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Autor: Looijenga, Eduard

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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Invariant Theory for Generalized Root Systems

Eduard Looijenga*

Mathematisch Instituut, Katholieke Universiteit, Toernooiveld, 6525ED Nijmegen, The Netherlands

Around 1967, Kac [5] and independently Moody [11] introduced a new class of Lie algebras including the finite dimensional semi-simple Lie algebras. Their point of departure was Serre's description of a semi-simple Lie algebra in terms of generators and relations by its Cartan matrix and the idea was to carry out the same construction with a so-called *generalized Cartan matrix*: this is an $l \times l$ matrix $(n_{\lambda\mu})$ with 2's on the diagonal and nonpositive integers outside the diagonal such that $n_{\lambda\mu}=0$ if and only if $n_{\mu\lambda}=0$. The first step is to introduce a (generalized) Weyl group, which goes as follows. Let V be a finite dimensional real vectorspace and let $B=\{\alpha_1,\ldots,\alpha_l\}\subset V$ resp. $\{\alpha_1^\vee,\ldots,\alpha_l^\vee\}\subset V^*$ be linearly independent vectors such that $\langle\alpha_\lambda,\alpha_\mu^\vee\rangle=n_{\lambda\mu}$. For each $\alpha_\lambda\in B$ we have a *fundamental reflection* s_{α_λ} in V defined by $s_{\alpha_\lambda}(x)=x-\langle x,\alpha_\lambda^\vee\rangle\alpha_\lambda$. If we denote by W the subgroup of $\mathrm{Aut}(V)$ generated by these reflections, then it can be shown that the pair $(W,\{s_{\alpha_1},\ldots,s_{\alpha_l}\})$ is a Coxeter system. As in the classical case, there is a *fundamental chamber* C, being defined as the set of $x\in V$ with $\langle x,\alpha_\lambda^\vee\rangle>0$ for $\lambda=1,\ldots,l$, such that each W-orbit meets $\bar{C}$ in at most one point. So $\bar{C}$ is a fundamental domain for the action of W on the W-orbit I of $\bar{C}$. It can be proved that I is a convex cone such that W acts properly discontinuously on its interior $\mathring{I}$. In this paper we will not occupy ourselves with the Lie algebra Kac and Moody associate to this situation. Our interest merely concerns the group W acting on $\mathring{I}$ and a complexification of this, which we proceed to describe.

First we note that W leaves invariant the (root) lattice Q in V spanned by $\alpha_1,\ldots,\alpha_l$. If we extend W by the translations in Q we obtain an affine transformation group $\tilde{W}$ which is a semi-direct product of W and Q. Clearly $\tilde{W}$ leaves invariant the domain $\Omega:=\{x+iy\in V_{\mathbf{C}}: y\in\mathring{I}\}$ and the action of $\tilde{W}$ on Ω is properly discontinuous. Therefore, the orbit space $\Omega/\tilde{W}$ inherits the structure of a normal complex-analytic space. It is not difficult to prove that the $\tilde{W}$-stabilizer of any point in Ω is generated by reflections. A theorem of Chevalley then implies that $\Omega/\tilde{W}$ is nonsingular (as an analytic space). The purpose of this paper is to describe a partial compactification of this orbit space. To some extent our

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construction imitates the one of Satake and Baily-Borel: we add to Ω lower-dimensional analytic manifolds (which are a substitute for the rational boundary components in the Baily-Borel theory) such that the $\tilde{W}$-action extends. The resulting space, denoted by $\hat{\Omega}$, maps naturally to the Coxeter complex of W. In general, $\hat{\Omega}$ is not locally compact, but the orbit space $\hat{\Omega}/\tilde{W}$ appears to be in a natural way an analytic Stein manifold. It also turns out that $\hat{\Omega}/\tilde{W}$ is the holomorphic hull of $\Omega/\tilde{W}$. An important feature of this partial compactification is the smoothness of $\hat{\Omega}/\tilde{W}$ and the fact that the locus at infinity, $\hat{\Omega}/\tilde{W}-\Omega/\tilde{W}$, comes equipped with a stratification whose strata arise in a similar way as $\Omega/\tilde{W}$ (they are smooth too).

If the Cartan matrix is of affine type (see (1.15) for the definition), this partial compactification is essentially the one described in [8]. Like this article, the present paper has its origin in our investigations of the deformation theory of certain minimally elliptic singularities. Applications in this area are briefly described in my Helsinki talk [9], in which lecture also a weaker form of our main result (3.1) is announced. (For this purpose, we will compute the fundamental group of the *regular* $\tilde{W}$-orbit space of Ω in a subsequent paper.) It turns out that this construction is also an ingredient of Shah's compactification of the coarse moduli space of polarized $K3$ surfaces of degree two.

In case the Weyl group W leaves invariant a nondegenerate quadratic form of hyperbolic type, the existence of a partial compactification of the type here discussed, was independently conjectured by Mumford (in a letter to Brieskorn, June 1978). Also, it was Mumford who pointed out (at an earlier occasion) that the main result of [8] could be understood as describing a partial compactification.

A substantial part of these results were obtained while the author was a guest of the IHES during the month of January, 1979. He is grateful to this institution for its hospitality and financial support.

List of Notations

Only notation occurring in more than one subsection is listed. For each symbol we give the subsection where it is introduced.

$\alpha(\alpha^\vee)$, (dual) root: (1.6)
$A=\bigoplus_{k=0}^{\infty} A_k$, dual weight algebra: (4.1)
$A^W(A^{-W})$, W-(anti-)invariants of A: (4.1)
$B(B^\vee)$, (dual) root basis: (1, 1)
B_0, maximal special subset: (2.1)
$C(C^\vee)$, (dual) fundamental chamber: (1.1), (1.4)
$C_X(C_X^\vee)$, (dual) fundamental chamber defined by $X\subset B$: (1.12)
$\hat{F}$, facet of $\hat{I}$ defined by a facet F of I: (2.10)
$F_X(F_X^\vee)$, (dual) fundamental facet defined by $X\subset B$: (1.1)
Φ_x, set of positive roots whose duals are negative on x: (1.10)
$\mathscr{H}(-)$, convex hull of $-$: (2.4)
$\mathscr{H}_+$, Poincaré's upper half plane
$I(I^\vee)$, (dual) Tits cone: (1.1)

$I_X(I_X^\vee)$, (dual) Tits cone defined by $X \subset B$: (1.12)
$I(X)$, projection of I in V/V_X: (2.7)
$\hat{I}$, partial compactification of I: (2.9)
in$(-)$, initial term of an element of A: (4.1)
J_p: (4.1)
M, $\tilde{W}$-orbit space of Ω: (2.18)
$M(X)$, $\tilde{W}_X$-orbit space of $\Omega(X)$: (2.18)
$N(N^\vee)$, $\mathbf{Z}$-module spanned by $n(n^\vee)$: (5.1)
$n(n^\vee)$: (4.1)
π_X, canonical projection $V \to V/V_X$: (2.8)
$P(P^\vee)$, semi-group of (dual) weights: (1.20)
$P_+(P_+^\vee)$, set of positive (dual) weights: (1.20)
$P_{++}(P_{++}^\vee)$, set of strictly positive (dual) weights: (1.20)
$Q(Q^\vee)$, (dual) root lattice: (1.6)
Q_X, lattice spanned by $X \subset B$: (2.17)
$Q_{V'} = Q \cap V'$: (2.18)
$Q(X) = Q/Q_X$: (2.16)
$R(R^\vee)$, set of (dual) roots: (1.6)
$R_\pm(R_\pm^\vee)$, set of (dual) positive resp. negative roots: (1.6)
S_p (4.1)
$\mathscr{S}_p$ (3.3)
$St(-)$, star of a facet $-$: (1.14)
*, as in X^*, Y^*: (2.1)
V^X, subspace of V orthogonal to $X^\vee$: (1.12)
V_X, subspace of V spanned by X: (2.1)
W, Weyl group of a root basis: (1.1)
$\tilde{W}$, extended Weyl group: (2.6)
W_X: (1.1)
$\tilde{W}_X = W_{X^*} \cdot Q(X)$: (2.16)
Ω, $\Omega(X)$, $\hat{\Omega}$: (2.18)

We write resp. $\mathbf{N}$, $\mathbf{Z}_+$, $\mathbf{Z}_-$ for resp. the set of positive, nonnegative, nonpositive integers. Similarly, $\mathbf{R}_+$ resp. $\mathbf{R}_-$ stands for the set of nonnegative resp. nonpositive real numbers. If S is a subset of a vector space V, then $\mathbf{Z}_+ S$ denotes the set of $x \in V$ of the form $x = \sum_{s \in S} x_s s$ with $s \in S$, $x_s \in \mathbf{Z}_+$ and $x_s = 0$ for all but a finite number of $s \in S$. We use a similar convention for $\mathbf{Z}_- \cdot S$, $\mathbf{R}_+ \cdot S$, etc.

1. Root Bases and Related Notions

In this section we examine the action of W on I. Among other things we find that virtually all the results of Chap. V, §4 of Bourbaki [4] remain valid in this situation. In particular, it is established that W is a Coxeter group, a result which was first proved by Kac (another proof, using the associated Lie algebra, was given by Moody and Teo [13] who attribute it to Solomon and Verma).

(1.1) Let V be a real vector space of finite dimension. A subset B of V together with an injection $^\vee: B \to V^*$, $\alpha \mapsto \alpha^\vee$, is called a *root basis* if the following conditions are satisfied:

(i) B (resp. $B^\vee$) is a linearly independent subset of V (resp. V^*),

(ii) $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in B$,

(iii) $\langle \alpha, \beta^\vee \rangle$ is a nonpositive integer for any distinct pair $\alpha, \beta \in B$,

(iv) $\langle \alpha, \beta^\vee \rangle = 0$ implies $\langle \beta, \alpha^\vee \rangle = 0$.

Note that we do not exclude the possibility that $B = \emptyset$. If $B \neq \emptyset$, we find it often convenient to order the elements of B: $B = \{\alpha_1, \ldots, \alpha_l\}$ with $l = |B|$. If it is clear what $^\vee$ should be, we omit it and simply say that (V, B) is a root basis. With this convention, (V, X) is a root basis for any $X \subset B$. Also, $(V^*, B^\vee)$ is a root basis, called the *dual* of (V, B).

Two root bases (V_1, B_1) and (V_2, B_2) are said to be isomorphic if there exists a linear isomorphism $f: V_1 \to V_2$ such that $f(B_1) = f(B_2)$ and $f^*(B_2^\vee) = B_1^\vee$. A simple exercise in linear algebra shows that a root basis (V, B) is determined up to isomorphism by its *Cartan matrix* $(\langle \alpha, \beta^\vee \rangle : \alpha, \beta \in B)$ and $\dim V$. Conversely, for any $l \times l$ matrix $(n_{\lambda\mu})$ satisfying $n_{\lambda\lambda} = 2$, $n_{\lambda\mu} \in \mathbf{Z}_-$ for $\lambda \neq \mu$ and $n_{\lambda\mu} = 0$ iff $n_{\mu\lambda} = 0$ and any real vector space V of finite dimension $\geqq l + \operatorname{corank}(n_{\lambda\mu})$ there exists a root basis $B = \{\alpha_1, \ldots, \alpha_l\}$ in V with $\langle \alpha_\lambda, \alpha_\mu^\vee \rangle = n_{\lambda\mu}$.

Given two root bases (V_1, B_1) and (V_2, B_2), then the direct sum $(V_1 \times V_2, B_1 \times \{0\} \cup \{0\} \times B_2)$ is also a root basis. A root basis isomorphic to a direct sum whose summands have positive dimension is called *reducible*, otherwise it is said to be *irreducible*. Note that for an irreducible root basis (V, B) we always have that $\dim V - |B|$ equals the corank of the matrix $(\langle \alpha, \beta^\vee \rangle)_{\alpha, \beta \in B}$. The cardinal of B is called the *rank* of (V, B). We sometimes call an element of $B(B^\vee)$ a *fundamental (dual) root*.

