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Titel: Vector Bundles on Cones over Projective Schemes.

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PURL: https://resolver.sub.uni-goettingen.de/purl?356556735_0062|log8

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Inventiones math. 62, 15–22 (1980)

Inventiones mathematicae
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Vector Bundles on Cones over Projective Schemes

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Introduction

Let C be the cone over a projective scheme $X \subseteq \mathbb{P}^n$. It is known that if X is projectively normal (i.e., C is normal), $\operatorname{Pic} C = 1$; thus we can ask ([1, p. 15]) whether $K^0(C) = \mathbb{Z}$? If the cone is not normal, there can be nontrivial line bundles, and, in fact, we can use $\operatorname{Pic} C$ to give a criterion for projective normality. In Sect. 1 we show that the comparison map $K^0(C) \to K_0(C)$ on Grothendieck groups from locally free to coherent sheaves factors through $\mathbb{Z}$, and we give some consequences. In Sect. 2 we relate the geometry of a singular surface to the properties of its K^0. In Sect. 3 we use the geometry to show that all vector bundles are trivial on the cone over a curve $X \subseteq \mathbb{P}^n$ embedded by a linear series of degree $d \geqq 3g+1$. This was previously known only for $g=0$. All the schemes considered are of finite type over an algebraically closed field k, and "points" will refer to "closed points".

0. Cones over Projective Schemes [5]

We will consider the cone over an embedded projective scheme $X \subseteq \mathbb{P}^n$ as well as the cone "over an ample line bundle $\mathscr{L}$" on a projective scheme. For $X \subseteq \mathbb{P}^n$ we take the cone $C \subseteq \mathbb{A}^{n+1}$, $C = \operatorname{Spec} R$ where $R = \bigoplus_{d \geqq 0} \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) / \Gamma(\mathbb{P}^n, \mathscr{I}_X(d))$ is the homogeneous coordinate ring of X. If $\mathscr{L}$ is an ample line bundle on a projective scheme X, we take the cone $C = \operatorname{Spec} S$ where $S = \bigoplus_{d \geqq 0} \Gamma(X, \mathscr{L}^{\otimes d})$; we write $0 \in C$ for the vertex (defined by S_+). Recall the relation between these cases: If $\mathscr{L} = \mathcal{O}_X(1)$ is the induced very ample line bundle on $X \subseteq \mathbb{P}^n$, $R_d \subseteq S_d$ with equality for d sufficiently large, thus there is a finite map v: $\operatorname{Spec} S \to \operatorname{Spec} R \subseteq \mathbb{A}^{n+1}$ which is an isomorphism off the vertices; if X is normal, S is normal

* Partially supported by National Science Foundation grants MCS77-18723(02) and MCS78-02428

0020-9910/80/0062/0015/$01.60

and v is the normalization of $\operatorname{Spec} R$. We write B for the total space of the dual line bundle $\mathscr{L}^v$ with $\pi: B \to X$ the projection. In either case, there is a proper map ("blowing-up the vertex") $\sigma: B \to C$ such that $\sigma^{-1}(0) = X$ identified with the 0-section of B and $\sigma: B - X \to C - 0$ is an isomorphism.

1. *K*-theory of Cones

We consider some of the formal aspects of the K-theory on cones. Let K^0 be the Grothendieck group of locally free sheaves and K_0 the Grothendieck group of coherent sheaves.

