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Index for Subfactors

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§1. Introduction

One of the first things Murray and von Neumann did with their theory of continuous dimension for subspaces affiliated with a type II_1 factor was to define an invariant for its action on a Hilbert space. This invariant has come to be known as the coupling constant and it measures the relative mobility of the factor and its commutant. To be precise, if M is a II_1 factor on $\mathscr{H}$ and M' is its commutant, the coupling constant C_M is infinite if M' is an infinite factor and if M' is finite, one takes any non-zero vector ξ in $\mathscr{H}$ and one considers the closed subspaces $\overline{M\xi}$ and $\overline{M'\xi}$ affiliated with M' and M, respectively. It is a nontrivial result of Murray and von Neumann that the ratio $C_M = \dim_{M'}(\overline{M}\xi)/\dim_M(\overline{M}'\xi)$ does not depend on ξ. This real number between 0 and ∞ is called the coupling constant.

From a more modern point of view, the coupling constant measures the dimension of a Hilbert space on which M acts and may be defined in terms of intertwining maps in the category of normal M-modules. This leads to the notation $C_M = \dim_M(\mathscr{H})$. This notation is more natural and has been useful in the study of the von Neumann algebra of a foliation where the dimensions of certain geometric Hilbert spaces are measured by a von Neumann algebra (see [8]). We will use this more suggestive notation while keeping the term "coupling constant." Two normal representations of M are unitarily equivalent iff they have the same coupling constant.

It is an easy observation that the coupling constant can be used to define a conjugacy invariant for subfactors of II_1 factors. We call this invariant the index since if the subfactor comes from a subgroup in the group constructions of II_1 factors, the conjugacy invariant is the index of the subgroup. The index is defined in general as $\dim_N(L^2(M, \mathrm{tr}))$ where N is the subfactor and tr is the trace on M. This definition was probably noticed by Murray and von Neumann and appears more or less explicitly in works by Goldman [14], Suzuki

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[26] and others. In fact Goldman proves that if a subfactor has index 2 then the whole factor may be expressed as the crossed product of the subfactor by a $\mathbb{Z}_2$ action. This is analogous to the fact that any subgroup of index 2 of a group is normal. It may be combined with Connes' classification of periodic automorphisms of the hyperfinite II_1 factor R ([6]) to yield the pleasing positive result that there is, up to conjugacy, only one subfactor of index 2 of R.

The question immediately arises: what possible values can the index take? The difficulty in answering this question is that there is very little to play with given an arbitrary subfactor of a II_1 factor. The only general result available is the existence of a conditional expectation onto the subfactor shown by Umegaki in [30]. In fact this tool will allow us to determine completely the possible values of the index for subfactors of R. Experience with dimension in II_1 factors suggests that the index will take on a continuum of values. This is indeed true but surprisingly one cannot turn on this effect until index 4 and even then it involves the fundamental group. It seems entirely plausible that for II_1 factors without full fundamental group, whose existence was shown by Connes in [9], the index may take only countably many values.

What, then, are the possible values of the index for subfactors of R? The answer is $\{4\cos^2\pi/n \mid n=3,4,\ldots\}\cup\{r\in\mathbb{R}\mid r\geqq 4\}\cup\{\infty\}$. The value ∞ and the real values $\geqq 4$ are easily obtained (2.1.19 and 2.2.5). The situation between 1 and 4 is more difficult. The basic idea of the analysis for index <4 is as follows: Let $N\subseteq M$ be II_1 factors. One represents M on $L^2(M,\mathrm{tr})$ and considers the extension e_N to $L^2(M,\mathrm{tr})$ of the conditional expectation onto N. One defines $\langle M, e_N\rangle$ to be the II_1 factor generated by M and e_N on $L^2(M,\mathrm{tr})$. (This construction appears also in [5], [24]).) The crucial observation at this point is that the index of M in $\langle M, e_N\rangle$ is the same as that of N in M. Thus one may iterate this extension process and one obtains a sequence of II_1 factors, each one obtained from the previous one by adding a projection. The inductive limit gives a II_1 factor and if the projections in the construction are numbered $e_1, e_2, \ldots,$ then they satisfy $e_i e_{i\pm 1} e_i = \tau e_i$, $e_i e_j = e_j e_i$ if $|i-j|\geqq 2$ and $\mathrm{tr}(e_{i_1} e_{i_2}\ldots e_{i_n})=\tau^n$ if $|i_j - i_k|\geqq 2$ for $j\neq k$. Here τ is the reciprocal of the index of N in M and tr denotes the trace on the inductive limit. Analysis of the algebra generated by the e_i's yields that if $\tau>1/4$ it can only be $\frac{1}{4}\sec^2\pi/n$ for $n=3,4,\ldots$. A large bonus of the analysis is that it shows how to construct subfactors with these values as τ.

It seems strange that the set of values contains a discrete and a continuous part, but this may yet be understood by the fact that, if the index is less than 4, the relative commutant is trivial while the constructions available for the continuous part all have nontrivial relative commutant. We have very little information on what happens to the values of the index if the relative commutant is required to be trivial, but note that a result of Popa in [23], together with Connes' result on injective factors [7] shows that any II_1 factor has a subfactor of infinite index with trivial relative commutant.

Before closing the introduction, I would like to pose 4 problems which I hope will lead to a more profound understanding of subfactors.

Problem 1 (due to Connes). What are the possible values of the index for subfactors of R with trivial relative commutant? Or, what is $\mathscr{C}_R$?

Problem 2. For each $n=3, 4, 5, \ldots$ are there only finitely many subfactors of R up to conjugacy with index $4\cos^2 \pi/n$?

Problem 3. If N is a subfactor of R, is N conjugate to $N \otimes R$ in a decomposition $R \cong R \otimes R$? ($[R:N] < \infty$)

Problem 4. If N is a subfactor of M which is regular and has trivial relative commutant, is M the crossed product of N by a group action? (True if $M=R$ by a result of Ocneanu [22], see [17].)

It would be impossible to thank everyone who helped me with this paper. I have tried to mention individual contributions in the text, but let me also thank especially B. Baker, A. Connes, F. Goodman, R. Powers, M. Takesaki, and A. Wassermann for many fruitful conversations. This paper is an extended version of the Comptes Rendus note [18].

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§ 2. Generalities

§ 2.1. The Global Index

If M is a finite factor acting on a Hilbert space $\mathscr{H}$ with finite commutant M', the coupling constant $\dim_M(\mathscr{H})$ of M is defined as $\mathrm{tr}_M(E_\xi^{M'})/\mathrm{tr}_{M'}(E_\xi^M)$ where ξ is a non-zero vector in $\mathscr{H}$, tr_A denotes the normalized trace and E_ξ^A is the projection onto the closure of the subspace $A\xi$. This definition, due to Murray and von Neumann in [20], is independent of ξ.

We recall some rules of calculation associated with $\dim_M$. (See [10, p. 263].)

$$\dim_M(\mathscr{H})>0 \tag{2.1.1}$$

$$\dim_M(\mathscr{H})=(\dim_{M'}(\mathscr{H}))^{-1} \tag{2.1.2}$$

If e is a projection in M', $\dim_{M_e}(e\mathscr{H})=\mathrm{tr}_{M'}(e)\dim_M(\mathscr{H})$ (2.1.3)

If E is a projection in M, $\dim_{M_e}(e\mathscr{H})=(\mathrm{tr}_M(e))^{-1}\dim_M(\mathscr{H})$ (2.1.4)

If $M\otimes 1$ is the amplification of M on $\mathscr{H}\otimes\mathscr{K}$,

$$\dim_M(\mathscr{H}\otimes\mathscr{K})=\dim_{\mathbb{C}}(\mathscr{K})\dim_M(\mathscr{H}) \tag{2.1.5}$$

$\dim_M(\mathscr{H})=1$ iff M is standard on $\mathscr{H}$, i.e. there is a cyclic trace vector for M. (2.1.6)

Agree to put $\dim_M(\mathscr{H})=\infty$ if M' is infinite.

Proposition 2.1.7. *Let M be as above and N be a subfactor. The number $\dim_N(\mathscr{H})/\dim_M(\mathscr{H})$ is independent of $\mathscr{H}$ provided M' is finite.*

Proof. Any two such representations differ by an amplification and an induction. By (2.1.3) and (2.1.5), both $\dim_M$ and $\dim_N$ are multiplied by the same constants in this process. Q.E.D.

Definition. If N is a subfactor of M, the number $\dim_N(\mathscr{H})/\dim_M(\mathscr{H})$ defined in 2.1.7 is called the (global) *index* of N in M and written $[M:N]$. Note that $[M:N]=\infty$ means that N' is infinite for any normal representation of M.

By (2.1.7) and (2.1.6), $[M:N]=\dim_N(L^2(M,\mathrm{tr}))$. Thus $[M:N]$ is a conjugacy invariant for N as a subfactor of M.

The rules of calculation for $\dim_M$ give some rules for $[M:N]$.

Proposition 2.1.8. *If $P\subseteq Q\subseteq M$ are II_1 factors then*

$$[M:M]=1 \tag{2.1.9}$$

$$[M:P]\geqq 1 \tag{2.1.10}$$

$$[M:P]=[M:Q][Q:P] \tag{2.1.11}$$

$$[M:P]\geqq[M:Q] \tag{2.1.12}$$

$$[M:P]=[M:Q] \quad \textit{implies } P=Q \tag{2.1.13}$$

$$[M:P]=[P':M'] \quad \textit{if } P' \textit{ is finite.} \tag{2.1.14}$$

Proof. The only nontrivial property is (2.1.13). To prove it first note that by (2.1.11) and (2.1.9) we may suppose that $M=Q$ and that M acts on $L^2(M,\mathrm{tr})$ with cyclic trace vector ξ. Then $P\xi$ is dense in $\mathscr{H}$ by hypothesis. But then for any $a\in M$ there is a net b_n of elements of P with $b_n\xi\to a\xi$ in $\mathscr{H}$, i.e. $\|b_n-a_n\|_2\to 0$. By [10, Lemma 1, p. 270], this implies $a\in P$. Q.E.D.

We next examine how the index behaves under tensor products.