Any $\alpha \in B$ determines a *fundamental reflection* s_α in V, defined by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$. (It is a reflection since $s_\alpha(\alpha) = -\alpha$ and s_α leaves the hyperplane orthogonal to $\alpha^\vee$ pointwise fixed.) The fundamental reflections generate a subgroup W of $\operatorname{Aut}(V)$, called the *Weyl group* of (V, B). We let W act on V^* in the contragradient way: $\langle x, w(\xi) \rangle := \langle w^{-1}(x), \xi \rangle$. The *fundamental chamber* $C \subset V$ is the cone defined by the inequalities $\alpha^\vee > 0$ $(\alpha \in B)$. Each subset $X \subset B$ defines a *fundamental face* $F_X = \{x \in V : \langle x, \alpha^\vee \rangle = 0$ for all $\alpha \in X$, $\langle x, \alpha^\vee \rangle > 0$ for all $\alpha \in B - X\}$. Note the two extremes: $F_\emptyset = C$ and F_B is a linear subspace of V. We further observe that the subgroup W_X of W generated by the $s_\alpha (\alpha \in X)$ leaves F_X pointwise fixed.

We call the W-orbit of the closure $\bar{C}$ of C, the *Tits cone* of (V, B) and denote it by I. We shall see that $\bar{C}$ is a fundamental domain for the action of W on I.

(1.2) By way of example, and also because we need it later, we discuss the rank two case in some detail. Let $B = \{\alpha, \beta\}$ be a root basis in V. Then $V' := V/F_B$ is a two-dimensional vector space and C (resp. I) is the inverse image of a quadrant C' (resp. a cone I') in V'. Also, s_α and s_β induce reflections s'_α and s'_β in V'. We put $u := -\langle \alpha, \beta^\vee \rangle$ and $v := -\langle \beta, \alpha^\vee \rangle$. In case $u = v = 0$, (V, B) is reducible. Therefore we assume that $u > 0$ (and hence $v > 0$). Consider the quadratic function q on V' defined by

$$q(x) = u\alpha^\vee(x)^2 + uv\alpha^\vee(x)\beta^\vee(x) + v\beta^\vee(x)^2.$$

It is easily verified that s'_α and s'_β leave q invariant. The discriminant d of q equals $uv(uv-4)$ and has therefore the same sign as $uv-4$. If $0<uv<4$, q is positive definite and so W is finite; the three possibilities $\{u,v\}=\{1,1\}$, $\{1,2\}$ and $\{1,3\}$ yield root bases of type resp. A_2, B_2 and G_2 and $s_\alpha s_\beta$ is of order resp. 3, 4 and 6. So in these cases W is a Weyl group in the usual sense and C and C' are chambers of W.

When $uv\geqq 4$, the set of $x\in V'$ where q is positive has two connected components, one of which contains $\bar{C}'-\{0\}$. This component (resp. $\bar{C}'$) determines an open interval J (resp. closed interval D) of the projective line $\mathbf{P}(V')$ associated to V'.

In case $uv=4$ (so $\{u,v\}=\{2,2\}$ or $\{1,4\}$), J is an affine line and the group W acts on J as an affine transformation group: s'_α and s'_β leave J invariant and induce the unique affine involutions of J having a fixed point on the boundary of D. It follows that D is a fundamental domain for the action of W on J, cf. [4], Chap. V, §2.4. This implies that I' is the cone on J and that $\bar{C}'$ (resp. $\bar{C}$) is a fundamental domain for the action of W on I' (resp. I).

When $uv>4$, then q is nondegenerate and indefinite and hence $\bar{J}$ is a proper interval of $\mathbf{P}(V')$. If t is an affine co-ordinate for $\mathbf{P}(V')$ such that $J=(0,\infty)$, then $\log(t)$ gives J the structure of an affine line. With respect to this affine structure, s_α and s_β induce the unique affine involutions of J having a fixed point in the boundary of D and so we arrive at the same conclusions as in the case $uv=4$.

Note that in the last two cases, $s_\alpha s_\beta$ is of infinite order.

Let (V,B) be a root basis. The following three results are basic for what will follow.

(1.3) **Theorem.** *The fundamental chamber $\bar{C}$ is a fundamental domain for the action of W on I in the following strong sense: if $w\in W$ and $X\subset B$ are such that $w(F_X)\cap\bar{C}\neq\emptyset$, then $w\in W_X$ and (hence) w leaves F_X pointwise fixed.*

(1.4) **Proposition.** *The pair $(W,\{s_\alpha\}_{\alpha\in B})$ is a Coxeter system. More precisely, the defining relations for W are $s_\alpha^2=1$ and if $\langle\alpha,\beta^\vee\rangle\langle\beta,\alpha^\vee\rangle=0,1,2,3$, then $s_\alpha s_\beta$ is of order 2, 3, 4, 6 respectively.*

For any $w\in W$, let $l(w)$ denote the length of w, that is, the smallest number of symbols necessary to write w as a word in the fundamental reflections.

(1.5) **Proposition.** *For $\alpha\in B$ and $w\in W$ we have $l(s_\alpha w)=l(w)+1$ if and only if $w(C)$ is contained in the half space defined by $\alpha^\vee>0$.*

That the fundamental reflections satisfy the relations mentioned in Prop. (1.4) was already clear from our separate discussion of the rank two case. The proofs of the remaining statements can be found in Bourbaki [4], Chap. V, §4.4, 5, 6. Although one is dealing there with a slightly different situation the proofs go through *verbatim*, with the sole exception of Lemma 1, §4.4, which can be replaced by the discussion (1.2).

(1.6) A subset F of I is called a *facet* if it is a W-translate of a fundamental facet. By (1.3), the facets form a partition of I. It also follows from (1.3) that for any facet F there exists a unique $X\subset B$ and a unique left coset $wW_X\subset W$ such that $F=w\cdot F_X$. We then say that F is *of type* X. The facets of type $\emptyset$ are also

called *chambers*. Notice that there is only one facet of type B (namely F_B). If F and F' are facets corresponding to left cosets wW_X resp. $w'W_{X'}$, then $F' \subset \overline{F}$ iff $wW_X \subset w'W_{X'}$. The dual root basis $(V^*, B^\vee)$ gives rise to a similar situation in V^*: a *fundamental dual chamber* $C^\vee$, a dual Tits cone $I^\vee$ etc. Since the facets of I and $I^\vee$ are labeled by the same left cosets, there is a bijective correspondence between the facets of I and those of $I^\vee$. This correspondence is W-equivariant, and preserves the incidence relations, but need not be induced by a linear isomorphism $V \to V^*$.

An element of $R := W \cdot B$ is called a *root*. It follows from (1.5) that any root α is either positive or negative on $C^\vee$. This sign determines a partition of R into *positive* (R_+) and *negative* (R_-) *roots*. The *root lattice* $Q := \mathbf{Z} \cdot B$ is invariant under W and so contains R. It follows that any positive root is in $\mathbf{Z}_+ \cdot B$. We could reformulate (1.5) by saying that a root $w(\alpha)$ ($\alpha \in B$, $w \in W$) is positive iff $l(ws_\alpha) = l(w) + 1$ (pass to w^{-1} and observe that $l(w^{-1}) = l(w)$).

Similarly, we have a set of *dual roots* $R^\vee := W \cdot B^\vee$ and a partition of $R^\vee$ into positive ($R^\vee_+$) and negative ($R^\vee_-$) dual roots.

(1.7) **Proposition.** *There is a unique W-equivariant bijection $R \to R^\vee$ which extends $^\vee : B \to B^\vee$. This bijection maps R_+ onto $R^\vee_+$.*

At the same time we prove

(1.8) **Proposition.** *If two positive roots are proportional, then they are equal.*

Proof. Suppose that $\alpha, \beta \in B$ and $w \in W$ are such that $w(\alpha)$ is a positive multiple of β. We prove that then $w(\alpha) = \beta$ and $w(\alpha^\vee) = \beta^\vee$. This will settle (1.8) and the first part of (1.7). Since α is indivisible in Q, so is $w(\alpha)$. Hence $w(\alpha) = \beta$. Now consider the element $s_\beta w s_\alpha w^{-1} \in W$. It leaves the fundamental dual facet $F^\vee_{\{\alpha\}}$ pointwise fixed and hence is by (1.3) either equal to $\mathbf{1}$ or s_α. Comparing determinants, we conclude that $s_\beta w s_\alpha w^{-1} = \mathbf{1}$. Since $ws_\alpha w^{-1}(x) = x - \langle x, w(\alpha^\vee)\rangle w(\alpha)$ and $s_\beta(x) = x - \langle x, \beta^\vee\rangle \beta$, it follows that $w(\alpha^\vee) = \beta^\vee$.

We observed in (1.6) that a root $w(\alpha)$ ($w \in W$, $\alpha \in B$) is positive iff $l(ws_\alpha) = l(w) + 1$. Combining this criterion with its dual form we find that $w(\alpha)$ is positive iff $w(\alpha^\vee)$ is.

(1.9) Note that if $\alpha \in R$, then s_α, $s_\alpha(x) := x - \langle x, \alpha^\vee\rangle \alpha$, is a reflection in W (this follows from the W-equivariance of the bijection $R \to R^\vee$).

(1.10) Let $\alpha \in B$. It follows from (1.8) that $R_+ - \{\alpha\}$ is just the set of roots which are positive on the fundamental dual facet $F^\vee_{\{\alpha\}}$. Since s_α leaves $F^\vee_{\{\alpha\}}$ invariant, this implies that s_α maps $R_+ - \{\alpha\}$ onto itself. So α is the only positive root which changes sign when applying s_α to it.

If $x \in V$, we let Φ_x denote the set of $\alpha \in R_+$ with $\langle x, \alpha^\vee\rangle < 0$.

(1.11) **Proposition.** *The set Φ_x is finite if and only if $x \in I$. In that case $|\Phi_x| = \min\{l(w)\colon w \in W$ and $x \in w(\overline{C})\}$ and if $x_0 \in \overline{C}$ is such that $x \in W \cdot \{x_0\}$, then $x_0 - x \in \mathbf{R}_+ \cdot B$.*

Proof. Let $x \in V$ be such that Φ_x is finite. We prove our assertions with induction on $|\Phi_x|$. If $\Phi_x = \emptyset$, then clearly $x \in \overline{C}$ and everything is obvious. So assume that $\Phi_x \neq \emptyset$. Then Φ_x contains a fundamental root $\alpha \in B$. Put $x' := s_\alpha(x)$. We show that

Φ_x is the disjoint union of $s_\alpha(\Phi_{x'})$ and $\{\alpha\}$. Since $\langle x', \alpha^\vee\rangle = \langle s_\alpha(x'), s_\alpha(\alpha^\vee)\rangle = -\langle x, \alpha^\vee\rangle > 0$ we have $\alpha \notin \Phi_{x'}$. If $\beta \in R_+ - \{\alpha\}$, then $s_\alpha(\beta) \in R_+ - \{\alpha\}$ by (1.10) and so $\langle x, s_\alpha(\beta^\vee)\rangle = \langle x', \beta^\vee\rangle$ implies that $\beta \in \Phi_{x'}$ iff $s_\alpha(\beta) \in \Phi_x$. Hence $\Phi_x = s_\alpha(\Phi_{x'}) \cup \{\alpha\}$ (disjoint union) and in particular $|\Phi_x| = |\Phi_{x'}| + 1$. By our induction hypothesis there exist $x_0 \in \bar{C}$ and $w' \in W$ such that $x' = w'(x_0)$, $l(w') = |\Phi_{x'}|$ and $x_0 - x' \in \mathbf{R}_+ \cdot B$. Clearly, $s_\alpha w'(x') = x$. Since $\langle x', \alpha^\vee\rangle > 0$, $\alpha^\vee$ is positive on $w'(C)$ and so $l(s_\alpha w') = l(w') + 1 = |\Phi_x|$, by (1.5). If $w \in W$ is such that $x \in w(\bar{C})$, then $x' \in s_\alpha w(\bar{C})$ and so $l(s_\alpha w) \geqq |\Phi_{x'}|$. Since $\alpha^\vee$ is positive on $s_\alpha w(C)$ we have $l(w) = l(s_\alpha s_\alpha w) = l(s_\alpha w) + 1$ and hence $l(w) \geqq |\Phi_{x'}| + 1 = |\Phi_x|$. This proves that $|\Phi_x| = \min\{l(w)\colon w \in W \text{ and } x \in w(\bar{C})\}$. As $x' - x = -\langle x, \alpha^\vee\rangle > 0$, it also follows that $x_0 - x = (x_0 - x') + (x' - x) \in \mathbf{R}_+ \cdot B$.

