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Titel: Calderón-Zygmund Kernels Carried by Linear Subspaces of Homogeneous Nilpotent Lie...

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Jahr: 1983

PURL: https://resolver.sub.uni-goettingen.de/purl?356556735_0073|log33

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Invent. math. 73, 467-489 (1983)

Inventiones mathematicae
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Calderón-Zygmund Kernels Carried by Linear Subspaces of Homogeneous Nilpotent Lie Algebras

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Introduction

In recent years Calderón-Zygmund kernels on nilpotent Lie groups and the corresponding convolution operators have been studied by several authors (see e.g. [4], [13], [15]). In [13] it was shown that the classical theory of Calderón-Zygmund kernels on $\mathbb{R}^n$ has very natural extensions to kernels on nilpotent Lie groups. However, if one wants to study more singular convolution operators, for example convolution with distributions which are extensions of Calderón-Zygmund kernels on linear subspaces, then many of the corresponding questions are open. In [3] D. Geller and E.M. Stein initiated research in this direction and showed that certain homogeneous distribution of degree $-2n$, carried by subspaces complementary to the center of the $(2n+1)$-dimensional Heisenberg group H_n, when extended to H_n, yield bounded convolution operators on $L^p(H_n)$ for $1<p<\infty$.

In this paper we will prove some quite general results about the L^2-boundedness of convolution operators on nilpotent Lie groups which arise from certain classes of Calderón-Zygmund kernels on linear subspaces, supplementary to the center, of the corresponding Lie algebra. The article is organized as follows:

In Sect. 1 we introduce some basic notations, and in Sect. 2 we define two classes (H1) and (H2) of Calderón-Zygmund kernels with weak homogeneity properties and study some properties of these kernels. In Sect. 3 we prove our first main result (Theorem 3.1) which essentially states that a Calderón-Zygmund kernel of class (H1) carried by a linear subspace which is supplementary to the center, and which contains a nontrivial central part, yields a bounded convolution operator on L^2, provided that certain means of the kernel over all affine subspaces which are translates of the central part are zero. Section 4 is a short review of results of Moore and Wolf about square integrable representations of nilpotent Lie groups and concludes with an estimate for twisted convolutions. In the following section we prove another estimate for twisted convolutions with compactly supported distributions on nilpotent Lie groups,

which is based on results of H. Leptin [7]. Finally, in Sect. 6 we present our second main result (Theorem 6.1) which states the following: If k is a Calderón-Zygmund kernel of class (H2) on a subspace supplementary to the center of the Lie algebra (of a homogeneous nilpotent Lie group), and if the group is "completely square integrable", then k yields a bounded convolution operator on L^2. This result generalizes the L^2-part of the result of Geller and Stein.

We would like to thank J. Cygan, A. Hulanicki, H. Leptin and L. Richardson for helpful conversations about this subject.

1. The General Set-Up

Let $\mathfrak{n}$ be a nilpotent Lie algebra with a one-parameter family of dilations $D=\{D_r\}_{r>0}$, satisfying the following conditions:

i) D_r is an automorphism of $\mathfrak{n}$ for any $r>0$,

ii) $D_r D_{r'}=D_{rr'}$ for any $r, r'>0$,

iii) D_r is diagonalizable for any $r>0$, and its eigenvalues are $r^{\alpha_1}, r^{\alpha_2}, \ldots, r^{\alpha_k}, r$, where

$$0<\alpha_1<\alpha_2<\ldots<\alpha_k<\alpha_{k+1}=1.$$

If we denote by $\mathfrak{n}_{\alpha_j}$ the eigenspace associated to the eigenvalues r^{α_j}, then the family $\{\mathfrak{n}_{\alpha_j}\}$ yields a gradation of $\mathfrak{n}$, since

$$[\mathfrak{n}_{\alpha_j}, \mathfrak{n}_{\alpha_l}]\subseteq \mathfrak{n}_{\alpha_j+\alpha_l} \tag{1}$$

(with the convention that $\mathfrak{n}_\beta=0$ if β is not one of the α_j).

If $\mathfrak{v}$ is a D-invariant linear subspace of $\mathfrak{n}$, then of course $\mathfrak{v}$ decomposes into eigenspaces $\mathfrak{v}_{\alpha_j}$, i.e.

$$\mathfrak{v}=\sum_j^{\oplus} \mathfrak{v}_{\alpha_j}, \quad \text{with } \mathfrak{v}_{\alpha_j}=\mathfrak{v}\cap\mathfrak{n}_{\alpha_j}, \tag{2}$$

where the summation is understood to be over all indices j with $\mathfrak{v}\cap\mathfrak{n}_{\alpha_j}\neq 0$.

Let $\mathfrak{z}$ denote the center of $\mathfrak{n}$, and let $\mathfrak{w}$ be a fixed D-invariant linear subspace of $\mathfrak{n}$ supplementary to $\mathfrak{z}$. Then, since $\mathfrak{z}$ is of course D-invariant,

$$\mathfrak{z}=\sum_j^{\oplus} \mathfrak{z}_{\alpha_j} \quad \text{and} \quad \mathfrak{w}=\sum_j^{\oplus} \mathfrak{w}_{\alpha_j}.$$

If we set $\mathfrak{z}_0=\mathfrak{w}\cap\mathfrak{z}$, and if we choose a linear subspace $\mathfrak{n}_0$ of $\mathfrak{w}$ which is D-invariant and complementary to $\mathfrak{z}_0$, and a D-invariant subspace $\tilde{\mathfrak{z}}$ of $\mathfrak{z}$ which is complementary to $\mathfrak{z}_0$, then we have

$$\mathfrak{w}=\mathfrak{n}_0\oplus\mathfrak{z}_0 \tag{3}$$

and

$$\mathfrak{n}=\mathfrak{w}\oplus\tilde{\mathfrak{z}},\ \mathfrak{z}=\mathfrak{z}_0\oplus\tilde{\mathfrak{z}}. \tag{4}$$

Let further N denote the connected, simply connected Lie group having $\mathfrak{n}$ as its Lie algebra. We will use the elements of $\mathfrak{n}$ as parameters of N via the exponential mapping, i.e. if $x, y \in \mathfrak{n}$, xy denotes the element of $\mathfrak{n}$ such that $\exp_N xy = \exp_N x \exp_N y$. In particular, $x^{-1} = -x$. In the same way as we identify N with $\mathfrak{n}$, we will also identify the center $Z = \exp_N \mathfrak{z}$ of N with $\mathfrak{z}$ and $W = \exp \omega$ with ω. Lebesgue measure on $\mathfrak{n}$ (resp. $\mathfrak{z}$) is the same as the Haar measure for the group N (resp. Z).

Now, since $\mathfrak{n}_0 \simeq \mathfrak{n}/\mathfrak{z}$ as linear spaces, we can regard $\mathfrak{n}_0$ as a Lie algebra. Since $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{z}$, we will often denote an element x of $\mathfrak{n}$ as a pair $x = (z, t)$ with $z \in \mathfrak{n}_0$, $t \in \mathfrak{z}$. Multiplication in $\mathfrak{n}$ may then be written

$$(z, s)(w, t) = (z w, s + t + P(z, w)) \quad \text{if } z, w \in \mathfrak{n}_0, s, t \in \mathfrak{z}, \tag{5}$$

where P is a polynomial cocycle from $\mathfrak{n}_0 \times \mathfrak{n}_0$ to $\mathfrak{z}$. To be more specific, P has the following properties:

$$P(-z, -w) = -P(w, z), \tag{6}$$

$$P(k_1 z, k_2 z) = 0 \quad \text{for any } k_1, k_2 \in \mathbb{R}, \tag{7}$$

$$P(z, w) + P(zw, y) = P(z, wy) + P(w, y), \tag{8}$$

$$P(D_r z, D_r w) = D_r P(z, w) \quad \text{for any } r > 0. \tag{9}$$

For $j = 1, \ldots, k+1$, we fix a norm $|\cdot|_{\alpha_j}$ on $\mathfrak{n}_{\alpha_j}$ and define a homogeneous gauge $|\cdot|$ on $\mathfrak{n}$ as

$$|x| = \sum_{j=1}^{k+1} |x_j|_{\alpha_j}^{1/\alpha_j}, \quad \text{if } x = \sum_j x_j \quad \text{with} \quad x_j \in \mathfrak{n}_{\alpha_j}.$$

If again $\mathfrak{v}$ denotes a D-invariant linear subspace of $\mathfrak{n}$, we also denote by $|\cdot|$ the restriction of $|\cdot|$ to $\mathfrak{v}$, and define a dual gauge, again denoted by $|\cdot|$, on $\mathfrak{v}^*$ as follows:

Let D_r act on $\mathfrak{v}^*$ by the adjoint action. Then $\mathfrak{v}^*$ also decomposes into eigenspaces $\mathfrak{v}_{\alpha_j}^*$, i.e.

$$\mathfrak{v}^* = \sum_j^{\oplus} \mathfrak{v}_{\alpha_j}^*.$$

Since obviously $\langle \lambda_i, x_j \rangle = 0$ for $\lambda_i \in \mathfrak{v}_{\alpha_i}^*$, $x_j \in \mathfrak{v}_{\alpha_j}$ and $i \neq j$, $\mathfrak{v}_{\alpha_j}^*$ may be canonically identified with the dual space of $\mathfrak{v}_{\alpha_j}$. If $|\cdot|_{\alpha_j}^*$ denotes the dual norm on $\mathfrak{v}_{\alpha_j}^*$, i.e.

$$|\lambda_j|_{\alpha_j}^* = \sup_{\substack{x \in \mathfrak{v}_{\alpha_j} \\ |x|_{\alpha_j} \leq 1}} |\langle \lambda_j, x \rangle|,$$

then we set

$$|\lambda| = \sum_j |\lambda_j|_{\alpha_j}^{*\, 1/\alpha_j}, \quad \text{if } \lambda = \sum_j \lambda_j \quad \text{with} \quad \lambda_j \in \mathfrak{v}_{\alpha_j}^*.$$