Proposition 1. *Let C be the cone over an embedded projective scheme $X \subseteq \mathbb{P}^n$ or the cone over an ample line bundle on a connected, projective scheme X. Then the map $K^0(C) \to K_0(C)$ factors through rank: $K^0(C) \to \mathbb{Z}$.*

Proof. First, the diagram

$$\begin{tikzcd} K^0(C) \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "\beta"] & K_0(C) \arrow[d, "\wr"] \\ K^0(C-0) \arrow[r] & K_0(C-0) \end{tikzcd}$$

commutes; hence α factors through $\mathbb{Z}$ if (and only if) β does. (Note that in the exact sequence $K_0(0) \to K_0(C) \to K_0(C-0) \to 0$ the image of $K_0(0)$ is 0 since the image in $K_0(\mathfrak{l})$ is 0 for a monomial curve $\mathfrak{l} \subset C$ through 0. Indeed, we can find such a curve $\mathfrak{l} = \operatorname{Spec} k[a_0 t^{d_0}, \ldots, a_n t^{d_n}]$, where $a_i \in k$ are not all zero, and then use the factorization $K_0(0) \to K_0(\tilde{\mathfrak{l}}) \to K_0(\mathfrak{l})$ where $\tilde{\mathfrak{l}} = \operatorname{Spec} k[t]$ is the normalization of $\mathfrak{l}$.) Next

$$\begin{tikzcd} K^0(C) \arrow[r, "r"] \arrow[d, "\beta"'] & K_0(B) \arrow[d] \\ K_0(C-0) \arrow[r, "\sigma^*", "\sim"'] & K_0(B-X) \end{tikzcd}$$

commutes; hence β factors through $\mathbb{Z}$ if γ does. Finally,

$$\begin{tikzcd} K^0(C) \arrow[r, "\gamma"] \arrow[d] & K_0(B) \arrow[d, "\wr i^*"] \\ K^0(0) \arrow[r] & K_0(X) \end{tikzcd}$$

commutes; hence γ factors through $\mathbb{Z}$. (For the homotopy isomorphism i^* see [3] or [7, p. 128].) Q.E.D.

Remark. Essentially the same proof shows that $K^i(C) \to K_i(C)$ factors through $K^i(C) \to K^i(0)$ for any $i \geqq 0$. (See [7] for definitions and properties of the higher K-groups.)

Corollary 1 (Murthy). *If C is normal,* $\operatorname{Pic} C = 1$.

Proof. If C is normal, the divisor class group $Cl(C)$ is defined, $\operatorname{Pic} C \to Cl\, C$ is injective, and there is a commutative diagramm

$$\begin{array}{ccc} K^0(C) & \longrightarrow & K_0(C) \\ {\scriptstyle \det}\downarrow & & \downarrow{\scriptstyle cl} \\ \operatorname{Pic}(C) & \longrightarrow & Cl(C) \end{array}$$

(For the map cl see [2, p. 501].) Then we apply the proposition. Q.E.D.

Corollary 2. *If X is smooth, the map $K^0(C) \to K^0(C-0)$ factors through rank: $K^0(C) \to \mathbb{Z}$.*

Proof. From the proposition we know $\beta\colon K^0(C) \to K_0(C-0)$ factors through $\mathbb{Z}$, and when X is smooth, $C-0$ is smooth so that $K^0(C-0) \xrightarrow{\sim} K_0(C-0)$. Q.E.D.

We now calculate $K_0(C)$ for any cone C. In particular, we can obtain a large class of examples for which $K^0 \to K_0$ is not surjective.

Proposition 2. $K_0(C) \cong K_0(X)/\lambda K_0(X)$ *where* $\lambda = 1 - [\mathscr{L}] \in K^0(X)$ (*which acts on* $K_0(X)$).

Proof. From the proof of Proposition 1 we have $K_0(C) \xrightarrow{\sim} K_0(C-0) \cong K_0(B-X)$. Now we compute $K_0(B-X)$ from the exact sequence $K_0(X) \xrightarrow{i_*} K_0(B) \to K_0(B-X) \to 0$. The map i_* is given by $[\mathscr{F}] \mapsto [\pi^*(\mathscr{F})] - [\pi^*(\mathscr{L}) \otimes \pi^*(\mathscr{F})]$ since $0 \to \pi^*(\mathscr{F}) \otimes \mathscr{I}_X \to \pi^*(\mathscr{F}) \to i_*(\mathscr{F}) \to 0$ is exact and $\mathscr{I}_X \cong \pi^*(\mathscr{L})$. Finally, we identify $K_0(B)$ with $K_0(X)$ via π^*. Q.E.D.