If $x \in w(\bar{C})$ $(w \in W)$, then the same proof shows that $|\Phi_x| \leqq l(w)$ and so Φ_x is finite.

(1.12) **Corollary.** *The Tits cone I is convex and we have $I = V$ if and only if W is finite.*

Proof (Compare [4], Chap. V, §4 exerc. 2). Given $x, y \in I$, we want to show that the interval $[x, y]$ is contained in I. For this purpose we may suppose that $x \in \bar{C}$. As for any $z \in [x, y]$, we have $\Phi_z \subset \Phi_y$ it follows from (1.11) that $[x, y] \subset I$.

Now assume that $I = V$. Then for any $x \in -C$ we have $\Phi_x = R_+$ and so R_+ is finite by (1.11). It also follows from (1.11) that $l(w) \leqq |R_+|$ for any $w \in W$. Hence W is finite.

Conversely, if W is finite, then $R = W \cdot B$ is finite and hence Φ_x is finite for any $x \in V$. It follows from (1.11) that then $I = V$.

As observed at the beginning, any subset X of B defines a root basis (V, X). We denote its fundamental chamber, resp. Tits cone by C_X, resp. I_X. We further let V^X denote the subspace of V which is orthogonal to $X^\vee$.

(1.13) **Lemma.** *We have $C_X = C + V^X$ and $I_X = I + V^X$.*

Proof. The first equality is obvious. It implies that

$$I_X = W_X \cdot \bar{C}_X = W_X(\bar{C} + V^X) = (W_X \cdot \bar{C}) + V^X \subset I + V^X.$$

For any $x \in V$, the set Φ_x (relative X) is a subset of Φ_x (relative B). Together with the criterion of (1.11) this implies that $I \subset I_X$.

(1.14) **Corollary.** *The topological interior $\mathring{I}$ of I is the union of facets with a finite stabilizer. In particular, W acts properly discontinuously on $\mathring{I}$.*

Proof. We must show that $F_X \subset \mathring{I}$ iff W_X is finite. If W_X is finite, then $I_X = V$ by (1.12) and hence $st(F_X)$ (the *star* of F_X, that is, the union of faces of I containing F_X in their closure) is open in V, thus implying $F_X \subset \mathring{I}$. If W_X is infinite, then it follows from (1.12) that I_X is contained in a half space whose boundary contains F_X. As $I \subset I_X$ by (1.13), this implies that $F_X \subset \partial I$.

(1.15) We say that a root basis (V, B) is of *affine type* if (i) it is irreducible, (ii) the Cartan matrix $(\langle \alpha, \beta^\vee\rangle)_{\alpha,\beta \in B}$ is of corank one and (iii) W_X is finite for any proper subset $X \subset B$. By an observation made in (1.1) we then have $\dim V = |B| + 1$. The classification of such root bases (which is fairly simple if one knows the

classification of the finite root systems) can be found in Kac [5] and Moody [11]. Note that this definition is self dual: (V,B) is of affine type iff $(V^*,B^\vee)$ is.

(1.16) **Proposition.** *Let (V,B) be an irreducible root basis such that $(-C)\cap(\mathbf{R}_+\cdot B)=\emptyset$. Then W is finite (and $F_B=\{0\}$) or (V,B) is of affine type (and $\dim F_B=1$). In both cases there exists a positive W-invariant quadratic form $(|)$ on V_B $(:=\mathbf{R}\cdot B)$ with kernel F_B such that every root has positive length.*

Proof. Since $(-C)\cap(\mathbf{R}_+\cdot B)=\emptyset$, there exists a $\xi\in V^*$ which is $\geqq 0$ on B and negative on $-C$. Then ξ must be of the form $\xi=\sum_{\alpha\in B}\xi_\alpha\alpha^\vee$ with $\xi_\alpha\geqq 0$, but not all $\xi_\alpha=0$, and $\xi\in\bar{C}^\vee$. We divide the proof into five steps.

Step 1. We have $\xi_\alpha>0$ for all $\alpha\in B$.

Proof. Put $X=\{\alpha\in B:\ \xi_\alpha>0\}$. If $\beta\in B-X$, then $0\leqq\langle\beta,\xi\rangle-\sum_{\alpha\in X}\langle\alpha,\beta^\vee\rangle\xi_\alpha\leqq 0$, which implies that $\langle\alpha,\beta^\vee\rangle=0$ for all $\alpha\in X$. Since $X\neq\emptyset$ and (V,B) is irreducible, it follows that $X=B$.

In what follows, we use the *Dynkin diagram* of (V,B). This is a graph on the set B, where α and β are connected by a $\langle\alpha,\beta^\vee\rangle\langle\beta,\alpha^\vee\rangle$-fold edge (so no edge at all if $\langle\alpha,\beta^\vee\rangle=0$) (usually one also directs the edge from α to β if $\langle\alpha,\beta^\vee\rangle<\langle\beta,\alpha^\vee\rangle$). Clearly an irreducible component of (V,B) corresponds to a connected component of its Dynkin diagram.

Step 2. The Dynkin diagram is either a tree (possibly with multiple edges), or a cycle without multiple edges.

Proof. Suppose the Dynkin diagram contains a cycle of period $k\geqq 3$, i.e. suppose that we have an injection $\mathbf{Z}/k\to B$, $i\mapsto\alpha_i$, such that $\langle\alpha_i,\alpha_j^\vee\rangle\neq\emptyset$ iff $j-i\in\{0,\pm 1\}$. We must show that then $\mathbf{Z}/k\to B$ is surjective and that $\langle\alpha_i,\alpha_{i\pm 1}^\vee\rangle=-1$ for all $i\in\mathbf{Z}/k$.

We have $0\leqq\langle\alpha_i,\xi\rangle\leqq 2\xi_{\alpha_i}-\xi_{\alpha_{i-1}}-\xi_{\alpha_{i+1}}$ and so $2\xi_{\alpha_i}\leqq\xi_{\alpha_{i-1}}+\xi_{\alpha_{i+1}}$ for all $i\in\mathbf{Z}/k$. None of these inequalities can be strict since summation over i yields an equality. As $\xi_\alpha>0$ for all α, this can only happen if $\langle\alpha_i,\alpha_{i\pm 1}^\vee\rangle=-1$ for all i and $\langle\alpha_i,\alpha^\vee\rangle=0$ for all $\alpha\in B$ distinct from α_i and $\alpha_{i\pm 1}$. Since (V,B) is irreducible, this implies that $\mathbf{Z}/k\to B$ is surjective.

Step 3. There exists a W-invariant bilinear form $(|)$ on V_B $(:=\mathbf{R}\cdot B)$ such that $(\alpha|\alpha)>0$ and $(\alpha|x)=\frac{1}{2}(\alpha|\alpha)\langle x,\alpha^\vee\rangle$ for all $\alpha\in B$.

Proof. In case the Dynkin diagram is a cycle without multiple edges we have $\langle\alpha,\beta^\vee\rangle=\langle\beta,\alpha^\vee\rangle$ for all $\alpha,\beta\in B$ and so the form defined by $(\alpha|\beta):=\langle\alpha,\beta^\vee\rangle$ is as required. If not, then the Dynkin diagram is a tree by step 2. In this case, we choose $\alpha\in B$ representing an end point of this tree. This means that $\langle\alpha,\beta^\vee\rangle\neq 0$ for precisely one $\beta\in B-\{\alpha\}$. If we assume inductively that $(|)$ has been defined on $V_{B-\{\alpha\}}$, then we extend $(|)$ to V_B by putting

$$(\alpha|\alpha)=\frac{\langle\alpha,\beta^\vee\rangle(\beta|\beta)}{\langle\beta,\alpha^\vee\rangle}\quad\text{and}\quad(\alpha|\gamma)=\tfrac{1}{2}(\gamma|\gamma)\langle\alpha,\gamma^\vee\rangle.$$

Then $(|)$ is invariant under $W_{B-\{\alpha\}}$ and s_α and hence under W.

Step 4. The form $(|)$ is positive and has kernel F_B. Either $F_B=\{0\}$ or F_B is of dimension one and spanned by an element $n=\sum_{\alpha\in B} n_\alpha\alpha$ with all $n_\alpha>0$.

Proof. In order to show that $(|)$ is positive, we restrict the function $x\mapsto(x|x)$ to the unit sphere in V_B (relative the basis B) and compute its minimum. Since $(\sum x_\alpha\alpha|\sum x_\alpha\alpha)\geqq(\sum|x_\alpha|\alpha|\sum|x_\alpha|\alpha)$ (we use that $(\alpha|\beta)\leqq 0$ if $\alpha\neq\beta$), the minimum is assumed in a point $x=\sum_{\alpha\in B} x_\alpha\alpha$ with $x_\alpha\geqq 0$ for all $\alpha\in B$. By the Lagrange principle, there exists a constant $c\in\mathbf{R}$ such that $(x|\alpha)=cx_\alpha$ for all $\alpha\in B$. Set $X=\{\alpha\in B\colon x_\alpha>0\}$. If $X\neq B$, then the assumption that (V,B) is irreducible implies that there exist $\alpha\in X$ and $\beta\in B-X$ such that $\langle\alpha,\beta^\vee\rangle<0$. This implies $\langle x,\beta^\vee\rangle<0$ and hence $(x|\beta)=\frac{1}{2}(\beta|\beta)\langle x,\beta^\vee\rangle<0$, thus contradicting the fact that $(x|\beta)=cx_\beta=0$. It follows that $X=B$ and hence either $x\in C$ $(c>0)$ or $x\in -C$ $(c<0)$. The assumption $(-C)\cap(\mathbf{R}\cdot B)=\emptyset$ excludes the last case and so $x\in C$ and $c>0$. In particular, we have $(x|x)=\sum_{\alpha\in B}(x|\alpha)x_\alpha=\sum_{\alpha\in B}cx_\alpha^2\geqq 0$, implying that $(|)$ is positive.

If $n=\sum_{\alpha\in B} n_\alpha\alpha$ is a nonzero element of the kernel K of $(|)$ then the above argument shows that then also $\sum_{\alpha\in B}|n_\alpha|\,\alpha\in K$ and that $|n_\alpha|>0$ for all α. This can only be if $\dim K\leqq 1$. Since $\alpha^\vee$ is a multiple of $(\ |\alpha)$, it follows that $K=F_B\cap V_B$. As (V,B) is irreducible, we must have $F_B\subset V_B$ (otherwise we could split off a supplement for $F_B\cap V_B$ in F_B) and so $K=F_B$.

Step 5. Proof of the proposition.

In case $F_B=\{0\}$, W is a subgroup of the compact group $\mathrm{Aut}(V,(|))$ and hence finite.

In case $F_B\neq\{0\}$, then it follows from step 4 that the restriction of $(|)$ the space spanned by any proper subset $X\subset B$ is positive definite and hence that W_X is finite. Since $\dim F_B=1$, the corank of $(\langle\alpha,\beta^\vee\rangle)_{\alpha,\beta\in B}$ equals one and so (V,B) is of affine type.

(1.17) **Corollary.** *Let (V,B) be an irreducible root basis. Then*

(i) *W is finite if and only if $\sum_{\alpha\in B} x_\alpha\alpha\in\bar{C}-\{0\}$ implies that all $x_\alpha>0$.*

(ii) *(V,B) is of affine type if and only if F_B contains an element of the form $n=\sum_{\alpha\in B} n_\alpha\alpha$ with $n_\alpha\in\mathbf{N}$ for all α. In this case we have $\bar{C}\cap V_B=F_B$ and n spans F_B.*

Proof. If W is finite, choose $y\in(\mathbf{R}_+\cdot B)-\{0\}$. Since $I=V$, there exists a $w\in W$ such that $w(y)\in\bar{C}$. Then $w(y)\in(\mathbf{R}_+\cdot B)-\{0\}$ by (1.11) and so $(\mathbf{R}_+\cdot B)\cap\bar{C}\neq\{0\}$. Now $(\mathbf{R}_+\cdot B)$ and $\bar{C}$ are both convex cones, so if not every element of $\bar{C}-\{0\}$ has positive coefficients on B, then there exists a *proper* subset $X\subset B$ and an element $x=\sum_{\alpha\in X} x_\alpha\alpha\in\bar{C}$ with all $x_\alpha>0$. As this implies $\langle\alpha,\beta^\vee\rangle=0$ for all $\alpha\in X$, $\beta\in B-X$, this contradicts the irreducibility of (V,B). This proves the 'only if' part of (i).

