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The Blocks of Finite General Linear and Unitary Groups

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Introduction

The study of the blocks of characters with respect to a prime r of finite Chevalley groups defined over fields of characteristic p divides naturally into the cases of equal characteristic $r=p$ and unequal characteristic $r \neq p$. This paper begins the study of the unequal characteristic case for the linear and the unitary groups when $r>2$. The results show a particularly close fit of the Deligne-Lusztig theory, the Brauer theory, and the combinatorial theory underlying the character theory of the symmetric groups.

The partition of the irreducible characters of the symmetric group of degree n into r-blocks was given an element combinatorial formulation in the Nakayama conjecture:

Let ϕ_λ, ϕ_μ be irreducible characters corresponding respectively to the partitions λ and μ of n. Then ϕ_λ and ϕ_μ are in the same r-block if and only if λ and μ have the same r-core.

We show that an analogue of the Nakayama conjecture holds for r-blocks of $G=GL(n,q)$ or $U(n,q)$. Indeed, let χ_λ and χ_μ be unipotent characters of G corresponding to the partitions λ and μ of n. We prove that χ_λ and χ_μ are in the same r-block of $GL(n,q)$ if and only if λ and μ have the same e-core, where e is the order of q modulo r. A similar result holds for $U(n,q)$ with e possibly replaced by $2e$. This result, however, is but a special case of a more general theorem on when two irreducible characters of G are in the same r-block. The irreducible characters of G can be parametrized as $\chi_{t,\psi}$, where t is a semisimple element of G and ψ is a unipotent character of $C_G(t)$. Here $\chi_{t,\psi}=\chi_{t',\psi'}$ if the pairs (t,ψ) and (t',ψ') are conjugate in G, and every such pair (t,ψ) does in fact label a character. We will show that the r-blocks of G are similarly parametrized as $B_{s,\varphi}$ by conjugacy classes of pairs (s,φ), where s is a semisimple r'-element of G and φ is a unipotent character of a canonically defined subgroup of $C_G(s)$. The more general theorem then states necessary and sufficient con-

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ditions for $\chi_{t,\psi}$ to be in $B_{s,\varphi}$ in terms of the pairs (t, ψ) and (s, φ). Since ψ and φ are in bijection with products of partitions, the conditions can be stated in combinatorial terms involving hooks and cores of Young diagrams. In the case of unipotent characters, these conditions reduce to the one in the preceding paragraph.

The case of blocks with cyclic defect groups plays an essential role in our work, and it is possible in turn to interpret the preceding classification of characters in a block as an extension of the cyclic theory of Brauer and Dade. Thus the characters in a block B with defect group R fall into families parametrized by equivalence classes of elements of R, and a natural definition of non-exceptional and exceptional characters in B arises from this parametrization. In the case where R is cyclic, these equivalence classes of elements of $R^{\#}$ correspond to the classes of irreducible characters of R used in the labeling of the exceptional characters of B in the cyclic theory. Moreover, the character formulas of the Deligne-Lusztig theory, when applied to the characters of B on r-sections, coincide with the formulas of Brauer and Dade.

The sections of this paper are as follows. Paragraph 1 gives the basic notation and the facts needed from the Deligne-Lusztig theory. Paragraph 2 concerns the characters of the linear and unitary groups. Two results, which we call of Curtis type and of Murnaghan-Nakayama type, are important for later use. These relate the values of irreducible characters on a specified element to values of characters of certain subgroups containing the element. Paragraph 3 describes the possible r-subgroups occurring as defect groups of blocks and introduces two basic configurations $B^{m,\alpha}$ and $B^{m,\alpha,\beta}$ of blocks which are the essential constituents of any block. Paragraph 4 contains the key step for a classification of blocks, namely a parametrization of the blocks $B^{m,\alpha}$ by a set $\mathscr{F}'$ of polynomials whose roots are r'-th roots of unity. The general classification of blocks is given in Paragraph 5. The theorem classifying characters in a block is proved by an inductive argument based on Brauer's Second Main Theorem. The difficult case is the first step of the induction when the center $Z(G)$ contains elements of order r. This is done in Paragraph 6. The general case is then proved by a simple induction argument in Paragraph 7. Finally, Paragraph 8 gives several consequences of the theorems. Among these is a proof that the height conjecture holds for blocks of linear and unitary groups.

As general references for the Deligne-Lusztig theory, the Brauer theory, and the character theory of the symmetric groups, we mention [21, 9, and 14].

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§1

The general linear and unitary groups can be viewed as the groups of automorphisms of an underlying vector space, or as the group of rational points of a Frobenius endomorphism of an algebraic group. We require both descriptions and shall use the following notation: V is a finite-dimensional vector space of dimension n over a finite field F of q elements and characteristic p, and $GL(V)$ its group of automorphisms. If $q=q_0^2$ is a square, F has a unique automorphism J of order 2, and V has up to equivalence a unique unitary group $U(V)$; here $U(V)$ can be replaced by any conjugate subgroup in $GL(V)$. We shall write $GL(n,F)$, $GL(n,q)$ and $U(n,F)$, $U(n,q)$ for the natural matrix representations of these groups.

Let $F[X]$ be the polynomial ring in the indeterminant X over F, and $\mathscr{F}_0$ the subset of monic irreducible polynomials different from X. In the case q is a square, say $q=q_0^2$, let $\sim$ be the permutation of $\mathscr{F}_0$ of order 2 defined by mapping

$$\Delta(X)=X^m+\alpha_{m-1}X^{m-1}+\ldots+\alpha_1 X+\alpha_0$$

onto $\tilde{\Delta}(X)=(\alpha_0^{-1})^J X^m \Delta^J(X^{-1})$. In particular, ω is a root of $\Delta(X)$ if and only if ω^{-q_0} is a root of $\tilde{\Delta}(X)$. Thus $\Delta=\tilde{\Delta}$ if and only if Δ has odd degree d and the roots of Δ have order dividing q_0^d+1. We let

$$\mathscr{F}_1=\{\Delta:\Delta\in\mathscr{F}_0,\Delta=\tilde{\Delta}\}$$

$$\mathscr{F}_2=\{\Delta\tilde{\Delta}:\Delta\in\mathscr{F}_0,\Delta\neq\tilde{\Delta}\}.$$

The polynomials in $\mathscr{F}_1$ and $\mathscr{F}_2$ have odd and even degrees respectively. The degree of a polynomial Λ will be denoted by d_Λ. In addition, we define a reduced degree δ_Λ for polynomials in $\mathscr{F}_0\cup\mathscr{F}_2$ by

$$\delta_\Lambda=\begin{cases} d_\Lambda & \text{if } \Lambda\in\mathscr{F}_0\\ \frac{1}{2}d_\Lambda & \text{if } \Lambda\in\mathscr{F}_2.\end{cases}$$

The conjugacy classes of elements in $GL(V)$ are described by elementary divisors. Given a power Δ^i of Δ in $\mathscr{F}_0$ and given g in $GL(V)$, let $m_{\Delta^i}(g)$ be the multiplicity of Δ^i as an elementary divisor of g. In addition, let (Λ) denote the companion matrix of the polynomial Λ, and let $m(\Lambda)$ denote the matrix direct sum of m copies of (Λ). Then

$$\prod_{\Delta\in\mathscr{F}_0}\prod_i m_{\Delta^i}(g)(\Delta^i)$$

is the rational canonical form of g.

A similar theory of elementary divisors holds for $U(V)$ (see [22], §2.6). Given $g\in GL(V)$, the intersection $g^{GL(V)}\cap U(V)$ is either empty or a conjugacy class of $U(V)$, and the second case occurs if and only if

$$m_{\Delta^i}(g)=m_{\tilde{\Delta}^i}(g) \qquad \text{for all } \Delta\in\mathscr{F}_0, \text{ all } i\in N.$$

In particular, given a power Λ^i of a polynomial $\Lambda\in\mathscr{F}_1\cup\mathscr{F}_2$, the multiplicity $m_{\Lambda^i}(g)$ may be defined as $m_{\Delta^i}(g)$, where Δ is any polynomial in $\mathscr{F}_0$ dividing Λ. We then call Λ^i an elementary divisor of g if $m_{\Lambda^i}(g)>0$. The corresponding canonical form of g as an element of $U(V)$ is then

$$\prod_{\mathscr{F}_1\cup\mathscr{F}_2}\prod_i m_{\Lambda^i}(g)(\Lambda^i).$$

In order to have notation which applies to both the linear and the unitary cases, we adopt the following convention: $G(V)=GL(V)$ or $U(V)$, and $\mathscr{F}=\mathscr{F}_0$ or $\mathscr{F}_1\cup\mathscr{F}_2$ according as V is a linear or a unitary space. Given $g\in G(V)$, its primary decomposition will be denoted by $g=\prod_{\Lambda\in\mathscr{F}} g_\Lambda$, where g_Λ is the primary component corresponding to Λ. Then V and $C_G(g)$ correspondingly decompose as $V=\sum_{\Lambda\in\mathscr{F}} V_\Lambda$ and $C_G(g)=\prod_{\Lambda\in\mathscr{F}} C(g)_\Lambda$. In the unitary case the V_Λ are non-singular unitary subspaces. We note the following special cases:

1) Suppose g_Λ is Semisimple. Replacing $U(V)$ by a conjugate we may assume $g_\Lambda=m(\Lambda)$. Let F_Λ be the matrix algebra of degree d_Λ generated over F by the matrix (Λ). Then F_Λ is isomorphic to an extension field E of degree δ_Λ over F. The isomorphism $\mathscr{I}$ from E to F_Λ gives a representation of E over F. If $\Lambda\in\mathscr{F}_0$, then $\mathscr{I}$ is equivalent to the regular representation $\mathscr{R}$ of E over F, and $C(g)_\Lambda$ is then represented in $GL(V_\Lambda)$ or $U(V_\Lambda)$ as $GL(m,F_\Lambda)$ or $U(m,F_\Lambda)$ respectively. Note that as such, $C(g)_\Lambda$ is a subgroup of $GL(md_\Lambda,F)$ or $U(md_\Lambda,F)$. If $\Lambda\in\mathscr{F}_2$, then $\mathscr{I}$ is equivalent to the direct sum representation $\mathscr{R}\oplus\mathscr{R}^J$, and $C(g)_\Lambda$ is then represented in $U(V_\Lambda)$ as $GL(m,F_\Lambda)$. Again we note that as such, $C(g)_\Lambda$ is a subgroup of $U(md_\Lambda,F)$.

2) Suppose g_Λ is Unipotent. We may assume $g_\Lambda=\prod m_i(\Lambda^i)$, where $\Lambda=X-1$ and $m_i=m_{\Lambda^i}(g)$. Let F_{Λ^i} be the algebra generated over F by (Λ^i), and let $F\otimes I$ be the embedding of F in the center of F_{Λ^i}, where I is the identity matrix of degree i. The group

$$D_\Lambda=\prod_i GL(m_i,F\otimes I) \quad \text{or} \quad \prod_i U(m_i,F\otimes I)$$

is contained as a subgroup of p-power index in $C(g)_\Lambda$. In particular, D_Λ contains a Sylow r-subgroup of $C(g)_\Lambda$ for every prime $r\neq p$.

Proposition (1A). *Let g be a semisimple element of $G=GL(n,F)$ or $U(n,F)$. Let $g=\prod_\Lambda g_\Lambda$ be the primary decomposition of g, and $\prod_\Lambda C(g)_\Lambda$ the corresponding decomposition of $C(g)$.*

i) If $G=GL(n,F)$, then $C(g)_\Lambda=GL(m_\Lambda(g),F_\Lambda)$, and $|F_\Lambda:F|=d_\Lambda=\delta_\Lambda$.

ii) If $G=U(n,F)$ and $\Lambda\in\mathscr{F}_1$, then $C(g)_\Lambda=U(m_\Lambda(g),F_\Lambda)$, and $|F_\Lambda:F|=d_\Lambda=\delta_\Lambda$.

iii) If $G=U(n,F)$ and $\Lambda\in\mathscr{F}_2$, then $C(g)_\Lambda=GL(m_\Lambda(g),F_\Lambda)$ and $|F_\Lambda:F|=\frac{1}{2}d_\Lambda=\delta_\Lambda$.

We shall say the elementary divisor Λ of g in (1A) is of linear type if $C(g)_\Lambda$ is linear, of unitary type if $C(g)_\Lambda$ is unitary.

The character theory of $G(V)$ is best described for our purposes in the language of Deligne-Lusztig. Let $\bar{F}$ be an algebraic closure of F, and let $\bar{G}$ be a reductive, connected algebraic group over $\bar{F}$ with an F-rational structure and associated Frobenius endomorphism σ. The group $G=\bar{G}^\sigma$ of fixed-points of σ on $\bar{G}$ is then a finite Chevalley group. We recall some terminology. A parabolic subgroup of G is a subgroup of the form $P=\bar{P}^\sigma$, where $\bar{P}$ is a σ-stable parabolic subgroup of $\bar{G}$. A maximal torus of G is a subgroup of the form T

$=\bar{T}^{\sigma}$; where $\bar{T}$ is a σ-stable maximal torus of $\bar{G}$. The Weyl group of T is $W_G(T)=(N_{\bar{G}}(\bar{T})/\bar{T})^{\sigma}$. The subscripts G and $\bar{G}$ will in general be omitted. A subgroup L of G is regular if $L=\bar{L}^{\sigma}$ for some σ-stable Levi subgroup $\bar{L}$ of a parabolic subgroup $\bar{P}$ of $\bar{G}$. If the $\bar{P}$ containing $\bar{L}$ is also σ-stable, then L is called a subparabolic subgroup of G, and $L\subseteq P$, where $P=\bar{P}^{\sigma}$.

Let L be a regular subgroup of G. Let R_L^G be the additive operator from $X(L)$ to $X(G)$ defined in the Deligne-Lusztig theory, where $X(L)$ and $X(G)$ are the character rings of representations of L and G respectively over $\bar{Q}_{\ell}$, an algebraic closure of the ℓ-adic field Q_{ℓ}. (This is an abuse of terminology in [7, 15], where R_L^G is written as $R_{\bar{L}}^{\bar{G}}$ or $R_{\bar{L}\subset\bar{P}}^{\bar{G}}$, and $X(L)$ and $X(G)$ are the Grothendieck rings of representations of L and G over $\bar{Q}_{\ell}$.) Among the properties developed in [7, 15], we note the following three:

(1.1) If $K\leqq L\leqq G$ are subgroups of G such that R_K^L, R_L^G, and R_K^G are defined, then

$$R_L^G(R_K^L(\theta))=R_K^G(\theta) \qquad \text{for } \theta\in X(K).$$

(1.2) If L and P are corresponding subparabolic and parabolic subgroups of G, then

$$R_L^G=\mathrm{Ind}_P^G\circ\mathrm{Inf}_L^P.$$

Here Inf_L^P is the inflation operator from $X(L)$ to $X(P)$, and Ind_P^G is the induction operator from $X(P)$ to $X(G)$.

(1.3) The degree of $R_L^G(\theta)$ is $\varepsilon_G\varepsilon_L|G:L|_{p'}\theta(1)$. Here $\varepsilon_G=(-1)^d$, where d is the dimension of a maximal F-split torus of $\bar{G}$.

We remark that the proof in [15], Paragraph 5 of (1.1) is for the case K is a maximal torus T, but the same proof applies for (1.1).

Let χ be a class function of G. The principal part $\chi_{(L)}$ of χ at a regular subgroup is the class function of L defined by the following adjoint condition:

(1.4) $$(\chi_{(L)},\theta)_L=(\chi,R_L^G(\theta))_G \qquad \text{for } \theta\in X(L).$$

In particular, $\chi_{(L)}\in X(L)$ if $\chi\in X(G)$. These functions were introduced for a maximal torus L by Green, [10] page 423, in the case $G=GL(n,q)$, and by Kilmoyer in the general case.

An irreducible character of G is unipotent if it occurs as a constituent of $R_T^G(1)$ for some maximal torus T of G. We will denote the set of irreducible characters of G by $\hat{G}$, and the subset of unipotent characters of G by $\hat{G}^u$. The irreducible constituents of $R_L^G(\theta)$ are in $\hat{G}^u$ if and only if $\theta\in\hat{L}^u$.

If T and T' are maximal tori of G, we write $T\sim_G T'$ or $T\nsim_G T'$ according as T and T' are conjugate or not conjugate in G. We use the same notation for subsets and elements of G. Then

(1.5) $$R_T^G(\theta)=R_{T'}^G(\theta')$$

if there exists $g\in G$ such that $T'=g^{-1}Tg$ and ${}^g\theta'=\theta$, where ${}^g\theta'$ is the function of T defined by ${}^g\theta'(t)=\theta'(g^{-1}tg)$. The $R_T^G(\theta)$ satisfy the following orthogonality relations:

$$(R_T^G(\theta), R_{T'}^G(\theta')) = \begin{cases} |W(T, \theta, \theta')| & \text{if } T = T' \\ 0 & \text{if } T \nsim_G T' \end{cases} \tag{1.6}$$

where $W(T, \theta, \theta') = \{w : w \in W(T),\ \theta' = {}^w\theta\}$. The value of $R_T^G(\theta)$ at a unipotent element u is independent of θ, and is denoted by $Q_T^G(u)$. The Q_T^G are then integer-valued functions on the subset G_u of unipotent elements of G. The Q_T^G are called Green functions and satisfy the following orthogonality relations:

$$\frac{1}{|G|} \sum_{u \in G_u} Q_T^G(u)\, Q_{T'}^G(u) = \begin{cases} |W(T)|/|T| & \text{if } T \sim_G T' \\ 0 & \text{if } T \nsim_G T' \end{cases}. \tag{1.7}$$

For the remaining part of Paragraph 1 we shall suppose $\bar{G} = GL(n, \bar{F})$ and σ is the mapping $a \mapsto a^{(q)}$ or $a \mapsto a^{-(q)^t}$, where $a^{(q)}$ is the matrix obtained from a by raising every entry of a to the q-th power, and t is the transpose operator. The group G is then $GL(n, q)$ or $U(n, q^2)$. This is a departure from the convention used at the beginning of Paragraph 1, where $G = GL(n, q)$ or $U(n, q)$. In the context of the character theory of $\bar{G}^\sigma$ and the "q to $-q$" phenomena, $GL(n, q)$ and $U(n, q^2)$ are the natural pairings of linear and unitary groups. We shall return to the first convention in Paragraph 3.

The Green functions satisfy a second orthogonality relation when $\bar{G} = GL(n, \bar{F})$.

$$\sum_{(T) \subset G} \frac{|T|}{|W(T)|} Q_T^G(u)\, Q_T^G(u') = \begin{cases} |C(u)| & \text{if } u \sim_G u' \\ 0 & \text{if } u \nsim_G u' \end{cases} \tag{1.8}$$

Here and elsewhere $(T) \subset G$ denotes a set of representatives for the G-conjugacy classes of maximal tori of G. The character formula for the $R_T^G(\theta)$ takes the following form: Let $x \in G$, and let $x = su$ be the Jordan decomposition of x, where s is semisimple and u is unipotent. Then

$$(R_T^G(\theta))(x) = \frac{1}{|C(s)|} \sum_{\substack{g \in G \\ gTg^{-1} \subset C(s)}} g\theta(s)\, Q_{gTg^{-1}}^{C(s)}(u). \tag{1.9}$$

An equivalent statement can be obtained as follows: Let $T_1, \ldots, T_k$ be representatives for the $C(s)$-classes of maximal tori of $C(s)$ conjugate in G to T, and let $g_i \in G$ be chosen so that $T_i = g_i T g_i^{-1}$. Let $K_i = C(s)\, g_i N(T)$. Then

$$(R_T^G(\theta))(x) = \frac{1}{|C(s)|} \sum_i \sum_{g \in K_i} \theta(s^g)\, Q_{gTg^{-1}}^{C(s)}(u). \tag{1.10}$$

Let $\bar{S}$ be the σ-stable maximal torus of $\bar{G}$ consisting of diagonal matrices of $\bar{G}$. We shall call $W = W(S)$ the Weyl group of G. As usual, W may be replaced by a conjugate W^g, where $g \in G$. The Weyl group W is represented as the symmetric group of degree n in its reflection representation, and as such, there is a unique involution w_0 in W which is the element of longest length. The G-conjugacy classes of maximal tori of G are in bijection with the σ-conjugacy classes of W. Two elements $w_1, w_2 \in W$ are σ-conjugate if $w_2 = w w_1 \sigma(w)^{-1}$ for some $w \in W$. In the linear case, σ-conjugacy is the same as conjugacy. In the unitary case $\sigma(w) = w_0 w w_0$ for all $w \in W$. Thus w_1, w_2 are σ-conjugate if and only if $w_1 w_0, w_2 w_0$ are conjugate. In both the linear and the unitary cases, the G-conjugacy classes of maximal tori of G are parameterized by conjugacy classes of W, and hence by partitions of n. We shall write T_w or T_λ for a torus representing the G-conjugacy class corresponding to the W-conjugacy class of w or the partition λ of n. Then

(1.11) $$|W(T_w)| = |C_W(w)|,$$

and if $\lambda = \{1^{r_1} 2^{r_2} \dots n^{r_n}\}$, then

(1.12) $$|T_\lambda| = \begin{cases} \prod_i (q^i - 1)^{r_i} & \text{in the linear case} \\ \prod_i (q^i - (-1)^i)^{r_i} & \text{in the unitary case} \end{cases}.$$

When $\lambda = \{n\}$, T_λ is called a Coxeter torus.

The irreducible characters ϕ of W are parametrized by partitions of n. We shall write ϕ_μ for the character in $\hat{W}$ corresponding to the partition μ. The unipotent characters of G in turn are parameterized by the irreducible characters of W. We shall write χ_ϕ for the character in $\hat{G}^u$ corresponding to the character ϕ in $\hat{W}$, and also write χ_μ for χ_ϕ if ϕ is ϕ_μ. The class function

(1.13) $$\chi^\phi = \frac{1}{|W|} \sum_{w \in W} \phi(w) R^G_{T_w}(1)$$

is $\pm\chi_\phi$, and if ϕ is ϕ_μ, χ^ϕ shall also be written as χ^μ. We introduce the χ^ϕ in order to simplify the appearance of certain formulas. The Eq. (1.13) can be inverted to give

(1.14) $$R^G_{T_w}(1) = \sum_{\phi \in \hat{W}} \phi(w) \chi^\phi.$$

The degree of χ^μ is given by a hook-length formula. Let $\mu = \{\mu_1, \mu_2, \dots, \mu_n\}$, and let $\mu' = \{\mu'_1, \mu'_2, \dots, \mu'_n\}$ be the conjugate partition. Let $P_\mu(x)$ be the polynomial

$$\frac{X^d (X^n - 1)(X^{n-1} - 1) \dots (X - 1)}{\prod_h (X^h - 1)},$$

where $d = \sum_j (\mu'_j)^2 - \sum_i i\mu_i$, and h runs over the hook lengths of μ. Then

(1.15) $$\chi^\mu(1) = \begin{cases} P_\mu(q) & \text{in the linear case} \\ P_\mu(-q) & \text{in the unitary case} \end{cases}.$$

This formula is well-known in the linear case; in the unitary case it follows from [17].

