Invent. math. 71, 365–379 (1983)

*Inventiones mathematicae*


# Singularities of Closures of $K$-orbits on Flag Manifolds

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## 1. Introduction

Let $G$ be a connected reductive group over $k=\bar{\mathbb{F}}_p$, and $\theta\colon G\to G$ an involutive automorphism. We fix a subgroup $K$ of $G$, having finite index in the fixed point set of $\theta$; and we assume that $p\neq 2$. Then $K$ has only finitely many orbits on the flag variety $\mathscr{B}$ of all Borel subgroups of $G$. If $x\in\mathscr{B}$, write $K_x$ for the isotropy group. Then the component group $K_x/(K_x)_0$ has exponent 2. Fix a prime $l\neq p$. A $K$-equivariant sheaf of $\bar{Q}_l$-vector spaces on the orbit $\mathcal{O}=K\cdot x$ is specified by the representation of $K_x/(K_x)_0$ on the stalk at $x$. Accordingly the sheaves with one dimensional stalk play a central role.

**Definition 1.1.** Let $\mathscr{D}$ be the set of all pairs $(\mathcal{O},\gamma)$, with $\mathcal{O}$ an orbit of $K$ on $\mathscr{B}$, an $\gamma$ an isomorphism class of $K$-equivariant sheaves of one dimensional $\bar{Q}_l$ vector spaces on $\mathcal{O}$. Since $\mathcal{O}$ is determined by $\gamma$, we may write simply $\gamma$ instead of $(\mathcal{O},\gamma)$. For $(\mathcal{O},\gamma)\in\mathscr{D}$, we write

$$l(\gamma)=\dim\mathcal{O}.$$

**Example 1.2.** Consider the group $G\times G$, with the involution $\theta(x,y)=(y,x)$; write $G_\Delta$ for the fixed point set, the diagonal subgroup of $G\times G$. The orbits of $G_\Delta$ on $\mathscr{B}\times\mathscr{B}$ are in one-to-one correspondence with the Weyl group $W$ of $G$, by $w\leftrightarrow\mathcal{O}_w=\{(B,B')$ in relative position $w\}$. The isotropy groups of $G_\Delta$ on $\mathscr{B}\times\mathscr{B}$ are all connected, so the set $\mathscr{D}_\Delta$ of Definition 1.1 may be identified with $W$.

**Example 1.3.** Suppose $G=SL(2,k)$, and

$$\theta g=\begin{pmatrix}1&0\\0&-1\end{pmatrix}g\begin{pmatrix}1&0\\0&-1\end{pmatrix}^{-1}.$$

Then $K$ is the usual torus, consisting of diagonal elements $\begin{pmatrix}z&0\\0&z^{-1}\end{pmatrix}$. The flag variety is

$$\mathscr{B}\cong\mathbb{P}^1=k\cup\{\infty\};$$

* Supported in part by grants from the National Science Foundation. (MCS-8104265 and MCS 8202127)

0020-9910/83/0071/0365/$03.00

and $K$ acts by

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \cdot y = z^2 y \qquad (z \in k^\times,\ y \in k \cup \{\infty\}).$$

Thus $K$ has three orbits $\{0\}$, $\{\infty\}$, and $k^\times$; the isotropy groups are $K$, $K$, and $\{\pm I\}$ respectively. The set $\mathscr{D}$ therefore has four elements: the constant sheaves on the orbits, and a "Möbius band" coming from the double cover of $k^\times$.

**Example 1.4.** Suppose $G = PGL(2, k)$, with the same automorphism as in Example 1.3. We take for $K$ the fixed point set of $\theta$, which is the normalizer of a torus in $G$. The other component of $K$ contains an element $n$ taking $y$ to $y^{-1}$ on $\mathbb{P}^1 = k \cup \{\infty\}$. This interchanges $\{0\}$ and $\{\infty\}$, so there are two orbits of $K$ on $\mathscr{B}$. The isotropy group of $\{0\}$ is $K_0$, and that of $\{1\}$ is $\{1, n\}$. Therefore there are three elements in $\mathscr{D}$: the constant sheaves on the orbits, and a second sheaf on $k^\times$ differing from the constant sheaf only in the action of $K$.

**Definition 1.5.** Suppose $(\mathcal{O}, \gamma) \in \mathscr{D}$. Let $\tilde{\gamma}$ be the Deligne-Goresky-MacPherson (hereafter DGM) extension of $\gamma$ to $\bar{\mathcal{O}}$ (see [5], [3], or [7]). This is an element of the derived category of $K$-equivariant constructible $l$-adic sheaves on $\bar{\mathcal{O}}$, and may be characterized by the following properties. Write $\tilde{\gamma}^i$ for its cohomology sheaves. Then

(a) $\tilde{\gamma}$ is self-dual

(b) $\tilde{\gamma}^i = 0$ for $i < 0$

(c) $\tilde{\gamma}^0|_{\mathcal{O}} \cong \gamma$

(d) If $i > 0$, then $\operatorname{supp}(\tilde{\gamma}^i)$ has codimension at least $i+1$ in $\bar{\mathcal{O}}$.

Regard $\tilde{\gamma}^i$ as a sheaf on all of $\mathscr{B}$ by extending it by zero off of $\bar{\mathcal{O}}$. Given $\gamma$, $\delta \in \mathscr{D}$, write $[\gamma : \tilde{\delta}^i]$ for the multiplicity of $\gamma$ in the Jordan-Hölder series for $\tilde{\delta}^i$. Our purpose is to compute the numbers $[\gamma : \tilde{\delta}^i]$. They are fairly delicate measures of the singularity of $\bar{\mathcal{O}}$, and the extendibility of $\delta$; if $\bar{\mathcal{O}}$ is smooth and $\delta$ extends nicely to it, then $\tilde{\delta}^i$ is zero for $i > 0$, and $\tilde{\delta}^0$ is the extension of $\delta$.

In the case of Example 1.2, this problem was solved by Kazhdan and Lusztig in [7]. (Actually that paper considered Schubert varieties, but the problems are equivalent.) The answer was formulated in a combinatorial way, and the proof was an application of the Weil conjectures as proved by Deligne.