If (V,B) is of affine type, then there exists a nonzero $n=\sum_{\alpha\in B} n_\alpha\alpha$, which is orthogonal to $B^\vee$ with all $n_\alpha\in\mathbf{Z}$. We assume that $n_\beta>0$ for some $\beta\in B$. We claim that then $n_\alpha>0$ for all $\alpha\in B$. For if we put $x:=n-n_\beta\beta$, then we have for any $\alpha\in B$

$-\{\beta\}$, $\langle x,\alpha^\vee\rangle \geqq \langle n,\alpha^\vee\rangle = 0$. Since $W_{B-\{\beta\}}$ is finite it follows from what we just proved, that either $x=0$ or $n_\alpha>0$ for all $\alpha\in B-\{\beta\}$ ($(V,B-\{\beta\})$ is not irreducible, but this doesn't matter). If $x=0$, β would be orthogonal to $B^\vee-\{\beta^\vee\}$, contradicting the irreducibility of (V,B), so $n_\alpha>0$ for all $\alpha\in B$. The same argument applied to $(V^*,B^\vee)$ gives a $n^\vee=\sum_{\alpha\in B} n_\alpha^\vee \alpha^\vee \in F_B^\vee$ with all $n_\alpha^\vee>0$. Since $n^\vee$ is positive on $\bar{C}-F_B$ and zero on B, it follows that $\bar{C}\cap V_B\subset F_B$. As $\dim F_B=1$, F_B is spanned by n.

Conversely, if F_B contains an element of the form $n=\sum_{\alpha\in B} n_\alpha\alpha$ with all $n_\alpha>0$, then n is positive on $C^\vee$ and zero on $B^\vee$, and it follows from (1.16) that either W is finite or that $(V^*,B^\vee)$ is of affine type. Since $F_B\neq\{0\}$, we are in the last case.

Finally, assume that for every nonzero $x=\sum_{\alpha\in B} x_\alpha\alpha\in\bar{C}$ we have $x_\alpha>0$ for all $\alpha\in B$. Then such an x is <0 on $-C^\vee$ and >0 on $B^\vee$ and it follows from (1.16) that W is finite or that $(V^*,B^\vee)$ is of affine type. In case $(V^*,B^\vee)$ is of affine type then the element $-n$ as defined above is in $\bar{C}$ but has negative coefficients on B, contradicting our assumption. So W is finite.

(1.18) **Corollary.** *Let (V,B) be an irreducible root basis. Then W is infinite if and only if $-\bar{C}$ contains an element $x=\sum_{\alpha\in B} x_\alpha\alpha$ with all $x_\alpha\in\mathbf{N}$.*

Proof. In the affine case, the element n of (1.17) is a required. If (V,B) is not of affine type (but has infinite W), then it follows from (1.16) that $-C$ contains an element $x'=\sum_{\alpha\in B} x_\alpha\alpha$ with all $x_\alpha\geqq 0$. Since $-C$ is open, we can choose $x'\in -C$ such that all x_α are positive rational numbers. Take for x an integral positive multiple of x'.

Conversely, if $\bar{C}$ contains $x=\sum_{\alpha\in B} x_\alpha\alpha$ with all $x_\alpha<0$, then W is infinite by (1.17).

(1.19) **Proposition.** *Let (V,B) be an irreducible root basis. Then the action of W on V_B is faithful, $F_B\subset V_B$ and V_B/F_B is a simple W-module. In case (V,B) is of affine type, W acts on V_B/F_B through a finite reflection group, otherwise the action of W on V_B/F_B is faithful, too.*

Proof. If $w\in W$ leaves B pointwise fixed, then $w(C)=C$ and so $w=\mathbf{1}$. This proves that V_B is a faithful W-module. Since (V,B) is irreducible, we have $F_B\subset V_B$. To show that V_B/F_B is a simple W-module, we prove that any W-invariant subspace $V'\subset V$, which contains F_B as a proper subspace, contains V_B. If $x\in V'-F_B$, then $\langle x,\alpha^\vee\rangle\neq 0$ for some $\alpha\in B$. Since we also have $s_\alpha(x)-x=-\langle x,\alpha^\vee\rangle\alpha\in V'$, it follows that $\alpha\in V'$. In the same way it is proved that $\beta\in V'$ for all $\beta\in B$ with $\langle\beta,\alpha^\vee\rangle\neq 0$. The irreducibility of (V,B) then implies that $B\subset V'$.

When (V,B) is of affine type, then it follows from (1.16) and (1.17) that W leaves invariant a positive definite quadratic form on V_B/F_B. Since W also leaves invariant a lattice in V_B/F_B (namely the image of the root lattice), it follows that W acts on V_B/F_B through a finite group.

When (V,B) is not affine, then (1.16) and (1.17) imply that $(V_B-F_B)\cap C\neq\emptyset$. Choose $x\in(V_B-F_B)\cap C$. If $w\in W$ is such that $w(x)\equiv x\,(\mathrm{mod}\,F_B)$, then $w(x)\in C$ (since $F_B\subset\bar{C}$) and so $w=\mathbf{1}$. This proves that V_B/F_B is a faithful W-module.

We consider affine root bases in more detail in section five.

(1.20) From Sect. 3 onwards, we will use the following notions. An element $x \in I$ satisfying $\langle x, \alpha^\vee \rangle \in \mathbf{Z}$ for all $\alpha \in B$ is called a *weight*. The weights form a semigroup (since I is convex), which we denote by P. An element of $P_+ := P \cap \bar{C}$ resp. $P_{++} := P \cap C$ is called a *positive* resp. *strictly positive weight*. Similarly we have a set of dual weights $P^\vee$, positive dual weights $P_+^\vee$ etc.

2. Partial Compactification of Certain Orbit Spaces

In this section we introduce a W-space $\hat{I}$ containing $\mathring{I}$ as an open-dense subset. Like I, $\hat{I}$ is partitioned into faces and in fact, the faces of $\hat{I}$ are in bijective correspondence with those of I. The topology on $\hat{I}$ looks a bit peculiar, but it's not too bad, for it appears that the orbit space $\hat{I}/W$ is locally compact Hausdorff. This prepares the way for a similar construction in a complex context, which is given at the end of this section.

(2.1) We call a subset X of B *special* if either $X = \emptyset$ or if for each connected component Y of the Dynkin diagram on X the corresponding Weyl group W_Y is infinite. We also use this adjective for the subspace V_X of V spanned be such a subset as well for any W-translate of V_X (we postulate $V_\emptyset = \{0\}$). It follows from (1.18) that $X \subset B$ is special iff there exists an $x \in -\bar{C}$ of the form $x = \sum_{\alpha \in X} x_\alpha \alpha$ with all $x_\alpha \in \mathbf{N}$ (if $X = \emptyset$, we take $x = 0$, of course). We call such an element a *positive X-root* (though x needn't be root in the earlier sense). Clearly a union of special subsets of B is again special. In particular, there is a maximal special subset, which we shall denote by B_0.

For any subset $X \subset B$, we let X^* denote the set of $\alpha \in B$ with $\langle \alpha, \beta^\vee \rangle = 0$ for all $\beta \in X$. The following lemma implies an important property of special subspaces.

(2.2) **Lemma.** *Let $X \subset B$ be special and let $\xi \in I^\vee$ be $\leqq 0$ on a positive X-root. Then $\xi \in W_{X^*} \cdot \bar{F}_X^\vee$ (and so ξ vanishes on V_X).*

Proof. We proceed with induction on $|\Phi_\xi|$. Let $x = \sum_{\alpha \in X} x_\alpha \alpha$ be a positive X-root with $\langle x, \xi \rangle \leqq 0$. If $\Phi_\xi = \emptyset$, then $\xi \in \bar{C}^\vee$. Since $0 \geqq \langle x, \xi \rangle = \sum_{\alpha \in X} x_\alpha \langle \alpha, \xi \rangle \geqq 0$, it follows that $\langle \alpha, \xi \rangle = 0$ for all $\alpha \in X$. If $\Phi_\xi \neq \emptyset$, then Φ_ξ contains a fundamental root $\alpha \in B$. If we put $\xi' := s_\alpha(\xi)$, then $|\Phi_{\xi'}| = |\Phi_\xi| - 1$. Since $\langle \alpha, \xi \rangle < 0$, we have $\langle x, \xi' \rangle = \langle x, \xi \rangle - \langle x, \alpha^\vee \rangle \langle \alpha, \xi \rangle \leqq \langle x, \xi \rangle \leqq 0$ and so our induction hypothesis applies: ξ' vanishes on X and $\xi' \in W_{X^*} \cdot \bar{F}_X^\vee$. It follows that $\langle x, \alpha^\vee \rangle \langle \alpha, \xi \rangle = \langle x, \xi \rangle \leqq 0$. As $\langle \alpha, \xi \rangle < 0$ and $\langle x, \alpha^\vee \rangle \leqq 0$ this implies $\langle x, \alpha^\vee \rangle = 0$. This can only be if $\alpha \in X \cup X^*$. Since $\langle \alpha, \xi' \rangle > 0$ and ξ' vanishes on X, we cannot have $\alpha \in X$. So $\alpha \in X^*$ and it also follows that $\xi = s_\alpha(\xi')$ vanishes on $s_\alpha(X) = X$ and is contained in $s_\alpha W_{X^*} \cdot \bar{F}_X^\vee = W_{X^*} \cdot \bar{F}_X^\vee$.

(2.3) **Corollary.** *If $X \subset B$ is special, then the W-stabilizer of V_X equals $W_{X \cup X^*} (\simeq W_X \times W_{X^*})$. Moreover, any special subspace V' contained in V_X is of the form $w(V_Y)$ for a unique special $Y \subset X$ and some $w \in W_X$.*

Proof. Choose $\xi \in F_X^\vee$ (in other words ξ vanishes on X and is positive on $B-X$). If $w \in W$ leaves V_X invariant, then $w^{-1}(\xi)$ also vanishes on X and so (2.2) and (1.3) imply that $w^{-1} \in W_{X^*} W_\xi = W_{X^* \cup X}$.

If $V' \subset V_X$ is special, then by definition, there exist a special $Y \subset B$ and a $w \in W$ such that $V' = w(V_Y)$. Since $w^{-1}(\xi)$ vanishes on Y, there exists by (2.2) a $w' \in W_{Y^*}$ such that $w' w^{-1}(\xi) \in \bar{F}_Y^\vee$. If we put $w'' := w w'^{-1}$, then (1.3) implies that $w'' \in W_X$. So $\xi \in \bar{F}_Y^\vee$ and hence $Y \subset X$. Moreover $w''(V_Y) = w(V_Y) = V'$. To prove uniqueness, assume that there is a special $Y' \subset B$ and a $w_0 \in W$ such that $w_0(V_{Y'}) = V_Y$. If $\eta \in F_{Y'}^\vee$, then $w_0(\eta)$ vanishes on $w_0(V_{Y'}) = V_Y$ and so by (2.2) we have $w_0 \in W_{Y \cup Y^*}$. This implies $V_Y = V_{Y'}$ and hence $Y = Y'$.

If $V' \subset V$ is a special subspace, then the above corollary shows that there exist facets F, F' of I such that W_F resp. $W_{F'}$ is the W-stabilizer of V' (resp. the set $w \in W$ acting trivially on V'). These facets are unique and we have $F \subset \bar{F}'$.

Simple examples show that corollary is definitely false we drop the assumption that X be special.

If $K \subset V$, denote by $\mathscr{H}(K)$ the convex hull of K.

(2.4) **Proposition.** *Let $x \in \bar{C} \cap \mathring{I}$. Then $\mathscr{H}(W \cdot \{x\})$ meets $\bar{C}$ in $(x + \mathbf{R}_- \cdot B) \cap \bar{C}$.*

Proof. Since $W \cdot \{x\}$ is contained in $x + \mathbf{R}_- \cdot B$ by (1.11) and $x + \mathbf{R}_- \cdot B$ is convex, we certainly have $\mathscr{H}(W \cdot \{x\}) \subset x + \mathbf{R}_- \cdot B$. We prove the converse with induction on $|B|$. The case $|B| = 1$ being trivial, we assume that the proposition holds for root bases of rank $< |B|$.