Proposition 2.1.15. *Let N_1 and N_2 be subfactors of the finite factors M_1 and M_2, respectively. Then $N_1\otimes N_2$ is a subfactor of $M_1\otimes M_2$ and $[M_1\otimes M_2:N_1\otimes N_2]=[M_1:N_1][M_2:N_2]$.*

Proof. Let M_1 and M_2 act with cyclic trace vectors $\xi_1 \in \mathscr{H}_1$ and $\xi_2 \in \mathscr{H}_2$ respectively. Then if e_1 and e_2 are the projections onto $\overline{N_1 \xi_1}$ and $\overline{N_2 \xi_2}$, $e_1 \otimes e_2$ is the projection onto $\overline{N_1 \otimes N_2(\xi_1 \otimes \xi_2)}$. Moreover $M_1 \otimes M_2$ is standard on $\mathscr{H}_1 \otimes \mathscr{H}_2$ and $(N_1 \otimes N_2)' = N_1' \otimes N_2'$. Thus $\mathrm{tr}_{(N_1 \otimes N_2)'}(e_1 \otimes e_2) = \mathrm{tr}_{N_1'}(e_1)\,\mathrm{tr}_{N_2'}(e_2)$ which gives the desired result. Q.E.D.

Proposition 2.1.16. *Let N_i, M_i be as above and suppose $N_i' \cap M_i = \mathbb{C}$, $i = 1, 2$. Then $(N_1 \otimes N_2)' \cap (M_1 \otimes M_2) = \mathbb{C}$.*

Proof. $(N_1 \otimes N_2)' \cap (M_1 \otimes M_2) = (N_1' \otimes N_2') \cap M_1 \otimes M_2$ (see [28, p. 227]) and $(N_1' \otimes N_2') \cap (M_1 \otimes M_2) = (N_1' \cap M_1) \otimes (N_2' \cap M_2)$. Q.E.D.

We now introduce two isomorphism invariants for II_1 factors.

Definition. If M is a finite factor let

$$\mathscr{I}_M = \{r \in \mathbb{R} \cup \{\infty\} \mid \text{there is a } \mathrm{II}_1 \text{ subfactor } N \text{ of } M \text{ with } [M:N] = r\}$$

$$\mathscr{C}_M = \{r \in \mathbb{R} \cup \{\infty\} \mid \text{there is a } \mathrm{II}_1 \text{ subfactor } N \text{ of } M \text{ with } [M:N] = r \text{ and } N' \cap M = \mathbb{C}\}.$$

The determination of $\mathscr{I}_M$ and $\mathscr{C}_M$ is in general rather difficult. We shall gather some immediate results about them.

Proposition 2.1.17. *If $M \cong M \otimes M$ then both $\mathscr{I}_M$ and $\mathscr{C}_M$ are subsemigroups (with 1) of $\{r \in \mathbb{R} \mid r \geqq 1\} \cup \{\infty\}$ under multiplication.*

Proof. This follows from 2.1.15 and 2.1.16. Q.E.D.

Lemma 2.1.18. *If N is a hyperfinite subfactor of M then $[M:N] < \infty$ implies that M is hyperfinite.*

Proof. If M acts in such a way that N′ is finite, then N′ is hyperfinite (e.g. [29]) and by [7], M' and so M is hyperfinite. Q.E.D.

Corollary 2.1.19. *For any II_1 factor M, $\infty \in \mathscr{I}_M$.*

Proof. If $M \cong R$ then $R \otimes 1 \subseteq R \otimes R$ is of infinite index. Otherwise by [21] there is a hyperfinite subfactor of M which is of infinite index by 2.1.18. Q.E.D.

Proposition 2.1.20. *For any separable II_1 factor M, $\infty \in \mathscr{C}_M$.*

Proof. A result of Popa in [23] says that we can find a maximal abelian subalgebra A and a unitary u with $uAu^* = A$ such that u and A generate a subfactor N of M, isomorphic to R. Since N contains a maximal abelian subalgebra, $N' \cap M = \mathbb{C}$.

So if M is not hyperfinite, combining this result with 2.1.18 gives the required subfactor. If $M = R$, see [17] or §2.3. Q.E.D.

In this paper we will show that

$$\mathscr{I}_M \cap [1, 4) = \mathscr{C}_M \cap [1, 4) \subseteq \{4 \cos^2 \pi/n \mid n = 3, 4, \ldots\}$$

and that

$$\mathscr{I}_R = \{4 \cos^2 \pi/n \mid n = 3, 4, \ldots\} \cup \{r \in \mathbb{R} \mid r \geqq 4\} \cup \{\infty\}.$$

§ 2.2. *The Local Index*

If the global index of a subfactor is finite, one may define a finer invariant, which I call the local index, obtained by restricting the trace on N' to $N' \cap M$. I would like to thank A. Connes for suggesting this approach.

Definition. Let $N \subseteq M$ be II_1 factors and let $p \in N' \cap M$ be a projection. The *index of N at p* will be $[M_p : N_p] = [M:N]_p$.

Lemma 2.2.1. *The index at p and the global index are related by the formula*

$$[M:N]_p = [M:N]\,\mathrm{tr}_M(p)\,\mathrm{tr}_{N'}(p).$$

Proof. If M begins in standard form on $\mathscr{H}$ then by 2.1.4, $\dim_{M_p}(p\mathscr{H}) = \mathrm{tr}_M(p)^{-1}$. Also $\dim_N(\mathscr{H}) = [M:N]$ so by 2.1.3, $\dim_{N_p}(p\mathscr{H}) = [M:N]\,\mathrm{tr}_{N'}(p)$. Thus $[M:N]_p = \dim_{N_p}(p\mathscr{H})/\dim_{M_p}(p\mathscr{H}) = [M:N]\,\mathrm{tr}_M(p)\,\mathrm{tr}_{N'}(p)$. Q.E.D.

Lemma 2.2.2. *If $\{p_i\}$ is a partition of unity in $N' \cap M$ then*

$$[M:N] = \sum_i \mathrm{tr}_M(p_i)^{-1}\,[M:N]_{p_i}.$$

Proof. For each i, $[M:N]\,\mathrm{tr}_{N'}(p_i) = \mathrm{tr}_M(p_i)^{-1}[M:N]_{p_i}$ and summing over i gives the result.

Corollary 2.2.3. *If $[M:N] < \infty$ then $N' \cap M$ is finite dimensional.*

Proof. If $N' \cap M$ were infinite dimensional we could find arbitrarily large partitions of unity and by 2.2.2 $[M:N] = \infty$. Q.E.D.

In fact with a little more care one may obtain the bound $[M:N] \geqq \dim_{\mathbb{C}}(N' \cap M)$.

Corollary 2.2.4. *If $[M:N] < 4$, $N' \cap M = \mathbb{C}$.*

Proof. If $N' \cap M \neq \mathbb{C}$, then it contains two mutually orthogonal non-zero projections and since $[M:N]_{p_i} \geqq 1$, $[M:N] \geqq \mathrm{tr}(p_1)^{-1} + \mathrm{tr}(p_2)^{-1} \geqq 4$. Q.E.D.

Corollary 2.2.5. *If M has fundamental group $= \mathbb{R}$, $\mathscr{I}_M$ contains $\{r \in \mathbb{R} \mid r \geqq 4\}$.*

Proof. We must exhibit for any $r \geqq 4$ a subfactor of index r. Choose $d \in (0, 1)$ with $1/d + 1/(1-d) = r$ and choose a projection p with $\mathrm{tr}_M(p) = d$. Then M_p and M_{1-p} are isomorphic so choose some isomorphism θ: $M_p \to M_{1-p}$ and let N be the subfactor $\{x + \theta(x) \mid x \in M_p\}$. Then $N_p = M_p$ and $N_{1-p} = M_{1-p}$ so by 2.2.2, $[M:N] = 1/d + 1/(1-d) = r$. Q.E.D.

§ 2.3. *Examples*

We give three examples of subfactors and their indices.

Example 2.3.1. Suppose $M = N \otimes P$ where P is a type I_n factor. Then $[M : N \otimes 1] = n^2$. This follows from 2.1.5.

Thus for any II_1 factor M, $\mathscr{I}_M$ always contains $\{n^2 | n \in \mathbb{Z} - \{0\}\} \cup \{\infty\}$. It is not inconceivable that there are II_1 factors for which $\mathscr{I}_M$ is no larger than this set.

Example 2.3.2. Let A be a von Neumann algebra and G a countable discrete group of automorphisms for which the crossed product $A \rtimes G$ is a finite factor. If H is a subgroup of G such that $A \rtimes H$ is a finite factor then $[A \rtimes G : A \rtimes H] = [G:H]$.

Proof. Write G as a disjoint union of cosets, $G = \coprod_{i \in I} H g_i$. Then let $V_i = \overline{(A \rtimes H)\, u_{g_i}}$. The V_i are subspaces of $L^2(A \rtimes G, \mathrm{tr})$ affiliated with $(A \rtimes H)'$ which are mutually orthogonal, where tr is the extension to $A \rtimes G$ of a faithful normal G-invariant trace on A and the u_g's are the implementing unitaries of the crossed product. Moreover if p_i is the projection onto V_i then $\sum_{i \in I} p_i = 1$ and the p_i are mutually equivalent in $(A \rtimes H)'$ since $V_i = J u_{g_i} J V_0$ where $V_0 = \overline{A \rtimes H}$ and J is the involution on $L^2(A \rtimes G, \mathrm{tr})$. This is because $J u_{g_i} J \in (A \rtimes G)' \subseteq (A \rtimes H)'$.

Thus if $[G:H] = \infty$, $(A \rtimes H)'$ is infinite so $[A \rtimes G : A \rtimes H] = \infty$. If $[G:H] < \infty$, $(A \rtimes H)'$ is finite since reduction by p_0 puts $A \rtimes H$ in standard form. So $[G:H]\, \mathrm{tr}_{(A \rtimes H)'}(p_0) = 1$ which establishes that $[A \rtimes G : A \rtimes H] = [G:H]$. Q.E.D.

This example is the justification for the name "index".

Example 2.3.3. If M is a II_1 factor and G is a finite group of outer automorphisms of M with fixed point algebra M^G, $[M : M^G] = |G|$.

Proof. This result could be established using 2.3.2 but I shall give a different proof which brings in the basic construction of Chapter 3.

Let M act on $L^2(M, \mathrm{tr})$ and let u_g be the unitaries extending the action of G on M. Then the u_g's act also on M' and it is established in [1] that $(M^G)'$ is isomorphic in the obvious way to $M' \rtimes G$. The projection onto $\overline{M^G}$ is $|G|^{-1} \sum_{g \in G} u_g$ and by the isomorphism with the crossed product its trace is $|G|^{-1}$. Thus $[M : M^G] = |G|$. Q.E.D.