In the present setting, we will again give the answer in a combinatorial form. However, we will replace the detailed study of the geometry with two kinds of extra information. The first is a generalization of Deligne's work, due to O. Gabber. The second is a representation theoretic interpretation of the $[\gamma : \tilde{\delta}^i]$, due to Belinson and Bernstein [1]. To $G$ and $K$ one can attach a real reductive Lie group $\mathscr{G}$. The elements of $\mathscr{D}$ parameterize certain irreducible representations of $\mathscr{G}$, and the $[\gamma : \tilde{\delta}^i]$ appear in character formulas for these representations. This is explained in [9]. This interpretation really matters for the calculation of $[\gamma : \tilde{\delta}^i]$: certain known facts about character formulas correspond to properties of $\tilde{\delta}$ which we were unable to prove geometrically. One of our main motivations, however, was to use this idea in the opposite direction, to get character formulas for $\mathcal{G}$. That problem, and the application of our results to it, is discussed in [9].

We turn now to the formulation of the main results.

**Definition 1.6.** Let $u$ be an indeterminate. Write $M$ for the free $\mathbb{Z}[u,u^{-1}]$ module with basis $\mathcal{D}$.

Let $(W,S)$ be the Weyl group of $G$. In Sect. 3 we will define for each $s\in S$ an endomorphism $T_s$ of $M$, given by explicit formulas on basis elements (as well as in a natural geometric way). Since this is rather messy, we will not repeat it here. Although the following fact is not used in what follows, it is certainly worth observing.

**Proposition 1.7** (cf. proof of Proposition 5.5). *The endomorphisms* $T_s$ (Definition 3.1 and Corollary 3.6) *make* $M$ *a module for the Hecke algebra of* $W$.

Combinatorially, our goal is to imitate the constructions of [6] using $M$ instead of the Hecke algebra. The function $l(\delta)$ on $\mathcal{D}$ (Definition 1.1) plays the role of the length function on $W$. We also need an analogue of the Bruhat ordering. The following definition is not the most natural one; but it does reduce to the Bruhat order in Example 1.2, and is adequate for our purposes.

**Definition 1.8.** The *Bruhat $\mathcal{G}$-order* on $\mathcal{D}$ is the smallest order with the following property. Suppose $\delta'\in\mathcal{D}$, $\delta$ appears in $T_s\delta'$ with non-zero coefficient, and $l(\delta)=l(\delta')+1$; that $\gamma$ and $\gamma'$ have the same relationship (with the same $s$); and that $\gamma'\leqq\delta'$. Then we require that $\gamma\leqq\delta$ and $\delta'\leqq\delta$.

The next ingredient for [6] was an anti-linear (with respect to $u\to u^{-1}$) automorphism of the Hecke algebra; it was defined by

$$D(T_w)=T_{w^{-1}}^{-1}\quad (w\in W). \tag{1.9}$$

In our case, we want an anti-linear automorphism of $M$ compatible with (1.9) and the Hecke algebra action on $M$. The existence of such a map is a combinatorial problem, but we must appeal to algebraic geometry (namely, Verdier duality) to solve it.

**Theorem 1.10.** *There exists a unique $\mathbb{Z}$-linear map $D$: $M\to M$, subject to the following conditions.*

(a) $D(um)=u^{-1}D(m)\quad (m\in M)$

(b) $D((T_s+1)m)=u^{-1}(T_s+1)D(m)\quad (m\in M, s\in S)$

(c) *If* $\delta\in\mathcal{D}$, *then*

$$D(\delta)=u^{-l(\delta)}[\delta+\sum_{\gamma<\delta}R_{\gamma,\delta}(u)\gamma].$$

*The $R_{\gamma,\delta}$ are actually polynomials in $u$, of degree at most $l(\delta)-l(\gamma)$.*

Because of Lemma 3.5, condition (b) of the theorem amounts to a system of equations for the $R_{\gamma,\delta}$. The proof of Lemma 6.8 of [9] explains an algorithm for solving these equations.

**Theorem 1.11.** *For each $\delta\in\mathscr{D}$ there is a unique element*

$$C_\delta=\sum_{\gamma\leqq\delta}P_{\gamma,\delta}(u)\,\gamma\in M,$$

*subject to the following conditions.*

(a) $D(C_\delta)=q^{-l(\delta)}C_\delta$.

(b) $P_{\delta,\delta}=1$

(c) *If* $\gamma\neq\delta$, $P_{\gamma,\delta}$ *is a polynomial in* $u$, *of degree at most* $\frac{1}{2}(l(\delta)-l(\gamma)-1)$.

Once the $R_{\gamma,\delta}$ are known, $P_{\gamma,\delta}$ may be computed exactly as for the Hecke algebra case (see [6]). The use of algebraic geometry in the definition of $D$ actually forces us to prove slightly stronger uniqueness theorems for $D$ and the $C_\delta$: they are unique over a larger coefficient ring than $\mathbb{Z}[u,u^{-1}]$. Here is the main result.

**Theorem 1.12.** *Suppose* $\gamma$, $\delta$ *are in* $\mathscr{D}$. *Define* $\tilde{\delta}$, $\tilde{\delta}^i$ *as in Definition* 1.5.

(a) $\tilde{\delta}^i=0$ if $i$ is odd

(b) $P_{\gamma,\delta}(u)=\sum[\gamma:\tilde{\delta}^{2i}]\,u^i$.

There is also a statement about the eigenvalues of a certain Frobenius map; this is stated precisely in Corollary 4.10.

One could ask for a more self-contained geometric proof of Theorem 1.12 along the lines of the study of Schubert varieties.

Let $x\in\mathscr{B}$ and let $\mathcal{O}$ be its $K$-orbit. One can construct an imbedding $\varphi$ of the affine space $k^N$ ($N=\operatorname{codim}\mathcal{O}$) into $\mathscr{B}$ with the following properties:

(a) $\varphi(0)=x$

(b) $\varphi(k^N)$ is transversal to $\mathcal{O}$

(c) There exists an action of the multiplicative group $G_m$ of $k$ on $k^N$ of the form

$$\lambda(x_1,\dots,x_N)=(\lambda^{a_1}x_1,\dots,\lambda^{a_N}x_N)\qquad(\lambda\in G_m,(x_1,\dots,x_N)\in k^N)$$

with $a_i>0$ and a homomorphism $h\colon G_m\to K$ such that $\varphi(\lambda x)=h(\lambda)\,\varphi(x)$ for all $\lambda\in G_m$, $x\in k^N$.