Then it is clear that

$$|D_r x| = r|x| \quad \text{and} \quad |D_r \lambda| = r|\lambda| \tag{10}$$

for any $x \in \mathfrak{n}$ and any $\lambda \in \mathfrak{v}^*$. Moreover, it is not hard to see that there exist constants $\gamma_1 > 0$

and $\gamma>0$ such that:

$$\gamma_1(|z|+|t|)\leqq|(z,t)|\leqq\gamma(|z|+|t|), \quad \text{if } z\in\mathfrak{n}_0,\ t\in\mathfrak{z}; \tag{11}$$

$$|x+y|\leqq\gamma(|x|+|y|); \tag{12}$$

$$|xy|\leqq\gamma(|x|+|y|); \tag{13}$$

any $\lambda\in\mathfrak{v}^*$ there exists an $x\in\mathfrak{v}$, such that

$$\langle\lambda,x\rangle=1 \quad \text{and} \quad |x|\leqq\gamma|\lambda|^{-1}. \tag{14}$$

Now, let f be an integrable function on $\mathfrak{n}$. For $z\in\mathfrak{n}_0$ and $\lambda\in\mathfrak{z}^*$ let

$$f_\lambda(z)=\int_{\mathfrak{z}} f(z,t)\,e^{-i\langle\lambda,t\rangle}\,dt. \tag{15}$$

If we extend this notion in the usual way also to square-integrable functions, then Plancherel's theorem yields

$$|f|^2_{L^2(\mathfrak{n})}=\int_{\mathfrak{z}^*}|f_\lambda|^2_{L^2(\mathfrak{n}_0)}\,d\lambda \tag{16}$$

for a suitable normalization of the Lebesgue measure $d\lambda$ on $\mathfrak{z}^*$. Moreover, if f and g are integrable functions on $\mathfrak{n}$, then a simple computation (compare for example [13]) shows that

$$(f*g)_\lambda(z)=\int_{\mathfrak{n}_0} f_\lambda(zw^{-1})\,g_\lambda(w)\,e^{i\langle\lambda,P(z,-w)\rangle}\,dw. \tag{17}$$

For $\lambda\in\mathfrak{z}^*$ we denote by ω_λ the cocycle

$$\omega_\lambda(z,w)=e^{i\langle\lambda,P(z,w)\rangle}.$$

If we define for integrable functions φ and ψ on $\mathfrak{n}_0$ as in [13] the λ-twisted convolution of ψ and φ as

$$\varphi*_\lambda\psi(z)=\int_{\mathfrak{n}_0}\varphi(zw^{-1})\,\psi(w)\,\omega_\lambda(z,-w)\,dw, \tag{18}$$

then (17) reads

$$(f*g)_\lambda=f_\lambda*_\lambda g_\lambda. \tag{19}$$

We also adapt the definition of a Calderón-Zygmund kernel from [13]:

Definition. A Calderón-Zygmund kernel on $\mathfrak{w}$ is a tempered distribution k on $\mathfrak{w}$ which is a locally integrable function away from the origin and satisfies the following conditions for some positive constant $A(k)$:

$$|\hat{k}|_{L^\infty(\mathfrak{w}^*)}\leqq A(k), \tag{20}$$

$$\int_{|x|>\beta|y|}|k(x-y)-k(x)|\,dx\leqq A(k) \quad \text{for every } y\neq 0, \tag{21}$$

where dx denotes Lebesgue measure on $\mathfrak{w}$, and where β is a fixed constant with $\beta>2\gamma$ for the constant γ which appears in (11) to (14).

In the following, let k be a Calderón-Zygmund kernel on $\mathfrak{w}$. We extend k to a tempered distribution K on $\mathfrak{n}$ by

$$\langle K, \varphi \rangle = \langle k, \varphi|_{\omega} \rangle, \quad \text{if } \varphi \in \mathcal{S}(\mathfrak{n}), \tag{22}$$

i.e., if δ denotes the point measure at 0 on $\tilde{x}$, then $K = k \otimes \delta$, if we write $\mathfrak{n} = \omega \oplus \tilde{x}$.

In the following, we will study conditions on k such that K yields a bounded convolution operator on $L^2(\mathfrak{n})$. For example, in case that $\mathfrak{y} = \mathfrak{n}$, it has been shown by F. Ricci in [13] that $K = k$ always yields a bounded convolution operator on $L^2(\mathfrak{n})$. In [3], D. Geller and E.M. Stein have shown that in case of the $(2n+1)$-dimensional Heisenberg group N and ω complementary to the center each kernel k on ω which is homogeneous of degree $-n$ (with respect to our notation) and which is smooth outside the origin, yields a bounded operator by convolution with K. They even showed more, namely that the corresponding operator is bounded on $L^p(\mathfrak{n})$ for $1 < p < \infty$. The advantage of our method is that it also works for more general groups, but on the other hand it seems not to be clear how one could also get results for $L^p(\mathfrak{n})$, $p \neq 2$.

With view to the results of Ricci in [13], we will restrict ourselves to the case $\omega \subsetneq \mathfrak{n}$, although our methods would also give a partially shorter proof of the results in [13].

Remark. Throughout the paper, for convenience we will frequently denote constants by C or C', even if sometimes different constants are meant.

2. Two Classes of Calderón-Zygmund Kernels

It will turn out that for our studies two special classes of Calderón-Zygmund kernels will be wellsuited.

We introduce the following weak homogeneity properties (let v be again a D-invariant subspace of $\mathfrak{n}$):

Definition. i) A Calderón-Zygmund kernel k on v is said to have property (H1), if there exists a constant $C_1(k)$ such that

$$\int_{R < |x| < 2R} |k(x)|\, dx \leqq C_1(k) \tag{H1}$$

for any $R > 0$.

ii) A Calderón-Zygmund kernel k on v is said to have property (H2), if there exists a constant $C_2(k)$ such that

$$\int_{|x| > R} |k(x)|^2\, dx \leqq C_2(k)^2\, R^{-q}, \tag{H2}$$

where $q = \Sigma\, \alpha_j \dim v_{\alpha_j}$ denotes the homogeneous dimension of v.

Remarks. 1. (H2) implies (H1), because, if (H2) holds, then

$$\begin{aligned} \int_{R < |x| < 2R} |k(x)|\, dx &\leqq \Big(\int_{|x| > R} |k(x)|^2\, dx \Big)^{1/2} \Big(\int_{R < |x| < 2R} dx \Big)^{1/2} \\ &\leqq C_2(k)\, R^{-q/2} \Big(R^q \int_{1 < |x| < 2} dx \Big)^{1/2} \\ &= C_2(k) \Big(\int_{1 < |x| < 2} dx \Big)^{1/2}. \end{aligned}$$

2. (H1) is equivalent to (H1)': For any constants σ_1, σ_2 with $0<\sigma_1<\sigma_2$ there exists a constant $C'>0$ such that

$$\int_{\sigma_1 R<|x|<\sigma_2 R} |k(x)|\,dx \leqq C' A_1(k)$$

for any $R>0$.

Of course we may assume that $\sigma_1=1$ and $\sigma_2>1$. If then $l\in\mathbb{N}$ is such that $2^l \geqq \sigma_2$, then obviously (H1) yields

$$\int_{R<|x|<\sigma_2 R} |k(x)|\,dx \leqq \sum_{j=1}^{l} \int_{2^{j-1}R<|x|<2^j R} |k(x)|\,dx \leqq l\, C_1(k).$$

3. Let ω be as in the preceding section. One can show that there exists a bounded Radon measure $d\sigma$ on the "unit sphere" $\Sigma=\{x\in\omega: |x|=1\}$ in ω such that for any $f\in L^1(\omega)$

$$\int_{\omega} f(x)\,dx = \int_0^{\infty} \int_{\Sigma} f(D_r\sigma)\, r^{m-1}\, d\sigma\, dr, \tag{23}$$

(compare [11]), where m denotes the homogeneous dimension of ω.

Now, let k be a homogeneous kernel of degree $-m$ on ω, and let us assume that k can be written as

$$k(x)=\frac{\Omega(x')}{|x|^m},$$

where $x'=D_{|x|^{-1}}x$, if $x\neq 0$, and where Ω is an L^∞-function on Σ which satisfies

$$\int_{\Sigma} \Omega(\sigma)\,d\sigma=0 \tag{24}$$

and the Dini condition

$$\int_0^1 \frac{\omega(t)}{t}\,dt<\infty. \tag{25}$$

Here $\omega(t)$ is the modulus of continuity,

$$\omega(t)=\sup\{|\Omega(\sigma_1)-\Omega(\sigma_2)|:\ \sigma_1,\sigma_2\in\Sigma, |\sigma_1-\sigma_2|<t\}.$$

Then it is known (see [4], [14]), that k is a Calderón-Zygmund kernel on ω. Moreover, applying (23) we see that k also satisfies condition (H2).

4. If a Calderón-Zygmund kernel on ν has property (H1) (resp. (H2)), we write

$$A_1(k)=A(k)+C_1(k) \quad \text{(resp. } A_2(k)=A(k)+C_2(k)\text{)}. \tag{26}$$

Now we will prove some properties of such Calderón-Zygmund kernels.

