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On Compactifiable Complex Analytic Surfaces

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0. Introduction and Statements of Results

Let M be a surface, i.e., a connected complex manifold of complex dimension 2. By a *compactification* of M, we mean a pair (S, C) consisting of a compact surface S and a (possibly reducible) curve C on S such that $S-C$ is biholomorphic to M. A surface M is called *compactifiable* if M has a compactification. A compactification (S, C) is called *algebraic* (resp. *non-algebraic*) if S is algebraic (resp. non-algebraic). In this paper, we shall study the following problems:

i) a classification of compactifiable surfaces by their *algebraic dimensions* (defined below),

ii) a classification of surfaces with non-algebraic compactifications,

iii) the determination of all (algebraic or non-algebraic) compactifications of surfaces having non-algebraic compactifications.

Now we shall describe our results on these problems.

1) Let M be a compact or non-compact surface. We define the *algebraic dimension* $a(M)$ of M to be the maximum of integers k with the following property:

There exist k meromorphic functions $f_1, \ldots, f_k$ on M such that $df_1 \wedge \ldots \wedge df_k$ does not vanish identically on M.

Then $0 \leqq a(M) \leqq 2$. When M is compact, our definition of $a(M)$ is equivalent to the usual definition of the algebraic dimension [2, I; p. 125]. Moreover, for a

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compact surface M, it is well known that $a(M)=2$ if and only if M is algebraic (Chow-Kodaira), and that $a(M)=1$ if and only if M is a non-algebraic elliptic surface (Kodaira). Anologously to the compact case, we have

Theorem A. *Let M be a compactifiable surface. Then:*

i) *$a(M)=2$ if and only if M is quasi-projective.*

ii) *$a(M)\leqq 1$ if and only if, for any compactification (S, C) of M, C is an exceptional set on S (i.e., the intersection matrix of C is negative definite) and $a(S)=a(M)\leqq 1$.*

Remark. In view of the property of exceptional sets, we can replace "for any ...," in the statement ii) of Theorem A by "there exists a ... such that".

2) Let M be a surface with a compactification (S, C). Then, for each connected component A of C, a sufficiently small neighborhood of A in S defines the end $e(A)$ of M canonically, and vice versa. We call an end $e(A)$ of M *exceptional* if the corresponding connected component A of C has a strongly pseudo-convex neighborhood in S, i.e. A is an exceptional set on S. Clearly this property is independent of the choice of the compactification (S, C) of M.

Now let E denote the union of the connected components of C which are exceptional. Set $D=C-E$. Then $S-D$ is a compact or compactifiable surface without exceptional ends, and $M=(S-D)-E$. Moreover, $S-D$ is obtained from a surface M^* without exceptional curves of the first kind by means of a finite succession of quadratic transformations. Thus we obtain a compact or compactifiable surface M^* with the following properties:

i) M^* has neither exceptional curves of the first kind nor exceptional ends,

ii) M can be written as $M=Q_k \dots Q_2 Q_1(M^*)-E$, where $Q_1, Q_2, \dots, Q_k$ are quadratic transformations and E is a (compact) exceptional set on $Q_k \dots Q_1(M^*)$.

Then M^* is determined by M uniquely up to finite succesion of quadratic transformations. In particular, M has a non-algebraic compactification if and only if so does M^*. Thus, to classify compactifiable surfaces M, we have only to consider M^* instead of M.

Theorem B. *Let M be a compactifiable surface without exceptional curves of the first kind or exceptional ends. Suppose M has non-algebraic compactifications. Then M is of one of the following types:*

i) *an affine $\mathbb{C}$-bundle of degree $\leqq 0$ over an elliptic curve.*

ii) *a surface obtained by deleting the 0-section from a line bundle of degree 0 over an elliptic curve.*

iii) *a surface obtained by deleting a finite number of fibres from a compact elliptic surface.*

3) A compactification (S, C) is called a *minimal normal* compactification if it satisfies the following two conditions:

i) C is of normal crossing on S.

ii) If S contains an exceptional curve E of the first kind on S, then E meets at least three other irreducible components of C.

The self-intersection number of any curve Θ on S will be denoted by $(\Theta)^2$. Let $\mathbb{P}^m$ denote the complex projective space of dimension m. Now, for surfaces of type i) (resp. type ii), type iii)) described in Theorem B, all their minimal normal compactifications will be given in the following Theorem C-I (resp. C-II, C-III).

Theorem C-I. *Let M be an affine $\mathbb{C}$-bundle of degree $-n \leqq 0$ over an elliptic curve (i.e., M is of type* i) *in* Theorem B). *Let (S, C) be a minimal normal compactification of M. Then:*

C-I_0. *When $n=0$, the following cases occur:*

a) *S is biholomorphic to $\mathbb{P}^2$ and C is the union of three lines in general position.*

b) *S is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and C is the union of two different fibres and two disjoint sections.*

c) *S is a Hopf surface and C is a non-singular elliptic curve.*

d) *S is a $\mathbb{P}^1$-bundle over an elliptic curve containing C as a section with $(C)^2 = 0$.*

C-I_+. *When $n>0$, the following cases occur:*

e) *S is a surface of Class VII_0 with $b_2(S)=n$ and C is the uniquely determined curve with $(C)^2=0$.*

f) *S is a $\mathbb{P}^1$-bundle over an elliptic curve containing C as a section with $(C)^2 = n$.*

Theorem C-II. *Let M be a surface obtained by deleting the* 0*-section from a line bundle of degree* 0 *over an elliptic curve (i.e., M is of type ii) in* Theorem B). *Let (S, C) be a minimal normal compactification of M. Then the following cases occur:*

g) *S is a Hopf surface and C is the union of two non-singular elliptic curves.*

h) *S is a $\mathbb{P}^1$-bundle over an elliptic curve and C is the union of two disjoint sections Γ_0, Γ_∞ with $(\Gamma_0)^2 = (\Gamma_\infty)^2 = 0$.*

Theorem C-III. *Let M be a surface obtained by deleting a finite number of fibres from a compact elliptic surface (i.e., M is of type* iii) *in* Theorem B). *Let (S, C) be a minimal normal compactification of M. Then S is an elliptic surface containing C as the union of a finite number of fibres.*

We give here a few comments on the above theorems.

i) The above cases a)–h) in Theorems C-I and C-II actually occur (cf. [2, II; pp. 694–695] and [1; p. 461, Remark]).

ii) The structure of the surfaces described in e) of Theorem C-I has been determined in [1]. They are called *parabolic Inoue* surfaces.

iii) All compactifications of $\mathbb{C}^* \times \mathbb{C}^*$ have been determined by Simha [6], Ueda [8] and Suzuki [7]. Since $\mathbb{C}^* \times \mathbb{C}^*$ has the structure of affine $\mathbb{C}$-bundle of degree 0 over an elliptic curve, our Theorem C-I can be regarded as a generalization of their results. In fact, our proof uses their arguments.

4) Finally we describe how our proofs go. To prove Theorem A, we shall show first that a surface M of type iii) described in Theorem B is quasi-projective (§2, Proposition (2.1)). Then Theorem A follows from this and

Theorem B. The proof of Theorem B needs the results of [1] and Kodaira's classification theory on compact surfaces. We shall prove Theorems C-I and C-II, using the Kodaira's classification theory on compact surfaces and the results of [1] (see § 1). We shall prove Theorems C-I and C-II, using the results on cluster sets of holomorphic maps due to Nishino and Suzuki [5], as well as the results on compactifications of $\mathbb{C}^* \times \mathbb{C}^*$ by Simha [6], Ueda [8] and Suzuki [7].

1. Surfaces with Non-Algebraic Compactifications; Proof of Theorem B

In our previous paper, [1; Main Th. and Prop. 3.4], we have proved

(1.1) **Fact.** *Let S be a VII_0-surface with $b_2(S)>0$. Suppose S has a divisor $D \neq 0$ with $(D)^2=0$ and let C be the support of D. Then:*

a) *C is the uniquely determined normal crossing curve on S.*

b) *$S-C$ is an affine $\mathbb{C}$-bundle of degree $-b_2(S)$ over an elliptic curve.*

We note here

(1.2) **Lemma.** *Let S be a compact surface having no meromorphic functions except constants and let C be a curve on S. Suppose that C and $S-C$ contain no exceptional curves of the first kind on S and that no connected component of C is exceptional on S. Then S is minimal.*

Proof. Suppose S has an exceptional curve E of the first kind. Then by hypothesis, we have a connected component A of C such that E intersects A but is not contained in A. Since S has no meromorphic functions except constants, the intersection matrix of the connected curve $A \cup E$ is negative semi-definite. Then by [8; p. 85, Lemma 2 iii)], the intersection matrix of A is negative definite. This contradicts our hypothesis that A is not exceptional. Q.E.D.

Now we prove Theorem B. Let M be as in Theorem B and (S, C) a non-algebraic compactification of M. By hypothesis, no connected component of C is exceptional.

1) Suppose first S has a non-constant meromorphic function. Then by [2, I; p. 131, Th. 4.1], S is an elliptic surface. Let $\Phi: S \to \Delta$ denote the canonical projection of S onto a curve Δ. Let $A_1, \ldots, A_j, \ldots$ denote the connected components of C. By [2, I; p. 138, Th. 4.3], each A_j is contained in $\Phi^{-1}(a_j)$ for some $a_j \in \Delta$. Since the intersection matrix of the fibre $\Phi^{-1}(a_j)$ is negative semi-definite, we know by [8; p. 85, Lemma 2 iii)] that, if $A_j \neq \Phi^{-1}(a_j)$, then the intersection matrix of A_j is negative definite. Since none of A_j's is exceptional by the hypothesis of Theorem B, it follows that $A_j = \Phi^{-1}(a_j)$ and hence $C = \bigcup_j \Phi^{-1}(a_j)$. Thus M is of type iii) of Theorem B.