Remarks. 1) Essentially the same calculations with the Chow group $A.$ show that $A.(C) \cong A.(X)/c_1(\mathscr{L}) \cdot A.(X)$ where $c_1(\mathscr{L}) \in A^1(X)$ is the Chern class of $\mathscr{L}$. (See [4] for notation.)

2) The only change in the above calculations over an arbitrary ground field k is that $K_0(0)$ may contribute $\mathbb{Z}/(m)$ to $K_0(C)$ if the projective scheme X does not have a k-rational point. We thank the referee for pointing this out by the example $X = \operatorname{Proj} \mathbb{R}[x_0, x_1]/(x_0^2 + x_1^2) \subset \mathbb{P}^1_{\mathbb{R}}$.

3) To get some idea of what Propositions 1 and 2 are about, note that a locally free sheaf $\mathscr{F}$ on X gives rise to a graded S-module $M = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes d})$ which is locally free except at the maximal ideal S_+ and hence a locally free sheaf on the punctured cone $C-0$ which does not extend to a locally free sheaf on C unless M is a free graded S-module. Thus from X we get elements of $K_0(C)$, not $K^0(C)$.

Proposition 3. *If $X \subseteq \mathbb{P}^n$ is normal,* $\operatorname{Pic} C \cong U(S/\mathfrak{c})/U(R/\mathfrak{c}) \cdot U(S)$ *where R is the homogeneous coordinate ring of X, S is the normalization, $\mathfrak{c}$ is the conductor of S into R, and U denotes the group of units. Further, for any $\mathscr{M}, \mathscr{M}' \in \operatorname{Pic} C$, $\mathscr{M} \oplus \mathscr{M}' \cong \mathscr{O}_C \oplus (\mathscr{M} \otimes \mathscr{M}')$, and in particular any $\mathscr{M} \in \operatorname{Pic} C$ can be generated by two global sections.*

Proof. We have a fibre product of rings $R = R/\mathfrak{c} \times_{S/\mathfrak{c}} S$, whence a Mayer-Vietoris sequence (see [2]) $U(R/\mathfrak{c}) \oplus U(S) \to U(S/\mathfrak{c}) \to \operatorname{Pic} R \to \operatorname{Pic}(R/\mathfrak{c}) \oplus \operatorname{Pic} S \to \ldots$. This gives the expression for $\operatorname{Pic} R$ once we check that $\operatorname{Pic} S = 1$ by Corollary 1 to Proposition 1, and $\operatorname{Pic} R/\mathfrak{c} = 1$ since $\operatorname{Supp} R/\mathfrak{c} = \{R_+\}$. If $\mathcal{M}$ and $\mathcal{M}' \in \operatorname{Pic} C$ are defined by units $u, u' \in U(S/\mathfrak{c})$, then $\mathcal{M} \oplus \mathcal{M}'$ is isomorphic to the fibre product $(R/\mathfrak{c})^2 \times_\theta S^2$ where $\theta = \begin{pmatrix} u & \\ & u' \end{pmatrix}$ and $\mathcal{O}_C \oplus \mathcal{M} \otimes \mathcal{M}'$ is isomorphic to the fibre product using $\psi = \begin{pmatrix} 1 & \\ & u u' \end{pmatrix}$. Since θ and ψ differ by elementary row operations (which lift to S^2), the fibre products are isomorphic. The final statement then follows from the isomorphism $\mathcal{M} \oplus \check{\mathcal{M}} \cong \mathcal{O}_C \oplus \mathcal{O}_C$. Q.E.D.

Corollary. *Assume $X \subseteq \mathbb{P}^n$ is normal and connected (hence integral). Then X is projectively normal (i.e., C is normal) if and only if* $\operatorname{Pic} C = 1$.