We pause in this summary of facts to note that the preceding formulas can be immediately extended to direct products $L=\prod_{\Gamma} L_{\Gamma}$, where each L_{Γ} is a linear or a unitary group of degree n_{Γ} over an extension field E_{Γ} of F. We shall adopt the following notation: As Weyl group of L, we take $W_L=\prod_{\Gamma} W_{\Gamma}$, where W_{Γ} is a Weyl group of L_{Γ}. The irreducible characters ϕ of W_L have the form $\prod_{\Gamma} \phi_{\Gamma}$, where $\phi_{\Gamma}\in\hat{W}_{\Gamma}$. If ϕ_{Γ} corresponds to the partition μ_{Γ} of n_{Γ}, we shall write $\phi_{\mu_{\Gamma}}$ for ϕ_{Γ}. We shall then write ϕ_{μ} for ϕ, where $\mu=\prod_{\Gamma} \mu_{\Gamma}$. A similar convention will be used for unipotent characters of L and their associated class functions, that is, we shall write χ_{ϕ}, χ_{μ} or χ^{ϕ}, χ^{μ} for the unipotent characters or functions corresponding to $\phi=\prod_{\Gamma} \phi_{\Gamma}$ or $\mu=\prod_{\Gamma} \mu_{\Gamma}$.

Finally, we come to the irreducible characters of G. In the linear case these were constructed by Green [10], and in the unitary case by Lusztig and Srinivasan [17]. To describe the construction we fix an isomorphism of $\bar{F}^{\times}$ into $\bar{Q}_{\ell}^{\times}$. Let s be a semisimple element in G and let $L=C_G(s)$. Such subgroups L are always regular. Then the fixed isomorphism induces an isomorphism (see [16], (7.4.2))

$$Z(L)\simeq \operatorname{Hom}(L/[L,L],\bar{Q}_{\ell}^{\times}). \tag{1.16}$$

The linear character of L corresponding to s under the isomorphism (1.16) will be denoted by $\hat{s}$. The irreducible characters χ of G are then in bijection with G-conjugacy classes of pairs (s,ψ), where s is a semisimple element of G and ψ is a unipotent character of $L=C_G(s)$. The bijection is given as $\chi=\varepsilon_G\varepsilon_L R_L^G(\hat{s}\psi)$. We note that if $\prod_{\Gamma} s_{\Gamma}$ is the primary decomposition of s and $\prod_{\Gamma} L_{\Gamma}$ is the corresponding decomposition of L, then L is a direct product of the type considered in the preceding paragraph. If ψ corresponds to the character ϕ in $\hat{W}_L$, we shall write χ_{ϕ} for ψ, and if ϕ is ϕ_{μ}, we shall also write χ_{μ} for ψ. Thus

$$\chi=\varepsilon_G\varepsilon_L R_L^G(\hat{s}\chi_{\phi})=\varepsilon_G\varepsilon_L R_L^G(\hat{s}\chi_{\mu})$$

and we shall write any of $\chi_{s,\psi}$, $\chi_{s,\phi}$ or $\chi_{s,\mu}$ for χ. As before, we introduce the class function

$$R_L^G(\hat{s}\chi^{\phi})=R_L^G(\hat{s}\chi^{\mu}), \tag{1.17}$$

and denote the function by $\chi^{s,\psi}$, $\chi^{s,\phi}$, or $\chi^{s,\mu}$.

The characters of the form $\chi_{s,\mu}$ for a fixed s form the geometric conjugacy class s_G. The following is an analogue of (1.13) for characters in s_G.

$$\chi^{s,\mu}=\frac{1}{|W_L|}\sum_{w\in W_L} \phi_{\mu}(w)\,R_{T_w}^G(\hat{s}). \tag{1.18}$$

Here $L=C_G(s)$, T_w is a torus of L corresponding to w^{W_L}, and $\hat{s}$ denotes the restriction of $\hat{s}$ to T_w. This formula follows from (1.17) by expressing χ^{μ} in the form (1.13) and using transitivity of the R-operators. We conclude this section with two results on inner products.

Proposition (1B). *Let H be a regular subgroup of G of the form $H=C_G(\rho)$, where ρ is a semisimple element of G. Let $\chi_{s,\lambda}$ and $\chi_{t,\mu}$ be irreducible characters of G and H respectively. Then the following hold:*

(i) If $t \nsim_G s$, then $(\chi^{s,\lambda}, R_H^G(\chi^{t,\mu}))_G=0$
(ii) If $t=s$, then $(\chi^{s,\lambda}, R_H^G(\chi^{s,\mu}))_G=(\chi^{\lambda}, R_{C_H(s)}^{C(s)}(\chi^{\mu}))_{C(s)}$

Proof: By definition

$$R_H^G(\chi^{t,\mu})=R_H^G(R_{C_H(t)}^H(\hat{t}\chi^{\mu}))=R_{C(t)}^G(\hat{t}R_{C_H(t)}^{C(t)}(\chi^{\mu})).$$

We note that $C_H(t)$ is a regular subgroup of $C(t)$ since $C_H(t)=C_{C(t)}(\rho)$. The classification of characters then implies the following: If $t \nsim_G s$, then $\chi^{s,\lambda}$ cannot occur in $R_H^G(\chi^{t,\mu})$ and i) holds. On the other hand, if $t=s$, then $(\chi^{s,\lambda}, R_H^G(\chi^{s,\mu}))$ is the multiplicity of χ^{λ} in $R_{C_H(t)}^{C(t)}(\chi^{\mu})$, which is the statement of ii).

Proposition (1C). *Let H be a regular subgroup of the form $C_G(\rho)$, where ρ is a semisimple element of G. Let $\prod_{\Gamma} H_{\Gamma}$ and $\sum_{\Gamma} V_{\Gamma}$ be the decompositions of H and the underlying space of G corresponding to the primary decomposition $\prod_{\Gamma} \rho_{\Gamma}$ of ρ. Suppose $H_{\Gamma}=G(V_{\Gamma})$ for all Γ. If χ_{ϕ} and $\chi_{\phi'}$ are respectively unipotent characters of G and H corresponding to $\phi\in\hat{W}$ and $\phi'\in\hat{W}_H$, then*

$$(\chi^{\phi}, R_H^G(\chi^{\phi'}))_G=(\phi, \mathrm{Ind}_{W_H}^W \phi')_W.$$

Proof: We note that W_H can be assumed to be a subgroup of W because of the hypothesis that $H_{\Gamma}=G(V_{\Gamma})$ for all Γ. By (1.13) applied to H, we have that

$$\chi^{\phi'}=\frac{1}{|W_H|}\sum_{w\in W_H}\phi'(w)\,R_{T_w}^H(1).$$

By (1.14), it follows that

$$R_H^G(\chi^{\phi'})=\frac{1}{|W_H|}\sum_{\phi\in\hat{W}}\sum_{w\in W_H}\phi(w)\,\phi'(w)\,\chi^{\phi},$$

whence $(\chi^{\phi}, R_H^G(\chi^{\phi'}))=(\phi|_H, \phi')_{W_H}=(\phi, \mathrm{Ind}_{W_H}^W(\phi'))_W$ as claimed.

§2

We continue with the notation from the end of the preceding section, so in particular, $G=GL(n,q)$ or $U(n,q^2)$. The main results of this section relate the values of irreducible characters of G on a given element to values of characters of certain subgroups containing the given element. Although the results are stated only for G, they extend immediately to direct products $L=\prod_{\Gamma} L_{\Gamma}$ of the form considered in Paragraph 1. The proofs of (2A) and (2B) are based on unpublished notes of Kilmoyer.

Lemma (2A). *Let χ be a class function of G and let $\chi_{(T)}$ be its principal part at the maximal torus T. Then*

$$\chi_{(T)}(t)=\frac{|T|}{|C(t)|}\sum_{v\in C(t)_u}\chi(tv)\,Q_T^{C(t)}(v) \tag{2.1}$$

for all $t\in T$.

Proof: It suffices to prove (2A) for the case where χ is the characteristic function of a conjugacy class of G, say the class x^G. Let $x=su$ be the Jordan decomposition of x, and let χ_0 denote the function on T given by the right hand side of (2.1). The term $\chi(tv)$ in (2.1) is non-zero only if $tv\sim_G su$. Moreover, the set of elements $v\in C(t)_u$ for which $tv\sim_G su$ is a conjugacy class of $C(t)$. Thus

$$\chi_0(t)=\begin{cases}\dfrac{|T|}{|C(x)|}\,Q_T^{C(t)}(u^g) & \text{if } t=s^g\\ 0 & \text{if } t\nsim_G s\end{cases}$$

and χ_0 is rational-valued. For any $\theta\in\hat{T}$, we have

$$\begin{aligned}(\theta,\chi_0)_T&=\frac{1}{|C(x)|}\sum_{\substack{t\in T\\ t=s^g}}\theta(t)\,Q_T^{C(t)}(u^g)\\ &=(R_T^G(\theta),\chi)_G\end{aligned}$$

by (1.9). Thus $(\theta,\chi_0)_T=(\theta,\chi_{(T)})_T$ for all $\theta\in\hat{T}$, and so $\chi_0=\chi_{(T)}$.

Lemma (2B). *Let χ be a class function of G, and let L be a regular subgroup of G. The following hold:*

i) $\chi=\sum_{(T)\subset G}\dfrac{1}{|W(T)|}R_T^G(\chi_{(T)})$

ii) $\chi_{(L)}=\sum_{(T)\subset L}\dfrac{1}{|W_L(T)|}R_T^L(\chi_{(T)})$.

Proof: Let θ, θ' be irreducible characters of the maximal tori T, T' respectively of G. Then

$$(R_T^G(\theta)_{(T')},\theta')=(R_T^G(\theta),R_{T'}^G(\theta')),$$

so by (1.6) we have

$$R_T^G(\theta)_{(T')}=\begin{cases}\sum_{w\in W(T)}{}^w\theta & \text{if } T=T'\\ 0 & \text{if } T\nsim_G T'\end{cases}. \tag{2.2}$$

By its definition $\chi_{(T)}$ is $W(T)$-invariant, so

$$R_T^G(\chi_{(T)})_{(T)}=|W(T)|\,\chi_{(T)}.$$

Let χ_0 be the right hand side of i). Then $\chi_{(T)}=\chi_{0(T)}$ for all T occurring in the sum $\sum_{(T)\subset G}$, and thus $(\chi-\chi_0, R_T^G(\theta))=0$ for all such T and all $\theta\in\hat{T}$. But $X(G)$ is spanned by such $R_T^G(\theta)$ by [17], Theorem 3.2. Thus $\chi=\chi_0$ and i) holds.

Now Eq. (2.2) holds with L in the place of G. Let χ_0 denote the right hand side of ii). By (1.1) and (1.4) $(\chi_{(L)})_{(T)}=\chi_{(T)}$ for any T occurring in the sum $\sum_{(T)\subset L}$. So as before, it follows that $\chi_{(L)}$ and χ_0 have the same principal parts at every such T. Again, $X(L)$ is spanned by the $R_T^L(\theta)$ for all such T and all $\theta\in\hat{T}$. Thus $\chi_{(L)}=\chi_0$ and ii) holds.

We remark that (2B)i) and (1.1) imply that

$$R_L^G(\theta)=\sum_{(T)\subset L}\frac{1}{|W_L(T)|}R_T^G(\theta_{(T)})$$

for any θ in $X(L)$. Thus the R_L^G are determined by the R_T^G. The following is the theorem of Curtis type.

Theorem (2C). *Let $x\in G$, and let L be a regular subgroup of G containing $C(s)$, where s is the semisimple part of x. Then*

$$\chi(x)=\sum_{\theta\in\hat{L}}(\chi, R_L^G(\theta))\,\theta(x)$$

for all $\chi\in\hat{G}$.

Proof: We need only show $\chi_{(L)}(x)=\chi(x)$ since

$$\chi_{(L)}=\sum_{\theta\in\hat{L}}(\chi_{(L)},\theta)\,\theta=\sum_{\theta\in\hat{L}}(\chi, R_L^G(\theta))\,\theta.$$

The character formula (1.10) implies that

$$R_T^L(\chi_{(T)})\,(x)=\sum_{\substack{(S)\subset C(s)\\ S\sim_L T}}\frac{|W_L(S)|}{|W_{C(s)}(S)|}\chi_{(S)}(s)\,Q_S^{C(s)}(u),$$

where u is the unipotent part of x. It follows by (2B) that

$$\begin{aligned}\chi_{(L)}(x)&=\sum_{(T)\subset C(s)}\frac{1}{|W_{C(s)}(T)|}\chi_{(T)}(s)\,Q_T^{C(s)}(u)\\ &=\sum_{(T)\subset C(s)}\frac{1}{|W_{C(s)}(T)|}\sum_{v\in C(s)_u}\frac{|T|}{|C(s)|}\chi(sv)\,Q_T^{C(s)}(v)\,Q_T^{C(s)}(u),\end{aligned}$$

the last equation following by (2A). If we interchange the order of summation and use the second orthogonality relation (1.8) for Green functions, we find that

$$\chi_{(L)}(x)=\sum_{v\in u^{C(s)}}\chi(sv)\frac{|C(sv)|}{|C(s)|}=\chi(su)$$

as desired.

Remark: (2C) was proved by Curtis [5], Theorem A, for an arbitrary $\bar{G}$ in the case L is subparabolic and χ is unipotent. A proof of (2C) for arbitrary $\bar{G}$ has been communicated to us by Lusztig.

We will need some facts on characters of symmetric groups. Let ϕ_λ and ϕ_τ be irreducible characters of S_l and S_t corresponding to the partitions λ and τ of l and t respectively. We shall write $\lambda \vdash l$ for a partition. The product $\phi_\lambda \phi_\tau$ is a character of $S_l \times S_t$. Let $\phi_\lambda \circ \phi_\tau$ be the character of S_{l+t} induced from $\phi_\lambda \phi_\tau$ on the subgroup $S_l \times S_t$ of S_{l+t}. The $\circ$ operation is associative and extends by linearity to an operation on class functions of S_l and S_t. Let Φ_t be the generalized character of S_t defined by

$$\Phi_t = \phi_{\{t\}} - \phi_{\{t-1,1\}} + \phi_{\{t-2,1^2\}} - \dots (-1)^{t-1} \phi_{\{1^t\}},$$

where the subscripts are the hook partitions of t. The sign of ϕ_τ in Φ_t is $(-1)^{l_\tau}$ where l_τ is the leg length of τ. The following two results are well-known. The first is a special case of the Littlewood-Richardson rule [13], 16.4; the second is the Murnaghan-Nakayama Formula [13], 21.1. In the formulas here and elsewhere, empty sums are interpreted as 0, and $\phi_{\{-\}} = 1$ if $\{-\}$ is the empty partition.

Proposition (2D). *Let $n = l + t$, and let $\lambda \vdash l$. Then*

$$\phi_\lambda \circ \Phi_t = \sum_\nu (-1)^{l_{\nu\lambda}} \phi_\nu,$$

where ν runs over all $\nu \vdash n$ gotten from λ by adding a t-hook, and $l_{\nu\lambda}$ is the leg length of the added hook.

Proposition (2E). *Let $n = l + t$, and let $\rho\sigma \in S_n$, where ρ is a t-cycle and σ is a permutation on the remaining l symbols. Let $\nu \vdash n$. Then*

$$\phi_\nu(\rho\sigma) = \sum_\lambda (-1)^{l_{\nu\lambda}} \phi_\lambda(\sigma),$$

where λ runs over all $\lambda \vdash l$ gotten from ν by deleting a t-hook, and $l_{\nu\lambda}$ is the leg length of the deleted hook.

We require generalizations of the preceding results. Let t be a fixed positive integer, and let $n = l + mt$, where $l \geqq 0$, $m \geqq 0$. If $\mu = \{1^{r_1} 2^{r_2} \dots k^{r_k}\}$ is a partition of m, let $\bar{\mu}$ denote the partition $\{t^{r_1} (2t)^{r_2} \dots (kt)^{r_k}\}$ of mt. We define $\Phi_{\bar{\mu}}$ by

$$\Phi_{\bar{\mu}} = \Phi_t^{r_1} \circ \Phi_{2t}^{r_2} \circ \dots \circ \Phi_{kt}^{r_k},$$

where $\Phi_{it}^{r_i}$ is the r_i-fold product $\Phi_{it} \circ \Phi_{it} \circ \dots \circ \Phi_{it}$. Given $\nu \vdash n$, we define $\mathscr{L}_\nu$ to be the set of $\lambda \vdash l$ gotten from ν by deleting a sequence of m t-hooks from ν. For each pair (ν, λ), where $\nu \vdash n$ and $\lambda \in \mathscr{L}_\nu$, Robinson and Farahat [8] have defined a character $\phi_{\nu/\lambda}$ of S_m and a sign $\varepsilon_{\nu\lambda}$ such that the following hold:

Let σ be a permutation on l symbols, ρ a permutation on the remaining mt symbols. Suppose the cycle type of ρ is $\bar{\mu}$, where $\mu \vdash m$. Then

$$\phi_\nu(\rho\sigma) = \sum_{\lambda \in \mathscr{L}_\nu} \varepsilon_{\nu\lambda} \phi_{\nu/\lambda}(\mu) \phi_\lambda(\sigma). \tag{2.3}$$

Farahat's theorem [8], page 312, is stated only for the case t is a prime p and σ is a p'-element. But neither hypothesis is used in his proof.

Lemma (2F). *Let the notation be as above, let $n=l+mt$, $\lambda\vdash l$, and $\mu\vdash m$. Then the following hold:*

i) $$\phi_\lambda\circ\Phi_{\bar\mu}=\sum_{\substack{\nu\vdash n\\ \lambda\in\mathscr{L}_\nu}}\varepsilon_{\nu\lambda}\,\phi_{\nu/\lambda}(\mu)\,\phi_\nu$$

ii) $$\Phi_{\bar\mu}=\sum_{\substack{\nu\vdash n\\ \{-\}\in\mathscr{L}_\nu}}\phi_\nu(\bar\mu)\,\phi_\nu$$

Proof. We have by (2D) that

(2.4) $$\phi_\lambda\circ\Phi_{\bar\mu}=\sum_{\nu\vdash n}C_{\nu\lambda}(\bar\mu)\,\phi_\nu.$$

Here ν runs over all partitions of n such that a sequence S of r_1 t-hooks, r_2 $(2t)$-hooks, ... can be successively added to λ to yield ν, and $C_{\nu\lambda}(\bar\mu)=\sum_S(-1)^{l_S}$, where S runs over all such sequences and l_S is the sum of the leg lengths of the hooks of S. If such sequences S linking λ and ν exist, then $\lambda\in\mathscr{L}_\nu$ by [18], I, Paragraph 5. Let σ be a permutation on l symbols, and ρ a permutation of cycle type $\bar\mu$ on the remaining mt symbols. Then

(2.5) $$\phi_\nu(\rho\sigma)=\sum_{\lambda\in\mathscr{L}_\nu}C_{\nu\lambda}(\bar\mu)\,\phi_\lambda(\sigma)$$

by (2D) and (2E). Hence i) follows by (2.3) and (2.5). Suppose $n=mt$ and ν is a partition of n such that $\mathscr{L}_\nu=\{-\}$. Then (2.5) becomes $\phi_\nu(\bar\mu)=C_{\nu,\{-\}}(\bar\mu)$, and ii) follows from (2.4).

We now come to the theorems of Murnaghan-Nakayama type for G. Let $\prod_\Gamma g_\Gamma$ be the primary decomposition of the element $g\in G$, and $\sum_\Gamma V_\Gamma$ the corresponding decomposition of the underlying space V of G. We fix an elementary divisor Λ of g and set

$$V_0=\sum_{\Gamma\neq\Lambda}V_\Gamma,\qquad V_1=V_\Lambda$$

$$\sigma=\prod_{\Gamma\neq\Lambda}g_\Gamma,\qquad \rho=g_\Lambda$$

Thus σ and ρ are respectively elements of $G_0=G(V_0)$ and $G_1=G(V_1)$. Let $t=d_\Lambda$, $m=m_\Lambda(\rho_s)$, where ρ_s is the semisimple part of ρ. Then $C_G(\rho_s)=H=H_0\times H_1$, where $H_0=G_0$, $H_1=C_{G_1}(\rho_s)$, and $H_1\simeq GL(m,q^t)$ or $U(m,q^{2t})$ according as Λ is of linear or unitary type. In particular, W_{H_1} is isomorphic to the symmetric group S_m. Finally let l be the dimension of V_0, so that $n=l+mt$. We shall continue with the notation used in (2F). The maximal tori of H_1, when viewed as maximal tori of G_1, correspond to partitions $\bar\mu\vdash mt$, where $\mu\vdash m$. Thus tori of G_1 of the form $T_{\bar\mu}$ can be taken as representations for the H_1-conjugacy classes of maximal tori of H_1.

Theorem (2G). *Let ρ, σ, H_1, and the notation be as above. Let $\nu \vdash n$. Then*

$$\chi^\nu(\rho\sigma) = \sum_{\lambda \in \mathscr{L}_\nu} a^\rho_{\nu\lambda} \chi^\lambda(\sigma),$$

where $\mathscr{L}_\nu$ is the set of partitions obtained from ν by removing a sequence of m t-hooks,

$$a^\rho_{\nu\lambda} = \varepsilon_{\nu\lambda} \sum_{\mu \vdash m} \frac{1}{|W_{H_1}(T_{\bar{\mu}})|} Q^{H_1}_{T\bar{\mu}}(\rho_u)\, \phi_{\nu|\lambda}(\mu),$$

and ρ_u is the unipotent part of ρ. If H_1 is linear, each $a^\rho_{\nu\lambda}$ is a polynomial in $Z[q^t]$ whose non-zero coefficients have the same sign, and in particular, $a^\rho_{\nu\lambda} \neq 0$ for $\lambda \in \mathscr{L}_\nu$.

