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Autor: Todorov, Andrei N.

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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A Construction of Surfaces with $p_g = 1$, $q = 0$ and $2 \leqq (K^2) \leqq 8$

Counter Examples of the Global Torelli Theorem

Andrei N. Todorov

Columbia University, Department of Mathematics, New York, NY 10027, USA

Introduction

The aim of this article is to give a construction of surfaces with $p_g = 1$, $q = 0$ and $2 \leqq (K^2) \leqq 8$. From the way we constructed surfaces with $p_g = 1$, $q = 0$ and $(K^2) = 2$ it follows that there exists two types of such surfaces; one of the types has a trivial fundamental group and the other type has $\pi_1 = \mathbb{Z}/2\mathbb{Z}$. We prove that the moduli space of the surfaces we constructed with $p_g = 1$, $q = 0$ and $(K^2) = 2$ has dimension 12 and its image in the period domain has dimension 9. So surfaces with $p_g = 1$, $q = 0$ and $(K^2) = 2$ give counter examples of the Global Torelli theorem, i.e. there exist surfaces with $p_g = 1$, $q = 0$ and $(K^2) = 2$ that are birationally distinct but have the same periods. The same is true for surfaces with $p_g = 1$, $q = 0$ and $3 \leqq (K^2) \leqq 8$. These surfaces have a "big" fundamental group. The calculation of the fundamental group of the surfaces we construct will be given in another paper.

Surfaces with $p = 1$ and $(K^2) = 1$ are studied in [C], [Ku] and [T]. In [C] and [T] it is proved that surfaces with $p_g = 1$ and $(K^2) = 1$ such that $|2K|$: $X \to \mathbb{P}^2$ is a Galois covering of $\mathbb{P}^2$, the Local Torelli theorem is not true. Such surfaces we will call canonical Galois coverings of $\mathbb{P}^2$. In [T] it is proved that the moduli space of all canonical Galois coverings of $\mathbb{P}^2$ with $p_g = 1$ and $(K^2) = 1$ has Dimension 12. Theorem 3 shows that the image of the moduli space of all canonical Galois coverings of $\mathbb{P}^2$ with $p_g = 1$ and $(K^2) = 1$ in the period domain has Dimension 10. This is an answer to a question of F. Catanese.

The surfaces with $p_g = 1$, $q = 0$ and $(K^2) = 8$ have a moduli space of $\dim \geqq 12$, while the period domain has $\dim = 11$, so Global Torelli fails generically for them.

I hope that the surfaces constructed here are the only surfaces of general type possessing pathological properties concerning the period map.

1. Geometry of the Double Points on a Kummer Surface

Let J be the jacobian of a non-singular curve of genus 2. The canonical involution i: $x \to -x$ of the two-dimensional torus J has 16 distinct fixed points and each of them determines an ordinary double point on the orbit space $J/i = X$. It can be proved that X can be embedded in $\mathbb{P}^3$ as a quartic with 16

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ordinary double points. See [M_1]. A quadratic transformation at each of these points desingularizes the orbit space X and the resulting non-singular surface $\hat{X}$ has 16 distinct projective lines E_i, such that $(E_i, E_i) = -2$ and $(E_i, E_j) = 0$ if $i \neq j$. X is called a Kummer surface.

1.1 *Definition.* Let $A_1, \ldots, A_k$ be k points in $\mathbb{P}^3$, where $k > 3$. We will say that these k points are in general position iff every four of them do not lie on a linear hypersurface in $\mathbb{P}^3$.

1.2 **Lemma.** *We can find* 6 *points among the double points* $\{P_i\}$ *of the Kummer surface* X *in* $\mathbb{P}^3$ *in general position.*

Proof. Let P_1 be any of the double points on X. Let us blow up the point P_1. We will obtain a surface $\tilde{X}$ with 15 double points. Consider the map r: $\tilde{X} \to \mathbb{P}^2$, obtained by the projection from P_0 to a linear hyperplane in $\mathbb{P}^3$. In [G and H] it is proved that: a) $\deg r = 2$ and b) the branch locus of r: $F \hookrightarrow \mathbb{P}^2$ consits of 6 distinct lines meeting in 15 distinct points. These double points on F are just the images $r(P_i)$, $i = 2, \ldots, 16$. From the calculation of the branch locus we get:

Fact 1. If a hyperplane section of X contains 4 or more double points, then this hyperplane section contains exactly 6 double points.

Proof. Let P_i be one of the double points that lies on the hyperplane section C. Project X onto $\mathbb{P}^2$ from P_i. Since C is a hyperplane section, it follows that $r(C)$ will be a line in $\mathbb{P}^2$. The line $r(C)$ will contain at least 3 double points. From the fact that the branch locus of $r\,F$ consists of 6 distinct lines meeting in 15 distinct points, we get that $r(C)$ must be one of the components of F. So $r(C)$ will contain exactly 5 of the double points of F. Remember that we are projecting from one of the points P_i on C, so C contains exactly 6 double points. Q.E.D.

Notation. Let me denote by L_i all hyperplane sections of X that contain exactly 6 double points. Their number is equal to 16. See [G and H].

Fact 2. Through each pair of double points $(P_i \neq P_j)$ there pass exactly two distinct such hyperplanes, say L_i and L_j. Every two distinct L_i and L_j intersect each other in two distinct double points of X.

Proof. Fact two is proved in [G and H]. This is exactly figure 21 on page 787 of [G and H]. Q.E.D.

Let L_1 and L_2 be distinct hyperplane sections of X, such that each of them contains exactly 6 double points. From Fact 2 we know that $L_1 \cap L_2$ contains exactly two distinct double points say P_9 and P_{10}, so $L_1 \cup L_2$ contains exactly 10 double points, say $P_1, P_2, \ldots, P_9$ and P_{10}. Let the remaining six points be $P_{11}, P_{12}, \ldots, P_{16}$.

Proposition: $P_{11}, P_{12}, \ldots, P_{16}$ *are in general position.*

Proof. Suppose that $P_{11}, \ldots, P_{16}$ are not in general position, which means that 4 of them lie on a hyperplane section say L_3 (this follows from fact 1). We know that L_3 contains exactly 6 double points. We have the following possibilities:

1) The six double points that lie on L_3 are $P_{11}, P_{12}, \ldots, P_{16}$. From here it follows that $L_3 \cap L_1 = L_3 \cap L_2 = \emptyset$, which contradicts Fact 2.

2) The double points that lie on L_3 are $P_{14}, P_{11}, P_{12}, P_{13}, P_9$, and P_{10}. From here it follows that $L_1 \cap L_2 = L_1 \cap L_3 = L_2 \cap L_3 = (P_9$ and $P_{10})$. This contradicts fact 2.

3) The other two points (except P_{14}, P_{11}, P_{12}, P_{13}) are both on L_1 or both on L_2 and these two points are different drom P_9 and P_{10}. If they are on L_1 then $L_2 \cap L_3 = \emptyset$. This contradicts fact 2.

4) L_3 contains except P_{14}, P_{11}, P_{12} and P_{13} one point on L_1 and the other on L_2 and one of these two points is different from P_9 and P_{10}. It follows that either $L_1 \cap L_3$ or $L_2 \cap L_3$ contains only one double point. This contradicts Fact 2.

5) L_3 contains 5 of the double points P_{16}, P_{11}, P_{12}, P_{13}, P_{14} and P_{15}. Then L_3 must contain one more double point. This follows from fact 1. We see now that $L_1 \cap L_2$ or $L_2 \cap L_3$ or both of them will contain only one double point. This contradicts fact 2. Q.E.D.