We then claim: If $z \in \bar{C}$, $\beta \in B$ and $z' \in (z + \mathbf{R}_- \cdot B) \cap \bar{C}$ are such that the coëfficient ξ_β of β in $z - z'$ is $\leqq \langle z, \beta^\vee \rangle$, then $z' \in \mathscr{H}(W \cdot \{z\})$. To see this, note that since $s_\beta(z) = z - \langle z, \beta^\vee \rangle \beta$ we have $z - \zeta_\beta \beta \in \mathscr{H}(W \cdot \{z\})$. For any $\alpha \in B - \{\beta\}$ we have $\langle z - \zeta_\beta \beta, \alpha^\vee \rangle \geqq \langle z, \alpha^\vee \rangle \geqq 0$, so $z - \zeta_\beta \beta \in \bar{C}_{B - \{\beta\}}$. Since $z' \in ((z - \zeta_\beta \beta) + \mathbf{R}_- \cdot (B - \{\beta\})) \cap \bar{C}_{B - \{\beta\}}$, our induction hypothesis implies that $z' \in \mathscr{H}(W \cdot \{z\})$.

Now let $y \in (x + \mathbf{R}_- \cdot B) \cap \bar{C}$ and assume first that the interval $[x, y]$ doesn't meet F_B. Then there exists a constant $c > 0$ such that $\max\{\langle z, \alpha^\vee \rangle : \alpha \in B\} \geqq c$ for all $z \in [x, y]$. If we put $x_t := (1-t)x + ty$, then $t \in [0,1]$, $t' \in [t, t+c]$ imply $x_{t'} \in \mathscr{H}(W \cdot \{x_t\})$ (by the above claim). This proves that $y = x_1 \in \mathscr{H}(W \cdot \{x\})$. It remains to treat the case when $[x, y] \cap F_B \neq \emptyset$. If $x \in F_B$, then W must be finite (since $x \in \mathring{I}$) and so $y \in (x + \mathbf{R}_- \cdot B) \cap \bar{C} = \{x\}$, by (1.17).

If $x \notin F_B$, then $y \in F_B$ and we claim that in this case, W is finite too. Let $X \subset B$ represent an irreducible component. We put $z := x - y$, so that $z = \sum_{\alpha \in B} z_\alpha \alpha$ with $z_\alpha \geqq 0$ and $W_z = W_x$ (hence finite). We further set $z_X = \sum_{\alpha \in X} z_\alpha \alpha$. If $z_X = 0$, then $W_X \subset W_z$ and so W_X is finite. If $z_X \neq 0$, then z_X is negative on $-C_X^\vee$ and $\geqq 0$ on $\mathbf{R}_+ \cdot X^\vee$ and so (1.16) implies that either W_X is finite or X is of affine type. In the last case, z_X is orthogonal to $X^\vee$ (by (1.17)) and hence W_z contains W_X, which would contradict the finiteness of W_z. We conclude that W_X is finite for any irreducible $X \subset B$ and hence that W is finite. To prove that $y \in \mathscr{H}(W \cdot \{x\})$, we assume without loss of generality that $y = 0$. Since W is finite, we have $\bar{C} \cap V_B \subset \mathbf{R}_+ \cdot B$ (by (1.17)) and so $(x + \mathbf{R}_- \cdot B) \cap \bar{C}$ is a neighbourhood of 0 in $\bar{C} \cap V_B$. So by what we have already proved, $\mathscr{H}(W \cdot \{x\})$ contains a neighbourhood of 0 in $\bar{C} \cap V_B$ with the possible exception of $\{0\}$ itself. Since $\mathscr{H}(W \cdot \{x\})$ is convex, it then also contains $y = 0$.

(2.5) **Corollary.** *If K is a subset of $\bar{C}\cap\mathring{I}$, then $\mathscr{H}(W\cdot K)\cap\bar{C}=(\mathscr{H}(K)+\mathbf{R}_-\cdot B)\cap\bar{C}$. Moreover, if K is compact in $\mathring{I}$, then $\mathscr{H}(W\cdot K)$ is closed in V.*

Proof. The first clause is immediate from the previous proposition. If K is compact in $\mathring{I}$, then $W\cdot K$ meets $\bar{C}\cap\mathring{I}$ in a compact subset K'. Then $\mathscr{H}(W\cdot K)\cap\bar{C}=(\mathscr{H}(K')+\mathbf{R}_-\cdot B)\cap\bar{C}$ is closed in V. Since each point of $\mathring{I}$ is in the closure of only finitely many chambers, it follows that $\mathscr{H}(W\cdot K)$ is closed in V.

(2.6) **Corollary.** *If x_1, $x_2\in\mathring{I}$ are such that $x_1-x_2\in V_B$, then $\mathscr{H}(W\cdot\{x_1\})$ and $\mathscr{H}(W\cdot\{x_2\})$ meet.*

Proof. If W is finite, then $0\in\mathscr{H}(W\cdot\{x_1\})\cap\mathscr{H}(W\cdot\{x_2\})$. In the general case, let $B_0\subset B$ denote the maximal special subset of B. Since W_{B-B_0} is finite, the above proves that $\mathscr{H}(W_{B-B_0}\cdot\{x_i\})\cap F_{B_0}\neq\emptyset$ $(i=1,2)$. We therefore assume that $B=B_0$. Let z be a positive B-root. Since $x_1-x_2\in V_B$, there exists a $t\geqq 0$ such that $x_1-x_2-tz\in\mathbf{R}_-\cdot B$. Then $x_1-tz\in(x_1+\mathbf{R}_-\cdot B)\cap(x_2+\mathbf{R}_-\cdot B)\cap\bar{C}$ and so $x-tz\in\mathscr{H}(W\cdot\{x_1\})\cap\mathscr{H}(W\cdot\{x_2\})$, by (2.4).

(2.7) Let $X\subset B$ be special. The elements of X^* map to linearly independent elements of V/V_X. Any $\alpha^\vee\in X^{*\vee}$ vanishes on V_X and the corresponding elements in the dual of V/V_X are linearly independent too. It follows that X^* determines a root basis in V/V_X. Since $I\subset I_{X^*}$ (we use the notation of (1.13)), the projection $\pi_X\colon V\to V/V_X$ maps I into the Tits cone $\pi_X(I_{X^*})$ of the root basis $(V/V_X,\pi_X(X^*))$. We put

$$I(X):=\pi_X(I)\quad\text{and}\quad\mathring{I}(X):=\pi_X(I)\cap\pi_X(\mathring{I}_{X^*})=\pi_X(I\cap\mathring{I}_{X^*}).$$

A facet of $I(X)$ is by definition the intersection of a facet of $\pi_X(I_{X^*})$ with $I(X)$. Clearly, $\mathring{I}(X)$ is then a union of facets.

(2.8) **Lemma.** *The projection $\pi_X\colon V\to V/V_X$ maps any facet of I resp. $\mathring{I}$ into a facet of $I(X)$ resp. $\mathring{I}(X)$. Each facet of $\mathring{I}(X)$ is of the form $\pi_X(F)$ for a unique face F of $St(F_{X^*})\cap\mathring{I}$. Moreover, π_X maps each face of $St(F_{X\cup X^*})\cap\mathring{I}$ onto a facet of $\mathring{I}(X)$.*

Proof. Since each facet F of I is contained in a facet F' of I_{X^*}, π_X maps F into the facet $\pi_X(F')$ of $\pi_X(I_{X^*})$. As the W-stabilizer of F contains the W_{X^*}-stabilizer of F', $F\subset\mathring{I}$ implies $F'\subset\mathring{I}_{X^*}$. This proves the first clause.

Next we show that for any $Y\subset X^*$ with W_Y finite, $\pi_X(F_Y)$ is a face of $I(X)$. Let y' be a point of the facet of $\mathring{I}(X)$ which contains $\pi_X(F_Y)$. Since $\mathring{I}$ meets $\pi_X^{-1}(y')$ in a W_X-invariant nonvoid subset, there exists a $y\in\bar{C}_X\cap I_0$ such that $\pi_X(y)=y'$. We can even arrange that $y\in C_X\cap I_0$: if $Z:=\{\alpha\in X\colon\langle z,\alpha^\vee\rangle=0\}$, then W_Z is finite and so there exists a $z\in(\mathbf{R}\cdot Z)\cap C_Z$, by (1.17). Replacing y by $y+z$, we may assume that $y\in C_X\cap\mathring{I}$, indeed. Now let x be a positive X-root. Then for all $t\geqq 0$ we have $y-tx\in C_X\cap\mathring{I}$. Taking t sufficiently large, we will also have $\langle y-tx,\alpha^\vee\rangle>0$ for all $\alpha\in B-(X\cup X^*)$. Since $\langle y-tx,\alpha^\vee\rangle=\langle y',\alpha^\vee\rangle$ if $\alpha\in X^*$, it follows that $y-tx\in F_Y$ and that $y'=\pi_X(y)$.

Since any facet F' of $\mathring{I}(X)$ is a W_{X^*}-translate of a facet as considered in the above paragraph, it follows that there is a unique face F of $St(F_{X^*})\cap\mathring{I}$ such that $F'=\pi_X(F)$. It also follows that π_X maps each facet of $St(F_{X\cup X^*})\cap\mathring{I}$ onto a face of $\mathring{I}(X)$, for $St(F_{X\cup X^*})\cap\mathring{I}$ is just the W_X-orbit of $St(F_{X^*})\cap\mathring{I}$.

(2.9) If $V' \subset V$ is a special subspace of V, then we let $I(V')$, resp. $\mathring{I}(V')$ denote the image of I resp. $\mathring{I}$ under the canonical projection $\pi_{V'}: V \to V/V'$. So if X is special, then $I(X)=I(V_X)$ and $\mathring{I}(X)=\mathring{I}(V_X)$ (the last equality by (2.8)). If F is the unique facet of I with the property that W_F is the W-stabilizer of V', then (2.8) implies that $\pi_{V'}$ maps (each face of) $St(F) \cap \mathring{I}$ onto (a face of) $\mathring{I}(V')$. We further note for that any inclusion $V'' \subset V'$ of special subspaces of V, the canonical projection $\pi_{V'}^{V''}: V/V'' \to V/V'$ maps $I(V'')$ resp. $\mathring{I}(V'')$ onto $I(V')$ resp. $\mathring{I}(V')$.

Let $\hat{I}$ denote the disjoint union of the cones $\{\mathring{I}(V'): V' \subset V \text{ special}\}$. Then W acts on $\hat{I}$ in an obvious way: $w \in W$ sends $x \in \mathring{I}(V')$ to $w(x) \in \mathring{I}(w(V'))$.

(2.10) **Lemma.** *For any facet F of I there is a unique face $\hat{F}$ of $\hat{I}$ with the same W-stabilizer. The assignment $F \mapsto \hat{F}$ sets up a bijective W-equivariant correspondence between the facets of I and those of $\hat{I}$.*

Proof. Let $Z \subset B$ and write (uniquely) $Z = X \cup Y$ with X the maximal special subset of Z and $Y \subset X^*$ with W_Y finite. Following (2.8), $\pi_X(F_Y)$ is then a facet of $\mathring{I}(X)$. Denoting this facet by $\hat{F}_Z$, then the W_{X^*}-stabilizer of $\hat{F}_Z$ equals W_Y. By (2.3), the W-stabilizer of $\hat{F}_Z$ is then $W_{X \cup Y} = W_Z$. Clearly $Z \neq Z'$ implies $\hat{F}_Z \neq \hat{F}_{Z'}$. By (1.3) resp. (2.3) and (2.8) any facet of I resp. $\hat{I}$ is a W-translate of a unique F_Z resp. $\hat{F}_Z$. So the assignment $F_Z \mapsto \hat{F}_Z$ extends to one with the required properties.

(2.11) Next we define a topology on $\hat{I}$. If $x \in \mathring{I}(V')$, then $\mathscr{V} \subset \hat{I}$ is a neighbourhood of x, if $\mathscr{V}$ contains a subset $\mathscr{V}'$ satisfying

(i) $x \in \mathscr{V}'$

(ii) $\mathscr{V}' \cap \mathring{I}$ is an open convex W_x-invariant subset of $\mathring{I}$

(iii) $\mathscr{V}' \cap \mathring{I}(V'') = \pi_{V''}(\mathscr{V}' \cap \mathring{I})$ if $V'' \subset V'$.