Proof. By definition of $\rho\sigma$, we have $C(\rho\sigma) \leqq C(\rho) \leqq H$, so by (2C)

(2.6) $$\chi^\nu(\rho\sigma) = \sum_{\theta \in \hat{H}} (\chi^\nu, R^G_H(\theta))\, \theta(\rho\sigma).$$

Here the sum need be taken only over $\theta \in \hat{H}^u$ since χ_ν is unipotent. Such a θ is of the form $\theta = \chi_\lambda \zeta$, where χ_λ is the unipotent character of $H_0 = G_0$ corresponding to $\lambda \vdash l$ and $\zeta \in \hat{H}^u_1$. We may replace χ_λ by χ^λ in θ without affecting (2.6). Moreover, $\zeta(\rho) = \zeta(\rho_u)$ since $\rho_s \in Z(H)$. Thus

(2.7) $$\chi^\nu(\rho\sigma) = \sum_{\lambda \vdash l} (\chi^\nu, R^G_H(\chi^\lambda \times \sum_{\zeta \in \hat{H}^u_1} \zeta(\rho_u)\, \zeta))\, \chi^\lambda(\sigma).$$

Let $a^\rho_{\nu\lambda}$ be the coefficient of $\chi^\lambda(\sigma)$ in the right-hand side of (2.7). Then by (1.1)

(2.8) $$a^\rho_{\nu\lambda} = (\chi^\nu, R^G_{G_0 G_1}(\chi^\lambda \times R^{G_1}_{H_1}(\sum_\zeta \zeta(\rho_u)\, \zeta)))$$

We compute $R^{G_1}_{H_1}(\sum_\zeta \zeta(\rho_u)\, \zeta)$. Since we may replace ζ by $-\zeta$ without affecting the expression, we shall take the ζ's to be the χ^ϕ, where ϕ runs over the characters in $\hat{W}_{H_1}$. By (1.13) applied to H_1, we have

$$\chi^\phi = \sum_{\mu \vdash m} \frac{1}{|W_{H_1}(T_{\bar{\mu}})|} \phi(\mu)\, R^{H_1}_{T\bar{\mu}}(1),$$

and in particular,

$$\chi^\phi(\rho_u) = \sum_{\mu \vdash m} \frac{1}{|W_{H_1}(T_{\bar{\mu}})|} \phi(\mu)\, Q^{H_1}_{T\bar{\mu}}(\rho_u).$$

Using the orthogonality relations for the characters of W_{H_1}, (1.11) applied to H_1, and (1.1), we find that

(2.9) $$R^{G_1}_{H_1}(\sum_\zeta \zeta(\rho_u)\, \zeta) = \sum_{\mu \vdash m} \frac{1}{|W_{H_1}(T_{\bar{\mu}})|} Q^{H_1}_{T\bar{\mu}}(\rho_u)\, R^{G_1}_{T\bar{\mu}}(1)$$

But $R^{G_1}_{T\bar{\mu}}(1) = \sum_{\alpha \vdash mt} \phi_\alpha(\bar{\mu})\, \chi^\alpha$ by (1.14) applied to G_1. Here α need run only over partitions of mt from which a sequence of m t-hooks can be removed. Thus by

(1C) and (2F) we have

$$(2.10)\qquad (\chi^\nu, R^G_{G_0G_1}(\chi^\lambda \times R^{G_1}_{T_{\bar\mu}}(1)) = (\phi_\nu, \phi_\lambda \circ \sum \phi_\alpha(\bar\mu)\,\phi_\alpha) = (\phi_\nu, \phi_\lambda \circ \Phi_{\bar\mu}) = \varepsilon_{\nu\lambda}\,\phi_{\nu/\lambda}(\mu)$$

The first part now follows by (2.8) and (2.9).

For simplicity of notation, we set

$$(2.11)\qquad b^\rho_\mu = Q^{H_1}_{T_{\bar\mu}}(\rho_u).$$

If H_1 is linear, the b^ρ_μ are known to be polynomials in $Z[q^t]$. Moreover, if $b^\rho_\mu = \sum_i b^\rho_{\mu i} q^{it}$, then the function f^ρ_i: $\mu \to b^\rho_{\mu i}$ is a character of S_m (see [11], page 122). Since

$$a^\rho_{\nu\lambda} = \varepsilon_{\nu\lambda} \sum_i (f^\rho_i, \phi_{\nu/\lambda})\, q^{it},$$

the second part of (2G) follows.

Remark. The b^ρ_μ are also polynomials in $Z[q^t]$ in any of the following cases: i) ρ is semisimple, ii) p and q are sufficiently large. Indeed, in i) $b^\rho_\mu = Q^{G_1}_{T_{\bar\mu}}(1)$, and in ii) Ennola's conjecture holds, i.e. the Green functions for $U(n,q^2)$ are polynomials in q obtained from the corresponding polynomials for $GL(n,q)$ by changing q to $-q$ (see [11]). The functions f^ρ_i in these cases are then generalized characters of S_m, so the $a^\rho_{\nu\lambda}$ are then polynomials in $Z[q^t]$.

We note several consequences of the proof of (2G) which will be needed in Paragraphs 6 and 7. Again, with the notation of (2G), we have

$$(2.12)\qquad (\chi^\nu, R^G_{H_0H_1}(\chi^\lambda\chi^\mu)) = \varepsilon_{\nu\lambda}(\phi_{\nu/\lambda}, \phi_\mu).$$

This follows from (2.10), since $\chi^\mu = \sum_{\kappa \vdash m} \frac{1}{|W_{H_1}(T_\kappa)|}\,\phi_\mu(\kappa)\, R^{H_1}_{T_\kappa}(1)$ and so

$$R^G_{H_0H_1}(\chi^\lambda\chi^\mu) = R^G_{G_0G_1}\Big(\chi^\lambda \times \sum_{\kappa \vdash m} \frac{1}{|W_{H_1}(T_\kappa)|}\,\phi_\mu(\kappa)\, R^{G_1}_{T_{\bar\kappa}}(1)\Big),$$

In particular, if the left-hand side of (2.12) is non-zero, then λ can be obtained from ν by deleting a sequence of m t-hooks.

This last remark can easily be extended to cover the case when ρ is the product of several primary factors and $C_G(\rho_s) = H$ has the form $H_0 \times H_1 \times \ldots \times H_k$, where $H_0 = G_0$ as before and H_i is isomorphic to $GL(m_i, q^{t_i})$ or $U(m_i, q^{2t_i})$ for $i \geqq 1$. Let $\nu \vdash n$, $\lambda \vdash l$, and $\mu_i \vdash m_i$ for $1 \leqq i \leqq k$. If $(\chi^\nu, R^G_H(\chi^\lambda\chi^{\mu_1}\ldots\chi^{\mu_k})) \neq 0$, then λ can be obtained from ν by deleting a sequence of m_k t_k-hooks, a sequence of m_{k-1} t_{k-1}-hooks, ..., and finally a sequence of m_1 t_1-hooks. This follows by induction on k.

Before stating the next theorem of Murnaghan-Nakayama type, we note the following lemma.

Lemma (2H). *Let χ be an irreducible character of G in the geometric conjugacy class s_G, and let $t \in Z(G)$. Then*

$$\chi(tg) = \hat{s}(t)\,\chi(g)$$

for all $g \in G$.

Proof. It suffices to show this for a class functions χ of the form $\chi^{s,\nu}$. Let $K = C(s)$ and apply (2B) to χ^ν with K in the place of G. Then

$$\chi^\nu = \sum_{(T) \subset K} \frac{1}{|W_K(T)|} R_T^K(\chi_{(T)}^\nu).$$

Since $s \in Z(K)$, it follows by (1.1) that

$$\chi^{s,\nu} = R_K^G(\hat{s}\chi^\nu) = \sum_{(T) \subset K} \frac{1}{|W_K(T)|} R_T^G(\hat{s}\chi_{(T)}^\nu).$$

But

$$R_T^G(\hat{s}\theta)(tg) = \hat{s}(t)\, R_T^G(\hat{s}\theta)(g)$$

for all $\theta \in \hat{T}$ by the character formula (1.9). Thus $\chi(tg) = \hat{s}(t)\,\chi(g)$ as claimed.

We will need an extension of (2G) which applies to an arbitrary $\chi \in \hat{G}$ under the additional hypotheses that the primary component ρ is semisimple and the elementary divisor Λ has degree $t=1$. We continue with the notation of (2G). Let $\chi = \chi_{s,\nu}$, and let s_α run over representatives for the H-conjugacy classes contained in $s^G \cap H$. By (1B) and (2C)

$$\chi^{s,\nu}(\rho\sigma) = \sum_\alpha \chi^{(s_\alpha)}(\rho\sigma),$$

where $\chi^{(s_\alpha)}$ is defined by

$$\chi^{(s_\alpha)}(\rho\sigma) = \sum_{\theta \in (s_\alpha)_H} (\chi^{s,\nu}, R_H^G(\theta))\,\theta(\rho\sigma). \tag{2.13}$$

For fixed α, let $s_\alpha = s_{\alpha 0} s_{\alpha 1}$, where $s_{\alpha i} \in H_i$. Let $\prod_\Gamma (s_{\alpha 1})_\Gamma$ be the primary decomposition of $s_{\alpha 1}$ as an element of G_1, and let $\prod_\Gamma \rho_\Gamma$ be the corresponding decomposition of ρ as an element of $C_{G_1}(s_{\alpha 1})$. The hypotheses on ρ and Λ imply that ρ_Γ is primary as an element of $C_{G_1}(s_{\alpha 1})_\Gamma$, its elementary divisor has degree 1, and $G_0 = H_0$, $G_1 = H_1$.

Theorem (2I). *Let the notation be as above. Then*

$$\chi^{(s_\alpha)}(\rho\sigma) = \hat{s}_\alpha(\rho) \sum_\lambda A_{\nu\lambda}^\alpha\, \chi^{s_{\alpha 0},\lambda}(\sigma)$$

where λ runs over products $\prod_\Gamma \lambda_\Gamma$ with $\lambda_\Gamma \vdash m_\Gamma(s_{\alpha 0})$. The coefficient $A_{\nu\lambda}^\alpha$ is given by

$$A_{\nu\lambda}^\alpha = |G_1 : C_{G_1}(s_{\alpha 1})|_{p'} \prod_\Gamma a_{\nu_\Gamma \lambda_\Gamma}^{\rho_\Gamma} \tag{2.14}$$

where the $a^{\rho_\Gamma}_{\nu_\Gamma \lambda_\Gamma}$ are the coefficients $a^{\rho}_{\nu\lambda}$ occuring in (2G) *when applied to the class function χ^{ν_Γ} of $C_G(s_\alpha)_\Gamma$ and the primary component ρ_Γ. In particular, λ_Γ is a partition of $m_\Gamma(s_{\alpha 0})$ obtained from ν_Γ by deleting a sequence of* 1*-hooks.*

Proof. We may take $s_\alpha = s$ without loss of generality. The characters θ in s_H have form $\theta_0 \theta_1$, where $\theta_i \in (s_i)_{H_i}$. Now s_0 and s_1, considered as elements of H_0 and H_1 respectively, have elementary divisors in the same set $\mathscr{F}$ used for elementary divisors of elements of G. Thus

$$\theta_0 = \chi_{s_0, \lambda}, \ \theta_1 = \chi_{s_1, \mu},$$

where $\lambda = \prod_\Gamma \lambda_\Gamma$, $\mu = \prod_\Gamma \mu_\Gamma$, $\lambda_\Gamma \vdash m_\Gamma(s_0)$, and $\mu_\Gamma \vdash m_\Gamma(s_1)$. Since (2.13) is unaffected when θ_i is replaced by $-\theta_i$, we may rewrite (2.13) using (2H) as

$$\chi^{(s)}(\rho\sigma) = \hat{s}(\rho) \sum_\lambda (\chi^{s,\nu}, R^G_H(\chi^{s_0,\lambda} \times \sum_\mu \chi^{s_1,\mu}(1)\, \chi^{s_1,\mu}))\, \chi^{s_0,\lambda}(\sigma).$$

Since $H = G_0 G_1$, we have by (1B) that

$$\chi^{(s)}(\rho\sigma) = \hat{s}(\rho) \sum_\lambda (\chi^{\nu}, R^K_{K_0 K_1}(\chi^{\lambda} \times \sum_\mu \chi^{s_1,\mu}(1)\, \chi^{\mu}))\, \chi^{s_0,\lambda}(\sigma),$$

where $K = C_G(s)$ and $K_i = C_{G_i}(s_i)$. Now

$$\chi^{s_1,\mu}(1) = |G_1 : C_{G_1}(s_1)|_{p'}\, \chi^{\mu}(1).$$

Thus

$$\chi^{(s)}(\rho\sigma) = \hat{s}(\rho)\, |G_1 : C_{G_1}(s_1)|_{p'} \sum_\lambda (\chi^{\nu}, R^K_{K_0 K_1}(\chi^{\lambda} \times \sum_\mu \chi^{\mu}(1)\, \chi^{\mu}))\, \chi^{s_0,\lambda}(\sigma)$$

But

$$(\chi^{\nu}, R^K_{K_0 K_1}(\chi^{\lambda} \times \sum_\mu \chi^{\mu}(1)\, \chi^{\mu})) = \prod_\Gamma (\chi^{\nu_\Gamma}, R^{K_\Gamma}_{(K_0 K_1)_\Gamma}(\chi^{\lambda_\Gamma} \times \sum_{\mu_\Gamma} \chi^{\mu_\Gamma}(1)\, \chi^{\mu_\Gamma}))$$

so (2I) holds by (2.8).

Remark. It is possible to state and prove (2I) without the hypotheses on ρ and Λ, but the weaker statement is sufficient in the applications.

§3

We return to the convention that G is $GL(n,q)$ or $U(n,q)$. Let r be an odd prime distinct from p. Let v be the exponential valuation of Z associated to r, normalized so that $v(r)=1$. If H is a finite group, we write $v(H)$ for $v(|H|)$. Let e be the order of q modulo r, and let $a = v(q^e - 1)$. The integers e and a will have this meaning for the rest of this paper. If $q = q_0^2$, we let e_0 be the order of q_0 modulo r. In particular, $e_0 = e$ if r divides $q_0^e - 1$, and $e_0 = 2e$ if r divides $q_0^e + 1$.

Lemma (3A). *Let v, e, a be as above, and let i be a positive integer. Then the following hold:*

i) *$v(q^i - 1) > 0$ if and only if e divides i, and if so, then $v(q^i - 1) = a + v(i)$.*

ii) *$v(q_0^i + 1) > 0$ if and only if e divides i, but e_0 does not divide i; and if so, then $v(q_0^i + 1) = a + v(i)$.*

Proof. This is immediate from the definitions and the hypothesis that r is odd. We note as a consequence that if G contains elements of order r, then G contains elements of order r^a.

We define two basic configurations which are essential constituents in the classification of blocks. For each non-negative integer α, let Λ_α be a polynomial in $\mathscr{F}$ having a primitive $r^{a+\alpha}$-th root of unity as root. The degree and reduced degree of Λ_α, defined in Paragraph 1, depend only on α and not on the choice of Λ_α; we denote them by d_α and δ_α respectively. Then d_α is the minimal dimension that the underlying space V of G must have in order that G contains elements of order $r^{a+\alpha}$, and $\delta_\alpha = er^\alpha$. In the unitary case, either $d_\alpha = \delta_\alpha$ for all α or $d_\alpha = 2\delta_\alpha$ for all α.

The first configuration arises as follows: Let m be a positive integer, and let $G^{m,\alpha}$ be $GL(md_\alpha, F)$ or $U(md_\alpha, F)$. Let (Λ_α) be the companion matrix of Λ_α. Replacing (Λ_α) by a conjugate if necessary, we may assume that $m(\Lambda_\alpha) \in G^{m,\alpha}$. The cyclic subgroup $R^{m,\alpha}$ of $G^{m,\alpha}$ generated by $m(\Lambda_\alpha)$ is then an r-subgroup of a Coxeter torus of $G^{m,\alpha}$, and $|R^{m,\alpha}| = r^{a+\alpha}$. $R^{m,\alpha}$ is a Sylow r-subgroup of this Coxeter torus if and only if $v(m) = 0$. We may replace $R^{m,\alpha}$ by conjugates under $G^{m,\alpha}$.

The second configuration arises as follows: Let β be a non-negative integer, S_β the group of permutation matrices of degree r^β, and X_β a Sylow r-subgroup of S_β. Here X_β may be taken as the wreath product $Z_r \wr \dots \wr Z_r$ of β copies of a cycle of order r. Let $G^{m,\alpha,\beta}$ be $GL(md_\alpha r^\beta, F)$ or $U(md_\alpha r^\beta, F)$, where $m \geqq 1$, $\alpha \geqq 0$. Then $R^{m,\alpha,\beta}$ is defined as the r-subgroup $R^{m,\alpha} \wr X_\beta$ of $G^{m,\alpha,\beta}$, and is a subgroup of the normalizer of r^β copies of a Coxeter torus of $G^{m,\alpha}$. Since $v(X_\beta) = v(r^\beta!)$, we have $v(R^{m,\alpha,\beta}) = (a+\alpha)\, r^\beta + v(r^\beta!)$, and $R^{m,\alpha,\beta}$ is a Sylow r-subgroup of this normalizer if and only if $v(m) = 0$. We may also replace $R^{m,\alpha,\beta}$ by conjugates under $G^{m,\alpha,\beta}$.

Proposition (3B). *Let $n = n_0 d_0 + l$, where n_0, l are non-negative integers with $0 \leqq l < d_0$. Let $n_0 = \sum c_\beta r^\beta$ be the r-adic expansion of n_0. Then*

$$I_l + \sum c_\beta R^{1,0,\beta}$$

is a Sylow r-subgroup of G. Here I_l is the identity matrix of degree l, and $c_\beta R^{1,0,\beta}$ is the direct product of c_β copies of $R^{1,0,\beta}$.

Proof. This is a result of Weir [23]. The symbols ε, e, q, q^2 in [22] correspond to our e, e_0, q_0, q.

Theorem (3C). *Let R be a defect group of an r-block of G. Then R is conjugate to*

$$\begin{pmatrix} R_0 & & & \\ & R_1 & & \\ & & \ddots & \\ & & & R_t \end{pmatrix} \tag{3.1}$$

where R_0 is an identity matrix of non-negative degree, and for $i \geqq 1$, there exist integers m_i, α_i, β_i such that $R_i = R^{m_i, \alpha_i, \beta_i}$.

Proof. We proceed by induction and assume $R \neq 1$. We may replace R by suitable conjugates in G at any stage if necessary.

Case I $\Omega_1(Z(R)) \leqq Z(G)$. Then $Z(R)$ is cyclic, $e=1$, and in the unitary case, r divides q_0+1. Now there exists an r'-element $g \in G$ such that R is a Sylow r-subgroup of $C_G(g)$. Thus R is a Sylow subgroup of $C_K(u)$, where $K=C_G(s)$ and s, u are the semisimple and unipotent parts of g. Let $\prod_\Gamma s_\Gamma$ be the primary decomposition of s, and $\prod_\Gamma K_\Gamma$ the corresponding decomposition of K. Now $Z(R) \geqq \prod_\Gamma O_r(Z(K_\Gamma))$, and $O_r(Z(K_\Gamma)) \neq 1$ if $K_\Gamma \neq 1$. Since $Z(R)$ is cyclic, it follows there is some Γ such that $s=s_\Gamma$ and $K=K_\Gamma$. Hence $K=GL(m, F_\Gamma)$ or $U(m, F_\Gamma)$, where $m=m_\Gamma(s)$ and $F_\Gamma=F[(\Gamma)]$ as defined in Paragraph 1.

We consider u as a unipotent element of K. Replacing K if necessary by a suitable conjugate, we may assume u has the form $\sum m_i(\Phi^i)$, where Φ^i is the polynomial $(X-1)^i$ considered as an element in $F_\Gamma[X]$, and (Φ^i) is the companion matrix of Φ^i over F_Γ. The field F_Γ is embedded as $F_\Gamma \otimes I_i$ in the center of $F_\Gamma[(\Phi^i)]$, where I_i is the identity matrix of degree i. Moreover, the subgroup $\prod_i GL(m_i, F_\Gamma \otimes I_i)$ or $\prod_i U(m_i, F_\Gamma \otimes I_i)$ of $C_K(u)$ contains a Sylow r-subgroup of $C_K(u)$. Since $Z(R)$ is cyclic, (3B) then implies u has the form $r^\beta(\Phi^i)$ for some β and i, and thus $v(C_K(u))=v(GL(r^\beta, F_\Gamma \otimes I)$ or $v(U(r^\beta, F_\Gamma \otimes I))$, where $I=I_i$.

We recall from Paragraph 1 that F_Γ is isomorphic to an extension field E of degree δ_Γ over F, the isomorphism $\mathscr{I}$ from E to F_Γ being the regular representation $\mathscr{R}$ of E over F if $d_\Gamma=\delta_\Gamma$, and $\mathscr{R} \oplus \mathscr{R}^J$ if $d_\Gamma=2\delta_\Gamma$. Let $v(\delta_\Gamma)=\alpha$, let E_r be the subfield of E of degree r^α over F, and let $F_{\Gamma,r}$ be the corresponding subalgebra of F_Γ. The restriction of $\mathscr{I}$ to E_r is then the direct sum of v copies of $\mathscr{R}_r$, where $v=d_\Gamma r^{-\alpha}$ and $\mathscr{R}_r$ is the regular representation of E_r over F. It now follows by (3A), (3B) that the Sylow subgroup $R^{vi,\alpha,\beta}$ of $GL(r^\beta v, F_{\Gamma,r} \otimes I)$ or $U(r^\beta v, F_{\Gamma,r} \otimes I)$ is a Sylow subgroup of $C_K(u)$, so (3C) holds.

Case II $\Omega_1(Z(R)) \nleqq Z(G)$. Let x be an element of order r in $Z(R)$, $x \notin Z(G)$. Let $\prod_\Gamma x_\Gamma$ be the primary decomposition of x, and $\prod C_\Gamma$ the corresponding decomposition of $C=C_G(x)$. For $\Gamma \neq X-1$, the C_Γ are either all linear groups or all unitary groups. Since $x \in Z(R)$, there exists a block b of C with defect group R such that $Br_C^G(b)=B$, where Br_C^G is the Brauer mapping. Let $b=\prod_\Gamma b_\Gamma$ and $R=\prod_\Gamma R_\Gamma$ be the corresponding decompositions of b and R, so b_Γ is then a block of C_Γ with defect group R_Γ. Since $C<G$, induction applies to each R_Γ. Let Ψ_α play the role with respect to $F_\Gamma[X]$ that Λ_α does with respect to $F[X]$. Then R_Γ is conjugate in C_Γ to a direct sum of an identity matrix, possibly zero, and matrix groups $S^{m,\alpha,\beta}$, where $S^{m,\alpha,\beta}=R^{m,\alpha,\beta}$ if $\Gamma=X-1$, and $S^{m,\alpha,\beta}=\langle m(\Psi_\alpha)\rangle \wr X_\beta$ if $\Gamma \neq X-1$. Here (Ψ_α) is the companion matrix of Ψ_α over F_Γ. We may view (Ψ_α) as a matrix of degree $r^\alpha d_0$ over F, since $F_\Gamma=F[(\Gamma)]$, and as such, (Ψ_α) is conjugate to (Λ_α). Thus $S^{m,\alpha,\beta}$, when viewed as a matrix group over F, is conjugate to $R^{m,\alpha,\beta}$. This completes the proof.

Remark. We shall write (3.1) as $R_0 \prod_i^t R_i$. (3C) was essentially proved by Olsson [19], Prop. 1.5 for the linear case, but the statement of his result omits some groups.

We next consider normalizers and centralizers of defect groups. Let $N^{m,\alpha}$ and $C^{m,\alpha}$ be the normalizer and centralizer of $R^{m,\alpha}$ in $G^{m,\alpha}$. Similarly, let $N^{m,\alpha,\beta}$ and $C^{m,\alpha,\beta}$ be the normalizer and centralizer of $R^{m,\alpha,\beta}$ in $G^{m,\alpha,\beta}$. We use the following notation: Suppose N is a matrix group of degree n, R is a normal subgroup of N, and Y is a permutation matrix group of degree r^β. Given $g \in N$ and $y \in Y$, let $gR \otimes y$ denote the set of matrices obtained from y by replacing the entries 1 in y by arbitrary elements in the coset gR, and the entries 0 in y by the zero matrix of degree n. We define

$$N/R \otimes Y = \bigcup_{\substack{g \in N \\ y \in Y}} (gR \otimes y).$$

Proposition (3D). *Let the notation be as above. Then the following hold:*

i) $N^{m,\alpha}/C^{m,\alpha}$ *is cyclic of order* d_α

ii) $N^{m,\alpha,\beta} = (N^{m,\alpha}/R^{m,\alpha}) \otimes Y_\beta$ *and* $C^{m,\alpha,\beta} = C^{m,\alpha} \otimes I_\beta$, *where* $Y_\beta = N_{S_\beta}(X_\beta)$ *and* I_β *is the identity element of* S_β.