Such transversals played a crucial role in [7]; and although there are some technical difficulties, it seems likely that they should allow using the methods of [7] in our case. If this is true, it would eliminate our use of representation theory, and of Gabber's extension of Deligne's work on the Weil coujecture.

The existence of these transversals is also of direct geometric significance. Very roughly speaking, it means that the singularities are (essentially) obtained as iterated cones. It is in that setting (over $\mathbb{C}$) that the Goresky-Macpherson intersection homology theory has been directly related to the $L^2$-cohomology theory of Cheeger, although the results which have been written down are not quite general enough to cover our case.

## 2. The Categories $\mathscr{C}$ and $\mathscr{C}'$

We want a geometric description of the module $M$. To get it, we choose an $\mathbb{F}_q$-rational structure on $G$ so that

(2.1) (a) each class of parabolic subgroups in $G$ is defined over $\mathbb{F}_q$;
(b) $K$ and $\theta$ are defined over $\mathbb{F}_q$;
(c) each $K$ orbit on $\mathscr{B}$ is defined over $\mathbb{F}_q$; and
(d) the Frobenius map $F$ acts trivially on $\mathscr{D}$.

**Definition 2.2.** Let $\mathscr{C}$ be the category of constructible, $K$-equivariant $l$-adic sheaves of $\bar{Q}_l$ vector spaces $\mathscr{I}$ over $\mathscr{B}$, endowed with a map

$$\Phi\colon F^*\mathscr{I}\to\mathscr{I}$$

having the following properties. If $x\in\mathscr{B}$, write $\Phi_x$ for the map on stalks.

(a) $\Phi$ is $K$-equivariant: that is, for all $x\in\mathscr{B}$ and $k\in K$,

$$\begin{array}{ccc} \mathscr{I}_{Fx} & \xrightarrow{\Phi_x} & \mathscr{I}_x \\ {\scriptstyle Fk}\downarrow & & \downarrow{\scriptstyle k} \\ \mathscr{I}_{Fk.Fx} & \xrightarrow[\Phi_{k\cdot x}]{} & \mathscr{I}_{k.x} \end{array}$$

is commutative.

(b) For any $x\in\mathscr{B}$, fix $n\geqq 1$ so that $F^n x=x$. Then all eigenvalues of $\Phi_x^n$ are of the form $\varepsilon_i q^{nd_i}$, with $\varepsilon_i$ roots of unity, and $d_i\in\mathbb{Z}$.

We identify $(\mathscr{I},\Phi)$ with $(\mathscr{I},\Phi')$ if $\Phi^n=(\Phi')^n$ for some $n$. The morphisms of $\mathscr{C}$ are morphisms of $K$-equivariant $l$-adic sheaves, compatible with the corresponding $\Phi^n$ for some $n$. Write $\mathscr{K}(\mathscr{C})$ for the Grothendieck group of $\mathscr{C}$.

Given $\gamma\in\mathscr{D}$ and $d\in\mathbb{Z}$, there is an isomorphism

$$\Phi\colon F^*\gamma\to\gamma$$

which satisfies (a) and (b) of the definition, with all $d_i=d$; $\Phi$ is unique up to a root of unity, so

$$u^d\gamma=(\gamma,\Phi)\in\mathscr{C}$$

is well defined. Clearly these elements form a $\mathbb{Z}$ basis of $\mathscr{K}(\mathscr{C})$; so we may identify

(2.3) $$M\cong\mathscr{K}(\mathscr{C}).$$

The form of the Weil conjectures which we will be using does not give conditions like 2.2(b); so we are forced to consider also a much larger category. Put

(2.4) $B_1$ = non-zero algebraic numbers (as a multiplicative subgroup of $\bar{Q}_l$)
$B_0$ = roots of unity in $B_1$
$B = B_1/B_0$

**Definition 2.5.** Write $\mathbb{Z}[B]$ for the group ring of $B$, and $M'$ for the free $\mathbb{Z}[B]$-module with basis $\mathscr{D}$. We identify $\mathbb{Z}[u,u^{-1}]$ as a submodule of $\mathbb{Z}[B]$ by sending $u$ to the image of $q$ in $B$, which we also write as $u$; thus $M\subseteq M'$. Define $\mathscr{C}'$ exactly like $\mathscr{C}$, except that the eigenvalues of $\Phi^n$ are only required to lie in $B_1$. Write $\mathscr{K}(\mathscr{C}')$ for the Grothendieck group of $\mathscr{C}'$.

Suppose we are given $\gamma \in \mathcal{D}$ and $b_1 \in B_1$; write $b$ for its image in $B$. Then there is a unique element $(\gamma, \Phi)$ of $\mathscr{C}'$ such that the eigenvalues of $\Phi^n$ are all of the form $\varepsilon b_1^n$, with $\varepsilon$ a root of unity. Write

$$b \cdot \gamma = (\gamma, \Phi) \in \mathscr{C}'.$$

These elements form a basis of $\mathscr{K}(\mathscr{C}')$; so we may identify

(2.6) $$M' \cong \mathscr{K}(\mathscr{C}').$$

## 3. The Operators $T_s$

**Definition 3.1.** Fix $s \in S$. Put

$$\mathcal{O}_s = \{(B, B') \in \mathscr{B} \times \mathscr{B} \text{ in relative position } s\}$$

$$\pi_1, \pi_2 \colon \mathcal{O}_s \to \mathscr{B} \text{ projections on factors.}$$

Suppose $\mathscr{I} \in \mathscr{C}$ (or $\mathscr{C}'$). We define

$$T_s(\mathscr{I}) \in \mathscr{K}(\mathscr{C}) \qquad (\text{or } \mathscr{K}(\mathscr{C}'))$$

as follows. Pull $\mathscr{I}$ back to $\pi_2^* \mathscr{I}$ on $\mathcal{O}_s$, and consider the higher direct images with compact support $R^i \pi_{1!}(\pi_2^* \mathscr{I})$. These are $K$-equivariant, constructible, $l$-adic sheaves on $\mathscr{B}$; they inherit the map $\Phi$, and we will verify the eigenvalue condition (2.2)(b) in Corollary 3.6. Put

$$T_s(\mathscr{I}) = \sum (-1)^i R^i \pi_{1!}(\pi_2^* \mathscr{I}).$$

As a potential aid to geometric intuition, we give another version of this definition, and a technical result needed later. Set

(3.2) $\mathscr{P}_s$ = variety of parabolic subgroups of $G$ of type $s$
$\pi_s = \mathscr{B} \to \mathscr{P}_s$ the natural projection
$L_x^s = \pi_s^{-1}(\pi_s(x)) \cong \mathbb{P}^1$ $(x \in \mathscr{B})$, the *line of type s* through $x$.

