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Autor: Adler, M.; Moerbeke, P. van

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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The Algebraic Integrability of Geodesic Flow on $SO(4)$

M. Adler[1]* and P. van Moerbeke[2]**

[1] Brandeis University, Waltham, MA 02254, USA
[2] University of Louvain, Louvain-La-Neuve, 1348, Belgium and Brandeis University, Waltham, MA 02254, USA

A Hamiltonian system in a $2s$-dimensional phase space is called completely integrable if it possesses s independent constants of motion which commute with respect to the associated Poisson bracket. By a classical theorem of Liouville, under a compactness assumption, the system typically will execute linear motion on a s-dimensional torus defined by its constants of motion. In most classical cases, the integrals are rational functions of the phase variables and hence, the torus is a real algebraic variety; but in most examples, the real torus above is part of a complex algebraic torus with algebraic addition law consistent with the linearization. Therefore, we shall call a Hamiltonian system *algebraically completely integrable* if it and its commuting flows can be linearized not only on a real algebraic torus but also on an abelian variety (complex algebraic torus with algebraic addition law).

This paper deals with a criterion (Proposition 1 of Sect. 1) for algebraic complete integrability, inspired by work of S. Kowalewski. In celebrated papers [5, 6], she has shown that the *only* algebraically completely integrable systems among the rigid body motions are Euler's rigid body, Lagrange's top and the famous Kowalewski top. Her method is based on the idea that, if the system is to be algebraically completely integrable, the phase variables of the problem (perhaps not the originally given ones) must be algebraic (abelian) functions and, meromorphic in time. This implies the existence of enough codimension one parameter families of (complex) pole solutions of the system so that all the (abelian) phase variables get a chance to blow up (not necessarily simultaneously). The latter requirement imposes algebraic conditions on the coefficients appearing in the differential equations. Although the sufficiency of this criterion is not established in general, this paper does this for a class of geodesic flow equations on $SO(4)$.

This procedure provides for the first time an effective way to check the algebraic complete integrability of a mechanical system; in addition the meth-

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od yields the algebraic torus and realizes the flow as a straight line motion on it. It also produces a way of computing the rational constants of the motion and the periods of the torus. Furthermore it provides a method of regularizing and completing the differential equations wherever they blow up. Propositions 1 and 2 of Sect. 1 contain some general statements to this effect while Theorem 1 of Sect. 2 deals with geodesic flow on $SO(4)$: it gives necessary and sufficient conditions in terms of the metric for algebraic complete integrability of geodesic flow on $SO(4)$ with regard to a diagonal left-invariant metric. This will now be explained.

Consider the group $SO(4)$ and its Lie algebra $so(4)$ paired with itself, via the customary inner product

$$\langle X, Y\rangle = -\tfrac{1}{2}\mathrm{Tr}(XY) \quad \text{where } X=(X_{ij})\in so(4). \tag{1}$$

A left invariant metric on $SO(4)$ is defined by a non-singular symmetric linear map Λ: $so(4)\to so(4)$: $X\curvearrowright \Lambda X$ and by the following inner product: given two vectors gX and gY in the tangent space to $SO(4)$ at the point $g\in SO(4)$

$$\langle gX, gY\rangle \equiv \langle X, \Lambda^{-1} Y\rangle$$

regardless of g. Then geodesic motion with regard to this metric takes on the form

$$\dot{X}=[X, \Lambda X]. \tag{2}$$

This flow is Hamiltonian with regard to the Kostant-Kirillov symplectic structure induced on the orbit formed by the co-adjoint action $\mathrm{Ad}^*_g(X)=g^{-1}Xg$ of the group $SO(4)$ on the dual Lie algebra $so(4)$; we define the following symplectic form between two vectors $\xi_1=[X, Z_1]$ and $\xi_2=[X, Z_2]$ (with $Z_1, Z_2\in so(4)$) tangent to the orbit of $X\in so(4)$,

$$(\xi_1, \xi_2)\equiv\langle X, [Z_1, Z_2]\rangle = \langle [X, Z_1], Z_2\rangle.$$

This orbit is clearly 4-dimensional and is defined by the two algebraic "orbit invariants"

$$\langle X, X\rangle = \sum_{1\leqq i<j\leqq 4} X_{ij}^2$$

and

$$\pm(\det X)^{1/2} = X_{12}X_{34} - X_{13}X_{24} + X_{23}X_{14}. \tag{3}$$

Functions H defined on this orbit lead to Hamiltonian vector fields

$$\dot{X}=[X, \nabla H].$$

In particular $H=\frac{1}{2}\langle X, \Lambda X\rangle$ with Λ defined above induces geodesic motion (2). The constants of the motion are given by the two orbit invariants (3) and H. Since the reduced phase space is 4-dimensional, we need one extra invariant to make the system completely integrable.

Throughout Λ is assumed to be *diagonal* and *non-degenerate* in the standard basis, i.e. $(\Lambda X)_{ij}=\Lambda_{ij}X_{ij}$ with Λ_{ij} all distinct. One of the main points of the paper is to prove the following statement.

Theorem 1. *The geodesic flow* (2) *on* $SO(4)$ *is algebraically completely integrable with the six abelian functions* X_{ij} $1 \leqq i < j \leqq 4$, *if and only if the diagonal metric* Λ *satisfies the commutation relation*

$$[X, \beta] + [\alpha, \Lambda X] = 0 \tag{4}$$

as an identity in X, *with diagonal matrices* $\alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_4)$ *with* α_i*'s all distinct and* $\beta = \operatorname{diag}(\beta_1, \ldots, \beta_4)$. *This identity amounts to:*

$$\Lambda_{ij} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j}.$$

Then the system has besides the orbit invariants and the Hamiltonian

$$\sum_{1 \leqq i < j \leqq 4} \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} X_{ij}^2, \tag{5}$$

the following invariant

$$\sum_{1 \leqq i < j \leqq 4} \frac{\gamma_i - \gamma_j}{\alpha_i - \alpha_j} X_{ij}^2, \qquad \text{with } \gamma_i \ (1 \leqq i \leqq 4) \text{ arbitrary} \in \mathbb{C}. \tag{6}$$

The four quadrics obtained by setting these four invariants equal to a constant, generically intersect in the affine part of an Abelian variety of genus 2.

We now explain the commutation relation (4): using this relation the Lax flow (2) can be transformed into the following Lax-type equation

$$(X + \alpha h)^{\cdot} = [X + \alpha h, \Lambda X + \beta h] \tag{7}$$

with an indeterminate h. As explained in [1] and [2], this flow is Hamiltonian on an orbit defined in a Kac-Moody Lie algebra; consider the Kac-Moody extension (for $n = 4$)

$$L = \{ \sum_{-\infty < i \leqq N} A_i h^i \mid A_i \in gl(n, \mathbb{C}), \ N \text{ arbitrary} \}$$

of $gl(n, \mathbb{C})$ with the bracket

$$[\sum A_i h^i, \sum B_j h^j] \equiv \sum_k \Big(\sum_{i+j=k} [A_i, B_j] \Big) h^k$$

and a nondegenerate, ad-invariant inner product given by

$$\langle \sum A_i h^i, \sum B_j h^j \rangle \equiv \sum_{i+j=-1} \operatorname{Tr}(A_i B_j).$$

This Lie algebra has a natural vector space decomposition $L = K + N$ into the two Lie subalgebras K and N of $(\geqq 0)$ and (< 0) powers of h respectively. With regard to the form above, $K^{\perp} = K$ and $N^{\perp} = N$. The infinite-dimensional Lie group underlying N acts coadjointly on $N^* \approx K^{\perp} = K$ and defines on its orbits a Kostant-Kirillov symplectic structure with Poisson bracket

$$\{H_1, H_2\}(a) = \langle a, [\nabla_{N^*} H_1, \nabla_{N^*} H_2] \rangle, \qquad a \in N^*, \ \nabla_{N^*} H \in N,$$

where $\langle\ ,\ \rangle$ also is the $N-N^*$ pairing. According to the Adler-Kostant-Symes theorem, the functions defined on this orbit which are infinitesimally invariant under the L-coadjoint action are all in involution; the corresponding Hamiltonian vector fields have the form

$$\dot{a}=[a, \operatorname{Proj}_K \nabla H].$$

The element $a=X+\alpha h\in N^*$ (as in Eq. (7)) defines an orbit in N^* and a symplectic structure; the L-invariant functions

$$\frac{1}{k}(ah^{-1})^k h^2 \qquad (k\geqq 3)$$

all lead to commuting Hamiltonian vector fields

$$\begin{aligned}(X+\alpha h)^{\cdot}&=[X+\alpha h, \operatorname{Proj}_K((Xh^{-1}+\alpha)^{k-1}h)]\\&=[X+\alpha h, Y+\beta h]\end{aligned}$$

where

$$Y_{ij}=\frac{\alpha_i^{k-1}-\alpha_j^{k-1}}{\alpha_i-\alpha_j}X_{ij} \quad \text{and} \quad \beta=\alpha^{k-1}.$$

Extending this fact to analytic functions, instead of merely powers, leads to the Lax equation (7).

Such equations can be linearized on a two-dimensional Prym variety in the Jacobi variety of the curve $\mathscr{X}$ of genus 3, defined by

$$\mathscr{X}\colon \det(X+\alpha h-zI)\equiv P(h, z)=0,$$

while the Prym variety is defined by the involution $(z, h)\curvearrowright(-z, -h)$ of $\mathscr{X}$. Note that the coefficients of the polynomial $P(h, z)$ are time-independent functions of the X_{ij} and therefore they also provide the constants of the motion.

The linearization of the flow proceeds as follows: the curve $\mathscr{X}$ has 4 simple branch points P_i covering $z=\infty$; the function h satisfies

$$(h)=-\sum_1^4 P_i+\sum_1^4 Q_i$$

for 4 other points Q_i on $\mathscr{X}$. Consider the eigenvector $f=(f_1, \ldots, f_4)$ of the matrix $X+\alpha h$ normalized at $f_1=1$; each f_k can be regarded as a meromorphic function on $\mathscr{X}$; then the minimal positive divisor D such that

$$(f_k)\geqq -D+P_1-P_k, \qquad 1\leqq k\leqq 4 \tag{8}$$

has order 3, while given such a divisor D, there is generically a unique matrix X (modulo conjugation by a complex diagonal matrix) such that the eigenvector of $X+\alpha h$ satisfies (8). Therefore the flow (2) makes D evolve on the Jacobi variety of $\mathscr{X}$ and more specifically, since the antisymmetry of X is preserved under (2), it evolves on the Prym variety described above, in a linear fashion. The map from the matrices X into the divisor D is algebraic, as D is given by the zeros of the common denominator of the rational function f_k. As a result,

the entries X_{ij}^2 of X are abelian functions on Prym $(\mathscr{X})$; in fact, the X_{ij} themselves are abelian functions on the intersection of the four quadrics, which is isogenic to Prym $(\mathscr{X})$.

As a by product, this roundabout way provides a proof that the intersection of the four quadrics mentioned in Theorem 1 is the affine part of an abelian surface. David Mumford has given in the appendix to this paper a direct algebro-geometrical proof that these four quadrics intersect in an abelian surface.

The reader may complain that the requirement that the X_{ij} be abelian functions is technical and that it is a priori conceivable that (2) is algebraically completely integrable for some Λ with a different set of abelian functions. The point is that, in studying completely integrable systems, one is always concerned with the class of the map which linearizes the equations, for every straight line motion is the same if one pays no attention to this class; a precise and workable way of dealing with this question of class is to specify which phase variables will be abelian functions, the choice being guided by experience and the nature of the equations at hand.

In [3], our methods are applied to a system governed by exponential nonnearest neighbor interactions, with Hamiltonian

$$H=\tfrac{1}{2}\sum_1^l m_i p_i^2+\sum_{i=1}^{l+1}\exp\left(\sum_{j=1}^{l} N_{ij}x_j\right),$$

parametrized by a real matrix $N=(N_{ij})$ of full rank. It will be algebraically completely integrable if and only if for all $i\neq j$, $2(NN^{\dagger})_{ij}(NN^{\dagger})_{jj}^{-1}$ is a nonpositive integer, i.e., the interactions correspond to the Toda systems for the Kac-Moody Lie algebras. In his thesis, Björn Birnir [8] studies the complex Korteweg-de Vries equation using the tool of regularization and blowing up (or down) to build the abelian variety going with the KdV equations. We wish to thank David Mumford, both for suggesting the study of geodesic motion in $SO(4)$ and for many extremely helpful observations. Our thanks also go to Luc Haine, for provocative discussions, which has led to many improvements of this paper.

Section 1. The Algebraic Integrability Criterion and an Algebraic Liouville Theorem

Consider a system of differential equations of the form

$$\dot{z}=f(z)=J\frac{\partial H}{\partial z},\qquad z\in\mathbb{C}^k, \tag{9}$$

rational in z, with $J=J(z)$ defining a real Hamiltonian structure for $z\in\mathbb{R}^k$; let φ^t be the corresponding phase flow. We shall assume that each one of the components of $f(z)$ are not identically zero. The system (9) is *algebraically completely integrable* if

(i) it is completely integrable with rational integrals in the real defining a compact variety; by Liouville's theorem, this defines a real algebraic torus, at least for most values of the integrals, and

(ii) the real torus is part of a complex algebraic torus A with algebraic addition law consistent with the linearization of φ^t and the other commuting flows.

Among the invariants we distinguish two kinds: the trivial ones satisfying $J\dfrac{\partial H_i}{\partial z}=0$, $(1\leqq i\leqq l)$ which define a symplectic manifold of dimension $k-l=2s$ by setting $H_i=c_i$ $(1\leqq i\leqq l)$ and the s remaining ones, which define commuting vector fields and a real torus by setting $H_i=c_i$ $(1\leqq i\leqq l+s)$; extended to $\mathbb{C}^k$, condition (ii) requires that the latter set of equations defines the affine part of an abelian variety A.