In order that this defines a topology, we must check that for any pair $\mathscr{V}_1$, $\mathscr{V}_2$ verifying these conditions, there is a $\mathscr{V}_3 \subset \mathscr{V}_1 \cap \mathscr{V}_2$ also verifying them. By (iii) $\mathscr{V}_i \cap \mathring{I}$ contains a point y_i with $\pi_{V'}(y_i) = x$. Then $\mathscr{V}_i$ also contains $\mathscr{H}(W_x \cdot \{y_i\})$. Since $y_1 - y_2 \in V'$, it follows from (2.6) that there exists a $y \in \mathscr{H}(W_x \cdot \{y_1\}) \cap \mathscr{H}(W_x \cdot \{y_2\})$. Since $\pi_{V'}: \mathring{I} \to \mathring{I}(V')$ is W_x-equivariant, we have $\pi_{V'}(y) = x$. So $x \in \pi_{V'}(\mathscr{V}_1 \cap \mathscr{V}_2 \cap \mathring{I})$. Hence, if we define $\mathscr{V}_3$ by $\mathscr{V}_3 \cap \mathring{I} = \mathscr{V}_1 \cap \mathscr{V}_2 \cap \mathring{I}$, $\mathscr{V}_3 \cap \mathring{I}(V'') = \pi_{V''}(\mathscr{V}_1 \cap \mathscr{V}_2 \cap \mathring{I})$ if $V'' \subset V'$ and $\mathscr{V}_3 \cap \mathring{I}(V'') = \emptyset$ if $V'' \not\subset V'$, then $\mathscr{V}_3$ is as required.

It is clear that with this topology the action of W on $\hat{I}$ is continuous. The closure of $\mathring{I}(V')$ is just the union of the $\mathring{I}(V'')$'s with $V' \subset V''$. So if we assign to each special $X \subset B$ the orbit space $S(X) := \mathring{I}(X)/W_{X^*}$, then we obtain a bijective correspondence between the collection of special subsets of B and the partition of $\hat{I}/W$ induced by the $\mathring{I}(V')$'s which reverses the lattice structures: $\overline{S}(X \cup Y) = \overline{S}(X) \cap \overline{S}(Y)$. The following proposition shows that the bijection $F \mapsto \hat{F}$ respects the incidence relation.

(2.12) **Proposition.** *Let F_1 and F_2 be facets of I. Then $\hat{F}_1$ meets the $\hat{I}$-closure of $\hat{F}_2$ if and only if $F_1 \subset \overline{F}_2$. If this is the case, then $\hat{F}_1 \subset (\hat{F}_2)^-$.*

Proof. Without loss of generality we may assume that $F_i = F_{Z_i}$ for certain $Z_i \subset B$ ($i = 1, 2$). Let X_i denote the maximal special subset of Z_i and put $Y_i = Z_i - Y_i$, so that $\hat{F}_i = \pi_{X_i}(F_{Y_i})$.

First suppose that $\hat{F}_1$ meets $(\hat{F}_2)^-$. Then clearly, $X_2 \subset X_1$. If $z \in St(F_{Y_1})$, then $\langle z, \alpha^\vee \rangle > 0$ for all $\alpha \in B - Y_1$. Since any $z' \in W_{X_1} \cdot St(F_{Y_1})$ is of the form $z - \sum_{\alpha \in X_1} \zeta_\alpha \alpha$

with $z\in St(F_{Y_1})$ and $\zeta_\alpha\geqq 0$, we have $\langle z',\alpha^\vee\rangle>0$ for all $\alpha\in B-Z_1$. Then this is also true for all $z'\in\mathscr{H}(W_{X_1}\cdot St(F_{Y_1}))$. Since $\mathscr{H}(W_{X_1}\cdot St(F_{Y_1}))$ is an open convex W_{Z_1}-invariant subset of $\mathring{I}$ which contains F_{Y_1}, $\pi_{X_2}\mathscr{H}(W_{X_1}\cdot St(F_{Y_1}))$ meets $\hat{F}_2$. In other words, $W_{X_1}\cdot St(F_{Y_1})$ contains a point z' such that $\langle z',\alpha^\vee\rangle=0$ for all $\alpha\in Y_2$. This implies $Y_2\subset Z_1$. So $Z_2=X_2\cup Y_2\subset Z_1$.

Conversely, assume that $Z_2\subset Z_1$. Since X_1 is the maximal special subset of Z_1, we then have $X_2\subset X_1$. Now, let $z\in\bar{C}\cap\mathring{I}$ be such that $\pi_{X_1}(z)\in F_1$ and let $\mathscr{V}$ be a neighbourhood of z in $\mathring{I}$. It will be enough to prove that $\mathscr{H}(W_{Z_1}\cdot\mathscr{V})$ meets F_{Y_2}.

Since $W_{X_1\cap Y_2}\subset W_{Y_2}$ is finite, there exists by (1.17) a nonnegative linear combination y of $X_1\cap Y_2$ such that $\langle y,\alpha^\vee\rangle=\langle z,\alpha^\vee\rangle$ for all $\alpha\in X_1\cap Y_2$. Then $z-y\in(Z+\mathbf{R}_-\cdot X_1)\cap\bar{C}\subset\mathscr{H}(W_{X_1}\cdot\{z\})$. Moreover, $\langle z-y,\alpha^\vee\rangle=0$ for all $\alpha\in(X_1\cap Y_2)\cup Y_1$. Since $Y_2\subset(X_1\cap Y_2)\cup Y_1$, it follows that $z-y\in\bar{F}_{Y_1}$ and so $\mathscr{H}(W_{Z_1}\cdot\mathscr{V})$, being a neighbourhood of $z-y$, meets F_{Y_1}.

(2.13) **Corollary.** *For any face $\hat{F}$ of $\hat{I}$, $St(\hat{F})$ is a neighbourhood of F whose facets correspond under the bijection of* (2.10) *with those of $St(F)$ and any W-orbit meets $St(\hat{F})$ along a W_F-orbit.*

Proof. Immediate from (2.10) and the above proposition.

Next we prove that $\hat{I}/W$ is a Hausdorff space. This will also follow from the existence of certain automorphic functions, constructed in the next section, and so the following proposition may be omitted.

(2.14) **Proposition.** *The orbit space $\hat{I}/W$ is Hausdorff.*

Proof. Let O_1 and O_2 be distinct W-orbits in $\hat{I}$. We must find disjoint W-invariant neighbourhoods of O_1 and O_2. Let $Z_i\subset B$ be such that $\hat{F}_{Z_i}\cap O_i\neq\emptyset$; then $\hat{F}_{Z_i}\cap O_i$ consists of a single point $\hat{x}_i$. If we write $Z_i=X_i\cup Y_i$, where $X_i\subset Z_i$ is the maximal special subset and $Y_i:=Z_i-X_i$, then by (2.8) we can choose $x_i\in F_{Y_i}$ such that $\hat{x}_i=\pi_{X_i}(x_i)$. Let us make the following

Hypothesis (H). The cones $x_i+V_{X_1\cap X_2}+\mathbf{R}_-\cdot X_i\,(i=1,2)$ are disjoint.

Then W-invariant disjoint neighbourhoods of O_1 and O_2 are obtained as follows. Since $x_i+V_{X_1\cap X_2}+\mathbf{R}_-\cdot X_i$ is closed, we can find a convex W_{Y_i}-invariant open neighbourhood $\mathscr{V}_i$ of x_i in $St(F_{Y_i})$ such that $\mathscr{V}_i+V_{X_1\cap X_2}+\mathbf{R}_-\cdot X_i\,(i=1,2)$ are disjoint. We define a neighbourhood $\hat{\mathscr{V}}_i$ of $\hat{x}_i$ by letting $\hat{\mathscr{V}}_i\cap\mathring{I}$ be the W_{X_i}-orbit of $\mathscr{V}_i$ and by putting $\hat{\mathscr{V}}_i\cap\mathring{I}(V')=\emptyset$ if $V'\not\subset V_{X_i}$ and $\hat{\mathscr{V}}_i\cap\mathring{I}(V')=\pi_{V'}(\mathscr{V}_i\cap\mathring{I})$ if $V'\subset V_{X_i}$. It follows from (2.5) that $\hat{\mathscr{V}}_i\cap\mathring{I}$ is a W_{Z_i}-invariant convex open subset of $\mathring{I}$ and $\pi_{X_1\cap X_2}(\hat{\mathscr{V}}_i\cap\mathring{I}_0)\,(i=1,2)$ are disjoint. We claim that the W-orbits of $\hat{\mathscr{V}}_1$ and $\hat{\mathscr{V}}_2$ don't meet. Otherwise there would exist a special $X'\subset B$ such that $W\cdot\hat{\mathscr{V}}_1\cap W\cdot\hat{\mathscr{V}}_2\cap\mathring{I}(X')\neq\emptyset$. It then follows from (2.3) that $X'\subset X_1\cap X_2$ and $\hat{\mathscr{V}}_1\cap\hat{\mathscr{V}}_2\cap\mathring{I}(X')\neq\emptyset$. Since $\hat{\mathscr{V}}_i\cap\mathring{I}(X')=\pi_{X'}(\hat{\mathscr{V}}_i\cap\mathring{I})$, this contradicts the fact that $\pi_{X_1\cap X_2}(\hat{\mathscr{V}}_i\cap\mathring{I})\,(i=1,2)$ are disjoint.

Finally, we show that we can always alter the choices of x_1 and x_2 to satisfy **(H)**. If $X_1=X_2$, then **(H)** is obviously satisfied, for $x_i+V_{X_i}=\pi_{X_i}^{-1}(\hat{x}_i)\,(i=1,2)$ are clearly disjoint. If $X_1\neq X_2$, say $X_1\not\subset X_2$, then we choose a positive X_1-root x. For $t\geqq 0$, we have $x_1-tx\in F_{Y_1}$, $\pi_X(x_1-tx)=\hat{x}_1$ and if t is sufficiently large x_1-tx and x_2 satisfy **(H)**.

(2.15) **Proposition.** *The orbit space $\hat{I}/W$ is locally compact and admits a countable basis for its topology.*

Proof. If $K \subset \mathring{\hat{I}}$ and $X \subset B$ is special, then we let $\hat{K}(X)$ denote the subset of $\hat{I}$ defined by $\hat{K}(X) \cap \mathring{\hat{I}} = \mathscr{H}(W_X \cdot K)$, $\hat{K}(X) \cap \mathring{\hat{I}}(V') = \emptyset$ if $V' \not\subset V_X$ and $\hat{K}(X) \cap \mathring{\hat{I}}(V') = \pi_{V'}(\hat{K}(X) \cap \mathring{\hat{I}})$ if $V' \subset V_X$. If K runs over a basis of $\bar{C} \cap \mathring{I}$ and X over the collection of special subsets of B, then $\{\hat{K}(X)\}$ maps to basis of $\hat{I}/W$. Hence $\hat{I}/W$ admits a countable basis. Local compactness follows from Lemma (2.16) below.

(2.16) **Lemma.** *If K is a compact convex subset of $\bar{C} \cap \mathring{I}$, then the image of $\hat{K}(X)$ in I/W is also compact.*

Proof. Since $\hat{I}/W$ is a Hausdorff space admitting a countable basis, it suffices to prove that the image of any sequence $\hat{z}_i \in \hat{K}(X)$ in $\hat{I}/W$ has an accumulation point in the image of $\hat{K}(X)$ in $\hat{I}/W$. Since $\hat{K}(X)$ is W_X-invariant, we may by (2.3) after passing to a subsequence assume that there exist a special $X' \subset X$ such that $\hat{z}_i \in \hat{K}(X) \cap \mathring{\hat{I}}(X')$. Choose $z_i \in \hat{K}(X) \cap \bar{C}_X$ that $\pi_{X'}(z_i) = \hat{z}_i$. By (2.5) we can write $z_i = z_i' - \sum_{\alpha \in X} \zeta_{i,\alpha} \alpha$ with $z_i' \in K$ and $\zeta_{i,\alpha} \geqq 0$. Passing to a subsequence once more, we may suppose that $z_i' \to z_\infty' \in K$ and that there exists a subset $X'' \subset X$ such that $\zeta_{i,\alpha} \to \infty$ if $\alpha \in X''$ and $\zeta_{i,\alpha} \to \zeta_{\infty,i} (< \infty)$ if $\alpha \in X - X''$. If x' is a positive X'-root, then we can always subtract a positive multiple of x' from z_i to ensure that X'' contains X'. Assuming that this is the case, we claim that X'' is special. For if $Y \subset X''$ is an irreducible component with W_Y-finite, then there exists by (1.17) an element $p \in \mathbf{R}_+ \cdot Y \cap C_Y^\vee$ and hence $\langle z_i, p \rangle = \langle z_i', p \rangle - \sum_{\alpha \in Y} \zeta_{i,\alpha} \langle \alpha, p \rangle \to -\infty$ as $i \to \infty$, which contradicts the fact that $\langle z_i, p \rangle \geqq 0$.