**Lemma 3.3.** *If $\mathscr{I}$ is in $\mathscr{C}$ or $\mathscr{C}'$,*

$$(T_s + 1)\mathscr{I} = \sum (-1)^i \pi_s^* (R^i \pi_{s!} \mathscr{I}).$$

*Proof.* Let $\tilde{\mathcal{O}}_s \subseteq \mathscr{B} \times \mathscr{B}$ be the fiber product

$$\begin{array}{ccc} \tilde{\mathcal{O}}_s & \xrightarrow{\pi_1} & \mathscr{B} \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_s} \\ \mathscr{B} & \xrightarrow[\pi_s]{} & \mathscr{P}_s. \end{array}$$

Then the set $\mathcal{O}_s$ of Definition 3.1 is the complement of the diagonal in $\tilde{\mathcal{O}}_s$. The lemma follows. Q.E.D.

**Lemma 3.4.** *Suppose $\mathscr{G}$ is a $K$-equivariant $l$-adic sheaf on $\mathscr{P}_s$, belonging to the category analogous to $\mathscr{C}$ (or $\mathscr{C}'$). Put $\mathscr{I}=\pi_s^*\mathscr{G}$. Then*

$$R^i\pi_{s!}\mathscr{I}=\begin{cases}\mathscr{G} & i=0\\ 0 & i=1\\ u^{-1}\mathscr{G} & i=2;\end{cases}$$

*here if $\Psi$ is the Frobenius map for $\mathscr{G}$, then $u^{-1}\mathscr{G}$ is an abbreviation for $(\mathscr{G},q^{-1}\Psi)$.*

*Proof.* The stalk of $R^i\pi_{s!}\mathscr{I}$ at $\pi_s(x)$ is obtained as the cohomology with compact supports of the fiber $L_x^s$ with coefficients in $\mathscr{I}$. Since $\mathscr{I}=\pi_s^*\mathscr{G}$, $\mathscr{I}$ looks like $\mathscr{G}_x$ on $L_x^s\cong\mathbb{P}^1$; so the cohomology lives in degrees zero and two, and has the indicated form. Q.E.D.

**Lemma 3.5.** *Suppose $(\mathcal{O},\gamma)\in\mathscr{D}$. Identify $\gamma$ with the corresponding element of $\mathscr{C}$ or $\mathscr{C}'$. Define*

$$\hat{\mathcal{O}}_s=\bigcup_{y\in\mathcal{O}}L_y^s$$

*Then $T_s\gamma$ lies in $\mathscr{K}(\mathscr{C})$, and in the identification* (2.3) *is given as follows.* (*We enumerate cases as in* [9], *Definition* 6.4, *where a more combinatorial separation of cases is given.*) *Fix $x\in\mathcal{O}$.*

(a) $L_x^s\subseteq\mathcal{O}$.

$$T_s\gamma=u\gamma$$

(b1) $L_x^s\cap\mathcal{O}=\{x\}$, *and $\hat{\mathcal{O}}-\mathcal{O}$ is a single $K$ orbit. Write $\hat{\gamma}$ for the unique locally constant extension of $\gamma$ to $\hat{\mathcal{O}}$.*

$$T_s\gamma=\hat{\gamma}|_{\hat{\mathcal{O}}-\mathcal{O}}.$$

(b2) $L_x^s\cap\mathcal{O}=L_x^s-\{point\}$. *Write $\hat{\gamma}$ for the unique extension of $\gamma$ to $\hat{\mathcal{O}}$. Necessarily $\hat{\mathcal{O}}-\mathcal{O}$ is a single $K$ orbit.*

$$T_s\gamma=(u-1)\gamma+u(\hat{\gamma}|_{\hat{\mathcal{O}}-\mathcal{O}}).$$

(c1) $L_x^s\cap\mathcal{O}=\{x,y\}$. *Then $\hat{\mathcal{O}}-\mathcal{O}$ is a single $K$ orbit, and $\gamma$ has two distinct extensions $\hat{\gamma}_i$ to $\hat{\mathcal{O}}$.*

$$T_s\gamma=\gamma+(\hat{\gamma}_1+\hat{\gamma}_2)|_{\hat{\mathcal{O}}-\mathcal{O}}.$$

(c2) $L_x^s\cap\mathcal{O}=L_x^s-\{$*two points in one $K$ orbit*$\}$, *and $\gamma$ extends to $\hat{\mathcal{O}}$. Write $\hat{\gamma}_1$ for the extension, $\gamma'=\hat{\gamma}|_{\hat{\mathcal{O}}-\mathcal{O}}$, and $\hat{\gamma}_2$ for the other extension of $\gamma'$ to $\hat{\mathcal{O}}$.*

$$T_s\gamma=(u-1)\gamma-\hat{\gamma}_2|_{\mathcal{O}}+(u-1)\gamma'.$$

(d1) $L_x^s\cap\mathcal{O}=\{x\}$, *and $\hat{\mathcal{O}}-\mathcal{O}$ is union of two orbits. Call these orbits $\mathcal{O}'$ and $\mathcal{O}''$, labelled so that*

$$\dim\mathcal{O}=\dim\mathcal{O}''=\dim\mathcal{O}'-1.$$

*Let $\hat{\gamma}$ be the unique extension of $\gamma$ to $\hat{\mathcal{O}}$.*

$$T_s\gamma=\hat{\gamma}|_{\mathcal{O}'}+\hat{\gamma}|_{\mathcal{O}''}.$$

(d2) $L^s_x \cap \mathcal{O} = L^s_x - \{2$ *points in two K orbits*$\}$, *and* $\gamma$ *extends to* $\hat{\mathcal{O}}$. *Call the orbits* $\mathcal{O}'$, $\mathcal{O}''$, *and the (unique) extension* $\hat{\gamma}$.

$$T_s \gamma = (u-2)\gamma + (u-1)(\hat{\gamma}|_{\mathcal{O}'} + \hat{\gamma}|_{\mathcal{O}''}).$$

(e) $L^s_x \cap \mathcal{O} = L^s_x - \{2$ *points*$\}$, *and* $\gamma$ *does not extend to* $\hat{\mathcal{O}}$.

$$T_s \gamma = -\gamma.$$

The proof is left to the reader; most of it takes place inside Examples 1.3 (for (d)) or 1.4 (for (c)).