Let k be a Calderón-Zygmund kernel on $\omega=\mathfrak{n}_0\oplus\mathfrak{z}_0$. For $\lambda_0\in\mathfrak{z}_0^*$, $z\in\mathfrak{n}_0$, we denote by $k_{\lambda_0}(z)$ the partial Fourier transform of k in direction of $\mathfrak{z}_0$, i.e., since $\hat{k}$ is in $L^\infty(\omega^*)$, for a.e. $\lambda_0\in\mathfrak{z}_0$ we have $\hat{k}_{\lambda_0}(\xi)=\hat{k}(\xi,\lambda_0)$, $\xi\in\mathfrak{n}_0^*$. Formally we may write

$$k_{\lambda_0}(z)=\int_{\varkappa_0} k(z,t)\,e^{-i\langle\lambda_0,t\rangle}\,dt. \tag{27}$$

The following lemma is an adaption of a result of F. Ricci in [13], Lemma 2, to our situation:

Lemma 2.1. *Let γ and γ_1 be as in* (11)–(14). *For almost every $\lambda_0\in\varkappa_0^*$, k_{λ_0} is a Calderón-Zygmund kernel on $\varkappa_0$, with the constant β in* (21) *replaced by $\beta'=\beta\gamma\gamma_1^{-1}$, and with $A(k_{\lambda_0})\leqq A(k)$. Moreover,*

$$\int_{|z|>\beta'|\lambda_0|^{-1}} |k_{\lambda_0}(z)|\,dz\leqq 2A(k).$$

We will omit the proof, because it is essentially the same given in [13].

Lemma 2.2. *Let k be a Calderón-Zygmund kernel on ω with property* (H1). *Let further $\chi\in C_0^\infty(\omega)$ with $0\leqq\chi(x)\leqq 1$, and $\chi(x)=1$ if $|x|\leqq 1$, and let $\chi_R(x)=\chi(D_{R^{-1}}x)$.*

Then for any $R>0$, $k\chi_R$ is a Calderón-Zygmund kernel with $A(k\chi_R)\leqq CA_1(k)$.

Proof. First, we have $|\widehat{k\chi_R}|_{L^\infty}\leqq|\hat{k}|_{L^\infty}|\hat{\chi}_R|_{L^1}$, and $|\hat{\chi}_R|_{L^1}=|\hat{\chi}|_{L^1}$, hence $|\widehat{k\chi_R}|_{L^\infty}\leqq|\hat{\chi}|_{L^1}A(k)$.

Now let $w\neq 0$. Then

$$\begin{aligned}
&\int_{|z|>\beta|w|} |(k\chi_R)(z-w)-(k\chi_R)(z)|\,dz\\
&\quad\leqq \int_{|z|>\beta|w|} |k(z-w)-k(z)|\,\chi_R(z-w)\,dz\\
&\quad+ \int_{|z|>\beta|w|} |k(z)|\,|\chi_R(z-w)-\chi_R(z)|\,dz\\
&\quad\leqq A(k)+\int_{|z|>\beta|w|} |k(z)|\,|\chi_R(z-w)-\chi_R(z)|\,dz.
\end{aligned}$$

Assume that $\chi(x)=0$ if $|x|\geqq\rho>1$:

We distinguish two cases:

1. $2\beta\gamma|w|<R$: If then $|z|\geqq(\rho\gamma+(2\beta)^{-1})R$, then we have $|z|\geqq\rho R$, hence $\chi_R(z)=0$, and

$$|z-w|\geqq\gamma^{-1}|z|-|w|)\geqq(\rho+(2\beta\gamma)^{-1})R-(2\beta\gamma)^{-1}R=\rho R,$$

hence also $\chi_R(z-w)=0$.

Moreover, if $|z|\leqq(2\gamma)^{-1}R$, then $|z|\leqq R$, hence $\chi_R(z)=1$, and

$$|z-w|\leqq\gamma(|z|+|w|\leqq R/2+R/2\beta\leqq R,$$

since $\beta>2\gamma\geqq 2$, hence also $\chi_R(z-w)=1$. Therefore in this case we have

$$\begin{aligned}
&\int_{|z|>\beta|w|} |k(z)|\,|\chi_R(z-w)-\chi_R(z)|\,dz\\
&\quad\leqq \int_{(\rho\gamma+\frac{1}{2\beta})R>|z|>\frac{1}{2\gamma}R} |k(z)|\,dz\leqq CA_1(k)\quad\text{by (H1)}'.
\end{aligned}$$

2. $2\beta\gamma|w| \geqq R$: If now $|z| > (2\rho+1)\beta\gamma^2|w|$, then $|z| > \rho R$, and also

$$|z-w| \geqq \gamma^{-1}|z| - |w| \geqq ((2\rho+1)\beta\gamma - 1)|w| \geqq 2\rho\beta\gamma|w| \geqq \rho R,$$

hence $\chi_R(z) = \chi_R(z-w) = 0$. Therefore

$$\int_{|z|>\beta|w|} |k(z)|\,|\chi_R(z-w) - \chi_R(z)|\,dz \leqq \int_{(2\rho+1)\beta\gamma^2|w|>|z|>\beta|w|} |k(z)|\,dz \leqq C' A_1(k) \quad \text{by (H1)'}.$$

Hence, $A(k\chi_R) \leqq A(k) + (C + C')A_1(k)$.

To formulate the following proposition, we first introduce a second homogeneous gauge on $\varkappa_0$ (compare with [11], Proposition 1); we assume here $\varkappa_0 \neq 0$:

Let $\| \ \|$ denote a Euclidean norm (or any "smooth" norm) on $\varkappa_0$. Then we define

$$|t|' = r, \quad \text{if } \|D_{r^{-1}} t\| = 1, \quad \text{for } t \in \varkappa_0. \tag{28}$$

It is not hard to show that $t \to |t|'$ is everywhere continuous and C^∞ on $\varkappa_0 \setminus \{0\}$ (which is in general not true for $|\ |$), and that $|\ |$ and $|\ |'$ are equivalent.

Proposition 2.3. *Let k be a Calderón-Zygmund kernel on ω having property* (H1), *and assume that for a.e. $z \in \varkappa_0$ and every a, b with $0 < a < b$,*

$$\int_{a<|t|'<b} k(z,t)\,dt = 0. \tag{29}$$

Then k_{λ_0} is in $L^1(\varkappa_0)$ for a.e. $\lambda_0 \in \varkappa_0^*$, and

$$|k_{\lambda_0}|_{L^1(\varkappa_0)} \leqq C A_1(k),$$

independently of λ_0.

Remark. If for example $R_1(x) = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}} \cdot \frac{x_1}{\|x\|^{n+1}}$ denotes a Riesz-kernel on $\mathbb{R}^n$, then Proposition 2.3 shows that $(R_1)^1_\lambda$ has uniformly bounded L^1-norm, if we denote by $(R_1)^1_\lambda$ the partial Fourier transform in direction of x_1, evaluated at λ. But, one can show easily that $(R_1)^j_\lambda$ is not even integrable, if $(R_1)^j_\lambda$ denotes the evaluation at λ of the partial Fourier transform of R_1 in direction of x_j, $j \neq 1$.

Before we prove Proposition 2.3, we state another lemma which is proved by the same method as Lemma 3 in [13]:

Lemma 2.4. *For each $\alpha > 0$ there exists a constant $C_\alpha > 0$, such that for any $R > 0$ and any Calderón-Zygmund kernel k on ω with support contained in the "ball" $B_R = \{x \in \omega : |x| \leqq R\}$ the following estimate holds:*

$$\int_\omega |k(x)|\,|x|^\alpha\,dx \leqq C_\alpha A(k) R^\alpha. \tag{30}$$

Proof. Let $x_0 \in \omega$, $x_0 \neq 0$, let $c>1$ and $x_n = D_{c^n} x_0$. For $r>0$, let $B_{x_n} = \{x \in \omega : |x-x_n| < r\}$. If $y_n = x_n - x_{n-1}$, it is clear that

$$\int_{B_{x_{n-1}}} |k(x)|\,dx \leqq \int_{B_{x_n}} |k(x-y_n)-k(x)|\,dx + \int_{B_{x_n}} |k(x)|\,dx. \tag{31}$$

If $x \in B_{x_n}$, then, by (12), $|x| \geqq \gamma^{-1}|x_n| - r$, i.e.

$$|x| \geqq \gamma^{-1} c^n |x_0| - r. \tag{32}$$

Furthermore, if $x_0 = \sum_j z_j$ with $z_j \in \omega_{\alpha_j}$, then

$$\begin{aligned} |y_n| &= |D_{C^n} x_0 - D_{C^{n-1}} x_0| \\ &= \sum_j |c^{n\alpha_j} z_j - c^{(n-1)\alpha_j} z_j|_{\alpha_j}^{1/\alpha_j} \\ &= c^{n-1} \sum_j |c^{\alpha_j} - 1|^{1/\alpha_j} |z_j|_{\alpha_j}^{1/\alpha_j} \\ &\leqq c^{n-1}(c-1)|x_0|. \end{aligned} \tag{33}$$

The middle integral in (30) can easily be estimated if $|x| > \beta |y_n|$ for $x \in B_{x_n}$. For this, by (32), (33), it suffices that $\gamma^{-1} c^n |x_0| - r > \beta c^{n-1}(c-1)|x_0|$, i.e.

$$r < c^{n-1}(\gamma^{-1} c - \beta(c-1))|x_0|. \tag{34}$$

Next we choose c small enough but greater than 1 such that $0<\mu<1$, where $\mu = \gamma^{-1} c - \beta(c-1)$. Then with $r = \mu |x_0|$ we have (34) for all x_0 and all $n \geqq 1$. Let us assume that c and r are chosen in such a way. Iterating (31) we get

$$\int_{B_{x_0}} |k(x)|\,dx \leqq n A(k) + \int_{B_{x_n}} |k(x)|\,dx,$$

since, by (21), $\int_{B_{x_j}} |k(x-y_j)-k(x)|\,dx \leqq A(k)$.