Proposition 1. *A necessary condition for* (9) *to be algebraically completely integrable with abelian functions* z_i, $1\leqq i\leqq k$, *is that* (9) *must have enough pole expansions in* t *such that each* z_i *blows up at least once, and such that the expansions depend on* $k-1$ *parameters.*

Proof. Pick a typical s-dimensional variety T in $\mathbb{C}^k$, whose compactification $\bar{T}=A$ is an abelian variety on which (9) is a linear motion, as required by the definition of algebraic complete integrability. Since z_i is an abelian function on $\bar{T}$, it blows up along one or several irreducible pieces of the divisor $D=\{p\in\bar{T}\mid z_i^{-1}=0, \text{ some } i\}$ in a meromorphic fashion. Let Δ be the closure of an irreducible piece in D along which some fixed z_i blows up. We claim that Δ cannot, for an open set of tori, be an invariant set of the flow. For if Δ is invariant and if ψ^u is one of the flows commuting with our given one and transversal to Δ, then $\psi^u(\Delta)\equiv\Delta^u$ would also be invariant. The existence of such a transversal ψ^s follows from the algebraic complete integrability assumption: condition (ii) implies that the flows generated by the s invariants $H_{l+1},\ldots,H_{l+s}$ are linear motions on $\bar{T}$, which therefore span the tangent space at all points of $\bar{T}$. Since Δ is a closed set in our torus it is compact, as is Δ^u. We also note that for $z\in\Delta^u$, $0<|u|<\varepsilon$, ε sufficiently small, there exists a constant M_u such that $|z_i|\leqq M_u$. For by compactness, and the transversality of ψ^u, there exists an open neighbourhood $O=O(u)$ about Δ which does not intersect Δ^u, for ε sufficiently small. If there did not exist such M_s as above, there would exist a sequence of points $\{z^{(l)}\}_{l=1}^{\infty}$ in Δ^u, such that $|z_i^{(l)}|>l$, and therefore these points would get arbitrarily close to Δ, which contradicts the existence of O. If a solution of φ^t starts on Δ^u, $0<|u|<\varepsilon$, it remains on Δ^u by its invariance, and so by the above $|z_i(t)|\leqq M_u$ for all complex t. But a bounded meromorphic function must be constant and so $z_i(t)=\text{constant}$. The same argument holds uniformly for any strip $0<\dfrac{\varepsilon}{2}<|u|<\varepsilon$, say. Therefore $\dot{z}_i=f_i(z)\equiv 0$ on an open set (using again the transversality of ψ) and hence $f_i(z)\equiv 0$ on the given torus, showing that z_i is a constant of the motion for that torus. If the divisor Δ were invariant under the flow for an open set of tori, $f_i(z)$ would be identically zero by analyticity, which contradicts our assumption on $f(z)$.

We have shown that for a typical torus A, there exists an open set B in D, such that the flow φ^t is transversal to B. Consider the family of solutions $z(t)$

of φ^t, with initial data on B. The solution $z(t)$ of (9) admits Laurent expansions in t, such that $z_i(t)$ blows up at $t=0$; the expansions may appear very different along the different irreducible pieces of D. The expansion will depend on the point where the trajectory hits D, accounting for $s-1$ free parameters and will also depend on the $l+s$ constants c_i in the $l+s$ equations $H_i=c_i$ defining T; together they depend on $l+2s-1=k-1$ parameters.

Remark. As a byproduct of Proposition 1, we have shown that for a Zariski open set of tori, the flow is transversal on a Zariski open set of an irreducible piece of D. Probably the flow is transversal on a Zariski open set of D.

Assume now $f(z)$ in the equation $\dot{z}=f(z)$ to be a homogeneous *quadratic* polynomial. Then by the majorant method [4], p. 57, any formal Laurent series automatically converges, by omitting the polar part. Thus searching for a $k-1$-parameter family of pole solutions of (9) is a formal matter.

Lemma 1. *Let $f_i(z)$, $1\leqq i\leqq k$, be k homogeneous quadratic polynomial in $z\in\mathbb{C}^k$ and assume that the equations $\dot{z}=f(z)$ do not admit harder than simple pole solutions with $k-1$ degrees of freedom, then a necessary requirement for algebraic complete integrability is that for every $1\leqq j\leqq k$, there exists a solution u of $u+f(u)=0$ with $u_j\neq 0$ and that for such solutions, the matrix $\dfrac{\partial f}{\partial z}=\left(\dfrac{\partial f_i}{\partial z_j}\right)_{1\leqq i,\,j\leqq k}$ has the property that*

$$L_u\equiv I+\frac{\partial f}{\partial z}(u) \tag{10}$$

is diagonalizable with $k-1$ nonnegative integer eigenvalues, in addition to the eigenvalue -1. Every 0 eigenvalue of L_u leads to a degree of freedom in the solution of $u+f(u)=0$.

Proof. Substituting the Laurent series

$$z=t^{-1}(z^{(0)}+z^{(1)}t+\ldots)$$

into $\dot{z}=f(z)$ and equating the t^{-2} terms, leads to the quadratic equation

$$F(z^{(0)})\equiv z^{(0)}+f(z^{(0)})=0. \tag{11}$$

Upon equating the coefficients of t^{j-2}, $j\geqq 1$, we find the recursion relation

$$(L-jI)\,z^{(j)}=a \text{ quadratic polynomial in } z^{(0)},\ldots,z^{(j-1)},\quad (j\geqq 1) \tag{12}$$

where $L=L_{z^{(0)}}$ depends on the solution $z^{(0)}$ of (11). For $j\geqq 1$, there are as many degrees of freedom in the determination of $z^{(j)}$ from the linear system (12) as the dimension of the null-space of $L-jI$; moreover there can be d degrees of freedom in the determinations of $z^{(0)}$ in the non-linear equation (11) only if its Jacobian L has at least a d-dimensional null-space. Note that by (11) and Euler's formula

$$0=2F(z^{(0)})=2z^{(0)}+\frac{\partial f}{\partial z}(z^{(0)})\,z^{(0)}=(L+I)\,z^{(0)}$$

and so -1 is automatically an eigenvalue of L with eigenvector $z^{(0)}$. Since each z_i must blow up at least once, there must be for every $1 \leqq j \leqq k$ some solution for which $z_j^{(0)} \neq 0$, finishing the proof of Lemma 1.

Remark 1. A similar statement can be made if $f(z)$ is a second degree polynomial, not necessarily homogeneous; this case, however, may involve pole solutions of degree two. The Toda equations [3] provide such an example, where half of the equations are homogeneous quadratic while the others are linear. The asymptotic expansions have a more complicated structure: one of the variables blows up as t^{-2}, while the other ones vanish up to order 3 at most, at least generically.

Remark 2. The necessary conditions for algebraic complete integrability, stated in Lemma 1 can be checked in principle; in many cases they are *effectively* computable in the form of algebraic conditions on the coefficients of the polynomial $f(z)$. Moreover the Laurent expansions provide the remaining integrals of the motion, by searching for polynomial expressions in the variables z which after substitution of the asymptotic expansion have no polar component. Such expressions would be bounded meromorphic functions on A and therefore constants.

We now come to an algebraic version of Liouville's theorem.

Proposition 2. *Assume that*

(i) *the Eqs.* (9) *have enough pole solutions in t such that each* z_i *blows*[1] *up at least once and such that the expansions depend on* $k-1$ *parameters;*

(ii) *the system has s commuting independent rational integrals* $H_{l+1}, \ldots, H_{l+s}$ *(real for real z) besides the trivial ones* $H_1, \ldots, H_l$, *whose associated Hamiltonian vector fields are everywhere independent on*

$$T = \bigcap_1^{l+s} \{H_i = c_i\} \subseteq \mathbb{C}^k.$$

Then, through the use of the asymptotic expansions, we can construct an augmented algebraic variety $\bar{T}$, *on which the flow* φ^t *is complete, at least for a generic set of* c_i.

(iii) *In addition, assume for a generic set of* c_i, *that* $\bar{T}$ *constructed above is compact and that* $\bar{T} \setminus T$ *is non-singular.*

Then $\dot{z} = f(z)$ *is algebraically completely integrable with abelian functions* z_i.

Proof. In a first step one constructs $\bar{T}$ from T, using the asymptotic expansions as explained below. To fix ideas and notation, let us discuss the case dealt with in Lemma 1 and we shall therefore rely on the observations made in the proof of that lemma. The differential equations (9) have a $k-1$-parameter family of solutions of the form

$$z(t) = t^{-1}(z^{(0)} + z^{(1)}t + \ldots), \tag{A}$$

[1] which is implied by algebraic complete integrability

where $z^{(0)}$ satisfies the set of non-linear equations $z^{(0)}+f(z^{(0)})\equiv F(z^{(0)})=0$ and $z^{(j)}$, $j\geqq 1$ satisfies

$$(L_{z^{(0)}}-jI)\,z^{(j)}=\text{polynomial in } z^{(0)},\ldots,z^{(j-1)},$$

with $L_{z^{(0)}}=\dfrac{\partial F}{\partial z}(z^{(0)})$. If j is a multiple eigenvalue of L, giving rise to m_j degrees of freedom, we may put

$$z^{(j)}=S_j(z^{(0)},\ldots,z^{(j-1)})+\sum_1^{m_j}\alpha_s^{(j)}\,V_s^{(j)}(z^{(0)}),$$

$\{V_s\}_{s=1}^{m_j}$ being an independent set of eigenvectors of L for the eigenvalue j; V_s and S_j are all rational function of its arguments. By induction, S_j itself becomes a rational function R_j of $z^{(0)}$ and the previously defined α's; so, letting $\alpha^{(j)}\equiv(\alpha_1^{(j)},\ldots,\alpha_{m_j}^{(j)})$, we have that

$$z^{(j)}=\sum_1^{m_j}\alpha_s^{(j)}\,V_s^{(j)}(z^{(0)})+R_j(z^{(0)},\alpha^{(1)},\ldots,\alpha^{(j-1)}). \tag{B}$$

To summarize, when a degree of freedom first make its appearance, it does so linearly and thereafter appears only rationally.

Suppose now the variety Y: $F(z^{(0)})=0$ is m_0-dimensional; clearly $\sum_{j\geqq 0} m_j=k-1$. Pick a smooth point on Y and a local parameter in a neighborhood U of this point, represented by m_0 rational functions $P_1(z^{(0)}),\ldots,P_{m_0}(z^{(0)})$ in $z^{(0)}$; let $P(z^{(0)})\equiv(P_1(z^{(0)}),\ldots,P_{m_0}(z^{(0)}))$. Then $z^{(0)}$ itself is an algebraic function $\mathscr{A}$ of $P(z^{(0)})$.

We are now in a position to specify the regularizing map. First observe that on the one hand

$$\left[\frac{d}{dt}\left(\frac{1}{z_i}\right)\right]^{-1}=-\frac{z_i^2}{f_i(z)}\equiv W_i(z)$$

is rational in z and on the other hand, using the expansion (A), equals

$$\begin{aligned}&=z_i^{(0)}+t\ \not\curlyvee\ (\text{Taylor series with rational coefficients in } \alpha^{(j)} \text{ and } z^{(0)}),\\&=z_i^{(0)}+O(t).\end{aligned} \tag{C}$$

Letting $W\equiv(W_1(z),\ldots,W_k(z))$, we may think of the m_0 rational functions $P_j(W(z))$, $j=1,\ldots,m_0$, as parametrizing U at $t=0$. By possibly some relabeling and by taking some appropriate linear combination of the variables z_s with rational coefficient in $z^{(0)}$, the parameter $\alpha_s^{(j)}$ or $\alpha_1^{(j)}$ can be made to appear for the first time as exactly the coefficient of t^{j-1} in that linear combination. This is achieved by finding a vector in $\mathbb{C}^k$, rational in $z^{(0)}$, which is perpendicular to the $j-1$ vectors $V_2^{(j)}(z^{(0)}),\ldots,V_{m_j}^{(j)}(z^{(0)})$ and has inner product 1 with $V_1^{(j)}(z^{(0)})$.

Therefore some linear combination $\sum_s \beta_s(z^{(0)})\, z_s$ has a Laurent expansion in t of the form

$$\sum_s \beta_s(z^{(0)})\, z_s = \sum \beta_s(z^{(0)})\, z_s^{(0)}\, t^{-1} + \ldots + \alpha_1^{(j)}\, t^{j-1} + \ldots,$$

so that

$$\frac{(\sum \beta(z^{(0)})\, z_s^{(0)})^2}{\sum \beta_s(z^{(0)})\, z_s} = (\sum \beta_s(z^0)\, z_s^0)\, t + r_2\, t^2 + \ldots + (r_{j+1} - \alpha_1^{(j)})\, t^{j+1} + \ldots,$$

where the r_i are rational functions and, in particular, for $i \leqq j+1$ rational in $z^{(0)}$ and the parameters $\alpha^{(k)}$, $k<j$; so, on the one hand, from the expansion above

$$\left(\frac{d}{dt}\right)^{j+1} \frac{(\sum \beta_s(z^{(0)})\, z_s^{(0)})^2}{\sum_s \beta_s(z^{(0)})\, z_s} = (j+1)!\,(r_{j+1} - \alpha_1^{(j)}) + O(t) \equiv \tilde{\alpha}_1^{(j)} + O(t)$$

and on the other hand, using the differential equations and the expansion $W(z) = z^{(0)} + O(t)$, the expression above equals

$$\begin{aligned} &= \text{rational function in } (z^{(0)}, z), \\ &= \text{rational function in } (W(z), z) + O(t) \\ &= \text{rational function in } z + O(t). \end{aligned}$$

Bringing all the terms of order $O(t)$ to one side, and taking account of the obvious birational equivalence between $\{\alpha_s^{(j)}\}$ and $\{\tilde{\alpha}_s^{(j)}\}$, we conclude in general that some rational function $Q_s^{(j)}(z)$ has the following behavior for small t

$$Q_s^{(j)}(z) = \alpha_s^{(j)} + O(t). \tag{D}$$

Since (after relabeling we may assume $z_1^{(0)} \neq 0$)

$$\tau \equiv z_1^{-1} = (z_1^{(0)})^{-1}\, t + O(t^2) \tag{E}$$

with rational coefficients in $z^{(0)}$ and $\alpha^{(j)}$, the variable t may be expanded in a Taylor series in τ with rational coefficients in α and $z^{(0)}$

$$t = z_1^{(0)}\, \tau + O(\tau^2); \tag{F}$$

we substitute (F) into (D) and we combine (F), (C) and $z^{(0)} = \mathscr{A}(P(z^{(0)}))$, to yield

$$\begin{aligned} \alpha_s^{(j)} &= Q_s^{(j)}(z) + O(\tau) \\ z^{(0)} &= \mathscr{A}(P(w(z))) + O(\tau). \end{aligned} \tag{G}$$

Let $\zeta_1, \ldots, \zeta_{k-1}$ denote the $k-1$ variables $P_i(w(z))$ and $Q_s^{(j)}(z)$ in that order. Then $R(z) \equiv (\tau, \zeta_1, \ldots, \zeta_{k-1})$ is a vector of rational functions of z, defining a rational map. Also we show the map is invertible, by writing z_i as a Laurent series in the variables $\tau, \zeta_1, \ldots, \zeta_{k-1}$. This is done as follows: first substitute (F) in (A) with $z^{(j)}$ defined by (B); then z will be a Laurent series in τ with rational coefficients in α and $z^{(0)}$. Next we replace α and $z^{(0)}$ by the power series (G) in an inductive fashion as follows: assume inductively that

$$z = \sum_{j \leqq k} \tau^j\, \theta_j(\zeta_1, \ldots, \zeta_{k-1}) + \tau^{k+1}\, \eta_{k+1}(\alpha, z^{(0)}) + O(\tau^{k+2}).$$

Then replacing in η_{k+1} the variables α and $z^{(0)}$ by (G), we arrive at an expansion for z, identical to the one above with k replaced by $k+1$; the series obtained in this way converges, by composition of analytic functions. As a result the rational map $z \to R(z)$ is one-to-one and its inverse is algebraic.