Proof. If X is projectively normal, then $\operatorname{Pic} C = 1$ by Corollary 1 to Proposition 1. For the other direction, observe that if X is not projectively normal, $R/\mathfrak{c} \subsetneqq S/\mathfrak{c}$ are finite-dimensional k-algebras so that $U(R/\mathfrak{c}) \subsetneqq U(S/\mathfrak{c})$. Under the hypotheses, S will be a (positively) graded domain with k in degree 0 so $U(S) = U(k)$, and we get $\operatorname{Pic} C \neq 1$ by the proposition. Q.E.D.

Remark. If $X \subseteq \mathbb{P}^n$ is normal, $R/\mathfrak{c} \subseteq S/\mathfrak{c}$ are finite-dimensional so $K^1(S/\mathfrak{c}) = U(S/\mathfrak{c})$ and $K^0(R/\mathfrak{c}) = \mathbb{Z}$, and the Mayer-Vietoris sequence $\ldots \to K^1(S/\mathfrak{c}) \to K^0(R) \to K^0(R/\mathfrak{c}) \oplus K^0(S)$ allows one often to reduce to working with $\operatorname{Spec} S$ (and consequently complete linear systems on X).

Examples. 0) The cone over n distinct points in $\mathbb{P}^1$ has non-trivial Picard group for $n \geqq 3$.

1) If X is not normal, $\operatorname{Pic} C$ may still be trivial as for X: $xy = 0$ in $\mathbb{P}^2$.

2) If X is normal, the map $\operatorname{Pic} C \to \operatorname{Pic}(C - 0)$ is trivial, for it factors through $\operatorname{Pic} S = 1$.

3) For the cone over the smooth rational quartic curve

$$\{(s^4, s^3 t, s t^3, t^4)\} \subset \mathbb{P}^3((s,t) \in \mathbb{P}^1), \operatorname{Pic} C \cong U(k[\varepsilon])/U(k) \cong \mathbb{G}_a$$

where $\varepsilon = s^2 t^2 (\operatorname{mod} \mathfrak{c})$, $\varepsilon^2 = 0$.

4) The multiplicative group $\mathbb{G}_m$ acts on C and therefore on $K^0(C)$ and $K_0(C)$. Clearly, it acts trivially on $K_0(C)$ (from the proof of Proposition 2). However, $\mathbb{G}_m$ need not act trivially on $K^0(C)$, for in the above example the action of $\mathbb{G}_m$ on $\operatorname{Pic} C \cong \mathbb{G}_a$ is the standard action.

2. K^0-theory of Singular Surfaces

On a scheme with singularities one does not have an isomorphism $K^0 \to K_0$ (*n.b.* Proposition 1) to enable one to use the strong formal properties (exact sequences, etc.) of K_0 for calculations. We would like to show that some standard constructions are still possible by extending to singular surfaces some results of Murthy-Swan ([6]). We begin with a proposition in arbitrary dimension (appli-

cable, for example, to cones over smooth projective schemes), then specialize to surfaces and finally to cones over curves.

For any quasi-projective scheme V, to a closed subscheme W which does not meet the singular locus of V, one can associate an element $[W]\in K^0(V)$. (For any $\mathcal{O}_V$-module admitting a (finite) resolution by locally free sheaves, the element $\Sigma(-1)^i[\mathcal{E}_i]\in K^0(V)$ is independent of the resolution $\mathcal{E}_*$. In particular, for any coherent $\mathcal{O}_V$-module $\mathcal{F}$ with $\operatorname{Supp}\mathcal{F}$ not meeting the singular locus, we can consider $[\mathcal{F}]\in K^0(V)$; to define $[W]$, apply this to $\mathcal{O}_W$.) Let $Z'(V)$ be the group of algebraic cycles on V not meeting the singular locus. (The group of algebraic cycles is the free abelian group on the integral closed subschemes of V.) Thus we obtain a map $\varphi\colon Z'(V)\to K^0(V)$.