**Corollary 3.6.** $T_s$ *is an endomorphism of* $\mathscr{K}(\mathscr{C})$ (*or* $\mathscr{K}(\mathscr{C}')$ *or* $M$, *or* $M'$) *satisfying*

$$(T_s - u)(T_s + 1) = 0$$

*Proof.* Clearly $T_s$ is additive for short exact sequences, $T_s(bm) = b\,T_s(m)$ for $b \in B$, $m \in M'$; so the eigenvalue condition need only be verified on generators. This was done in Lemma 3.5. So $T_s$ is well defined in the Grothendieck group. The conclusion of Lemma 3.4 can now be written as

$$(T_s + 1)(\pi_s^* \mathscr{G}) = (u+1)\pi_s^* \mathscr{G}$$

or

$$(T_s - u)(\pi_s^* \mathscr{G}) = 0. \tag{3.7}$$

If $\mathscr{I}$ is an arbitrary object of $\mathscr{C}$, then Lemma 3.3 shows that $(T_s + 1)\mathscr{I}$ is a combination of various $\pi_s^* \mathscr{G}$; so (3.7) gives

$$(T_s - u)(T_s + 1)\mathscr{I} = 0. \quad \text{Q.E.D.}$$

This identity for $T_s$ may easily be checked case by case in Lemma 3.5 as well.

Another approach is sketched at the end of Sect. 5.

## 4. Proofs of Main Theorems

Suppose $\mathscr{I} \in \mathscr{C}'$. Following Verdier [8] one can associate to $\mathscr{I}$ a complex $\mathbb{R}\,\mathrm{Hom}\,(\mathscr{I}$, dualizing complex of $\mathscr{B})$ of $l$-adic sheaves on $\mathscr{B}$, well defined up to quasi-isomorphism. According to Deligne [2], its cohomology sheaves $D^{-i}(\mathscr{I})$ are constructible. (In the analytic category,

$$D^{-i}(\mathscr{I})_x = \mathrm{Hom}(H^i_c(U, \mathscr{I}), \mathbb{C}) \qquad (x \in \mathscr{B}),$$

for any small ball $U$ containing $x$.) These are again $K$-equivariant sheaves. They inherit a map $\Phi$ from $\mathscr{I}$: it is obtained as a limit of contragredients of maps induced by $\Phi$. There are natural long exact sequences in $D^{-i}$ attached to short exact sequences in $\mathscr{C}'$, so we get a map

$$D\colon \mathscr{K}(\mathscr{C}') \to \mathscr{K}(\mathscr{C}'), \qquad D(\mathscr{I}) = \sum (-1)^i D^{-i}(\mathscr{I}). \tag{4.1}$$

The corresponding map on $M'$ is also written $D$.

**Lemma 4.2.** *Suppose $m \in M'$, $\delta \in \mathscr{D}$, and $b \in B$. Then*

(a) $D(bm) = b^{-1} D(m)$

(b) $D((T_s+1)m) = u^{-1}(T_s+1)D(m)$

(c) *If $\delta \in \mathscr{D}$, then*

$$D(\delta) = u^{-l(\delta)}\delta + \sum_{\gamma<\delta} a_\gamma \gamma, \qquad a_\gamma \in \mathbb{Z}[B].$$

*Proof.* Part (a) is clear. For (b), we use Lemma 3.3. Taking higher direct images of the proper map $\pi_s$ commutes with $D$ in the derived category ([8], p. 195); and $D$ commutes with inverse image by a smooth map of relative dimension $d$, up to a Tate twist $(d)$. Now $\pi_s$ is proper and smooth, of relative dimension 1; so, in $\mathscr{K}(\mathscr{C}')$,

$$\sum_{i,j}(-1)^{i+j}\pi_s^*(R^i\pi_{s'}(D^{-j}\mathscr{I}))\otimes \bar{Q}_l(1)$$
$$=\sum_{i,j}(-1)^{i+j}D^{-j}(\pi_s^*(R^i\pi_{s'}\mathscr{I})).$$

By Lemma 3.3, this is (b). For (c), we use the representation theoretic interpretation of $\mathscr{D}$ ([9], Theorem 3.5 and Proposition 4.1). Then (c) is contained in Corollary 5.12 of [9] (which is a weak cohomology vanishing result for group representations.) Q.E.D.

**Lemma 4.3** ([9], Lemma 6.8). *There is at most one endomorphism $D'$ of $M'$ or $M$ satisfying conditions* (a)–(c) *of Lemma* 4.2. *If $D'$ is such an endomorphism, and we write*

$$D'(\delta) = u^{-l(\delta)}\sum_{\gamma\in\mathscr{D}} R_{\gamma,\delta}\gamma,$$

*then $R_{\gamma,\delta}$ is a polynomial in $u$, of degree at most $l(\delta)-l(\gamma)$. In particular, $D'$ must preserve $M$.*

This result is proved by a purely combinatorial analysis of the three conditions (a)–(c). Theorem 1.10 is a consequence of these two lemmas.

We turn now to Theorem 1.11. Fix $\delta \in \mathscr{D}$. If $\delta$ is regarded as an element of $\mathscr{C}$, then $\tilde{\delta}^i$ is in a natural way an object of $\mathscr{C}'$. Write $\tilde{\Phi}$ for corresponding Frobenius map. Define

(4.4)(a) $A_d^1 = \{\lambda \in \bar{Q}_l^* \mid \lambda$ is algebraic, and all complex conjugates of $\lambda$ have absolute value $q^d\} \subseteq B_1$

(notation 2.4). The sets $A_d^1$ are cosets of $A_0^1$. Put

(4.4)(b) $$A_d = A_d^1/B_0.$$

Suppose $x \in \mathscr{B}$; choose $n$ so that $F^n x = x$. By a deep theorem of Deligne and Gabber (see [4]),

(4.5) eigenvalues of $\tilde{\Phi}^n$ on $\tilde{\delta}_x^i \subseteq A_{0\cdot n}^1 \cup A_{\frac{1}{2}\cdot n}^1 \cup \ldots \cup A_{i/2\cdot n}^1$.