2) Suppose next S has no meromorphic functions except constants. Blowing down exceptional curves of the first kind contained in C, we may assume by Lemma (1.2) that S is minimal. Then S is either a VII_0-surface, a complex torus or a $K3$ surface (see [3, I; p. 759, Th. 11]). Now we shall check each case.

2-1) *The case where S is* VII_0.

2-1_0) Suppose $b_2(S)=0$. Then by [3, II; p. 699, Th. 34 and p. 700, Lemma 10], S is a Hopf surface which has at most two irreducible curves. In case S has only one irreducible curve, $M=S-C$ is an affine $\mathbb{C}$-bundle of degree 0 over an elliptic curve. This is of type i) in Theorem B. In case S has exactly two irreducible curves Θ_1, Θ_2, each $S-\Theta_i$, $i=1$ or 2, is a line bundle of degree 0 over an elliptic curve containing $\Theta_j, j\neq i$, as the 0-section (cf. [3, II; p. 698, Th. 32]). Thus M is of type i) or of type ii) in Theorem B according as M is $S-\Theta_i$ or $S-\Theta_1\cup\Theta_2$.

2-1_+) Suppose $b_2(S)>0$. Since S has no meromorphic functions except constants, the intersection matrix of each connected component of C is negative semi-definite (but not definite by the hypothesis of Theorem B). Hence, in view of [8; p. 85, Lemma 2], we have a divisor $D\neq 0$ on S with $(D)^2=0$ whose support is C. Thus by Fact (1.1) b), $M=S-C$ is of type i) in Theorem B.

2-2) *The case where S is a complex torus.* This case does not occur. In fact, applying translations to the curve C, we have infinitely many irreducible curves on S. By [2, I; p. 139, Th. 5.1], this contradicts our hypothesis that S has no meromorphic functions except constants.

2-3) *The case where S is a $K3$ surface.* This case does not occur. In fact, letting $C=\bigcup_{j=1}^{m}\Theta_j$ denote the decomposition of C into the irreducible components Θ_j ($\Theta_i\neq\Theta_j$ if $i\neq j$), define a divisor D_r on S by

$$D_r=\sum_{j=1}^{m} r_j\Theta_j \quad \text{for } r=(r_1,\ldots,r_j,\ldots,r_m)\in\mathbb{Z}^m.$$

Then, since S has no meromorphic functions except constants,

$$1\geqq\dim H^0(S,\mathcal{O}(D_r))+\dim H^0(S,\mathcal{O}(-D_r)) \quad \text{for } r\neq 0.$$

Since the canonical bundle of S is trivial, by the Riemann-Roch inequality:

$$\dim H^0(S,\mathcal{O}(D_r))+\dim H^0(S,\mathcal{O}(-D_r))\geqq\tfrac{1}{2}(D_r)^2+2.$$

Taking these inequality together, we have $(D_r)^2\leqq -2$ for $r\neq 0$, i.e., the intersection matrix of C is negative definite. This contradicts the hypothesis of Theorem B.

2. Quasi-Projectivity of Non-Compact Elliptic Surfaces

This section is devoted to the proof of the following

(2.1) **Proposition.** *Let M be a surface obtained by deleting a finite number of fibres from a compact elliptic surface. Then M is quasi-projective.*

a) We begin by recalling some facts from Kodaira's theory on elliptic surfaces. We shall keep the almost same notations and terminology as used in [2, II–III].

Let Δ be a (compact or non-compact) Riemann surface. By *an elliptic surface* over Δ we shall mean a surface V together with a proper holomorphic map Φ of V onto Δ such that the inverse image $\Phi^{-1}(u)$ of any general point

$u \in \Delta$ is a non-singular elliptic curve. Φ is called the *canonical projection* of V onto Δ. For any open subset E of Δ, we write $V|E$ for the elliptic surface $\Phi^{-1}(E)$ over E with the canonical projection $\Phi|\Phi^{-1}(E)$. V is called an *extension* of $V|E$. By the *fibre* C_u over $u \in \Delta$, we mean the divisor of the holomorphic function $\tau_u \circ \Phi$, where τ_u is a local holomorphic coordinate of Δ around u such that $\tau_u(u)=0$. Letting $\Phi^{-1}(u)=\bigcup_j \Theta_j$ denote the decomposition of $\Phi^{-1}(u)$ into the irreducible components $\Theta_j (\Theta_i \neq \Theta_j$ if $i \neq j)$, we write C_u as

$$C_u = \sum_j n_j \Theta_j,$$

where n_j are positive integers. The greatest common divisor m of n_j's is called the *multiplicity* of C_u; the fibre C_u is called *multiple* if $m \geqq 2$. When $\Phi^{-1}(u)$ contains no exceptional curves of the first kind on V, the fibre C_u is classified into *types* I_b, I_b^*, II, II*, III, III*, IV, and IV* (see [2, II; p. 546, Th. 6.2]). In view of this classification, we see that, if $C_u = \sum_j n_j \Theta_j$ is *not* multiple, then $n_j = 1$ for some j. Thus:

(2.2) **Fact.** *Let $\Phi: V \to \Delta$ be an elliptic surface. Suppose the fibre C_u over $u \in \Delta$ is not multiple. Then there is an open neighborhood E of u in Δ such that $V|E$ admits a holomorphic section over E.*

By a *basic elliptic surface* over Δ we shall mean a pair $(\Phi: B \to \Delta, e)$ consisting of an elliptic surface B over Δ with the canonical projection Φ and a holomorphic section e of B over Δ such that no fibre of Φ contains exceptional curves of the first kind on B. We call the section e the *identity section* of this basic elliptic surface. When there is no necessity of specifying Φ or e, we write (B, e) or B simply for $(\Phi: B \to \Delta, e)$. Note that no basic elliptic surface has multiple fibres.

For a basic elliptic surface $(\Phi: B \to \Delta, e)$, let $B^\#$ denote the open subset of B consisting of all points of B where Φ is of maximal rank. Then $B^\#$ is an analytic fibre space over Δ of abelian groups $B^\# \cap \Phi^{-1}(u)$, $u \in \Delta$, with the identity element $e(u)$ (see [2, II; p. 603, Th. 9.1]). The collection of the identity component of $B^\# \cap \Phi^{-1}(u)$, $u \in \Delta$, forms an analytic fibre subspace $B_0^\#$ of $B^\#$. Let $\mathcal{O}(B_0^\#)$ denote the sheaf of germs of holomorphic sections of $B_0^\#$ over Δ.

Given $\eta \in H^1(\Delta, \mathcal{O}(B_0^\#))$, take a sufficiently fine open covering $\{E_j\}$ of Δ so that none of $B|E_j \cap E_k$ has singular fibres and η is represented by a 1-cocycle $\{\eta_{jk}\}$, $\eta_{jk} \in H^0(E_j \cap E_k, \mathcal{O}(B_0^\#))$. Then the elliptic surface B^η over Δ is, by definition [2, II; p. 613, Def. 9.2], obtained from the collection $\{B|E_j\}$ by identifying $z_j \in B|E_j$ with $z_k \in B|E_k$ if and only if

$$z_j = \eta_{jk}(u) + z_k, \qquad u = \Phi(z_j) = \Phi(z_k)$$

where the symbol $+$ denotes the group operation on the abelian group C_u with the identity element $e(u)$.

(2.3) **Fact.** *Let V be an elliptic surface over Δ. Suppose that V has no multiple fibres and no fibre of V contains exceptional curves of the first kind on V. Then:*

i) *$V = B^\eta$ for some basic elliptic surface B over Δ and $\eta \in H^1(\Delta, \mathcal{O}(B_0^\#))$.*

ii) *When Δ is non-compact, V is a basic elliptic surface over Δ.*

Proof. i) See the proof of [2, II; p. 623, Th. 10.1].

ii) Let B be an arbitrary basic elliptic surface over Δ. By i), it suffices to show that $H^1(\Delta, \mathcal{O}(B_0^\#))=0$. Corresponding to B, we have the sheaf G and the line bundle L over Δ, which fit into the exact sequence

$$\ldots \to H^1(\Delta, \mathcal{O}(L)) \to H^1(\Delta, \mathcal{O}(B_0^\#)) \to H^2(\Delta, G) \to \ldots$$

(cf. [2, III, p. 2, Th. 11.2]). Since Δ is a non-compact Riemann surface and hence is Stein, $H^1(\Delta, \mathcal{O}(L))=0$. By the definition [2, II; p. 579] of G, there are countably many points $a_\rho \in \Delta$ such that $G|\Delta - \{a_\rho\}$ is a locally constant sheaf. It follows $H^2(\Delta, G)=0$, since Δ is homotopically equivalent to a subset of Δ which is a one-dimensional CW-complex containing all a_ρ's. Thus, by the above exact sequence, $H^1(\Delta, \mathcal{O}(B_0^\#))=0$. Q.E.D.

(2.4) **Fact** [2, III; p. 6, Th. 11.5]. *Let B be a basic elliptic surface over a compact curve Δ. Then B^η is algebraic if and only if $\eta \in H^1(\Delta, \mathcal{O}(B_0^\#))$ is of finite order.*

Let $\Phi: B \to E$ be an elliptic surface and let X be a non-compact closed curve on B. When X intersects a general fibre C_τ of B at n points $z_\tau^1, \ldots, z_\tau^n \in C_\tau$ (counting with multiplicity), for any point $o_\tau \in C_\tau$ we set

$$s(X, o_\tau) = z_\tau^1 + \ldots + z_\tau^n,$$

where the symbol $+$ denotes the group operation on the abelian group C_τ with the identity element o_τ.