2. A Construction of Surfaces with $p_g = 1$, $q = 0$ and $2 \leqq (K^2) \leqq 8$

From Lemma 1.2 we know that there exist 6 double points on X (Kummer surface in $\mathbb{P}^3$) in general position. We can find a quadric Q in $\mathbb{P}^3$ such that:

a) $Q \cap X$ contains i of the singular points of X that are in general position, where $0 \leqq i \leqq 6$. This can be done because the space of all quadrics in $\mathbb{P}^3$ is a projective space of dim $= 9$ and the double points are in general position.

b) $X \cap Q$ contains exactly i double points and the curve $Q \cap X$ has exactly i singular double points. It follows from Bertini theorem that we can find Q with the above properties.

2.1 **Lemma.** *Let me denote by C_i the proper transform of $Q \cap X$ on $\hat{X}$, $0 \leqq i \leqq 6$, then*

a) $(C_i, C_i) = 16 - 2i$

b) $C_i + \sum E_j \equiv 0 \pmod 2$ *in* $H_2(X\mathbb{Z})$ *(the sum is taken over all E_j such that $(C_i, E_j) = 0$).*

Proof. a) Let $p\colon \hat{X} \to X$ be the map that blows up each of the double points of X. Clearly we have:

$$p^*(Q \cap X) = C_i + E_1 + \ldots + E_i = 2p^*(H), \tag{2.1.1}$$

H is a hyperplane section of X that does not contain any of the singular points on X.

From (2.1.1) it follows that:

$$C_i = 2p^*(H) - (E_1 + E_2 + \ldots + E_i) \tag{2.1.2}$$

From (2.1.2) we get:

$$(C_i, C_i) = 4(p^*(H), p^*(H)) + (E_1, E_1) + (E_2, E_2) + \ldots + (E_i, E_i). \tag{2.1.3}$$

This is true because $(p^*(H), E_k)=0$ and $(E_k, E_1)=-2\delta_{k1}$. So from (2.1.3) and the fact that $\deg X=4$ we get that $(C_i, C_i)=16-2i$. Q.E.D.

The Proof of b). From (2.1.2) we get

$$(2.1.4)\quad \begin{aligned} &C_i+E_{i+1}+\ldots+E_{16}\\ &=2p^*(H)-(E_1+\ldots+E_i)+(E_{i+1}+\ldots+E_{16})\\ &=2p^*(H)-(E_1+\ldots+E_i+E_{i+1}+\ldots+E_{16})+2(E_{i+1}+\ldots+E_{16}).\end{aligned}$$

Now b) follows from Lemma 2.2. Q.E.D.

2.2 **Lemma.** *$E_1+E_2+\ldots+E_{16}\equiv 0 \pmod 2$ in $H_2(\hat{X},\mathbb{Z})$.*

Proof. Let $P: \hat{J}\to J$ be the map that blows up each of the fixed points of i. Clearly the involution i lifts to an involution $\hat{i}$ on $\hat{J}$. As i acts as $-id$ on the tangent space of J at any of its fixed points, the fixed locus of $\hat{i}$ is just the union of the exceptional curves on $\hat{J}$. Note that $\hat{J}/\hat{i}=\hat{X}$, i.e. $\hat{J}$ is a double covering of $\hat{X}$ with the ramification divisor $(E_1+\ldots+E_{16})$. From here we get that $(E_1+\ldots+E_{16})\equiv 0 \pmod 2$ in $H_2(\hat{X},\mathbb{Z})$. Q.E.D.

Remark. C_i is a non-singular curve on X.

From Lemma 2.1 b) it follows that there exists a double covering $\hat{Y}$ of $\hat{X}$ with a ramification divisor $C_i+E_{i+1}+\ldots+E_{16}$. Let $\hat{p}: \hat{Y}\to\hat{X}$ be the canonical map. Notice that $\hat{p}^*(C_i)=2C_i'$ and $\hat{p}^*(E_j)=2E_j'$ for $j=i+1,\ldots,16$.

2.3 **Lemma.** *$(E_j', E_j')=-1$ and $(C_i', C_i')=8-i$.*

Proof. From the well known formula $(\hat{p}^*(E_j), \hat{p}^*(E_j))=\deg\hat{p}x(E_j,E_j)$ (see [H]) and the fact that $\hat{p}^*(E_j)=2E_j'$ we get

$$(\hat{p}^*(E_j), \hat{p}^*(E_j))=4(E_j', E_j')=2(E_j', E_j')=-4,$$

so $(E_j, E_j)=-1$. Using the same arguments we get that $(C_i' C_i')=8-i$. Q.E.D.

From Lemma 2.3 and Castelnuovo's theorem it follows that we can blow down all E_j' on $\hat{Y}$ and we will get a non-singular surface Y.

2.4 **Lemma.** a) *Y is a minimal model, i.e. it does not contain exceptional curves of the first kind.*

b) *$p_g(Y)=1$, $q(Y)=0$ and $(K_Y^2)=8-i$.*

Proof. The following lemma is proved in [M_2] on p. 110.

Lemma. *Let $f: X^r\to Y^r$ be a regular dominating morphism of smooth r dimensional varieties with branch locus B. Then for all rational r-forms w on Y $(f^*(w))=B+f^{-1}((w))$.*

From this lemma and the fact that X is a K-3 surface, i.e. $K_X=0$, we get:

$$(2.4.1)\qquad K_{\hat{Y}}=C_i'+E_{i+1}'+\ldots+E_{16}'.$$

From the fact that $(E_j', E_j')=-1$ (this is lemma 2.3) we get that

$$(2.4.2)\qquad K_Y=C_i' \quad\text{and so}\quad (K_Y^2)=8-i.$$

From here we get that Y is a minimal model, otherwise K_Y must contain all the exceptional curves of the first kind. Q.E.D.

The Proof of b)

Step 1. The topological Euler characteristic $\chi_{\text{top}}(\hat{Y})=16+i$.

Proof. First we will compute $\chi_{\text{top}}(\hat{Y})$ from the well known formula:

$$\chi_{\text{top}}(\hat{Y})=2\chi_{\text{top}}(\hat{X})-\chi_{\text{top}}(B), \tag{2.4.3}$$

where B is the branch locus of $P\colon \hat{Y}\to\hat{X}$. From the fact that $\hat{X}$ is a K-3 surface it follows that:

$$\chi_{\text{top}}(\hat{X})=24. \tag{2.4.4}$$

The branch locus B consists of the disjoint union of non-singular curves. i.e. $B=C_i'+E_{i+1}'+\ldots+E_{16}'$, so.

$$\chi_{\text{top}}(B)=\chi(C_i')+(16-i)\chi(\mathbb{P}^1)=\chi(C_i')+2(16-i). \tag{2.4.5}$$

Note that

$$\chi(C_i')=2-2p_g(C_i'). \tag{2.4.6}$$

From the adjunction formula $2p_g(C_i')-2=(c_i', C_i'+K_X)$ we get:

$$\chi(C_i')=2i-16 \tag{2.4.7}$$

(here we use the fact that $C_i'=C_i$). So

$$\chi(B)=2i-16+32-2i=16 \tag{2.4.8}$$

$$\chi(\hat{Y})=2.24-16=48-16=32. \tag{2.4.9}$$

On the other hand we have $\chi(\hat{Y})=\chi(Y)+16-i$, so

$$\chi(Y)=16+i. \tag{2.4.10}$$

Q.E.D.

Step 2. $\chi(0_Y)=p_g-q+1=2$.