Proof. i) follows from the description of conjugacy classes in $G^{m,\alpha}$, since the generator $m(\Lambda_\alpha)$ of $R^{m,\alpha}$ is conjugate to $(m(\Lambda_\alpha))^q$ in the linear case, and to $(m(\Lambda_\alpha))^{-q_0}$ in the unitary case.

As a wreath product, $R^{m,\alpha,\beta} = R^{m,\alpha} \wr X_\beta$ has a base subgroup A of the form $\prod A_i$, where A_i is a copy of $R^{m,\alpha}$. Let $\sum V_i$ be the corresponding decomposition of the underlying space. If $a_i \in A_i^\#$, then $[V_i, a_i] = V_i$ and $[V_j, a_i] = 0$ for $i \neq j$ by the definition of $R^{m,\alpha}$. In particular, the elements in $\bigcup_i A_i^\#$ are characterized among the elements $a \in A^\#$ by the property that $\dim[V, a] = \dim V_1$. Since $R^{m,\alpha}$ is cyclic and r is odd, A is the unique normal abelian subgroup of its order in $R^{m,\alpha,\beta}$ by a theorem of Alperin, [1] Theorem 2. Hence $N^{m,\alpha,\beta}$ normalizes A, and $N^{m,\alpha,\beta}$ then acts as a permutation group on the set $\{V_i\colon 1 \leqq i \leqq r^\beta\}$. Thus $N^{m,\alpha,\beta} \leqq G^{m,\alpha} \wr S_\beta$. Each $g \in G^{m,\alpha} \wr S_\beta$ can be viewed as the element obtained by replacing the entries 0 and 1 of a permutation matrix $\pi(g)$ in S_β by zero matrices and matrices in $G^{m,\alpha}$. So g normalizes $R^{m,\alpha,\beta}$ only if $\pi(g) \in Y_\beta$. But since X_β is a transitive subgroup of S_β, $g \in N^{m,\alpha,\beta}$ if and only if $g \in x R^{m,\alpha} \otimes \pi(g)$ with $x \in N^{m,\alpha}$ and $\pi(g) \in Y_\beta$. Thus $N^{m,\alpha,\beta} = N^{m,\alpha}/R^{m,\alpha} \otimes Y_\beta$. If $g \in C^{m,\alpha}$, then $\pi(g) = I_\beta$ since $g \in C(A_i)$ and $V_i = [V, A_i]$. It then follows easily that $C^{m,\alpha,\beta} = C^{m,\alpha} \otimes I_\beta$. This completes the proof.

We can now consider normalizers and centralizers in the general case. Let R be the subgroup of G given in (3.1), where $R_i = R^{m_i,\alpha_i,\beta_i}$ for $i \geqq 1$. Let V_i be the underlying space of R_i, and let $G_i = G(V_i)$.

Proposition (3E). *Let the notation be as above. Then* $N(R)$ *acts by conjugation as a permutation group on* $\Omega = \{R_0, R_1, \ldots, R_t\}$. *Let* $\Omega_0, \Omega_1, \ldots, \Omega_s$ *be the orbits, and suppose the* R_i *are numbered so that* $R_i \in \Omega_i$ *for* $0 \leqq i \leqq s$. *Then*

i) $\Omega_0 = \{R_0\}$, *and for* $i, j \geqq 1$, R_i *and* R_j *are in the same orbit if and only if* $m_i = m_j$, $\alpha_i = \alpha_j$, *and* $\beta_i = \beta_j$.

ii) $N(R)=G_0 \prod_{i=1}^{s} (N^{m_i,\alpha_i,\beta_i} \wr S(\Omega_i))$, *where* $S(\Omega_i)$ *is the symmetric group on* Ω_i.

iii) $C(R)=G_0 \prod_{i=1}^{t} C_i$, *where* $C_i=C_{G_i}(R_i)=C^{m_i,\alpha_i,\beta_i}$.

Proof. Let $V_a=[V,a]$ for $a\in Z(R)^{\#}$, and let $\mathscr{V}=\{V_a\colon a\in Z(R)^{\#}\}$. We partially order $\mathscr{V}$ by inclusion. The elements in $\bigcup_i Z(R_i)^{\#}$ are then characterized among the elements a in $Z(R)^{\#}$ by the property that V_a is minimal in the partial ordering on $\mathscr{V}$, since $[V_i,a]=V_i$ for $a\in Z(R_i)^{\#}$ and since $[V_i,G_j]=0$ for $i\neq j$. Thus $N(R)$ induces a permutation action on $\{V_1,\dots,V_t\}$. Also, $N(R)$ fixes V_0 since $V_0=C_V(R)$. Thus $N(R)$ acts as a permutation group on $\{V_0,V_1,\dots,V_t\}$.

Suppose $g\in N(R)$ and $V_j=V_i g$ for $i,j\geqq 1$. Then $V_j=[V,R_i]\,g=[V,R_i^g]$ so that $R_i^g\leqq R_j$. A similar argument with $V_i=V_j g^{-1}$ shows $R_j^{g^{-1}}\leqq R_i$. Thus $R_i^g=R_j$, and $N(R)$ is represented as a permutation group on Ω. Here we can add R_0 to the set, since $N(R)$ fixes the subgroup R_0. The two permutation representations of $N(R)$ are equivalent if V_i is identified with R_i for $0\leqq i\leqq t$. If R_i and R_j are in the same orbit of $N(R)$ and $i,j\geqq 1$, then their base subgroups are conjugate by the forementioned theorem of Alperin, from which it follows $m_i=m_j$, $\alpha_i=\alpha_j$, and $\beta_i=\beta_j$. Conversely, if R_i and R_j are so related in the linear case, then there exists $g\in N(R)$ such that $R_i^g=R_j$. This is also true in the unitary case since V_i and V_j are then non-singular subspaces of the same dimension. In both the linear and the unitary case g can be chosen so that it acts as a transposition on Ω. But $S(\Omega_i)$ is generated by its transpositions. Thus i), ii) hold. Finally, $C(R)\leqq C(R_i)$ for all i, so $C(R)\leqq \prod_{i=0}^{t} G_i$. Thus $C(R)=\prod_{i=0}^{t} C_{G_i}(R_i)$ and iii) holds.

§4

We consider blocks of $G^{m,\alpha}$ with defect group $R^{m,\alpha}$ in this section. Quantities associated with the configuration $G^{m,\alpha}$ will need the superscripts m,α in later sections, but we shall drop these superscripts in this section. Let m and α be fixed. We set $G=G^{m,\alpha}$, $R=R^{m,\alpha}$, $N=N^{m,\alpha}$, and $C=C^{m,\alpha}$.

Proposition (4A). *Let* G, R, C *be as above. The following hold:*

i) *If* Λ_α *is of linear type, then* $C=GL(m,F_{\Lambda_\alpha})$.

ii) *If* Λ_α *is of unitary type, then* $C=U(m,F_{\Lambda_\alpha})$.

In each case, F_{Λ_α} *is an extension field of degree* $\delta_\alpha=er^\alpha$ *over* F, *represented as a matrix algebra of degree* d_α, *and* R *is a Sylow subgroup of* $Z(C)$.

Proof. This is essentially a restatement of (1A).

Let $T=T^{m,\alpha}$ be a Coxeter torus of G containing R. Then

$$T=C_G(T)\leqq C,\ N_G(T)=T\langle\sigma\rangle,$$

where $\sigma=\sigma^{m,\alpha}$ acts on T by $t\mapsto t^q$ in the linear case and by $t\mapsto t^{-q_0}$ in the unitary case. In each case σ may be chosen so that $\langle\sigma\rangle\cap T=1$. In particular,

$$|N_G(T)\colon T|=|\langle\sigma\rangle|=md_\alpha.$$

Since R is characteristic in T, σ normalizes R and induces an automorphism of order d_α of R. Hence by (3D), $N=C\langle\sigma\rangle$, $\sigma^{d_\alpha}\in C$, and $N_C(T)=T\langle\sigma^{d_\alpha}\rangle$. In particular,

$$|N_C(T)\colon T|=m.$$

As usual we shall say an element $t\in T$ is regular (with respect to C) if $C_C(t)=T$.

Proposition (4B). *C has blocks with defect group R if and only if $(r,m)=1$ and Coxeter tori of C contain regular elements.*

Proof. Suppose $(r,m)=1$ and s is a regular element in T. Since $R\leqq Z(C)$, the elements sy, where $y\in R$, are also regular elements of T. Thus $\theta_y=\pm R_T^C(\widehat{sy})$ is an irreducible character of C for all $y\in R$. The $r^{a+\alpha}$ characters θ_y obtained in this way have the same degree $d=|C\colon T|_{p'}$. Since R is a Sylow subgroup of T, we may assume s is an r'-element by replacing s by sy for some $y\in R$. The character formula (1.9) implies that θ_1 is r-rational, and that

$$\theta_y(g)=\begin{cases}\hat{y}(g_r)\,\theta_1(g_{r'}) & \text{if } g_r\in R\\ 0 & \text{if } g_r\notin R,\end{cases} \tag{4.1}$$

where $g_r, g_{r'}$ are the r-part, r'-part respectively of g. In particular, θ_1 is the unique r-rational character among the θ_y, and

$$\nu(\sum_y \theta_y(g))\geqq \nu(R)=\nu(T).$$

By (4.1) the coefficients of the idempotent

$$\frac{d}{|C|}\sum_{g\in G}\sum_{y\in R}\theta_y(g)\,g^{-1}$$

are $\mathfrak{r}$-local integers in an algebraic number field K for any prime ideal $\mathfrak{r}$ of K dividing r. This implies that $\mathfrak{b}=\{\theta_y\colon y\in R\}$ is a union of blocks $\mathfrak{b}_i$ of C. But the defect groups of every $\mathfrak{b}_i$ contain R. Since the θ_y have degree d and $\nu(d)=\nu(C\colon R)$, each $\mathfrak{b}_i$ in $\mathfrak{b}$ has defect group R. But $R\leqq Z(C)$, so by a theorem of Reynolds [20], Theorem 3, the following holds: $\mathfrak{b}_i$ contains exactly $|R|$ characters, all of the same degree. Moreover, $\mathfrak{b}_i$ contains a unique r-rational character which is the canonical character in $\mathfrak{b}_i$. Since $\mathfrak{b}$ contains only $|R|$ characters, $\mathfrak{b}$ is necessarily a single block of C with defect group R.

Conversely, suppose C has a block $\mathfrak{b}$ with defect group R. By Reynold's Theorem the characters in $\mathfrak{b}$ have the same degree d, and necessarily $\nu(d)=\nu(C\colon R)$. Express the canonical character θ of $\mathfrak{b}$ in the form

$$\theta=\pm R_L^C(\hat{s}\psi), \tag{4.2}$$

where s is a semisimple element of $C, L=C_C(s)$, and $\psi\in\hat{L}^u$. Then $d=|C\colon L|_{p'}\,\psi(1)$, so $\nu(\psi(1))=\nu(L\colon R)$. By a theorem of Ito [12], $\psi(1)$ divides $|L\colon Z(L)|$. Hence R is a Sylow subgroup of $Z(L)$. Let $\prod_\Delta s_\Delta$ be the primary decomposition of s as an element of C, and let $\prod_\Delta L_\Delta$ be the corresponding decomposition of L. Here Δ runs over appropriate polynomials in $F_{\Lambda_\alpha}[X]$. Let $m_\Delta(s)$, d_Δ, δ_Δ be the corresponding quantities for the situation where G is replaced by C. Then $m=\sum_\Delta m_\Delta(s)\, d_\Delta$, and each L_Δ is isomorphic to a linear or unitary group of degree $m_\Delta(s)$ over an extension field of degree δ_Δ over F_{Λ_α}. By (3A) $v(Z(L_\Delta)) \geqq a+\alpha+v(\delta_\Delta)$ for each non-trivial L_Δ. But since $v(Z(L))=a+\alpha$, there exists a unique Δ such that

$$L=L_\Delta,\ m=m_\Delta(s)\, d_\Delta,\ v(\delta_\Delta)=0. \tag{4.3}$$

Let λ be the partition of $m_\Delta(s)$ corresponding to ψ. The degree formula (1.15), applied to L, implies that λ consists of a single node, since $v(\psi(1))=v(L\colon Z(L))$. Thus $m_\Delta(s)=1$, so by (4.3), $C_C(s)$ is a Coxeter torus, $m=d_\Delta$, and $v(m)=0$. This completes the proof.

The preceding proof actually proves more than the statement of (4B). If $\theta_{\mathfrak{b}}$ is the canonical character of a block $\mathfrak{b}$ of C with defect group R, then $\theta_{\mathfrak{b}}= \pm R_T^G(\hat{s})$, where $T=C_G(s)$ and s is regular with respect to C. This follows from (4.2) since ψ is necessarily the trivial character of T. The first half of the proof shows $\mathfrak{b}=\{\pm R_T^C(\widehat{sy})\colon y\in R\}$. Now there exists $y\in R$ such that sy is an r'-element, so that $\pm R_T^C(\widehat{sy})$ is then an r-rational character. Since $\theta_{\mathfrak{b}}$ is the unique r-rational character in $\mathfrak{b}$, s must be an r'-element. Hence the mapping $\mathfrak{b}\mapsto s^C$ induces a bijection between the set $\mathscr{A}=\mathscr{A}^{m,\alpha}$ of blocks of C with defect group R and the set $\mathscr{S}=\mathscr{S}^{m,\alpha}$ of C-conjugacy classes of regular r'-elements of C. By its definition the bijection is preserved by automorphisms of C. We summarize these remarks in

Proposition (4C). *Suppose $(r,m)=1$. For each $\mathfrak{b}\in\mathscr{A}$, there exists a sign $\varepsilon_{\mathfrak{b}}$ and a class $(s_{\mathfrak{b}})^C$ in $\mathscr{S}$ such that*

$$\mathfrak{b}=\{\varepsilon_{\mathfrak{b}}\, R_T^C(\widehat{s_{\mathfrak{b}}y})\colon\ T=C_C(s_{\mathfrak{b}}),\ y\in R\}.$$

In particular, $\varepsilon_{\mathfrak{b}}\, R_T^C(\hat{s}_{\mathfrak{b}})$ is the canonical character of $\mathfrak{b}$. The mapping $\mathfrak{b}\mapsto(s_{\mathfrak{b}})^C$ is a bijection of $\mathscr{A}$ onto $\mathscr{S}$ preserved under automorphisms of C.

The group $N=N_G(R)$ acts on $\mathscr{A}$ by conjugation. Given $\mathfrak{b}\in\mathscr{A}$, let the stabilizer of $\mathfrak{b}$ be denoted by $N_{\mathfrak{b}}$, so that

$$N_{\mathfrak{b}}=\{g\in N\colon \mathfrak{b}^g=\mathfrak{b}\}=\{g\in N\colon \theta_{\mathfrak{b}}^g=\theta_{\mathfrak{b}}\}.$$

The second equality holds since $\mathfrak{b}$ contains a unique canonical character. The index $|N_{\mathfrak{b}}\colon C|$ is by definition the inertial index $e_{\mathfrak{b}}$ of $\mathfrak{b}$. We call $\mathfrak{b}$ a root block if $(r,e_{\mathfrak{b}})=1$. The connection with the term "root" in [4] (2J) is clear, since $\mathfrak{b}$ is a root block of C if and only if $Br_C^G(\mathfrak{b})$ is a block of G with defect group R.

Proposition (4D). *Let $\mathfrak{b}$ and $(s_{\mathfrak{b}})^C$ be corresponding elements in $\mathscr{A}$ and $\mathscr{S}$. Let $T=C_C(s_{\mathfrak{b}})$, and let $\Sigma=\langle\sigma\rangle$, where σ is the element of order md_α generating $N_G(T)$ over T. Then the following hold:*

i) $N_{\mathfrak{b}}=C\,C_\Sigma(s_{\mathfrak{b}})$, $N_{\mathfrak{b}}\cap\Sigma=(C\cap\Sigma)\times C_\Sigma(s_{\mathfrak{b}})$

ii) $e_{\mathfrak{b}}=|C_\Sigma(s_{\mathfrak{b}})|$, $(e_{\mathfrak{b}},m)=1$

iii) $(s_{\mathfrak{b}})^G\cap T=(s_{\mathfrak{b}})^\Sigma$

iv) *There exists a unique Γ in $\mathscr{F}$ such that $s_{\mathfrak{b}}$ and $e_{\mathfrak{b}}(\Gamma)$ are conjugate in G. In particular, $md_\alpha=e_{\mathfrak{b}}d_\Gamma$.*

Proof. The two parts in i) are equivalent since $N = C\Sigma$ and $N_{\mathfrak{b}} \geqq C$. We consider the second statement in i). The product $(C \cap \Sigma) \times C_{\Sigma}(s_{\mathfrak{b}})$ is direct since $T \cap \Sigma = 1$. Let $g \in N_{\mathfrak{b}} \cap \Sigma$. Then $\theta_{\mathfrak{b}} = \theta_{\mathfrak{b}}^{g}$, or equivalently,

$$R_T^C(\hat{s}_{\mathfrak{b}}) = R_T^C(\hat{s}_{\mathfrak{b}})^g \tag{4.4}$$

Thus $(s_{\mathfrak{b}})^g = (s_{\mathfrak{b}})^h$ for some $h \in N_C(T)$. Since $N_C(T) = T(C \cap \Sigma)$, we may choose h in $C \cap \Sigma$. Then $g \in (C \cap \Sigma)\, C_{\Sigma}(s_{\mathfrak{b}})$. Conversely, let $g \in C_{\Sigma}(s_{\mathfrak{b}})$. Then (4.4) holds, $\theta_{\mathfrak{b}} = \theta_{\mathfrak{b}}^{g}$, and $g \in N_{\mathfrak{b}} \cap \Sigma$. Since the factor $C \cap \Sigma$ is in $N_{\mathfrak{b}} \cap \Sigma$, i) holds.

The first statement in i) implies $e_{\mathfrak{b}} = |C_{\Sigma}(s_{\mathfrak{b}})|$, since $C \cap C_{\Sigma}(s_{\mathfrak{b}}) \leqq T \cap \Sigma = 1$. The second statement in i) implies $(e_{\mathfrak{b}}, m) = 1$, since $N_{\mathfrak{b}} \cap \Sigma$ is cyclic and $|C \cap \Sigma| = m$. Thus ii) holds.

Suppose $t \in T$ and $t^g = s_{\mathfrak{b}}$ for some $g \in G$. Then T and T_g, being Coxeter tori of G, are such in $K = C_G(s_{\mathfrak{b}})$. Thus there exists $k \in K$ such that $T^{gk} = T$, and so $gk \in N_G(T) = T\Sigma$. In particular, there exists $\tau \in \Sigma$ such that $t^{\tau} = t^{gk} = (s_{\mathfrak{b}})^k = s_{\mathfrak{b}}$, and so $\tau \in (s_{\mathfrak{b}})^{\Sigma}$. This proves the non-trivial inclusion in iii) and iii) holds.

Finally, let $\prod_{\Gamma} s_{\Gamma}$ be the primary decomposition of $s_{\mathfrak{b}}$ as an element of G, and let $\prod_{\Gamma} K_{\Gamma}$ be the corresponding decomposition of $K = C_G(s_{\mathfrak{b}})$. A generator ρ of R decomposes as $\prod_{\Gamma} \rho_{\Gamma}$, where $\rho_{\Gamma} \in K_{\Gamma}$, and by definition of R, $\rho_{\Gamma} \neq 1$ whenever $K_{\Gamma} \neq 1$. But

$$\prod_{\Gamma} \langle \rho_{\Gamma} \rangle \leqq C_K(\rho) = C_C(s_{\mathfrak{b}}) = T$$

and T is cyclic. Thus there exists a unique Γ in $\mathscr{F}$ such that $s = s_{\Gamma}$ and $K = K_{\Gamma}$. Hence $K = GL(m_{\Gamma}(s_{\mathfrak{b}}), F_{\Gamma})$ or $U(m_{\Gamma}(s_{\mathfrak{b}}), F_{\Gamma})$, and $m_{\Gamma}(s_{\mathfrak{b}}) d_{\Gamma} = m d_{\alpha}$. But $m_{\Gamma}(s_{\mathfrak{b}}) = |N_K(T) : T|$, and $N_K(T) = T(K \cap \Sigma) = T C_{\Sigma}(s_{\mathfrak{b}})$. Thus $m_{\Gamma}(s_{\mathfrak{b}}) = e_{\mathfrak{b}}$ by ii), and iv) holds.

Proposition (4E). *Let $\mathfrak{b}$ be a root block in $\mathscr{A}$, let Γ be the unique polynomial in $\mathscr{F}$ determined by $\mathfrak{b}$, and let η_{Γ} be the additive order of δ_{Γ} modulo e. Then*

$$e_{\mathfrak{b}} = \begin{cases} 2\eta_{\Gamma} & \text{if } \Gamma \text{ is unitary, } \Lambda_{\alpha} \text{ is linear} \\ \eta_{\Gamma} & \text{in all other cases.} \end{cases}$$

Proof. Since $\mathfrak{b}$ is a root block of $\mathscr{A}$, we have by (4D) and (4B) that

$$e_{\mathfrak{b}} d_{\Gamma} = m d_{\alpha}, \qquad (e_{\mathfrak{b}}, m) = (e_{\mathfrak{b}}, r) = (r, m) = 1. \tag{4.5}$$

If G is linear, then $d_{\Gamma} = \delta_{\Gamma}$, $d_{\alpha} = \delta_{\alpha}$, so by (4.5) $e_{\mathfrak{b}} \delta_{\Gamma} = m\delta_{\alpha} = mer^{\alpha}$ and $e_{\mathfrak{b}} = \eta_{\Gamma}$. If G is unitary and d_{Γ} and d_{α} have the same parity, the same argument applies. Suppose G is unitary. If $d_{\Gamma} = 2\delta_{\Gamma}$ is even and $d_{\alpha} = \delta_{\alpha} = er^{\alpha}$ is odd, then e is odd, so by (4.5) $2e_{\mathfrak{b}}\delta_{\Gamma} = mer^{\alpha}$ and $e_{\mathfrak{b}} = \eta_{\Gamma}$. If $d_{\Gamma} = \delta_{\Gamma}$ is odd and $d_{\alpha} = 2\delta_{\alpha}$ is even, then (4.5) implies $e_{\mathfrak{b}}\delta_{\Gamma} = 2mer^{\alpha}$, $e_{\mathfrak{b}}$ is even, and $\frac{1}{2} e_{\mathfrak{b}} = \eta_{\Gamma}$. Thus (4E) holds.