Now, if $\gamma^{-1} c^n |x_0| - r > R$, then, by (32), $|x| > R$ for any $x \in B_{x_n}$, hence $k(x)=0$ for $x \in B_{x_n}$. But, this happens if

$$c^n > \gamma(R\,|x_0|^{-1} + \mu). \tag{35}$$

For such an n, we have

$$\int_{B_{x_0}} |k(x)|\,dx \leqq n A(k). \tag{36}$$

Now, if $|x_0|$ is of order $2^{-j}R$, then the right hand side of (35) is of order 2^j, so we can choose an integer n in (35) which is of order j.

Let $\Gamma_j = \{x \in \omega : 2^{-j-1} R \leqq |x| < 2^{-j} R\}$. Using dilations, one can show that there is a certain number N, such that each Γ_j may be covered by N "balls" of the form $B_m^j = \{x \in \omega : |x - x_m^j| < \mu |x_m^j|\}$ with $x_m^j \in \Gamma_j$. Hence, by (36),

$$\int_{\Gamma_j} |k(x)|\,dx \leqq \sum_{m=1}^{N} \int_{B_m^j} |k(x)|\,dx \leqq Cj A(k). \tag{37}$$

Take now $\alpha>0$. We have, since $\operatorname{supp} k\subset B_R$,

$$\begin{aligned}\int_{\omega}|k(x)|\,|x|^{\alpha}\,dx&=\sum_{j\geqq 0}\int_{\Gamma_j}|k(x)|\,|x|^{\alpha}\,dx\\&\leqq\sum_{j\geqq 0}2^{-j\alpha}R^{\alpha}\int_{\Gamma_j}|k(x)|\,dx\\&\leqq C\,A(k)\,R^{\alpha}\sum_{j\geqq 0}j2^{-j\alpha}\\&=C_{\alpha}A(k)\,R^{\alpha}.\end{aligned}$$

Let us now prove Proposition 2.3.

Let $\chi_1\in C_0^{\infty}(\varkappa_0)$ such that $0\leqq\chi_1\leqq 1$, $\chi_1(z)=1$ if $|z|\leqq\rho_1$ with $\rho_1>0$, let $\chi_2^+\in C_0^{\infty}(\mathbb{R}^+)$ such that $0\leqq\chi_2^+\leqq 1$, $\chi_2^+(r)=1$ if $0\leqq r\leqq\rho_2$, with $\rho_2>0$, and let $\chi_2(t)=\chi_2^+(|t|')$ for $t\in\varkappa_0$. Write $\chi(z,t)=\chi_1(z)\,\chi_2(t)$ for $(z,t)\in\omega$. Then, obviously $\chi\in C_0^{\infty}(\omega)$, and, since $|\ |$ and $|\ |'$ are equivalent on $\varkappa_0$, we can choose ρ_1 and ρ_2 big enough such that $\chi(x)=1$ if $|x|\leqq 1$, which we will suppose from now on.

We write $\chi_R=\chi\circ D_{R^{-1}}$ for $R>0$, and decompose k as

$$k=k'+k'',$$

where

$$k'=k\chi_{\beta\gamma|\lambda_0|^{-1}}. \tag{38}$$

By Lemma 2.2, k' is a Calderón-Zygmund kernel with

$$A(k')\leqq C\,A_1(k), \tag{39}$$

and

$$\operatorname{supp} k'\subset B_{\rho\beta\gamma|\lambda_0|^{-1}}, \tag{40}$$

if $\rho>1$ is chosen such that $\operatorname{supp}\chi\subset B_{\rho}$.

Now, after remark 3 there exists a bounded Radon measure $d\mu(t)$ on $\Sigma'=\{t\in z_0\colon |t|'=1\}$, such that for any $f\in L^1(\varkappa_0)$

$$\int_{\varkappa_0}f(t)\,dt=\int_0^{\infty}\int_{\Sigma'}f(D_r t)\,r^{p-1}\,d\mu(t)\,dr, \tag{41}$$

where $p=\Sigma\alpha_j\dim(\varkappa_0\cap\varkappa_{\alpha_j})$ denotes the homogeneous dimension of $\varkappa_0$.

Hence, for a.e. $z\in\varkappa_0$

$$\begin{aligned}&\int_{\varkappa_0}k'(z,t)\,dt\\&\quad=\chi_1(D_{1/(\beta\gamma|\lambda_0|^{-1})}z)\int_0^{\infty}\chi_2^+((\beta\gamma|\lambda_0|^{-1})^{-1}r)\,k(z,D_r t)\,r^{p-1}\,d\mu(t)\,dr.\end{aligned}$$

But, (29) implies that for any $\psi^+\in C_0^{\infty}(\mathbb{R}^+)$ with $0\notin\operatorname{supp}\psi^+$

$$0=\int_0^{\infty}\int_{\Sigma'}k(z,D_r t)\,\psi^+(r)\,r^{p-1}\,d\mu(t)\,dr,$$

and therefore, for a.e. $z\in\varkappa_0$

$$\int_{\varkappa_0}k'(z,t)\,dt=0.$$

So we get

$$k'_{\lambda_0}(z)=\int_{\varkappa_0} k'(z,t)(e^{-i\langle\lambda_0,t\rangle}-1)\,dt. \tag{42}$$

Now, if $t=\sum_j t_j$ with $t_j\in\varkappa_0\cap\varkappa_{\alpha_j}$, and $\lambda_0=\sum_j\lambda_{0,j}$ with $\lambda_{0,j}\in\varkappa^*_{0,\alpha_j}$, then

$$\begin{aligned}|e^{-i\langle\lambda_0,t\rangle}-1|&\leqq|\langle\lambda_0,t\rangle|\\&=|\sum_j\langle\lambda_{0,j},t_j\rangle|\leqq\sum_j|\lambda_{0,j}|^*_{\alpha_j}|t_j|_{\alpha_j}=\sum_j|\lambda_{0,j}|^{\alpha_j}|t_j|^{\alpha_j},\end{aligned}$$

which, together with (42), yields

$$\begin{aligned}\int_{\varkappa_0}|k'_{\lambda_0}(z)|\,dz&\leqq\sum_j|\lambda_{0,j}|^{\alpha_j}\iint|k'(z,t)|\,|t_j|^{\alpha_j}\,dt\,dz\\&\leqq\sum_j|\lambda_{0,j}|^{\alpha_j}\int|k'(x)|\,|x|^{\alpha_j}\,dx.\end{aligned}\tag{43}$$

By Lemma 2.4 and (39), (40),

$$\int|k'(x)|\,|x|^{\alpha_j}\,dx\leqq C_{\alpha_j}A_1(k)(\beta\gamma\rho|\lambda_0|^{-1})^{\alpha_j}\leqq C'A_1(k)|\lambda_0|^{-\alpha_j},$$

hence

$$\int_{\varkappa_0}|k'_{\lambda_0}(z)|\,dz\leqq C'A_1(k)\sum_j|\lambda_{0,j}|^{\alpha_j}|\lambda_0|^{-\alpha_j}\leqq CA_1(k). \tag{44}$$

Next we choose $s\in\varkappa_0$ such that $\langle\lambda_0,s\rangle=1$ and $|s|\leqq\gamma|\lambda_0|^{-1}$ (compare (14)), and define

$$H(z,t)=k''(z,t-s)-k''(z,t).$$

We claim that H is in fact an integrable function on ω:

Since by (39) k'' is a Calderón-Zygmund kernel with $A(k'')\leqq CA_1(k)$, we have

$$\iint_{|(z,t)|>\beta|s|}|H(z,t)|\,dz\,dt\leqq CA_1(k). \tag{45}$$

Suppose now $|(z,t)|\leqq\beta|s|$. Then $|(z,t)|\leqq\beta\gamma|\lambda_0|^{-1}$, hence, by (38), $k''(z,t)=0$, i.e. $H(z,t)=k''(z,t-s)$. Therefore,

$$\begin{aligned}\iint_{|(z,t)|\leqq\beta|s|}|H(z,t)|\,dz\,dt&=\iint_{|(z,t)|\leqq\beta|s|}|k''(z,t-s)|\,dz\,dt\\&\leqq\int_{|z|\leqq\beta|s|}\int_{|t+s|\leqq\beta|s|}|k''(z,t)|\,dz\,dt.\end{aligned}$$

But, if $|z|\leqq\beta|s|$ and $|t+s|\leqq\beta|s|$, then

$$|(z,t)|\leqq\gamma(|z|+|t+s-s|)\leqq\gamma(|z|+\gamma|t+s|+\gamma|s|)\leqq\tau|\lambda_0|^{-1}$$

with

$$\tau=\gamma^2(\beta+\gamma+\beta\gamma).$$

Moreover, $k''(z,t)=0$, if $|(z,t)|\leqq\beta\gamma|\lambda_0|^{-1}$. It follows that

$$\iint_{|(z,t)|\leqq\beta|s|}|H(z,t)\,dz\,dt\leqq\int_{\beta\gamma|\lambda_0|^{-1}<|x|<t|\lambda_0|^{-1}}|k(x)|\,dx\leqq CA_1(k) \tag{46}$$

by (H1)′.

So, from (45) and (46) we derive

$$|H|_{L^1(\omega)}\leqq CA_1(k). \tag{47}$$

An argument analogues to the one used in the proof of Lemma 2 in [13] shows that

$$k''_{\lambda_0}(z)=(e^{-i\langle\lambda_0,s\rangle}-1)^{-1}\int_{\varkappa_0}H(z,t)\,e^{-i\langle\lambda_0,t\rangle}\,dt.$$

But, since $\langle\lambda_0,s\rangle=1$, this, together with (47), immediately gives us that

$$\int_{\varkappa_0}|k''_{\lambda_0}(z)|\,dz\leqq|(e^{-i}-1)^{-1}|\,|H|_{L^1(\omega)}\leqq C'A_1(k). \tag{48}$$

Together with (44), this finishes the proof.