Proof. From Noether formula we get:

$$12(p_g-q+1)=c_1^2+c_2, \tag{2.4.11}$$

so

$$12(p_g-q+1)=8-i+16+i=24 \tag{2.4.12}$$

$$p_g-q+1=2. \quad \text{Q.E.D.} \tag{2.4.13}$$

Step 3. $q(Y)=0$.

Proof. Since $q(Y)$ is a birational invariant (see [H]) it is enough to prove that $q(\hat{Y})=0$. From the Hodge decomposition of Kahler manifolds and the Poincaré duality we get that $q(\hat{Y}))=\dim H^1(\Omega^2_{\hat{Y}})$. If we prove that $H^1(\Omega^2_{\hat{Y}})=0$, then $q(Y)=0$. We have the following exact sequence:

(2.4.14) $$0\to\Omega^2_{\hat{Y}}\xrightarrow{w}\Omega^2_{\hat{Y}}(C'_i)\xrightarrow{\text{Res}}\Omega^1_{C'_i}\to 0$$

(Res means Poincaré residue).

(2.4.15) $$\begin{aligned}0\to H^0(\Omega^2_{\hat{Y}})\to H^0(\Omega^2_{\hat{Y}}(C'_i))\xrightarrow{\text{Res}} H^0(\Omega^1_{C'_i})\\ \to H^1(\Omega^2_{\hat{Y}})\to H^1(\Omega^2_{\hat{Y}}(C'_i)).\end{aligned}$$

Step 3 will follow from (2.4.15) and the following two propositions:

(2.4.16) **Proposition.** $H^1(\Omega^2_{\hat{Y}}(C'_i))=0.$

(2.4.17) **Proposition.** *Res in* (2.4.15) *is a map onto.*

Proof of (1.4.16). From the Serre duality we get:

(2.4.18) $$\dim H^1(\Omega^2_{\hat{Y}}(C'_i))=\dim H^1(0_{\hat{Y}}(-C'_i)).$$

Let F: $\hat{Y}\to Y$ be the map which blows down all the exceptional curves on $\hat{Y}$ of the first kind. From the definition of $R^i f_*(0_{\hat{Y}}(-C'_i))$ we get:

(2.4.19) $$\begin{gathered}f_*0_{\hat{Y}}(-C'_i)=0_{\hat{Y}}(-C'_i),\\ R^1f_*(0_{\hat{Y}}(-C'_i))=R^2f_*(0_{\hat{Y}}(-C'_i))=0.\end{gathered}$$

From the Leray spectral sequence

$$E_2^{p,q}=H^p(Y,R^qf_*(0(-C'_i)) \Rightarrow H^{p+q}(\hat{Y},0_{\hat{Y}}(-C'_i))$$

we get:

(2.4.20) $$\dim H^1(\hat{Y},0_{\hat{Y}}(-C'_i))=\dim H^1(Y,0_Y)(-C'_i))$$

(see (2.4.19)). From Serre duality and the fact that $C'_i=K_Y$ we get:

(2.4.21) $$\dim H^1(Y,0_Y(-C'_i))=\dim H^1(Y,0_Y(2K_Y)).$$

Kodaira proved in [K] that if Y is a surface of general type then $\dim H^1(Y,O_Y(nK_Y))=0$ if $n\geqq 2$. So (2.4.16) is proved. Q.E.D.

Proof of (2.4.17). On $\hat{Y}$ a canonical involution $\hat{j}$ acts and $\hat{Y}/\hat{j}=\hat{X}$, where $\hat{X}$ is a K-3 surface. The fixed point locus of $\hat{j}$ is $C'_i+E'_{i+1}+\ldots+E'_{16}$. On X we have the following exact sequence:

(2.4.22) $$0\to H^0(\Omega^2_{\hat{X}})\to H^0(\Omega^2_{\hat{X}}(C_i))\xrightarrow{\text{Res}} H^0(\Omega^1_{C_i})\to H^1(\Omega^2_{\hat{X}})=0$$

($\hat{X}$ is a K-3 surface). From (2.4.22) we get that Res: $H^0(\Omega^2_{\hat{Y}}(C_i))\to H^0(\Omega^1_{C_i})$ is a surjective map. Let $w\in H^0(\Omega^2_{\hat{Y}}(C_i))$ such that $\text{Res}(w)\neq 0$, then we will prove that:

a) $\hat{p}^*(w)\in H^0(\Omega^2_{\hat{Y}}(C'_i))$,

b) $\text{Res}(\hat{p}^*(w))\neq 0$.

$\hat{p}$: $\hat{Y}\to\hat{X}$ is the canonical map.

The Proof of a). Let U be some affine open neighborhood of the point $P \in C_i$. Let (x, y) be local coordinates in U, where y is the local equation of C_i in U. Then we have:

$$w|_U = f(x,y)\frac{dx \wedge dy}{y} \quad (w \in H^0(\Omega^2_{\hat{X}}(C_i))). \tag{2.4.23}$$

Because $\hat{p}$: $\hat{Y} \to \hat{X}$ is a finote morphism, then $\hat{p}^{-1}(U)$ will be an affine open set in $\hat{Y}$. It is clear that we can choose $(\hat{p}^*(x), y')$ as a local coordinate system in $\hat{p}^{-1}(U)$, where y is the local equation of C'_i in $\hat{p}^{-1}(U)$. Notice that $\hat{p}^*(y) = y'^2$. Let me denote by x $p^*(x)$

$$\begin{aligned} \hat{p}^*(w)|_{\hat{p}^{-1}(U)} &= f(x, \hat{p}^*(y))\frac{dx \wedge d\hat{p}^*(y)}{\hat{p}^*(y)} \\ &= f(x, y'^2)\frac{dx \wedge dy'^2}{y'^2} \\ &= 2f(x, y'^2)\frac{dx \wedge dy'}{y'}. \end{aligned}$$

(2.4.24) proves a). Q.E.D.

The Proof of b). Notice that when we restrict the function x on $U \cap C_i$, then we get a local coordinate in $U \cap C_i$. The map Res is given by the following formula:

$$\operatorname{Res}(w)|_{U \cap C_i} = f(x, 0)\,dx \quad \text{and} \quad \operatorname{Res}(\hat{p}^*(w)) = f(\hat{p}^*(x), 0)\,dx. \tag{2.4.25}$$

From (2.4.25) b) follows immediately. Q.E.D.

Notice that $\hat{p}^*$: $H^0(\Omega^2_{\hat{X}}(C_i)) \hookrightarrow H^0(\Omega^2_{\hat{Y}}\, C'_i))$ is an injective map. So from this fact a) and b) it follows that Res in (2.4.15) is a surjective map. Step 3 is proved. Q.E.D.

From Step 3 Lemma 2.4 follows immediately. Q.E.D.

So we have constructed minimal surfaces with $p_g = 1$, $q = 0$ and $2 \leqq (K^2) \leqq 8$. The canonical divisor of these surfaces are non-hyperelliptic curves. All of them are non-singular curves.

2.5 **Lemma.** *C_i is not a hyperelliptic curve on the* K-3 *surface $\hat{X}$.*

Proof. In Chapter 10 in [Sh] the following lemma is proved: Let X be a K-3 surface and C a non-singular curve on X, then 1) the complete linear system $|C|$ has no fixed components and no fixed points, 2) $|C|$ gives an embedding iff C is a non-hyperelliptic curve. From this lemma it follows that if we prove that the complete linear system $|C_i|$ gives an embedding then C_i will be a non-hyperelliptic curve. It is easy to see that $|C_i|$ is just the space of all quadrics in $\mathbb{P}^3$ passing through $p_{11}, \ldots, p_{16}$. In order to prove that $|C_i|$ gives an embedding for X we must prove that if $x \neq y$ are two points on X different from $p_{11}, \ldots, p_{16}$ we can always find a quadric passing say through $p_{11}, \ldots, p_{16}$ and x and not passing through y. But this follows immediately from the fact that the dimension of the family of all quadrics passing through $p_{11}, \ldots, p_{16}$ and X is equal to two, while the dimension of all quadrics passing through $p_{11}, \ldots, p_{16}$, x and y is equal to 1, so from here it follows that $|C_i|$ gives an embedding and so Lemma 2.5 is proved. Q.E.D.