Then $R(z) \equiv (\tau, \zeta_1, \ldots, \zeta_{k-1})$ defines one-dimensional fibres, each one parametrized by τ, for τ in a small enough disk $D \subseteq \mathbb{C}$, over the base space of parameters $(\zeta_1, \ldots, \zeta_{k-1})|_{\tau=0} = (P(z^{(0)}), \alpha) \in U \times \mathbb{C}^{k-1-m_0}$; these fibres are the trajectories of the original differential equations expressed in the rational expressions τ and ζ_i in the variable z. We now embed part of the variety $\bar{T}$ in $D \times U \times \mathbb{C}^{k-1-m_0}$ via the new variables R as follows: in the old variables z_i, the variety T is defined by intersecting the hypersurfaces $H_i(z_i, \ldots, z_k) = c_i$, $1 \leqq i \leqq 1 + s$ in $\mathbb{C}^k$. Substituting into the rational expression $H_i(z_1, \ldots, z_k)$ the power series of z in terms of $(\tau, \zeta_1, \ldots, \zeta_{k-1})$ yields $l+s$ algebraic expressions

$$G_i(\tau, \zeta_1, \ldots, \zeta_{k-1}) = c_i \qquad 1 \leqq i \leqq l+s$$

whose intersection is in the completion $\mathcal{T}$ of T in $D \times U \times \mathbb{C}^{k-1-m_0}$; clearly $\mathcal{T} \setminus T$ is a codimension 1 subvariety in $\bar{T}$ given in $D \times U \times \mathbb{C}^{k-1-m_0}$ by intersecting T with $\tau = 0$:

$$G_i(0, \zeta_1, \ldots, \zeta_{k-1}) = c_1, \qquad 1 \leqq i \leqq l+s.$$

We must show that the variety T is non-singular along $\bar{T} \setminus T$, i.e. that the matrix

$$J = \begin{pmatrix} \dfrac{\partial G_1}{\partial \tau} & \dfrac{\partial G_1}{\partial \zeta_1} & \cdots & \dfrac{\partial G_1}{\partial \zeta_{k-1}} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial G_{l+s}}{\partial \tau} & \dfrac{\partial G_{l+s}}{\partial \zeta_1} & \cdots & \dfrac{\partial G_{l+s}}{\partial \zeta_{k-1}} \end{pmatrix}_{\tau=0}$$

has maximal rank. Since the Taylor expansions of ζ_i in τ respect the flow, we have that

$$0 = \left.\frac{dG_i}{d\tau}\right|_{\tau=0} = \left.\left(\frac{\partial G_i}{\partial \tau} + \sum_{j=1}^{k-1} \frac{\partial G_i}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial \tau}\right)\right|_{\tau=0},$$

expressing a dependence between the columns of J, involving at least the first column. Therefore

$$\operatorname{rank} J = \operatorname{rank} \begin{pmatrix} \dfrac{\partial G_1}{\partial \zeta_1} & \cdots & \dfrac{\partial G_1}{\partial \zeta_{k-1}} \\ \dfrac{\partial G_{l+s}}{\partial \zeta_1} & \cdots & \dfrac{\partial G_{l+s}}{\partial \zeta_{k-1}} \end{pmatrix}_{\tau=0}.$$

This matrix has maximal rank, since by assumption the variety $\bar{T} \setminus T$ is non-singular. This is to say that the tangent space to $\bar{T}$ at $p \in \bar{T} \setminus T$ is spanned by the tangent space to $\bar{T} \setminus T$ and the transversal direction $\partial/\partial\tau$.

We now take all possible asymptotic solutions $z(t) = t^{-1}(z^{(0)} + z^{(1)}t + \ldots)$ with $k-1$ degrees of freedom; it defines various charts $D \times U \times \mathbb{C}^{k-1-m_0}$ which together with the affine chart $\mathbb{C}^k$ provides an algebraic atlas: indeed, among the overlaps between charts, we distinguish three kinds: the transition from the affine chart $\mathbb{C}^k$ to a non-affine one is clearly algebraic because $R(z)$ is rational

in z and the map $z \to R(z)$ is one-to-one. The transitions between non-affine charts belonging to the same curve Y are also algebraic, because they inherit the rational relations between the two m_0-uples of meromorphic functions of type $P(z^{(0)})$ defined in their respective neighborhood. Finally those intersections of charts about the intersection of two different varieties Y require a more careful argument, but along the same lines as before. This atlas also regularizes the flow, as in the non-affine charts the flow proceeds along the fibres constructed above.

In the event the $k-1$ parameter families of asymptotic solutions of (9) do not have the form of Lemma 1, but, say, are of the form (with perhaps distinct integers m)

$$z = \frac{1}{t^m}(z^{(0)} + z^{(1)}t + \ldots), \qquad m \geqq 1, \; z^{(0)} \neq 0,$$

as indeed you might think if $f(z)$ in (9) was not purely quadratic, then indeed as one checks by equating coefficients in (9) you again have both algebraic and linear degrees of freedom coming up in the asymptotic expansion in a *recursive* fashion as before, and so the arguments used above can be taken over. For these cases one interesting difference is that now, z_1^{-1}, say, could not serve as time parameter but instead an expression of the form $\left(\frac{d}{dt}\right)^m \left(\frac{1}{z_1}\right)$. We shall not dwell on the other differences as they are inessential. This concludes the construction of $\bar{T}$.

We now prove the second assertion in the theorem. The phase flow $\varphi_1^t(p) \equiv \varphi^t(p)$ corresponding to the differential equations (9) is holomorphic everywhere on $\bar{T}$: on T by the analyticity with regard to t of the solution of the differential equation and on $\bar{T}$, because of the way T is constructed. For a different invariant the phase flow $\varphi_i^t(p)$ is also holomorphic for $p \in T$. We now show this function is well-defined and holomorphic in $\bar{T}$.

Let $p \in \bar{T} \setminus T$ and then by the Laurent development of $z(t)$, we have that $\varphi_1^\varepsilon(p) \in T$. Then we may define φ_i^t on T by $\varphi_i^t(p) = \varphi_1^{-\varepsilon}(\varphi_i^t(\varphi_1^\varepsilon(p)))$; since $\varphi_i^t(\varphi_1^\varepsilon(p))$ is holomorphic in t around $t=0$ and since φ_1^ε maps bi-holomorphically a neighborhood of q onto one of $\varphi_1^\varepsilon(q)$, it follows that also $\varphi_i^t(p)$ is holomorphic in t around $t=0$. Therefore, since the coordinate functions $R(z)$ are holomorphic in t around $t=0$ and the variables z_k or z_k^{-1} are analytic in $R(z)$ on a given chart

$$z_k(t_1, \ldots, t_s) \equiv z_k(\varphi_1^{t_1}(p), \ldots, \varphi_s^{t_s}(p))$$

or its inverse is holomorphic in $t_1, \ldots, t_s$.

Showing that $\bar{T}$ is a torus proceeds along the same lines as in the classical Liouville theorem by considering the map

$$\mathbb{C}^s \to \bar{T}\colon (t_1, \ldots, t_s) \curvearrowright (\varphi_1^{t_1}(p), \ldots, \varphi_s^{t_s}(p)),$$

by dividing out $\mathbb{C}^s$ by the lattice of points $(t_1, \ldots, t_s)$ which is mapped into the base point p and by using the compactness of $\bar{T}$. Therefore $\bar{T}$ is a complex torus with straight line motions given by the flows and the functions z_i are meromorphic on $\bar{T}$.

Now we use the classical fact (see Mumford [9]) that a torus which is given locally by algebraic equations with algebraic transition functions is an abelian variety, as a consequence of Chows Lemma. As a result, the functions z_i are abelian on $\bar{T}$, finishing the proof of Proposition 2.

Remark 1. In the sequel we shall not use the full force of Proposition 2, but Luc Haine, in a forthcoming paper, will carry out the program implicit in the proposition for the case of Theorem 1, thus providing an alternate and direct method to the linearization discussed in the introduction using the Kac-Moody linearization. This will also provide a proof that the intersection of the four quadrics in that problem is generically the affine part of an abelian variety.

Section 2. Geodesic Flow on $SO(4)$

This section deals with the proof of Theorem 1; it is done by applying Lemma 1 to the geodesic flow equations on $SO(4)$ with diagonal metric Λ:

$$\dot{X}=[X, \Lambda X], \quad X \in so(4). \tag{13}$$

Set

$$X=(X_{ij})_{1\leqq i,\, j\leqq 4}\equiv \sum x_i e_i \equiv \begin{pmatrix} 0 & -x_3 & x_2 & -x_4 \\ x_3 & 0 & -x_1 & -x_5 \\ -x_2 & x_1 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{pmatrix}$$

and

$$\Lambda X=(\Lambda_{ij} X_{ij})_{1\leqq i,\, j\leqq 4}=\sum \lambda_i x_i e_i.$$

Then upon setting $A_{ij}=(\lambda_i-\lambda_j)$, (13) takes the form

$$\begin{aligned} \dot{x}_1 &= x_2 x_3 A_{32}+x_5 x_6 A_{65} \\ \dot{x}_2 &= x_1 x_3 A_{13}+x_4 x_6 A_{46} \\ \dot{x}_3 &= x_1 x_2 A_{21}+x_4 x_5 A_{54} \\ \dot{x}_4 &= x_3 x_5 A_{35}+x_2 x_6 A_{62} \\ \dot{x}_5 &= x_3 x_4 A_{43}+x_1 x_6 A_{16} \\ \dot{x}_6 &= x_2 x_4 A_{24}+x_1 x_5 A_{51}. \end{aligned} \tag{14}$$

The notations of Lemma 1 are used here in the following context

$$\mathbb{C}^6 \supseteq \mathbb{R}^6 \equiv so(4), \quad z \equiv X \quad \text{and} \quad f(z) \equiv [X, \Lambda X].$$

The non-linear equations

$$F(X^0) \equiv X^0+[X^0, \Lambda X^0]=0$$

determining X^0 in the asymptotic expansion

$$X=\frac{1}{t}(X^0+X^1 t+\ldots) \tag{15}$$

have the following form

$$
\begin{aligned}
0&=x_1^0+x_2^0x_3^0A_{32}+x_5^0x_6^0A_{65}\\
0&=x_2^0+x_1^0x_3^0A_{13}+x_4^0x_6^0A_{46}\\
0&=x_3^0+x_1^0x_2^0A_{21}+x_4^0x_5^0A_{54}\\
0&=x_4^0+x_3^0x_5^0A_{35}+x_2^0x_6^0A_{62}\\
0&=x_5^0+x_3^0x_4^0A_{43}+x_1^0x_6^0A_{16}\\
0&=x_6^0+x_2^0x_4^0A_{24}+x_1^0x_5^0A_{51}.
\end{aligned}
\tag{16}
$$

Setting

$$\mathscr{L}^\varepsilon(Y)=[Y,\Lambda X^0]+[X^0,\Lambda Y]+\varepsilon Y,$$

the operator L of Lemma 1 is $\mathscr{L}=\mathscr{L}^1$; although the subscript used in Lemma 1 will be omitted the operators $\mathscr{L}$ and $\mathscr{L}^\varepsilon$ clearly depend on X^0. For future use we display the operator $\mathscr{L}^\varepsilon$ with respect to the basis e_i:

$$
\mathscr{L}^\varepsilon=\begin{pmatrix}
\varepsilon & x_3^0A_{32} & x_2^0A_{32} & 0 & x_6^0A_{65} & x_5^0A_{65}\\
x_3^0A_{13} & \varepsilon & x_1^0A_{13} & x_6^0A_{46} & 0 & x_4^0A_{46}\\
x_2^0A_{21} & x_1^0A_{21} & \varepsilon & x_5^0A_{54} & x_4^0A_{54} & 0\\
0 & x_6^0A_{62} & x_5^0A_{35} & \varepsilon & x_3^0A_{35} & x_2^0A_{62}\\
x_6^0A_{16} & 0 & x_4^0A_{43} & x_3^0A_{43} & \varepsilon & x_1^0A_{16}\\
x_5^0A_{51} & x_4^0A_{24} & 0 & x_2^0A_{24} & x_1^0A_{51} & \varepsilon
\end{pmatrix}.
\tag{17}
$$

The main result of this section, Theorem 1, depends on the following sequence of lemmas, but most heavily on Lemma 2, the main lemma of this section.

Lemma 2. *If all* $x_i^0\neq 0$, *then*

$$\det(\mathscr{L}-lI)=(l-2)^2(l^2-1)[(l-1)^2-4x_1^0x_2^0x_3^0K-1],$$

with

$$K=\lambda_1\lambda_4(\lambda_2+\lambda_5-\lambda_3-\lambda_6)+\lambda_2\lambda_5(\lambda_3+\lambda_6-\lambda_1-\lambda_4)+\lambda_3\lambda_6(\lambda_1+\lambda_4-\lambda_2-\lambda_5).$$

Lemma 3. *A Zariski open set of the algebraic variety in* $\mathbb{C}^6$ *defined by* $K=0$ *can be rationally parametrized by*

$$\Lambda_{ij}=\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j},\qquad \alpha_i\neq\alpha_j,\ 1\leqq i,\ j\leqq 4. \tag{18}$$

The latter condition amounts to the existence of diagonal matrices α *and* β *satisfying the following commutation relation*

$$[X,\beta]+[\alpha,\Lambda X]=0,$$

identically in X, *with the entries* α_i *of* α *being all distinct.*

Lemma 4. *If all* $x_i^0\neq 0$ *and* $K=-(4x_1^0x_2^0x_3^0)^{-1}$, *then the only diagonalizable case for* L *is when*

$$A_{14}=A_{25}=A_{36}=0, \tag{19}$$

which is clearly inadmissible[2] *by the non-degeneracy assumption on the metric* Λ.

[2] In fact, this inadmissible case does lead to an integrable system, as will later be explained

Then most of the manifold given by the variety $K=-(4x_1^0 x_2^0 x_3^0)^{-1}$ does not lead to an integrable system in spite of the fact that the matrix $\mathscr{L}$ has the required number of nonnegative integer eigenvalues. Finally, we need the following slightly tedious lemmas.