§ 3. The Basic Construction

§ 3.1. Extending Finite von Neumann Algebras by Subalgebras

Let M be a finite von Neumann algebra with faithful normal normalized trace tr and let N be a von Neumann subalgebra. By [30] there is a conditional expectation $E_N : M \to N$ defined by the relation $\mathrm{tr}(E_N(x)y) = \mathrm{tr}(xy)$ for $x \in M$, $y \in N$. The map E_N is normal and has the following properties:

$$E_N(axb) = aE_N(x)b \quad \text{for } x \in M,\ a, b \in N \text{ (the bimodule property)} \tag{3.1.1}$$

$$E_N(x^*)=E_N(x)^* \quad \text{for all } x\in M \tag{3.1.2}$$

$$E_N(x^*)E_N(x)\leqq E_N(x^*x) \quad \text{and} \quad E_N(x^*x)=0 \quad \text{implies } x=0. \tag{3.1.3}$$

Let ξ be the canonical cyclic trace vector in $L^2(M,\mathrm{tr})$. Identify M with the algebra of left multiplication operators on $L^2(M,\mathrm{tr})$. The conditional expectation E_N extends to a projection e_N on $\mathscr{H}$ via $e_N(x\xi)=E_N(x)\xi$. Let J be the involution $x\xi\mapsto x^*\xi$.

Proposition 3.1.4.

(i) *For* $x\in M$, $e_N x e_N=E_N(x)e_N$.

(ii) *If* $x\in M$ *then* $x\in N$ *iff* $e_N x=x e_N$.

(iii) $N'=\{M'\cup\{e_N\}\}''$.

(iv) J *commutes with* e_N.

Proof. (i) If y is an arbitrary element of M then $e_N x e_N(y\xi)=e_N x(E_N(y)\xi)$ $=E_N(x)E_N(y)\xi$ by 3.1.1, and $E_N(x)e_N(y\xi)=E_N(x)E_N(y)\xi$. But the vectors $y\xi$ are dense in $L^2(M,\mathrm{tr})$.

(ii) Relation 3.1.1 shows that e_N commutes with N as in (i). Moreover if $x\in M$ and $e_N x=x e_N$ then $(e_N x)\xi=E_N(x)\xi=(xe)\xi=x\xi$. Since ξ is separating, $x=E_N(x)$.

(iii) It suffices to show that $\{M'\cup\{e_N\}\}'=N$. This follows from (i).

(iv) This follows from 3.1.2. Q.E.D.

These calculations lead to the following.

Definition. Let $\langle M,e_N\rangle$ be the von Neumann algebra on $L^2(M,\mathrm{tr})$ generated by M and e_N. This is the basic construction.

Proposition 3.1.5. (i) $\langle M,e_N\rangle=JN'J$.

(ii) *Operators of the form* $a_0+\sum_{i=1}^{n}a_i e_N b_i$ *with* a_i, $b_i\in M$, *give a dense* $*$*-subalgebra of* $\langle M,e_N\rangle$.

(iii) $x\mapsto x e_N$ *is an isomorphism of* N *onto* $e_N\langle M,e_N\rangle e_N$.

(iv) *The central support of* e_N *in* $\langle M,e_N\rangle$ *is* 1.

(v) $\langle M,e_N\rangle$ *is a factor iff* N *is.*

(vi) $\langle M,e_N\rangle$ *is finite iff* N' *is.*

Proof. (i) and (ii) follow immediately from 3.1.4. For (iii), to show that $e_N\langle M,e_N\rangle e_N\subseteq N e_N$ it suffices by (ii) to show that $e_N(a e_N b)e_N\in N e_N$. This follows from (i) of 3.1.4. Moreover if $x e_N=0$ then $x e_N\xi=x\xi=0$ and ξ is separating. Thus $x\mapsto x e_N$ is injective. Affirmations (iv), (v) and (vi) are now easy. Q.E.D.

We want to consider special traces on $\langle M,e_N\rangle$.

Definition. If P is a subalgebra of $\langle M,e_N\rangle$, a trace Tr on $\langle M,e_N\rangle$ is called a (τ,P) trace if Tr extends tr and $\mathrm{Tr}(e_N x)=\tau\,\mathrm{tr}(x)$ for $x\in P$.

Lemma 3.1.6. *A* (τ,N) *trace is a* (τ,M) *trace.*

Proof. If $x \in M$, $\mathrm{Tr}(x e_N) = \mathrm{Tr}(e_N x e_N) = \mathrm{Tr}(E_N(x) e_N) = \tau\,\mathrm{tr}(E_N(x)) = \tau\,\mathrm{tr}(x)$. Q.E.D.

We shall now concentrate on the case where M and N are factors.

Proposition 3.1.7. *If M and N are factors then $[M:N] < \infty$ iff $\langle M, e_N \rangle$ is finite and in this case the canonical trace* Tr *on $\langle M, e_N \rangle$ is a (τ, M) trace where $\tau = [M:N]^{-1}$. In particular* $\mathrm{Tr}(e_N) = [M:N]^{-1}$. *Also* $[\langle M, e_N \rangle : N] = [M:N]$.

Proof. By 3.1.6 it suffices to show that Tr is a (τ, N) trace. But consider the map $y \to \mathrm{Tr}(e_N y)$ defined on N. This is a trace by (ii) of 3.1.4 so since N is a factor there is a constant K such that $\mathrm{Tr}(e_N y) = K\,\mathrm{tr}(y)$. Moreover the trace of e_N in N' is by definition τ so by (iv) of 3.1.4, (i) of 3.1.5 and uniqueness of the trace, this is the same as $\mathrm{Tr}(e_N)$. Thus $K = \tau$, and Tr is a (τ, N) trace.

To prove this last assertion note that

$$\begin{aligned}[\langle M, e_N \rangle : M] &= \dim_{M'}(L^2(M, \mathrm{tr}))/\dim_{N'}(L^2(M, \mathrm{tr})) = [\dim_{N'}(L^2(M, \mathrm{tr}))]^{-1} \\ &= \dim_N(L^2(M, \mathrm{tr})) = [M:N]. \quad \text{Q.E.D.}\end{aligned}$$

For some general results about projections onto finite subalgebras see Skau's paper [24].

Finally in this section I show that the basic construction is generic for subfactors of finite index in the sense that all such subfactors are of the form $M \subseteq \langle M, e_N \rangle$, although not canonically.

Lemma 3.1.8. *Let N be a II_1 factor acting on $L^2(N, \mathrm{tr})$ and let M be a II_1 factor containing N. Then $[M:N]$ is finite and there is a subfactor P of N such that $M = \langle N, e_P \rangle$.*

Proof. Since M' is a II_1 factor, $[M:N] < \infty$. Let $P = JM'J$. Then $[N:P] = [M:P]$ as in 3.1.7 and $\langle N, e_P \rangle$ is a subfactor of M with $[M : \langle N, e_P \rangle] = 1$. Thus by 2.1.13, $M = \langle N, e_P \rangle$. Q.E.D.

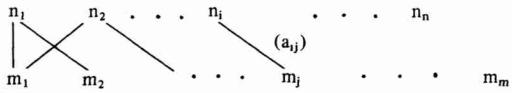
Corollary 3.1.9. *Let $N \subseteq M$ be II_1 factors with $[M:N] < \infty$ then there is a subfactor P of N and an isomorphism θ: $M \to \langle N, e_P \rangle$ with $\theta|_N = \mathrm{id}$.*

Proof. Represent M on $L^2(M, \mathrm{tr})$. Choose a projection $p \in M'$ with $\mathrm{tr}(p) = [M:N]^{-1}$. Then by [20], on $pL^2(M, \mathrm{tr})$, M and N act with N in standard form. By 3.1.8 we are through. Q.E.D.

§ 3.2. Inclusions of Complex Semisimple Algebras

Let $N \subseteq M$ be finite dimensional complex semisimple algebras and $N = \bigoplus_{i=1}^{n} N_i$, $M = \bigoplus_{j=1}^{m} M_j$ be their canonical decompositions as direct sums of simple algebras, $N_i \cong M_{n_i}(C)$, $M_j \cong M_{m_j}(C)$. The inclusion of N in M is specified up to conjugacy by an $n \times m$ matrix $A_N^M = (a_{ij})$ where (a_{ij}) is the number of simple components of a simple M_j module viewed as an N_i module. This may be zero or any positive integer. If $\{p_i\}$ are the central idempotents for the N_i and q_j those for the M_j, $a_{ij} = 0$ iff $p_i q_j = 0$. We will call the matrix A_N^M the inclusion matrix.

The inclusion can also be described diagramatically as follows:

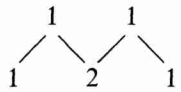


Here there are a_{ij} lines between n_i and m_j. This diagram will be called the Bratteli diagram after [4].

If the identity of M is the same as that of N we have the obvious relation $m_j = \sum_{i=1}^{n} a_{ij} n_i$ which we shall write as

$$\vec{m} = \vec{n} A_N^M. \tag{3.2.1}$$

A concrete example is



which is the diagram for the inclusion of $\mathbb{C}S_2$ in $\mathbb{C}S_3$.

If $N \subseteq M \subseteq P$, note the formula

$$A_N^P = A_N^M A_M^P. \tag{3.2.2}$$

If V is a faithful M-module, the centres of M and M'' are identical so that (if 3.2.1 holds) in the decompositions of M' and N' as simple algebras we may write $N' = \bigoplus_{i=1}^{n} N_i'$ and $M' = \bigoplus_{j=1}^{m} M_j'$. The following formula is in [3, §5, ex. 17].

$$A_{M'}^{N'} = (A_N^M)^T. \tag{3.2.3}$$

Since there is only one normalized trace on $M_n(\mathbb{C})$, a trace on $M = \bigoplus_j M_j$ may be specified by a column vector $\vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}$ where t_j is the trace of a minimal idempotent in M_j. The trace of the identity is the product $\vec{m} \cdot \vec{t}$, where $\vec{m} = (m_1, m_2, \ldots, m_m)$.

If $\vec{t}$ defines a trace on M whose restriction to N is defined by the vector $\vec{s}$ then the following relation is immediate.

$$\vec{s} = A_N^M \vec{t}. \tag{3.2.4}$$

Conversely if $\vec{s}$ and $\vec{t}$ define traces on N and M, respectively, then they agree on N if 3.2.4 holds.

§3.3. Finite Dimensional C-Algebras*

A finite dimensional C^*-algebra is of course semisimple so the discussion and notation of §3.2 applies. The presence of the $*$-operation allows us to perform the basic construction of §3.1. The main question to be answered in this

section is: given a faithful trace tr on the finite dimensional C^*-algebra M, when does there exist a (faithful) (τ, M) trace on $\langle M, e_N\rangle$? All traces in this section will be positive, i.e. $\vec{t}$ is a positive vector in the natural ordering of $\mathbb{R}^n$.

We begin with a lemma giving more information on the basic construction in finite dimensions.