This is the "Riemann hypothesis" part of the Weil conjectures for DGM cohomology. So we can write

$$\tilde{\delta}^i \leftrightarrow \sum_{\gamma\in\mathscr{D}} P^i_{\gamma,\delta} \in M',$$

with

(4.6)(a) $$P^i_{\gamma,\delta}\in\mathbb{Z}[A_{0\cdot n}\cup A_{\frac{1}{2}\cdot n}\cup\ldots\cup A_{i/2\cdot n}]$$

(the free $\mathbb{Z}$ module with the indicated generators). Define

(4.6)(b) $$P_{\gamma,\delta}=\sum_i(-1)^i P^i_{\gamma,\delta}\in\mathbb{Z}[B]$$

$$C_\delta=\sum_{\gamma\in\mathscr{D}} P_{\gamma,\delta}\gamma\in M'$$

$$\leftrightarrow\sum(-1)^i\tilde{\delta}^i\in\mathscr{K}(\mathscr{C}').$$

**Lemma 4.7**

(a) $D(C_\delta)=u^{-l(\delta)}C_\delta$

(b) $P_{\delta,\delta}=1$

(c) *If* $\gamma\neq\delta$, *put* $m=\frac{1}{2}(l(\delta)-l(\gamma)-1)$. *Then*

$$P_{\gamma,\delta}\in\mathbb{Z}[A_{0\cdot n}\cup A_{\frac{1}{2}\cdot n}\cup\ldots\cup A_{m/2\cdot n}].$$

*Proof.* Parts (a), (b), and (c) are reformulations of conditions (a), (c), and (d) (respectively) in Definition 1.5. Q.E.D.

**Lemma 4.8.** *Fix* $\delta\in\mathscr{D}$. *Then there is at most one element*

$$C'_\delta=\sum_{\gamma\in\mathscr{D}} P'_{\gamma,\delta}\gamma$$

*of* $M'$, *satisfying conditions* (a)–(c) of *Lemma* 4.7. *If* $C'_\delta$ *is such an element, then* $P'_{\gamma,\delta}$ *is a polynomial in* $q$ *(with integer coefficients).*

*Proof.* We proceed by downward induction on $l(\gamma)$. Condition (a), when written out in terms of the $R$ polynomials of Lemma 4.3, is a recursion formula for the $P_{\gamma,\delta}$ in terms of the $R$ and the $P_{\gamma',\delta}$ with $l(\gamma')>l(\gamma)$. We leave the details to the reader (compare [6], proof of Theorem 1.1). Q.E.D.

These two lemmas prove Theorem 1.11.

**Corollary 4.9.**

$$P_{\gamma,\delta}(l)=\sum_i(-1)^i[\gamma:\tilde{\delta}^i].$$

The remainder of the proof of Theorem 1.12 is representation theoretic, and may be described as follows. In [10], there is an algorithm which (in light of [1]) computes the numbers $[\gamma:\tilde{\delta}^i]$, modulo a technical conjecture (Conjecture 2.5 of [10]). The recursion step in this algorithm is based on formulas which are also satisfied by the $P_{\gamma,\delta}$. So to prove Theorem 1.12, one only has to prove the technical conjecture. At each step of the algorithm, one can use Corollary

4.9 to deduce enough of the conjecture to justify the following step. Details of the argument are given in [9].

**Corollary 4.10.** *Fix $\delta \in \mathscr{D}$, and choose a representative $(\delta, \Phi) \in \mathscr{C}'$ for the corresponding element of $\mathscr{K}(\mathscr{C})$. (Thus we are assuming that if $x \in \mathscr{B}$, and $F^n x = x$, then the eigenvalues of $\Phi_x^n$ are roots of unity.) Let $\tilde{\Phi}$ be the induced map on $\tilde{\delta}^{2i}$. Suppose $x \in \mathscr{B}$, and $F^n x = x$. Then the eigenvalues of $\tilde{\Phi}$ on $\tilde{\delta}_x^{2i}$ are of the form $\varepsilon q^{ni}$, with $\varepsilon$ a root of unity.*

*Proof.* We want to show that

$$P_{\gamma,\delta}^{2i} = [\gamma : \tilde{\delta}^{2i}]\, u^i.$$

By Theorem 1.12(b) and (4.6)(a),

$$\sum_i [\gamma : \tilde{\delta}^{2i}]\, u^i = \sum_i P_{\gamma,\delta}^{2i},$$

$$P_{\gamma,\delta}^{2i} \in \mathbb{N}[A_{0\cdot n} \cup \ldots \cup A_{i\cdot n}] \qquad (\mathbb{N} = \{0, 1, 2, \ldots\}),$$

$$[\gamma : \tilde{\delta}^{2i}] = P_{\gamma,\delta}^{2i}(1).$$

(The coefficients of $P_{\gamma,\delta}^{2i}$ are dimensions of eigenspaces, and therefore non-negative.) We proceed by downward induction on $i$; so suppose

$$P_{\gamma,\delta}^{2i} = P_{\gamma,\delta}^{2i}(1)\, u^i, \qquad i > i_0.$$

Comparing coefficients of $u^{i_0}$ in

$$\sum_i P_{\gamma,\delta}^{2i} = \sum P_{\gamma,\delta}^{2i}(1)\, u^i$$

gives

$$\text{coefficient of } u^{i_0} \text{ in } P_{\gamma,\delta}^{2i_0} = \text{sum of all coefficients in } P_{\gamma,\delta}^{2i_0}.$$

Since the coefficients are non-negative,

$$P_{\gamma,\delta}^{2i_0} = P_{\gamma,\delta}^{2i_0}(1)\, u^{i_0}$$

as required. Q.E.D.

## 5. Complements

We propose to define a $W$ graph ([6]) attached to $M$, analogous to the one defined using the Hecke algebra in [6]. The main ingredients are contained in

**Definition 5.1.** Suppose $\gamma, \delta \in \mathscr{D}$, and $l(\gamma) < l(\delta)$. Set $\mu(\gamma, \delta)$ = coefficient of $q^{\frac{1}{2}(l(\delta) - l(\gamma) - 1)}$ in $P_{\gamma,\delta}$, and $\mu(\delta, \gamma) = \mu(\gamma, \delta)$.