(2.5) **Fact.** *Let $(\Phi: B \to E, e)$ be a basic elliptic surface. Then, for each non-compact closed curve X on $B^\#$, the map*

$$s(X, e): \tau \mapsto s(X, e(\tau)) \in C_\tau \subset B$$

extends to a holomorphic section of $B^\#$ over E.

b) Here we state a lemma, whose proof will be given in the next subsection c).

Let $(\Phi: B \to E, e)$ be a basic elliptic surface over the unit disc $E = \{\tau \in \mathbb{C} \mid |\tau| < 1\}$. For any fibre C_τ of B, define an integer $\nu(C_\tau)$ according to the type of C_τ as follows:

(2.6)

type	I_0	I_0^*	I_b, I_b^* $(b>0)$	II, II*	III, III*	IV, IV*
$\nu(C_\tau)$	1	2	2	6	4	3

Let $D = \{\sigma \in \mathbb{C} \mid |\sigma| < 1\}$ denote the unit disc with the coordinate σ. Set $D' = D - \{0\}$ and $E' = E - \{0\}$. For each positive integer m, define a holomorphic map $p_m: D \to E$ by

$$p_m: \sigma \in D \mapsto \tau = \sigma^m \in E.$$

Then we have the elliptic surface $\Psi: F'_m \to D'$ induced from $B|E'$ by p_m, and the m-fold unramified covering

$$\pi: F'_m \to B|E'$$

so that $\Phi \circ \pi = p_m \circ \Psi$.

(2.7) **Lemma.** *Let $(\Phi: B \to E, e)$ be a basic elliptic surface over the unit disc E. Suppose $B|E'$ has no singular fibres and let $\nu = \nu(C_0)$. Then, for each holomorphic section φ of $B|E'$ over E' and each positive integer m, there exist a basic elliptic surface $(\Psi: F \to D, \psi)$ over the unit disc D and a biholomorphic fibre map λ of $F|D'$ onto $F'_{\nu m}$ with the following properties:*

i) *Each covering transformation g of $F'_{\nu m}$ with respect to $B|E'$ induces a biholomorphic automorphism $\tilde{g}$ of F given by $\tilde{g} = \lambda^{-1} \circ g \circ \lambda$ on $F|D'$.*

ii) *For each $\sigma \in D'$,*

$$\underbrace{\pi\lambda\psi(\sigma) + \ldots + \pi\lambda}_{m}\psi(\sigma) = \varphi(p_{\nu m}(\sigma)) \tag{2.8}$$

where the symbol $+$ denotes the group operation on the abelian group $\Phi^{-1}(p_{\nu m}(\sigma))$ with the identity element $e(p_{\nu m}(\sigma))$.

c) To prove Lemma (2.7), we recall the construction of B and F'_m in [2, II; § 8]. We assume that $B|E'$ has no singular fibres.

The universal covering space of E' is the upper half $\mathbb{H}$ of $\mathbb{C}$ with the projection

$$l \in \mathbb{H} \mapsto \tau = \exp 2\pi i l \in E'.$$

By [2, II; p. 573, Th. 7.1], we have a holomorphic function ω on $\mathbb{H}$ and an integral unimodular matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with the following properties:

i) $\operatorname{Im} \omega(l) > 0$.

ii) $(\omega(l), 1)$ are the periods of the elliptic curve C_τ, $\tau = \exp 2\pi i l$.

iii) A represents a generator of the monodromy group acting on $H_1(C_\tau, \mathbb{Z})$ $(\tau \neq 0)$ so that

$$\omega(l+1) = (\alpha\omega(l) + \beta)/(\gamma\omega(l) + \delta) \quad \text{for } l \in \mathbb{H}.$$

Setting

$$A^k = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \quad \text{for } k \in \mathbb{Z},$$

define a holomorphic functions $f_k(l)$, $k \in \mathbb{Z}$, on $\mathbb{H}$ by

$$f_k(l) = (\gamma_k\, \omega(l) + \gamma_k)^{-1}.$$

Define holomorphic automorphisms $g(k, n_1, n_2)$, $k, n_1, n_2 \in \mathbb{Z}$, of $\mathbb{H} \times \mathbb{C}$ by

$$g(k, n_1, n_2): (l, \zeta) \mapsto (l + k, f_k(l)(\zeta + n_1\, \omega(l) + n_2)).$$

For each positive integer m, set

$$G_m = \{g(mk, n_1, n_2) \mid k, n_1, n_2 \in \mathbb{Z}\}.$$

Then G_1 is a properly discontinuous group and G_m is a normal subgroup of G_1. We have

$$F'_m = \mathbb{H} \times \mathbb{C}/G_m, \qquad B|E' = \mathbb{H} \times \mathbb{C}/G_1.$$

For any $(l, \zeta) \in \mathbb{H} \times \mathbb{C}$, we denote the corresponding point on $\mathbb{H} \times \mathbb{C}/G_m$ by $((l, \zeta))_m$. Then Φ and e are written as

$$\Phi: ((l, \zeta))_1 \in B|E' \mapsto \tau = \exp 2\pi i l \in E',$$
$$e: \tau \in E' \mapsto ((l, 0))_1 \in B|E', \qquad \tau = \exp 2\pi i l.$$

Note that $B^{\#}|E' = B|E'$. The group operation on $\Phi^{-1}(\tau)$, $\tau = \exp 2\pi i l$, is defined by the formula:

$$((l, \zeta_1))_1 + ((l, \zeta_2))_1 = ((l, \zeta_1 + \zeta_2))_1.$$

Let C_0 denote the fibre of B over the origin $0 \in E$ and let $\nu = \nu(C_0)$. We shall describe B as well as an extension $F_{\nu m}$ over D of $F'_{\nu m}$.

Case A: *C_0 is neither of type I_b nor of type I_b^*, $b > 0$.* In this case, the order of the matrix A is ν (cf. [2, II; p. 604, Table I]) and we have

$$\begin{pmatrix} \alpha_{\nu k} & \beta_{\nu k} \\ \gamma_{\nu k} & \delta_{\nu k} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } k \in \mathbb{Z}. \tag{2.9}$$

Fix a positive integer m and let

$$\sigma = \exp(2\pi i l/\nu m), \qquad \tau = p_{\nu m}(\sigma) = \sigma^{\nu m}.$$

Then $\omega(l)$ and hence $f_k(l)$ are single-valued holomorphic functions of σ, $|\sigma| < 1$. Hence we may write $\omega(\sigma)$ and $f_k(\sigma)$ for $\omega(l)$ and $f_k(l)$ respectively. We note that $\operatorname{Im} \omega(\sigma) > 0$ for $\sigma \in D$ since $\operatorname{Im} \omega(l) > 0$. In particular,

$$f_k(\sigma) \neq 0 \quad \text{for } \sigma \in D,\ k \in \mathbb{Z}.$$

Let $H_{\nu m}$ denote the properly discontinuous group of holomorphic automorphisms

$$h(n_1, n_2): (\sigma, \zeta) \mapsto (\sigma, \zeta + n_1 \omega(\sigma) + n_2), \qquad n_1, n_2 \in \mathbb{Z},$$

of $D \times \mathbb{C}$. We form the quotient space

$$F_{\nu m} = D \times \mathbb{C}/H_{\nu m}.$$

Then $F_{\nu m}$ is a basic elliptic surface over D with the identity section defined by the equation: $\zeta = 0$. For any point $(\sigma, \zeta) \in D \times \mathbb{C}$, we denote the corresponding point of $F_{\nu m}$ by $[\sigma, \zeta]$.

Since $f_{\nu k}(l) = 1$, $k \in \mathbb{Z}$, by (2.9), we have

$$g(\nu m k, n_1, n_2): (l, \zeta) \mapsto (l + \nu m k, \zeta + n_1 \omega(l) + n_2) \quad \text{for } k \in \mathbb{Z}. \tag{2.10}$$

Therefore

$$F'_{\nu m} = \mathbb{H} \times \mathbb{C}/G_{\nu m} = F_{\nu m}|D'.$$

The quotient group $Z_{vm}=G_1/G_{vm}$ is identified with the cyclic group of order vm generated by the holomorphic automorphism

$$g\colon [\sigma,\zeta]\mapsto[e_{vm}\sigma, f_1(\sigma)\zeta] \tag{2.11}$$

of F_{vm}, where $e_{vm}=\exp(2\pi i/vm)$. Now consider the case: $m=1$. Then B is the minimal desingularization of the normal complex analytic space F_v/Z_v (see [2, II; pp. 582–596]).