Lemma 3.3.1. *Let $N \subseteq M$ be finite dimensional C^*-algebras and let* tr *be a faithful positive normalized trace on M. Let e_N and $\langle M, e_N\rangle$ be as in* §3.1. *Suppose $\{p_i | i=1,2,\ldots,n\}$ are the minimal central projections of N. Then*

(i) *Jp_iJ are the minimal central projections of $\langle M, e_N\rangle$*

(ii) *$A_M^{\langle M, e_N\rangle} = (A_N^M)^T$ (with the obvious identification of the indices, $p_i \leftrightarrow Jp_iJ$)*

(iii) *$e_N Jp_iJ = e_N p_i$*

(iv) *$x \to e_N x Jp_iJ$ is an isomorphism from p_iN onto $(e_N Jp_iJ)\langle M, e_N\rangle(e_N Jp_iJ)$.*

Proof. (i) The p_i are the minimal central projections of N' and $\langle M, e_N\rangle = JN'J$.

(ii) This follows from 3.2.3 and (i).

(iii) If $x \in M$, $(e_N Jp_iJ)(x\xi) = e_N(xp_i\xi) = E_N(x)p_i\xi$, and

$$(e_N p_i)(x\xi) = e_N(p_i x\xi) = p_i E_N(x)\xi = E_N(x)p_i\xi.$$

(iv) Injectivity follows from (iii), and (iii) of 3.1.5. If $y \in p_i e_N \langle M, e_N\rangle p_i e_N$ then $y = e_N z$ for $z \in N$ and $p_i z = z$ by (iii) of 3.1.5 so $y = e_N p_i z$. Q.E.D.

Theorem 3.3.2. *Let M, N and* tr *be as above and let* tr *be given on M by the vector $\vec{t}$ and on N by the vector $\vec{s}$. Then there is a (τ, M) trace* Tr *on $\langle M, e_N\rangle$ iff*

(i) $A^T A\vec{t} = (1/\tau)\vec{t}$

(ii) $AA^T\vec{s} = (1/\tau)\vec{s}$

where $A = A_N^M$.

Proof. ($\Rightarrow$) Let Tr be given on $\langle M, e_N\rangle$ by $\vec{r}$, r_i being the trace of a minimal projection in $Jp_iJ\langle M, e_N\rangle$. By (iv) of 3.3.1, such a minimal projection may be chosen of the form $e_N q$, q being a minimal projection in p_iN. Since Tr is a (τ, M) trace, $r_i = \tau s_i$ so $\vec{r} = \tau\vec{s}$. But by (ii) of 3.3.1, 3.2.2 and 3.2.4, $\vec{s} = AA^T\vec{r} = \tau AA^T\vec{s}$, i.e. $AA^T\vec{s} = (1/\tau)\vec{s}$. Also $\vec{t} = A^T\vec{r} = \tau A^T AA^T\vec{r} = \tau A^T A\vec{t}$.

($\Leftarrow$) Define a trace Tr on $\langle M, e_N\rangle$ by the vector $\vec{r} = \tau\vec{s}$. It is a faithful positive trace since tr is. By 3.1.6 it suffices to show that it extends tr and that it is a (τ, N) trace. For the former, by 3.2.4 we need $A^T\vec{r} = \vec{t}$. But $A\vec{t} = \vec{s}$ and by (i), $A^T\vec{s} = (1/\tau)\vec{t}$ so $A^T\vec{r} = \vec{t}$. For the latter let q be a minimal projection in p_iN. Then $e_N q$ is a minimal projection in $Jp_iJ\langle M, e_N\rangle$ and by definition $\mathrm{Tr}(e_N q) = \tau\,\mathrm{tr}(q)$. The map $x \to \mathrm{Tr}(e_N x)$ is a trace on p_iN so by uniqueness and linearity, $\mathrm{Tr}(e_N x) = \tau\,\mathrm{tr}(x)$ for all $x \in N$. Q.E.D.

§3.4. Two Extensions; Goldman's Theorem

Let N be a proper von Neumann subalgebra of the finite von Neumann algebra M with faithful normal normalized trace tr. Suppose there is a faithful normal (τ, M) trace Tr on $\langle M, e_N\rangle$. Then we may form the extension $\langle\langle M, e_N\rangle, e_M\rangle$.

Proposition 3.4.1.

(i) $e_M e_N e_M = \tau e_M$

(ii) $e_N e_M e_N = \tau e_N$

(iii) $e_M \wedge e_N = e_M \wedge e_N^\perp = e_M^\perp \wedge e_N = 0$.

Proof. (i) By (i) of 3.1.4, $e_M e_N e_M = E_M(e_N) e_M$ and since Tr is a (τ, M) trace, $E_M(e_N) = \tau$.

(ii) By (ii) of 3.1.5 it suffices to verify the relation on vectors of the form $a\xi$ and $a e_N b \xi$ where $a, b \in M$. But

$$e_N e_M e_N(a\xi) = e_N(E_M(e_N a)\xi) = \tau e_N(a\xi),$$

and

$$e_N e_M e_N(a e_N b \xi) = e_N e_M(E_N(a) e_N b \xi) = e_N(\tau E_N(a) b \xi)$$

while $\tau e_N(a e_N b \xi) = \tau e_N E_N(\tau) b \xi$.

(iii) $e_N \wedge e_M = s - \lim_{n \to \infty} (e_N e_M e_N)^n = 0$ since $\tau < 1$.

The other relations follow from $(1-\tau) < 1$. Q.E.D.

Now suppose there is a faithful $(\tau, \langle M, e_N \rangle)$ trace Tr on $\langle \langle M, e_N \rangle, e_M \rangle$.

Corollary 3.4.2. *If $\tau \neq 1/2$, the von Neumann algebra generated by e_N and e_M is isomorphic to $M_2(\mathbb{C}) \oplus \mathbb{C}$. If $\tau = 1/2$ it is isomorphic to $M_2(\mathbb{C})$ and we have the relation $e_M + e_N - e_M e_N - e_N e_M = 1/2$.*

Proof. The relations of 3.4.1 show immediately that the von Neumann algebra generated by e_N and e_M has dimension at most 5 and is not abelian. But $\mathrm{Tr}(e_M \vee e_N) = 2\tau - \mathrm{Tr}(e_M \wedge e_N) = 2\tau$. This is enough to prove the affirmations about the structure of the algebra. An easy calculation shows that $p = (1-\tau)^{-1}(e_M + e_N - e_M e_N - e_N e_M)$ is a projection with $p e_N = e_N$ and $p e_M = e_M$. Thus $p = e_1 \vee e_2$ and if $\tau = 1/2$, $p = 1$. Q.E.D.

Corollary 3.4.3. (Goldman's theorem [14]). *Let N be a subfactor of the II_1 factor M with $[M:N] = 2$. Then M decomposes as the crossed product of N by an outer action of $\mathbb{Z}_2$.*

Proof. By 3.1.9 we know that M is of the form $\langle N, e_P \rangle$ for a subfactor of index 2 of N. Thus N is generated by N and $u = 2e_P - 1$. Moreover since tr is a $(1/2, N)$ trace on M, $\mathrm{tr}(ux) = 0$ for $x \in N$, and the relationship of 3.4.2 implies that, if $v = 2e_N - 1$, $uv = -vu$. Thus for $x \in N$, $vuxu^* = uxu^*v$. So uxu^* commutes with e_N and by (ii) of 3.1.4, $uxu^* \in N$. Thus by [25], M is the crossed product of N by a $\mathbb{Z}_2$ action which is necessarily outer since M is a factor. Q.E.D.

§4. Possible Values of the Index

§4.1. Certain Algebras Generated by Projections

Chapter 4 will be largely devoted to proving the following result.

Theorem 4.1.1. *Let M be a von Neumann algebra with faithful normal normalized trace* tr. *Let $\{e_i \mid i = 1, 2, \ldots\}$ be projections in M satisfying*

a) $e_i e_{i\pm 1} e_i = \tau e_i$ *for some* $\tau \leqq 1$

b) $e_i e_j = e_j e_i$ *for* $|i-j| \geqq 2$

c) $\operatorname{tr}(w e_i) = \tau \operatorname{tr}(w)$ *if* w *is a word on* $1, e_1, e_2, \ldots, e_{i-1}$

Then if P *denotes the von Neumann algebra generated by the* e_i's,

(i) $P \cong R$ *(the hyperfinite* II_1 *factor)*

(ii) $P_\tau = \{e_2, e_3, \ldots\}''$ *is a subfactor of* P *with* $[P:P_\tau] = \tau^{-1}$.

(iii) $\tau \leqq 1/4$ *or* $\tau = \frac{1}{4}\sec^2 \pi/n$, $n = 3, 4, \ldots$.

In this section we will prove (i) and (ii). We begin the proof with some notation.

Definition. Let $A_{m,n}$ be the $*$-algebra generated by $1, e_m, e_{m+1}, \ldots, e_n$ for $1 \leqq m \leqq n \leqq \infty$. Let A_n be $A_{1,n}$, $A_0 = \mathbb{C}$. Thus $A_\infty = A_{1,\infty}$ is the $*$-algebra generated by 1 and the e_i's.

Next some combinatorial results. If w is an (associative) word on the e_i's, call it *reduced* if it is of minimal length for the grammatical rules $e_i e_{i\pm 1} e_i \leftrightarrow e_i$, $e_i e_j \leftrightarrow e_j e_i$ for $|i-j| \geqq 2$, $e_i^2 \leftrightarrow e_i$.

Lemma 4.1.2. *Let* $e_{i_1} e_{i_2} \ldots e_{i_k}$ *be a reduced word. Then if* $m = \max\{i_1, i_2, \ldots, i_k\}$, m *occurs only once in the list* $i_1, i_2, \ldots, i_k$.

Proof. By induction on the length of a reduced word. It is trivial for words of length $\leqq 1$. Suppose true for words of length $\leqq n$ and let w be a reduced word of length $n+1$. Suppose $w = w_1 e_m w_2 e_m w_3$ where m is the maximum index and w_2 does not contain e_m. Then there are 2 possibilities.

a) w_2 does not contain e_{m-1}. In this case e_m commutes with all the e_i's in w_2 so the length of w may be shortened using $e_m^2 \to e_m$.

b) w_2 contains e_{m-1}. Then since w is reduced, so is w_2 and by induction $w_2 = v_1 e_{m-1} v_2$ where v_1 and v_2 are words on $e_1, e_2, \ldots, e_{m-2}$. But then e_m commutes with v_1 and v_2 so that the length of w may be reduced using $e_m e_{m-1} e_m \to e_m$. Q.E.D.

It is clear that in the algebra A, any word on the e_i's is proportional to a reduced word.