3. The Moduli Space of the Surfaces we Constructed with $p_g=1$, $q=0$ and $(K^2)=2,3,\ldots,8$

The sutudy of the moduli space of the surfaces we constructed in § 2 is based on the following theorem;

Theorem 1. *Let Y be a minimal surface of general type with the following properties:*

1) $p_g(Y)=1$, $q(Y)=0$ *and* $(K^2)=2$,
2) K_Y *is a non-singular and non-hyperelliptic curve*
3) K_Y *is an ample divisor and*
4) *there exists an involution* $j\colon X\to X$ *such that* $j/_C=id$.

Then a) *the linear system* $|2K_Y|$ *gives a holomorphic map* $f\colon Y\to \mathbb{P}^3$, $f(Y)=X$ *is K-3 surface with* 10 *ordinary double points, i.e. X is a quartic with ten ordinary double points).*

b) $\deg f=2$.

Proof

3.1 **Proposition.** $|2K_Y|$ *gives a holomorphic map.*

Proof. We have the following exact sequences:

$$0\to\Omega_Y^2\xrightarrow{w}\Omega_Y^2(K_Y)\xrightarrow{\text{Res}}\Omega_{K_Y}^1\to 0 \tag{3.1.1}$$

$$0\to H^0(\Omega_Y^2)\to H^0(\Omega_Y^2(K_Y))\xrightarrow{\text{Res}}H^0(\Omega_{K_Y}^1)\to 0. \tag{3.1.2}$$

a) Suppose that $|2K_Y|$ has a fixed component D. Since K_Y is an ample divisor $(D,K_Y)\neq 0$. So the restriction of the linear system $|2K_Y|$ on K_Y will not give a holomorphic map on K_Y. From (3.1.1) we get that the restriction of $|2K_Y|$ is the canonical system of K_Y. It is a well known fact that on a non-hyperelliptic curve the canonical map always gives a holomorphic embedding. Since K_Y is a non-singular and non-hyperelliptic curve we get a contradiction. This means that $|2K_Y|$ does not have fixed components.

b) Suppose that $|2K_Y|$ has a base point x_0. Since $2K_Y\in|2K_Y|$ this point x_0 must be on K_Y. Now repeating the same arguments as in a) we get that this is impossible. Q.E.D.

From the Serre duality we get that $H^2(\Omega_Y^2(K_Y))=0$. Kodaira proved that for all surfaces of general type we have $H_1(X,0_X(nK_X))=0$ for $n>1$. See [K]. Using these facts and Riemann-Roch we get that $\dim H^0(\Omega_Y^2(K_Y))=4$, so we get a map $f_{|2K_Y|}\colon Y\to\mathbb{P}^3$ and the map is holomorphic. We will denote this map by f.

3.2 **Proposition.** a) $\deg f(Y)=4$, *i.e.* $f(Y)=X$ *is a hypersurface of degree* 4 *in* $\mathbb{P}^3$,

b) *the degree of the map f is equal to* 2.

Proof. a) Since $(2K_Y,2K_Y)=8$ and $|2K_Y|$ is a complete linear system, it follows that $\deg f(Y)$ can be 2, 4 or 8. The proof of the fact that $\deg X=4$ is based on the following well known fact: Let C be a non-singular non-hyperelliptic curve of genus 3, then $|K_C|$: $C \to \mathbb{P}^2$ is an isomorphism onto a non-singular plane curve of degree 4. (See [H].) From the adjunction formula and the fact that $(K_Y^2)=2$ we get that the genus of K_Y is equal to 3. We assumed that K_Y is a non-singular and non-hyperelliptic curve of genus 3. Since $|2K_Y|$ restricted to K_Y is the canonical map and $f(K_Y)$ is a hyperplane section of degree 4 (because $j|_C = id$) we get that $\deg f(Y)=4$.

b) From $(2K_Y, K_Y 2)=8$ and $\deg f(Y)=4$ it follows that the degree of the map f is two. Q.E.D.

Since $j|_{K_Y}=id$, we get that $j^*(w_Y(2,0))=w_Y(2,0)$, i.e. the only holomorphic two form on Y is invariant under the action of j. Indeed let U be a neighborhood of a point p on K_Y. In U we can choose a local coordinate system (x,y) such that $x^j=x$ and $y^j=-y$. Notice that y is the local equation of K_Y in Y. From the fact that the divisor of $w_Y(2,0)$ is K_Y, we obtain: $w_Y(2,0)_U = y\,dx\wedge dy$, so $w_Y^j(2,0)_U = -y\,dx \wedge d(-y) = y\,dx\wedge dy$. From these local calculations it follows that $w_Y^j(2,0)=w_Y(2,0)$.

3.3 **Proposition.** *j can have only isolated fixed points outside K_Y.*

Proof. Suppose that $j(p)=p$ and $p \notin K_Y$. Let U be a neighborhood of p. We have proved that $w_Y^j(2,0)=w_Y(2,0)$, and so the representation of $\mathbb{Z}/2\mathbb{Z}=(1,j)$ in $T_{p,Y}$ (the tangent space of Y at p) must preserve the skew-symmetric form $w_Y(2,0)$. This means that we can find a local coordinate system in $U(x,y)$ such that $x^j = -x$ and $y^j=-y$, so from here it follows that p is an isolated fixed point of j.

3.4 **Proposition.** *The number of fixed points of the involution j is equal to* 10.

Proof. From Proposition 3.3 it follows that the orbit space $Y/j\,4f(Y)$ has only ordinary double points and since $\deg f(Y)$ in $\mathbb{P}^3$ is equal to 4 it follows from the famous results of M. Artin that after we blow up the double points on $Y/j=f(Y)=X$ we will get a K-3 surface $\hat{X}$. Let k be the number of the fixed points of j. Since $f(Y)=Y/j$ is a K-3 surface with k ordinary double points it follows that $\chi_{\text{top}}(Y/j)=24-k$. We know that $\chi_{\text{top}}(Y)=22$. Comparing the two Euler characteristics we get that $k=10$. Q.E.D.

Theorem 1 is proved.

Remark. Repeating word by word the proof of Theorem 1 one can prove the following theorem: Let Y be a surface with $p_g=1$, $q(Y)=0$ and $3 \leqq (K_Y^2) \leqq 8$. Suppose that K_Y is a non-singular and non-hyperelliptic curve, then the complete linear system $|2K_Y|$ gives us a holomorphic map f onto a K-3 surface with $8+(K_Y^2)$ simple double points.

$$f\colon Y \to \mathbb{P}^{(K_Y^2)+1} \deg f = 2.$$

In order to calculate the moduli space of all surfaces with $p_g=1$, $q=0$ and $2 \leqq (K^2) \leqq 8$ and K_y a non-singular an non-hyperelliptic curve we need the following two propositions:

3.5 **Proposition.** *Let Y_1 and Y_2 be two surfaces with the properties stated in Theorem* 1 *and let the images $f_{|2K_{Y_1}|}(Y_1)$ and $f_{|2K_{Y_2}|}(Y_2)$ are birationally distinct K-3 surfaces. then Y_1 and Y_2 are birationally distinct.*

Proof. Suppose that Y_1 and Y_2 are isomorphic and let g: $Y_1 \to Y_2$ be an isomorphism. g induces an isomorphism

$$g^*: H^0(Y_2, 0(2K_{Y_2})) \to H^0(Y_1, 0_{Y_1}(2K_{Y_1})).$$

From here we get that there exists an element of the group $PGL(N)$, g_1 such that $g_1(X_2) = X_1$, where $N = \dim P(H^0(0(2K_{Y_1}))$, $X_i = f_{|2K_i|}(Y_i)$ and $i=1$ and 2.