Lemma 5. *If some of the x_i^0 are zero, then either*

$$K=0 \quad \text{or} \quad A_{14}=A_{25}=A_{36}=0,$$

or else the Laurent series (15) *has less than five degrees of freedom in its determination.*

Lemma 6. *A Laurent series solution to* (13) *with a pole harder than t^{-1} can have at most four degrees of freedom in its determination.*

Proof of Theorem 1, assuming the above lemmas. Assume geodesic flow to be algebraically completely integrable with abelian functions x_i $(1\leqq i\leqq 6)$. According to Lemma 6, expansions with 5 degrees of freedom can only be found among those with a pole not any worse than t^{-1}. Lemma 1 implies that $\mathscr{L}$ with X^0, a solution of the non-linear system (16), must be diagonalizable with 5 nonnegative integer eigenvalues besides -1. Assuming all x_i^0 to be $\neq 0$, $\mathscr{L}$ already has 4 eigenvalues -1, 2, 2, 1 and since $\operatorname{Tr}\mathscr{L}=6$, as is seen from (17), two possibilities remain for the spectrum of $\mathscr{L}$.

Either (i) -1, 2, 2, 1, 0, 2, implying with Lemma 2 that $K=0$; according to Lemma 3 this condition on the metric implies the rational parametrization of Λ_{ij} and the commutation relation (4) in Theorem 1. To find a fourth independent invariant, one uses either the asymptotic expansions or directly the differential equations. If $\sum \mu_i x_i^2$ is to be an invariant, then each of the coefficients of $x_1x_2x_3$, $x_1x_5x_6$, $x_2x_4x_6$ and $x_3x_4x_5$ appearing on the right hand side of the identity

$$\frac{d}{dt}\frac{1}{2}\left(\sum \mu_i x_i^2\right)=\det\begin{vmatrix}1&1&1\\ \lambda_1&\lambda_2&\lambda_3\\ \mu_1&\mu_2&\mu_3\end{vmatrix}x_1x_2x_3+\det\begin{vmatrix}1&1&1\\ \lambda_1&\lambda_5&\lambda_6\\ \mu_1&\mu_5&\mu_6\end{vmatrix}x_1x_5x_6$$

$$-\det\begin{vmatrix}1&1&1\\ \lambda_2&\lambda_4&\lambda_6\\ \mu_2&\mu_4&\mu_6\end{vmatrix}x_2x_4x_6+\det\begin{vmatrix}1&1&1\\ \lambda_3&\lambda_4&\lambda_5\\ \mu_3&\mu_4&\mu_5\end{vmatrix}x_3x_4x_5$$

must vanish; they provide a linear system of 4 equations in 6 unknowns $\mu_1,\ldots,\mu_6$ with matrix

$$M\equiv\begin{pmatrix}A_{32}&A_{13}&A_{21}&0&0&0\\ A_{65}&0&0&0&A_{16}&A_{51}\\ 0&A_{46}&0&A_{62}&0&A_{24}\\ 0&0&A_{54}&A_{35}&A_{43}&0\end{pmatrix}.$$

The condition $K=0$ implies that rank $M=3$ and hence M has, besides the null vectors $(1, \ldots, 1)$ and $(\lambda_1, \ldots, \lambda_6)$, a third independent null vector $(\mu_1, \ldots, \mu_6)$. Then one checks the general solution μ of $M\mu=0$ has the form indicated by (6).

Conversely we now have to show that if the metric satisfies the condition $K=0$, then the geodesic flow is completely integrable. As explained in the introduction we saw that the commutation relation (4) implies the Lax-type equation (7), that this flow can be realized as a Hamiltonian flow on an orbit in a Kac-Moody Lie algebra, and that as a result the system is completely integrable and linearizable on an abelian variety A isogenic to a Prym variety P in the Jacobi variety of the curve $\det(X+\alpha h-zI)=0$.

Scrutiny of the linearization procedure shows that only the x_i^2 are abelian functions in P. This fact combined with the arguments at the end of Proposition 2, show that the x_i themselves are abelian functions on A which is isogenic to P. This ends the proof in case (i).

Or (ii) $-1, 2, 2, 1, 1, 1$; this implies by Lemma 2, that $K=-(4x_1^0 x_2^0 x_3^0)^{-1}$ and by Lemma 4, that $A_{14}=A_{25}=A_{36}=0$ an inadmissible case.

Recall it was assumed that all $x_i^0 \neq 0$, $1 \leqq i \leqq 6$; according to Lemma 5, if this were not the case and if the expansions were to have 5 degrees of freedom, then $K=0$ or $A_{14}=A_{25}=A_{36}=0$ must hold and we are back in the case discussed above. This ends the proof of Theorem 1.

Remark. In the case $A_{14}=A_{25}=A_{36}=0$, the system of differential equations decouples into two systems

$$
\begin{aligned}
(x_1 \pm x_4)^{\cdot} &= A_{32}(x_2 \pm x_5)(x_3 \pm x_6), \\
(x_2 \pm x_5)^{\cdot} &= A_{13}(x_3 \pm x_6)(x_1 \pm x_4), \\
(x_3 \pm x_6)^{\cdot} &= A_{21}(x_1 \pm x_4)(x_2 \pm x_5).
\end{aligned}
$$

This is nothing but two copies of geodesic flow on $SO(3)$ in the variables $x_1 \pm x_4$, $x_2 \pm x_5$ and $x_3 \pm x_6$ corresponding to the product decomposition of $SO(4)$ into two copies of $SO(3)$; the latter is completely integrable also with quadratic invariants on a two-dimensional abelian variety, which is the product of two elliptic curves.

Before we proceed to the proof of Lemmas 2-6, it is necessary to discuss the automorphisms A of $SO(4)$, which merely permute its basis (possibly with sign changes), for these will play an important role. They all derive from the $SO(3)$-decomposition of $SO(4)$. Specifically, if we define

$$
f_i^{\pm} = \tfrac{1}{2}(e_i \pm e_{i+3}), \qquad 1 \leqq i \leqq 3,
$$

then upon using

$$
\begin{aligned}
&[e_i, e_j] = [e_{i+3}, e_{j+3}] = \varepsilon_{ijk} e_k, \\
&[e_{i+3}, e_j] = [e_i, e_{j+3}] = \varepsilon_{ijk} e_{k+3},
\end{aligned}
\qquad 1 \leqq i, j \leqq 3,
$$

we find

$$[f_i^{\pm}, f_j^{\pm}] = \varepsilon_{ijk} f_k^{\pm}, \quad [f_i^{\pm}, f_j^{\pm}] = 0, \quad 1 \leqq i, j \leqq 3,$$

from which it follows that $SO(4) = SO(3) \oplus SO(3)$; this decomposition respects the Killing form. The following list generates A, modulo sign changes:

τ_1	τ_2	τ_3	τ_4
$\tau_1(e_1) = e_4$	$\tau_2(e_1) = e_1$	$\tau_3(e_i) = e_{i-1}$	$\tau_4(e_1) = e_3$
$\tau_1(e_2) = e_5$	$\tau_2(e_2) = e_5$	$i \bmod 3$	$\tau_4(e_2) = e_2$
$\tau_1(e_3) = e_3$	$\tau_2(e_3) = e_6$		$\tau_4(e_3) = -e_1$
$\tau_1(e_4) = e_1$	$\tau_2(e_4) = e_4$	$\tau_3(e_{i+3}) = e_{i+2}$	$\tau_4(e_4) = e_6$
$\tau_1(e_5) = e_2$	$\tau_2(e_5) = e_2$	$i \bmod 3$	$\tau_4(e_5) = e_5$
$\tau_1(e_6) = e_6$	$\tau_2(e_6) = e_3$		$\tau_4(e_6) = -e_4$

and componentwise

$$\begin{aligned}
\tau_1(x_1, \ldots, x_6) &= (x_4, x_5, x_3, x_1, x_2, x_6) \\
\tau_2(x_1, \ldots, x_6) &= (x_1, x_5, x_6, x_4, x_2, x_3) \\
\tau_3(x_1, \ldots, x_6) &= (x_2, x_3, x_1, x_5, x_6, x_4) \\
\tau_4(x_1, \ldots, x_6) &= (-x_3, x_2, x_1, -x_6, x_5, x_4).
\end{aligned}$$

The automorphisms above change the metric, unless analogous permutations are performed in the λ_i's. Let τ_i also denote the corresponding permutation on the λ_i's, i.e.,

$$\begin{aligned}
\tau_1(\lambda_1, \ldots, \lambda_6) &= (\lambda_4, \lambda_5, \lambda_3, \lambda_1, \lambda_2, \lambda_6) \\
\tau_2(\lambda_1, \ldots, \lambda_6) &= (\lambda_1, \lambda_5, \lambda_6, \lambda_4, \lambda_2, \lambda_3) \\
\tau_3(\lambda_1, \ldots, \lambda_6) &= (\lambda_2, \lambda_3, \lambda_1, \lambda_5, \lambda_6, \lambda_4) \\
\tau_4(\lambda_1, \ldots, \lambda_6) &= (\lambda_3, \lambda_2, \lambda_1, \lambda_6, \lambda_5, \lambda_4).
\end{aligned} \tag{20}$$

The Eq. (14) clearly depends only on the differences of λ_i's; the system is invariant under the τ_i provided the λ_i's are permuted accordingly, as expected. Multiplying the λ_i's with a common factor amounts to rescaling the time. As a consequence, any condition for algebraic complete integrability must be one or several *homogeneous* expressions in the λ_i's, depending on the differences $\lambda_i - \lambda_j$ and invariant (except maybe for the sign) under the permutations τ_i.

We consider therefore the space $\mathcal{S}_n$ of homogeneous polynomials of degree n in, $\lambda_1 \ldots \lambda_6$, which are invariant under τ_1, τ_2 and τ_3 and under the map $\lambda_i \to \lambda_i + \alpha (\alpha \neq 0)$.

Lemma 7. $\dim \mathcal{S}_1 = 0$, $\dim \mathcal{S}_2 = 2$ *and* $\dim \mathcal{S}_3 = 5$. *Moreover, for* $i = 1, 2$, *we have that* $\mathcal{S}_i$ *is the same as* $\mathcal{S}_i = \{P \mid \text{degree } P = i,$ *invariant under* $\lambda_j \to \lambda_j + \alpha$, $\tau_k(P) = \pm P$, *all* $1 \leqq k \leqq 4\}$ *and*

$$\mathcal{S}_3 = \mathcal{S}_3^+ \oplus \mathcal{S}_3^-, \quad \dim \mathcal{S}_3^+ = 3, \ \dim \mathcal{S}_3^- = 2,$$

where

$$\mathscr{S}_3^{\pm} = \mathscr{S}_3 \cap \{P \mid \tau_4(P) = \pm P\}.$$

The space $\mathscr{S}_3^-$ *is generated by*

$$K \equiv Q_1 - Q_2 \quad \textit{and} \quad S \equiv Q_3 - Q_4,$$

with

$$Q_1 = \sum_1^3 \lambda_i \lambda_i' (\lambda_{i+1} + \lambda_{i+1}'),$$

$$Q_2 = \sum_1^3 \lambda_i \lambda_i' (\lambda_{i-1} + \lambda_{i-1}'),$$

$$Q_3 = \sum_1^3 (\lambda_i^2 + \lambda_i'^2)(\lambda_{i+1} + \lambda_{i+1}'),$$

$$Q_4 = \sum_1^3 (\lambda_i^2 + \lambda_i'^2)(\lambda_{i-1} + \lambda_{i-1}');$$

all indices in the above must be read mod 3 *and* $\lambda_i' \equiv \lambda_{i+3}$.

Proof of Lemma 7. Let $\mathscr{T}_n$ be the set of polynomials of homogeneous degree n, invariant under τ_1, τ_2 and τ_3 except maybe for the sign. Clearly $\dim \mathscr{T}_1 = 1$, because $\sum_1^6 \lambda_i \in \mathscr{T}_1$ and $\dim \mathscr{S}_1 = 0$. A basis for $\mathscr{T}_2$ is given by the polynomials

$$P_1 = \sum_1^6 \lambda_i^2, \qquad P_2 = \sum_1^3 (\lambda_i + \lambda_i')(\lambda_{i+1} + \lambda_{i+1}'), \qquad P_3 = \sum_1^3 \lambda_i \lambda_i'$$

generated respectively by λ_1^2, $\lambda_1 \lambda_2$ and $\lambda_1 \lambda_4$. Moreover

$$P_1(\lambda + \alpha) = P_1(\lambda) + 2\alpha P_1 + 6\alpha^2,$$
$$P_2(\lambda + \alpha) = P_2(\lambda) + 4\alpha P_1 + 12\alpha^2,$$
$$P_3(\lambda + \alpha) = P_3(\lambda) + \alpha P_1 + 3\alpha^2,$$

showing that $\dim \mathscr{S}_2 = 2$. Finally $\mathscr{T}_3$ contains in addition to Q_1, Q_2, Q_3, Q_4, the polynomials

$$Q_5 = \lambda_1 \lambda_2 \lambda_3 + \sum_1^3 \lambda_i \lambda_{i+1}' \lambda_{i+2}',$$

$$Q_6 = \lambda_1' \lambda_2' \lambda_3' + \sum_1^3 \lambda_i \lambda_{i+1} \lambda_{i+2}',$$

$$Q_7 = \sum_1^3 (\lambda_i^2 + \lambda_i'^2)(\lambda_i + \lambda_i'),$$

$$Q_8 = \sum_1^3 \lambda_i^3,$$

and one verifies

$$\begin{aligned}
Q_1(\lambda+\alpha)&=Q_1(\lambda)+\alpha(P_2+2P_3)+3\alpha^2P_1+6\alpha^3,\\
Q_2(\lambda+\alpha)&=Q_2(\lambda)+\alpha(P_2+2P_3)+3\alpha^2P_1+6\alpha^3,\\
Q_3(\lambda+\alpha)&=Q_3(\lambda)+\alpha(2P_1+2P_2)+6\alpha^2P_1+12\alpha^3,\\
Q_4(\lambda+\alpha)&=Q_4(\lambda)+\alpha(2P_1+2P_2)+6\alpha^2P_1+12\alpha^3,\\
Q_5(\lambda+\alpha)&=Q_5(\lambda)+\alpha P_2+2\alpha^2P_1+4\alpha^3,\\
Q_6(\lambda+\alpha)&=Q_6(\lambda)+\alpha P_2+2\alpha^2P_1+4\alpha^3,\\
Q_7(\lambda+\alpha)&=Q_7(\lambda)+\alpha(4P_1+4P_3)+6\alpha^2P_1+12\alpha^3,\\
Q_8(\lambda+\alpha)&=Q_8(\lambda)+3\alpha P_1+3\alpha^2P_1+6\alpha^3.
\end{aligned}$$

Therefore the polynomials

$$Q_1-Q_2,\quad Q_3-Q_4,\quad Q_5-Q_6,\quad Q_3+Q_1-3Q_5-\tfrac{1}{2}Q_7,$$

and

$$Q_8+3Q_5-\tfrac{3}{2}Q_3,$$

form a basis for $\mathscr{S}_3$. One checks that τ_4 leaves invariant Q_5, Q_6, Q_7 and Q_8 and that it interchanges $Q_1\leftrightarrow Q_2$ and $Q_3\leftrightarrow Q_4$ and therefore that

$$Q_1-Q_2 \quad \text{and} \quad Q_3-Q_4\in\mathscr{S}_3^-;$$

the three remaining polynomials belong to $\mathscr{S}_3^+$.