Proposition 2.5. *If k is a Calderón-Zygmund kernel on ω having property* (H2), *then for almost every $\lambda_0\in\varkappa_0^*$ k_{λ_0} is a Calderón-Zygmund kernel having property* (H2) *on $\varkappa_0$, and there exists some constant $C>0$, independent of k and λ_0, such that $A_2(k_{\lambda_0})\leqq CA_2(k)$.*

Proof. By Lemma 2.1, k_{λ_0} is a Calderón-Zygmund kernel such that $A(k_{\lambda_0})\leqq A(k)$ for a.e. $\lambda_0\in\varkappa_0^*$. So, it remains to prove property (H2) for k_{λ_0}.

Let $R>0$. By Minkowsky's inequality we have

$$\Big(\int_{|z|>R}|k_{\lambda_0}(z)|^2\,dz\Big)^{1/2}\leqq\int_{\varkappa_0}\Big(\int_{|z|>R}|k(z,t)|^2\,dz\Big)^{1/2}dt. \tag{49}$$

One should realize here that we assumed that k_{λ_0} agrees with the vector-valued integral $\int k(\cdot,t)e^{-i\langle\lambda_0,t\rangle}\,dt$ as an element of the Hilbert space $L^2(\varkappa_0,d\mu_R)$, where $d\mu_R$ is the measure on $\varkappa_0$ which is defined by $\int\varphi\,d\mu_R=\int_{|z|>R}\varphi(z)\,dz$. But, since the right hand side of the inequality (49) will turn out to be finite, one can show that in the fact our assumption is true for almost every $\lambda_0\in\varkappa_0^*$.

Let

$$h(t)=\Big(\int_{|z|>R}|k(z,t)|^2\,dz\Big)^{1/2}.$$

After Fubini's theorem, h is Lebesgue measurable. Let m denote the homogeneous dimension of ω. From (H_2) we conclude that for any $\varepsilon>0$

$$\begin{aligned}\int_{|t|>\varepsilon}h^2(t)\,dt&=\int_{|t|>\varepsilon}\int_{|z|>R}|k(z,t)|^2\,dz\,dt\\&\leqq\int_{|(z,t)|>\gamma_1(\varepsilon+R)}|k(z,t)|^2\,dz\,dt\\&\leqq CA_2(k)^2(\varepsilon+R)^{-m},\end{aligned} \tag{50}$$

since after (11) $|z|>R$ and $|t|>\varepsilon$ imply

$$|(z,t)|\geqq\gamma_1(|z|+|t|)\geqq\gamma_1(\varepsilon+R).$$

So, for $\varepsilon>0$ and $n\in\mathbb{N}_0$, by Schwartz' inequality

$$\int\limits_{2^n\varepsilon<|t|<2^{n+1}\varepsilon}h(t)\,dt\leqq\Big(\int\limits_{2^n\varepsilon<|t|<2^{n+1}\varepsilon}h^2(t)\,dt\Big)^{1/2}\Big(\int\limits_{2^n\varepsilon<|t|<2^{n+1}\varepsilon}dt\Big)^{1/2}$$
$$\leqq C^{1/2}A_2(k)(R+2^n\varepsilon)^{-\frac{m}{2}}\Big(\int\limits_{1<|t|<2}dt(2^n\varepsilon)^p\Big)^{1/2},$$

where $p=\sum\limits_j\alpha_j\dim(\varkappa_0\cap\nu_{\alpha_j})$ denotes the homogeneous dimension of $\varkappa_0$. So, for some new constant $C>0$ we have

$$\int\limits_{2^n\varepsilon<|t|<2^{n+1}\varepsilon}h(t)\,dt\leqq CA_2(k)\frac{(2^n\varepsilon)^{p/2}}{(R+2^n\varepsilon)^{m/2}},$$

hence

$$\int\limits_{\varepsilon<|t|}h(t)\,dt=\sum_{n\geqq0}\int\limits_{2^n\varepsilon<|t|<2^{n+1}\varepsilon}h(t)\,dt$$
$$\leqq CA_2(k)\sum_{n\geqq0}\frac{(2^n\varepsilon)^{p/2}}{(R+2^n\varepsilon)^{m/2}}.\tag{51}$$

Let $\sigma=\log_2\dfrac{R}{\varepsilon}$. Then

$$\sum_{n\geqq0}\frac{(2^n\varepsilon)^{p/2}}{(R+2^n\varepsilon)^{m/2}}$$
$$\leqq\sum_{n\leqq\sigma}\frac{(2^n\varepsilon)^{p/2}}{R^{m/2}}+\sum_{n>\sigma}\frac{(2^n\varepsilon)^{p/2}}{(2^n\varepsilon)^{m/2}}$$
$$\leqq R^{-m/2}\varepsilon^{p/2}\frac{2^{(\sigma+1)p/2}-1}{2^{p/2}-1}+\varepsilon^{(p-m)/2}2^{\sigma(p-m)/2}\frac{1}{1-2^{(p-m)/2}}$$
$$\leqq C'R^{\frac{p-m}{2}},$$

where C' is independent of R and ε.

Hence, by (51),

$$\int\limits_{\varkappa_0}h(t)\,dt\leqq CC'A_2(k)\,R^{\frac{p-m}{2}},$$

and so, by (49),

$$\int\limits_{|z|>R}|k_{\lambda_0}(z)|^2\,dz\leqq\Big(\int\limits_{\varkappa0}h(t)dt\Big)^2\leqq C''A_2(k)^2R^{-(m-p)}.\tag{52}$$

But, $m-p$ is just the homogeneous dimension of ν_0. So, (52) proves that $A_2(k_{\lambda_0})\leqq CA_2(k)$ for some $C>0$.

3. The case where $\varkappa_0 = \omega \cap \varkappa \neq 0$

In case that $\varkappa_0 \neq 0$, we have the following application of Proposition 2.3:

Theorem 3.1. *Let k be a Calderón-Zygmund kernel on ω having property* (H1), *and assume that for a.e. $z \in \mathfrak{n}_0$ and every a, b with $0 < a < b$,*

$$\int_{a < |t|' < b} k(z,t)\,dt = 0. \tag{29}$$

Let K denote the extension of k to $\mathfrak{n}$ as described in (22), *i.e. $K = k \otimes \delta$. Then the operator T: $C_0^\infty(N) \to C^\infty(N)$ given by $Tf = f * K$ extends to a bounded operator from $L^2(N)$ to $L^2(N)$, and $\|T\| \leqq C A_1(k)$.*

Proof. By (16) we have for $f \in C_0^\infty(\mathfrak{n})$

$$|Tf|^2_{L^2(\mathfrak{n})} = \int_{\varkappa^*} |(Tf)_\lambda|^2_{L^2(\mathfrak{n}_0)}\,d\lambda, \tag{53}$$

and (19) yields

$$(Tf)_\lambda = f_\lambda *_\lambda K_\lambda. \tag{54}$$

Now, if we write $\lambda = (\lambda_0, \lambda_1)$ with $\lambda_0 \in \varkappa_0^*$, $\lambda_1 \in \tilde{\varkappa}^*$ (compare (4)), then for $\xi \in \mathfrak{n}_0^*$

$$\hat{K}_\lambda(\xi) = \hat{K}(\xi, \lambda_0, \lambda_1) = \hat{k}(\xi, \lambda_0) = \hat{k}_{\lambda_0}(\xi),$$

hence

$$K_\lambda = k_{\lambda_0}. \tag{55}$$

But, after Proposition 2.3, $|k_{\lambda_0}|_{L_1(\mathfrak{n}_0)} \leqq C A_1(k)$, and so

$$\begin{aligned} &|f_\lambda *_\lambda K_\lambda|_{L^2(\mathfrak{n}_0)} \\ &\quad = |f_\lambda *_\lambda k_{\lambda_0}|_{L^2(\mathfrak{n}_0)} \leqq \| |f_\lambda| * |k_{\lambda_0}| \|_{L^2(\mathfrak{n}_0)} \\ &\quad \leqq |k_{\lambda_0}|_{L^1(\mathfrak{n}_0)} |f_\lambda|_{L^2(\mathfrak{n}_0)} \leqq C A_1(k) |f_\lambda|_{L^2(\mathfrak{n}_0)}, \end{aligned}$$

which finally yields

$$|Tf|^2_{L^2(\mathfrak{n})} \leqq \int C^2 A_1(k)^2 |f_\lambda|^2_{L^2(\mathfrak{n}_0)}\,d\lambda = C^2 A_1^2(k) |f|^2_{L^2(\mathfrak{n})},$$

i.e. $\|T\| \leqq C A_1(k)$.

Remarks. In [13] it has been proved that in case $\omega = \mathfrak{n}$ we can dispense with property (H1) and (29) in Theorem 3.1. Moreover, in the next sections we will show that for a certain class of groups we can also omit property (29), if we replace property (H1) by (H2). So, it is not clear whether property (29) is superfluous at all. Moreover, it is an open question whether the operator T is bounded in general if $\omega \cap \varkappa = 0$. However, for "completely square integrable" groups this is true, as will also be shown in the following sections.