Corollary 4.1.3. (i) A_n *is finite dimensional.*

(ii) *For* $x \in A_n$, $e_{n+1} x e_{n+1} = E_{A_{n-1}}(x) e_{n+1}$ *(here* $E_{A_{n-1}}$: $A_n \to A_{n-1}$ *is with respect to the restriction of* tr).

(iii) $x \mapsto x e_{n+1}$ *is an isomorphism of* A_{n-1} *onto* $e_{n+1} A_{n+1} e_{n+1}$.

Proof. (i) If there are only finitely many reduced words, A_n is finite dimensional. But this follows immediately by induction from 4.1.2.

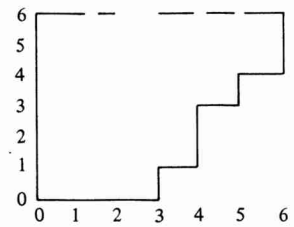
(ii) First of all $\operatorname{tr}(x e_{n+1}) = \tau \operatorname{tr}(x)$ for $x \in A_n$ follows immediately from 4.1.1(c) so that $E_{A_{n-1}}(e_n) = \tau$. By 4.1.2 and linearity it suffices to consider x of the form $w e_n w'$ with w and w' in A_{n-1}. But then $e_{n+1} x e_{n+1} = \tau w w' e_{n+1}$ by 4.1.1(a) and (b), and $E_{A_{n-1}}(x) e_{n+1} = \tau w w' e_{n+1}$ by the bimodule property of $E_{A_{n-1}}$.

(iii) If w is a reduced word on $e_1, e_2, \ldots, e_{n+1}$ write $w = x e_{n+1} y$ with $x, y \in A_n$. Then by (ii), $e_{n+1} w e_{n+1} = t e_{n+1}$ for $t \in A_{n-1}$. Thus $e_{n+1} A_{n+1} e_{n+1} \subseteq A_{n-1} e_{n+1}$. To show that the map is an isomorphism, suppose $x e_{n+1} = 0$, $x \in A_{n-1}$. Then $x x^* e_{n+1} = 0$ so $\operatorname{tr}(x x^*) = 0$, i.e. $x = 0$. Q.E.D.

Aside 4.1.4. In fact it is possible to uniquely order reduced words by pushing $e_{\max}$ to the right as far as possible. It is easy to show that such an ordered reduced word is of the form

$$(e_{j_1} e_{j_1-1} \dots e_{k_1})(e_{j_2} e_{j_2-1} \dots e_{k_2}) \dots (e_{j_p} e_{j_p-1} \dots e_{k_p})$$

where j_p is the maximum index, $j_i \geqq k_i$ and $j_{i+1} > j_i$, $k_{i+1} > k_i$. To each such word we may associate an increasing path on the integer lattice between $(0,0)$ and $(n+1, n+1)$, which does not cross the diagonal. For instance $(e_3 e_2 e_1)(e_4 e_3)(e_5 e_4)$ in A_5 would correspond to the path (I owe this observation to H. Wilf):



It is well known that such paths are counted by the Catalan numbers $1/(n+2)\binom{2(n+1)}{n+1}$ so we obtain $\dim A_n \leqq 1/(n+2)\binom{2(n+1)}{n+1}$. Uniqueness and linear independence of ordered reduced words would follow from $\dim A_n = 1/(n+2)\binom{2(n+1)}{n+1}$ which we will prove in §5.1 for $\tau \leqq 1/4$. See Aside 5.1.1.

We now want to study traces on A_∞. For this define a *totally reduced word* to be a reduced word on the e_i's where we also allow cyclic permutations. The following result is obvious.

Remark 4.1.5. Any trace on A_n is determined by its effect on totally reduced words.

Lemma 4.1.6. Any totally reduced word is of the form $w = e_{i_1} e_{i_2} \dots e_{i_n}$ with $|i_j - i_k| \geqq 2$, $j \neq k$, and $\operatorname{tr}(w) = \tau^n$.

Proof. The last assertion is immediate from 4.1.1(c). We prove the first assertion by induction on the length of a totally reduced word. It suffices to prove that if $m = \max\{i_1, i_2, \dots, i_n\}$ in a totally reduced word $e_{i_1} e_{i_2} \dots e_{i_n}$, then e_{m-1} does not occur. For this note that by a cyclic permutation we may suppose the word is of the form $e_m w$ and that e_{m-1} occurs at most once in w (since a totally reduced word is reduced). But then we may proceed to $e_m w e_m$ using $e_m^2 = e_m$ and then eliminate e_{m-1} from w using (a) and (b) of 4.1.1. Q.E.D.

We shall now show that any normal normalized trace on P is equal to tr. For this it suffices to show that it equals tr on completely reduced words. For each subset $I \subseteq N$ with $|i-j| \geqq 2$ whenever $i, j \in I$, $i \neq j$, define A_I to be the algebra generated by $\{e_i \mid i \in I\}$.

Lemma 4.1.7. *For any finite permutation σ of I, there is a unitary $u \in A_\infty$ such that $u e_k u^* = e_{\sigma(k)}$ for all $k \in I$.*

Proof. It suffices to show that any transposition $e_i \leftrightarrow e_j$ $(j \geqq i)$ can be effected by a unitary. We may even suppose that no k strictly between i and j is in I. If we can find a unitary $u \in A_{i,j}$ with $u e_i u^* = e_j$ and $u e_j u^* = e_i$ then this u will do since it commutes with all the other e_k's, $k \in I$. But for this it suffices to show that e_i and e_j are equivalent in $A_{i,j}$. And $e_i e_{i+1} \dots e_j$ is a multiple of a partial isometry v with $vv^* = e_i$, $v^* v = e_j$. Q.E.D.

Corollary 4.1.8. *Any normal normalized trace on P is equal to* tr *on A_I''.*

Proof. Since all the e_i's are independent for tr, A_I'' may be identified with an infinite tensor product Bernoulli shift algebra. By 4.1.7 the normalizer induces the obvious action of S_∞ on A_I''. This action is well known to be ergodic. Hence any invariant measure which is absolutely continuous with respect to tr is proportional to tr. Q.E.D.

Corollary 4.1.9. *P is a II_1 factor isomorphic to R.*

Proof. By 4.1.4, 4.1.5 and 4.1.8 there is only one normal normalized trace on P. Thus P is a factor. By [21] and (a) of 4.1.3, $P \cong R$. Q.E.D.

Note that our proof that P is a factor follows similar lines to the scheme laid out in [27].

Corollary 4.1.10. *P_τ is a subfactor of P.*

Proof. Writing $f_i = e_{i+1}$, the f_i satisfy the same relations as the e_i. Q.E.D.

Lemma 4.1.11. *For each n, the map $e_i \to e_{n-i}$ extends to a* tr*-preserving $*$-automorphism σ_n of A_n.*

Proof. The map obviously extends to a $*$-automorphism of the free involutive monoid on the self-adjoint e_i. It thus suffices to know that if w is a word on e_1, $e_2, \dots, e_n$ then $\mathrm{tr}(\sigma_n(w)) = \mathrm{tr}(w)$. For then if $x = \sum_i c_i w_i$ for $c_i \in \mathbb{C}$,

$$\mathrm{tr}(xx^*) = \mathrm{tr}(\sum_{i,j} c_i \bar{c}_j w_i w_j^*) = \mathrm{tr}(\sum_{i,j} c_i \bar{c}_j \sigma(w_i) \sigma(w_j^*))$$

so that $\sum_i c_i w_i \to \sum_i c_i \sigma(w_i)$ is a well defined isometry for the definite hermitian scalar product defined by tr. But the trace of w is determined by 4.1.1(a) and (b) and the formula of 4.1.6, all of which are invariant under the interchange $e_i \leftrightarrow e_{n-i}$. Q.E.D.

Corollary 4.1.12.

(i) $E_{A_{2,n}}(e_1) = \tau$

(ii) *for* $x \in A_{2,n}$, $e_1 x e_1 = E_{A_{3,n}}(x) e_1$

(iii) $E_{P_\tau}(e_1) = \tau$, $e_1 x e_1 = E_{A_{3,\infty}'}(x) e_1$ *for* $x \in P_\tau$.

Proof. (i) and (ii) follow from σ_n applied to (ii) of 4.1.3, and (iii) is just the limit as $n \to \infty$ of (i) and (ii). Q.E.D.

Proof of (ii) of 4.1.1 (calculation of $[P:P_\tau]$). Do the basic construction to obtain $\langle P, e_{P_\tau}\rangle$. Then $e_{P_\tau} e_1 e_{P_\tau} = \tau e_{P_\tau}$ follows from (i) of 4.1.12. We further claim that $e_1 e_{P_\tau} e_1 = \tau e_1$. By (iii) of 4.1.12, elements of the form $a_0 + \sum_{i=1}^{n} a_i e_1 b_i$ with a_i, $b_i \in P_\tau$ are dense in P so it suffices to verify $e_1 e_{P_\tau} e_1 = \tau e_1$ on $x\xi$ and $x e_1 y \xi$ with $x, y \in P_\tau$. But $e_1 e_{P_\tau} e_1 (x\xi) = \tau e_1 x \xi = (\tau e_1)(x\xi)$, and

$$e_1 e_{P_\tau} e_1 (x e_1 y \xi) = e_1 (E_{P_\tau}(e_1 E_{A_{3,\infty}'}(x) y)\xi) = \tau e_1 E_{A_{3,\infty}'}(x) y \xi = \tau e_1 (x e_1 y \xi).$$

These two relations imply that e_1 and e_{P_τ} are equivalent in $\langle P, e_{P_\tau}\rangle$. Now by (iii) of 3.1.5, e_{P_τ} is a finite projection and e_1 is in a II_1 factor so that $\langle P, e_{P_\tau}\rangle$ is necessarily a finite factor, i.e. $[P:P_\tau] < \infty$. But we know that $\mathrm{tr}(e_1) = \tau$ so $[P:P_\tau] = \mathrm{tr}(e_{P_\tau})^{-1} = \tau^{-1}$. Q.E.D.

§4.2. Restrictions on τ

We shall now prove (iii) of 4.1.1. We keep the notation of §4.1. Also define $s_n = e_1 \vee e_2 \vee \ldots \vee e_n$.