Proposition 4. *If V has isolated singularities, $\varphi\colon Z'(V)\to K^0(V)$ is surjective.*

Proof. Let S be the finitely many singular points, and let $\mathcal{L}$ be an ample line bundle on V. Take $[\mathcal{E}]\in K^0(V)$ where $\mathcal{E}$ is locally free of rank r. For n sufficiently large, we can take r global sections of $\mathcal{E}\otimes\mathcal{L}^{\otimes n}$ which are linearly independent over S. This gives a map $(\mathcal{L}^{v\otimes n})^r\to\mathcal{E}$ with kernel and cokernel supported off S. Likewise, we can get a section of $\mathcal{L}^{\otimes n}$ not vanishing on S so that $0\to\mathcal{L}^{v\otimes n}\to\mathcal{O}_V\to\mathcal{O}_W\to 0$ is exact. Now it remains to prove that for any coherent $\mathcal{O}_V$ module $\mathcal{F}$ with $\operatorname{Supp}\mathcal{F}\cap S=\emptyset$, $[\mathcal{F}]\in\operatorname{im}\varphi$. To do this we can use K_0, and the proof is standard. Q.E.D.

Corollary. *Suppose V is a quasi-projective surface with isolated singularities. Then if every nonsingular point of V is the complete intersection of locally principal (Cartier) divisors on V, $K^0(V)$ is generated by* $\operatorname{Pic}(V)$.

Proof. By the proposition it suffices to show that for any nonsingular point $p\in V$, $[p]\in K^0(V)$ is in the subgroup generated by $\operatorname{Pic}(V)$. Since p is assumed to be the intersection of locally principal divisors D, E on V, we have an exact sequence $0\to\mathcal{K}\to\mathcal{I}_D\oplus\mathcal{I}_E\to\mathfrak{m}_p\to 0$. By the following lemma $\mathcal{K}\cong\mathcal{I}_D\otimes\mathcal{I}_E$ is locally free (of rank 1), and hence $[\mathcal{O}_p]$ is in the subgroup generated by $\operatorname{Pic}(V)$. Q.E.D.

We include a variant of the technique of Serre for constructing rank-2 vector bundles from certain codimension-2 subschemes.

Lemma 1. *Let $\mathfrak{m}\subset\mathcal{O}_V$ be the ideal of a 0-dimensional local complete intersection (e.g., a nonsingular point) on a surface V. For any surjection $f\colon\mathcal{E}\to\mathfrak{m}$ where $\mathcal{E}$ is locally free of rank 2, $\ker f\cong\Lambda^2(\mathcal{E})$. If $\mathcal{K}$ is a line bundle on V with $H^2(\mathcal{K})=0$, there is an exact sequence $0\to\mathcal{K}\to\mathcal{E}\to\mathfrak{m}\to 0$ where $\mathcal{E}$ is locally free of rank 2.*

Proof. Given $f\colon\mathcal{E}\to\mathfrak{m}$, define $i\colon\Lambda^2(\mathcal{E})\to\mathcal{E}$ by $s\wedge t\mapsto f(s)t-f(t)s$. Now one checks locally (Koszul complex) that $i\colon\Lambda^2(\mathcal{E})\to\ker f$ is an isomorphism. Given $\mathcal{K}$ with $H^2(\mathcal{K})=0$, we have a surjection $\operatorname{Ext}^1(\mathfrak{m},\mathcal{K})\to H^0(\mathcal{E}xt^1(\mathfrak{m},\mathcal{K}))$ since the next term is $H^2(\mathcal{H}om(\mathfrak{m},\mathcal{K}))$ and $\mathcal{H}om(\mathfrak{m},\mathcal{K})\cong\mathcal{K}$. Now $\mathcal{E}xt^1(\mathfrak{m},\mathcal{K})$ is a free $\mathcal{O}_V/\mathfrak{m}$ module so that we can find an element $\xi\in\operatorname{Ext}^1(\mathfrak{m},\mathcal{K})$ which generates $\mathcal{E}xt^1(\mathfrak{m},\mathcal{K})$, and therefore ξ gives the desired exact sequence with $\mathcal{E}$ locally free. Q.E.D.