As this point it is appropriate to give the

Proof of Lemma 3. From

$$\lambda_1=\frac{\beta_2-\beta_3}{\alpha_2-\alpha_3},\qquad \lambda_2=\frac{\beta_1-\beta_3}{\alpha_1-\alpha_3},\qquad \lambda_4=\frac{\beta_1-\beta_4}{\alpha_1-\alpha_4},$$

we have that

$$\lambda_3=\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}=\frac{(\alpha_1-\alpha_3)\lambda_2-(\alpha_2-\alpha_3)\lambda_1}{(\alpha_1-\alpha_3)-(\alpha_2-\alpha_3)}$$

and

$$\lambda_6=\frac{\beta_3-\beta_4}{\alpha_3-\alpha_4}=\frac{(\alpha_1-\alpha_4)\lambda_4-(\alpha_1-\alpha_3)\lambda_2}{(\alpha_1-\alpha_4)-(\alpha_1-\alpha_3)};$$

from these expressions, $\dfrac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}$ and $\dfrac{\alpha_1-\alpha_3}{\alpha_1-\alpha_4}$ can be expressed in terms of the λ_i's and put into

$$\lambda_5=\frac{\beta_2-\beta_4}{\alpha_2-\alpha_4}=\frac{\lambda_4(\alpha_1-\alpha_4)-\lambda_2(\alpha_1-\alpha_3)+\lambda_1(\alpha_2-\alpha_3)}{(\alpha_1-\alpha_4)-(\alpha_1-\alpha_3)+(\alpha_2-\alpha_3)}$$

to give

$$\begin{aligned}
0=&(\lambda_4-\lambda_5)(\lambda_6-\lambda_2)(\lambda_3-\lambda_1)+(\lambda_5-\lambda_2)(\lambda_6-\lambda_4)(\lambda_3-\lambda_4)\\
&+(\lambda_1-\lambda_5)(\lambda_3-\lambda_2)(\lambda_6-\lambda_4)=K.
\end{aligned}$$

The map

$$(\beta_2-\beta_3,\ \beta_1-\beta_3,\ \beta_1-\beta_4,\ \alpha_2-\alpha_3,\ \alpha_1-\alpha_3,\ \alpha_1-\alpha_4)\in\mathbb{P}^5,$$

with

$$\alpha_i-\alpha_j\neq 0 \qquad (i\neq j),$$

into $(\lambda_1,\ldots,\lambda_6)\in\mathbb{C}$, such that $K=0$, is invertible if the λ_i's are distinct. Therefore a Zariski open subset of the algebraic variety $K=0$ can be parametrized by (18), proving the lemma.

We now prove the crucial lemma of this section:

Proof of Lemma 2. To this end, we introduce

$$L_i^\varepsilon\equiv\mathscr{L}^\varepsilon(x_i^0\,e_i),\qquad 1\leqq i\leqq 6, \tag{21}$$

$$X_i^0\equiv x_i^0\,e_i+x_{i+3}^0\,e_{i+3},\qquad 1\leqq i\leqq 3, \tag{22}$$

and

$$U_i^\varepsilon\equiv\mathscr{L}^\varepsilon(X_i^0)=L_i^\varepsilon+L_{i+3}^\varepsilon,\qquad 1\leqq i\leqq 3. \tag{23}$$

A few eigenvalues are easy to obtain. To begin with, as has been observed, -1 is an eigenvalue of $\mathscr{L}$ with eigenvector X^0. Moreover, using (16) and (17) we have that

$$U_i^\varepsilon=\mathscr{L}^\varepsilon(X_i^0)=-X^0+(1+\varepsilon)\,X_i^0,\qquad 1\leqq i\leqq 3 \tag{24}$$

and, for $\varepsilon=1$

$$U_i\equiv\mathscr{L}(X_i^0)=-X^0+2X_i^0,\qquad 1\leqq i\leqq 3.$$

Therefore, $X_i^0-X_j^0$ $(i\neq j)$ provide two independent eigenvectors of $\mathscr{L}$ with eigenvalue 2. We have therefore obtained three eigenvalues -1, 2, 2 for the operator $\mathscr{L}$; hence

$$\det(\mathscr{L}-lI)=-(\textstyle\prod x_i^0)^{-1}(l-2)^2(l+1)\,\Delta(l),$$

where

$$\Delta(l)=\left(\prod_1^6 x_i^0\right)[(1-l)^3+B(1-l)^2+C(1-l)+D]. \tag{25}$$

Compute from (24),

$$U_i^\varepsilon+U_j^\varepsilon-(1-\varepsilon)\,U_k^\varepsilon=\mathscr{L}^\varepsilon(X_i^0+X_j^0-(1-\varepsilon)\,X_k^0)=(\varepsilon-2)(\varepsilon+1)\,X_k^0,$$
$$i,\ j,\ k \text{ all distinct} \tag{26}$$

and for $\varepsilon=1$ we have

$$\mathscr{L}(X_i^0+X_j^0)=-2X_k^0. \tag{27}$$

Define

$$Y_1^0\equiv x_2^0\,e_2+x_3^0\,e_3+x_4^0\,e_4,$$
$$Y_2^0\equiv x_3^0\,e_3+x_1^0\,e_1+x_5^0\,e_5,$$
$$Y_3^0\equiv x_1^0\,e_1+x_2^0\,e_2+x_6^0\,e_6.$$

One then computes $\mathscr{L}(Y_i^0)$ for $i=3$ and the rest is obtained by applying τ_3; this is done, for instance, using the matrix (17) and the relations (16). Then the volume form

$$\mathscr{L}(X_2^0+X_3^0)\wedge\mathscr{L}(X_1^0+X_3^0)\wedge\mathscr{L}(X_1^0+X_2^0)\wedge\mathscr{L}(Y_1^0)\wedge\mathscr{L}(Y_2^0)\wedge\mathscr{L}(Y_3^0)$$

equals, on the one hand,

$$= -8 \prod_1^6 x_i^0 \, \mathscr{L}(e_1) \wedge \mathscr{L}(e_2) \wedge \mathscr{L}(e_3) \wedge \mathscr{L}(e_4) \wedge \mathscr{L}(e_5) \wedge \mathscr{L}(e_6)$$
$$= -8 \prod_1^6 x_i^0 \det \mathscr{L},$$

and on the other hand, from (27),

$$= -2^6 \det \begin{pmatrix} x_1^0 & 0 & 0 & x_2^0 x_3^0 A_{32} & 0 & 0 \\ 0 & x_2^0 & 0 & 0 & x_3^0 x_1^0 A_{13} & 0 \\ 0 & 0 & x_3^0 & 0 & 0 & x_1^0 x_2^0 A_{21} \\ x_4^0 & 0 & 0 & 0 & x_3^0 x_5^0 A_{35} & x_2^0 x_6^0 A_{62} \\ 0 & x_5^0 & 0 & x_3^0 x_4^0 A_{43} & 0 & x_1^0 x_6^0 A_{16} \\ 0 & 0 & x_6^0 & x_2^0 x_4^0 A_{24} & x_1^0 x_5^0 A_{51} & 0 \end{pmatrix}. \tag{28}$$

This determinant is a homogeneous polynomial of degree 9 in x_i^0, and cubic in A_{ij}, vanishing doubly for $x_i^0 = 0$ $(1 \leqq i \leqq 3)$ (because then three columns are proportional) and simply whenever $x_i^0 = 0$ $(4 \leqq i \leqq 6)$ (because then two columns are proportional). Therefore this expression equals

$$= -2^6 \prod_1^6 x_i^0 \prod_1^3 x_i^0 \times \text{a homogeneous cubic polynomial in } A_{ij}.$$

Putting $x_i^0 = 1$ in this determinant and subtracting row i from row $i+3$ $(1 \leqq i \leqq 3)$, (28) equals

$$= -2^6 \prod_1^6 x_i^0 \prod_1^3 x_i^0 \det \begin{pmatrix} -A_{32} & A_{35} & A_{62} \\ A_{43} & -A_{13} & A_{16} \\ A_{24} & A_{51} & -A_{21} \end{pmatrix}. \tag{29}$$

Notice that $\det \mathscr{L}$ does not change sign under the automorphism τ_4, since its action on $\mathscr{L}$ merely changes the basis. Also notice that $\prod_1^6 x_i^0$ does not change sign and $\prod_1^3 x_i^0$ does change sign under τ_4. In addition, the condition $\det \mathscr{L} = 0$ is invariant under the automorphisms defined above. Therefore the latter determinant in (29) belongs to $\mathscr{S}_3^-$. It is easy to see that it contains $2\lambda_1 \lambda_2 \lambda_4$ and is square free. Therefore it equals $2K$ (see Lemma 7) and

$$\det \mathscr{L} = 2^4 \prod_1^3 x_i^0 \, K. \tag{30}$$

Observe as before that since $\det \mathscr{L}$ and K are invariant under the automorphisms, (30) implies

$$\det \mathscr{L} = 2^4 x_1^0 x_2^0 x_3^0 K = 2^4 x_1^0 x_5^0 x_6^0 K = 2^4 x_3^0 x_4^0 x_5^0 K = 2^4 x_2^0 x_4^0 x_6^0 K \tag{31}$$

and from (30) and (31),

$$\Delta(0) = -4\left(\prod_1^6 x_i^0\right)\left(\prod_1^3 x_i^0\right) K. \tag{32}$$

For later use, we replace the third column of (28) by the sum of all columns and use the relations (16) to yield

$$\det \mathscr{L} = -2^3 \left(\prod_1^6 x_i^0\right)^{-1} \times \det \begin{pmatrix} x_1^0 & 0 & x_5^0 x_6^0 A_{65} & x_2^0 x_3^0 A_{32} & 0 & 0 \\ 0 & x_2^0 & x_4^0 x_6^0 A_{46} & 0 & x_3^0 x_1^0 A_{13} & 0 \\ 0 & 0 & x_5^0 x_5^0 A_{54} & 0 & 0 & x_1^0 x_2^0 A_{21} \\ x_4^0 & 0 & 0 & 0 & x_3^0 x_5^0 A_{35} & x_2^0 x_6^0 A_{62} \\ 0 & x_5^0 & 0 & x_3^0 x_4^0 A_{43} & 0 & x_1^0 x_6^0 A_{16} \\ 0 & 0 & 0 & x_2^0 x_4^0 A_{24} & x_1^0 x_5^0 A_{51} & 0 \end{pmatrix}.$$

This determinant is a homogeneous polynomial of degree 10 in x_i^0 and 4 in A_{ij}, vanishing doubly for $x_1^0 = x_2^0 = x_4^0 = x_5^0 = 0$ and simply for $x_3^0 = x_6^0 = 0$, as in the first case three and in the second case two columns are proportional. Therefore using the same method as in (28, 29),

$$\begin{aligned} \det \mathscr{L} &= -2^3 x_1^0 x_2^0 x_4^0 x_5^0 \times \text{a homogeneous quartic polynomial in } A_{ij} \\ &= -2^3 x_1^0 x_2^0 x_4^0 x_5^0 \det \begin{pmatrix} A_{54} & 0 & 0 & A_{21} \\ -A_{65} & -A_{32} & A_{35} & A_{62} \\ -A_{46} & A_{43} & -A_{13} & A_{16} \\ -A_{54} & A_{24} & A_{51} & -A_{21} \end{pmatrix} \\ &= -2^3 x_1^0 x_2^0 x_4^0 x_5^0 (A_{54} + A_{21})\, 2K, \end{aligned}$$

by replacing, for instance, the first column by the sum of the three first columns in the determinant above and using (29). Using the automorphisms, we find

$$\begin{aligned} \det \mathscr{L} &= -2^3 x_1^0 x_2^0 x_4^0 x_5^0 (A_{54} + A_{21})\, 2K \\ &= -2^3 x_2^0 x_3^0 x_5^0 x_6^0 (A_{65} + A_{32})\, 2K \\ &= -2^3 x_1^0 x_3^0 x_4^0 x_6^0 (A_{46} + A_{13})\, 2K, \end{aligned}$$

which combined with (30) leads to a relation, useful in the future:

$$x_1^0 x_2^0 x_3^0 = -x_2^0 x_3^0 x_5^0 x_6^0 (A_{65} + A_{32}). \tag{33}$$

We now compute $\det(\mathscr{L} - 2I)$. We proceed as follows: making the threefold exterior product of (26) for $k = 1, 2, 3$, we find

$$\begin{aligned} &(\varepsilon - 2)^3 (\varepsilon + 1)^3 X_1^0 \wedge X_2^0 \wedge X_3^0 \\ &\quad = (U_2^\varepsilon + U_3^\varepsilon - (1-\varepsilon) U_1^\varepsilon) \wedge (U_1^\varepsilon + U_3^\varepsilon - (1-\varepsilon) U_2^\varepsilon) \wedge (U_1^\varepsilon + U_2^\varepsilon - (1-\varepsilon) U_3^\varepsilon) \\ &\quad = (\varepsilon - 2)^2 (\varepsilon + 1)\, U_1^\varepsilon \wedge U_2^\varepsilon \wedge U_3^\varepsilon. \end{aligned} \tag{34}$$

Therefore, using this fact

$$\begin{aligned}\det \mathscr{L}^\varepsilon &= \left(\prod_1^6 x_i^0\right)^{-1} L_1^\varepsilon \wedge L_2^\varepsilon \wedge \ldots \wedge L_6^\varepsilon \\ &= \left(\prod_1^6 x_i^0\right)^{-1} L_1^\varepsilon \wedge L_2^\varepsilon \wedge L_3^\varepsilon \wedge U_1^\varepsilon \wedge U_2^\varepsilon \wedge U_3^\varepsilon \quad \text{(by (23))} \\ &= \left(\prod_1^6 x_i^0\right)^{-1} (\varepsilon-2)(\varepsilon+1)^2\ L_1^\varepsilon \wedge L_2^\varepsilon \wedge L_3^\varepsilon \wedge X_1^0 \wedge X_2^0 \wedge X_3^0 \quad \text{(by (34))} \qquad (35)\end{aligned}$$

but also, by the definition of Δ

$$\det L^\varepsilon = \left(\prod_1^6 x_i^0\right)^{-1} (\varepsilon-2)(\varepsilon+1)^2\, \Delta(1-\varepsilon).$$