Write $\mathcal{O}$ for the support of $\delta$, and define $\hat{\mathcal{O}}_s$ as in Lemma 3.5. Put

$$\tau(\delta) = \{s \in S \mid \mathcal{O} \text{ is open in } \hat{\mathcal{O}}_s, \text{ and } \delta \text{ extends to } \hat{\mathcal{O}}_s\}.$$

If $s \notin \tau(\delta)$, set

$$s \cdot \delta = \{\delta' \in \mathscr{D} \mid l(\delta') = l(\delta) + 1, \text{ and } \delta' \text{ appears in } T_s \delta\}.$$

**Lemma 5.2.** *Suppose $\delta \in \mathscr{D}$, $s \in S$, and $s \in \tau(\delta)$. Then $(T_s - u) C_\delta = 0$.*

*Proof*. We begin with a different construction of $\tilde{\delta}$. Recall the notation of Sect. 3:

$$\pi_s \colon \mathscr{B} \to \mathscr{O}_s, \quad \hat{\mathscr{O}}_s = \bigcup_{x \in \mathscr{O}} L_x^s.$$

Write

$$\mathscr{V} = \pi_s(\mathscr{O}_s).$$

Since $s \in \tau(\delta)$, $\mathscr{O}$ is open in $\hat{\mathscr{O}}_s$, and $\delta$ extends to $\mathscr{I}$ on $\hat{\mathscr{O}}_s$, such that

$$\mathscr{I} = \pi_s^* \mathscr{G}$$

for some $K$-equivariant $l$-adic sheaf $\mathscr{G}$ on $\mathscr{V}$. Write $\tilde{\mathscr{G}}$ for the DGM extension of $\mathscr{G}$ to $\bar{\mathscr{V}} = \pi_s(\bar{\mathscr{O}})$. Since $D$ commutes (up to a twist) with inverse image by a smooth map, the characterization in Definition 1.5 gives

$$\tilde{\delta} \cong \pi_s^*(\tilde{\mathscr{G}}).$$

Now apply (3.7). Q.E.D.

**Lemma 5.3.** *Suppose $\delta \in \mathscr{D}$, $s \in S$, and $s \notin \tau(\delta)$. Then*

$$(T_s + 1) C_\delta = \sum_{\delta' \in s \cdot \delta} C_{\delta'} + \sum_{\substack{\gamma \leqq \delta \\ s \in \tau(\gamma)}} \mu(\gamma, \delta) u^{\frac{1}{2}(l(\delta) - l(\gamma) + 1)} C_\gamma.$$

*Proof*. Define

$$C_\delta^s = (T_s + 1) C_\delta - \sum_{\substack{\gamma < \delta \\ s \in \tau(\delta)}} \mu(\gamma, \delta) u^{\frac{1}{2}(l(\delta) - l(\gamma) + 1)} C_\gamma.$$

By Theorem 1.11(a) and Theorem 1.10(b),

$$D(C_\delta^s) = u^{-(l(\delta) + 1)} C_d^s.$$

The rest of the proof follows [6] exactly: $C_\delta^s$ has the same leading terms as $\sum_{\delta' \in s \cdot \delta} C_{\delta'}$, and satisfies the vanishing condition of Theorem 1.11(b) by construction. We leave the details to the reader. Q.E.D.

**Lemma 5.4.** (a) *Suppose $s \notin \tau(\delta)$, and $\delta' \in s \cdot \delta$. Then $\mu(\delta', \delta) = 1$.*

(b) *Suppose $\gamma \geqq \delta$, $s \notin \tau(\delta)$, $s \in \tau(\gamma)$, and $\mu(\gamma, \delta) \neq 0$. Then $\gamma \in s \cdot \delta$.*

*Proof*. For (a), consider the coefficient of $\delta'$ in the identity of Lemma 5.2 for $\delta'$. The claim follows by inspection of Lemma 3.5. For (b), the assumptions mean first of all that $P_{\delta,\gamma}$ has degree $\frac{1}{2}(l(\gamma) - l(\delta) - 1)$. Consider the occurrence of $\delta$ in the identity of Lemma 5.2 for $\gamma$. Using Lemma 3.5, we can write this as

$$\begin{aligned} u P_{\delta',\gamma} &= u P_{\delta,\gamma} \quad \text{(case (b1), } \delta'=s\cdot\delta) \\ P_{\delta,\gamma}+(u-1)(P_{\delta_1,\gamma}+P_{\delta_2,\gamma}) &= u P_{\delta,\gamma} \quad \text{(case (c1), } s\cdot\delta=\{\delta_1,\delta_2\}) \\ P_{\delta'',\gamma}+(u-1)P_{\delta',\gamma} &= u P_{\delta,\gamma} \quad \text{(case (d1), } s\cdot\delta=s\cdot\delta''=\{\delta'\}) \\ -P_{\delta,\gamma} &= u P_{\delta,\gamma} \quad \text{(case (e))}. \end{aligned}$$

The last of these is obviously impossible. The others force $P_{\delta',\gamma}$ to have degree

$$\tfrac{1}{2}(l(\gamma)-l(\delta)-1)=\tfrac{1}{2}(l(\gamma)-l(\delta'))$$

for some $\delta'\in s\cdot\delta$. So $\gamma=\delta'$. Q.E.D.

This result is also obvious representation theoretically.

**Proposition 5.5.** *Consider the graph with vertex set $\mathscr{D}$, and edges*

$$\{(\gamma,\delta)\in\mathscr{D}\,|\,\mu(\gamma,\delta)\neq 0\}.$$

*considered with multiplicity $\mu(\gamma,\delta)$. Label each $\gamma\in\mathscr{D}$ by $\tau(\gamma)\subseteq S$. Then $\mathscr{D}$ is a W-graph* (cf. [6]). *That is, if we let $\mathscr{E}$ be the free $\mathbb{Z}[u^{\frac{1}{2}},u^{-\frac{1}{2}}]$ module with basis $\{e_\gamma|\gamma\in\mathscr{D}\}$, then*

$$T_s\cdot e_\delta=\begin{cases} -e_\delta, & s\in\tau(\delta)\\ u\,e_\delta+u^{\frac{1}{2}}\sum\limits_{\substack{\gamma\in\mathscr{D}\\ s\in\tau(\gamma)}}\mu(\gamma,\delta)\,e_\gamma, & s\notin\tau(\delta)\end{cases}$$