For applications, we give a stronger version of Proposition 4 for affine surfaces. Lemma 5 in Sect. 3 gives an example of how this can be useful.

Proposition 5. *Suppose V is an affine surface with isolated singularities and $Y \subset V$ is a divisor. Then $K^0(V)$ is generated by $[p]$ for nonsingular points $p \in V - Y$ together with* $\mathrm{Pic}(V)$.

Proof. It suffices by Proposition 4 to show that all nonsingular points are in the subgroup generated by $[p]$ for $[p] \in V - Y$ together with $\mathrm{Pic}(V)$. We may assume that Y contains the singular points. By Lemma 1, there exists an exact sequence $0 \to \mathcal{O}_V \to \mathcal{E} \to \mathfrak{m}_p \to 0$ where $\mathcal{E}$ is locally free of rank 2 and $\Lambda^2(\mathcal{E}) \cong \mathcal{O}$. Now $\mathcal{E}|_Y \cong \mathcal{O}_Y^2$ since Y is affine of dimension 1 and $\Lambda^2(\mathcal{E}|_Y) \cong \mathcal{O}_Y$. Hence we can get a map $\mathcal{O}_V^2 \to \mathcal{E}$ with kernel and cokernel supported off Y, and we can finish as in the proof of Proposition 4. (Here, in the affine case, any module supported off Y clearly has a filtration with the desired quotients.) Q.E.D.

There is a simple criterion for detecting an intersection of locally principal divisors.

Lemma 2. *Suppose V is an affine surface and δ is an effective Cartier divisor (e.g., a nonsingular point) on a curve $Z \subset V$. Then δ is the complete intersection of Z with an effective Cartier divisor Z' on V if and only if the class $[\delta] = \mathcal{O}_Z(\delta) \in \mathrm{Pic}\, Z$ extends to* $\mathrm{Pic}\, V$. *Further, Z' can be taken to be principal if and only if the class* $[\delta] = 1$.

Proof. If Z' exists, then $\mathcal{O}_V(Z')$ extends $\mathcal{O}_Z(\delta)$. Now, for the converse, suppose $\mathcal{O}_Z(-\delta) \subset \mathcal{O}_Z$ extends to a line bundle $\mathcal{L}$ on V. Let $\mathfrak{m} \subset \mathcal{O}_V$ be the ideal of δ as a subscheme of V and $\mathcal{I} \subset \mathfrak{m}$ the ideal of Z. There is some isomorphism $\mathcal{L}/\mathcal{I}\mathcal{L} \to \mathfrak{m}/\mathcal{I}$ which we lift to a (nonzero) homomorphism $f: \mathcal{L} \to \mathfrak{m}$. Then $\mathcal{I}' = f(\mathcal{L})$ defines the desired $Z' \subset V$. The last statement is obvious. Q.E.D.

Remark. On an arbitrary surface V, to get Z' one must add the assumption that a section of $\mathcal{O}_Z(\delta)$ defining δ extend.

Now we apply the cancellation theorem of Murthy-Swan for K^0 on affine surfaces (over an algebraically closed field).

Proposition 6. *Let V be an integral affine surface with isolated singularities. The following are equivalent:*

i) *Every vector bundle on V is isomorphic to $\mathcal{M} \oplus \mathcal{M}' \oplus \mathcal{O}_V^n$ for some $\mathcal{M}, \mathcal{M}' \in \mathrm{Pic}\, V$ and $n \geqq 0$.*

ii) *Every nonsingular point $p \in V$ is the complete intersection of locally principal divisors on V.*