Lemma 4.2.1. *If $1 - s_n \neq 0$ then it is a minimal projection in A_n which belongs to $Z(A_n)$.*

Proof. If w is a word on $e_1, e_2, \ldots, e_n$ then

$$(1 - e_1 \vee e_2 \vee \ldots \vee e_n) w = 0. \quad \text{Q.E.D.}$$

Lemma 4.2.2. *If $s_n \neq 1$ then $e_{n+1} \wedge s_n = e_{n+1} s_{n-1}$.*

Proof. By 4.1.3(ii), $e_{n+1} s_n e_{n+1} = E_{A_{n-1}}(s_n) e_{n+1}$. But by 4.2.1 and the bimodule property of $E_{A_{n-1}}$, $E_{A_{n-1}}(s_n) \in Z(A_{n-1})$. Let $p_0, p_1, \ldots, p_k$ be the minimal projections in $Z(A_{n-1})$ with $p_0 = 1 - s_{n-1}$, and write $E_{A_{n-1}}(s_n) = \sum_{i=0}^{k} \lambda_i p_i$. Then since $e_{n+1} \wedge s_n \geqq e_{n+1} s_{n-1}$ and $\lim_{m\to\infty} (e_{n+1} s_n e_{n+1})^m = e_{n+1} \wedge s_n$, and by (iii) of 4.1.3, $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 1$. Since $s_n \neq 1$, $E_{A_n}(s_n) \neq 1$ so by (iii) of 4.1.3, $\lambda_k < 1$. Thus $e_{n+1} \wedge s_n = e_{n+1} s_{n-1}$. Q.E.D.

Corollary 4.2.3. *If* $\mathrm{tr}(s_k) > 0$, $\mathrm{tr}(1 - s_{k+1}) = \mathrm{tr}(1 - s_k) - \tau\,\mathrm{tr}(1 - s_{k-1})$. *And* $\mathrm{tr}(1 - s_1) = 1 - \tau$, $\mathrm{tr}(1 - s_2) = 1 - 2\tau$.

Proof. The trace tr satisfies $\mathrm{tr}(p \vee q) = \mathrm{tr}(p) + \mathrm{tr}(q) - \mathrm{tr}(p \wedge q)$. So by 4.2.2 and (c) of 4.1.1, $\mathrm{tr}(1 - s_{k+1}) = \mathrm{tr}(1 - s_k) - \tau\,\mathrm{tr}(1 - s_{k-1})$. Q.E.D.

For this reason we define the polynomials $P_n(x)$ by $P_0(x) = 0$, $P_1(x) = 1$ and $P_{n+1} = P_n - x P_{n-1}$, so that if $P_{k+2}(\tau) > 0\ \forall k \leqq n$ then $P_{k+2}(\tau) = \mathrm{tr}(1 - s_k)$.

Lemma 4.2.4. *Let* $\sigma = \dfrac{1 + \sqrt{1 - 4x}}{2}$, $\tilde{\sigma} = \dfrac{1 - \sqrt{1 - 4x}}{2}$. *Then*

(i) $P_n(x) = \dfrac{\sigma^n - \tilde{\sigma}^n}{\sigma - \tilde{\sigma}}$

(ii) $P_n(\frac{1}{4}\sec^2\theta)=\sin n\theta/(2^{n-1}\cos^{n-1}\theta\sin\theta)$

(iii) $\deg P_n=\left[\frac{n-1}{2}\right]$.

Proof. (i) The general solution to the difference equation is $P_n=A\sigma^n+B\tilde{\sigma}^n$. The initial conditions give $A+B=0$, $A(\sigma-\tilde{\sigma})=1$.

(ii) Putting $\sigma=re^{i\theta}$, $\tilde{\sigma}=re^{-i\theta}$, $x=\frac{1}{4}\sec^2\theta$, $r=\frac{1}{2}\sec\theta$, $\sigma^n-\tilde{\sigma}^n=2ir^n\sin n\theta$, $\sigma-\tilde{\sigma}=2ir\sin\theta$.

(iii) Follows easily by induction from the difference equation. Q.E.D.

Corollary 4.2.5. (i) *The smallest root of* P_n *is* $\frac{1}{4}\sec^2\frac{\pi}{n}$.

(ii) $P_n(\tau)>0$ *for* $\tau<\frac{1}{4}\sec^2\frac{\pi}{n}$.

(iii) $P_{n+1}(\tau)<0$ *for* τ *between* $\frac{1}{4}\sec^2\frac{\pi}{n+1}$ *and* $\frac{1}{4}\sec^2\frac{\pi}{n}$.

Proof. (i) By counting the number of distinct values of $\frac{1}{4}\sec^2\frac{m\pi}{n}$ (which are roots of P_n by 4.2.4) we find that all roots of P_n are real and they are the numbers $\frac{1}{4}\sec^2\frac{m\pi}{n}$ with $\frac{m\pi}{n}<\frac{\pi}{2}$. The smallest is $\frac{1}{4}\sec^2\frac{\pi}{n}$.

(ii) By induction the coefficient of $x^{[(n-1)/2]}$ in $P_n(x)$ is positive when $\left[\frac{n-1}{2}\right]$ is even and negative when $\left[\frac{n-1}{2}\right]$ is odd.

(iii) $P_{n+1}(\tau)$ must be negative between its first and second real roots, and $\sec^2\pi/(n+1)<\sec^2\pi/n<\sec^2 2\pi/(n+1)$. Q.E.D.

Proof of (iii) of 4.1.1. Suppose $\tau>1/4$ and $\tau\neq\frac{1}{4}\sec^2\frac{\pi}{n}$, $n=3,4,5,\ldots$. Then there is a $k\geqq 3$ with $\frac{1}{4}\sec^2\pi/(k+1)<\tau<\frac{1}{4}\sec^2\pi/k$. But then $P_n(\tau)>0$ for all $n\leqq k$ so $P_{n+1}(\tau)=\mathrm{tr}(1-s_{n-1})$ by 4.2.3. But by (iii) of 4.2.5, $P_{n+1}(\tau)<0$ which is impossible since $1-s_{n-1}$ is a projection. Q.E.D.

§4.3. Values of the Index

Theorem 4.3.1. *If* N *is a subfactor of the* II_1 *factor* M *then either* $[M:N]\geqq 4$ *or* $[M:N]=4\cos^2\pi/n$ *for some* $n\geqq 3$.

Proof. If $[M:N]<\infty$, define the increasing sequence M_i, $i=0,1,2,\ldots$ of II_1 factors by the relations $M_0=M$, $M_1=\langle M,e_N\rangle$, $M_{i+1}=\langle M_i,e_{M_{i-1}}\rangle$ for $i\geqq 1$. The inductive limit becomes a II_1 factor with faithful normal normalized trace tr (by uniqueness of the trace - see also [19]). Moreover if $\tau=[M:N]^{-1}$ and $e_i=e_{M_i}$ then the e_i satisfy the conditions of 4.1.1 by 3.4.2 and 3.1.7. By Theorem 4.1.1 either $[M:N]\geqq 4$ or $[M:N]=4\cos^2\pi/n$ for some $n\in\mathbb{Z}$, $n\geqq 3$. Q.E.D.

Theorem 4.3.2. *For each $n=3,4,\ldots$ there is a subfactor P_τ of R with $[R:P_\tau]=4\cos^2\pi/n$ and $P'\cap R=\mathbb{C}$. For each $r\geqq 4$, $r\in\mathbb{R}$, there is a subfactor P of R with $[R:P]=r$.*

Proof. The existence of subfactors with index $r\geqq 4$ was shown in 2.2.5 and the assertion about the relative commutant when $[R:P]=4\cos^2\pi/n$ follows from 2.2.4. Thus we only need to construct subfactors with index $4\cos^2\pi/n$.

To do this note that the conditions of 3.3.2 for finite dimensional C^*-algebras $N\subseteq M$ together with (ii) of 3.3.1 show, by interchanging A and A^T, $\vec{s}$ and $\vec{t}$, that if there is a (τ, M) trace on $\langle M, e_N\rangle$ then there is a $(\tau,\langle M,e_N\rangle)$ trace on $\langle\langle M,e_N\rangle,e_M\rangle$ and so on. Thus we may iterate the basic construction once we have started it. Once the construction has been iterated, the inductive limit has a faithful normal normalized trace on it and the e_i's resulting from the iteration satisfy the conditions of 4.1.1 so by the result of 4.1.1 we may choose P_τ as the subfactor of R.

Thus it suffices to find $N\subseteq M$, finite dimensional C^*-algebras with inclusion matrix A and positive vectors $\vec{s}$ and $\vec{t}$ with $A^TA\vec{t}=(4\cos^2\pi/n)\vec{t}$ and $AA^T\vec{s}=(4\cos^2\pi/n)\vec{s}$. In fact the matrix A is enough since the subalgebra N can then be taken as the direct sum of as many copies of $\mathbb{C}$ as there are rows in the matrix.

Let A be the square $n\times n$ matrix (a_{ij}) with $a_{ij}=1$ if $|i-j|=1$ and 0 otherwise, e.g. $\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$. We leave it to the reader to check the linear algebra. This choice of A was suggested by F. Goodman. Q.E.D.

Remark 4.3.3. If we had started with any $m\times n$ non-negative integer valued matrix we could have made the construction of 4.3.2. Thus 4.1.1 implies that for such a matrix, if $\|A\|\leqq 2$ then $\|A\|=2\cos\pi/n$, $n=3,4,\ldots$. This can also be proved using the well-known result of Kronecker which asserts that if z is an algebraic integer all of whose conjugates have absolute value equal to 1, then z is a root of unity.

§5. The Bratteli Diagram; the Relative Commutant

§5.1. The Bratteli Diagram when $\tau\leqq 1/4$

Let $\{e_1,e_2,\ldots\}$, M and tr be as in §4.1. Let $A_n=\{e_1,e_2,\ldots,e_n\}''$ and B_n be the algebra generated by $\{e_1,e_2,\ldots,e_n\}$ (without 1). We know that A_n and B_n are finite dimensional C^*-algebras. We want to determine the Bratteli diagram for them and the value of tr on minimal projections. In this section we suppose $\tau\leqq 1/4$. This means that

$$P_{n+2}(\tau)=\mathrm{tr}(1-e_1\vee e_2\vee\ldots\vee e_n)>0 \qquad \text{for all } n\geqq 1$$

(where P_n is as in §4.2).

Let $\left\{ {n \atop b} \right\} = \binom{n}{b} - \binom{n}{b-1}$ (ordinary binomial symbols with the convention $\binom{n}{-1} = 0$). We shall show by induction that

(a) $A_n = \bigoplus_{k=0}^{[\frac{n+1}{2}]} Q_k^n$ where $Q_k^n \cong M_{\left\{ {n+1 \atop k} \right\}}(\mathbb{C})$.

(b) $Q_0^n = (1 - e_1 \vee e_2 \vee \ldots \vee e_n)\mathbb{C}$ so that

(c) $B_n = \bigoplus_{k=1}^{[\frac{n+1}{2}]} Q_k^n$

(d) The trace of a minimal projection in Q_k^n is $\tau^k P_{n+2-2k}(\tau)$ for $k = 0, 1, \ldots, \left[\frac{n+1}{2}\right]$.