It will be convenient to have the following notation: Given $\Gamma \in \mathscr{F}$, let

$$e_{\Gamma} = \begin{cases} 2\eta_{\Gamma} & \text{if } \Gamma \text{ is unitary, } \Lambda_{\alpha} \text{ is linear} \\ \eta_{\Gamma} & \text{in all other cases.} \end{cases} \tag{4.6}$$

We recall that the Λ_α for $\alpha \geqq 0$ are either all linear or all unitary, so that e_Γ is well-defined. Furthermore, let

$$(4.7) \qquad n_\Gamma = \begin{cases} \min\{m : r \text{ divides } |GL(m, q^{\delta_\Gamma})|\} & \text{if } \Gamma \text{ is linear} \\ \min\{m : r \text{ divides } |U(m, q^{\delta_\Gamma})|\} & \text{if } \Gamma \text{ is unitary.} \end{cases}$$

Proposition (4F). *Let e_Γ, n_Γ be as above. Then the following hold:*
i) $(e_\Gamma, r) = 1$
ii) $e_\Gamma = 1$ *if the roots of Γ have order divisible by r*
iii) $e_\Gamma = n_\Gamma$.

Proof. i) holds since e_Γ divides $2e$. If Γ is as in ii), then $\eta_\Gamma = 1$. If Γ is unitary and the roots of Γ have multiplicative order $cr^{a+\alpha}$, where $(c, r) = 1$ and $\alpha \geqq 0$, then δ_Γ is odd and $cr^{a+\alpha}$ divides $q_0^{\delta_\Gamma} + 1$. Since $F_{q^{\delta_\alpha}} \supseteq F_{q^{\delta_\Gamma}}$ and δ_Γ is odd, it follows that δ_α divides δ_Γ and $r^{a+\alpha}$ divides $q_0^{\delta_\alpha} + 1$. Thus Λ_α is also unitary, $e_\Gamma = 1$, and ii) holds. iii) is clear if Γ is of linear type. Suppose Γ is of unitary type, so that δ_Γ is odd. If η_Γ is even, then $n_\Gamma = 2\eta_\Gamma$. But e and d_α are also even, so $e_\Gamma = 2\eta_\Gamma = n_\Gamma$. If η_Γ is odd, then e is odd, since e divides $\delta_\Gamma \eta_\Gamma$. But $q_0^e \equiv \pm 1 \pmod r$, so $q_0^e \equiv q_0^{\delta_\Gamma \eta_\Gamma} \pmod r$, and $e_\Gamma = n_\Gamma$. Thus iii) holds.

§5

We now prove the main results on the classification of blocks. We consider first the basic configurations $G^{m,\alpha}$ for varying m and α. In particular, the quantities introduced in Paragraph 4 will now carry superscripts m, α. Let $\mathscr{B}^{m,\alpha}$ be the set of blocks of $G^{m,\alpha}$ with defect group $R^{m,\alpha}$, and let $\mathscr{B} = \bigcup_{m,\alpha} \mathscr{B}^{m,\alpha}$. Let $\mathscr{F}'$ be the set of polynomials in $\mathscr{F}$ whose roots have order prime to r. We define a mapping $\mathscr{E}$ from $\mathscr{B}$ to $\mathscr{F}'$ as follows: Given $B \in \mathscr{B}^{m,\alpha}$, let $\mathfrak{b}$ be a root of B in the sense of Brauer. Thus $\mathfrak{b} \in \mathscr{A}^{m,\alpha}$ and $\mathfrak{b}$ induces B. The $C^{m,\alpha}$-class in $\mathscr{S}^{m,\alpha}$ corresponding to $\mathscr{A}^{m,\alpha}$ by (4D) is represented by an element $s_\mathfrak{b}$ which as an element of $G^{m,\alpha}$, has only one elementary divisor Γ and $\Gamma \in \mathscr{F}'$. Since $\mathfrak{b}$ is determined up to conjugacy in $N^{m,\alpha}$ and the defect groups of B are determined up to conjugacy in $G^{m,\alpha} \setminus N^{m,\alpha}$, the elementary divisor Γ depends only on B. We define the mapping $\mathscr{E}: \mathscr{B} \to \mathscr{F}'$, where $\mathscr{E}(B)$ is the Γ just defined.

Theorem (5A). *$\mathscr{E}$ is a bijection from $\mathscr{B}$ onto $\mathscr{F}'$.*

Proof: Let $B \in \mathscr{B}$, say $B \in \mathscr{B}^{m,\alpha}$, and let $\mathscr{E}(B) = \Gamma$ be defined using the root $\mathfrak{b} \in \mathscr{A}^{m,\alpha}$, and the class of $s_\mathfrak{b}$ in $\mathscr{S}^{m,\alpha}$. Given q and Γ, the integers d_Γ and δ_Γ can be computed. We claim that the triple (q, r, Γ) determines in turn each of the integers α, δ_α, d_α, $e_\mathfrak{b}$, and m. Indeed, $\alpha = \nu(d_\Gamma)$ by (4.5), since $\alpha = \nu(d_\alpha)$. The degrees δ_α and d_α are then determined by q, r, and α. Next, $e_\mathfrak{b}$ is determined by (4E), and finally m is given by (4.5). This establishes the claim.

Suppose B' is a block in $\mathscr{B}$ such that $\mathscr{E}(B') = \mathscr{E}(B) = \Gamma$. By the preceding paragraph B' is necessarily in $\mathscr{B}^{m,\alpha}$. Let $\mathfrak{b}'$ be a root of B', so $\mathfrak{b}' \in \mathscr{A}^{m,\alpha}$ and $\mathfrak{b}'$ induces B'. We may choose the representative $s_{\mathfrak{b}'}$ of the corresponding class in $\mathscr{S}^{m,\alpha}$ so that $s_{\mathfrak{b}'} \in T^{m,\alpha}$, where $T^{m,\alpha}$ is the Coxeter torus $C_{C^{m,\alpha}}(s_\mathfrak{b})$. Since $s_{\mathfrak{b}'}$ and $s_\mathfrak{b}$

are conjugate in $G^{m,\alpha}$ to $e_{\mathfrak{b}}(\Gamma)$, it follows by (4D) iii) that $s_{\mathfrak{b}'}\in(s_{\mathfrak{b}})^{\Sigma}$ where $\Sigma=\langle\sigma^{m,\alpha}\rangle$. Thus the canonical characters of $\mathfrak{b}'$ and $\mathfrak{b}$ are conjugate under $N^{m,\alpha}$, and so $B'=B$. We have proved that $\mathscr{E}$ is injective.

Conversely, let $\Gamma\in\mathscr{F}'$. We determine in turn the integers α, δ_α, d_α, e_Γ, m as follows: Let $\alpha=v(d_\Gamma)$. Then δ_α, d_α are determined by definition from q, r, and α. Let e_Γ be as in (4.6). Lastly, define m by the equation

(5.1) $$md_\alpha=e_\Gamma d_\Gamma.$$

We note that m is an integer. Indeed, d_α is er^α or $2er^\alpha$, and er^α divides $e_\Gamma d_\Gamma$ by the definition of α and e_Γ. Thus m is an integer by (5.1) except possibly in the case $d_\alpha=2er^\alpha$. In that case, if $d_\Gamma=2\delta_\Gamma$, then (5.1) becomes $mer^\alpha=e_\Gamma\delta_\Gamma$ and we argue as before. If $d_\Gamma=\delta_\Gamma$, then $2e$ divides $e_\Gamma\delta_\Gamma$ since $e_\Gamma=2\eta_\Gamma$, and m is again an integer.

With α, δ_α, d_α, e_Γ, m determined as above, let T be a Coxeter torus of $C^{m,\alpha}$. We claim there exists an r'-element $s\in T$ conjugate to $e_\Gamma(\Gamma)$ in $G^{m,\alpha}$. Since $\Gamma\in\mathscr{F}'$, the roots of Γ are d-th roots of unity, where d is a divisor of $q^{\delta_\Gamma}-1$ relatively prime to r. So it suffices by the definition of a Coxeter torus and (5.1) to show d divides $|T|$. In the linear case, $|T|=q^{md_\alpha}-1$ and $md_\alpha=e_\Gamma d_\Gamma=e_\Gamma\delta_\Gamma$, so d divides $|T|$. In the unitary case, $|T|=q_0^{md_\alpha}-(-1)^{md_\alpha}$. Suppose md_α is even, so that $|T|=q^{\frac{1}{2}md_\alpha}-1$. If $d_\Gamma=2\delta_\Gamma$, then δ_Γ divides $\frac{1}{2}md_\alpha$ by (5.1); if $d_\Gamma=\delta_\Gamma$, then δ_Γ is odd, and again δ_Γ divides $\frac{1}{2}md_\alpha$ by (5.1). Thus d divides $|T|$ in these cases. Lastly, suppose md_α is odd, so that $|T|=q_0^{md_\alpha}+1$. Then e_Γ and d_Γ are odd by (5.1), $\Gamma\in\mathscr{F}_1$, and d is necessarily a divisor of $q_0^{d_\Gamma}+1$. But $q_0^{d_\Gamma}+1$ divides $|T|$, and so the claim holds.

We can show that $\mathscr{E}$ is surjective. Let $K=C_{G^{m,\alpha}}(s)$, so $K=GL(e_\Gamma,F_\Gamma)$ or $U(e_\Gamma,F_\Gamma)$. By (4F) iii) $R^{m,\alpha}$ is a Sylow subgroup of K and $C_K(R^{m,\alpha})=T^{m,\alpha}$. In particular, s is a regular r'-element of $C^{m,\alpha}$ since $C_{C^{m,\alpha}}(s)=C_K(R^{m,\alpha})=T$. Moreover, $(r,m)=1$ by (5.1). By (4C) there exists a block $\mathfrak{b}\in\mathscr{A}^{m,\alpha}$ corresponding to the $C^{m,\alpha}$-conjugacy class of s. Let $\Sigma=\langle\sigma^{m,\alpha}\rangle$ have the meaning given in (4D). By (4D) ii) $e_{\mathfrak{b}}=|C_\Sigma(s)|$. But

$$|C_\Sigma(s)|=|K\cap\Sigma|=|N_K(T):T|=e_\Gamma$$

since T is a Coxeter torus of K. Since $(r,e_\Gamma)=1$, $\mathfrak{b}$ is a root block of $C^{m,\alpha}$. Let B be the block of $G^{m,\alpha}$ induced by $\mathfrak{b}$. Then $B\in\mathscr{B}^{m,\alpha}$ and $\mathscr{E}(B)=\Gamma$. This completes the proof of (5A).

The following is a consequence of the last part of the preceding proof.

Corollary (5B). *Let $\mathfrak{b}$ be a block of $C^{m,\alpha}$ with defect group $R^{m,\alpha}$, and let s be a representative of the $C^{m,\alpha}$-conjugacy class corresponding to $\mathfrak{b}$. Then $R^{m,\alpha}$ is a Sylow subgroup of $C_{C^{m,\alpha}}(s)$.*

We consider next the basic configurations $G^{m,\alpha,\beta}$. We recall by (3D) that

(5.2) $$\begin{aligned} R^{m,\alpha,\beta}&=R^{m,\alpha}\wr X_\beta\\ N^{m,\alpha,\beta}&=N^{m,\alpha}/R^{m,\alpha}\otimes Y_\beta\\ C^{m,\alpha,\beta}&=C^{m,\alpha}\otimes I_\beta,\end{aligned}$$

where X_β is a Sylow r-subgroup of S_β, $Y_\beta = N_{S_\beta}(X_\beta)$, and I_β is the identity of S_β. In particular, blocks of $C^{m,\alpha}$ can be identified with blocks of $C^{m,\alpha,\beta}$. Given a block $\mathfrak{b}$ of $C^{m,\alpha}$, we will write $\mathfrak{b}\otimes I_\beta$ for the corresponding block of $C^{m,\alpha,\beta}$. Moreover, if $\theta_\mathfrak{b}$ is the canonical character of $\mathfrak{b}$, we will write $\theta_\mathfrak{b}\otimes I_\beta$ for the canonical character of $\mathfrak{b}\otimes I_\beta$. Let $\mathscr{A}^{m,\alpha,\beta}$ be the set of blocks of $C^{m,\alpha,\beta}R^{m,\alpha,\beta}$ with defect group $R^{m,\alpha,\beta}$.

Lemma (5C). *There exists a bijection between the sets $\mathscr{A}^{m,\alpha,\beta}$ and $\mathscr{A}^{m,\alpha}$ given as follows: b and $\mathfrak{b}$ are corresponding blocks of $\mathscr{A}^{m,\alpha,\beta}$ and $\mathscr{A}^{m,\alpha}$ if the restriction of the canonical character θ_b* of b to $C^{m,\alpha,\beta}$ satisfies

$$\theta_b|_{C^{m,\alpha,\beta}} = \theta_\mathfrak{b}\otimes I_\beta,$$

or equivalently, if

$$b = (\mathfrak{b}\otimes I_\beta)^{R^{m,\alpha,\beta}C^{m,\alpha,\beta}}.$$

Moreover, the stabilizers of b and $\mathfrak{b}$ in $N^{m,\alpha,\beta}$ and $N^{m,\alpha}$ are related by

$$N_b^{m,\alpha,\beta} = (N_\mathfrak{b}^{m,\alpha}/R^{m,\alpha})\otimes Y_\beta.$$

Proof. The bijection follows from a theorem of Brauer [3] I, (5A). Now the stabilizer $N_b^{m,\alpha,\beta}$ of b in $N^{m,\alpha,\beta}$ is also the stabilizer of θ_b in $N^{m,\alpha,\beta}$. But since θ_b is trivial on $R^{m,\alpha,\beta}$, it follows that $N_b^{m,\alpha,\beta}$ is then the stabilizer of $\theta_b|_{C^{m,\alpha,\beta}} = \theta_\mathfrak{b}\otimes I_\beta$. Let $N^{m,\alpha} = C^{m,\alpha}\Sigma$, where Σ has the meaning given in (4D). Now $C^{m,\alpha}/R^{m,\alpha}\otimes Y_\beta$ stabilizes $\theta_\mathfrak{b}\otimes I_\beta$, so by (5.2) $N^{m,\alpha,\beta}$ acts on b as Σ does on $\mathfrak{b}$. The remaining assertions of (5C) easily follow.

We now come to the general theorem. Let $G = GL(n,F)$ or $U(n,F)$. Let B be a block of G with defect group R. By (3C) we may assume

(5.3) $$R = R_0 \prod_{i=1}^{t} R_i,$$

where R_0 is the identity matrix of degree $l \geqq 0$, and $R_i = R^{m_i,\alpha_i,\beta_i}$ for $1 \leqq i \leqq t$. Let the corresponding decompositions of $C(R)$ and $C(R)R$ be $\prod C_i$ and $\prod C_i R_i$, where $C_0 = GL(l,F)$ or $U(l,F)$, and $C_i = C^{m_i,\alpha_i,\beta_i}$ for $1 \leqq i \leqq t$. Let b be a block of $C(R)R$ which is a root of B, and let $b = \prod_i b_i$, where b_i is a block of $C_i R_i$ with defect group R_i.

The block b_0, if it occurs, has defect 0. The unique character θ_0 in b_0 has the form $\chi_{s_0,\lambda}$ in the notation of Paragraph 1, that is $\theta_0 = \pm R_{K_0}^{C_0}(\hat{s}_0\chi_\lambda)$, where s_0 is a semisimple element of C_0, $K_0 = C_{C_0}(s_0)$, and χ_λ is the unipotent character of K_0 corresponding to λ. Here $\lambda = \prod_\Gamma \lambda_\Gamma$, where $\lambda_\Gamma \vdash m_\Gamma(s_0)$. Since θ_0 has defect 0 and $\theta_0(1) = |C_0 : K_0|_{p'}\,\chi_\lambda(1)$, it follows that χ_λ also has defect 0. In particular, $O_r(K_0) = 1$ and s_0 is an r'-element of C_0. Moreover, λ_Γ has no e_Γ-hooks for all Γ, where e_Γ is defined as in (4.6). Otherwise χ_{λ_Γ} would not be of defect 0 by the degree formula (1.15) and (4F) iii). It is clear that λ depends only on B, and not on the choice of R and b used in defining λ. We call λ the unipotent factor of B.

Each block b_i for $1 \leqq i \leqq t$ corresponds by (5B) to a block $\mathfrak{b}_i$ in $\mathscr{A}^{m_i,\alpha_i}$, and $\mathfrak{b}_i$ in turn corresponds by (4C) to a C^{m_i,α_i}-conjugacy class represented by a semisimple r'-element s_i. The r'-element of $C(R)$ defined by

$$s = s_0 \prod_{i=1}^{t} (s_i \otimes I_{\beta_i}) \tag{5.4}$$

will be called a semisimple factor of B. It is clear that the conjugacy class s^G depends only on B and not on the choice of R and b. Moreover, every element in s^G arises as a semisimple factor of B.

Given a semisimple r'-element s of G and $\Gamma \in \mathscr{F}'$, let $\mathscr{C}_\Gamma(s)$ be the set of e_Γ-cores of partitions of $m_\Gamma(s)$. We recall that the e_Γ-core of a partition μ_Γ is the partition obtained from μ_Γ by successively deleting e_Γ-hooks until a partition is reached which has no e_Γ-hooks. The e_Γ-core of μ_Γ is well-defined by a theorem of Nakayama [18], I, Paragraph 4. Let $\mathscr{C}(s) = \prod_\Gamma \mathscr{C}_\Gamma(s)$. Two pairs (s, λ) and (s', λ'), where $\lambda \in \mathscr{C}(s)$ and $\lambda' \in \mathscr{C}(s')$, are G-conjugate if $s' = s^g$ for some $g \in G$ and $\lambda' = \lambda$.

Theorem (5D). *Let B be a block of G, and let s and λ be the semisimple and unipotent factors of B. Then the mapping $B \mapsto (s, \lambda)$ induces a bijection $\mathscr{J}$ from the set of blocks of G onto the set of G-conjugacy classes of pairs (s, λ), where s is a semisimple r'-element and $\lambda \in \mathscr{C}(s)$.*

Proof. Let R be a defect group of B, and let b be a block of $C(R)R$ inducing B. We may suppose $R = \prod_{i=0} R_i$, $C(R)R = \prod_{i=0} C_i R_i$, and $b = \prod_{i=0} b_i$ are the decompositions used in the definition of s and λ. For $i \geqq 1$,

$$b_i = (\mathfrak{b}_i \otimes I_{\beta_i})^{C_i R_i}, \tag{5.5}$$

where $\mathfrak{b}_i \in \mathscr{A}^{m_i, \alpha_i}$ and m_i, α_i, β_i are the parameters for R_i. We normalize the $\mathfrak{b}_i$ occurring in (5.5) as follows: Let $\mathscr{A}_0^{m,\alpha}$ be a set of representatives for the orbits of $N^{m,\alpha}$ on $\mathscr{A}^{m,\alpha}$. By replacing b by a conjugate if necessary, we may assume that $\mathfrak{b}_i \in \mathscr{A}_0^{m_i, \alpha_i}$ for $i \geqq 1$. Given m, α, β, let

$$I^{m,\alpha,\beta} = \{i\colon 1 \leqq i \leqq t, R_i = R^{m,\alpha,\beta}\}.$$

It follows by (3E) that

$$N(R) = C_0 \prod_{m,\alpha,\beta} (N^{m,\alpha,\beta} \wr S(I^{m,\alpha,\beta}))$$

where $S(I^{m,\alpha,\beta})$ is the symmetric group on $I^{m,\alpha,\beta}$. Given $\mathfrak{b} \in \mathscr{A}_0^{m,\alpha}$, let

$$I_{\mathfrak{b}}^{m,\alpha,\beta} = \{i\colon i \in I^{m,\alpha,\beta}, \mathfrak{b}_i = \mathfrak{b}\}, t_{\mathfrak{b}}^{m,\alpha,\beta} = |I_{\mathfrak{b}}^{m,\alpha,\beta}|.$$

Then the stabilizer of b in $N(R)$ is

$$N(R)_b = C_0 \prod_{m,\alpha,\beta} \prod_{\mathfrak{b} \in \mathscr{A}_0^{m,\alpha}} (N_{\mathfrak{b}}^{m,\alpha}/R^{m,\alpha} \otimes Y_\beta) \wr S(I_{\mathfrak{b}}^{m,\alpha,\beta})$$

by (5C). Thus b has inertial index

$$|N(R)_b\colon C(R)R| = \prod_{m,\alpha,\beta} \prod_{\mathfrak{b} \in \mathscr{A}_0^{m,\alpha}} (e_{\mathfrak{b}}(Y_\beta : X_\beta))^{t_{\mathfrak{b}}^{m,\alpha,\beta}} (t_{\mathfrak{b}}^{m,\alpha,\beta})!, \tag{5.6}$$

where for $\mathfrak{b}\in\mathscr{A}_0^{m,\alpha}$, $e_{\mathfrak{b}}$ is the inertial index of $\mathfrak{b}$ in $N^{m,\alpha}$. Since b is a root of B, (5.6) is relatively prime to r. In particular,

$$(5.7)\qquad (r, e_{\mathfrak{b}})=1, \quad t_{\mathfrak{b}}^{m,\alpha,\beta}<r$$

for those $\mathfrak{b}$'s occurring in (5.5), and such $\mathfrak{b}$'s are root blocks.

Brauer's First Main Theorem implies each root block $\mathfrak{b}$ in $\mathscr{A}_0^{m,\alpha}$ corresponds in $1-1$ fashion to a block of $\mathscr{B}^{m,\alpha}$, which in turn corresponds by (5 A) to a Γ in $\mathscr{F}'$. By abuse of notation we shall write $\mathscr{E}(\mathfrak{b})=\Gamma$. Given $\Gamma\in\mathscr{F}'$, let

$$I^{\Gamma}=\{i\colon 1\leqq i\leqq t, \mathscr{E}(\mathfrak{b}_i)=\Gamma\}.$$

By (5 A) Γ determines a block in $\mathscr{B}$, say $B^{m,\alpha}$ in $\mathscr{B}^{m,\alpha}$. In particular, if $i\in I^{\Gamma}$, then $m_i=m$, $\alpha_i=\alpha$, and $\mathfrak{b}_i$ is the unique root block in $\mathscr{A}_0^{m,\alpha}$ inducing $B^{m,\alpha}$. Moreover, $e_{\mathfrak{b}_i}=e_{\Gamma}$ by the proof of (5 A), and the $C^{m,\alpha}$-conjugacy class corresponding to $\mathfrak{b}_i$ is represented by an element whose canonical form in $G^{m,\alpha}$ is $e_{\Gamma}(\Gamma)$. We shall denote this element by s_{Γ}. Given $\Gamma\in\mathscr{F}'$ and β, let

$$I^{\Gamma,\beta}=\{i\colon i\in I^{\Gamma}, \beta_i=\beta\}, \qquad t^{\Gamma,\beta}=|I^{\Gamma,\beta}|.$$

In particular, I^{Γ} is the disjoint union $\bigcup_{\beta} I^{\Gamma,\beta}$, and

$$(5.8)\qquad I^{\Gamma,\beta}=I_{\mathfrak{b}}^{m,\alpha,\beta}, \qquad t^{\Gamma,\beta}=t_{\mathfrak{b}}^{m,\alpha,\beta},$$

where $\mathfrak{b}$ is the block of $\mathscr{A}_0^{m,\alpha}$ determined by Γ. The semisimple factor (5.4) of B can now be rewritten as

$$s=s_0\prod_{\Gamma,\beta}\prod_{i\in I^{\Gamma,\beta}}(s_{\Gamma}\otimes I_{\beta})$$

and so

$$m_{\Gamma}(s)=m_{\Gamma}(s_0)+e_{\Gamma}\sum_{\beta}t^{\Gamma,\beta}r^{\beta}.$$

Thus e_{Γ} divides $m_{\Gamma}(s)-m_{\Gamma}(s_0)$, and so λ_{Γ} is necessarily in $\mathscr{C}_{\Gamma}(s)$ and $\lambda\in\mathscr{C}(s)$. Moreover, $\Sigma t^{\Gamma,\beta}r^{\beta}$ is the r-adic expansion of $e_{\Gamma}^{-1}(m_{\Gamma}(s)-m_{\Gamma}(s_0))$ by (5.7) and (5.8).