Equating both expressions and evaluating Δ at $\varepsilon=-1$, we have

$$\begin{aligned}\Delta(2) &= 2L_1^\varepsilon \wedge L_2^\varepsilon \wedge L_3^\varepsilon \wedge \frac{1}{2}\left(X_1^0 + \sum_1^3 L_i^\varepsilon\right) \wedge X_2^0 \wedge X_3^0|_{\varepsilon=-1} \\ &= 2\left(L_1^\varepsilon - \frac{1}{2}\left(X^0 + \sum_1^3 L_i^\varepsilon\right)\right) \wedge \left(L_2^\varepsilon - \frac{1}{2}\left(X^0 + \sum_1^3 L_i^\varepsilon\right)\right) \wedge \left(L_3^\varepsilon - \frac{1}{2}\left(X^0 + \sum_1^3 L_i\right)\right) \\ &\quad \wedge \frac{1}{2}\left(X^0 + \sum_1^3 L_i^\varepsilon\right) \wedge X_2^0 \wedge X_3^0|_{\varepsilon=-1};\end{aligned}$$

for $\varepsilon=-1$,

$$\frac{1}{2}\left(X^0 + \sum_1^3 L_i^\varepsilon\right)\Big|_{\varepsilon=-1} = (x_2^0 x_3^0 A_{32},\ x_1^0 x_3^0 A_{31},\ x_1^0 x_2^0 A_{21},\ 0,\ 0,\ 0)^t$$

and, by a somewhat tedious computation,

$$\Delta(2) = 2 \det \begin{pmatrix} x_5^0 x_6^0 A_{65} & 0 & 0 & x_2^0 x_3^0 A_{32} & 0 & 0 \\ 0 & x_4^0 x_6^0 A_{46} & 0 & x_1^0 x_2^0 A_{31} & x_2^0 & 0 \\ 0 & 0 & x_4^0 x_5^0 A_{54} & x_1^0 x_2^0 A_{21} & 0 & x_3^0 \\ 0 & x_2^0 x_6^0 A_{62} & x_3^0 x_5^0 A_{35} & 0 & 0 & 0 \\ x_1^0 x_6^0 A_{16} & 0 & x_3^0 x_4^0 A_{43} & 0 & x_5^0 & 0 \\ x_1^0 x_5^0 A_{51} & x_2^0 x_4^0 A_{42} & 0 & 0 & 0 & x_6^0 \end{pmatrix}$$

$$\begin{aligned}&= -2\left(\prod_1^6 x_i^0\right) K[(x_2^0)^2 (x_3^0)^2 A_{32} + (x_2^0)^2 (x_6^0)^2 A_{62} \\ &\quad + (x_3^0)^2 (x_5^0)^2 A_{35} + (x_5^0)^2 (x_6^0)^2 A_{65}] \\ &= -2\left(\prod_1^6 x_i^0\right) K[-x_2^0 x_3^0 x_5^0 x_6^0 2(A_{65} + A_{32}) \\ &\quad - x_1^0 x_2^0 x_3^0 - x_2^0 x_4^0 x_6^0 - x_3^0 x_4^0 x_5^0 - x_1^0 x_5^0 x_6^0] \\ &= 2\left(\prod_1^6 x_i^0\right) K[2x_2^0 x_3^0 x_5^0 x_6^0 (A_{65} + A_{32}) + 4x_1^0 x_2^0 x_3^0] \\ &= 4\left(\prod_1^6 x_i^0\right)\left(\prod_1^3 x_i^0\right) K,\end{aligned}$$

using the relations (16) in the second line, using (31) in the third line and using (33) in the fourth line. Comparing this result with $\Delta(0)$ in (32), we find

$$\Delta(2) = -\Delta(0).$$

Remembering the form (25) of $\Delta(l)$ and observing that $B=0$ from a computation on the matrix appearing in (35), we have that

$$\left(\prod_1^6 x_i^0\right)^{-1} \Delta(0) = 1 + C + D,$$
$$\left(\prod_1^6 x_i^0\right)^{-1} \Delta(2) = -1 - C + D.$$

Adding and subtracting these two equations yields

$$D = 0$$

and

$$C = -4\left(\prod_1^3 x_i^0\right) K - 1. \tag{36}$$

Hence from (25) and the above we find

$$\det(\mathscr{L} - lI) = (l-2)^2(l+1)(l-1)[(1-l)^2 - 4x_1^0 x_2^0 x_3^0 K - 1]$$

which establishes Lemma 2.

We now proceed with the

Proof of Lemma 4. Notice that $K \neq 0$ since $4x_1^0 x_2^0 x_3^0 K + 1 = 0$, and that our first nonlinear system $\mathscr{L}(X_0) = -X_0$ is locally uniquely solvable, since its Jacobian determinant does not vanish. From (31), since $K \neq 0$ and $\prod_1^6 x_i^0 \neq 0$,

$$\frac{x_1^0}{x_4^0} = \frac{x_2^0}{x_5^0} = \frac{x_3^0}{x_6^0} = \frac{x_4^0}{x_1^0} = \frac{x_5^0}{x_2^0} = \frac{x_6^0}{x_3^0}.$$

Therefore $x_i^{02} = x_{i+3}^{02}$ $(1 \leqq i \leqq 3)$. Multiplying the three first equations of (16) with x_1^0, x_2^0, and x_3^0 respectively, and using (31), we find

$$x_1^{02} = -x_1^0 x_2^0 x_3^0 (A_{32} + A_{65}),$$
$$x_2^{02} = -x_1^0 x_2^0 x_3^0 (A_{13} + A_{46}),$$
$$x_3^{02} = -x_1^0 x_2^0 x_3^0 (A_{21} + A_{54}),$$

and multiplying these three equations,

$$x_1^0 x_2^0 x_3^0 = -(A_{32} + A_{65})^{-1}(A_{13} + A_{46})^{-1}(A_{21} + A_{54})^{-1}.$$

Putting these equations into the expression (36), we see that C is a homogeneous rational function in A_{ij}, whose numerator, for the same reason given before, must be in $\mathscr{S}_3^-$ and it must therefore be a linear combination of K and S. Since $x_i^{02} = x_{i+3}^{02}$, we have $x_i^0 = \varepsilon_i x_{i+3}^0$ with $\varepsilon_i = \pm 1$ $(1 \leqq i \leqq 3)$ and in view of

(31), $\varepsilon_i = \pm 1$ simultaneously. According to (35),

$$\det(\mathscr{L} - I) = -\left(\prod_1^6 x_i^0\right)^{-1} 2L_1^0 \wedge L_2^0 \wedge L_3^0 \wedge X_1^0 \wedge X_2^0 \wedge X_3^0$$

holds and the eigenvalue problem for the latter matrix turns into solving the following linear system of equations,

$$\begin{aligned}
&x_2^0 x_3^0 A_{32}(y_2 + y_3) + x_1^0 y_4 = 0, \quad && x_2^0 x_3^0 (A_{62} y_2 + A_{35} y_3) + x_1^0 y_4 = 0,\\
&x_1^0 x_3^0 A_{13}(y_1 + y_3) + x_2^0 y_5 = 0, \quad && x_1^0 x_3^0 (A_{16} y_1 + A_{43} y_3) + x_2^0 y_5 = 0,\\
&x_1^0 x_2^0 A_{21}(y_1 + y_2) + x_3^0 y_6 = 0, \quad && x_1^0 x_2^0 (A_{51} y_1 + A_{24} y_2) + x_3^0 y_6 = 0.
\end{aligned}$$

Subtracting the $i+3^{\text{rd}}$ from the ith equation ($1 \leqq i \leqq 3$), we get

$$\begin{aligned}
&A_{36} y_2 + A_{52} y_3 = 0,\\
A_{63} y_1 \qquad &+ A_{14} y_3 = 0,\\
A_{25} y_1 + A_{41} y_2 \qquad &= 0;
\end{aligned}$$

then y_4, y_5, y_6 are uniquely determined from the three last equations; so the eigenvalue 1 will correspond to a 3-dimensional eigenspace if and only if the matrix of this system has rank 1, i.e. $A_{36} = A_{52} = A_{14} = 0$. This concludes the proof of Lemma 4.

Proof of Lemma 5. Consider a solution $\{x_1^0, \ldots, x_6^0\}$ of the non-linear set of Eqs. (16) and assume there is a variety of solutions around this one. Then the Jacobian determinant $\det \mathscr{L} = 0$ along this variety. If at least one of the x_i^0's vanishes, then one moves away from that point on the solution manifold so that $\prod_1^6 x_i^0 \neq 0$ and all the formulas in Sect. 2 hold by a continuity argument.

Suppose this were not possible, then there is a situation where, say, $x_1^0 = 0$ and stays like that throughout that manifold; the equations become

a) $0 = \quad x_2^0 x_3^0 A_{32} + x_5^0 x_6^0 A_{65}$,
b) $0 = x_2^0 \qquad + x_4^0 x_6^0 A_{46}$,
c) $0 = x_3^0 \qquad + x_4^0 x_5^0 A_{54}$,
d) $0 = x_4^0 + x_3^0 x_5^0 A_{35} + x_2^0 x_6^0 A_{62}$,
e) $0 = x_5^0 + x_3^0 x_4^0 A_{43}$,
f) $0 = x_6^0 + x_2^0 x_4^0 A_{24}$.

To begin with, assume all $A_{ij} \neq 0$ in these equations. If both determinants

$$\begin{aligned}
&1 - A_{24} A_{46} x_4^{02}\\
&1 - A_{43} A_{54} x_4^{02}
\end{aligned} \tag{27}$$

of the systems of equations b) and f) in x_2^0 and x_6^0 and equations c) and e) in x_3^0 and x_5^0 vanish, then equations b), c), e), and f) allow nonzero solutions for x_2^0,

x_6^0, x_3^0 and x_5^0:

$$\frac{x_2^0}{x_6^0} = -A_{46} x_4^0, \quad \frac{x_3^0}{x_5^0} = -A_{54} x_4^0.$$

Putting this into the first equation

$$x_5^0 x_6^0 (A_{46} A_{54} A_{32} x_4^{02} + A_{65}) = 0;$$

since generically $x_5^0 x_6^0 \neq 0$, upon combining the latter with (37) we find after some manipulation that some of the A_{ij} vanish, which contradicts the assumption: If one of the expressions (37) does not vanish, then $x_2^0 = x_6^0 = 0$ or $x_3^0 = x_5^0 = 0$, and the remaining system has a unique solution, which is now not under consideration.

Next, we assume, say, $A_{24} = 0$, implying $x_6^0 = 0$ and hence $x_2^0 = 0$; equations c), d) and e) again have a unique solution in x_3^0, x_4^0 and x_5^0. Assume now $A_{32} = 0$; then $x_5^0 x_6^0 A_{65} = 0$: if x_5^0 or $x_6^0 = 0$, the system is again uniquely solvable, but the case $A_{65} = 0$ requires some discussion. The matrix of the operator $\mathscr{L} - lI$ reads

$$\begin{pmatrix} 1-l & 0 & 0 & 0 & 0 & 0 \\ x_3^0 A_{13} & 1-l & 0 & x_6^0 A_{46} & 0 & x_4^0 A_{46} \\ x_2^0 A_{21} & 0 & 1-l & x_5^0 A_{54} & x_4^0 A_{54} & 0 \\ 0 & x_6^0 A_{62} & x_5^0 A_{35} & 1-l & x_3^0 A_{35} & x_2^0 A_{62} \\ x_6^0 A_{16} & 0 & x_4^0 A_{43} & x_3^0 A_{43} & 1-l & 0 \\ x_5^0 A_{51} & x_4^0 A_{24} & 0 & x_2^0 A_{24} & 0 & 1-l \end{pmatrix};$$

again $l = 1$ is an eigenvalue and it is easy to see, by roughly the same manipulation as before, that the eigenvalues are $l = -1, 0, 1, 2, 2, 2$. In fact $x_1^0 = 0$, and whenever the A_{ij}'s on the same row vanish, the condition $K = 0$ is trivially satisfied. Other cases can be considered and can be dealt with along similar lines.

On the contrary, now assume the solutions to the nonlinear system to be discrete with $\prod x_i^0 = 0$, say $x_1^0 = 0$ with the other $x_i^0 \neq 0$. Then 0 is an eigenvalue of $\mathscr{L}$ because the matrix of

$$\begin{aligned} 4L_2^0 \wedge \ldots \wedge L_6^0 = &-(L_2^0 + L_3^0 + L_5^0 + L_6^0) \wedge (L_3^0 + L_4^0 + L_6^0) \wedge (L_2^0 + L_4^0 + L_5^0) \\ &\wedge (L_2^0 + L_3^0 + L_4^0) \wedge (L_3^0 + L_5^0) \end{aligned}$$

has two proportional columns

$$L_2^0 + L_3^0 + L_5^0 + L_6^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2x_4^0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad L_3^0 + L_5^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2x_3^0 x_5^0 A_{35} \\ 0 \\ 0 \end{pmatrix},$$

and there are only 4 eigenvalues left excluding -1 and 0; thus only 4 possible degrees of freedom.

Next we consider the case where the solution is unique and at least two of the x_i^0's are zero. We assume $x_1^0=0$ throughout. If $x_4^0=0$, then a), ..., f) imply that all $x_i^0=0$, and so we must assume $x_4^0\neq 0$. Suppose $x_2^0=0$, then $x_6^0=0$, and we have

$$x_3^0+x_4^0x_5^0A_{54}=0,$$
$$x_4^0+x_3^0x_5^0A_{35}=0,$$
$$x_5^0+x_3^0x_4^0A_{43}=0.$$

Observe that $A_{35}\neq 0$ since $x_4^0\neq 0$. Assume $A_{54}A_{43}\neq 0$; note, in this case, all the x_i^0 are uniquely determined, and so only the case of 1 being a triple eigenvalue of $\mathscr{L}$ can lead to five degrees of freedom. We thus must examine the matrix $\mathscr{L}-1$. After some elementary nonsingular operations, the matrix $\mathscr{L}-1$ turns into

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where

$$D_1=\operatorname{diag}(x_3^0, x_4^0, x_5^0),$$
$$D_2=\begin{pmatrix} x_5^0A_{51} & x_4^0A_{24} & 0 \\ 0 & x_3^0A_{32} & x_5^0A_{65} \\ x_3^0A_{13} & 0 & x_4^0A_{46} \end{pmatrix}.$$

Thus we pick up three degrees of freedom if and only if $D_2=0$, i.e., $A_{51}=A_{24}=A_{32}=A_{65}=A_{13}=A_{46}=0$, which implies $A_{14}=A_{25}=A_{36}$, which corresponds to the case already discussed.