iii) $\mathrm{Pic}\, V$ *generates $K^0(V)$.*

Proof. Assume i), and fix a nonsingular point $p \in V$. By Lemma 1 we can get a surjection $f: \mathcal{E} \to \mathfrak{m}_p$ where $\mathcal{E}$ is locally free of rank 2. By assumption, $\mathcal{E} \cong \mathcal{M} \oplus \mathcal{M}'$ for line bundles $\mathcal{M}, \mathcal{M}'$ on V. Now the images $f(\mathcal{M}), f(\mathcal{M}') \subset \mathfrak{m}_p$ give the locally principal ideals for divisors D, D' with intersection p, whence ii). That ii) implies iii) is covered by the corollary to Proposition 4. Finally, given iii), every vector bundle is stably isomorphic to a direct sum of line bundles and hence of the form i) by the cancellation theorem. Q.E.D.

An example (with no singularities) for which these conditions hold is the product of two smooth curves.

Corollary. *Let C be the cone over a smooth projective curve. The following are equivalent:*

i) *Every vector bundle on C is isomorphic to $\mathcal{M}\oplus\mathcal{O}_C^n$ for some $\mathcal{M}\in\operatorname{Pic}C$ and $n\geqq 0$ (and therefore trivial if C is normal).*

ii) *All points $p\neq 0$ on C are complete intersections in C.*

Proof. Assuming i), it only remains to notice that in the above proof if we assume $\Lambda^2(\mathcal{E})\cong\mathcal{O}$ by Lemma 1, we must have $\mathcal{E}\cong\mathcal{O}^2$; hence the divisors are principal. For ii) implies i), we use the isomorphism $\mathcal{M}\oplus\mathcal{M}'\cong\mathcal{O}\oplus(\mathcal{M}\otimes\mathcal{M}')$ from Proposition 3. Q.E.D.

Remarks. 1) It is enough to verify either condition in the normal case. For we can use on i) the Mayer-Vietoris sequence, as in the remark after Proposition 3, together with cancellation.

2) It suffices, by Proposition 5, to verify, instead of ii), that all points in any (nonempty) open set are complete intersections *in* C. Of course, one can remove a divisor from C so that the complement is isomorphic to the product of a smooth affine curve with $\mathbb{A}^1-0$, and on such a surface all points are complete intersections (by precisely the technique of i) implies ii)).

3. Cones over Curves

Let $C=\operatorname{Spec}S$ be the (normal) cone over an ample line bundle $\mathcal{L}$ on a smooth projective curve X, i.e., $S=\bigoplus_{d\geqq 0}\Gamma(X,\mathcal{L}^{\otimes d})$; $0\in C$ denotes the vertex and $\pi\colon C-0\to X$ the projection. We will describe a technique which enables us to show that all points $p\neq 0$ on C are complete intersections when the complete linear series on X cut out by $S_1=\Gamma(X,\mathcal{L})$ is sufficiently ample.

Take a point $p\neq 0$ on C. To show p is a complete intersection, it is necessary and sufficient, according to Lemma 2, to have a principal divisor Z on C nonsingular at p with $[p]=1$ in $\operatorname{Pic}Z$. One can test this in the following way: if $\bar{Z}$ is a complete model containing Z (e.g., consider $\bar{Z}$ on the projective cone $\bar{C}$) and nonsingular on the complement of Z, then $[p]=1$ in $\operatorname{Pic}Z$ if and only if $[p]=[\delta']$ in $\operatorname{Pic}\bar{Z}$ for some divisor δ' on $\bar{Z}$ supported on the complement of Z in $\bar{Z}$.