4. Nilpotent Lie groups Having Square Integrable Irreducible Unitary Representations

Let us first give a short summary of some well-known facts about square integrable unitary representations of nilpotent Lie groups, which all can be

found in [10]. For convenience, we will state these results only for the special groups considered above and we will use the same notation:

Let π be a unitary factor representation of N on a Hilbert space $\mathfrak{H}_\pi$. Then there exists a unitary character χ of Z with $\pi(t)=\chi(t)\cdot\mathbb{1}$ for $t\in Z$. Thus the absolute values of the coefficients $c_{\xi,\eta}(x)=\langle\pi(x)\xi,\eta\rangle$ for $\xi,\eta\in\mathfrak{H}_\pi$ of π may be considered as functions on N/Z. So, if we again parametrize N by $\mathfrak{n}$, and identify $\mathfrak{n}/\mathfrak{z}$ with $\mathfrak{n}_0$, then we say that π is *square integrable*, if for some nonzero coefficient $c_{\xi,\eta}$

$$\int_{\mathfrak{n}_0}|c_{\xi,\eta}(z)|^2\,dz<\infty. \tag{56}$$

It is known (see [10], Theorem B) that if π is irreducible and square integrable, then there exists a constant $d(\pi)>0$, called the *formal degree* of π, such that

$$\int_{\mathfrak{n}_0}c_{\xi_1,\eta_1}(z)\overline{c_{\xi_2,\eta_2}}(z)\,dz=d(\pi)^{-1}\langle\xi_1,\xi_2\rangle\overline{\langle\eta_1,\eta_2\rangle} \tag{57}$$

for any $\xi_1,\xi_2,\eta_1,\eta_2\in\mathfrak{H}_\pi$.

One should realize that $d(\pi)$ of course depends on the choice of the Haar measure dz on $\mathfrak{n}_0$.

According to the Kirillov theory (see [6]), the irreducible unitary representation of N are parametrized by the space of N-orbits in the dual space $\mathfrak{n}^*$ of $\mathfrak{n}$ under the coadjoint action. We shall write π_λ for the representation corresponding to an orbit $\mathfrak{o}$ containing $\lambda\in\mathfrak{n}^*$, and by b_λ we denote the skew symmetric bilinear form

$$b_\lambda(x,y)=\langle\lambda,[x,y]\rangle$$

on $\mathfrak{n}$. Since the singular subspace of b_λ clearly contains $\mathfrak{z}$, b_λ may be viewed as a skew two-form on $\mathfrak{n}/\mathfrak{z}\simeq\mathfrak{n}_0$. Finally, by χ_λ we will denote the character $\chi_\lambda(t)=\exp(i\langle\lambda,\log t\rangle)$ on Z, and the representation which is induced by χ_λ on N will be written as $\operatorname{ind}_Z^N\chi_\lambda$. If $\mathfrak{z}^\perp$ denotes the annihilator of $\mathfrak{z}$ in $\mathfrak{n}^*$, then Theorem 1 in [10] reads as follows:

Theorem A (Moore, Wolf). *For a linear functional λ on $\mathfrak{n}$ with orbit $\mathfrak{o}$ and corresponding representation $\pi=\pi_\lambda$ the following conditions are equivalent:*

i) *π is square integrable.*
ii) *$\operatorname{ind}_Z^N\chi_\lambda$ is a factor representation.*
iii) *$\mathfrak{o}=\lambda+\mathfrak{z}^\perp$.*
iv) *b_λ is nondegenerate on $\mathfrak{n}_0$.*

Now, we choose a fixed volume element α on $\mathfrak{n}_0$, and take the Haar measure dz on $\mathfrak{n}_0$ to be the measure defined by α. We define the Pfaffian $P(\lambda)$ of b_λ as follows:

Let r be the dimension of $\mathfrak{n}_0$. If r is odd, then let $P(\lambda)=0$, and if $r=2l$ is even, then we define $P(\lambda)$ by $b_\lambda^l=P(\lambda)\alpha$, where b_λ^l denotes the l-th exterior power of the restriction of b_λ to $\mathfrak{n}_0$. So, b_λ is nondegenerate on $\mathfrak{n}_0$ if and only if $P(\lambda)\neq 0$. Moreover, from Theorem A it follows easily that $P(\lambda)$ in fact depends only on the restriction of λ to $\mathfrak{z}$. Therefore, we may consider P as a function on $\mathfrak{z}^*$, and the theorem shows that the square integrable irreducible

representations of N are in one-to-one correspondence to the set of functionals $\lambda \in \mathfrak{z}^*$ with $P(\lambda) \neq 0$.

It is not surprising that P is homogeneous with respect to the dilations D_r. In fact, since D_r is an automorphism of $\mathfrak{n}$, one easily shows that $D_r b_\lambda = b_{D_r \lambda}$, and so

$$P(D_r\lambda)\alpha = b^l_{D_r\lambda} = D_r(b^l_\lambda) = P(\lambda)D_r\alpha = (\det D_r|_{\mathfrak{n}_0})P(\lambda)\alpha,$$

where D_r also denotes the canonical extension of D_r to the exterior algebra of $\mathfrak{n}_0$. Since

$$\det D_r|_{\mathfrak{n}_0} = r^q,$$

where q denotes the homogeneous dimension of $\mathfrak{n}_0$, we get

$$P(D_r\lambda) = r^q P(\lambda). \tag{58}$$

The Pfaffian on $\mathfrak{z}^*$ and the formal degree of square integrable representations are related as follows:

Theorem B (Moore, Wolf). *If for $\lambda \in \mathfrak{z}^*$ with $P(\lambda) \neq 0$ the corresponding irreducible representation of N is denoted by π_λ, then the formal degree of π_λ is $d(\pi_\lambda) = P((2\pi)^{-1}\lambda)$.*

The factor $(2\pi)^{-1}$ here stems from another normalization of the Fourier transform than in [10].

From these results, we draw the following conclusion:

Corollary 4.1. *Let $\lambda \in \mathfrak{z}^*$ with $P(\lambda) \neq 0$. Then, for any f and g in $L^2(\mathfrak{n}_0)$,*

$$|f *_\lambda g|_{L^2(\mathfrak{n}_0)} \leqq |P((2\pi)^{-1}\lambda)|^{-1/2} |f|_{L^2(\mathfrak{n}_0)} |g|_{L^2(\mathfrak{n}_0)}. \tag{59}$$

Proof. Since N is of type I, Theorem A shows that $\sigma = \mathrm{ind}_Z^N \chi_\lambda$ is unitary equivalent to a multiple of π_λ. This, together with (57), easily implies that

$$\int_{\mathfrak{n}_0} |\langle \sigma(z)\xi, \eta\rangle|^2 \, dz \leqq d(\pi_\lambda)^{-1} |\xi|^2 |\eta|^2 \tag{60}$$

for any $\xi, \eta \in \mathfrak{H}_\sigma$.

But, if f and g are in $L^2(\mathfrak{n}_0)$, then the functions $\xi(z,t) = f(z)\bar{\chi}_\lambda(t)$ and $\eta(z,t) = g(z)\bar{\chi}_\lambda(t)$ are in $\mathfrak{H}_\sigma$ by the definition of induced representations, and $|\xi|_{\mathfrak{H}_\sigma} = |f|_{L^2(\mathfrak{n}_0)}$, $|\eta|_{\mathfrak{H}_\sigma} |g|_{L^2(\mathfrak{n}_0)}$. Moreover, an easy computation, using (5)–(8), shows that for $z \in \mathfrak{n}_0$

$$\langle \sigma(z)\xi, \eta\rangle = \langle \xi, \sigma(z^{-1})\eta\rangle = \int_{\mathfrak{n}_0} \bar{g}(zw^{-1}) f(w^{-1}) e^{i\langle \lambda, P(z,-w)\rangle} \, dw,$$

since, by the definition of induced representations,

$$(\sigma(z^{-1})\eta)(w,t) = \eta((z,0)(w,t)).$$

So, if we write $\check{f}(w) = f(w^{-1})$, then, by (18),

$$\langle \sigma(z)\xi, \eta\rangle = (\bar{g} *_\lambda \check{f})(z),$$

and (60) reads as

$$|\bar{g} *_\lambda \check{f}|_{L^2(\mathfrak{n}_0)} \leqq d(\pi_\lambda)^{-1/2} |f|_{L^2(\mathfrak{n}_0)} |g|_{L^2(\mathfrak{n}_0)}.$$

But, since $|\check{f}|_{L^2(\pi_0)}=|f|_{L^2(\pi_0)}$ and $|\bar{g}|_{L^2(\pi_0)}=|g|_{L^2(\pi_0)}$, this together with Theorem B implies (59).

5. An Estimate for Twisted Convolutions

Let N be a nilpotent Lie group as above, but not necessarily having dilations. Let $X_1,\dots,X_n$ be a Jordan-Hölder basis for the Lie algebra $\mathfrak{n}$ of N, such that $[X,X_j]$ is always contained in the span $\mathfrak{g}_{j-1}$ of $X_1,\dots,X_{j-1}$ for any $X\in\mathfrak{n}$, and such that $\mathfrak{x}=\mathfrak{g}_r$ for some r.

We consider $X\in\mathfrak{n}$ as a left-invariant differential operator on N, and define a left invariant differential operator D on N as

$$D=(X_1+1)(X_2+1)\dots(X_n+1). \tag{61}$$

Similarly, if $G_j=\exp_N\mathfrak{g}_j$ denotes the closed subgroup of N corresponding to $\mathfrak{g}_j$, and if we consider X_i for $i\leqq j$ also as a left-invariant differential operator on G_j, we define D_j on G_j to be $D_j=(X_1+1)(X_2+1)\dots(X_j+1)$.

In the following, we will use a well-known method to construct a fundamental solution for D (compare for example to the proof of Lemma 3.1 in [1]).

Let $h(t)=e^{-t}$, if $t\geqq 0$, and $h(t)=0$ for $t<0$. Then h is a fundamental solution for the differential operator $\frac{d}{dt}+1$ on $\mathbb{R}$. We will now define inductively fundamental solutions ω_j on G_j for D_j.