Case B: C_0 *is either of type* I_b *or of type* I_b^* ($b>0$). In this case, $v=2$. By [2, II; p. 578, (7.7)], $\omega(l)$ is written as

$$\omega(l)=bl+f(\tau) \quad \text{for } l\in\mathbb{H},\ \tau=\exp 2\pi i l,$$

where $f(\tau)$ is a holomorphic function on E and b is a positive integer. We have

$$A=\begin{cases}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} & \text{if } C_0 \text{ is of type } \mathrm{I}_b,\\ \begin{pmatrix}-1 & -b\\ 0 & -1\end{pmatrix} & \text{if } C_0 \text{ is of type } \mathrm{I}_b^*.\end{cases} \tag{2.12}$$

Let $\mathbb{P}^1$ denote the complex projective line with the inhomogeneous coordinate w. Set

$$\begin{aligned} &W_0=D\times\mathbb{P}^1, && \Theta_0=\{0\}\times\mathbb{P}^1,\\ &\Gamma_0=D\times\{0\}, && \Gamma_\infty=D\times\{\infty\},\\ &p_0=(0,0)\in W_0, && p_{-1}=(0,\infty)\in W_0.\end{aligned}$$

Moreover, inductively we define non-compact surfaces W_k, curves $\Theta_{\pm k}$ on W_k and points p_k, p_{-k-1} on W_k, $k=1,2,3,\ldots$ as follows:

i) W_k is the blowing-up of W_{k-1} at p_{k-1} and p_{-k},

ii) Θ_k and Θ_{-k} are the non-singular rational curves on W_k corresponding to p_{k-1} and p_{-k} respectively,

iii) $\{p_k\}=\Gamma_0\cap\Theta_k$, $\{p_{-k-1}\}=\Gamma_\infty\cap\Theta_{-k}$, where we denote the proper transforms of Γ_0 and Γ_∞ by the same symbols respectively. Identifying $W_k-\Gamma_0\cup\Gamma_\infty$ with the open submanifold of $W_{k+1}-\Gamma_0\cup\Gamma_\infty$ canonically, we define a non-compact surface $\tilde{F}$ to be their inductive limit:

$$\tilde{F}=\operatorname{ind}\lim_k(W_k-\Gamma_0\cup\Gamma_\infty).$$

For each positive integer n define a bimeromorphic automorphism h_n of W_0 by

$$h_n\colon(\sigma,w)\mapsto(\sigma, w\exp 2\pi i\omega(l)),\qquad \sigma=\exp 2\pi i l/n.$$

Then h_n induces a biholomorphic automorphism $\tilde{h}_n$ of $\tilde{F}$, which generates a properly discontinuous group H_n free from fixed points (cf. [2, II; pp. 596–600]). Let F_n denote the quotient space $\tilde{F}/H_n$. Then F_n is a basic elliptic surface over D whose central fibre is of type I_{nb}. We note that the canonical projection is induced by $(\sigma,w)\mapsto\sigma$, and the identity section is given by the equation: $w=1$.

Fix a positive integer m and let $N_{\nu m}$ denote the normal subgroup of $G_{\nu m}$ consisting of $g(\nu m k, 0, n_2)$, $k, n_2 \in \mathbb{Z}$. Then we have

$$\mathbb{H} \times \mathbb{C}/N_{\nu m} = D' \times \mathbb{C}^* = \tilde{F} - \bigcup_{j \in \mathbb{Z}} \Theta_j,$$

where the projection of $\mathbb{H} \times \mathbb{C}$ onto $D' \times \mathbb{C}^*$ is given by

$$(l, \zeta) \mapsto (\sigma, w), \qquad \sigma = \exp 2\pi i\, l/\nu m, \qquad w = \exp 2\pi i\, \zeta.$$

The quotient group $G_{\nu m}/N_{\nu m}$ is identified with the infinite cyclic group generated by the holomorphic automorphism $\tilde{h}_{\nu m}$ of $D' \times \mathbb{C}^*$. Thus

$$F'_{\nu m} = F_{\nu m}|D'.$$

For any point $(\sigma, w) \in D' \times \mathbb{C}^*$, we denote the corresponding point of $F_{\nu m}|D'$ by $[\sigma, w]^*$.

B_1) Suppose that the central fibre C_0 of B is of type I_b. Then $B = F_1$ as a basic elliptic surface. The covering transformation group of $F'_{\nu m}$ with respect to $B|E'$ is generated by the automorphism

$$g: [\sigma, w]^* \to [e_{\nu m}\, \sigma, w]^*, \qquad e_{\nu m} = \exp(2\pi i/\nu m). \tag{2.13}$$

We note that g extends to a holomorphic automorphism of $F_{\nu m}$.

B_2) Suppose the central fibre C_0 of B is of type I_b^*. Consider the bimeromorphic automorphism r of W_0 defined by

$$r: (\sigma, w) \mapsto (-\sigma, w^{-1}).$$

Then r induces a biholomorphic automorphism of F_2, which generates the cyclic group Z_2 of order 2. B is the minimal desingularization of the normal complex analytic space F_2/Z_2 (see [2, II; pp. 600–602]). The covering transformation group of $F'_{\nu m}$ with respect to $B|E'$ is generated by the automorphism

$$g: [\sigma, w]^* \mapsto [e_{\nu m}\, \sigma, w^{-1}]^*, \qquad e_{\nu m} = \exp(2\pi i/\nu m). \tag{2.14}$$

We note that g extends to a holomorphic automorphism of $F_{\nu m}$.

Proof of Lemma (2.7). Let φ be a holomorphic section of $B|E'$ over E'. Considering a lift of φ to $\mathbb{H} \times \mathbb{C}$, we have a holomorphic function $\xi(l)$ on $\mathbb{H}$ such that

$$\varphi(\tau) = ((l, \xi(l)))_1 \in B|E', \qquad \tau = \exp 2\pi i\, l.$$

Then, for each integer k, there are integers n_1, n_2 such that

$$(l + k, \xi(l + k)) = g(k, n_1, n_2)\,(l, \xi(l)). \tag{2.15}$$

Now fix a positive integer m and let $\nu = \nu(C_0)$. Then $f_\nu(l) = 1$ by (2.9) or (2.12). Hence by (2.15),

$$\xi(l + \nu) = \xi(l) + n_1\, \omega(l) + n_2$$

for some integers n_1, n_2. It follows

$$\xi(l+m\nu)=\begin{cases}\xi(l)+m n_1\,\omega(l)+m n_2 & \text{(Case A)}\\ \xi(l)+m n_1\,\omega(l)+m n_2+m(m-1)\,b\,n_1 & \text{(Case B)},\end{cases}$$

where b is a positive integer. Thus we have a holomorphic section ψ' of $F'_{\nu m}$ over D' defined by

$$\psi'\colon \sigma\in D'\mapsto((l,\xi(l)/m))_{\nu m}\in F'_{\nu m},\qquad \sigma=\exp(2\pi i\,l/\nu m).$$

Clearly ψ' satisfies

$$\underbrace{\pi\psi'(\sigma)+\ldots+\pi\psi'(\sigma)}_{m}=\varphi(p_{\nu m}(\sigma))\qquad \text{for } \sigma\in D'. \tag{2.16}$$

Set $F=F_{\nu m}$ and let ψ denote the identity section of F over D. As in the proof of [2, II; p. 618, Lemma 10.3], define a biholomorphic fibre map λ of $F|D'$ onto $F'_{\nu m}=F_{\nu m}|D'$ by

$$\lambda\colon z\in F|D'\mapsto z+\psi'(\sigma)\in F'_{\nu m},\qquad \sigma=\Psi(z).$$

Then $\lambda\circ\psi=\psi'$ on D'. Hence (2.7) follows from (2.16).

Let g be the generator of the covering transformation group of $F'_{\nu m}$ with respect to $B|E'$ given by (2.11), (2.13) or (2.14). We write $\lambda^{-1}\circ g\circ\lambda$ explicitly. First λ is written as

$$\lambda\colon[\sigma,\zeta]\mapsto\left[\sigma,\zeta+\frac{1}{m}\xi(l)\right]\qquad\text{(Case A)}$$

$$\lambda\colon[\sigma,w]^*\mapsto[\sigma,w\exp(2\pi i\,\xi(l)/m)]^*\qquad\text{(Case B)},$$

where $\sigma=\exp(2\pi i\,l/\nu m)$. Note that, by (2.15),

$$\xi(l+1)=f_1(l)(\xi(l)-m_1\,\omega(l)-m_2)$$

for some integers m_1, m_2, and that, by (2.12),

$$f_1(l)=\begin{cases}1 & \text{in Case } B_1\colon\ C_0 \text{ is of type } I_b,\\ -1 & \text{in Case } B_2\colon\ C_0 \text{ is of type } I_b^*.\end{cases}$$

Using these, we obtain that $\lambda^{-1}\circ g\circ\lambda$ is given by

$$[\sigma,\zeta]\mapsto\left[e_{\nu m}\sigma, f_1(\sigma)\left(\zeta+\frac{m_1}{m}\omega(\sigma)+\frac{m_2}{m}\right)\right]\qquad\text{(Case A)}$$

$$[\sigma,w]^*\mapsto[e_{\nu m}\sigma, w\,\sigma^{m_1\nu b}\exp(2\pi i\,m_2/m)]^*\qquad\text{(Case } B_1)$$

$$[\sigma,w]^*\mapsto[e_{\nu m}\sigma, w^{-1}\sigma^{m_1\nu b}\exp(2\pi i\,m_2/m)]^*\qquad\text{(Case } B_2)$$

where $e_{\nu m}=\exp(2\pi i/\nu m)$. Thus $\lambda^{-1}\circ g\circ\lambda$ extends to a holomorphic automorphism of F. Q.E.D.

d) Now we shall prove Proposition (2.1). Let M be a sin Proposition (2.1). By hypothesis, there are a compact elliptic surface S with the canonical projection $\Phi: S \to \Delta$ onto a curve Δ and a finite number of points $a_1, \ldots, a_\lambda, \ldots, a_l \in \Delta$ such that

$$M = S - \bigcup_{\lambda=1}^{l} \Phi^{-1}(a_\lambda).$$

For our purpose, we may assume that

i) no fibre of S contains exceptional curves of the first kind on S,

ii) none of $\Phi^{-1}(a_\lambda)$, $1 \leqq \lambda \leqq l$, is multiple, applying the inverse of logarithmic transformations if necessary (cf. [3, I; pp. 766–771, §4]). For each a_λ, take a sufficiently small disc E_λ in Δ with centre a_λ and set $E'_\lambda = E_\lambda - \{a_\lambda\}$ so that

i) $S|E_\lambda$ admits a holomorphic section e_λ over E_λ (Fact (2.2)),

ii) $S|E'_\lambda$ has no singular fibres.

Thus $(S|E_\lambda, e_\lambda)$, $1 \leqq \lambda \leqq l$, are basic elliptic surfaces.