3.6 **Proposition.** *Suppose that Y_1 and Y_2 are surfaces constructed in the way described in Theorem* 1, *that they are constructed from the same K-3 surface X but that the ramification divisors are not isomorphic. Then Y_1 and Y_2 are birationally distinct.*

Proof. Suppose that g: $Y_1 \to Y_2$ is an isomorphism. Then f will induce an isomorphism f: $K_{Y_1} \xrightarrow{\sim} K_{Y_2}$, since $p_g(Y_1) = p_h(Y_2) = 1$. This contradicts the assumption that K_{Y_1} and K_{Y_2} are not isomorphic. Q.E.D.

3.7 **Lemma.** *Suppose that X is a K-3 surface embedded in $\mathbb{P}^N$ and X has at most ordinary double points, then the number of moduli of all non-isomorphic hyperplane sections of X is equal to N, i.e. the dimension of the space where X is embedded.*

Proof. This is a standard fact about the number of moduli of curves on a fixed K-3 surface. For the completeness of this article we will prove this fact for K-3 surfaces embedded in $\mathbb{P}^3$. First some notes about the automorphisms of X induced by $PGL(3)$. Notice that all automorphisms of X induced by $PGL(3)$ formed a compact algebraic group G with a Lie algebra contained in $H^0(X, \Theta_X) = 0$, so G is a finite group. Now let C be a curve cut by a hyperplane section. We suppose that C is a non-singular curve. Notice that C is canonically embedded in $\mathbb{P}^2$, i.e. from the adjunction formula it follows that $H \cdot C = K_C$, H is a hyperplane section. Since C is a canonical curve in $\mathbb{P}^3$ it follows that all the automorphisms of C are induced by $PGL(3)$. Let G' be the group of automorphisms of C. Now let C' be a non-singular hyperplane section on X different from all images of C by the action of the finite group G. We want to prove that C' is not isomorphic to C. Suppose that C and C' are isomorphic curves. From the fact that $H \cdot C = K_C$ it follows that the isomorphism f: $C \to C'$ is induced by $g \in PGL(3)$ and $g \in G$ (this is because of the way we choose C'). So $C' = g(X) \cap g(H) = X \cap H'$, $C' = X \cap H'$. From this fact it follows that $C' = g(X) \cap X$, but this is impossible because it is easy to see that $g(X)$ intersects X transversally and $g(X) \cap X$ is an irreducible curve. From here it follows that he space of all hyperplanes in $\mathbb{P}^3$ which is isomorphic to $\mathbb{P}^3$, defines a family of curves $F \subset \mathbb{P}^3 \times X$, where $F = (x, H_x \cap X)$ (H_x is the hyperplane section defined by the point x). The fibers of this family are non-isomorphic curves, so this family has Dimension 3.

Theorem 2. *The moduli space of all surfaces with the properties stated in Theorem* 1 *is isomorphic to $U x(\Gamma \setminus SO(2,9)/SO(2) \times SO(9))$, where U is an open subset in $\mathbb{P}^3$ and $\Gamma \setminus SO(2,9)/SO(2) \times SO(9)$ corresponds to the moduli space of all K-3 surfaces from which we contructed the surfaces with the propertie stated in Theorem* 1. *Γ is an arithmetic subgroup of $SO(2,9)$.*

Proof. From Theorem 1, Proposition 3.6 and Lemma 1 it follows that the moduli space of all surfaces with the same properties as in theorem 1 will be isomorphic to $U \times M_i$ where M_i is the moduli space of all K-3 surfaces from which we construct the surfaces with the properties stated in Theorem 1 and U corresponds to the moduli space of the ramification divisors on the K-3 surface from which we construct the surface with the properties stated in Theorem 1. The proof of the fact that $M = \Gamma SO(2,9)/SO(2) \times SO(9)$ will ge given in Appendix 1, because we need some facts about Hodge structures and these facts will be introduced in the next paragraph.

Remark. From now on, if we say that a surface S has the propertes stated in Theorem 1, we will understand that the surface S has the following properties:

1) $p_g(S)=1$, $q(S)=0$ and $2 \leqq (K_S^2) \leqq 8$.
2) K_S is a non-singular and non-hyperelliptic curve.

4. General Facts About Hodge Structures on Surfaces with $p_g=1$ and the Period Mapping for the Surfaces we have Constructed in 2

In [B] Bombieri proved that $|5K_X|$ (K_X is the canonical class of the surface X) gives an embedding modulo rational double points for all surfaces X of general type. From now on we will consider only those surfaces of general type with an ample canonical divisor K_X. Next we must define what is a polarized Hodge structure on a surface X of general type with $p_g \geqq 1$.

Let X be a surface of general type with $p_g \geqq 1$ and ample canonical class. From Bombieri's theorem it follows that $|5K_X|$ gives a non-singular embedding of X in $\mathbb{P}^N$. It is a standard fact that the Poincaré dual of $|5K_X|$ is a $(1,1)$ form that comes from the restriction of the form of Fubini-Study metric of $\mathbb{P}^N$ on X. Let me denote this form by w. Let $H^2(X,\mathbb{Z})'$ be the torsion free part of $H^2(X,\mathbb{Z})$. On $H^2(X,\mathbb{Z})'$ there is an inner product induced by the cup product, so $H^2(X,\mathbb{Z})'$ is an Euclidean lattice and we will denote this Euclidean lattice by L. It is a standard fact that the signature of the bilinear form is equal to $(2p_g+1, h^{1,1}-1)$ where $h^{1,1}=\dim H^1(\Omega_X^1)$. Let $(L\otimes\mathbb{C})_w=(x\in L\otimes\mathbb{C}\,|\,(x,w)=0)$.

4.1 *Definition.* A polarized Hodge structure on L with a polarization class w is defined as a filtration $H^{2,0} \subset H^{2,0}+H^{1,1} \subset (L\otimes\mathbb{C})_w$, which has the following properties: a)

a) $\dim H^{2,0}=p_g$;
b) $(H^{1,1})^{\perp}=(H^{2,0}+H^{0,2})$, where $H^{0,2}=\overline{H^{2,0}}$;
c) $(x,x)=0$ for all $x\in H^{2,0}$ and
d) $(x,\bar{x})>0$ for all $x\in H^{2,0}$ and $x\neq 0$.

Griffiths proved that the space

$$SO(2p_g, h^{1,1}-1)/U(p_g)\times SO(h^{1,1}-1)/\Gamma$$

parametrizes all admissible Hodge structure on L with $\dim H^{2,0}=p_g$. $\Gamma=(g\cdot \mathrm{Aut}(L)\,|\,g(w)=w)$.

3.2 *Definition.* The space $SO(2p_g, h^{1,1}-1)/U(p_g)\times SO(h^{1,1}-1)/\Gamma$ is called the period domain.