First, of course $\omega_1(\exp tX_1)=h(t)$ is a fundamental solution for $D_1=X_1+1$. So, let us assume that we already defined a fundamental solution ω_j for D_j on G_j. If we identify $\exp\mathbb{R}X_{j+1}$ with $\mathbb{R}$, and if we let $\mathbb{R}$ operate on G_j by $x^t=\exp(-tX_{j+1})X\exp(tX_{j+1})$, then we may write G_{j+1} as $\mathbb{R}XG_j$, with multiplication $(t,x)(s,y)=(t+s,x^sy)$. We define ω_{j+1} by

$$\omega_{j+1}(t,x)=h(t)\,\omega_j(x^{-t}).$$

Then

$$\begin{aligned}(X_{j+1}+1)\,\omega_{j+1}(t,x)&=\int_{\mathbb{R}}\omega_{j+1}((t,x)(-s,e))(X_{j+1}+1)\,\delta(s)\,ds\\&=\int_{\mathbb{R}}h(t-s)\,\omega_j(x^{-t})\left(\frac{d}{ds}+1\right)\delta(s)\,ds\\&=\left(\frac{d}{dt}+1\right)h(t)\,\omega_j(x^{-t})=\delta(t)\,\omega_j(x^{-t})=\delta(t)\,\omega_j(x),\end{aligned}$$

where e denotes the neutral element in G_j, δ the Dirac distribution on $\mathbb{R}\simeq\exp\mathbb{R}X_{j+1}$, and where the integrals have to be interpreted as applications of distributions. So, if we apply our induction hypothesis, then

$$D_{j+1}\omega_{j+1}(t,x)=D_j(\delta(t)\,\omega_j(x))=\delta(t)\,\delta(x).$$

Thus $\omega=\omega_n$ is a fundamental solution for D.

Moreover, it is not hard to see that there exist polynomial functions $p_j(t_n, t_{n-1}, \ldots, t_{j+1})$ for $j=1,\ldots,n-2$, not even depending on $t_1,\ldots,t_r$ (since $\varkappa$ is spanned by $X_1,\ldots,X_r$), such that

$$\begin{aligned}\omega(\exp t_n X_n \exp t_{n-1} X_{n-1} \ldots \exp t_1 X_1) \\ = h(t_n)\, h(t_{n-1}) \prod_{j=1}^{n-2} h(t_j + p_j(t_n, \ldots, t_{j+1})),\end{aligned} \tag{62}$$

see e.g. [12] as general reference.

Since we may choose the Haar measure dx on N such that

$$\int_N \varphi(x)\,dx = \int_{\mathbb{R}^n} \varphi(\exp t_n X_n \ldots \exp t_1 X_1)\,dt_1 \ldots dt_n,$$

(62) shows that

$$|\omega|_{L^p(N)} = |h|^n_{L^p(\mathbb{R})} \tag{63}$$

for any $p \in [1, \infty)$.

We again write (compare (15))

$$\begin{aligned}&\omega_\lambda(z_n, \ldots, z_{r+1}) \\ &= \int \omega(\exp(z_n X_n + \ldots + z_1 X_1)) e^{-i\langle \lambda, z_r X_r + \ldots + z_1 X_1\rangle}\, dz_r \ldots dz_1\end{aligned}$$

for $\lambda \in \varkappa^*$.

Since there exist polynomial functions $q'_j(z_n, z_{n-1}, \ldots, z_{j+1})$ for $j=1,\ldots,n-2$, also not depending on $z_1,\ldots,z_r$, such that

$$\begin{aligned}&\exp(z_n X_n + \ldots + z_1 X_1) = \exp(z_n X_n) \exp(z_{n-1} X_{n-1})\ x \\ &\exp((z_{n-2} + q'_{n-2}(z_n, z_{n-1})) X_{n-2}) \ldots \exp((z_1 + q'_1(z_n, \ldots, z_2)) X_1),\end{aligned}$$

(62) shows that

$$\omega(\exp(z_n X_n + \ldots + z_1 X_1)) = h(z_n)\, h(z_{n-1}) \prod_{j=1}^{n-2} h(z_j + q_j(z_n, \ldots, z_{j+1})),$$

where the q_j are polynomial functions having the same properties as the functions p_j or q'_j. So we have

$$\begin{aligned}\omega_\lambda(z_n, \ldots, z_{r+1}) = h(z_n)\, h(z_{n-1}) \ldots h(z_{r+1} + q_{r+1}(z_n, \ldots, z_{r+2})) \\ \cdot \prod_{j=1}^{r} \hat{h}(\langle \lambda, X_j\rangle) e^{i\langle \lambda, X_j\rangle q_j(z_n, \ldots, z_{r+1})};\end{aligned}$$

realize here that for $j=1,\ldots,r$ the function q_j is in fact a function $q_j(z_n, \ldots, z_{r+1})$.

If we now again write $N_0 = \exp \mathfrak{n}_0$, where $\mathfrak{n}_0$ denotes the span of $X_n, \ldots, X_{r+1}$, and endowe N_0 with the group structure from N/Z, then the same argument as above also shows that

$$|\omega_\lambda|_{L^p(N_0)} \leqq |\hat{h}|^r_{L^\infty(\mathbb{R})} |h|^{n-r}_{L^p(\mathbb{R})} \tag{65}$$

for every $\lambda \in \mathfrak{z}^*$, if Haar measure on N_0 is chosen appropriatly. But, $\hat{h}(\xi) = \dfrac{1}{1+i\xi}$ for every $\xi \in \mathbb{R}$, and thus $|\hat{h}|_{L^\infty(\mathbb{R})} \leqq 1$, and it is also easily seen that $|h|_{L^p(\mathbb{R}^n)} \leqq 1$ for every $p \in [1, \infty]$.

So, we have proved the following

Lemma 5.1. *There exists a left-invariant differential operator D on N, which has a fundamental solution ω, such that ω and ω_λ are in $L^p(N)$ and $L^p(N_0)$ resp., for all p with $1 \leqq p \leqq \infty$, and such that*

$$|\omega|_{L^p(N)} \leqq 1 \quad \text{and} \quad |\omega_\lambda|_{L^p(N_0)} \leqq 1$$

for all $\lambda \in \mathfrak{z}^*$ and $1 \leqq p \leqq \infty$.

In the following, for convenience of the notation we will assume that N is again the group which has been considered in the first sections.

For any $\lambda \in \mathfrak{z}^*$ and any compactly supported distribution $v \in \mathscr{E}'(\mathfrak{n}_0)$ we will denote by T_v^λ the twisted convolution operator

$$(T_v^\lambda \varphi)(z) = (\varphi *_\lambda v)(z) = \int_{\mathfrak{n}_0} \varphi(z w^{-1}) \omega_\lambda(z, -w)\, v(w)\, dw, \tag{66}$$

where the integral has again to be interpreted as application of a distribution.

Further, let G_λ denote the group which is $N_0 \times T$ as a manifold, where T denotes the circle group, and with multiplication

$$(z, \xi)(w, \eta) = (z w, \omega_\lambda(z w)\, \xi \eta). \tag{67}$$

We take the Haar measure on G_λ to be $dz\, d\xi$, where $d\xi$ on T is normed to have total mass one.

Let $A_p(G_\lambda)$ denote the Figa-Talamanca-Herz algebra of G_λ (see e.g. [2]), and let $\rho \in A_p(G_\lambda)$ be any function such that $\rho(z, \xi) = \xi$ for all z in a neighborhood of the support of v and all $\xi \in T$.

If $\|T_v^\lambda\|_p$ denotes the norm of T_v^λ as an operator on $L^p(N_0)$, then in [7] H. Leptin has shown that $\|T_v^\lambda\|_p$ can be estimated by the norm of the untwisted convolution operator T_v^0 as

$$\|T_v^\lambda\|_p \leqq |\rho|_{A_p(G_\lambda)} \|T_v^0\|_p, \tag{68}$$

for $1 < p < \infty$.

Proposition 5.2. *If v is a distribution with compact support on $N_0 \simeq \mathfrak{n}_0$, if $\chi \in C_0^\infty(N_0)$ is such that $\chi(z) = 1$ for all z in a neighborhood of* supp v, *and if D is a left-invariant differential operator on N as described in Lemma* 3, *then*

$$\|T_v^\lambda\|_p \leqq |(D\delta)_\lambda *_\lambda \chi|_{L^q(N_0)} \|T_v^0\|_p, \tag{69}$$

where δ denotes the Dirac distribution on N, and where $\dfrac{1}{q} = 1 - \dfrac{1}{p}$ for $1 < p < \infty$.

Proof. Let ^{d} denote a fundamental solution of D as described in Lemma 5.1. Then $D\,{}^d = \delta$, or ${}^d * (D\delta) = \delta$. So, after (19), since for any $\lambda \in \mathfrak{z}^*$ $\delta_\lambda = \delta$ on $N_0 \simeq \mathfrak{n}_0$,

$$\omega_\lambda *_\lambda (D\delta)_\lambda = \delta. \tag{70}$$

This yields

$$\chi=\delta *_\lambda \chi=\omega_\lambda *_\lambda ((D\delta)_\lambda *_\lambda \chi). \tag{71}$$

Now, let $f=\omega_\lambda$, $g=(D\delta)_\lambda *_\lambda \chi$, and let $\tilde{f}(z,\xi)=f(z)\xi$ and $\tilde{g}(z,\xi)=g(z)\xi$ on G_λ. A short calculation shows that $(\tilde{f}*\tilde{g})(z,\xi)=(f*_\lambda g)(z)\xi=\chi(z)\xi$, after (71).

So, we may choose $\rho=\tilde{f}*\tilde{g}$ in (68). But, then

$$|\rho|_{A_p(G_\lambda)}\leqq|\tilde{f}|_{L^p(G_\lambda)}|\check{\tilde{g}}|_{L^q(G_\lambda)}=|f|_{L^p(N_0)}|g|_{L^q(N_0)},$$

where, after Lemma 5.1, $|f|_{L^p(N_0)}=|\omega_\lambda|_{L^p(N_0)}\leqq 1$.

So, $|\rho|_{A_p}(G_\lambda)\leqq|(D\delta)_\lambda *_\lambda \chi|_{L^q(N_0)}$, which, together with (68), proves (69).