We put $\hat{z}_\infty := \pi_{X''}(z_\infty - \sum_{\alpha \in X - X''} \zeta_{\infty,\alpha} \alpha)$ and claim that $\hat{z}_\infty$ is in $\hat{K}(X)$ and $\lim_{i \to \infty} \hat{z}_i = \hat{z}_\infty$. Since $\hat{K}(X) \cap \bar{C}_X = (K + \mathbf{R}_- \cdot X) \cap \bar{C}_X$ is closed in V, its image under $\pi_{X''}$ is closed in V/V''. Hence $\hat{z}_\infty$, being the limit of $\{\pi_{X''}(z_i)\}$, is in $\pi_{X''}(\hat{K}(X) \cap \bar{C}_X) \subset \hat{K}(X)$. Next, let $y \in \bar{C}_X \cap \mathring{\hat{I}}$ be such that $\pi_{X''}(y) = \hat{z}_\infty$. If $\mathscr{V}$ is any neighbourhood of y in $\bar{C}_X \cap \mathring{\hat{I}}$, then for i sufficiently large we can find $\{y_i \in \mathscr{V}\}$ and $\{\eta_{i,\alpha} \in \mathbf{R}\}$ such that $z_i = y_i - \sum_{\alpha \in X''} \eta_{i,\alpha} \alpha$ with $\eta_{i,\alpha} \to \infty$ as $i \to \infty$. So z_i is in the convex hull of $W_{X''} \cdot \mathscr{V}$ for $i \gg 0$. In particular, $\hat{z}_i \in \pi_{X'}(\mathscr{H}(W_{X''} \cdot \mathscr{V}))$ for $i \gg 0$. This proves that $\hat{z}_\infty = \lim_{i \to \infty} \hat{z}_i$.

Paraphrasing the above, we may say that $\hat{I}/W$ is obtained from $\bar{C}$ ($\simeq I/W$) by first removing the facets with infinite stabilizer and then attaching the same number of facets in a different manner to the remaining $\bar{C} \cap \mathring{I}$. Presumably, there is a homeomorphism of $\bar{C}$ onto $\hat{I}/W$ which preserves facets.

We now introduce a complex analogue of $\hat{I}$. If $X \subset B$ is special, then we put

$$\Omega(X) := \{x + iy \in V_{\mathbf{C}}/V_{X,\mathbf{C}} : x \in V/V_X \text{ and } y \in \mathring{\hat{I}}(X)\}.$$

Clearly, $\Omega(X)$ is invariant under W_{X^*} and W_{X^*} acts properly discontinuously on $\Omega(X)$ by (1.14). Let Q denote the lattice in V generated by B and write $Q(X)$ ($\simeq \mathbf{Z} \cdot B/\mathbf{Z} \cdot X$) for the image of Q in V/V_X. If $q \in Q(X)$ (and also if $q \in Q$), we let $t(q)$ denote the translation $z \in V/V_X \mapsto z + q \in V/V_X$. The extension of W_{X^*} by $t(Q(X))$ is semi-direct. We denote this extension by $\tilde{W}_X$, although we often write $\tilde{W}$ instead of $\tilde{W}_\emptyset$. Note that $\tilde{W}_X$ also acts in a properly discontinuous manner on $\Omega(X)$.

(2.17) **Lemma.** *The $\tilde{W}_X$-stabilizer of any $\omega\in\Omega(X)$ is a finite reflection group; it is nontrivial if and only if $\langle\omega,\alpha^\vee\rangle\in\mathbf{Z}$ for some $\alpha\in W_{X^*}\cdot X^*$.*

Proof. Write $\omega=x+iy$ with $x\in V/V_X$ and $y\in\mathring{I}$. By applying an element of W_{X^*} to ω we may assume that $y\in\pi_X(F_Y)$ for some $Y\subset X^*$. The $\tilde{W}_X$-stabilizer of y equals $W_Y\cdot Q(X)$. The elements in $W_Y\cdot Q(X)$ which also fix x are clearly contained $W_Y\cdot Q_Y$ (where $Q_Y:=\mathbf{Z}\cdot Y$). Since $W_Y\cdot Q_Y$ acts as an affine Weyl group on V/V_X, it follows from [4], Chap. VI §2, that the stabilizer of ω is generated by reflections and is nontrivial iff $\langle\omega,\alpha^\vee\rangle\in\mathbf{Z}$ for some $\alpha\in W_Y\cdot Y$.

(In fact the above proof leads to a precise determination of the $\tilde{W}_X$-conjugacy class of the stabilizer $\tilde{W}_{X,\omega}$ of ω. By *loc. cit.* such a class comes from a (unique) subset $Y\subset X^*$ with W_Y finite and a face Φ of the *fundamental alcove* Δ_Y of $W_Y\cdot Q_Y$. The face Φ need not be unique however: If we project Q parallel to the fixed point subspace of W_Y (in V) onto $\mathbf{R}\cdot Y$, the image is a lattice in Q'_Y in $\mathbf{R}\cdot Y$ which contains Q_Y and is contained in the weight lattice P_Y of Y. Following *loc. cit.* Q'/Q_Y acts on Δ_Y (even faithfully). Then two faces Φ, Φ' of Δ_Y determine the same conjugacy class iff they are equivalent under the Q'/Q_Y-action.)

(2.18) In view of Chevalley's theorem [4], Chap. V, §5, the above lemma implies that the orbit space $M(X):=\Omega(X)/\tilde{W}_X$ is (relative the canonical analytic structure) an analytic manifold.

If $V'\subset V$ is special, then we let $\Omega(V')$, resp. $Q_{V'}$ denote the set of $\omega\in V_{\mathbf{C}}/V'_{\mathbf{C}}$ whose imaginary part lies in $\mathring{I}(V')$, resp. the lattice $Q\cap V'$. Note that $Q_{V'}$ spans V'. We let $\hat{\Omega}$ stand for the disjoint union of the $\Omega(V')$'s. The group $\tilde{W}$ acts in an evident manner on $\hat{\Omega}$. There is a canonical projection $\hat{\Omega}\to\hat{I}$, for obvious reasons denoted by Im, which is equivariant relative the canonical projection $\tilde{W}\to W$. Thus Im induces a map from $M:=\hat{\Omega}/\tilde{W}$ onto $\hat{I}/W$. Clearly, $\hat{M}$ is partitioned into the subsets $\{M(X)\colon X \text{ special}\}$ and $M(X)$ is mapped onto $S(X)\,(=\mathring{I}(X)/W_{X^*})$ by Im.

We define a topology on $\hat{\Omega}$: If $\omega\in\Omega(V')$ then we call a subset $\mathscr{V}$ of $\hat{\Omega}$ a neighbourhood of ω if it contains a subset $\mathscr{V}'$ satisfying

(i) $\omega\in\mathscr{V}'$,

(ii) $\mathscr{V}'\cap\Omega$ is an open $\tilde{W}_\omega$-invariant subset of Ω,

(iii) $\mathscr{V}'\cap\Omega(V'')=\pi_{V''}(\mathscr{V}'\cap\Omega)$ if $V''\subset V'$.

To see that this defines a topology, let $\mathscr{V}_1$ and $\mathscr{V}_2$ satisfy these three conditions. By (i) and (iii) there exist $\omega_i\in\mathscr{V}_i\cap\Omega$ with $\pi_{V'}(\omega_i)=\omega$. It follows from (2.6) that we may assume that $\mathrm{Im}(\omega_1)=\mathrm{Im}(\omega_2)$. Since $\tilde{W}_{\omega_i}$ contains the translation group $Q_{V'}$ $\mathscr{V}_i$ contains the convex hull of $\omega_i+Q_{V'}$ which is just ω_i+V'. Since $\omega_1-\omega_2\in V'$, it follows that $\omega_1\in\mathscr{V}_1\cap\mathscr{V}_2$. So if we define $\mathscr{V}_3\subset\hat{\Omega}$ by $\mathscr{V}_3\cap\Omega=\mathscr{V}_1\cap\mathscr{V}_2\cap\Omega$, $\mathscr{V}_3\cap\Omega(V'')=\emptyset$ if $V''\not\subset V'$ and $\mathscr{V}_3\cap\Omega(V'')=\pi_{V''}(\mathscr{V}_1\cap\mathscr{V}_2\cap\Omega)$ if $V''\subset V'$, then $\mathscr{V}_3\subset\mathscr{V}_1\cap\mathscr{V}_2$ and $\mathscr{V}_3$ satisfies the three conditions above.

Clearly, the projection Im: $\hat{\Omega}\to I$ is continuous. If F is a face of I and $V'\subset V$ is the (unique) special subspace which contains F then we have a canonical $W_F\cdot Q$-equivariant projection

$$\hat{\pi}_F\colon\ \mathrm{Im}^{-1}(St(\hat{F}))\to\mathrm{Im}^{-1}(St(\hat{F}))\cap\Omega(V'),$$

coinciding with $\pi_{V'}^{V''}$ on $\mathrm{Im}^{-1}(St(\hat{F}))\cap\Omega(V'')$, which is also continuous.

(2.19) **Proposition.** *The orbit space $\hat{M}=\hat{\Omega}/\tilde{W}$ is Hausdorff.*

Proof. Let O_1 and O_2 be distinct $\tilde{W}$-orbits. If $\mathrm{Im}(O_1)\neq\mathrm{Im}(O_2)$, then it follows from (2.14) that $\mathrm{Im}(O_1)$ and $\mathrm{Im}(O_2)$ possess distinct W-invariant neighbourhoods. If $\mathrm{Im}(O_1)=\mathrm{Im}(O_2)$, let X be the unique special subset of B such that $O_i\cap\Omega(X)\neq\emptyset$. Since $O_i\cap\Omega(X)$ is a $\tilde{W}_X$-orbit and $\tilde{W}_X$ acts properly discontinuously on $\Omega(X)$ there exists $\tilde{W}_X$-invariant open neighbourhoods $\mathscr{V}_i$ of $O_i\cap\Omega(X)$ with $\mathscr{V}_1\cap\mathscr{V}_2=\emptyset$. By (2.3) each W-orbit meets $St(\mathring{I}(X))$ along a $W_{X\cup X^*}$-orbit. There is a canonical continuous $W_{X\cup X^*}\cdot Q$-equivariant retraction $\mathrm{Im}^{-1}(St(\mathring{I}(X)))\to\Omega(X)$ which we denote by ρ_X. Then the $\tilde{W}$-orbits of $\rho_X^{-1}(\mathscr{V}_1)$ and $\rho_X^{-1}(\mathscr{V}_2)$ are disjoint open subsets of $\hat{\Omega}$. If K is a subset of Ω and $X\subset B$ is special, then we define $\hat{K}(X)\subset\hat{\Omega}$ by putting $\hat{K}(X)\cap\Omega=\mathscr{H}(W_XQ_X\cdot K)$, $\hat{K}(X)\cap\hat{\Omega}(V'')=\emptyset$ if $V''\not\subset V_X$ and $\hat{K}(X)\cap\hat{\Omega}(V'')=\pi_{V''}(\hat{K}(X)\cap\Omega)$ if $V''\cap V_X$.

Denote the orbit map $\hat{\Omega}\to\hat{M}$ by $\hat{p}$.