Let p: $V\to D$ be a family of non-singular surfaces with $p_g\geqq 1$, where D is a complex manifold. There exists a canonical map P:

$$D\to SO(2p_g, h^{1,1}-1)/U(p_g)\times SO(h^{1,1}-1)/\Gamma.$$

P is defined in the following manner: to every point $y\in D$, $P(y)$ is the admissible polarized Hodge structure of the surface $p^{-1}(y)$, defined by the complex structure on $p^{-1}(y)$. In [G] Griffiths proved that P is a holomorphic map.

Remark. The period domain of the surfaces we have constructed in §2 is: $SO(2,11+i)/SO(2)\times SO(11+i)/\Gamma$, where $16-i$ is the number of all E_k that do not intersect C_i on X. $\Gamma=SO(2,11+i;\mathbb{Z})$.

Proof. In §2 we proved that $\chi_{\text{top}}(Y)=16+i$ and $q(Y)=0$, so $\dim H^2(X,\mathbb{Z})=b_2=14+i$. From $\dim H^{2,0}=1$ it follows that $h^{1,1}(Y)=16+i$ and $q(Y)=0$. Now our remark follows from the result of Griffiths mentioned above. Q.E.D.

2) It is not difficult to prove that the surfaces we contructed in §2 have ample canonical divisors.

Theorem 3. *Suppose that Y and Y′ are surfaces with the following properties:* 1) $p_g(Y)=p_g(Y')=1$, $q(Y)=q(Y')=0$ *and* 2) $2\leqq(K_Y^2)=(K_{Y'}^2)\leqq 8$, 3) K_Y *and* $K_{Y'}$ *are non-singular and non-isomorphic curves,* 4) *Y and Y′ are obtained from the same K-3 surface, i.e. from theorem* 1 *we know that on Y and Y′ involutions i and i′ act in such a way that* $i|_{K_Y}=id=i'|_{K_{Y'}}$ *and the orbit spaces* $Y/i=Y'/i'$ *are K-3 surfaces, we suppose that these two K-3 surfaces are isomorphic.*

Then there exists an isomorphism g: $H^2(Y,q)\xrightarrow{\sim} H^2(Y',\mathbb{Q})$ *which preserves the inner product induced by the cup product and the Hodge filtrations. g is defined over* $\mathbb{Z}$.

Proof. From Theorem 1 we know that on Y and Y' the involutions i and i' act and so they induce an action of $\mathbb{Z}/2\mathbb{Z}$ on $H^2(Y,\mathbb{Q})$ and $H^2(Y',\mathbb{Q})$ respectively. So $H^2(Y,\mathbb{Q})=H_Y^+ + H_{Y'}^-$ where

$$H_Y^+=(x\in H^2(Y.\mathbb{Q})\,|\,i(x)=x \quad\text{and}\quad H_Y^-=(x\in H^2(Y',\mathbb{Q})\,|\,i(x)=-x).$$

The same is true for $H^2(Y',\mathbb{Q})$, i.e. $H^2(Y',\mathbb{Q})=H_{Y'}^+ + H_{Y'}^+$.

4.3 **Proposition.** H_Y^+ *is orthogonal to* H_Y^- *with respect to the quadratic form on* $H^2(Y,\mathbb{Q})$ *induced by the cup product. The same is true for* Y'.

Proof. Let $x\in H_Y^+$ and $z\in H_{Y'}^-$.

$$(x,z)=(i(x),i(z))=(x,-z)=-(xz)=0.\quad \text{Q.E.D.}$$

3.4 **Proposition.** $H^{2,0}(Y)+H^{0,2}(Y)\subset H_Y^+\otimes\mathbb{C}$ *and* $H_Y^-\otimes\mathbb{C}=H^{1,1}(Y)^-$.

Proof. From the proof of Theorem 1 we know that the form $w_Y(2,0)$ is invariant under the action of i. The same is true for the anti-holomorphic form $w_Y(2,0)$. So $H^{2,0}+H^{0,2}\subset H_Y^+\otimes\mathbb{C}$.

From definition 3.1 we know that $H^{1,1}=(H^{2,0}+H^{0,2})^{\perp}$. From here it follows that $H^{1,1}$ is invariant under the action of i, so $H^{1,1}(Y)=H^{1,1}(Y)^{+}+H^{1,1}(Y)^{-}$. $H^{1,1}(Y)^{-}=H_{Y}^{-}\otimes\mathbb{C}$ follows immediately from Proposition 3.3 and the fact $(H^{2,0}+H^{0,2})\subset H_{\mathscr{Y}}\otimes\mathbb{C}$. Q.E.D.

We may assume that Y and Y' are obtained from the same K-3 surface as double coverings, but with non-isomorphic divisors. From this and the proof of Proposition 3.7 it follows that we can find a family $Y''\xrightarrow{q}D$ of non-singular surfaces with the properties stated in Theorem 3 such that a) $D\subset\mathbb{C}$ is simply connected; b) $Y=q^{-1}(y_0)$ and $Y'=q^{-1}(y_1)$, where y_0 and y_1 are two points in D. Since D is simply connected it follows that Y and Y' are diffeomorphic, even more we can find a diffeomorphism $f\colon Y\to Y'$ with the following property: $f(i(x))=i'f(x)$. On the other hand the Hodge structures on $H^2(Y,\mathbb{Q})^{+}=p^*(H^2(X.\mathbb{Q}))$ are isomorphic because these two structures are induced from the same K-3 surface X. Now our theorem follows immediately from Propositions 3.3 and 3.4, i.e. the diffeomorphism f induces a Hodge isometry. Since this isometry of Hodge structures is induced by a diffeomorphism f it follows that this isometry is defined over $\mathbb{Z}$. Q.E.D.

Remark. Notice that we have proved the following lemma: let $\gamma\in H_2(Y,Z)^{-}$, i.e. $i(\gamma)=-\gamma$ then $\int_{\gamma} w_Y(2{,}0)=0$ and if $\beta\in H_2(Y,Z)^{+}$, then $\int_{\beta} w_Y(2{,}0)=\int_{p^*(\beta')} p^*(w_X(2{,}0))=\int_{\beta'} w_X(2{,}0)$. From here it follows that all surfaces constructed in the way described in Theorem 1 from the same K-3 surface have the same periods, i.e. they are mapped to the same point of the period domain.

Corollary. *The dimension of the image under the period map of the moduli space of all surfaces with the properties stated in Theorem 1 is equal to the dimension of the moduli space of the K-3 surfaces from which they are obtained, i.e.* $SO(2,3+i)SO(2)\times SO(3+i)/\Gamma$, *so this dimension is strictly less than the dimension of the moduli space of the surfaces with the properties stated in Theorem 1.*

Proof. The corollary follows immediately from Theorem 1, Theorem 2 and the remark on the preceeding page. Q.E.D.

From this corollary it follows that there exists birationally different surfaces with the properties stated in Theorem 1 which have the same periods.

Appendix 1. Moduli of K-3 Surfaces

We need some standard facts about K-3 surfaces, which can be found in [Sh and P].

Definition. A K-3 surface is a simply connected two dimensional complex manifold with a trivial canonical class.

If X is a K-3 surface, then $H^2(X,\mathbb{Z})$ is a free abelian group of rank 22. The cup product defines in $H^2(X,\mathbb{Z})$ a scalar product in $\mathbb{Z}$. Thus $H^2(X,\mathbb{Z})$ is an Euclidean lattice, which we will denote by H_X. In [Sh] Chapter 10 it is proved for every K-3 surface X, H_X is an even, unimodular lattice with a signature (3, 19). In [Se] it is proved that all such lattices are isomorphic. Let me fix one of them and call it L.