We now turn to the proof $\mathscr{J}$ is a bijection. Let s be a semisimple r'-element of G, and let $\lambda\in\mathscr{C}(s)$. Then e_{Γ} divides $m_{\Gamma}(s)-|\lambda_{\Gamma}|$ for $\Gamma\in\mathscr{F}'$. Let $\sum_{\beta}t^{\Gamma,\beta}r^{\beta}$ be the r-adic expansion of $e_{\Gamma}^{-1}(m_{\Gamma}(s)-|\lambda_{\Gamma}|)$. Each Γ in $\mathscr{F}'$ determines integers m and α, and a root block $\mathfrak{b}_{\Gamma}$ in $\mathscr{A}_0^{m,\alpha}$. Following the convention for (3 C), we define

$$R_{\Gamma}=I_{\Gamma}\prod_{\beta}(R^{m,\alpha,\beta})^{t^{\Gamma,\beta}},$$

where I_{Γ} is the identity matrix of degree $|\lambda_{\Gamma}|d_{\Gamma}$. Since $md_{\alpha}=e_{\Gamma}d_{\Gamma}$ by (5.1) and

$$|\lambda_{\Gamma}|d_{\Gamma}+\sum_{\beta}t^{\Gamma,\beta}md_{\alpha}r^{\beta}=|\lambda_{\Gamma}|d_{\Gamma}+\sum_{\beta}t^{\Gamma,\beta}e_{\Gamma}d_{\Gamma},$$

the degree of R_{Γ} is $m_{\Gamma}(s)d_{\Gamma}$. Next, we set $R=\prod_{t}R_{\Gamma}$. Since $n=\sum_{\Gamma}m_{\Gamma}(s)d_{\Gamma}$, R is a subgroup of G of the form $\prod_{i=0}^{t}R_i$ given in (2C), where R_0 has degree l

$=\sum_{\Gamma}|\lambda_\Gamma|d_\Gamma$ and $R_i=R^{m_i,\alpha_i,\beta_i}$ for $i\geqq 1$. Let $\prod_{i=0}^{t} C_iR_i$ be the corresponding decomposition of $C(R)R$, and let

$$J_\Gamma=\{i\colon 1\leqq i\leqq t, R_i \text{ occurs in } R_\Gamma\}.$$

Thus $\{1,2,\ldots,t\}$ is the disjoint union $\bigcup_{\Gamma} J_\Gamma$ of the J_Γ.

We define a block $b=\prod_{i=0}^{t} b_i$ of $C(R)R$, where b_i is a block of C_iR_i with defect group R_i, as follows: Firstly, let $s_0\in C_0$ be chosen so that its canonical form is $\prod_{\Gamma}|\lambda_\Gamma|(\Gamma)$. Since λ_Γ is a partition of $m_\Gamma(s_0)$ with no e_Γ-hooks, λ determines a unipotent character χ_λ of defect 0 of $C_{C_0}(s_0)$. The corresponding character $\theta_0=\chi_{s_0,\lambda}$ of C_0 is then of defect 0, and we may take b_0 to be the block of C_0 containing θ_0. Next, if $i\in J_\Gamma$ and $\beta_i=\beta$, we take $b_i=b_{\Gamma,\beta}$, where $b_{\Gamma,\beta}$ is the block of $\mathscr{A}^{m,\alpha,\beta}$ defined by

$$b_{\Gamma,\beta}=(\mathfrak{b}_\Gamma\otimes I_\beta)^{R^{m,\alpha,\beta}C^{m,\alpha,\beta}}.$$

By (5C) b_i has defect group R_i. The calculations preceding (5.6) show that the inertial index of b is relatively prime to r, since the integer $t_{\mathfrak{b}}^{m,\alpha,\beta}$ in (5.6) is $t^{\Gamma,\beta}$, where $\mathfrak{b}=\mathfrak{b}_\Gamma$, and $t^{\Gamma,\beta}<r$. Thus b induces a block of B of G with defect group R. Moreover, the construction shows that if $b_i=b_{\Gamma,\beta}$, then the contribution of b_i to the semisimple factor of B is $s_\Gamma\otimes I_\beta$. Thus the semisimple factor of B has canonical form $\prod_{\Gamma} c_\Gamma(\Gamma)$, where

$$c_\Gamma=m_\Gamma(s_0)+e_\Gamma\sum_{\beta} t^{\Gamma,\beta}r^\beta,$$

and $\mathscr{J}(B)=(s,\lambda)^G$. This proves $\mathscr{J}$ is surjective. But the argument also shows $\mathscr{J}$ is injective. For if $\mathscr{J}(B)=(s,\lambda)^G$ is computed using the defect group R and the root b of B, then R and b are determined in turn up to the usual conjugacies by (s,λ). This completes the proof of (5D).

$\mathscr{J}(B)$ can be described alternatively as follows: We note first that the canonical character $\theta_{\mathfrak{b}}$ of a block $\mathfrak{b}$ in $\mathscr{A}^{m,\alpha}$ can be written in the form $\chi_{s_{\mathfrak{b}},\{1\}}$ by (4C). Secondly, if b in $\mathscr{A}^{m,\alpha,\beta}$ and $\mathfrak{b}$ in $\mathscr{A}^{m,\alpha}$ are corresponding blocks in the sense of (5C), then the canonical character θ_b of b restricts to

$$(\theta_{\mathfrak{b}}\otimes I_\beta)=(\chi_{s_{\mathfrak{b}},\{1\}}\otimes I_\beta)$$

on $C^{m,\alpha,\beta}$. Now let b be a root of B with defect group R. In the notation of the preceding proof, the canonical character θ of b decomposes as $\prod_{i=0}\theta_i$ where θ_i is the canonical character of b_i. The character θ_0 has the form $\chi_{s_0,\lambda}$. The characters θ_i for $i\geqq 1$ restrict to $\chi_{s_i,\{1\}}\otimes I_{\beta_i}$ on C_i. Then

$$s=s_0\prod_{i=1}^{t}(s_i\otimes r^{\beta_i}) \quad\text{and}\quad \lambda$$

are the semisimple and unipotent factors of B. In particular, the factor s can be read from the canonical character θ of the root b.

The mapping $\mathscr{J}$ behaves well with respect to major subsections of a block. Let B be a block of G with defect group R, and let $\mathscr{J}(B)=(s,\lambda)^G$, where s and λ are obtained by means of the canonical character θ of a root b of B with defect group R as described above. Now let $\rho\in Z(R)$ and $H=C(\rho)$. By Brauer [3], II, (4A) there exist blocks of H with defect group R which induce B. These blocks of H are the major subsections of B in H with defect group R and are obtained as follows. The $N(R)$-orbit $b^{N(R)}=\{b^n\colon n\in N(R)\}$ decomposes as a union of $N_H(R)$-orbits, say as $\bigcup_\alpha (b_\alpha)^{N_H(R)}$, where $b_\alpha=b^{n_\alpha}$ for some $n_\alpha\in N(R)$. Then $B_\alpha=Br^H_{C(R)R}(b_\alpha)$ is a major subsection of B in H with defect group R, and the mapping $b_\alpha\mapsto B_\alpha$ is a bijection between the $N_H(R)$-orbits on $b^{N(R)}$ and such major subsections of B. Moreover, the canonical character of b_α is θ^{n_α} and the semisimple factor determined by b_α is $s_\alpha=s^{n_\alpha}$.

To interpret this in H, let $H=\prod_\Lambda H_\Lambda$, $B_\alpha=\prod_\Lambda B_{\alpha\Lambda}$, $R=\prod_\Lambda R_\Lambda$ be the decompositions corresponding to the primary decomposition $\rho=\prod_\Lambda \rho_\Lambda$. So $B_{\alpha\Lambda}$ is a block of H_Λ with defect group R_Λ. Since $C(R)R\leqq H$, there are corresponding decompositions $C(R)R=\prod C_\Lambda R_\Lambda$ and $b_\alpha=\prod b_{\alpha\Lambda}$, and $b_{\alpha\Lambda}$ is a block of $C_\Lambda R_\Lambda$ with defect group R_Λ inducing $B_{\alpha\Lambda}$. Since $C_\Lambda=C_{H_\Lambda}(R_\Lambda)$, $b_{\alpha\Lambda}$ is a root of $B_{\alpha\Lambda}$. The canonical character θ_α of b_α and the element s_α also decompose as $\prod\theta_{\alpha\Lambda}$ and $\prod s_{\alpha\Lambda}$, where $\theta_{\alpha\Lambda}$ is the canonical character of $b_{\alpha\Lambda}$ and $s_{\alpha\Lambda}$ is the corresponding semisimple factor determined by $\theta_{\alpha\Lambda}$. Let $\mathscr{J}_\Lambda$ be the mapping of (5D) for the group H_Λ.

i) If $\Lambda\neq X-1$, then $Z(H_\Lambda)$ contains r-elements, R_Λ has no trivial direct summand, and $\mathscr{J}_\Lambda(B_{\alpha\Lambda})=(s_{\alpha\Lambda},\{-\})^{H_\Lambda}$.

ii) If $\Lambda=X-1$, then R_Λ contains R_0 as a direct summand, H_Λ is a group of the same type as G over F, and $\mathscr{J}_\Lambda(B_{\alpha\Lambda})=(s_{\alpha\Lambda},\lambda)^{H_\Lambda}$. We shall write $\mathscr{J}(B_\alpha)$ for $\prod_\Lambda \mathscr{J}_\Lambda(B_{\alpha\Lambda})$.

The following is a consequence of the proof of (5D).

Corollary (5E). *Let B be a block of G with defect group R, and let $\mathscr{J}(B)=(s,\lambda)^G$ where $s\in C(R)$. Let $\prod_\Gamma K_\Gamma$ and $\prod_\Gamma R_\Gamma$ be the decompositions of $K=C_G(s)$ and R corresponding to the primary decomposition $\prod_\Gamma s_\Gamma$ of s. If $\lambda_\Gamma=\{-\}$, then R_Γ is a Sylow r-subgroup of K_Γ.*

Proof. If $\lambda_\Gamma=\{-\}$, then $m_\Gamma(s)=e_\Gamma\sum_\beta t^{\Gamma,\beta}r^\beta$ and $R_\Gamma=\prod_\beta (R^{m,\alpha,\beta})^{t^{\Gamma,\beta}}$, where m,α are the integers determined by Γ. Thus

$$v(R_\Gamma)=\sum_\beta t^{\Gamma,\beta}((a+\alpha)\,r^\beta+v(r^\beta\,!)).$$

By (3B) this is also $v(K_\Gamma)$.

§6

In this section we shall classify the characters in a block under the assumption that $Z(G)$ contains elements of order r. Thus throughout this section $e=1$ and in the case G is unitary, r divides q_0+1. The bijection $\mathcal{J}$ of (5D) then reduces to a correspondence between blocks of G and conjugacy classes of semisimple r'-elements of G. We shall simplify notation and write $\mathcal{J}(B)=s^G$ if B and s^G are corresponding pairs. Moreover, if R is a defect group of B and $s\in C(R)$, then R is a Sylow subgroup of $C_G(s)$ by (5E). The following is the main result.

Theorem (6A). *Let B be a block of G with defect group R, and let $\mathcal{J}(B)=s^G$, where $s\in C(R)$. Then $B=\bigcup_{y\in R}(sy)_G$, where $(sy)_G$ is the geometric conjugacy class of G determined by sy.*

The proof of (6A) will require a number of preliminary lemmas. We note the following weaker result contained in (6A).

Theorem (6B). *Let B be a block of G with defect group R, and let $\mathcal{J}(B)=s^G$, where $s\in C(R)$. Then $B\supseteq s_G$.*

In fact, (6B) *implies* (6A). *Namely, let $\chi_{sy,\mu}\in(sy)_G$ so that $\chi^{sy,\mu}=R^G_{C(sy)}(\widehat{sy}\chi^\mu)$. Then*

$$R^G_{C(sy)}(\widehat{sy}\chi^\mu)=R^G_{C(sy)}(\hat{s}\chi^\mu)$$

on the set $G_{r'}$ of r'-elements of G. Indeed, by (2B) i) *applied to $C(sy)$, it suffices to show*

$$R^G_T((\widehat{sy}\chi^\mu)_{(T)})=R^G_T((\hat{s}\chi^\mu)_{(T)})$$

on $G_{r'}$ for all $T\subseteq C(sy)$, and this is so by (1.10) *and* (2A).

Now

$$R^G_{C(sy)}(\hat{s}\chi^\mu)=R^G_{C(s)}(\hat{s}R^{C(s)}_{C(sy)}(\chi^\mu))$$

by (1.1) and $R^{C(s)}_{C(sy)}(\chi^\mu)$ is a linear integral combination of unipotent characters χ_ν of $C(s)$. Hence by (6B) the restriction of $\chi_{sy,\mu}$ to $G_{r'}$ is a corresponding linear combination of characters in B, and so $\chi_{sy,\mu}\in B$. Since B is arbitrary and every geometric conjugacy class of G is of the form $(sy)_G$ for some semisimple r'-element s of G and some $y\in R$, where R is a Sylow r-subgroup of $C_G(s)$, it follows that (6A) holds.

It will then be enough to prove (6B). However, in any inductive situation, we may assume that (6A) holds. In particular, this will be assumed for proper subgroups of G of the form $H=C(\rho)$, where ρ is an element of order a power of r. Such subgroups are then of the form $H=H_1\times H_2\times\ldots\times H_k$, where the H_i are linear groups if G is linear, and unitary groups if G is unitary. We introduce the following notation: Let t be a semisimple element of G, and suppose $t^G\cap H=\bigcup_\alpha t^H_\alpha$, where the union is disjoint. Given $\chi\in t_G$, let

$$\chi^{(t_\alpha)}(\rho\sigma)=\sum_{\zeta\in(t_\alpha)_H}(\chi,R^G_H(\zeta))\,\zeta(\rho\sigma) \tag{6.1}$$

for all $\sigma \in H_{r'}$. The $\chi^{(t_\alpha)}$ are then functions defined on the r-section of H determined by ρ. By induction the ζ in $(t_\alpha)_H$ belong to a block b of H, where $\mathcal{J}(b)=((t_\alpha)_{r'})^H$. Let $\Delta(b)=\{\alpha\colon (t_\alpha)_H \subseteq b\}$. Since $(\chi, R_H^G(\zeta))=0$ by (1B) for any $\zeta \in b$ with $\zeta \notin \bigcup_\alpha (t_\alpha)_H$, it follows that

$$\sum_{\alpha \in \Delta(b)} \chi^{(t_\alpha)}(\rho\sigma) = \sum_{\zeta \in b} (\chi, R_H^G(\zeta))\, \zeta(\rho\sigma). \tag{6.2}$$

The $\zeta(\rho\sigma)$ are expressible in terms of the generalized decomposition numbers $d^{\rho}_{\zeta\varphi}$ and the Brauer characters φ in b. Since $\chi(\rho\sigma)=\sum_\alpha \chi^{(t_\alpha)}(\rho\sigma)$, it follows by the linear independence of Brauer characters, [2], I, (3E), that

$$\sum_{\alpha \in \Delta(b)} \chi^{(t_\alpha)}(\rho\sigma) = \chi^{(b)}(\rho\sigma), \tag{6.3}$$

where $\chi^{(b)}(\rho\sigma)$ is the function $\sum_{\varphi \in b} d^{\rho}_{\chi\varphi}\varphi(\sigma)$ introduced by Brauer in [3], II, (1.4). We recall that

$$\chi(\rho\sigma) = \sum_b \chi^{(b)}(\rho\sigma). \tag{6.4}$$

If t is an r'-element, then induction and (5D) imply $|\Delta(b)|=0$ or 1, and we shall write B_α for the block of H corresponding to t_α^H.

Lemma (6C). *Let s be a semisimple r'-element of G with only one elementary divisor Γ, let ρ be an element of order r in $C(s)$, and let $\chi \in s_G$. Let $\chi^{(s)}$ be defined as above for the H-conjugacy class s^H, where $H=C(\rho)$. Then there exists a unipotent element σ in H such that $\chi^{(s)}(\rho\sigma) \neq 0$.*

Proof. Let $H=H_1 \times H_2 \times \ldots \times H_k$, where $H_i = GL(m_i, F)$ if G is linear, and $H_i = U(m_i, F)$ if G is unitary. We may then express $s=\prod_i s_i$, where $s_i \in H_i$. We set $K=C_G(s)$, $L=C_H(s)$, and $L_i = C_{H_i}(s_i)$. Thus $H \cap K = L = \prod_i L_i$. As an element of H_i, s_i also has Γ as its only elementary divisor. Let n_i be the multiplicity of Γ in s_i. Then $m_i = n_i d$, where $d=d_\Gamma$. We claim that two maximal tori of L conjugate in H are already conjugate in L. Indeed, since $H=\prod H_i$ and $L=\prod L_i$, it suffices to show that two maximal tori of L_i conjugate in H_i are already conjugate in L_i. The conjugacy classes of maximal tori in L_i are parametrized by partitions of n_i, since $L_i \simeq GL(n_i, q^d)$ or $U(n_i, q^d)$. Suppose T is a maximal torus of L_i in the L_i-class corresponding to the partition $\mu = \{1^{r_1} 2^{r_2} \ldots\}$ of n_i. Then T, viewed as a torus of H_i, is in the H_i-class corresponding to the partition $\bar{\mu} = \{d^{r_1} (2d)^{r_2} \ldots\}$. Since distinct μ's give distinct $\bar{\mu}$'s, the claim follows.

Let $\zeta_1, \zeta_2, \ldots, \zeta_k$ be the characters in s_H, and let $\sigma_1, \ldots, \sigma_l$ be representatives for the conjugacy classes of unipotent elements of H. The number k is also the number of L-conjugacy classes of maximal tori of L, so $k \leq l$ by the preceding paragraph. Consider the system of equations

$$\begin{aligned}\chi^{(s)}(\rho\sigma_j) &= \sum_i (\chi, R_H^G(\zeta_i))\, \zeta_i(\rho\sigma_j)\\ &= \hat{s}(\rho) \sum_i (\chi, R_H^G(\zeta_i))\, \zeta_i(\sigma_j)\end{aligned}$$

for $1 \leqq j \leqq l$. To complete the proof of (6C), it will suffice to show that the $k \times l$ matrix $(\zeta_i(\sigma_j))$ has rank k, and that the k-tuple $((\chi, R_H^G(\zeta_i))$ is not the zero vector.

Let $\zeta_i = \chi_{s,\phi_i}$, where ϕ_i is an irreducible character of W_L. By (1.18) and (1.3) there exists a sign ε such that

$$\zeta_i = \frac{\varepsilon}{|W_L|} \sum_{w \in W_L} \phi_i(w) R_{T_w}^H(\hat{s})$$

for $1 \leqq i \leqq k$. Since $R_{T_w}^H(\hat{s})(\sigma_j) = R_{T_w}^H(1)(\sigma_j) = Q_{T_w}^H(\sigma_j)$, we have

$$\zeta_i(\sigma_j) = \varepsilon \sum_\alpha \frac{\phi_i(w_\alpha)}{|C_{W_L}(w_\alpha)|} Q_{T_{w_\alpha}}^H(\sigma_j),$$

where w_α runs over a set of representatives for the conjugacy classes of W_L. The orthogonality relations for the irreducible characters of W_L and for the Green functions of H then imply that the matrix $(\zeta_i(\sigma_j))$ has rank k. Finally, if $\chi = \chi_{s,\phi}$, where ϕ is an irreducible character of the Weyl group W_K, then

$$\begin{aligned}(\chi^{s,\phi}, R_H^G(\chi^{s,\phi_i})) &= (\chi^\phi, R_L^K(\chi^{\phi_i})) \\ &= (\phi, \operatorname{Ind}_{W_L}^{W_K}(\phi_i))\end{aligned}$$

by (1B) and (1C). Thus there exists i such that $(\chi, R_H^G(\zeta_i)) \neq 0$, and the proof of (6C) is complete.

Lemma (6D). *Let B be a block of G, and suppose $\mathcal{J}(B) = s^G$, where s has at least two distinct elementary divisors. If $s_G \cap B \neq \phi$, then $s_G \subseteq B$.*

Proof. Let $\mathcal{G}$ be the set of elementary divisors of s. We define a relation $\sim$ on s_G as follows: $\chi_{s,\mu} \sim \chi_{s,\mu'}$ if there exists $\Gamma \in \mathcal{G}$ such that $\mu_\Gamma = \mu'_\Gamma$. The relation $\sim$ is reflexive and symmetric, and since $|\mathcal{G}| \geqq 2$, it follows that the transitive extension of $\sim$ is an equivalence relation with one equivalence class, namely s_G. Hence it will be enough to prove the following: If $\chi_{s,\mu} \in B$ and $\chi_{s,\mu} \sim \chi_{s,\mu'}$, then $\chi_{s,\mu'} \in B$. We set $\chi = \chi_{s,\mu}$, $\chi' = \chi_{s,\mu'}$ for simplicity of notation. Let

$$\mathcal{G}_0 = \{\Gamma \in \mathcal{G}\colon \mu_\Gamma = \mu'_\Gamma\}$$
$$\mathcal{G}_1 = \{\Gamma \in \mathcal{G}\colon \mu_\Gamma \neq \mu'_\Gamma\}.$$

We may assume $\mathcal{G}_1 \neq \phi$ as well as $\mathcal{G}_0 \neq \phi$.

A defect group R of B has the form $\prod_{\Gamma \in \mathcal{G}} R_\Gamma$, where by (5E) R_Γ is a Sylow subgroup of $C(s)_\Gamma$. Let ω be a fixed primitive r-th root of unity in F, and let ρ be the element of order r in $Z(R)$ defined by

$$\rho_\Gamma = \begin{cases} I_\Gamma & \text{if } \Gamma \in \mathcal{G}_0 \\ \omega I_\Gamma & \text{if } \Gamma \in \mathcal{G}_1, \end{cases}$$

where I_Γ is the identity matrix in R_Γ. In particular, $C(\rho) = H = H_0 \times H_1$ is a proper subgroup of G. Let $s^G \cap H = \bigcup_\alpha (s_\alpha)^H$, and for each α, write $s_\alpha = s_{\alpha 0} \times s_{\alpha 1}$, where $s_{\alpha i} \in H_i$. We may assume s has been chosen as one of the s_α's, and that for $s = s_0 \times s_1$, the elementary divisors of s_i are in $\mathcal{G}_i$ for $i = 0, 1$. Let b be the block of H corresponding to the H-class s^H. Then b is a major subsection of B in H with defect group R by the remarks following (5D).