Suppose now that $A_{54}A_{43}=0$, say $A_{54}=0$; then $X^0=0$, and similarly for $A_{43}=0$, concluding the proof of Lemma 5.

Finally we conclude this section with the

Proof of Lemma 6. If (1) has a solution $X(t)$ with a harder than simple pole, then

$$X=t^{-k}\sum_{i=0}^{\infty}X^it^i, \qquad k>1,\ X^0\neq 0.$$

Equating the coefficients of t^{-k+i-1}, and, setting $l=i+k-1$, yields the linear system of equations

$$(-2k+l+1)X^{l-k+1}=\sum_{\substack{\alpha\geqq 0\\ \beta\geqq 0\\ \alpha+\beta=l\geqq 0}}[X^\alpha, \Lambda X^\beta],$$

with the understanding that $X^i=0$ for $i<0$. Since $k>1$, the first equation corresponding to $l=0$, reads

$$[X^0, \Lambda X^0]=0. \qquad (*)$$

and the $l+1$st equation for $l>0$ reads

$$\mathscr{L}^0(X^l)+\text{terms involving } X^i(0\leqq i<l)=0. \qquad (**)$$

Assume $\mathscr{L}^0$ has nullvectors $Y_1,\ldots,Y_s$; then the solution to $(**)$ reads $X^j+\sum_{i=1}^{s}\gamma_i^j Y_i$. Upon multiplying this formula by t^{j-k} and summing from $j=0$ to ∞, the solution of (13) must have the form (with $g_i(t)=0$ or free)

$$X+\sum_{i=1}^{s} g_i(t)\,Y_i=(X+\sum_{i\neq m} g_i(t)\,Y_i)+g_m(t)\,Y_m\equiv\tilde{X}+g_m(t).$$

Substituting this expression into (13) and using that $\tilde{X}$ is a solution, imply

$$\dot{g}_m(t)\,Y_m=g_m(t)([\tilde{X},\Lambda Y_m]+[Y_m,\Lambda\tilde{X}])+g_m^2(t)\,[Y_m,\Lambda Y_m].$$

Applying $\mathscr{L}^0$ to this equation yields

$$0=g_m(t)\,\mathscr{L}^0([\tilde{X},\Lambda Y_m]+[Y_m,\Lambda\tilde{X}])+g_m^2(t)\,\mathscr{L}^0([Y_m,\Lambda Y_m]);$$

unless the coefficients of $g_m(t)$ and $g_m^2(t)$ both vanish[3] and so $g_m(t)$ is uniquely defined and therefore degrees of freedom in the solution can only come from the equation $(*)$; thus we have at most s of them. In what follows we show $s\leqq 4$.

The nonlinear equations $(*)$ have the form

a′) $x_2^0x_3^0A_{32}+x_5^0x_6^0A_{65}=0$,
b′) $x_1^0x_3^0A_{13}+x_4^0x_6^0A_{46}=0$,
c′) $x_1^0x_2^0A_{21}+x_4^0x_5^0A_{54}=0$,
d′) $x_3^0x_5^0A_{35}+x_2^0x_6^0A_{62}=0$,
e′) $x_3^0x_4^0A_{43}+x_1^0x_6^0A_{16}=0$,
f′) $x_2^0x_4^0A_{24}+x_1^0x_5^0A_{51}=0$.

First assume all $A_{ij}\neq 0$. Then according to equations a′) and d′)

$$\left(\frac{x_2^0}{x_5^0}\right)^2=\frac{A_{65}A_{35}}{A_{32}A_{62}},\qquad \left(\frac{x_3^0}{x_6^0}\right)^2=\frac{A_{62}A_{65}}{A_{35}A_{32}};$$

according to b′) and e′)

$$\left(\frac{x_1^0}{x_4^0}\right)^2=\frac{A_{46}A_{43}}{A_{13}A_{16}},\qquad \left(\frac{x_3^0}{x_6^0}\right)^2=\frac{A_{46}A_{16}}{A_{43}A_{13}},$$

and according to c′) and f′)

$$\left(\frac{x_1^0}{x_4^0}\right)^2=\frac{A_{54}A_{24}}{A_{21}A_{51}},\qquad \left(\frac{x_2^0}{x_5^0}\right)^2=\frac{A_{54}A_{51}}{A_{24}A_{21}}.$$

---

[3] If the coefficients vanish, then in particular $[Y_m,\Lambda Y_m]$ is a nullvector of $\mathscr{L}^0$, which a computation shows is not so

These expressions are compatible if and only if $K=0$ as is clear from some manipulation. Since $\mathscr{L}^0(X_i^0)=0$, i $=1, 2, 3$,

$$\operatorname{rank} \mathscr{L}^0 \leqq 3.$$

In fact, we have equality since the upper 3×3 minor in the matrix (17). Sect. 3, of $\mathscr{L}^0$ does not vanish:

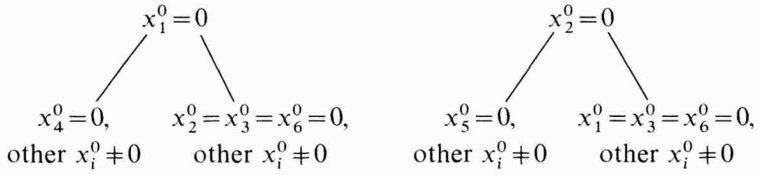
$$\det\begin{pmatrix} 0 & x_3^0 A_{32} & x_2^0 A_{32} \\ x_3^0 A_{13} & 0 & x_1^0 A_{13} \\ x_2^0 A_{21} & x_1^0 A_{21} & 0 \end{pmatrix} = 2x_1^0 x_2^0 x_3^0 A_{32} A_{13} A_{21} \neq 0.$$

Since only the ratios $(x_i/x_{i+3})^2$ are determined, the most general solution for the case above of $\mathscr{L}^0(Z)=0$ has the form

$$Z=\sum_{i=1}^{3} \alpha_i X_i^0,$$

yielding only three degrees of freedom.

Next, we assume, say, $A_{54}=0$ and all other $A_{ij}\neq 0$, implying $x_1^0 x_2^0 A_{21}=0$. We then have the following sequence of posibilities



Let us examine the two alternatives $x_1^0=x_4^0=0$ and $x_1^0=x_2^0=x_3^0=x_6^0=0$, the two other ones being similar. For the first alternative, the matrix of $\mathscr{L}^0$ reads

$$\begin{pmatrix} 0 & x_3^0 A_{32} & x_2^0 A_{32} & 0 & x_6^0 A_{65} & x_5^0 A_{65} \\ x_3^0 A_{13} & 0 & 0 & x_6^0 A_{46} & 0 & 0 \\ x_2^0 A_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_6^0 A_{62} & x_5^0 A_{35} & 0 & x_3^0 A_{35} & x_2^0 A_{62} \\ x_6^0 A_{16} & 0 & 0 & x_3^0 A_{43} & 0 & 0 \\ x_5^0 A_{51} & 0 & 0 & x_2^0 A_{24} & 0 & 0 \end{pmatrix}.$$

An eigenvector $u=(u_1, \ldots, u_6)$ must have $u_1=u_4=0$. There remains a 4-dimensional eigenspace and therefore we cannot expect more than 4 degrees of freedom. For the second alternative, the matrix of $\mathscr{L}^0$ has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x_5^0 A_{65} \\ 0 & 0 & 0 & 0 & 0 & x_4^0 A_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_5^0 A_{35} & 0 & 0 & 0 \\ 0 & 0 & x_4^0 A_{43} & 0 & 0 & 0 \\ x_5^0 A_{51} & x_4^0 A_{24} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

implying for the eigenvector u, that $u_3=u_6=0$, allowing therefore at most 4 degrees of freedom.

If $A_{54}=A_{21}=0$, then, if the reader wishes, he can argue as in the beginning of this proof. He can check that the remaining alternatives never lead to more than 4 degrees of freedom, concluding the proof of the Lemma 6.

References

1. Adler, M., van Moerbeke, P.: Completely integrable systems, Euclidean lie algebras, and curves. Adv. in Math. **38**, 267–317 (1980)
2. Adler, M., van Moerbeke, P.: Linearization of Hamiltonian systems, Jacobi varieties, and representation theory. Adv. in Math. **38**, 318–379 (1980)
3. Adler, M., van Moerbeke, P.: Kowalewski's asymptotic method, Kac-Moody Lie algebras and regularization. Comm. Math. Physics **83**, 83–106 (1982)
4. Hille, E.: Ordinary differential equations in the complex domain. NY: Wiley-Interscience 1976
5. Kowalewski, S.: Sur le probleme de la rotation d'un corps solide autour d'un point fixe. Acta Mathematica **12**, 177–232 (1889)
6. Kowalewski, S.: Sur une propriété du système d'équations differentielles qui définit la rotation d'un corps solide autor d'un point fixe. Acta Mathematica **14**, 81–83 (1889)
7. Mischenko, A.S., Fomenko, A.T.: Euler's equation on finite-dimensional Lie groups. Izv. Akad. Nauk. SSSR Ser. Mat. **42**, (1978)
8. Birnir, B.: Complex Hill's equations, complex Korteweg-de Vries flows and Jacobi varieties. PhD Dissertation-NYU 1981
9. Mumford, D.: Abelian varieties. Oxford University press: Bombay-Oxford: 1974

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Appendix

D. Mumford
Harvard University, Cambridge, MA 02138, USA

In connection with the preceding article, Adler and van Moerbeke asked me whether there was a direct algebro-geometric proof that the loci defined by setting their 4 quadratic integrals equal to constants was the affine part of an abelian surface. We give here such a proof. The precise statement is this:

Proposition. *Let* $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $1 \leqq i \leqq 3$ *be given such that the* α_i *are distinct, non-zero and*

$$\det \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \neq 0.$$

Let

$$\lambda_1=\frac{\beta_2-\beta_3}{\alpha_2-\alpha_3}, \quad \mu_1=\frac{\gamma_2-\gamma_3}{\alpha_2-\alpha_3},$$

$$\lambda_2=\frac{\beta_1-\beta_3}{\alpha_1-\alpha_3}, \quad \mu_2=\frac{\gamma_1-\gamma_3}{\alpha_1-\alpha_3},$$

$$\lambda_3=\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}, \quad \mu_3=\frac{\gamma_1-\gamma_2}{\alpha_1-\alpha_2},$$

$$\lambda_4=\beta_1/\alpha_1, \quad \mu_4=\gamma_1/\alpha_1,$$

$$\lambda_5=\beta_2/\alpha_2, \quad \mu_5=\gamma_2/\alpha_2,$$

$$\lambda_6=\beta_3/\alpha_3, \quad \mu_6=\gamma_3/\alpha_3.$$

In $\mathbb{C}^6$, let F_c be the affine variety defined by

$$\sum_1^6 X_i^2 = c_1,$$

$$\sum_1^6 \lambda_i X_i^2 = c_2,$$

$$\sum_1^6 \mu_i X_i^2 = c_3,$$

$$X_1 X_4 + X_2 X_5 + X_3 X_6 = c_4$$

where $c_i \in \mathbb{C}$, $1 \leqq i \leqq 4$. Then for c_i's in a Zariski-open subset of $\mathbb{C}^4$, F_c is an affine open piece of an abelian surface.

Proof. Let $\bar{F}_c$ be the projective variety on $\mathbb{P}^6$ defined by

$$\sum_1^6 X_i^2 = c_1 X_0^2, \quad \sum_1^6 \lambda_i X_i^2 = c_2 X_0^2, \quad \sum_1^6 \mu_i X_i^2 = c_3 X_0^2, \quad X_1 X_4 + X_2 X_5 + X_3 X_6 = c_4 X_0^2.$$

Since F_c is the fibre of a morphism from $\mathbb{C}^6$ to $\mathbb{C}^4$ over $\check{c} \in \mathbb{C}^4$, for almost all c, F_c is a smooth affine surface. The problem is to describe the singularities at ∞ of $\bar{F}_c$. Let $C = \bar{F}_c \cap (X_0 = 0)$, the scheme defined by the 4 equations $\sum X_i^2 = \sum \lambda_i X_i^2 = \sum \mu_i X_i^2 = X_1 X_4 + X_2 X_5 + X_3 X_6 = 0$, which is independent of $\check{c}$. We consider the Jacobian matrix of the 4 quadratic polynomials defining C:

$$\begin{pmatrix} 2X_1 & 2X_2 & 2X_3 & 2X_4 & 2X_5 & 2X_6 \\ 2\lambda_1 X_1 & 2\lambda_2 X_2 & 2\lambda_3 X_3 & 2\lambda_4 X_4 & 2\lambda_5 X_5 & 2\lambda_6 X_6 \\ 2\mu_1 X_1 & 2\mu_2 X_2 & 2\mu_3 X_3 & 2\mu_4 X_4 & 2\mu_5 X_5 & 2\mu_6 X_6 \\ X_4 & X_5 & X_6 & X_1 & X_2 & X_3 \end{pmatrix}.$$

First, consider this matrix at a point where $\prod_1^6 X_i \neq 0$. By elementary column operations, it is equivalent to:

$$\begin{pmatrix} \alpha_2 - \alpha_3 & \alpha_1 - \alpha_3 & \alpha_1 - \alpha_2 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_2 - \beta_3 & \beta_1 - \beta_3 & \beta_1 - \beta_2 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_2 - \gamma_3 & \gamma_1 - \gamma_3 & \gamma_1 - \gamma_2 & \gamma_1 & \gamma_2 & \gamma_3 \\ (\alpha_2 - \alpha_3)\dfrac{X_4}{X_1} & (\alpha_1 - \alpha_3)\dfrac{X_5}{X_2} & (\alpha_1 - \alpha_2)\dfrac{X_6}{X_3} & \alpha_1 \left(\dfrac{X_4}{X_1}\right)^{-1} & \alpha_2 \left(\dfrac{X_5}{X_2}\right)^{-1} & \alpha_3 \left(\dfrac{X_6}{X_3}\right)^{-1} \end{pmatrix}$$

hence if $u = x_4/x_1$, $v = x_5/x_2$, $w = x_6/x_3$, by row operations it is equivalent to:

$$\begin{pmatrix} 0 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & 0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 & \gamma_1 & \gamma_1 & \gamma_2 \\ (\alpha_2 - \alpha_3)u & (\alpha_1 - \alpha_3)v & (\alpha_1 - \alpha_2)w & \alpha_1 u^{-1} & \alpha_2 v^{-1} & \alpha_3 w^{-1} \\ -\alpha_2 v^{-1} + \alpha_3 w^{-1} & -\alpha_1 u^{-1} + \alpha_3 w^{-1} & -\alpha_1 u^{-1} + \alpha_2 v^{-1} & & & \end{pmatrix}. \quad (1)$$