Set $\Delta' = \Phi(M)$. Let $m_1, m_2, \ldots, m_l$ be natural numbers. Assume that $\Phi: M \to \Delta'$ has multiple fibres C_{a_ρ}, $a_\rho \in \Delta'$, $l+1 \leqq \rho \leqq r$, and other fibres are not multiple. Let m_ρ, $l+1 \leqq \rho \leqq r$, be the multiplicities of C_{a_ρ}, respectively. Let m_0 denote the least common multiple of the integers $m_1, m_2, \ldots, m_l, \ldots, m_r$, and let $d = m_1 m_2 \ldots m_r$. Fixing an arbitrary point $a_0 \in \Delta - \{a_\lambda\}_{\lambda=1}^{r}$ such that $\Phi^{-1}(a_0)$ is non-singular, we construct the d-fold covering $p: \Gamma \to \Delta$ of Δ which is unramified over $\Delta - \{a_0, a_1, \ldots, a_r\}$, and has d/m_λ branch points of order $m_\lambda - 1$ over each point a_λ, $\lambda = 0, 1, \ldots, r$. Setting $\Gamma' = p^{-1}(\Delta')$, let N denote the elliptic surface over Γ' with the canonical projection $\Psi': N \to \Gamma'$ induced from $\Phi: M \to \Delta'$ by p. Let $\pi: N \to M$ denote the natural projection. Then π is a ramified covering and N has no multiple fibres.

First we shall see that M has a non-compact closed curve X with the following property:

(2.17) For each $x \in X$, either Φ is of maximal rank at x or x is a non-singular point of the support of a certain multiple fibre C_{a_ρ}.

Take m_λ, $1 \leqq \lambda \leqq l$, arbitrarily. By Fact (2.3) ii), N is a basic elliptic surface. Let W be the image of the identity section. Then $X = \pi(W)$ is the desired one.

By Fact (2.5), the curve X determines a holomorphic section,

$$\varphi_\lambda(u) = s(X, e_\lambda(u)) \quad \text{for } u \in E'_\lambda, \tag{2.18}$$

of $S|E'_\lambda$ over E'_λ. Let n denote the intersection number of X with a fibre C_u, $u \in \Delta'$, of M. Note that n is independent of u. Now we set (and fix)

$$m_\lambda = n\, \nu(C_{a_\lambda}) \quad \text{for } \lambda = 1, \ldots, l,$$

where $\nu(C_{a_\lambda})$ is the number defined by (2.6). Then, in view of Lemma (2.7), we can construct a compact elliptic surface $\Psi: \Sigma \to \Gamma$ with the following properties:

i) $\Sigma|\Gamma' = N$ and $\Psi|N = \Psi'$.

ii) Every covering transformation of N with respect to M extends to a holomorphic automorphism of Σ.

iii) $\Sigma|p^{-1}(E_\lambda)$ has a holomorphic section ψ_λ over $p^{-1}(E_\lambda)$ such that

$$\pi\psi_\lambda(v)+\ldots+\underbrace{\pi\psi_\lambda(v)}_{n}=\varphi_\lambda(p(v)) \quad \text{for } v\in p^{-1}(E'_\lambda). \tag{2.19}$$

Let G denote the covering transformation group of N with respect to M. By property ii), G acts on Σ as a group of holomorphic automorphisms. Since G is a finite group, the quotient space Σ/G is a normal complex analytic space. Moreover, since $\Sigma|\Gamma'=N$ and hence $(\Sigma|\Gamma')/G=M$, the desingularization S^* of Σ/G is a compactification of M. We shall show that Σ *is algebraic*. Then it follows that S^* is algebraic and hence M is quasi-projective.

Let $\{D_\mu\}$ be a finite open covering of Γ by sufficiently small discs D_μ such that

i) $\Sigma|D_\mu$ admits a holomorphic section o_μ over D_μ for each μ (see Fact (2.2)),

ii) $\Sigma|D_\mu\cap D_\nu$ has no singular fibres,

iii) if $p^{-1}(a_\lambda)\cap D_\mu\neq\emptyset$ for some $\lambda=1,\ldots,l$, then D_μ is one of the connected components of $p^{-1}(E_\lambda)$.

We assume

$$o_\mu=\psi_\lambda \quad \text{if } D_\mu\subset p^{-1}(E_\lambda). \tag{2.20}$$

Let $Y=\pi^{-1}(X)$. Then by (2.17), $Y\cap\Psi^{-1}(D_\mu\cap\Gamma')$ is a non-compact closed curve on $(\Sigma|D_\mu\cap\Gamma')^{\#}$. Hence by Fact (2.5), $s(X,o_\mu)$ is a holomorphic section of $\Sigma|D_\mu\cap\Gamma'$ over $D_\mu\cap\Gamma'$. Note that

$$\pi\, s(Y,o_\mu(v))=s(X,\pi\, o_\mu(v)),$$

since $\pi\colon \Psi^{-1}(v)\to\Phi^{-1}(\pi(v))$ is biholomorphic. Moreover, for $v\in D_\mu\cap p^{-1}(E'_\lambda)$,

$$s(X,e_\lambda\,\pi(v))=s(X,\pi\, o_\mu(v))+\pi\,\underbrace{o_\mu(v)+\ldots+\pi\, o_\mu(v)}_{n-1},$$

where the symbol $+$ denotes the group operation on the abelian group $\Phi^{-1}(\pi(v))$ with the identity element $e_\lambda\,\pi(v)$. Therefore, when $D_\mu\cap p^{-1}(a_\lambda)\neq\emptyset$ for some $\lambda=1,\ldots,l$, we see by (2.18)–(2.20) that

$$s(Y,o_\mu(v))=o_\mu(v) \quad \text{for } v\in D_\mu\cap p^{-1}(E'_\lambda).$$

Thus, for every μ we have a holomorphic section y_μ of $\Sigma|D_\mu$ over D_μ such that

$$y_\mu(v)=s(Y,o_\mu(v)) \quad \text{for each general point } v\in D_\mu\cap\Gamma'. \tag{2.21}$$

By Fact (2.3), there is a basic elliptic surface $(\chi\colon B\to\Gamma,o)$ so that $\Sigma=B^\eta$ for some $\eta\in H^1(\Gamma,\mathcal{O}(B_0^{\#}))$. In our situation, η is represented by a 1-cocycle $\{\eta_{\mu\nu}\}$, $\eta_{\mu\nu}\in H^0(D_\mu\cap D_\nu,\mathcal{O}(B_0^{\#}))$, and there are biholomorphic fibre maps Λ_μ of $\Sigma|D_\mu$ onto $B|D_\mu$ satisfying

$$\Lambda_\mu(z)=\eta_{\mu\nu}(\Psi(v))+\Lambda_\nu(z) \quad \text{for } z\in\Sigma|D_\mu\cap D_\nu, \tag{2.22}$$

$$\Lambda_\mu\, o_\mu(v)=o(v) \quad \text{for } v\in D_\mu. \tag{2.23}$$

Now define fibre-preserving automorphisms $L_{\mu\nu}$ of $B|D_\mu \cap D_\nu$ by

$$L_{\mu\nu}: z \mapsto \eta_{\mu\nu}(\chi(z)) + z \quad \text{for } z \in B|D_\mu \cap D_\nu.$$

Then by (2.22):

$$\Lambda_\mu(Y) = L_{\mu\nu}\Lambda_\nu(Y) \quad \text{on } B|D_\mu \cap D_\nu.$$

By (2.21) and (2.23):

$$\Lambda_\mu y_\mu(v) = s(\Lambda_\mu(Y), o(v)) \quad \text{for } v \in D_\mu \cap \Gamma'.$$

Combining these, we obtain

$$\Lambda_\mu y_\mu(v) = n \cdot \eta_{\mu\nu}(v) + \Lambda_\nu y_\nu(v) \quad \text{for } v \in D_\mu \cap D_\nu,$$

since the intersection number of $\Lambda_\mu(Y)$ with any fibre of $B|D_\mu \cap D_\nu$ is n.

Set $C_v^\# = \chi^{-1}(v) \cap B^\#$ and $C_{0v}^\# = \chi^{-1}(v) \cap B_0^\#$. Then, for every $v \in \Gamma$ the order $m(v)$ of the quotient group $C_v^\# / C_{0v}^\#$ is finite. Also the set $\{b_\rho\}$ of all points $b_\rho \in \Gamma$ such that $m(b_\rho) \geqq 2$, is finite. Let m be the least common multiple of $m(b_\rho)$'s. Then

$$\xi_\mu: v \mapsto m \cdot \Lambda_\mu y_\mu(v)$$

is a holomorphic section of $B_0^\#$ over D_μ satisfying

$$m n \cdot \eta_{\mu\nu}(v) = \xi_\mu(v) - \xi_\nu(v) \quad \text{for } v \in D_\mu \cap D_\nu.$$

This means that $mn\eta$ vanishes. Hence we conclude by Fact (2.4) that Σ is algebraic.

3. Structure of Compactifiable Surfaces; Proof of Theorem A

In this section, we shall prove Theorem A. Let M be a compactifiable surface. Then we have a compact or compactifiable surface M^* without exceptional curves of the first kind or exceptional ends such that M is obtained by applying a finite succesion of quadratic transformations to M^*, and then by deleting (compact) exceptional sets (see 2) of § 0). Note that any meromorphic function on M induces one on M^* naturally. In particular, $a(M) = a(M^*)$.

1) *The case where M^* is compact.* In this case, i) of Theorem A follows from the corresponding theorem [2, I; p. 125, Th. 3.1] on compact surfaces, and ii) if Theorem A is trivial, since $a(M) = a(M^*)$ and M^* is a blowing-down of a compactification of M.