Definition. A marked K-3 surface is called a pair (X,f), where X is a K-3 surface and $f: H_X \to L$ is an isomorphism of lattices.

Definition. An admissible Hodge structure on L of type (1, 20, 1) is defined as a filtration $H^{2,0} \subset H^{2,0}+H^{1,1} \subset L\otimes \mathbb{C}$, with the following properties: a) $\dim H^{2,0}=1$; b) for any $w \in H^{2,0}(ww)=0$ and $(w,\bar{w})>0$ if $w \neq 0$; c) $H^{1,1}=H^{2,0}+H^{0,2})^{\perp}$, where $H^{0,2}=\overline{H^{2,0}}$.

It is not difficult to prove that $\Omega = SO(3,19)/SO(2)\times SO(1,19)$ parametrizes all admissible Hodge structures of type (1, 20, 1) on $L\otimes \mathbb{C}$. Ω can be represented by the following formulas in $P(L\otimes \mathbb{C}) = \mathbb{P}^{21}(\mathbb{C})$:

$$z_1^2+z_2^2+z_3^2-z_4^2-\ldots-z_{22}^2=0$$
$$|z_1|^2+|z_2|^2+|z_3|^2-|z_4|^2-\ldots-|z_{22}|^2>0.$$

We define the period map in the following way: Let (X,f) be a marked K-3 surface. Then $\tau(X,f)$ is the admissible Hodge structure $f(H^{2,0}(X)) \subset f(H^{2,0}(X)+H^{1,1}(X) \subset L\otimes \mathbb{C}$.

Theorem 4. *The moduli space of all K-3 surfaces that are images* $g_{|2K_Y|}$ *of surfaces* Y *with the properties stated in Theorem 1 is isomorphic to:*

$$SO(2,3+i)/SO(2)\times SO(3+i)/\Gamma$$

Γ *is an arithmetic subgroup of* $SO(2,3+i)$ *which will be defined at the end of the proof.* i *is defined as follows* $(K_Y^2)=8-i$, *where* $i=0,1,2,3,4,5$ *and* 6.

Proof. From Theorem 1 we know that the image $g_{|2K_Y|}(Y)$ is a K-3 surface X with $16-i$ different simple double points and $g_{|2K_Y|}(K_Y)$ is a non-singular curve C_i on X isomorphic to K_Y. Let me blow up all the simple double points on X and denote by $E_1, E_2, \ldots, E_{16-i}$ the exceptional curves on $\hat{X}$ of the second kind. Of course we have $(C_i^2)=16-2i$, $(E_k, E_1)=-2\delta_{k1}$ for all k and 1 and $(C_i, E_j)=0$. Now let me fix a marking of $\hat{X}$, i.e. an isomorphism of the lattices $H_{\hat{X}} \xrightarrow{f} L$. Let me denote by $c_i, e_1, \ldots, e_{16-i}$ the images $f(DC_i), f(De_1), \ldots, f(De_{16-i})$ in L, where D is the Poincare duality operator, $D: H_2(\hat{X},\mathbb{Z}) \xrightarrow{\sim} H^2(\hat{X},\mathbb{Z})$.

Definition. Let M_i be the subspace in Ω that corresponds to all marked K-3 surfaces (S,f) for which

$$f^{-1}(c_i), f^{-1}(e_1), \ldots, f^{-1}(e_{16-i})$$

are algebraic cycles on S, this means that the (Poincaré duals) of $f^{-1}(c_i)$, $f^{-1}(e_1), \ldots, f^{-1}(e_{16-i})$ can be realized as an algebraic cycle on S.

Lemma. 1) *M_i is isomorphic to* $SO(2,3+i)/SO(2)\times SO(3+i)$.

2) *Every point of M_i corresponds to a marked K-3 surface (S,f) with the following properties:*

a) $D^{-1}(f^{-1}(c_i)), D^{-1}(f^{-1}(e_1)), \ldots, D^{-1}(f^{-1}(e_{16-i}))$ *can be realized as a non-singular curves* b)

$$D^{-1}(f^{-1}(c_i))+D^{-1}(f^{-1}(e_1))+\ldots+D^{-1}(f^{-1}(e_{16-i}))\equiv 0 \pmod 2$$

in $H_2(S,\mathbb{Z})$.

Proof. First we will prove Condition 1). The proof of Condition 1) is based on the following criterion of Lefschetz for a cycle C in $H_2(S,\mathbb{Z})$ to be an algebraic one: A cycle C is an algebraic one if $DC\in H^{1,1}(S)\cap H^2(S,\mathbb{Z})$. This is equivalent to the following conditions: 1) $DC\in H^2(S,\mathbb{Z})$ and 2) $(w_S(2,0), DC)=0$ for all $w_S(2,0)\in H^{2,0}(S)$. For the proof of this fact look at [G and H]. From this criterion it follows that the image of the space of all marked K-3 surfaces (s,f) for which $f^{-1}(c_i), f^{-1}(e_1), \ldots, f^{-1}(e_{16-i})$ are algebraic cycles, under the period map must lie on $M_{c_i}\cap M_{e_1}\cap\ldots\cap M_{e_{16-i}}$, where M_{c_i} is defined as $\mathbb{P}(H_{c_i})\cap\Omega$;

$$H_{c_i}=(v\in L\otimes\mathbb{C}\,|\,(v,c_i)=0),$$
$$M_{e_j}=\mathbb{P}(H_{e_j})\cap\mathbb{P}(H_{e_j})\cap\Omega;$$
$$H_{e_j}=(v\in L\otimes C\,|\,(v,e_j)=0).$$

In [To] it is proved that every point of corresponds to a marked K-3 surface and in [L and P] it is proved that any two K-3 surfaces are isomorphic is they have isometric Hodge strictures, so form these two theorems it follows that the moduli space of all marked K-3 surface (S,f) for which $f^{-1}(c_i)$, $f^{-1}(e_1)$, $\ldots$, $f^{-1}(e_{16-i})$ are algebraic cycles on S is isomorphic to $M_{c_i}\cap M_{e_1}\cap\ldots\cap M_{e_{16-i}}$. Notice that we have fixed the vectors c_i, e_1, $\ldots$, e_{16-i} in L. Let $H_{c_i,e_1,\ldots,e_{16-i}}$ be the subspace in $L\otimes R$ generated by $c_i e_1, \ldots, e_{16-i}$. It is easy to see that the group that precerves the inner product in $L\otimes R$ and acts as id on $H_{c_i,e_1,\ldots,e_{16-i}}$ is isomorphic to $SO(2,3+i)$. From this fact and the fact that Ω parametrizes all oriented two dimensional subspaces in $L\otimes R$ for which $(,)$ is strictly positive (for the proof of this fact see [To]) it follows that $SO(2,3+i)$ acts transitively on $M_{c_i}\cap M_{e_1}\cap\ldots\cap M_{e_{16-i}}$. It is an obvious fact that the stationary subgroup of

$$SO(2,3+i) \quad \text{is } SO(2)\times SO(3+i),$$

so

$$M_{c_i}\cap M_{e_1}\cap\ldots\cap M_{e_{16-i}}\cong SO(2,3+i)/SO(2)+SO(3+i).$$

This proves Condition 1). Q.E.D.