Remark. One could use Proposition 5.2 to give another proof of Satz 1 in [8].

6. Singular Kernels on Groups Having Square Integrable Representations

In the following we will only consider nilpotent Lie groups N such that all irreducible unitary representations which are not trivial on the center are square integrable. Such groups will be called *completely square integrable groups*. From the results of Moore and Wolf it follows that these are just the groups such that

$$P(\lambda)\neq 0 \quad \text{for all } \lambda\in\mathfrak{z}^*-\{0\}. \tag{72}$$

It is known that there are nilpotent Lie groups of any nilpotent degree having square integrable representation. For example, in [9] J. Ludwig gave concrete constructions of n-step nilpotent Lie groups with exclusively flat coadjoint orbits and one-dimensional center for any positive integer n, which are completely square integrable. Moreover, it is easily seen that these groups admit dilations. Also, all generalized Heisenberg groups (see [5]) are homogeneous completely square integrable groups, which may have centers of any positive dimension.

Let ω, $\mathfrak{z}_0,\ldots$, and k be as described in Sect. 1 (so especially $\omega+\mathfrak{z}=\mathfrak{n}$).

Then we have the following main result:

Theorem 6.1. *Assume that N is completely square integrable, and let k be a Calderón-Zygmund kernel on ω having property* (H2). *If $K=k\otimes\delta$ denotes the extension of k to $\mathfrak{n}$ as described in* (22), *and if T denotes the operator T: $C_0^\infty(N)\to C^\infty(N)$ given by $Tf=f*K$, then T extends to a bounded operator from $L^2(N)$ to $L^2(N)$, and $\|T\|\leqq CA_2(k)$.*

Proof. A look at the proof of Theorem 3.1 shows that it is enough to prove that there exists a constant $C>0$, such that all twisted convolution operators $T^\lambda_{k_{\lambda_0}}$ on $\mathfrak{n}_0$, given by

$$T^\lambda_{k_{\lambda_0}}(\varphi)=\varphi *_\lambda k_{\lambda_0},$$

are bounded for all $\lambda=(\lambda_0,\lambda_1)\in\mathfrak{z}^*$, and $\|T^\lambda_{k_{\lambda_0}}\|_2\leqq CA_2(k)$.

Here, we make the convention that $k_{\lambda_0}=k$, if $\mathfrak{z}_0=\omega\cap\mathfrak{z}=0$.

After Proposition 2.3, k_{λ_0} is a Calderón-Zygmund kernel having property (H2), and $A_2(k_{\lambda_0}) \leqq C A_2(k)$ for every $\lambda_0 \in \mathfrak{z}_0^*$.

So, it will be even enough to prove the following

Claim. There exists a constant $C'>0$, such that $\|T_h^\lambda\|_2 \leqq C' A_2(k)$ for any Calderón-Zygmund kernel h on $\mathfrak{n}_0$ with $A_2(h) \leqq C A_2(k)$.

Let $\varphi \in C_0^\infty(\mathfrak{n}_0)$, and h as above. Then, by (9), for any $\lambda \in \mathfrak{z}^*$ and $r>0$,

$$\begin{aligned}
\varphi *_{D_r\lambda} h(z) &= \int_{\mathfrak{n}_0} \varphi(z w^{-1})\, e^{i\langle D_r\lambda,\, P(z,\,-w)\rangle}\, h(w)\, dw \\
&= \int_{\mathfrak{n}_0} \varphi(z w^{-1})\, e^{i\langle \lambda,\, P(D_r z,\,-(D_r w))\rangle}\, h(w)\, dw \\
&= r^{-q} \int_{\mathfrak{n}_0} (\varphi \circ D_{r^{-1}})(D_r z \cdot w^{-1})\, e^{i\langle \lambda,\, P(D_r z,\,-w)\rangle}\, h \circ D_{r^{-1}}(w)\, dw \\
&= r^{-q}[(\varphi \circ D_{r^{-1}}) *_\lambda (h \circ D_{r^{-1}})](D_r z),
\end{aligned}$$

where q denotes the homogeneous dimension of $\mathfrak{n}_0$. This calculation shows that

$$\|T_h^{D_r\lambda}\|_2 = r^{-q}\, \|T^\lambda_{h \circ D_{1/r}}\|_2 \tag{73}$$

for any $\lambda \in \mathfrak{z}^*$ and $r>0$.

Let $\lambda \neq 0$, and let $\chi \in C_0^\infty(\mathfrak{n}_0)$ be such that $0 \leqq \chi \leqq 1$, $\chi(z)=1$ if $|z| \leqq 1$, and $\chi(z) = 0$ if $|z| \geqq 2$. Define

$$h' = h(\chi \circ D_{|\lambda|}), \quad \text{and} \quad h'' = h - h'. \tag{74}$$

After Lemma 2.2, h' is a Calderón-Zygmund kernel with $A_2(h') \leqq C A_2(h) \leqq C' A_2(k)$. So, after (73),

$$\|T_h^\lambda\|_2 = |\lambda|^{-q}\, \|T^{\lambda'}_{h' \circ D_{1/|\lambda|}}\|_2, \tag{75}$$

where $\lambda' = D_{1/|\lambda|}\lambda$.

But, $h' \circ D_{1/|\lambda|} = (h \circ D_{1/|\lambda|})\chi$, hence $h' \circ D_{1/|\lambda|}(z) = 0$ if $|z| \geqq 2$. So, if we write $\chi' = \chi \circ D_{1/3}$, then $\chi' = 1$ on a neighborhood of the support of $h' \circ D_{1/|\lambda|}$, hence an application of Proposition 5.2 yields

$$\|T^{\lambda'}_{h' \circ D_{1/|\lambda|}}\|_2 \leqq |(D\delta)_{\lambda'} *_{\lambda'} \chi'|_{L^2(\mathfrak{n}_0)}\, \|T^0_{h' \circ D_{1/|\lambda|}}\|_2, \tag{76}$$

where D is a left-invariant differential operator on N as described in Lemma 5.1. We may write $(D\delta)(z,t)$ as

$$(D\delta)(z,t) = \sum_{\alpha,\beta} a_{(\alpha,\beta)}\, \delta^{(\alpha)}(z)\, \delta^{(\beta)}(t),$$

corresponding to the decomposition of $\mathfrak{n}$ as $\mathfrak{n}_0 \oplus \mathfrak{z}$, where the sum is a finite sum over multi-indices $\alpha \in \mathbb{N}_0^{\dim \mathfrak{n}_0}$ and $\beta \in \mathbb{N}_0^{\dim \mathfrak{z}}$, and where the $a_{(\alpha,\beta)}$ are constants. Thus,

$$(D\delta)_{\lambda'}(z) = \sum_{\alpha,\beta} a_{(\alpha,\beta)}(-i\lambda')^\beta\, \delta^{(\alpha)}(z).$$

Since $|\lambda'|=1$, it is clear that there exists a constant $a>0$, such that $|a_{(\alpha,\beta)}(-i\lambda')^{\beta}|\leqq a$ for all α and β, independently of $\lambda\in\mathfrak{x}^*$.

Moreover, if $z\in\operatorname{supp}\chi'$, then $|z|\leqq 6$. Since $P(z,w)$ is a polynomial function, these observations together make clear that

$$|(D\delta)_{\lambda'} *_{\lambda'} \chi'|_{L^2(\mathfrak{n}_0)}\leqq C \tag{77}$$

for some constant $C>0$, independently of $\lambda\in\mathfrak{x}^*-\{0\}$. From (75), (76) and (77) we get

$$\|T^{\lambda}_{h'}\|_2\leqq|\lambda|^{-q}\, C\,\|T^0_{h'\circ D_{1/|\lambda|}}\|_2 = C\,\|T^{\lambda}_{h'}\|_2. \tag{78}$$

But, either by induction over the degree of $\mathfrak{n}$, or by applying Theorem 1 of [13], we may assume that $\|T^0_{h'}\|_2\leqq C A_2(h')\leqq C' A_2(k)$, since T^0_h is just the operator of convolution with h' from the right on N_0.

So, finally we obtain

$$\|T^{\lambda}_{h'}\|_2\leqq C A_2(k). \tag{79}$$

To estimate $\|T^{\lambda}_{h''}\|_2$, we will apply Corollary 4.1. We have

$$\begin{aligned}|h''|^2_{L^2(\mathfrak{n}_0)} &= \int_{\mathfrak{n}_0} |1-\chi(D_{|\lambda|}z)|^2\,|h(z)|^2\,dz\\ &\leqq \int_{|z|>|\lambda|^{-1}} |h(z)|^2\,dz\leqq A_2(h)^2\,|\lambda|^q.\end{aligned} \tag{80}$$

If we let $\varkappa=\inf\{|P((2\pi)^{-1}\lambda)|\colon |\lambda|=1\}$, then, by (72), $\varkappa>0$, since P is continuous. If we write $\lambda'=D_{1/|\lambda|}\lambda$, then (80) together with (58) and (59) yields

$$\begin{aligned}\|T^{\lambda}_{h''}\|_2 &\leqq |P((2\pi)^{-1}\lambda)|^{-1/2}\,|h''|_{L^2(\mathfrak{n}_0)}\\ &\leqq (|\lambda|^q\,|P((2\pi)^{-1}\lambda')|)^{-1/2}\,A_2(h)\,|\lambda|^{q/2}\\ &\leqq C'\varkappa^{-1/2}\,A_2(k).\end{aligned} \tag{81}$$

Let us remark that we could have obtained (82) also without using the homogeneity of P, but instead by applying (76) also to h''.

(80) and (81) together yield

$$\|T^{\lambda}_{h}\|_2\leqq(C+C'\varkappa^{-1/2})\,A_2(k), \qquad \text{q.e.d.}$$

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Oblatum 10-III-1983