2) *The case where M^* is non-compact.* In this case, by Theorem B and Proposition (2.1), M^* and hence M are quasi-projective. Thus $a(M) = 2$ and we obtain Theorem A.

4. Compactifications of Affine $\mathbb{C}$-Bundles over Elliptic Curves; Proofs of Theorems C-I, C-II and C-III

Throughout this section, we use the following notations:

i) (S, C) is a compactification of a surface M.

ii) $\#(C)$ denotes the number of irreducible components, $\Theta_1, \ldots, \Theta_b$, $b = \#(C)$, of C.

iii) $r(C)$ denotes the rank of the intersection matrix of $[(\Theta_i \cdot \Theta_j)]$ of C.

a) We begin by recalling some known results. Consider the following condition:

(4.1) For each irreducible component Θ_j of C, there is a holomorphic map $f_j: \Delta^* \to S - C$ of a punctured disc $\Delta^* = \{z \in \mathbb{C} | 0 < |z| < 1\}$ to $S - C$ such that $\Theta_j \subset f_j(0; S) \subset C$,

where $f_j(0; S)$ is the *cluster set* of f_j at 0 defined by

$$f_j(0; S) = \bigcap_{\rho > 0} \overline{f_j(\Delta_\rho^*)}, \qquad \Delta_\rho^* = \{z \in \mathbb{C} | 0 < |z| < \rho\},$$

and $\overline{f_j(\Delta_\rho^*)}$ is the closure in S of $f_j(\Delta_\rho^*)$. We note

(4.2) **Fact** [5; p. 466, Prop. 3]. *Let $f: \Delta^* \to S - C$ be a holomorphic map with $f(0; S) \subset C$. Suppose $f(0; S)$ contains at least two points of $\Theta_j - \bigcup_{j \neq k} \Theta_k$. Then $\Theta_j \subset f(0; S)$.*

In [5], Nishino and Suzuki determined the topological type and the intersection matrix of the curve C satisfying (4.1). In view of their results, [5; p. 484, Th. 5] and [5; p. 487, Cor. 2], we have

(4.3) **Fact.** *Let (S, C) be a minimal normal compactification with C connected. Suppose (S, C) satisfies* (4.1). *Then $0 \leqq b_1(C) \leqq 2$. Moreover,*

a) *if $b_1(C) = 0$, then $r(C) \geqq \#(C) - 1$),*

b) *if $b_1(C) = 1$ but S is not ruled, then $(C)^2 = 0$ and $r(C) = \#(C) - 1$,*

c) *if $b_1(C) = 2$, then C is a non-singular elliptic curve.*

(4.4) **Fact** [7; pp. 461–462, § 3]. *Suppose (S, C) is a minimal normal rational compactification of a surface M with the following properties:*

i) *M has no exceptional curves of the first kind.*

ii) *(S, C) satisfies* (4.1).

iii) *C is connected and $b_1(C) = 1$.*

iv) *There exists an exact sequence of homology with real coefficients:*

$$0 \to \mathbb{R}^2 \to H_2(C, \mathbb{R}) \xrightarrow{i_*} H_2(S, \mathbb{R}) \to 0,$$

where i_* is induced by the inclusion map $i: C \hookrightarrow S$.

Then (S, C) satisfies either

a) *S is $\mathbb{P}^2$ and C is the union of three lines in general position, or*

b) *S is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and C is the union of a 0-section, an ∞-section and two distinct fibres.*

In both cases, M is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

In fact, Suzuki proved in [7; § 3] Fact (4.4) under the assumption: $M = \mathbb{C}^* \times \mathbb{C}^*$, which was used in his argument only for showing that (S, C) satisfies i)–iv); the rest of his proof does not use the assumption: $M = \mathbb{C}^* \times \mathbb{C}^*$.

(4.5) **Proposition.** *Let M be either*

i) *an affine $\mathbb{C}$-bundle over an elliptic curve, or*

ii) *a surface obtained by deleting the 0-section from a line bundle over an elliptic curve.*

Then all pluri-genera of any compactification of M vanish.

Proof. Let S be a compactification of M. In any case, i) or ii), the universal covering surface of M is $\mathbb{C}^2$. Let f denote the canonical projection of $\mathbb{C}^2$ onto M. We regard f as a holomorphic map of $\mathbb{C}^2$ to S. Let v be a smooth volume form on S. Set

$$\deg(f|\Delta_r^2)=\int_{\Delta_r^2} f^*v/\int_S v$$

where Δ_r^2 is a polydisc of radius r in $\mathbb{C}^2$. Then we can easily verify that $\deg(f|\Delta_r^2)=O(r^2)$ or $O(r^3)$ according as M is of type i) or of type ii). In particular, we have

$$\liminf_{r\to\infty}\deg(f|\Delta_r^2)/r^4=0.$$

Thus, by [4; p. 41, Th. 3], all pluri-genera of S vanish. Q.E.D.

b) Here we shall prove two lemmas.

(4.6) **Lemma.** *Let (S, C) be a compactification of an affine $\mathbb{C}$-bundle M of degree $-n\leqq 0$ over an elliptic curve. Then:*

a) *C is connected and*

$$r(C)=\begin{cases}\#(C)+b_1(C)-3 & \text{if } n=0,\\ \#(C)+b_1(C)-2 & \text{if } n\neq 0.\end{cases}$$

b) *If C is a non-singular elliptic curve, then $(C)^2=n$.*

Proof. Blow up S, $\sigma: S'\to S$, at points on C properly so that $C'=\sigma^{-1}(C)$ is of simple normal crossing and (S', C') is a compactification of M. Then

$$r(C')=\#(C')=r(C)-\#(C),\qquad b_1(C')=b_1(C). \tag{4.7}$$

Let N be a tubular neighborhood of C' in S', i.e., N is a neighborhood of C' with the following properties:

i) C' is a strong deformation retract of N.

ii) $\overline{N}-C'$ is homeomorphic to $\partial N\times(0,1]$.

Since C' is of simple normal crossing, the tubular neighborhood N of C' always exists and satisfies

(4.8) the homotopy type of ∂N is determined uniquely by M,

$$r(C')-\#(C')=b_1(C')-b_1(\partial N). \tag{4.9}$$

(See [8; pp. 83–84].)

On the other hand, M can be compactified into a $\mathbb{P}^1$-bundle Σ over an elliptic curve admitting the ∞-section $\Gamma=\Sigma-M$ with $(\Gamma)^2=n$. Let T be the tubular neighborhood of Γ in Σ. Then by (4.8), ∂N is homotopically equivalent

to ∂T, which is a circle bundle of degree $\pm n$ over Γ. In particular, ∂N is connected. It follows the connectedness of N, C' and hence C. Moreover, using the Gysin exact sequence, we see

$$H_1(\partial N, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}). \tag{4.10}$$

Combining this with (4.7) and (4.9), we obtain a). Suppose now that C is a non-singular elliptic curve. Then ∂N is a circle bundle of degree $\pm(\Gamma)^2$ over C. Since $(\Gamma)^2 \geqq 0$, we have that Γ and C are not exceptional. In particular, $(C)^2 \geqq 0$. Thus b) follows from (4.10) Q.E.D.

(4.11) **Lemma.** *Let M be an affine $\mathbb{C}$-bundle of degree $\leqq 0$ over an elliptic curve. Then, every compactification (S, C) of M satisfies* (4.1).

Proof. Let $\tilde{M}$ denote $\mathbb{C}^2$ with coordinates (ζ, η). We write M as the quotient surface $\tilde{M}/\langle\sigma, \tau\rangle$ of $\tilde{M} = \mathbb{C}^2$ by the group $\langle\sigma, \tau\rangle$ generated by holomorphic automorphisms

$$\sigma: (\zeta, \eta) \mapsto (\zeta + \omega, a(\zeta)\eta + b(\zeta)),$$
$$\tau: (\zeta, \eta) \mapsto (\zeta + 1, \eta),$$

where ω is a constant with $\operatorname{Im}\omega > 0$, $a(\zeta)$ and $b(\zeta)$ are holomorphic functions on $\mathbb{C}$ satisfying

$$a(\zeta + 1) = a(\zeta), \qquad b(\zeta + 1) = b(\zeta)$$

and $a(\zeta)$ never vanishes on $\mathbb{C}$. The quotient surface $M^* = \tilde{M}/\langle\tau\rangle$ of $M = \mathbb{C}^2$ by the group $\langle\tau\rangle$ generated by τ is biholomorphic to $\mathbb{C}^* \times \mathbb{C}$ with the coordinates (w, ξ) given by

$$w = \exp 2\pi i \zeta, \qquad \xi = \eta.$$

Then σ induces the holomorphic automorphism

$$g: (w, \xi) \mapsto (\alpha w, a^*(w)\xi + b^*(w))$$

of $M^* = \mathbb{C}^* \times \mathbb{C}$, where

$$\alpha = \exp 2\pi i \omega \qquad (0 < |\alpha| < 1)$$
$$a^*(\exp 2\pi i \zeta) = a(\zeta), \qquad b^*(\exp 2\pi i \zeta) = b(\zeta).$$

Moreover, since the degree of the affine $\mathbb{C}$-bundle M is not positive, we may assume that g extends to a holomorphic contraction of $\mathbb{C}^2$ (with the coordinates (w, ξ)) such that $g(0) = 0$. Let Δ_ρ denote the ρ-disc $\Delta_\rho = \{t \in \mathbb{C} \mid |t| < \rho\}$ and set $\Delta_\rho^* = \Delta_\rho - \{0\}$. For simplicity, we write Δ and Δ^*, respectively, for Δ_1 and Δ_1^*.