Proof of Condition 2). First we will prove Condition b). We started with a surface Y which has the properties stated in Theorem 1. We know from Theorem 1 that $\hat{Y}$ is a double covering of a K-3 surface $\hat{X}$ with ramification divisor on $\hat{X}$, $C_i+E_1+E_2+\ldots+E_{16-i}$. From this it follows that $C_i+E_1+\ldots+E_{16-i}\equiv O \pmod 2$ in $H_2(\hat{X},\mathbb{Z})$ and so $c_i+e_1+\ldots+e_{16-i}\equiv o \pmod 2$ in L. So it follows that we have on S,

$$D^{-1}f^{-1}(c_i)+D^{-1}f^{-1}(e_1)+\ldots+D^{-1}f^{-1}(e_{16-i})\equiv O \pmod 2$$

in $H_2(S,\mathbb{Z})$. This proves Condition b). Q.E.D.

The proof of Condition a) will be given in two steps.

Step 1. $D^{-1}f^{-1}(c_i)$ can be realized as a non-singular curve on S.

Proof. Notice that $(D^{-1}f^{-1}(c_i), D^{-1}f^{-1}(c_i))=(c_i, c_i)>0$. From the Lefschetz criterion we know that we can find an algebraic cycle C on S such that C is homological to $D^{-1}f^{-1}(c_i)$ and $(C, C)=(c_i, c_i)>0$. Step 1 follows immediately from the following lemma proved in [Sh] Chap. 10 and Bertini's theorem.

Lemma. *Let S be a surface of type K-3 and let C be an algebraic cycle on S with the following property: $(C, C)>0$, then the complete linear system $|C|$ has no fixed components and no fixed points.* Q.E.D.

Step 2. $D^{-1}f^{-1}(e_j)$ can be realized on a non-singular rational curve on S.

Proof. From Lefschetz criterion it follows that we can find an algebraic cycle E_j homological to $D^{-1}f^{-1}(e_j)$. Step 2 follows immediately from the fact that $(E_j, E_j)=(e_j, e_j)=-2$, the adjunction formula, i.e. $p_g(E_j)=1/2(e_j, E_j)+1=0$, Riemann-Roch theorem and the fact that the sublattice generated by e_j in L has rank 1. For more details see [Sh and P]. Q.E.D.

This proves our lemma. Q.E.D.

This lemma shows that each point s of

$$M_{c_i}\cap M_{e_1}\cap\ldots\cap M_{e_{16-i}}\cong SO(2,3+i)/SO(2)\times SO(3Ti)$$

corresponds to a marked K-3 surface (S,f) for which we can repeat the construction described in § 2 and so we will get a surface Y_S with the properties stated in Theorem 1. If we forget about the marking of the K-3 surfaces we will get immediately that the moduli space of all K-3 surfaces that are images of $g_{|2K_Y|}(Y)$, where Y are surfaces with the properties stated in Theorem 1, is isomorphic to:

$$SO(2,3+i)/SO(2)\times SO(3+i)/\Gamma,$$

Where Γ is defined as follows:

$$\Gamma=(g\in \mathrm{Aut}(L)\,|\,g(c_i)=c_i, g(e_1)=e_1, \ldots, g(e_{16-i})=e_{16-i}).\quad \text{Q.E.D.}$$

Appendix 2. Some Remarks About the fundamental Group of the Surfaces with $p_g=1$, $q=0$ and $(K^2)=2$ Constructed in § 2

(The full details will appear in another paper.)

Remark 1. One can prove that a surface with the following properties: $p_g(Y)=1$, $q(Y)=0$, $(K^2)=2$ and K_Y is a non-singular and non-hyperelliptic curve, has an abelian fundamental group. Outline of the proof: Notice that if Y has the properties stated above then Y is a Galois covering of $\mathbb{P}^2$ with a Galois group $\mathbb{Z}_2\times\mathbb{Z}_2$. From Theorem 1 we know that an involution j acts on Y with the following properties: 1) $j_{|K_Y|}=id$, 2) outside $K_Y j$ has 10 fixed points, 3) $Y/J=X$ can be embedded as a quartic in P^3 with 10 ordinary doubple points. A projection of X from one of its double points onto $\mathbb{P}^2$ shows that X is a double covering of $\mathbb{P}^2$ with a ramification divisor F, a plane curve in P^2 of degree 6 with 9 double points with distinct tangents. Now it is easy to see using Zariski theorem, recently proved by Deligne in [D], that the fundamental group of Y is an abelian one.

Remark 2. If the fundamental group is abelian then $\pi_1(Y)=\mathrm{Tor}(\mathrm{Pic}(Y))$. Bombieri proved in [B] that if $p_g(Y)=1$, $q(Y)=0$ and $(K_Y^2)=2$, then $\mathrm{Tor}(\mathrm{Pic}(Y))$ is either 0 or $\mathbb{Z}_2$. We will sow that from the way we choose the points $p_{11}, p_{12}, p_{13}, p_{14}, p_{15}$ and p_{16} in § 1 and the quadric Q with the properties stated in § 2 passing through these six points it follows that $\mathrm{Tor}(\mathrm{Pic}(Y))=\mathbb{Z}_2$. Y is constructed in the same way as in § 2. We will use the same notations as in § 1 and § 2. Suppose that the quadrics L_1 and L_2 (see for the definition of L_1 and L_2 in § 1) contain respectively p_1, p_2, p_3, p_4, p_9 and p_{10}; p_5, p_6, p_7, p_8, p_9 and p_{10}. Since X is a double covering of $\mathbb{P}^2$ with a ramification divisor F consisting of 6 distinct lines in $\mathbb{P}^2$ and $L_1=p^*(L'_1)$; $L_2=p^*(L'_2)$. (We suppose that L'_1 and L'_2 are components of F) we get than on $\hat{X}$ we have:

$$(*) \qquad 2L_1+E_1+E_2+E_3+E_4+E_9+E_{10} = 2L_2+E_5+E_6+E_7+E_8+E_9+E_{10}=H$$

($H=p^*(L)$, where L is a line in $\mathbb{P}^2$ not contained in F and not passing through the double points of F.)

From (*) we get:

$$(**) \qquad (e_1+E_2+E_3+E_4+\ldots+E_7+E_8=2H-2(L_1+L_2)-2(E_9+E_{10}),$$

so

$$(***) \qquad E_1+E_2+\ldots+E_8 \equiv \mathrm{mod}\, 2) \quad \text{in } H_2(\hat{X},\mathbb{Z}).$$

From (***) and Bombieri's result it follows immediately that $\mathrm{Tor}(\mathrm{Pic}(Y))=\mathbb{Z}_2$. Indeed let Y' be a double covering of X with a ramification divisor (***), then it is easy to see that $Y' \times_X Y$ is an etale covering of Y. $Y' \times_X Y$ means desingularized manifold.

Remark 3. If we choose the quadric Q to pass through the points $p_1, p_{12}, p_{13}, p_{14}, p_{15}$ and p_{16}, then one can prove that a) $p_1, p_{12}, \ldots, p_{16}$ are in general position, b) among $E_2, E_3, E_4, \ldots, E_{11}$ there are no relations of type $\sum_{i=1}^{k} E_{j_i} \equiv (\mathrm{mod}\, n)$ in $H_2(\hat{X},\mathbb{Z})$ for any k and n. It is not difficult to prove that n can be only 2 and k can be only 8. Now let us repeat the construction in § 2. We will get Y'' with the properties stated in Theorem 1. It is not difficult to prove that $\mathrm{Pic}(Y'')$ has no torsion and so from Remark 1 will follow that Y'' will be simply connected.

Remark 4. Notice that the surfaces, constructed in § 2 with $p_g=1$ & $(K^2)=8$ have the following property: the moduli space has Dimension 12, while the period domain $SO(2,11)/SO(2)\times SO(11$ has Dimension 11, so for these surfaces global Torelli theorem is not true generically, i.e. the moduli space has a greater dimension than the period space.

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