11}$ such that $(\cup N^0_k)' \cap e^{n-1}_{11} M e^{n-1}_{11} = \mathbb{C} e^{n-1}_{11}$. Then the finite dimensional subfactors N_k generated in M by N^0_k and M_{n-1} satisfy $(\cup N_k)' \cap M = \mathbb{C}$ so that $\bigcup\limits_k (N_k \cup M')''$ is a dense *-subalgebra in $\mathscr{B}(\mathscr{H})$. By the Kaplansky density theorem there exist $x^n_1, \ldots, x^n_n$ in the unit ball of $\bigcup\limits_k (N_k \cup M')''$ such that $\|x^n_i \xi_j - y_i \xi_j\| < 2^{-n}$, $1 \leqq i, j \leqq n$. Let k be large enough such that all x^n_i, $1 \leqq i \leqq n$, are in N_k and denote $M^0_n = N_k$. Further let $\{f_{ij}\}$ be a matrix unit for M^0_n such that $\{f_{ii}\}$ is a refinement of e^{n-1}_{jj} and more generally such that each e^{n-1}_{ij} is the sum of some f_{rs}. Let also B^0 be a maximal abelian subalgebra in $f_{11} M f_{11}$, generated by an increasing sequence of finite dimensional subalgebras B^0_m. Then $B_m = \sum\limits_i f_{i1} B^0_m f_{1i}$ increase to a maximal abelian subalgebra in M so that by Lemma 1 there exist an m such that $\|[B_m \xi] \xi_j - z \xi_j\| < 2^{-n}$, $1 \leqq j \leqq n$, for some z in the unit ball of $(B_m \cup M')''$. Take $A_n = B_m$, $z_n = z$ and choose any matrix unit $\{g_{pq}\}_{p.q}$ in $f_{11} M f_{11}$ with $\text{span}\{g_{pp}\}_p = B^0_m = A_n f_{11}$. Then the matrix algebra M_n generated in M by M^0_n and $\{g_{pq}\}_{p.q}$, with the set of matrix units $\{e^n_{ij}\}_{i,j} = \{f_{rs} g_{pq}\}$ satisfy the conditions.

Let $A = \overline{\bigcup\limits_n A^w_n}$ and $N = \overline{\bigcup\limits_n M^w_n}$. Then by (2) and Lemma 1, A is maximal abelian in M. Moreover the normalizer of A in M generates an algebra that contains N by (3), $\bigcup\limits_k (M_k \cup M')''$ is dense in $\mathscr{B}(\mathscr{H})$, so that $(N \cup M')'' = \mathscr{B}(\mathscr{H})$ which means that $N' \cap M = \mathbb{C}$. In particular N is a factor. This shows that A is semiregular in M.

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