Fix an irreducible components Θ_j of $C = \bigcup_k \Theta_k$. We shall find a holomorphic map $f_j: \Delta^* \to S - C$ such that $\Theta_j \subset f_j(0; S) \subset C$. For this purpose, take a holomorphic embedding h of $\Delta \times \Delta$ into S such that

$$h(\{0\} \times \Delta) \subset \Theta_j - \bigcup_{j \neq k} \Theta_k, \qquad h(\Delta^* \times \Delta) \subset M.$$

We write $h_t(s)$ for $h(s,t)$. Let γ be a circle: $|s|=$constant with positive orientation on Δ. We regard γ as a generator of the fundamental group $\pi_1(\Delta^* \times \Delta) \cong \mathbb{Z}$ of $\Delta^* \times \Delta$. Identifying the automorphisms σ, τ of $\tilde{M}$ with the generators of the fundamental group $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$ of M canonically, we have

$$(h_t)_* \gamma = m_1 \sigma + m_2 \tau \tag{4.12}$$

for some integers m_1, m_2 (independent of $t \in \Delta$). Then, for each $t \in \Delta$, the first coordinate ζ of the (local) lifts of h_t to $\tilde{M} = \mathbb{C}^2$ define a multi-valued holomorphic function $\zeta_t = \zeta_t(s)$ on Δ^* so that the analytic continuation along γ induces the transformation:

$$\zeta_t(s) \mapsto \zeta_t(s) + m_1 \omega + m_2 \qquad (\operatorname{Im} \omega > 0).$$

Let $\tilde{\omega}$ denote the canonical projection of $\tilde{M} = \mathbb{C}^2$ onto M. Define a holomorphic map $\tilde{v}$ of $\mathbb{C}$ into $\tilde{M}$ by

$$\tilde{v}: t \mapsto (\zeta, \eta) = (0, t) \qquad \text{for } t \in \mathbb{C}.$$

Set $v = \tilde{\omega} \circ \tilde{v}: \mathbb{C} \to M$. Then we have the following two cases:

1) $v(\mathbb{C})$ and $h_t(\Delta^*)$ intersects infinitely many points for each $t \in \Delta$.

2) There exists $t_0 \in \Delta^*$ such that $v(\mathbb{C})$ and $h_{t_0}(\Delta^*)$ intersect at most finitely many points.

Case 1. In this case, we define $f_j: \Delta^* \to S - C$ by $f_j(z) = v(1/z)$ for $z \in \Delta^*$. Then, since $v: \mathbb{C} \to S - C$ is an embedding, $h(\{0\} \times \Delta) \subset f_j(0; S) \subset C$. Thus by Fact (4.2), $\Theta_j \subset f_j(0; S) \subset C$, as desired.

Case 2. In this case, replacing Δ^* by a sufficiently small punctured disc Δ_ρ^* if necessary, we have

$$\zeta_{t_0}(s) \neq n_1 \omega + n_2 \qquad \text{for } n_1, n_2 \in \mathbb{Z},\ 0 < |s| < 1.$$

It follows by [2, II; p. 613, Lemma 10.1] that $m_1 = m_2 = 0$ in (4.12). Therefore, we can find a lift $h^*: \Delta^* \times \Delta \to M^*$ of h to M^*. In the coordinates (w, ξ) of $M^* = \mathbb{C}^* \times \mathbb{C}$, we write h^* as $h^*(s,t) = (w h^*(s,t),\ \xi h^*(s,t))$. First we claim that $w h^*(s,t)$ *does not converge to* 0 *as* $s \to 0$ *for any (fixed)* $t \in \Delta$. Suppose not. Then $w h^*(s,t)$ is holomorphic on $\Delta \times \{t\}$. Hence the integer m_2 in (4.12) is the order of zero of $w h^*(s,t)$ at $s = 0$, which is positive. This is a contradiction.

Noting that $w h^*(s,t)$ is holomorphic on $\Delta^* \times \Delta$, we expand $w h^*(s,t)$ into the Laurent power series

$$w h^*(s,t) = \sum_{\mu \in \mathbb{Z}} c_\mu(t)\, s^\mu$$

in s, where $c_\mu(t)$, $\mu \in \mathbb{Z}$, are holomorphic functions on Δ. Then, the set of all points $t \in \Delta$ such that $\{w h^*(s,t)\}_{s \in \Delta^*}$ does not converge to any point of $\mathbb{C}$ is

$$\bigcup_{\mu < 0} \{t \in \Delta \mid c_\mu(t) \neq 0\}$$

and hence an open subset of Δ. Therefore we can find two distinct points t_1, t_2 of Δ so that we have either

2-a) each $w h^*(s, t_i)$, $i=1$ or 2, converges to a point of $\mathbb{C}^*$ as $s \to 0$,

2-b) neither $w h^*(s, t_1)$ nor $w h^*(s, t_2)$ converges to any point of $\mathbb{C}$ as $s \to 0$.

Now let π denote the canonical projection of M^* onto M. Set $h_i^*(s) = h^*(s, t_i)$, $h_i = \pi \circ h_i^*$ and $p_i = h_i(0)$ for $i=1,2$.

Subcase 2-a). We note that, since $h_i(0; S) \subset C$, $\xi h_i^*(s)$ has a pole at $s=0$ for $i=1,2$. Define a holomorphic map μ of $\mathbb{C}^*$ into $M^* = \mathbb{C}^* \times \mathbb{C}$ by

$$\mu: s \mapsto (w, \xi) = (s, \xi_0) \quad \text{for } s \in \mathbb{C}^*,$$

where ξ_0 is a non-zero constant. Set $f_j = \pi \circ \mu$ and $c = |\xi_0|$.

First we shall show that $f_j(0; S)$ *contains the two points* p_1, p_2 *of* $\Theta_j - \bigcup_{j \neq k} \Theta_k$. Fix $0<r<1$ and $0<\rho<1$ arbitrarily. Then, since g is a contraction of $\mathbb{C}^2$ with $g(0)=0$, there is a positive number λ_0 such that

$$\xi(g^\lambda h_i^*(\partial \Delta_r) \subset \Delta_c \quad \text{for } \lambda \geqq \lambda_0$$

where $\partial \Delta_r$ is the boundary of Δ_r in $\mathbb{C}$. Note that $\xi g^\lambda h_i^*(s)$ $(\lambda > 0)$ has a pole at $s=0$ since so does $\xi h_i^*(s)$. Thus

$$\mathbb{C} - \Delta_c \subset \xi(g^\lambda h_i^*(\Delta_r^*)) \quad \text{for } \lambda \geqq \lambda_0.$$

Moreover, since $g^* w = \alpha w (0<|\alpha|<1)$ and $\lim_{s \to 0} w h_i^*(s) \in \mathbb{C}$, we have

$$w(g^\lambda h_i^*(\Delta_r)) \subset \Delta_\rho^* \quad \text{for } \lambda \geqq \lambda_0,$$

replacing λ_0 by a larger constant if necessary. Taking these together, we have

$$g^\lambda(h_i^*(\Delta_r^*)) \cap \mu(\Delta_\rho^*) \neq \emptyset \quad \text{for } \lambda \geqq \lambda_0.$$

It follows $h_i(\Delta_r) \cap f_j(\Delta_\rho^*) \neq \emptyset$. Therefore, since $0<r<1$ and $0<\rho<1$ are arbitrary, we obtain $p_i \in f_j(0; S)$.

Next we shall show that $f_j(0; S) \subset C$. Take an open neighborhood U of C in S arbitrarily. Then we have a relatively compact subset Ω of $M^* = \mathbb{C}^* \times \mathbb{C}$ such that

$$w(\Omega) \subset \Delta - \Delta_{|\alpha|}, \quad \pi(\Omega) = S - U.$$

Since g is a contraction of $\mathbb{C}^2$ with $g(0)=0$, there is a positive number λ_0 such that

$$\xi(g^\lambda(\Omega)) \subset \Delta_c \quad \text{for } \lambda \geqq \lambda_0.$$

Therefore, since $g^* w = \alpha w$ $(0<|\alpha|<1)$,

$$\mu(\Delta_r^*) \cap \bigcup_{k \in \mathbb{Z}} g^k(\Omega) = \emptyset, \quad r = |\alpha|^{\lambda_0}.$$

Thus we conclude that, for any open neighborhood U of C in S, there is $0<r<1$ such that $f_j(\Delta_r^*) \subset U$. It follows that $f_j(0; C) \subset C$.

Consequently, by Fact (4.2), we have $\Theta_j \subset f_j(0; S) \subset C$.

Subcase 2-b). Define a holomorphic map v of $\mathbb{C}$ into M^* by

$$v: t \mapsto (w, \xi) = (1, t) \quad \text{for } t \in \mathbb{C}.$$

and set $v = \pi \circ v$. Define f_j by $f_j(z) = v(1/z)$ for $z \in \Delta^*$. Then we have $f_j(0; S) \subset C$.

By the hypothesis of Subcase 2-b), we have either

2-b^0) $wh_i^*(s)$ has essential singularity at $s=0$, or

2-b$^-$) $wh_i^*(s)$ has a pole at $s=0$,

for each $i=1$ or 2. Then there is a sequence $\{s_k\}_{k=1}^{\infty}$ of points $s_k \in \Delta^*$ with $\lim_{k\to\infty} s_k = 0$ satisfying

$$h_i(s_k) \in \begin{cases} v(\mathbb{C}) & \text{in case 2-b}^0) \\ g^{-k}(v(\mathbb{C})) & \text{in case 2-b}^-) \end{cases}$$

for each $i=1$ or 2. Hence there are sequences $\{p_{ik}\}_{k=1}^{\infty}$, $i=1,2$ of points $p_{ik} \in v(\mathbb{C})$ such that $\lim_{k\to\infty} p_{ik} = p_i$ in S. Therefore, since $v: \mathbb{C} \to S - C$ is an embedding, $f_j(0; S)$ contains the two points p_1, p_2 of $\Theta_j - \bigcup_{j \neq k} \Theta_k$. Thus, by Fact (4.2), we have $\Theta_j \subset f_j(0; S) \subset C$. Q.E.D.

c) In the following, we shall use Lemma (4.11) without reference.