(e) The inclusion matrix of A_{n-1} in A_n is

(i) When n is even

$$A = (a_{ij}) \text{ with } a_{ij} = \begin{cases} 1 & \text{if } j = i \text{ or } i+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$i, j = 0, 1, \ldots, \left[\frac{n+1}{2}\right]$$

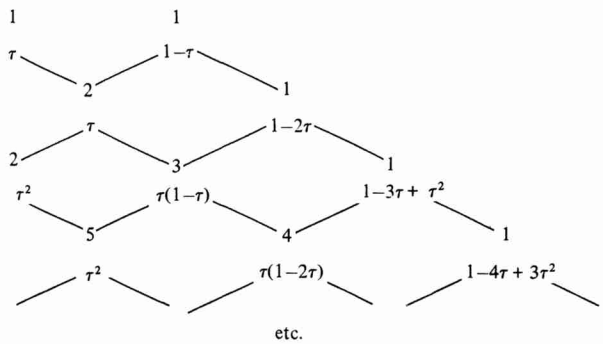
(here the indices i and j refer to the subscript of Q).

(ii) when n is odd

$$A = (a_{ij}) \quad a_{ij} = \begin{cases} 1 & \text{if } j = i \text{ or } i+1 \\ 0 & \text{otherwise} \end{cases}$$

$$i = 0, ., \ldots, (n+1)/2; \; j = 0, 1, \ldots, (n+3)/2.$$

All this information is summed up by the following diagram (which also appears on p. 118 of [31]).



The numbers are the $\left\{ {n+1 \atop k} \right\}$ and the polynomials, which give the trace on minimal projections in the corresponding matrix algebra, change by multiplication by τ each step down a vertical column.

Proof. The proof will use the basic construction for the inclusion $A_{n-1} \subseteq A_n$ to obtain an almost faithful representation of A_{n+1}.

The truth of assertions (a)→(e) for $n=2$ follows immediately from §3.4. For the inductive step we shall treat only the case $n=2m$, the odd case being essentially identical. Suppose (a)→(e) are true for all $k \leqq n$ and apply the basic construction of §3.1 with $M=A_n$, $N=A_{n-1}$ with respect to tr. Let $E=E_{A_{n-1}}$, $e_{A_{n-1}}=e$. By induction and (ii) of 3.3.1 we know that we have the following Bratteli diagram:

$$\begin{array}{lccccccccccc}
A_{n-1} & \cdots & \left\{\begin{matrix} n \\ m \end{matrix}\right\} & & \left\{\begin{matrix} n \\ m-1 \end{matrix}\right\} & & \cdots & & \left\{\begin{matrix} n \\ 0 \end{matrix}\right\} & \\
\cap\!| & & \diagdown & & \diagdown & & & \diagup & \diagdown & \\
A_n & \cdots & & \left\{\begin{matrix} n+1 \\ m \end{matrix}\right\} & & \left\{\begin{matrix} n+1 \\ m-1 \end{matrix}\right\} & \cdots & \left\{\begin{matrix} n+1 \\ 1 \end{matrix}\right\} & & \left\{\begin{matrix} n+1 \\ 0 \end{matrix}\right\} \\
\cap\!| & & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \\
\langle A_n, e\rangle & \cdots & \left\{\begin{matrix} n+2 \\ m+1 \end{matrix}\right\} & & \left\{\begin{matrix} n+2 \\ m \end{matrix}\right\} & & \cdots & & \left\{\begin{matrix} n+2 \\ 1 \end{matrix}\right\}
\end{array}$$

The numbers on the bottom line follow from 3.7.1, (ii) of 3.3.1 and the identities

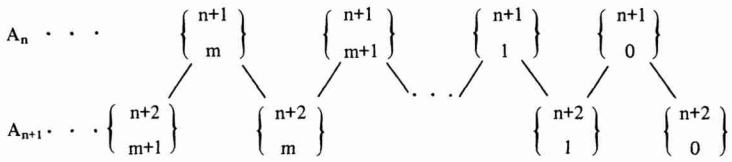
$$\left\{\begin{matrix} a \\ b \end{matrix}\right\} + \left\{\begin{matrix} a \\ b-1 \end{matrix}\right\} = \left\{\begin{matrix} a+1 \\ b \end{matrix}\right\}, \quad \left\{\begin{matrix} 2m+2 \\ m+1 \end{matrix}\right\} = \left\{\begin{matrix} 2m+1 \\ m \end{matrix}\right\}.$$

By assertion (c), the algebra generated by $\{e_1, e_2, \ldots, e_n\}$ is the direct sum of the first m terms on the A_n line so if we define the faithful (non-normalized) trace Tr on $\langle A_n, e\rangle$ by the rule $\mathrm{Tr}(ep_k)=\tau\,\mathrm{tr}(p_k)$ where p_k is a minimal projection in Q_{k-1}^{n-1} $(k=1,2,\ldots,m+1)$, the identity $P_j=P_{j+1}+\tau P_{j-1}$ ensures that Tr agrees with tr on B_n. Moreover as in 3.3.2, uniqueness of the trace and linearity show that $\mathrm{Tr}(ex)=\tau\,\mathrm{tr}(x)$ for all x in A_{n-1} and hence all x in A_n. Also Tr is a faithful positive trace since all the polynomials $P_n(\tau)$ are positive.

But now consider the algebras B_{n+1} and $\langle A_n, e\rangle$. B_{n+1} is generated by B_n and e_{n+1} so that any element can be written $a_0+\sum_{i=1}^{i} a_i e_{n+1} b_i$ with $a_0 \in B_n$, a_i, $b_i \in A_n$ for $i=1,2,\ldots,k$. Multiplication is defined by $e_{n+1} x e_{n+1}=E(x)e_{n+1}$ for $x \in A_n$ (see 4.1.3) and the faithful trace tr defined by $\mathrm{tr}(x e_{n+1})=\tau\,\mathrm{tr}(x)$ for $x \in A_n$. In $\langle A_n, e\rangle$, sums of the form $a_0+\sum_{i=1}^{j} a_i e b_i$ with $a_0 \in B_n$, a_i, $b_i \in A_n$ for $i \geqq 1$, form a 2-sided ideal. Since the central support of e is 1 ((iv) of 3.1.5), any element of $\langle A_n, e\rangle$ can be written in this form. Also $exe=E(x)e$ for $x \in A_n$ and the faithful trace Tr on $\langle A_n, e\rangle$ satisfies $\mathrm{Tr}(aeb)=\mathrm{tr}(aeb)$ for a, $b \in A_n$ and $\mathrm{Tr}(b)=\mathrm{tr}(b)$ for $b \in B_n$. Thus we may define a map from B_{n+1} to $\langle A_n, e\rangle$ by $a_0+\sum_i a_i e_{n+1} b_i \mapsto a_0+\sum_i a_i e b_i$ which is a surjective isometry for the definite hermitian scalar products defined by tr and Tr (and hence is well defined).

At this stage (†) we have obtained assertion (c) for $n+1$ and the values of tr on the minimal projections in Q_1^{n+1}, Q_2^{n+1}, ..., Q_{m+1}^{n+1}. But A_{n+1} is just $\{B_{n+1} \cup \{1\}\}''$ and since $\mathrm{tr}(1-e_1 \vee \ldots \vee e_{n+1})=P_{n+3}(\tau)>0$, A_{n+1} is $B_{n+1} \oplus (1-e_1 \vee e_2 \vee \ldots \vee e_{n+1})\mathbb{C}$. Moreover $x(1-e_1 \vee e_2 \vee \ldots \vee e_{n+1})=0$ for any $x \in B_n$

and $(1-e_1 \vee \ldots \vee e_{n+1})(1-e_1 \vee \ldots \vee e_n) \neq 0$ so the Bratteli diagram for $A_n \subseteq A_{n+1}$ is forced to be



This proves assertions (a) and (e) for $n+1$, and assertion (d) follows from $\operatorname{tr}(1-e_1 \vee \ldots \vee e_{n+1}) = P_{n+3}(\tau)$. This ends the proof. Q.E.D.

Aside 5.1.1. The binomial identity

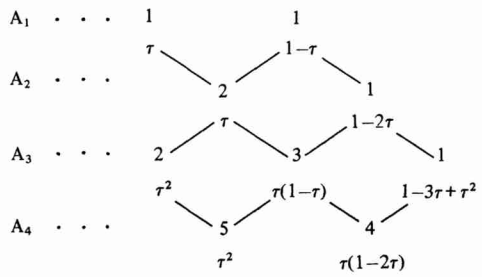
$$\sum_{i=0}^{[\frac{n+1}{2}]} \binom{n+1}{i}^2 = \frac{1}{n+2}\binom{2(n+1)}{n+1}$$

follows from [13, p. 63]. This shows that $\dim A_n$ is the same as the number of ordered reduced words which are thus linearly independent. See 4.1.4.

§ 5.2. The Bratteli Diagrams $\tau = \frac{1}{4}\sec^2\frac{\pi}{n}$

If $\tau = \frac{1}{4}\sec^2 \pi/(n+2)$ then $P_k(\tau) > 0$ for all $k \leqq n+1$ so the inductive argument of § 5.1 goes through until the point marked (†) for assertions (a)→(e) up to step n. At this stage we find that $e_1 \vee e_2 \vee \ldots \vee e_n = 1$ so $B_n = A_n$. From this point on the basic construction for the pair $A_k \subseteq A_{k+1}$ will give an isometric (so faithful) surjective representation of A_{k+2} and the Bratteli diagram will not grow any wider.

To convince the reader of these assertions without boring him with the details, we treat the case $n=4$. The argument of § 5.1 shows that we have the following diagram

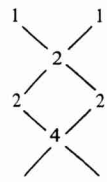


Now if $\vec{s} = \begin{pmatrix} \tau \\ 1-2\tau \end{pmatrix}$, $\vec{t} = \begin{pmatrix} \tau^2 \\ \tau(1-\tau) \\ 1-3\tau+\tau^2 \end{pmatrix}$, and $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\vec{s}$ is an eigenvector for $A^T A$ with eigenvalue $3 = \tau^{-1} = 4\cos^2\frac{\pi}{6}$ and $\vec{t}$ is an eigenvector for AA^T with

the same eigenvalue. They give normalized traces on A_2 and A_3 so by §3.3 the basic construction will continue to give faithful representations of the A_i by the same argument as §5.1, since $A_n = B_n$ for $n \geqq 4$.

The Bratteli diagrams will be:

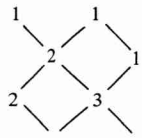
For $n=2$ ($\tau = 1/2$)



etc.