We apply (2I) to $\chi(\rho)$ and $\chi'(\rho)$. Thus $\chi(\rho)=\sum_\alpha \chi^{(s_\alpha)}(\rho)$, where

(6.5) $$\chi^{(s_\alpha)}(\rho)=\hat{s}_\alpha(\rho)\sum_\lambda A^\alpha_{\mu,\lambda}\chi^{s_{\alpha 0},\lambda}(1).$$

For fixed α, λ runs over all $\prod_\Gamma \lambda_\Gamma$ such that λ_Γ is a partition of $m_\Gamma(s_{\alpha 0})$ obtained from μ_Γ by deleting a sequence of $m_\Gamma(s_{\alpha 1})$ 1-hooks. The coefficients $A^\alpha_{\mu,\lambda}$ are the quantities in (2.14). We have an analogous formula for χ'.

(6.6) $$\chi'^{(s_\alpha)}(\rho)=\hat{s}_\alpha(\rho)\sum_{\lambda'} A^\alpha_{\mu',\lambda'}\chi^{s_{\alpha 0},\lambda'}(1),$$

where λ' runs over all $\prod_\Gamma \lambda'_\Gamma$ such that λ'_Γ is a partition of $m_\Gamma(s_{\alpha 0})$ obtained from μ'_Γ by deleting a sequence of $m_\Gamma(s_{\alpha 1})$ 1-hooks. By our choice of ρ it follows that for $s_\alpha=s$ in (6.5) and (6.6), the λ and λ' in these expressions are unique since $s_{\alpha 0}$ and $s_{\alpha 1}$ have no Γ in common, and indeed,

$$\lambda_\Gamma=\mu_\Gamma,\ \lambda'_\Gamma=\mu'_\Gamma \quad \text{for } \Gamma\in\mathscr{G}_0.$$

Thus $\lambda=\lambda'$ in these expressions. Moreover, $\chi^{(s)}=\chi^{(b)}$, $\chi'^{(s)}=\chi'^{(b)}$ by induction, so (6.5) and (6.6) become

(6.7) $$\chi^{(b)}(\rho)=\hat{s}(\rho)\,A^\alpha_{\mu,\lambda}\chi^{s_0,\lambda}(1)$$ $$\chi'^{(b)}(\rho)=\hat{s}(\rho)\,A^\alpha_{\mu',\lambda}\chi^{s_0,\lambda}(1)$$

when $s_\alpha=s$. Now

$$A^\alpha_{\mu,\lambda}=|H_1: C_{H_1}(s_1)|_{p'}\prod_\Gamma a^{\rho_\Gamma}_{\mu_\Gamma,\lambda_\Gamma},$$

where the $a^{\rho_\Gamma}_{\mu_\Gamma,\lambda_\Gamma}$ are the coefficients occurring in

$$\chi^{\mu_\Gamma}(\rho_\Gamma)=a^{\rho_\Gamma}_{\mu_\Gamma,\lambda_\Gamma}\chi^{\lambda_\Gamma}(1).$$

If $\Gamma\in\mathscr{G}_0$, then $\rho_\Gamma=\mathrm{I}_\Gamma$, $\mu_\Gamma=\lambda_\Gamma$, and $a^{\rho_\Gamma}_{\mu_\Gamma,\lambda_\Gamma}=1$. If $\Gamma\in\mathscr{G}_1$, then $\rho_\Gamma=\omega\mathrm{I}_\Gamma$, $\lambda_\Gamma=\phi$, and $a^{\rho_\Gamma}_{\mu_\Gamma,\lambda_\Gamma}=\chi^{\mu_\Gamma}(1)$. Thus

$$A^\alpha_{\mu,\lambda}=|H_1: C_{H_1}(s_1)|_{p'}\prod_{\Gamma\in\mathscr{G}_1}\chi^{\mu_\Gamma}(1).$$

Similarly

$$A^\alpha_{\mu',\lambda}=|H_1: C_{H_1}(s_1)|_{p'}\prod_{\Gamma\in\mathscr{G}_1}\chi^{\mu'_\Gamma}(1)$$

Hence by (6.7)

$$\chi^{(b)}(\rho)=N\chi'^{(b)}(\rho), \qquad \text{where } N=\prod_{\Gamma\in\mathscr{G}_1}\frac{\chi^{\mu_\Gamma}(1)}{\chi^{\mu'_\Gamma}(1)}.$$

But $\chi^{(b)}(\rho)\neq 0$ by [3], II(4C), since b is a major subsection of B in H. Thus $\chi'^{(b)}(\rho)\neq 0$ and $\chi'\in B$ by Brauer's Second Main Theorem. This completes the proof of (6D).

We also require two results related to the minimal configurations $G^{m,\alpha}$ and $G^{m,\alpha,\beta}$.

Lemma (6E). *Let $G=G^{m,\alpha}$, let B be a block of G with defect group $R=R^{m,\alpha}$, and let $\mathcal{J}(B)=s^G$, where $s\in C(R)$. Then $B=\bigcup_{y\in R}(sy)_G$.*

Proof. We note that (6E) is just (6A) for a block with cyclic defect group. Since $G=G^{m,\alpha}$ and $R=R^{m,\alpha}$, it follows by (4D) that s has only one elementary divisor Γ, of multiplicity $e_\Gamma=1$, and $C_G(s)=T$ is a Coxeter torus of G. If $R\leqq Z(G)$, then

$$B=\{\varepsilon_s R_T^G(\widehat{sy})\colon y\in R\}$$

by (4C). But $(sy)_G=\{\varepsilon_s R_T^G(\widehat{sy})\}$ since $C_G(sy)=T$ for all $y\in R$. Thus (6E) holds. Suppose then $R\not\leqq Z(G)$ and let $H=C(R)=C(\rho)$, where ρ is a generator of R. Suppose $\chi\in B$ and $\chi\in t_G$. Let $t^G\cap H=\bigcup_\alpha (t_\alpha)^H$. For each block b of H, let $\Delta(b)=\{\alpha\colon (t_\alpha)_H\subseteq b\}$ as before. Now there exists some r'-element σ in G with defect group R such that $\omega_\chi(\sigma)\not\equiv 0\,(\mathrm{mod}\,\mathrm{r})$, where $\omega_\chi(\sigma)$ denotes the value of the algebra homomorphism ω_χ associated to χ on the G-class sum of σ. Since R is cyclic, χ has height 0, and so $\chi(\sigma)\not\equiv 0\,(\mathrm{mod}\,\mathrm{r})$. But $\sigma\in C(\rho)$ and $\chi(\rho\sigma)\equiv\chi(\sigma)\,(\mathrm{mod}\,\mathrm{r})$. Thus $\chi(\rho\sigma)\neq 0$, so by (6.3) and (6.4) there exists a block b of H and $\alpha\in\Delta(b)$ such that $\chi^{(b)}(\rho\sigma)\neq 0$ and $\chi^{(t_\alpha)}(\rho\sigma)\neq 0$. The first inequality implies $Br_H^G(b)=B$ by Brauer's Second Main Theorem [2], II, (6A). But all subsections of B in H are major subsections with defect group R since R is cyclic and normal in H. So by the remarks following (5D), $\mathcal{J}(b)=(s^n)^H$ for some $n\in N(R)$. The second inequality implies by (6.1) that there exists ζ in $(t_\alpha)_H$ such that $\zeta\in b$. So by induction $(t_\alpha)_{r'}\sim_H s^n$. Thus $t\sim_G sy$ for some $y\in R$, since R is a Sylow subgroup of $C_G(s)$. We have shown then that $B\subseteq\bigcup_{y\in R}(sy)_G$. On the other hand, the Brauer-Dade theory [6], Theorem 1, implies B contains one non-exceptional character and $|R|-1$ exceptional characters. But the class $(sy)_G$ contains only one character since $C_G(sy)=T$, and the number of such classes is at most $|R|$. Hence $B=\bigcup_{y\in R}(sy)_G$ and the proof of (6E) is complete.

Let $G=G^{m,\alpha,\beta}$, $R=R^{m,\alpha,\beta}$, $C=C^{m,\alpha,\beta}$, where $\beta>0$, and let B be a block of G with defect group R. By (5C) a root b of B with defect group R has the form $Br_C^{RC}(\mathrm{b}\otimes I_\beta)$, where b is a block of $C^{m,\alpha}$ with defect group $R^{m,\alpha}$, I_β is the identity matrix of degree r^β, and $\mathrm{b}\otimes I_\beta$ is the corresponding block of C. Moreover, the defect group R is of the form $R^{m,\alpha}\wr X_\beta$, where X_β is a Sylow r-subgroup of the group S_β of permutation matrices of degree r^β. We choose for X_β a group of the form $X_{\beta-1}\wr X_1$, so that

$$R=(R^{m,\alpha}\wr X_{\beta-1})\wr X_1=R^{m,\alpha,\beta-1}\wr X_1.$$

Let ρ be an element of order r in the base group of the wreath product $R^{m,\alpha,\beta-1}\wr X_1$ of the form

$$\rho=\begin{pmatrix}\rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \rho_r\end{pmatrix}$$

Here $\rho_i = \omega_i I$, where ω_i is an r-th root of unity, possibly 1, I is the identity matrix of $R^{m,\alpha,\beta-1}$, and $\omega_i \neq \omega_j$ for $i \neq j$. Then $C(\rho) = H = H_1 \times H_2 \times \ldots \times H_r$, where $H_i = G^{m,\alpha,\beta-1}$. Let B_i be the block of H_i induced by $\mathrm{b} \otimes \mathrm{I}_{\beta-1}$, and denote the defect group $R^{m,\alpha,\beta-1}$ of B_i as R_i. Finally, let B_H be the block of H with defect group $R_H = \prod_i R_i$ defined by $B_H = \prod_i B_i$.

Lemma (6F). *Let the notation be as above. Then* $Br_H^G(B_H) = B$.

Proof. Let $N = N_G(H)$, so $N = HS$ where S is isomorphic to the symmetric group of degree r and $S \cap H = 1$. Let $B_N = Br_H^N(B_H)$ and let R_N be a defect group of B_N. The block B_N is defined since $C(R_H) \leqq H$. Since N stabilizes B_H, it follows by [3], I, (4C), (4D), (4G) that $R_N \cap H = R_H$ and that $R_N / R_N \cap H$ is isomorphic to a Sylow r-subgroup of S. Thus

$$\begin{aligned} v(R_N) &= r[r^{\beta-1}(a+\alpha) + v(r^{\beta-1}!)] + v(r!) \\ &= r^{\beta}(a+\alpha) + v(r^{\beta}!) \end{aligned}$$

and $v(R_N) = v(R)$.

We claim that R is a defect group of B_N. Indeed, let ψ_1 be a fixed character in B_1 of height 0, and for $2 \leqq i \leqq r$, let ψ_i be the character in B_i corresponding to ψ_1 under the natural identification of B_i, H_i with B_1, H_1. Then $\psi = \psi_1 \psi_2 \ldots \psi_r$ is a character in B_H of height 0 invariant under N. Let χ be a character in B_N such that $\chi|_H$ contains ψ as a constituent, and let ω_χ and ω_ψ be the algebra homomorphisms corresponding to χ and ψ. The choice of such a χ is possible by [3], I, (4A). Suppose $\mathscr{K}$ is a conjugacy class of N contained in H, and $\mathscr{K}$ decomposes as $\mathscr{L}^{(1)} \cup \mathscr{L}^{(2)} \cup \ldots \cup \mathscr{L}^{(s)}$, where the $\mathscr{L}^{(i)}$ are conjugacy classes of H. Since $\chi|_H = m\psi$ for some integer m, it follows that $\omega_\chi([\mathscr{K}]) = s\omega_\psi([\mathscr{L}^{(1)}])$, where $[\mathscr{K}]$ and $[\mathscr{L}^{(1)}]$ denote the class sums over $\mathscr{K}$ and $\mathscr{L}^{(1)}$ respectively. In particular, we define a class $\mathscr{K}$ with defect group R as follows: Let $\mathscr{L}_1$ be an r-regular class of H_1 with defect group R_1 such that $\omega_{\psi_1}([\mathscr{L}_1]) \not\equiv 0 \pmod{\mathrm{r}}$, let $\mathscr{L}_i$ be the class of H_i corresponding to $\mathscr{L}_1$, and let $\mathscr{L} = \mathscr{L}_1 \times \mathscr{L}_2 \times \ldots \times \mathscr{L}_r$. Then $\mathscr{L}$ is also a conjugacy class of N, and we take $\mathscr{K} = \mathscr{L}$. But

$$\omega_\chi([\mathscr{K}]) = \omega_{\psi_1}([\mathscr{L}_1])^r \not\equiv 0 \pmod{\mathrm{r}}.$$

Thus R is a defect group of B_N.

The root b of B has the form $Br_C^{CR}(\mathrm{b} \otimes I_\beta)$. Since $CR \leqq N$, $Br_{CR}^N(b)$ is defined as a block of N with defect group R. We set $b^N = Br_{CR}^N(b)$. The transitivity properties of the Brauer mappings imply

$$b^N = Br_{CR}^N(b) = Br_C^N(\mathrm{b} \otimes I_\beta), \qquad B = Br_N^G(b^N).$$

Hence, to complete the proof of (6F), it will be enough to show $B_N = b^N$. But since B_N and b^N have defect group R, it will be enough to show that the algebra characters ω_{B_N} and ω_{b^N} associated to B_N and b^N satisfy

$$\omega_{B_N}([\mathscr{K}]) = \omega_{b^N}([\mathscr{K}]) \tag{6.8}$$

for all r-regular classes $\mathscr{K}$ of N with R as a defect group. Let $\mathscr{K}$ be such a class, so there exists $g \in \mathscr{K}$ with defect group R. Then $g \in C \subseteq H$, so on the one hand, $g = c \otimes I_\beta$ for some $c \in C^{m,\alpha}$, and on the other hand, $g = g_1 g_2 \ldots g_r$, where $g_i = c \otimes I_{\beta-1}$ is in H_i.

Now $b^N = Br_C^N(\mathrm{b} \otimes I_\beta)$, so $\omega_{b^N}([\mathscr{K}]) = \omega_{\mathrm{b} \otimes I_\beta}([\mathscr{K} \cap \mathrm{C}])$. Since $\mathscr{K}$ has R as a defect group, it follows by [2], I, (10A) that $\mathscr{K} \cap C$ is a single $N_N(R)$-conjugacy class, namely $g^{N_N(R)}$. But

$$N_G(R) = N^{m,\alpha}/R^{m,\alpha} \otimes N_{S_\beta}(X_\beta)$$
$$= (N^{m,\alpha}/R^{m,\alpha} \otimes I_\beta)(I \otimes N_{S_\beta}(X_\beta))$$

and $N \supseteq (N^{m,\alpha}/R^{m,\alpha} \otimes I_\beta)$. Thus

$$N_N(R) = (N^{m,\alpha}/R^{m,\alpha} \otimes I_\beta)(I \otimes Y)$$

for some subgroup Y where $X_\beta \leqq Y \leqq N_{S_\beta}(X_\beta)$. Since $I \otimes Y$ centralizes g, it follows that

$$\mathscr{K} \cap C = g^{N_N(R)} = \{c^n \otimes I_\beta \colon n \in N^{m,\alpha}\}.$$

If $\mathscr{C} = c^{N^{m,\alpha}}$, we have then that $\mathscr{K} \cap C = \mathscr{C} \otimes I_\beta$. In particular,

$$\omega_{b^N}([\mathscr{K}]) = \omega_{\mathrm{b} \otimes I_\beta}([\mathscr{K} \cap C]) = \omega_{\mathrm{b}}([\mathscr{C}]). \tag{6.9}$$

On the other hand, if $\mathscr{L}_i = g_i^{H_i}$, then $\mathscr{K} = \mathscr{L}_1 \times \mathscr{L}_2 \times \ldots \times \mathscr{L}_r$ so that $\mathscr{K} \cap H = \mathscr{K}$. Thus

$$\omega_{B_N}([\mathscr{K}]) = \omega_{B_H}([\mathscr{K}]) = \omega_{B_1}([\mathscr{L}_1])^r = \omega_{B_1}([\mathscr{L}_1]).$$

But $B_1 = Br_{C_1}^{H_1}(\mathrm{b} \otimes I_{\beta-1})$, where $C_1 = C_{H_1}(R_1) = C^{m,\alpha,\beta-1}$. Thus $\omega_{B_1}([\mathscr{L}_1]) = \omega_{\mathrm{b} \otimes I_{\beta-1}}([\mathscr{L}_1 \cap \mathrm{C}_1])$. Let $N_1 = N_{H_1}(R_1) = N^{m,\alpha,\beta-1}$. Since $\mathscr{L}_1$ is an H_1-conjugacy class with R_1 as a defect group, $\mathscr{L}_1 \cap C_1$ is a single N_1-conjugacy class, namely $g_1^{N_1}$. Just as before, we conclude that $g_1^{N_1} = \mathscr{C} \otimes I_{\beta-1}$. Thus

$$\omega_{B_N}([\mathscr{K}]) = \omega_{\mathrm{b}}([\mathscr{C}]). \tag{6.10}$$

Now (6.9) and (6.10) imply (6.8), and the proof of (6F) is complete.

We now prove (6B). We partition the blocks of G into two disjoint sets. A block B with defect group R is of type I if $\Omega_1(Z(R)) \leqq Z(G)$, and of type II if $\Omega_1(Z(R)) \nleqq Z(G)$. Blocks B of type I occur only when there exist integers m, α, β such that $G = G^{m,\alpha,\beta}$, $R = R^{m,\alpha,\beta}$, and B is a block of G with defect group R. In particular, the semisimple r'-element s corresponding to B has only one elementary divisor.

Suppose B is a block of G of type I with defect group R, and $\mathscr{J}(B) = s^G$ where $s \in C(R)$. If R is cyclic, then $s_G \subseteq B$ by (6E). If R is non-cyclic, let $\rho \in \Omega_1(R)$ and $H = C(\rho)$ be as in (6F). In particular, $H < G$. Let $s^G \cap H = \bigcup_\alpha (s_\alpha)^H$, where s is one of the representatives s_α. Given $\chi \in s_G$, we consider the particular sum

$$\chi^{(s)}(\rho\sigma) = \sum_{\zeta \in s_H} (\chi, R_H^G(\zeta))\, \zeta(\rho\sigma)$$

for all $\sigma \in H_{r'}$. If B_H is the block of H corresponding to the H-conjugacy class s^H, then

$$\chi^{(s)}(\rho\sigma) = \chi^{(B_H)}(\rho\sigma)$$

by (6.3). So by (6C) there exists $\sigma \in H_{r'}$ such that $\chi^{(B_H)}(\rho\sigma) \neq 0$. Thus $\chi \in Br_H^G(B_H) = B$ by (6F) and Brauer's Second Main Theorem. Hence $s_G \subseteq B$.

Now suppose B is a block of G of type II with defect group R, and $\mathscr{J}(B) = s^G$ where $s \in C(R)$. Choose $\rho \in \Omega_1(Z(R))$ with $\rho \notin Z(G)$, and let $H = C(\rho)$. We claim $B \subseteq \bigcup_{y \in R} (sy)_H$. Indeed, suppose $\chi \in B \cap t_G$, and let $t^G \cap H = \bigcup_{\alpha} t_\alpha^H$. For each block b of H, let $\Delta(b) = \{\alpha \colon (t_\alpha)_H \subseteq b\}$. Since $\rho \in Z(R)$, there exists by (6.3), (6.4), and [3], II (4C) a major subsection b of B in H with defect group R and an $\alpha \in \Delta(b)$ such that $\chi^{(b)}(\rho) \neq 0$, $\chi^{(t_\alpha)}(\rho) \neq 0$. But $\mathscr{J}(b) = (s^n)^H$ for some $n \in N(R)$ by the remarks following (5D). Thus $(t_\alpha)_{r'} \sim_G s$, and since R is a Sylow r-subgroup of $C_G(s)$, $t \sim_G sy$ for some $y \in R$. Hence χ is of the form $\chi_{sy,\mu}$. Now on r'-elements of G,

$$\chi^{sy,\mu} = R_{C(sy)}^G(\hat{s}\chi^\mu) = R_{C(s)}^G(\hat{s} R_{C(sy)}^{C(s)}(\chi^\mu)).$$

In particular, there exists a unipotent character ψ of $C(s)$ occurring as a constituent of $R_{C(sy)}^{C(s)}(\chi^\mu)$ such that $R_{C(s)}^G(\hat{s}\psi)$ contains a Brauer character in B with non-zero multiplicity. Thus $s_G \cap B \neq \phi$.

If s has more than one elementary divisor, then $s_G \subseteq B$ by (6D). Suppose then s has only one elementary divisor, and let $s^G \cap H = \bigcup_\alpha s_\alpha^H$, where s is one of the s_α. The block b of H corresponding to the H-conjugacy class s^H is then a major subsection of B in H with defect group R. Moreover, for any $\chi \in s_G$,

$$\chi^{(b)}(\rho\sigma) = \chi^{(s)}(\rho\sigma)$$

for all $\sigma \in H_{r'}$. By (6C) there exists σ in $H_{r'}$ such that $\chi^{(b)}(\rho\sigma) \neq 0$. So by Brauer's Second Main Theorem, $\chi \in B$. We have shown then $s_G \subseteq B$ for blocks of type II as well. This completes the proof of (6B) and hence of (6A).

§ 7.

We now prove the classification of characters in a block in the general case.

Theorem (7A). *Let B be a block of G with defect group R, and let $\mathscr{J}(B) = (s, \lambda)^G$, where $s \in C(R)$ and $\lambda \in \mathscr{C}(s)$. Let χ be an irreducible character of G of the form $\chi_{t,\nu}$. Then $\chi \in B$ if and only if*

i) *$t \sim_G sy$ for some $y \in R$*

ii) *for every $\Gamma \in \mathscr{F}$, ν_Γ has e_Γ-core λ_Γ.*

Proof. Suppose $\chi_{t,\nu} \in B$. If $R = 1$, then $B = \{\chi_{s,\lambda}\}$ and (7A) holds. Suppose $R \neq 1$ and choose ρ in $\Omega_1(Z(R))$ such that ρ has one other elementary divisor other than $X - 1$ and such that $C_V(\rho) = C_V(R)$, V being the underlying space of G. Let $V_0 = C_V(\rho)$, and let $V_1 = [V, \rho] = [V, R]$. Then $C(\rho) = H = H_0 \times H_1$, where H_i

acts on V_i. We may assume $H<G$, for otherwise (7A) follows from (6A). By the remarks following (5D) there exists a major subsection B_α of B in H with defect group R such that $\mathscr{J}(B_\alpha)=(s,\lambda)^H$. Here, if $B_\alpha=B_{\alpha 0}\times B_{\alpha 1}$ and $s=s_0 s_1$, where $B_{\alpha i}$ is a block of H_i and $s_i\in H_i$, then $\mathscr{J}(B_{\alpha 0})=(s_0,\lambda)^{H_0}$ and $\mathscr{J}(B_{\alpha 1})=(s_1,\{-\})^{H_1}$, the last equality holding since $Z(H_1)$ contains elements of order r. Moreover, by the choice of ρ, $B_{\alpha 0}$ must have defect 0, so $B_{\alpha 0}$ consists of only one character ζ_0, say of the form $\chi_{s_0,\lambda}$.