For any $z\in S_1$, let Z be the "hyperplane section of C through p": $z-z(p)=0$ on C. If $z(p)\neq 0$, note that the vertex $0\notin Z$ and $\pi\colon C-0\to X$ induces an isomorphism of Z with an open subset of X; the complement of $\pi(Z)$ in X is the support of the divisor δ: $z=0$ on X. Now $[p]=1$ in $\operatorname{Pic}Z$ if and only if $\pi(p)\equiv\delta'$ on X where δ' is some divisor with $\operatorname{Supp}\delta'\subseteq\operatorname{Supp}\delta$. Writing $P=\pi(p)\in X$ and $\mathcal{L}=\mathcal{O}_X(D)$ for some divisor D on X, we have proved:

Lemma 3. *To show that $p\neq 0$ on C is a complete intersection, it suffices to solve the following problem concerning the linear series $|D|$ on X: $(*)$ Given $P\in X$, find $\delta\in|D|$ such that $P\equiv\delta'$ where $\operatorname{Supp}\delta'\subseteq\operatorname{Supp}\delta$ and $P\notin\operatorname{Supp}\delta$.*

Moreover, if this can be done, one of the equations used to cut out $p\in C$ can be taken to be linear (nonhomogeneous).

Lemma 4. *If X has genus g and D has degree $d\geqq 3g+1$, then $(*)$ can be solved for any $P\in X$.*

Proof. First we take some $Q\neq P\in X$ such that $|P+gQ|$ is base point free, so we get $P+gQ\equiv E$ (effective) where $P\notin\operatorname{Supp}E$. We let $\delta'=E-gQ$. Now it suffices to

have $|D-E-Q|$ base point free, for then we can take $\delta'' \in |D-E-Q|$, $P \notin \operatorname{Supp} \delta''$ so that $\delta = \delta'' + E + Q$ has the desired properties. If $\deg(D-E-Q) \geqq 2g$, i.e., $\deg D \geqq 3g+2$, $|D-E-Q|$ has no base points. To treat the case of degree $3g+1$, note that we can find E containing some point with multiplicity >1, and by removing the multiplicity before considering $|D-E-Q|$, we can continue as before. Q.E.D.

Remark. It is possible to improve the restriction on the degree somewhat; however, we have not been able to give an argument valid for all divisors of degree $d \geqq 2g+1$, at which point the divisor is necessarily very ample and gives a projectively normal embedding of X (Mumford).

Lemma 5. *If X is hyperelliptic and* $\deg D \geqq 2g+1$, *then* $(*)$ *can be solved for any non-Weierstrass point $P \in X$.*

Proof. If P is not a Weierstrass point, then there is $Q \neq P$ such that $|P+Q|$ is the g^1_2. Now we can take a Weierstrass point $R \in X$ such that $P+Q \equiv 2R$, and let $\delta' = 2R - Q$. Since $|D-Q-R|$ has degree $\geqq 2g-1$, it does not have any base points unless we have equality. When $\deg(D-Q-R) = 2g-1$, P is a base point exactly when $D-Q-R-P \equiv K$ the canonical divisor; by changing R we can assume this does not happen. Then we take $\delta'' \in |D-Q-R|$ with $P \notin \delta''$ and set $\delta = \delta'' + Q + R$. Q.E.D.

Putting together these lemmas with the corollary to Proposition 6, we have obtained the following:

Theorem. *Let C be the cone over an ample line bundle $\mathscr{L}$ on a smooth projective curve X of genus g over an algebraically closed field. If* $\deg \mathscr{L} \geqq 3g+1$, *then all vector bundles on C are trivial. The same holds if X is hyperelliptic and* $\deg \mathscr{L} \geqq 2g+1$.

Remarks. 1) For the obvious omitted case of a line bundle of degree 3 on an elliptic curve, $(*)$ can be solved for any P as in the hyperelliptic case. Thus, all vector bundles are trivial on the cone over any very ample line bundle on a curve of genus $g \leqq 2$.

2) By the same techniques one can consider set-theoretic complete intersections on C and the corresponding implications for K^0.

3) Clearly our method of using elements of S_1 is very special, for one could have $S_1 = 0$ for an ample line bundle. Even for the smooth plane curves of degree $\geqq 4$, it does not seem to work. We hope to develop some additional techniques to handle the arbitrary case.

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Received March 15, 1979/Revised July 2, 1980