*defines a Hecke algebra action on $\mathscr{E}$.*

*Proof.* Embed $\mathscr{E}$ in $M'$ by

$$u^{m/2}\,e_\delta\to u^{-\frac{1}{2}(l(\delta)+m)}\,C_\delta.$$

Then (if we write $E_\delta=u^{-\frac{1}{2}l(\delta)}\,C_\delta$

$$T_s e_\delta\to T_{s^{-1}}^{-1}E_\delta,\qquad u^{\frac{1}{2}}e_\delta\to u^{-\frac{1}{2}}E_\delta$$

by Lemmas 5.2–5.4. Since $T_s\to T_{s^{-1}}^{-1}$, $u^{\frac{1}{2}}\to u^{-\frac{1}{2}}$ defines an automorphism of the Hecke algebra, we need only show that the operators $T_s$ and multiplication by $u$ make $M'$ a module for the Hecke algebra; that is, we must prove Proposition 1.7 for $M'$. To do this, we first construct the other operators $T_w$ $(w\in W)$ directly. Put (for $\mathscr{I}\in\mathscr{C}'$)

$$\begin{gathered} \mathcal{O}_w=\{(B,B')\in\mathscr{B}\times\mathscr{B}\text{ in relative position }w\}\\ \pi_i\colon\ \mathcal{O}_w\to\mathscr{B}\text{ projections on factors}\\ T_w\mathscr{I}=\sum(-1)^i R^i\pi_{1!}(\pi_2^*\mathscr{I})\in\mathscr{K}(\mathscr{C}'). \end{gathered}$$

To see that this is a Hecke algebra action, two relations must be verified. The first is

$$(T_s+1)(T_s-u)=0\qquad(s\in S),$$

which is contained in Corollary 3.6. For the other, suppose $w, w' \in W$, and $l(ww') = l(w) + l(w')$. We must show that $T_{ww'} = T_w T_{w'}$. Now $\mathcal{O}_{ww'}$ may be identified with

$$\{(B, B', B'') \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \mid (B, B') \in \mathcal{O}_w,\ (B', B'') \in \mathcal{O}_{w'}\}.$$

Write $\tilde{\pi}_i$ $(i = 1, 2, 3)$ for the three projections on $\mathcal{B}$, and $\tilde{\pi}_{12}$, $\tilde{\pi}_{23}$ for those on $\mathcal{O}_w$, $\mathcal{O}_{w'}$. Define $\pi_j$, $\pi'_j$ $(j = 1, 2)$ as above for $w$, $w'$. Then

$$T_{ww'}(\mathcal{I}) = \sum (-1)^i R^i \tilde{\pi}_{1!}(\tilde{\pi}_3^* \mathcal{I})$$

$$T_w(T_{w'} \mathcal{I}) = T_w(\sum (-1)^i R^i \pi'_{1!}(\pi'^*_2 \mathcal{I}))$$

$$= \sum_{j, j'} (-1)^{j+j'} R^{j'} \pi_{1!}(\pi_2^*(R^j \pi'_{1!}(\pi'^*_2 \mathcal{I})))$$

$$= \sum_{j, j'} (-1)^{j+j'} R^{j'} \pi_{1!}(R^j \tilde{\pi}_{12!}(\tilde{\pi}_{23}^*(\pi'^*_2 \mathcal{I})))$$

(since
$$\begin{array}{ccc} \mathcal{O}_{ww'} & \longrightarrow & \mathcal{O}_w \\ \downarrow & & \downarrow \\ \mathcal{O}_{w'} & \longrightarrow & \mathcal{B} \end{array}$$
is a Cartesian diagram)

$$= \sum (-1)^{j+j'} R^{j'} \pi_{1!}(R^j \tilde{\pi}_{12} \tilde{\pi}_3^* \mathcal{I})$$

(since $\pi_2 \tilde{\pi}_{23} = \tilde{\pi}_3$)

$$= \sum (-1)^i R^i \pi_{1!}(\tilde{\pi}_3^* \mathcal{I}) = T_{ww}(\mathcal{I})$$

(using the spectral sequence of $\pi_1 \tilde{\pi}_{12} = \tilde{\pi}_1$). Q.E.D.

When specialized to $u = 1$, the Hecke algebra representation of Proposition 5.5 becomes the "coherent continuation" representation of $W$ on the lattice of characters of the corresponding real Lie group $\mathcal{G}$, in the basis given by the irreducible characters.

It is possible to define the action of the entire Hecke algebra at once, rather than just defining the action of a basis. To do this, we consider the group $G \times G$ of Example 1.2. Then the analogue of $M$ for $(G \times G, G_\Delta)$ (Definition 1.6) is just

$$\mathbb{Z}[u, u^{-1}][W] = \mathcal{H},$$

the underlying space of the Hecke algebra. Write $\mathcal{C}_\Delta$ for the analogue of $\mathcal{C}$ (Definition 2.2); then

$$\mathcal{K}(\mathcal{C}_\Delta) \cong \mathcal{H}.$$

The multiplication on $\mathcal{H}$ may be defined directly on $\mathcal{K}(\mathcal{C}_\Delta)$, as follows. Suppose $\mathcal{I}$ and $\mathcal{I}'$ are in $\mathcal{C}_\Delta$. Write

$$\pi_{ij}\colon \mathcal{B} \times \mathcal{B} \times \mathcal{B} \to \mathcal{B} \times \mathcal{B} \quad (\mathrm{i} \neq \mathrm{j} \in \{1, 2, 3\})$$

for the obvious projections. Then we can define

$$\mathscr{I} \times \mathscr{I}' = \sum (-1)^i R^i_{\pi_{13!}} [(\pi_{12}^* \mathscr{I}) \otimes (\pi_{23}^* \mathscr{I}')] \in \mathscr{K}(\mathscr{C}_\Delta).$$

This gives the usual Hecke algebra structure on $\mathscr{H}$. If now $\mathscr{G} \in \mathscr{C}$, we can define (writing $\pi_1, \pi_2$ for the projections of $\mathscr{B} \times \mathscr{B}$ on $\mathscr{B}$)

$$\mathscr{I} \cdot \mathscr{G} = \sum (-1)^i R^i_{\pi_1,} [(\mathscr{I} \otimes (\pi_2^* \mathscr{G})] \in \mathscr{K}(\mathscr{C});$$

and this recovers the action of $\mathscr{H}$ on $\mathscr{K}(\mathscr{C})$ already defined. These ideas in turn are merely "Euler characteristics" of operations on derived categories; but the significance of this is still not clear.

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Oblatum 28-I-1981 & 9-VI-1982