We now argue as we did in Paragraph 6. Namely, we have

$$\chi(\rho\sigma)=\sum_b \sum_{\zeta\in b} (\chi, R_H^G(\zeta))\,\zeta(\rho\sigma)$$

for all $\sigma\in H_{r'}$, where b runs over the blocks of H. The ζ's in b can be expressed in terms of the Brauer characters φ in b, and the linear independence of the Brauer characters implies as before that

$$\chi^{(b)}(\rho\sigma)=\sum_{\zeta\in b} (\chi, R_H^G(\zeta))\,\zeta(\rho\sigma).$$

Since $\rho\in Z(R)$ and B_α is a major subsection, $\chi^{(B\alpha)}(\rho)\neq 0$ by [3], II, (4C). In particular, there exists $\zeta\in B_\alpha$ such that $(\chi, R_H^G(\zeta))\neq 0$. We write $\zeta=\zeta_0\zeta_1$, where $\zeta_0=\chi_{s_0,\lambda}$ is the unique character of $B_{\alpha 0}$ and $\zeta_1\in B_{\alpha 1}$. By (6A) ζ_1 has the form $\chi_{s_1 y,\mu}$ for some $y\in R$. Thus $\zeta\in(sy)_H$, $t\sim_G sy$ by (1B), and i) holds.

We may assume then that $\chi_{t,\nu}=\chi_{sy,\nu}$. Now

$$\begin{aligned}(\chi^{sy,\nu}, R_H^G(\zeta)) &= (\chi^{sy,\nu}, R_H^G(\chi^{s_0,\lambda}\times\chi^{s_1 y,\mu}))\\ &=(\chi^\nu, R_L^K(\chi^\lambda\times\chi^\mu))\end{aligned}$$

by (1B), where $K=C_G(sy)$, $L=C_H(sy)$. Thus $(\chi^\nu, R_L^K(\chi^\lambda\chi^\mu))\neq 0$. Let $sy=\prod_\Gamma (sy)_\Gamma$ be the primary decomposition of sy, and let $\prod_\Gamma K_\Gamma$ and $\prod_\Gamma \rho_\Gamma$ be the corresponding decompositions of K and ρ, where $\rho_\Gamma\in K_\Gamma$. In particular, if we set $L_\Gamma=C_{K_\Gamma}(\rho_\Gamma)$, then $L=\prod_\Gamma L_\Gamma$. Now ρ_Γ need not be primary as an element of K_Γ. But the elementary divisors Δ of ρ_Γ relative to K_Γ, with the exception of $X-1$, are algebraic conjugates and hence of the same degree $d_{\Gamma,\Delta}$. Here Δ and $d_{\Gamma,\Delta}$ play the role for K_Γ that Γ and d_Γ play for G. For simiplicity of notation, let D_Γ be this common degree $d_{\Gamma,\Delta}$. Since

$$(\chi^\nu, R_L^K(\chi^\lambda\chi^\mu))=\prod_\Gamma (\chi^{\nu_\Gamma}, R_{L_\Gamma}^{K_\Gamma}(\chi^{\lambda_\Gamma}\chi^{\mu_\Gamma}))$$

it follows that

$$(\chi^{\nu_\Gamma}, R_{L_\Gamma}^{K_\Gamma}(\chi^{\lambda_\Gamma}\chi^{\mu_\Gamma}))\neq 0$$

for each Γ. By (2.12) and the remark following it, λ_Γ is thus obtained from ν_Γ by deleting a sequence of D_Γ-hooks. Let Γ be an elementary divisor of sy. If $\Gamma\in\mathscr{F}'$, then $D_\Gamma=e_\Gamma$, since both are the integer n_Γ of (4.7). For such Γ, λ_Γ is obtained from ν_Γ by deleting a sequence of e_Γ-hooks, and so λ_Γ is the e_Γ-core of ν_Γ. In particular, this is so for the elementary divisors of s_0. If $\Gamma \notin \mathscr{F}'$, then $Z(K_\Gamma)$ contains elements of order r, so $e_\Gamma = D_\Gamma = 1$. Thus the e_Γ-core of ν_Γ is $\{-\}$. But $\lambda_\Gamma = \{-\}$ as well, so (ii) holds.

Conversely, let $\chi = \chi_{sy,\nu}$, where $y \in R$ and ν satisfies ii). Suppose χ is in the block B' with defect group R', and let $\mathscr{J}(B') = (s', \lambda')^G$, where $s' \in C(R')$. The first part of (7A) implies that $(sy) \sim_G (s'y')$ for some $y' \in R'$, and so $s \sim_G s'$. Moreover, for each Γ, the e_Γ-core of ν_Γ is λ'_Γ. But ii) implies $\lambda'_\Gamma = \lambda_\Gamma$ for each Γ. Thus $(s', \lambda)^G = (s, \lambda)^G$, $B' = B$ by (5D), and $\chi \in B$. This completes the proof.

The preceding theorem classifying the characters in B can be viewed as an extension of the Brauer-Dade theory for blocks with cyclic defect groups. Indeed, we may define the characters in $B \cap s_G$ as the non-exceptional characters of B. As will be shown in (8A), the restrictions of the characters in $B \cap s_G$ to $G_{r'}$ form a basic set for B, so as in the cyclic theory, the number of Brauer characters in B is $|B \cap s_G|$. We may define the remaining characters in B as the exceptional characters of B. These fall into families $B \cap (sy)_G$, where $y \in R^{\#}$. Their values on the r-sections of G which meet R can be expressed in terms of the character formulas of Paragraph 2. In the case where R is cyclic, we recover the formulas of the cyclic theory.

§ 8.

This section contains consequences of the theorems (5D) and (7A) classifying blocks and characters of $G = GL(n,q)$ or $U(n,q)$.

Theorem (8A). *Let B be a block G with defect group R, and let $\mathscr{J}(B) = (s, \lambda)^G$, where $s \in C(R)$. Let M_B be the Z-module generated by the restrictions of the χ in B to the set $G_{r'}$ of r'-elements of G, and let M'_B be the submodule of M_B generated by the restrictions of the χ in $B \cap s_G$ to $G_{r'}$. Then $M_B = M'_B$. Moreover, the number l_B of Brauer characters in B is $|B \cap s_G|$.*

Proof. A character in B has the form $\chi_{sy,\mu}$, where $y \in R$ and μ_Γ has e_Γ-core λ_Γ for all $\Gamma \in \mathscr{F}$. Now on $G_{r'}$,

$$\chi^{sy,\mu} = R^G_{C(sy)}(\hat{s}\chi^\mu) = R^G_{C(s)}(\hat{s} R^{C(s)}_{C(sy)}(\chi^\mu)). \tag{8.1}$$

Let $\prod_\Gamma s_\Gamma$ be the primary decomposition of s, and let $\prod_\Gamma K_\Gamma$ be corresponding decomposition of $C_G(s) = K$. Since $y \in C(s)$, we may write $y = \prod_\Gamma y_\Gamma$, where $y_\Gamma \in K_\Gamma$. As an element of K_Γ, y_Γ is the product $\prod_{\Delta \in \mathscr{F}_\Gamma} y_{\Gamma,\Delta}$ of its primary factors, where Δ runs over the set $\mathscr{F}_\Gamma$ of elementary divisors appropriate to K_Γ. Moreover, the factors $y_{\Gamma,\Delta}$ and $y_{\Gamma',\Delta'}$ are different if $(\Gamma, \Delta) \neq (\Gamma', \Delta')$. Let $y_{\Gamma,0}$ be the primary factor of y_Γ corresponding to the elementary divisor $X - 1$ in $\mathscr{F}_\Gamma$. The degrees of the remaining elementary divisors Δ of y_Γ are then divisible by e_Γ. Now $C_{K_\Gamma}(y_\Gamma) = L_\Gamma = \prod_{\Delta \in \mathscr{F}_\Gamma} L_{\Gamma,\Delta}$, and $C(sy) = L = \prod_{\Delta,\Gamma} L_{\Gamma,\Delta}$. The class function χ^μ in (8.1) is then of the form $\prod_{\Delta,\Gamma} \chi^{\mu_{\Gamma,\Delta}}$, where $\mu_{\Gamma,\Delta}$ is a partition of $m_\Delta(y_\Gamma)$. In particular, the partition $\mu_{\Gamma,0}$ corresponding to the factor $y_{\Gamma,0}$ is a partition of $m_{X-1}(y_\Gamma)=m_\Gamma(sy)$, and $\mu_{\Gamma,0}$ has e_Γ-core λ_Γ. Now

$$(8.2) \qquad \begin{aligned} R^{C(s)}_{C(sy)}(\chi^\mu) &= \prod_\Gamma R^{K_\Gamma}_{L_\Gamma}\Big(\prod_{\Delta\in\mathscr{F}_\Gamma}\chi^{\mu_{\Gamma,\Delta}}\Big) \\ &= \prod_\Gamma R^{K_\Gamma}_{L_\Gamma}\Big(\chi^{\mu_{\Gamma,0}}\times \prod_{\substack{\Delta\in\mathscr{F}_\Gamma \\ \Delta\neq X-1}}\chi^{\mu_{\Gamma,\Delta}}\Big). \end{aligned}$$

A unipotent character χ_ν of K occurs as a constituent of (8.2) only if for each Γ, $\mu_{\Gamma,0}$ is obtained from ν_Γ by deleting a sequence of e_Γ-hooks. Indeed, this follows by the second remark following (2G) and the fact that the Δ's different from $X-1$ occurring as elementary divisors of y_Γ have degrees divisible by e_Γ. In particular, the e_Γ-core of ν_Γ is λ_Γ. The right-hand side of (8.1) hence belongs to M'_B and $M_B=M'_B$. We have shown then that $l_B\leqq|B\cap s_G|$. Now $|s_G|$ is the number of unipotent conjugacy classes in $C_G(s)$. Thus $|s_G|$ is the number of r'-conjugacy classes of G having a representative whose semisimple part is s, and $\sum_{s_G}\sum_B|B\cap s_G|$ is the total number of r'-conjugacy classes of G. But this last number is also $\sum_B l_B$. Thus $l_B=|B\cap s_G|$ for all B and (8 A) holds.

We next consider the height conjecture for blocks of $G=GL(n,q)$ or $U(n,q)$.

Theorem (8B). *Let B be a block of G with defect group R, and let $\mathscr{J}(B)=(s,\lambda)^G$, where $s\in C(R)$.*

i) *If R is non-abelian, then there exists an irreducible character in $B\cap s_G$ of positive height.*

ii) *If R is abelian, then all irreducible characters in B have zero height.*

Proof. We begin with some remarks on the characters of the symmetric groups (see [14]). Let t be a fixed positive integer, let λ be the t-core of some partition of n, and let $w=t^{-1}(n-|\lambda|)$. Each partition μ of n with t-core λ then determines a quotient or skew partition of w consisting of t disjoint proper partitions, some of which may be empty. Conversely, given t proper partitions of w nodes altogether, there exists a partition μ of n with t-core λ such that μ/λ is the given union. The hooks of μ/λ are in bijection with the hooks of μ having length divisible by t, the bijection mapping a j-hook of μ/λ onto a jt-hook of μ. The character $\phi_{\mu/\lambda}$ of S_w corresponding to μ/λ has degree $w!/H_{\mu/\lambda}$, where $H_{\mu/\lambda}$ is the product of all hook lengths, multiplicities counted, in μ/λ. Moreover, if $\nu_1,\nu_2,\ldots,\nu_r$ are the proper partitions occurring as constitutents of μ/λ, then $\phi_{\mu/\lambda}=\phi_{\nu_1}\circ\phi_{\nu_2}\circ\ldots\circ\phi_{\nu_r}$, where $\circ$ is the operation introduced in Paragraph 2. If $w\geqq r$, it follows by induction on w that there exists a partition μ of n with t-core λ such that $\phi_{\mu/\lambda}(1)$ is divisible by r. On the other hand, if $w<r$, then $\phi_{\mu/\lambda}(1)$ is relatively prime to r for all such μ

Let $\prod_\Gamma s_\Gamma$ be the primary decomposition of s, and let $\prod_\Gamma K_\Gamma$ be the corresponding decomposition of $C_G(s)=K$. As was shown in the proof of (5D), $R=\prod_\Gamma R_\Gamma$, where R is the subgroup of K_Γ defined by

$$(8.3) \qquad R_\Gamma=I_\Gamma\cdot\prod_\beta(R^{m_{\Gamma,\alpha_\Gamma,\beta}})^{t_{\Gamma,\beta}}.$$

Here I_Γ is the identity matrix of degree $|\lambda_\Gamma| d_\Gamma$, m_Γ and α_Γ are the integers determined by Γ in the proof of (5A), and the $t^{\Gamma,\beta}$ are the coefficients occurring in the r-adic expansion $\sum t^{\Gamma,\beta} r^\beta$ of

$$w_\Gamma = e_\Gamma^{-1}(m_\Gamma(s) - |\lambda_\Gamma|).$$

In particular, R_Γ is abelian if and only if $t^{\Gamma,\beta}=0$ for all $\beta>0$, or equivalently, if and only if $w_\Gamma<r$. Moreover,

$$\begin{aligned} v(R_\Gamma) &= \sum_\beta t^{\Gamma,\beta}\, v(R^{m_\Gamma,\alpha_\Gamma,\beta}) \\ &= \sum_\beta t^{\Gamma,\beta}[(a+\alpha_\Gamma) r^\beta + v(r^\beta!)] \\ &= w_\Gamma(a+\alpha_\Gamma) + v(w_\Gamma!) \end{aligned} \tag{8.4}$$

Consider first the case $s=1$. We fix $\Gamma = X-1$, the unique elementary divisor of s. Then $m_\Gamma=1$, $\alpha_\Gamma=0$, and λ is the e_Γ-core of a partition of n. Here $e_\Gamma=e$ or $2e$, and $e_\Gamma=2e$ if and only if G is unitary and r divides q_0^e-1. Set $w=e_\Gamma^{-1}(n-|\lambda|)$. Then

$$R = I_{|\lambda|} \cdot \prod_\beta (R^{1,0,\beta})^{t^{\Gamma,\beta}}$$

$$v(R) = wa + v(w!)$$

by (8.3), (8.4), and R is abelian if and only if $w<r$. Now the unipotent characters in B have the form $\chi_\mu = \chi_{1,\mu}$, where μ has e_Γ-core λ. Let $\mathscr{H}_{\mu/\lambda}$ and $\mathscr{H}_\mu$ denote the set of hook lengths, multiplicities counted, occurring in μ/λ and μ respectively. We claim the following, which is an analogue of a formula for the symmetric group, holds for all χ_μ in B.

$$v(\chi_\mu(1)) = v(G:R) + v(\phi_{\mu/\lambda}(1)) \tag{8.5}$$

In particular, χ_μ has height $v(\phi_{\mu/\lambda}(1))$. Indeed, $\chi_\mu(1)$ is given by (1.15). If G is linear, then

$$\begin{aligned} v(\chi_\mu(1)) &= v(G) - \sum_{h\in\mathscr{H}_\mu} v(q^h-1) \\ &= v(G) - \sum_{h\in\mathscr{H}_{\mu/\lambda}} v(q^{he_\Gamma}-1) \\ &= v(G) - wa - \sum_{h\in\mathscr{H}_{\mu/\lambda}} v(h) \qquad \text{by (3A)} \\ &= v(G:R) + v(w!) - v(H_{\mu/\lambda}) \qquad \text{by (8.4).} \end{aligned}$$

Thus (8.5) holds, since $v(\phi_{\mu/\lambda}(1)) = v(w!) - v(H_{\mu/\lambda})$. If G is unitary, then

$$v(\chi_\mu(1)) = v(G) - \sum_{h\in\mathscr{H}_\mu} v(q_0^h - (-1)^h).$$

But $v(q_0^h-(-1)^h)>0$ if and only if e_Γ divides h, in which case $v(q_0^h-(-1)^h)=a+v(h)$. This follows from (3A) and (4F)iii). Thus (8.5) holds in the unitary case as well. We can now show (8B) for the case of unipotent characters. Indeed, if R is non-abelian, then $w \geqq r$, and there exists a partition μ of n with e_Γ-core λ such that $\nu(\phi_{\mu/\lambda}(1))>0$. The corresponding character χ_μ in B is then of positive height. If R is abelian, then $w<r$, and all unipotent characters in B necessarily have zero height.

The proof of (8B) in the general case now follows by reduction to the preceding case. Suppose R is non-abelian. Since $R=\prod_\Gamma R_\Gamma$, there exists a fixed Γ such that R_Γ is non-abelian. For this fixed Γ, we may choose a partition μ_Γ of $m_\Gamma(s)$ such that μ_Γ has e_Γ-core λ_Γ and such that χ_{μ_Γ} has positive height. Let $\chi_{s,\mu}$ be any character in $B \cap s_G$ such that μ_Γ is the chosen partition. Then

$$\begin{aligned}\nu(\chi_{s,\mu}(1)) &= \nu(G:K)+\sum_\Gamma \nu(\chi_{\mu_\Gamma}(1))\\ &= \nu(G:K)+\sum_\Gamma \nu(K_\Gamma:R_\Gamma)+\sum_\Gamma \nu(\phi_{\mu_\Gamma/\lambda_\Gamma}(1))\\ &= \nu(G:R)+\sum_\Gamma \nu(\phi_{\mu_\Gamma/\lambda_\Gamma}(1)).\end{aligned}$$

Here, we have used (8.5) for each χ_{μ_Γ}. Thus $\chi_{s,\mu}$ has positive height and i) holds.

Finally, suppose R is abelian. Then R_Γ is abelian, $w_\Gamma<r$, and R_Γ has exponent $a+\alpha_\Gamma$ for each Γ. Let $\chi_{sy,\mu} \in B$, where $y \in R$, and let $y=\prod_\Gamma y_\Gamma$, where $y_\Gamma \in R_\Gamma$. We note that the elementary divisors of $s_\Gamma y_\Gamma$ are disjoint from those of $s_{\Gamma'} y_{\Gamma'}$ if $\Gamma \neq \Gamma'$. Let $\mathscr{E}_\Gamma$ be the set of elementary divisors of $s_\Gamma y_\Gamma$. If $\Delta \in \mathscr{E}_\Gamma$ and $\Delta \neq \Gamma$, then $\delta_\Delta=\eta_\Gamma \delta_\Gamma$, where η_Γ is the additive order of δ_Γ modulo e. This follows since the order of y_Γ divides $r^{a+\alpha_\Gamma}$. We claim that $d_\Delta=e_\Gamma d_\Gamma$. Indeed, if G is linear, then $d_\Delta=\delta_\Delta$, $d_\Gamma=\delta_\Gamma$, $e_\Gamma=\eta_\Gamma$, and the claim holds. Suppose G is unitary. Let c be the multiplicative order of a root of Γ, and let cr^γ be the multiplicative order of a root of Δ. We have that $0<\gamma \leqq a+\alpha_\Gamma$. Suppose Γ is of unitary type. Then δ_Γ is odd and c divides $q_0^{\delta_\Gamma}+1$. If $e_\Gamma=\eta_\Gamma$, then η_Γ is odd and r^γ divides $q_0^{\delta_\Gamma \eta_\Gamma}+1$. So $\delta_\Gamma \eta_\Gamma$ is odd, cr^γ divides $q_0^{\delta_\Gamma \eta_\Gamma}+1$, and Δ is of unitary type. Thus $d_\Delta=e_\Gamma d_\Gamma$. If $e_\Gamma=2\eta_\Gamma$, then either η_Γ is even, or η_Γ is odd and r^γ divides $q_0^{\delta_\Gamma \eta_\Gamma}-1$. In the first case δ_Δ is even; in the second case cr^γ does not divide $q_0^{\delta_\Gamma \eta_\Gamma}+1$. In both cases, $d_\Delta=2\delta_\Delta=e_\Gamma d_\Gamma$. Suppose then Γ is of linear type, so $d_\Gamma=2\delta_\Gamma$ and $e_\Gamma=\eta_\Gamma$. If δ_Γ is even, so is δ_Δ, and then $d_\Delta=2\delta_\Delta=e_\Gamma d_\Gamma$. Suppose δ_Γ is odd. Then c does not divide $q_0^{\delta_\Gamma}+1$. If Δ were of unitary type, then η_Γ would be odd, and cr^γ would divide $q_0^{\delta_\Gamma \eta_\Gamma}+1$. This would imply the odd part $c_{2'}$ of c divides $q_0^{\delta_\Gamma}+1$. Thus the 2-part c_2 of c is not 1, q_0 is odd, and c_2 does not divide $q_0^{\delta_\Gamma}+1$. But since c_2 divides $q_0^{\delta_\Gamma \eta_\Gamma}+1$, η_Γ must be even. This is impossible, so Δ is of linear type, $d_\Delta=2\delta_\Delta=e_\Gamma d_\Gamma$, and the claim holds.

Since $m_\Gamma(s)\, d_\Gamma=\sum_{\Delta \in \mathscr{E}_\Gamma} m_\Delta(s_\Gamma y_\Gamma)\, d_\Delta$, it follows by the preceding claim that

$$m_\Gamma(s)=m_\Gamma(s_\Gamma y_\Gamma)+\sum_{\Delta \in \mathscr{E}_\Gamma-\{\Gamma\}} m_\Delta(s_\Gamma y_\Gamma)\, e_\Gamma.$$

But $m_\Gamma(s)=|\lambda_\Gamma|+e_\Gamma w_\Gamma$, and hence

$$r>w_\Gamma=e_\Gamma^{-1}(m_\Gamma(s_\Gamma y_\Gamma)-|\lambda_\Gamma|)+\sum_{\Delta \in \mathscr{E}_\Gamma-\{\Gamma\}} m_\Delta(s_\Gamma y_\Gamma). \tag{8.6}$$

Let $L=C_G(sy)$, and let $\prod_{\Delta} L_{\Delta}$, $\prod_{\Delta} R_{\Delta}$, $\prod_{\Delta} \chi_{\mu_\Delta}$ be the decompositions of L, R, χ_μ corresponding to the primary decomposition of sy. We note that $R \leqq L$ since R is abelian. Now $\chi_{sy,\mu}(1)=|G:L|_{p'}\chi_\mu(1)$, so that

$$v(\chi_{sy,\mu}(1))=v(G:L)+\sum_{\Delta} v(\chi_{\mu_\Delta}(1)).$$

But for each Δ, $v(\chi_{\mu_\Delta}(1))=v(L_\Delta:R_\Delta)$ by the preceding case and (8.6). Thus $v(\chi_{sy,\mu}(1))=v(G:R)$ and $\chi_{sy,\mu}$ has zero height.

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