The singular locus of C is defined by the condition that the rank of this matrix is less than 4, i.e.,

$$(\alpha_2 - \alpha_3)u - \alpha_2 v^{-1} + \alpha_3 w^{-1} = 0,$$

$$(\alpha_1 - \alpha_3)v - \alpha_1 u^{-1} + \alpha_3 w^{-1} = 0,$$

$$(\alpha_1 - \alpha_2)w - \alpha_1 u^{-1} + \alpha_2 v^{-1} = 0.$$

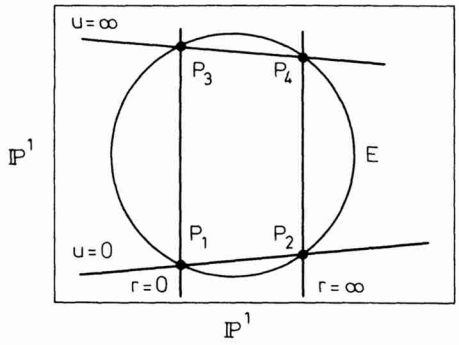
Call these 3 equations $f(u, v, w)$, $g(u, v, w)$, and $h(u, v, w)$. Note that

$$\alpha_1 vwf - \alpha_2 uwg + \alpha_3 uvh \equiv 0,$$

hence we have here in effect only 2 equations. These define in $(\mathbb{C}^*)^3$ an affine piece of an elliptic curve. In fact,

$$(\alpha_2 - \alpha_3) u - \alpha_2 v^{-1} = (\alpha_1 - \alpha_3) v - \alpha_1 u^{-1}$$

is a smooth elliptic curve E in $\mathbb{P}^1 \times \mathbb{P}^1$ (having degree 2 in each of the 2 variables), and one solves for w via $h=0$. With a little checking, one finds that one has the picture



where the 3 rational functions u, v, w on E have zeroes and poles:

$$(u) = P_1 + P_2 - P_3 - P_4,$$
$$(v) = P_1 + P_3 - P_2 - P_4,$$
$$(w) = P_1 + P_4 - P_2 - P_3.$$

Choosing X_1, X_2, X_3 suitably as functions of u, v, w, can we lift this curve to C and is it in fact all of C? To analyze $C \cap \left(\prod_1^6 X_i \neq 0\right)$, let $u = X_4/X_1$, $v = X_5/X_2$, $w = X_6/X_3$ and rewrite the defining equations of C as

$$X_1^2(1+u^2) + X_2^2(1+v^2) + X_3^2(1+w^2) = 0,$$
$$X_1^2(\lambda_1 + \lambda_4 u^2) + X_2^2(\lambda_2 + \lambda_5 v^2) + X_3^2(\lambda_3 + \lambda_6 w^2) = 0,$$
$$X_1^2(\mu_1 + \mu_4 u^2) + X_2^2(\mu_2 + \mu_5 v^2) + X_3^2(\mu_3 + \mu_6 w^2) = 0,$$
$$X_1^2 u + X_2^2 v + X_3^2 w = 0.$$

Then the projection of C into u, v, w-space is given by

$$rk \begin{pmatrix} 1+u^2 & 1+v^2 & 1+w^2 \\ \lambda_1 + \lambda_4 u^2 & \lambda_2 + \lambda_5 v^2 & \lambda_3 + \lambda_6 w^2 \\ \mu_1 + \mu_4 u^2 & \mu_2 + \mu_5 v^2 & \mu_3 + \mu_6 w^2 \\ u & v & w \end{pmatrix} \leqq 2.$$

But replacing the 3rd column by

$$\alpha_1(\alpha_2 - \alpha_3) u^{-1} \times (1^{\text{st}} \text{ col.}) - \alpha_2(\alpha_1 - \alpha_3) v^{-1} \times (2^{\text{nd}} \text{ col.}) + \alpha_3(\alpha_1 - \alpha_2) w^{-1} \times (3^{\text{rd}} \text{ col.})$$

we see after a small computation that this is equivalent to

$$rk\begin{pmatrix} 1+u^2 & 1+v^2 & \alpha_1 f-\alpha_2 g+\alpha_3 h \\ \lambda_1+\lambda_4 u^2 & \lambda_2+\lambda_5 v^2 & \beta_1 f-\beta_2 g+\beta_3 h \\ \mu_1+\mu_4 u^2 & \mu_2+\mu_5 v^2 & \gamma_1 f-\gamma_2 g+\gamma_3 h \\ u & v & 0 \end{pmatrix} \leqq 2.$$

The 1st and 2nd columns are easily seen to be everywhere independent, so this means the 3rd column is zero, i.e., $f=g=h=0$ again. Retracing our steps, we have proved:

$$C\cap\left(\prod_1^6 X_i \neq 0\right)$$
$$= \text{the curve } X_0=0$$
$$X_1=\sqrt{\alpha_1(\alpha_2-\alpha_3)}\,u^{-1/2}$$
$$X_2=\sqrt{\alpha_2(\alpha_3-\alpha_1)}\,v^{-1/2}$$
$$X_3=\sqrt{\alpha_3(\alpha_1-\alpha_2)}\,w^{-1/2}$$
$$X_4=uX_1$$
$$X_5=vX_2$$
$$X_6=wX_3$$

with

$$(u, v, w)\in E-\{P_1, P_2, P_3, P_4\}.$$

What does this imply about the *scheme* $C\cap(\prod X_i\neq 0)$? Since the Jacobian matrix is everywhere singular, it has multiplicity. But since the 4×4-minors of the Jacobian, plus the equations defining C define a non-singular *reduced* elliptic curve C_{red}, it follows that $C\cap(\prod X_i\neq 0)$ has multiplicity two, and locally near every point, 3 of its 4 equations have independent differentials, defining a smooth surface, and the fourth equation may be taken to be the square of the fourth equation defining C_{red}.

What happens if $\prod X_i=0$? Take, for instance, points where $X_1X_2X_3\neq 0$ but X_4, X_5, X_6 may be zero. Here we can still change coordinates to X_1, X_2, X_3, u, v, w and as above it follows easily that C contains the above points *plus* 4 more points corresponding to $P_i\in E$. These are given by

$$X_1=\pm\sqrt{\alpha_2-\alpha_3},$$
$$X_2=\pm\sqrt{\alpha_3-\alpha_1},$$
$$X_3=\pm\sqrt{\alpha_1-\alpha_2},$$
$$X_4=u=0,$$
$$X_5=v=0,$$
$$X_6=w=0.$$

We can argue similarly[4] in the other coordinate patches (e.g., where $X_1X_2X_6\neq 0$, so X_1, X_2, X_6, u, v, w^{-1} are coordinates, or where $X_1X_3X_5\neq 0$, so X_1, X_3, X_5, u, v^{-1}, w are coordinates, etc.) What comes out is that the whole curve C_{red} is a 4-sheeted unramified covering of E with covering group $(\mathbb{Z}/2\mathbb{Z})^2$, hence it must be E again, with covering map

$$2_E: E\to E.$$

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[4] A special argument is needed to see that

$$C\cap(X_1=X_4=0)=\emptyset$$

(similarly, where $X_2=X_5=0$ or where $X_3=X_6=0$). The simplest way to argue seems to be to note that all components of C have dimension at least 1, that the above curve E doesn't meet $X_1=X_4=0$ and that by inspection of the equations, at least

As for the singularities of C at the 16 new points added where C meets $\prod X_i = 0$, they must be of the same type as the rest, in view of the fact that a) C_{red} is a smooth curve, and b) C itself is a complete intersection.

This completes our description of C. What does this imply about the singularities of $\bar{F}_c$ along C? Call the four defining equations of C f_1, f_2, f_3, f_4. Then $\bar{F}_c$ is defined by

$$c_i X_0^2 = f_i.$$

At any point $P \in C_{\text{red}}$, there is exactly one combination of the $df_i(P)$ which is zero:

$$\lambda_1 df_1(P) + \lambda_2 df_2(P) + \lambda_3 df_3(P) + \lambda_4 df_4(P) = 0.$$

This gives a kind of "Gauss map"

$$\gamma\colon C_{\text{red}} \to \mathbb{P}^3$$
$$\gamma(P) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

We see from the matrix (1) that in our case γ is given by

$$C_{\text{red}} \longrightarrow E \xrightarrow{\gamma'} \mathbb{P}^3$$

where $\gamma'(u, v, w)$ has as its coordinates the

$$4\ 3\times 3 \text{ minors of } \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \alpha_1 u^{-1} & \alpha_2 v^{-1} & \alpha_3 w^{-1} \end{pmatrix}$$

i.e., changing coordinates in $\mathbb{P}^3$,

$$\gamma'(u, v, w) = (1, u^{-1}, v^{-1}, w^{-1}).$$

Let $H_c \subset \mathbb{P}^3$ be the hyperplane $\sum c_i \lambda_i = 0$. Then if $\gamma(P) \notin H_c$, making a linear change in the equations of $\bar{F}_c$, they look like:

$$\begin{aligned} c_i X_0^2 &= f_i, \quad 1 \leqq i \leqq 3, \\ c_4 X_0^2 &= f_4, \quad c_4 \neq 0, \\ f_4 &\equiv g^2 (\operatorname{mod} f_1, f_2, f_3) \\ df_1, df_2, df_3, dg &\quad \text{independent.} \end{aligned}$$

Thus C is an ordinary double curve of $\bar{F}_c$ at P. On the other hand, if the c_i are chosen so that $\gamma^{-1}(H_c)$ has no multiplicities, it is easy to see that $\bar{F}_c$ has ordinary pinch points at all $P \in \gamma^{-1}(H_c)$ (i.e., $\bar{F}_c$ has a local analytic equations $x^2 = yz^2$). The number of these points is $\deg \gamma^{-1}(H_c)$, which in our case is $\deg [C_{\text{red}}: E] \cdot \deg \gamma'(E)$, i.e., $4 \cdot 4 = 16$.

Now let $\pi\colon \tilde{F}_c \to \bar{F}_c$ be the normalization of $\bar{F}_c$, and let $\tilde{C}$ be the (reduced) curve in $\tilde{F}_c$ over C. $\tilde{F}_c$ is non-singular, and we compute its invariants as follows:

$$\begin{aligned} K_{\tilde{F}_c} &= \pi^*(K_{\bar{F}_c}) - \tilde{C} \\ &= \pi^*((\deg f_1 + \deg f_2 + \deg f_3 + \deg f_4) \cdot H + \bar{F}_c \cdot K_{\mathbb{P}^6}) - \tilde{C} \\ &\qquad (H = \text{hyperplane in } \mathbb{P}^6) \\ &= \pi^*(H) - V(X_0) \quad (\text{since } X_0 \text{ vanishes to order 1 along } \tilde{C}) \\ &= 0 \end{aligned}$$

while

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{F}_c}) &= \chi(\pi_* \mathcal{O}_{\tilde{F}_c}) \\ &= \chi(\mathcal{O}_{\bar{F}_c}) + \chi(\pi_* \mathcal{O}_{\tilde{F}_c}/\mathcal{O}_{\bar{F}_c}) \\ &= \chi(\mathcal{O}_{\bar{F}_c}) + \chi(\pi_* \mathcal{O}_C/\mathcal{O}_{C_{\text{red}}}). \end{aligned}$$

Now

$$\tilde{C} \to C_{\text{red}}$$

is a double cover, branched over the 16 pinch points, hence $\tilde{C}$ has genus 9. Thus $\chi(\mathcal{O}_{\tilde{C}}) = -8$, $\chi(\mathcal{O}_{C_{\text{red}}}) = 0$, hence $\chi(\pi_* \mathcal{O}_{\tilde{C}}/\mathcal{O}_{C_{\text{red}}}) = -8$. But the Koszul complex:

$$0 \to \mathcal{O}_{\mathbb{P}^6}(-8) \to \mathcal{O}_{\mathbb{P}^6}(-6)^4 \to \mathcal{O}_{\mathbb{P}^6}(-4)^6 \to \mathcal{O}_{\mathbb{P}^6}(-2)^4 \to \mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_{F_c} \to 0$$

gives $\chi(\mathcal{O}_{F_c}) = 8$, hence

$$\chi(\mathcal{O}_{\tilde{F}_c}) = 0.$$

By Enriques' classification of surfaces, it follows that $\tilde{F}_c$ is an abelian surface. QED

We can go further and describe what abelian surfaces arise this way and how to construct these embeddings in theta functions. In fact, note that

$$\alpha: (X_0, X_1, \ldots, X_6) \mapsto (-X_0, X_1, \ldots, X_6)$$

is an automorphism of $\bar{F}_c$, hence of $\tilde{F}_c$, of order 2 with 16 isolated fixed points at the points of $\tilde{C}$ over the pinch points of $\bar{F}_c$. Whenever an automorphism α of order 2 on an abelian variety X has an isolated fixed point P, α must be -1_X (the eigenvalues of $d\alpha_P$ are all -1, hence α, lifted to the universal cover of X, must be $-\text{id.}$). Now let $\mathbb{L} = \pi^*(\mathcal{O}(1))$ be the line bundle on $\tilde{F}_c$, 7 of whose sections give us the morphism

$$\pi: \tilde{F}_c \to \bar{F}_c \to \mathbb{P}^6.$$

Then $c_1(\mathbb{L}) = \pi^*(H)$ and because $\deg \bar{F}_c = 16$, $(c_1(\mathbb{L})^2) = 16$. By the classification theory of ample line bundles on abelian varieties

$$\tilde{F}_c \cong \mathbb{C}^2/L_\Omega$$

where either

i) $$\Omega = \begin{pmatrix} 8 & 0 & a & b \\ 0 & 1 & b & c \end{pmatrix}$$

or

ii) $$\Omega = \begin{pmatrix} 4 & 0 & a & b \\ 0 & 2 & b & c \end{pmatrix}$$

and $\mathbb{L}$ is the canonical associated bundle. In both cases, $h^0(\mathbb{L}) = 8$ but in the first, $\mathbb{L}$ has 5 even sections, 3 odd ones; in the second, $\mathbb{L}$ has 6 even sections, 2 odd ones. Since $X_1, \ldots, X_6$ are 6 even ones, we conclude that (ii) holds, hence $\mathbb{L} = \mathbb{M}^2$, $(c_1(\mathbb{M})^2) = 4$, and that π is defined by a subspace $V \subset \Gamma(\mathbb{L})$,

$$V = \Gamma(\mathbb{L})_+ \oplus \mathbb{C} \cdot s$$

where $s \in \Gamma(\mathbb{L})_-$. The affine piece which occurs in the dynamical system is exactly

$$\mathbb{C}^2/L_\Omega - V(s).$$