Note that this is the same as the complex Clifford algebras (see [16, p. 148]).

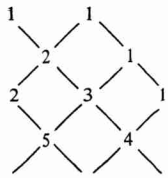
For $n=3$ ($\tau = 1/\varphi^2$, φ = golden ratio)



etc.

This diagram already appears in [4], [11].

For $n=4$



etc.

The pattern is now clear. The traces on minimal projections are the same as for the case $\tau < 1/4$ where this makes sense.

§5.3. The Relative Commutant when $\tau \leqq 1/4$

One of the main motivations of this paper was to decide whether there is a continuum of values of the index realized by subfactors with trivial relative commutant (see problem 1). It was originally thought that the subfactors P_τ, $\tau \leqq 1/4$, had trivial relative commutant. In this section we shall show that this is only true when $\tau = 1/4$. The calculations were first finished by *A*. Wassermann to whom the author is grateful. The difficulty is the absence of an orthonormal basis which makes it impossible to exhibit an element of the relative commutant.

The case $\tau < 1/4$. Let us adopt the notation of §4.1 so R is presented as the algebra generated by the e_i's. We intend to show that $E_{R \cap P'_\tau}(e_1)$ is not a scalar, i.e. $E_{R \cap P'_\tau}(e_1) \neq \tau$. For this we will show that $\|E_{R \cap P'_\tau}(e_1) - \tau\|_1 \neq 0$ (remember $\|x\|_1 = \mathrm{tr}(|x|)$). We shall need some lemmas.

Lemma 5.3.1. $E_{R\cap P'_\tau}(x)=s-\lim_{n\to\infty} E_{R\cap A'_{2,n}}(x)$ *for* $x\in R$.

Proof. Since $P_\tau=\left(\bigcup_{n=1}^{\infty} A_{2,n}\right)''$, the algebras $R\cap A'_{2,n}$ are a decreasing sequence of von Neumann algebras with intersection $P'_\tau\cap R$. The rest is well known (e.g. it is true in $L^2(R,\mathrm{tr})$). Q.E.D.

Lemma 5.3.2. $\|E_{A'_{2,n+1}\cap R}(e_1)-\tau\|_1=\|E_{A'_n\cap A_{n+1}}(e_{n+1})-\tau\|_1$.

Proof. Let du denote Haar measure on the unitary group of a finite dimensional C^*-algebra. Then

$$\|E_{A'_{2,n+1}\cap R}(e_1-\tau)\|_1=\|\int_{U(A_{2,n+1})}(ue_1u^*-\tau)\,du\|_1.$$

Applying the isomorphism σ_{n+1} of 4.1.11 we find that this expression equals

$$\|\int_{U(A_n)}(ue_{n+1}u^*-\tau)\,du\|_1$$

which equals $\|E_{A'_n\cap A_{n+1}}(e_{n+1})-\tau\|_1$. Q.E.D.

Thus to show that $P'_\tau\cap R\neq C$, it suffices to show that

$$\lim_{n\to\infty}\|E_{A'_n\cap A_{n+1}}(e_{n+1})-\tau\|_1\neq 0.$$

Since we know that the limit exists, it suffices to consider n odd; say $n=2m-1$. From §5.1 let $p_0,p_1,\ldots,p_{m-1}$ be the minimal central projections in A_{n-1} corresponding to $Q_0^{n-1},Q_1^{n-1},\ldots,Q_{m-1}^{n-1}$, similarly $q_0,q_1,\ldots,q_m$ for A_n and $r_0,r_1,\ldots,r_m$ for A_{n+1}. So $p_0=1-e_1\vee e_2\vee\ldots\vee e_{n-1}$, $q_0=1-e_1\vee e_2\vee\ldots\vee e_n$ and $r_0=1-e_1\vee e_2\vee\ldots\vee e_{n+1}$. We also know from §5.1 and (iii) of 3.3.1 that

$$e_{n+1}r_i=e_{n+1}p_{i-1}\qquad\text{for } i\geqq 1. \tag{5.3.3}$$

Since all the embeddings on the Bratteli diagram are of multiplicity at most one, the relative commutant of A_n in A_{n+1} is the abelian algebra generated by the mutually orthogonal projections $q_0r_0,q_0r_1,q_1r_0,\ldots,q_mr_m$. Thus

$$E_{A'_n\cap A_{n+1}}(e_1)=\frac{\mathrm{tr}(q_0r_0e_{n+1})}{\mathrm{tr}(q_0r_0)}q_0r_0+\frac{\mathrm{tr}(q_0r_1e_{n+1})}{\mathrm{tr}(q_0r_1)}q_0r_0+\ldots+\frac{\mathrm{tr}(q_mr_me_{n+1})}{\mathrm{tr}(q_mr_m)}q_mr_m$$

so that

$$\begin{aligned}\|E_{A'_n\cap A_{n+1}}(e_{n+1})-\tau\|_1&=\sum_{i=0}^{m}|\mathrm{tr}(q_ir_ie_{n+1})-\tau\,\mathrm{tr}(q_ir_i)|\\&\quad+\sum_{i=0}^{m-1}|\mathrm{tr}(q_ir_{i+1}e_{n+1}-\tau\,\mathrm{tr}(q_ir_i)|\\&\geqq\sum_{i=1}^{m}\tau|\mathrm{tr}(q_ip_{i-1})-\mathrm{tr}(q_ir_i)|\qquad\text{by 5.3.3}\\&\geqq\tau\left|\sum_{i=1}^{m}\mathrm{tr}(q_ip_{i-1})-\mathrm{tr}(q_ir_i)\right|.\end{aligned}$$

But from the Bratteli diagram, $q_i p_{i-1}$ is a projection of rank $\left\{ {2m-1 \atop i-1} \right\}$ in the matrix algebra Q_i^{2m-1} $(=q_i A_n)$ and $q_i r_i$ is of rank $\left\{ {2m \atop i} \right\}$ in $Q_i^{2m}(=r_i A_{n+1})$. So the last sum may be written

$$L=\tau\left|\sum_{i=1}^{m}\left(\left\{ {2m-1 \atop i-1} \right\}\tau^i P_{2(m-i)+1}(\tau)-\left\{ {2m \atop i} \right\}\tau^i P_{2(m-i)+2}(\tau)\right)\right|.$$

We saw in 4.2.4 that $P_n(\tau)=(\sigma^n-\tilde{\sigma}^n)/(\sigma-\tilde{\sigma})$ where $\sigma=(1+\sqrt{1-4\tau})/2$ and $\tilde{\sigma}=(1-\sqrt{1-4\tau})/2$. Note that for $\tau<1/4$, $\sigma>1/2$, $\tilde{\sigma}<1/2$ and $\tilde{\sigma}+(\tau/\tilde{\sigma})=1$, $\sigma+(\tau/\sigma)=1$. Let us now calculate the relevant limits.

Lemma 5.3.6. (a) $\lim_{m\to\infty}\left(\sum_{i=1}^{m}\left\{ {2m-1 \atop i-1} \right\}\tilde{\sigma}^{2(m-i)+1}\tau^i\right)=0$

(b) $\lim_{m\to\infty}\left(\sum_{i=1}^{m}\left\{ {2m \atop i} \right\}\tilde{\sigma}^{2(m-i)+2}\tau^i\right)=0$

(c) $\lim_{m\to\infty}\left(\sum_{i=1}^{m}\left\{ {2m-1 \atop i-1} \right\}\sigma^{2(m-i)+1}\tau^i\right)=\tau(1-\tau/\sigma^2)$

(d) $\lim_{m\to\infty}\left(\sum_{i=1}^{m}\left\{ {2m \atop i} \right\}\sigma^{2(m-i)+2}\tau^i\right)=\sigma^2(1-\tau/\sigma^2)$.

Proof. (a) and (c). Note that $\sigma^{2(m-i)+1}\tau^i=\tau(\tau/\sigma)^{i-1}\sigma^{2m-i}$ so if $x=\sigma$ or $\tilde{\sigma}$, in both cases we have to evaluate

$$\lim_{m\to\infty}\tau\left(\sum_{i=1}^{m}\binom{2m-1}{i-1}\left(\frac{\tau}{x}\right)^{i-1}x^{2m-i}-\frac{\tau}{x^2}\sum_{i=1}^{m}\binom{2m-1}{i-2}\left(\frac{\tau}{x}\right)^{i-2}x^{2m-i+1}\right).$$

Since $x+\tau/x=1$, we recognize the probability of $\geqq m$ successes in $2m-1$ Bernoulli trials. If $x=\sigma$, the probability of success is $>1/2$ so by the de Moivre-Laplace central limit theorem the limit is $\tau(1-\tau/\sigma^2)$ and if $x=\tilde{\sigma}$, the probability of success is $<1/2$ so the limit is 0.

(b) and (d) are proved in the same way with the substitution $x^{2(m-i)+2}\tau^i=x^2(\tau/x)^i x^{2m-i}$. Q.E.D.

Expanding L and using 4.3.5 we see that

$$(\sigma-\tilde{\sigma})/L=\tau|(1-\tau/\sigma^2)(\tau-\sigma^2)|\neq 0 \quad \text{for } \tau\neq 1/4.$$

This shows that $P_\tau'\cap R\neq\mathbb{C}$ for $\tau<1/4$.

The case $\tau=1/4$. In this case we contend that $P_\tau'\cap R=\mathbb{C}$. Luckily we can use another model of $P_{1/4}$. Let R be realized as the closure of the Fermion algebra $\bigotimes_{i=1}^{\infty}(M_2(\mathbb{C}))_i$ with respect to the trace tr. The fixed point algebra of the obvious infinite product action of $U(2)$ is generated by the representation of S_∞ coming from interchanging the tensor product components. The transpositions between successive components may be written $2e_i-1$ with $\mathrm{tr}(e_i)=1/4$ and it is a matter of calculation to show that the e_i's satisfy $e_i e_{i\pm 1} e_i=\frac{1}{4}e_i$, $e_i e_j=e_j e_i$ for $|i-j|\geqq 2$. Thus the e_i algebra is $R^{U(2)}$ and the subfactor $P_{1/4}$ is $M_2(C)'_1\cap R^{U(2)}$.

But it is shown in [31] that $(R^{U(2)})' \cap R = \mathbb{C}$ so that $P'_{1/4} \cap R = M_2(\mathbb{C})_1$ and $M_2(\mathbb{C})_1 \cap R^{U(2)} = \mathbb{C}$. Thus $P'_{1/4} \cap R^{SU(2)} = \mathbb{C}$. For $R^{U(2)}$ see also [12], [15], [31], [2]. Q.E.D.

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