Proof of Theorem C-I. Let M be an affine $\mathbb{C}$-bundle of degree $-n \leqq 0$ over an elliptic curve and (S, C) a minimal normal compactification of M. Consider the exact homology sequence of the pair (S, C) with real coefficients:

$$\begin{aligned} (4.13) \qquad & 0 \to H_3(S) \to H_3(S, C) \to H_2(C) \xrightarrow{i_*} H_2(S) \\ & \to H_2(S, C) \to H_1(C) \to H_1(S) \to 0. \end{aligned}$$

where i_* is induced by the inclusion map $i: C \hookrightarrow S$. Since $M = S - C$ is homotopically equivalent to a real 2-torus, we have $\dim H_3(S, C) = \dim H^1(S - C) = 2$ and $\dim H_2(S, C) = \dim H^2(S - C) = 1$ by duality. $b_2(C)$ is equal to the number $\#(C)$ of irreducible components of C. Combining these, we have the following five case:

α) $b_1(C)=0$, $b_1(S)=0$, $b_2(S)=\#(C)-1$.
β) $b_1(C)=1$, $b_1(S)=0$, $b_2(S)=\#(C)-2$.
γ) $b_1(C)=1$, $b_1(S)=1$, $b_2(S)=\#(C)$.
δ) $b_1(C)=2$, $b_1(S)=1$, $b_2(S)=\#(C)-1$.
ε) $b_1(C)=2$, $b_1(S)=2$, $b_2(S)=\#(C)+1$.

Case α). This case does not occur. In fact, by Lemma (4.6) a), we have $r(C) \leqq \#(C)-2$, contradicting Fact (4.3) a).

Case β). Let I denote the intersection form on $H_2(S)$. Then $\operatorname{rank}(I|\operatorname{Im} i_*) = r(C)$ and hence $b_2(S) \geqq r(C)$. Thus, using Lemma (4.6) a), we see that $n=0$ and $r(C) = \#(C)-2$. Since $b_1(C)=1$ but $r(C) \neq \#(C)-1$, S is ruled by Fact (4.3) b). Moreover, since $b_1(S)=0$, S is rational. From (4.13) we have the exact sequence:

$$0 \to \mathbb{R}^2 \to H_2(C, \mathbb{R}) \xrightarrow{i_*} H_2(S, \mathbb{R}) \to 0.$$

Thus (S, C) satisfies the conditions i)–iv) in Fact (4.4). Hence we obtain the case a) or the case b) of Theorem C-I.

Case γ). By blowing down S, we obtain from (S, C) a compactification (S^*, C^*) of M such that C^* contains no exceptional curves of the first kind on S^*. Then, $b_2(S^*)$ is equal to the number of irreducible components of C^*, which is positive (since M has no exceptional ends). Since $b_1(S)=1$, S^* is not ruled. Hence, by Fact (4.3) b), $(C)^2=0$ and therefore $(C^*)^2=0$. Thus we conclude by Lemma (1.2) that S^* is a VII_0-surface with $b_2(S^*)>0$ and $(C^*)^2=0$. Moreover, by Fact (1.1) a), we have that $S=S^*$ and $C=C^*$. Then, in view of Fact (1.1) b), we can compactify $M=S-C$ into a $\mathbb{P}^1$-bundle Σ over an elliptic curve admitting the ∞-section $\Gamma=\Sigma-M$ with $(\Gamma)^2=b_2(S)$. Consequently, by Lemma (4.6) b), we obtain $b_2(S)=n$. This is the case e) of Theorem C-I.

Case δ). By Fact (4.3) c), we have $\#(C)=1$ and hence $b_2(S)=r(C)=0$. Thus S is VII_0. Moreover, by Proposition (4.5), all pluri-genera of S vanish. Therefore S is a Hopf surface and C is a non-singular elliptic curve (see [3, II; p. 699, Th. 34 and p. 707, Th. 35]). By Lemma (4.6) a), we have $n=0$. This is the case c) of Theorem C-I.

Case ε). By Proposition (4.5), all pluri-genera of S vanish. Therefore, since $b_1(S)=2$, S is algebraic ([3, I; p. 758, Th. 10]). Moreover, by Enriques' theorem, S is ruled over an elliptic curve E (see [3, IV; p. 1068, Th. 52]). Let $\alpha: S\to E$ denote the projection. By Fact (4.3) c), C is a non-singular elliptic curve. In particular, $\#(C)=1$ and hence $b_2(S)=2$. Therefore S contains no exceptional curves of the first kind. Thus $\alpha: S\to E$ is a $\mathbb{P}^1$-bundle.

Now, since $\pi_1(S-C)\cong\mathbb{Z}^2$, we can adapt the argument in [6; 4.7] as follows. Since C and E are elliptic curves, $\alpha: C\to E$ is an unramified covering, say d-fold. Therefore $\alpha: S-C\to E$ is a smooth fibre bundle. Let F denote the typical fibre of $\alpha: S-C\to E$. Then F is obtained by deleting d points from $\mathbb{P}^1$. Since $\pi_2(E)=0$ and $\pi_1(S-C)=\pi_1(E)\cong\mathbb{Z}^2$, it follows from the homotopy sequence of the fibration $\alpha: S-C\to E$ that $\pi_1(F)=0$. This implies $d=1$. Thus C defines a section of the $\mathbb{P}^1$-bundle $\alpha: S\to E$. By Lemma (4.6)b), we have $(C)^2=n$. This is the case d) or the case f) of Theorem C-I. Q.E.D.

Proof of Theorem C-II. Let M be a surface obtained by deleting the 0-section from a line bundle of degree 0 over an elliptic curve E. Then M has a compactification (Σ, Γ) such that Σ is a $\mathbb{P}^1$-bundle over E and Γ is the union of the 0-section Γ_1 and the ∞-section Γ_2 of Σ over E. Now let (S, C) be a minimal normal compactification of M. Then C consists of two connected components C_1, C_2 corresponding to Γ_1, Γ_2, respectively. Moreover, for each $i=1$ or 2, there is a compactification S_i of $\Sigma-\Gamma_i$ such that $S_i-\Gamma_j$, $i\neq j$, is biholomorphic to S_j-C_j and $S_i-(\Sigma-\Gamma_i)$ is identified with C_i. By Theorem C-I, we know that C_i is a non-singular elliptic curve with $(C_i)^2=0$ for $i=1, 2$, since $\Sigma-\Gamma_i$ is a line bundle of degree 0 over the elliptic curve E.

Consider the exact homology sequence of the pair (S, C) with real coefficients:

$$\ldots\to H_i(C)\to H_i(S)\to H_i(S, C)\to \ldots$$

Since $M=S-C$ is homotopically equivalent to a real 3-torus, we have

$\dim H_3(S, C)=\dim H^1(S-C)=3$, $\dim H_2(S, C)=\dim H^2(S-C)=3$ aim $H_1(S, C)$ $=\dim H^3(S-C)=1$ by duality. Combining these, we have only two cases:

$\zeta)$ $b_1(S)=1$, $b_2(S)=0$.
$\eta)$ $b_1(S)=b_2(S)=2$.

Case ζ). In this case, the arguments in Case δ) of the proof of Theorem C-I are valid verbatim and we obtain that S is a Hopf surface. This is the case g) of Theorem C-II.

Case η). By the first part of the arguments in Case ε), we know that S is a $\mathbb{P}^1$-bundle over an elliptic curve E'. Moreover, the remaining part of Case ε) is valid almost verbatim as follows. Let $\alpha: S\to E'$ denote the projection of the $\mathbb{P}^1$-bundle. Since E' and the connected components C_1, C_2 of C are elliptic, $\alpha: C_i\to E'$ is a d_i-fold unramified covering. Therefore $\alpha: S-C\to E'$ is a smooth fibre bundle whose typical fibre F is obtained by deleting (d_1+d_2) points from $\mathbb{P}^1$. Since $\pi_2(E')=0$, $\pi_1(S-C)=\mathbb{Z}^3$ and $\pi_1(E')=\mathbb{Z}^2$, it follows from the homotopy exact sequence of the fibration $\alpha: S-C\to E'$ that $\pi_1(F)=\mathbb{Z}$. This implies that $d_1+d_2=2$ and hence $d_1=d_2=1$. Thus each C_i defines a section of the $\mathbb{P}^1$-bundle S over E'. This is the case h) of Theorem C-II. Q.E.D.

Proof of Theorem C-III. Let M be as in Theorem C-III. Then we have a compact elliptic surface $\Phi: \Sigma\to\Delta$ and a finite number of points $a_\rho\in\Delta$, $1\leqq\rho\leqq k$, such that

$$M=\Sigma-\bigcup_\rho \Phi^{-1}(a_\rho).$$

We write Φ' for $\Phi|M$. Let (S, C) be a compactification of M. Take local holomorphic coordinates τ_ρ of Δ around a_ρ. Then holomorphic functions $\tau_\rho\circ\Phi'$ are bounded on a neighborhood of C. Hence, by Riemann's extension theorem, Φ' extends to a holomorphic map Ψ of S onto Δ so that C $=\bigcup_\rho \Psi^{-1}(a_\rho)$. Thus, since $\Phi': M\to\Delta-\{a_\rho\}$ is an elliptic surface, $\Psi: S\to\Delta$ is also an elliptic surface containing C as the union of the fibres $\Psi^{-1}(a_\rho)$. Q.E.D.

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