(2.20) **Proposition.** *The orbit space $\hat{M}$ is locally compact and admits a countable basis for this topology.*

Proof. The collection $\{\hat{p}(\hat{K}(X))\}$ is a basis for $\hat{M}$ if K runs over a basis of $\mathrm{Im}^{-1}(\bar{C}\cap\mathring{I})$ and X runs over the special subsets. Hence M has a countable basis. The local compactness follows from:

(2.21) **Lemma.** *If K is a convex compact subset of $\mathrm{Im}^{-1}(\bar{C}\cap\mathring{I})$, then $\hat{p}(K(X))$ is compact.*

Proof. Let $\{O_j\}_{j=1}^{\infty}$ be a sequence of $W_X\cdot Q_X$-orbits in $\hat{K}(X)$. By passing to a subsequence we may by (2.15) assume that there is a special $X'\subset X$ and $\omega_j\in O_j\cap\Omega(X')$ such that $\mathrm{Im}(\omega_j)$ converges to some $y_\infty\in\mathring{I}(X'')$ with $X'\subset X''\subset X$. Let K' be a compact subset of V_X such that $V_X=Q_X\cdot K'$. Since $\mathrm{Re}(\omega_j)\in\pi_{X'}(K+V_X)$, we may suppose (by taking an element in the same Q_X-orbit as ω_j) that $\mathrm{Re}(\omega_j)\in\pi_{X'}(K+K')$. Since $K+K'$ is compact, there is an accumulation point $x_\infty\in\pi_{X'}(K+K')$ of $\{\mathrm{Re}(\omega_j)\}$. We leave it to the reader to verify that $\omega_\infty:=\pi_{X''}^{X'}(x_\infty)+iy_\infty$ is in $\hat{K}(X)$ and $\hat{p}(\omega_\infty)$ is an accumulation point of $\{p(O_j)\}_{j=1}^{\infty}$.

3. Proof that $\hat{\Omega}/\tilde{W}$ is a Stein Space

In this section we prove the larger part of our main result:

(3.1) **Theorem.** *The orbit space $\hat{M}=\hat{\Omega}/\tilde{W}$ is in a canonical way a Stein manifold inducing on each stratum $M(X)$, $X\subset B$ special, the given analytic structure. The assignment $X\mapsto M(X)$ defines an isomorphism between the dual of the lattice of special subsets of B and the incidence lattice of the stratification $\{M(X)\}$ (so $\bar{M}(X)\cap\bar{M}(Y)=\bar{M}(X\cup Y)$).*

Since $M(X)$ is of codimension $\geqq 2$ in M if $X\neq\emptyset$, the theorem shows that M may be characterized as the *holomorphic hull* of $M:=M(\emptyset)=\Omega/\tilde{W}$.

Example. Let (V,B) be an irreducible root basis. We say that (V,B) is of *hyperbolic type* if the following three conditions are satisfied:

(i) (V,B) is not of affine type,

(ii) W leaves invariant a nondegenerate quadratic form $(|)$,

(iii) W is infinite and for every proper subset $X\subset B$, W_X is finite or (V,X) is of affine type.

Since $(\alpha|\alpha)\langle x,\alpha^\vee\rangle=2(x|\alpha)$ for all $\alpha\in B$, it follows that $(|)$ has constant (nonzero) sign on B. After multiplying the form with a scalar we may assume that $(\alpha|\alpha)\in\mathbf{N}$ for all $\alpha\in B$. Then it is not difficult to show that $(|)$ has signature $(l-1,1)$, where $l=\dim V=|B|$. Moreover, $\mathring{I}$ is one of the (two) components of $\{x\in V\colon (x|x)<0\}$ and W is of finite index in the group of automorphisms of Q which preserve $(|)$. For all these remarks, see [4], Chap. V, §4 exerc. 10–19, which in essence also contain a classification of these root bases. Following our theorem, $\hat{M}$ is obtained by adding to M a punctured disc $M(X)$ for each subset $X\subset B$ of affine type and the singleton $M(B)$. In this case $M\subset\hat{M}$ coincides with the Baily-Borel construction; indeed this is one of the rare instances where the latter gives a nonsingular variety.

We prove the theorem via a number of propositions.

(3.2) **Lemma.** *Let K be a compact subset of $\mathring{I}$ and let p be a dual weight. If $A_{K,p}(N)$ denotes the number of $p'\in W\cdot\{p\}$ whose maximum on K is $\geqq -N$, then $A_{K,p}(N)=O(N^l)$, $N\to\infty$.*

Proof. Choose a finite polyhedron Σ in $\mathring{I}$ with $K\subset\Sigma$. Then $A_{K,p}(N)\leqq A_{\Sigma,p}(N)$. If Σ_0 denotes the set of vertices of Σ, then $A_{\Sigma,p}(N)\leqq A_{x\in\Sigma_0}A_{\{p\},p}(N)$ (simply because any $\xi\in V^*$ assumes on Σ its maximum at a vertex). Of course, we can choose Σ in such a way that each $x\in\Sigma_0$ lies in an open chamber. Hence it suffices to treat the case when K consists of a single point $x\in C$. We also may assume that $p\in P_+^\vee$. Any $p'\in W\cdot p$ is then of the form $p-\sum_{\alpha\in B}\xi_\alpha\alpha^\vee$ with $\xi_\alpha\in\mathbf{Z}_+$. If moreover, $\langle x,p'\rangle\geqq -N$, then the inequalities

$$-N\leqq\langle x,p'\rangle=\langle x,p\rangle-\sum_{\alpha\in B}\xi_\alpha\langle x,\alpha^\vee\rangle$$

and $\langle x,\alpha^\vee\rangle\geqq 0\,(\alpha\in B)$ imply that $\xi_\alpha=O(N)$, $N\to\infty\,(\alpha\in B)$ and hence that $A_{\{x\},p}(N)=O(N^l)$, $l\to\infty$.

(3.3) For any $p\in P_+^\vee$ and $\omega\in\Omega(V')$ we define the series

$$\mathscr{S}_p(\omega)=\sum_{p'\in W\cdot\{p\},\,p'|V'\equiv 0}\exp(-2\pi i\langle\omega,p'\rangle).$$

We shall prove below that $\mathscr{S}_p(\omega)$ converges. If $V'=V_X$ (with $X\subset B$ special), then the set over which we sum is by (2.2) void if $p|V_X\not\equiv 0$ (and so $\mathscr{S}_p(\omega)=0$) and a W_{X^*}-orbit otherwise.

(3.4) **Proposition.** *For any special $V'\subset V$, the series $\mathscr{S}_p$ converges uniformly and absolutely on compact subsets of $\Omega(V')$ and the resulting function on $\hat{\Omega}$ (which we also denote by $\mathscr{S}_p$) is continuous and $\tilde{W}$-invariant.*

Proof. Let $K\subset\mathring{I}$ be compact. Then for any $\omega\in\Omega$ with $\mathrm{Im}(\omega)\in K$ we have

$$\sum_{p'\in W\cdot\{p\}}|\exp(-2\pi i\langle\omega,p'\rangle)|\leqq\sum_{N\in\mathbf{Z}}A_{K,p}(N)\exp(-2\pi N).$$

Since $A_{K,p}(N)=0$ if $N \ll 0$ and $A_{K,N}(N)=O(N^l)$, $N \to \infty$, it follows that the last series converges. So $\mathscr{S}_p$ converges uniformly and absolutely on compact subsets of Ω. This is then also the case on any $\Omega(V')$, for

$$\mathscr{S}_p \circ \pi_{V'}(\omega) = \sum_{p' \in W \cdot \{p\},\, p'|V' \equiv 0} \exp(-2\pi i \langle \omega, p' \rangle)$$

is simply a subseries of $\mathscr{S}_p(\omega)$. Thus $\mathscr{S}_p$ defines a function on $\hat{\Omega}$ which is clearly $\tilde{W}$-invariant.

To prove that $\mathscr{S}_p$ is continuous, let $X \subset B$ be a special subset, $V' \subset V_X$ a special subspace, $K \subset \bar{C} \cap \mathring{I}$ a compact set and x a positive X-root. We put $K_t := K - tx\ (t \geqq 0)$. We want to estimate $\mathscr{S}_p \circ \pi_{V'} - \mathscr{S}_p \circ \pi_X$ on $\mathrm{Im}^{-1}(K_t)$. Clearly,

$$(*) \qquad |\mathscr{S}_p \circ \pi_{V'}(\omega) - \mathscr{S}_p \circ \pi_X(\omega)| \leqq \sum_{p' \in W \cdot \{p\},\, p'|V_X \not\equiv 0} \exp(2\pi \langle \mathrm{Im}\, \omega, p' \rangle).$$

Choose any $N \in \mathbf{R}$. If $\omega \in \mathrm{Im}^{-1}(\mathscr{H}(W_X \cdot K_t))$ and $p' \in W \cdot \{p\}$ are such that $\langle \mathrm{Im}(\omega), p' \rangle \geqq -N$, then there must exist $y \in K$ and $w \in W_X$ such that $\langle y - tx, w(p') \rangle \geqq -N$. By Lemma (2.2) we either have $\langle x, w(p') \rangle = 0$ (in which case $w(p')$ vanishes on X) or $\langle x, w(p') \rangle > 0$ and then $\langle x, w(p') \rangle \in \mathbf{Z}$. So if we assume that in addition $p'|V_X \equiv 0$ (which is equivalent to $w(p')|V_X \not\equiv 0$, since $w \in W_X$) it follows that $\langle y, w(p') \rangle \geqq -N + t$. As the number of $p' \in W \cdot \{p\}$ satisfying this inequality equals $A_{K,p}(N-t)$, it follows that the right hand side of (*) is estimated by

$$\sum_{N \in t + \mathbf{Z}} A_{K,p}(N-t) \exp(-2\pi N) = \exp(-2\pi t) \cdot \sum_{N \in \mathbf{Z}} A_{K,p}(N) \exp(-2\pi N)$$

which tends to zero as t goes to ∞, by (3.4). This proves that $\mathscr{S}_p$ is continuous at every point of $\Omega(X)$.

Next we want to show that the functions $\mathscr{S}_p$ separate the W-orbits of Ω. We do this with the help of the following

(3.5) **Lemma.** *Let $Z \subset B$ and let $p \in P_+^\vee$ vanish on the maximal special subset X of Z. Then for any compact $K \subset \hat{F}_Z$, the sequence $\{\mathscr{S}_{kp}(\omega) \exp 2\pi i \langle \omega, kp \rangle : k \in \mathbf{N}\}$ converges uniformly on $\mathrm{Im}^{-1}(\mathscr{H}(W_{X^*} \cdot K) \cap \hat{F}_Z)$ to a positive integer (which will turn out to be the cardinal of $W_Y \cdot \{p\}$, where $Y := Z - X$).*

Proof. Let $\omega \in \mathrm{Im}^{-1}(\hat{F}_Z)$. Then for all $p' \in W_{X^*} \cdot \{p\}$ we have $\langle \mathrm{Im}\, \omega, p' \rangle \leqq \langle \mathrm{Im}\, \omega, p \rangle$. With induction on $|\Phi_{p'}|$ one easily proves that equality holds iff $p' \in W_Y \cdot \{p\}$. Hence

$$|\mathscr{S}_p(\omega) \exp(2\pi i k \langle \omega, p \rangle) - \sum_{p' \in W_Y \cdot \{p\}} 1|$$

$$\leqq \sum_{p' \in W_{X^*} \cdot \{p\},\, p' - p < 0 \text{ on } \hat{F}_Z} \exp 2\pi k \langle \mathrm{Im}\, \omega, p' - p \rangle.$$

Fix $\varepsilon > 0$ and $N_0 \in \mathbf{N}$. We collect the terms of the last sum according to whether $p' - p$ is $\leqq -N_0$ on K or not. The subsum corresponding to the $p' \in W_{X^*} \cdot \{p\}$ with $p' - p \leqq -N_0$ on K is estimated by

$$\sum_{N \geqq N_0} A_{K,\{p\}}(k(N+d)) \exp(-2\pi k N),$$

where $d := \max\{\langle y, p\rangle : y \in K\}$. Taking N_0 sufficiently large, it will be $< \frac{1}{2}\varepsilon$. The remaining terms are finite in number and each of them is the k-th power of a positive number < 1. So by taking k sufficiently large, their sum will be also $< \frac{1}{2}\varepsilon$.

(3.6) **Corollary.** *The functions* $\{\mathscr{S}_p : p \in P_+^\vee\}$ *separate the* $\tilde{W}$*-orbits of* $\hat{\Omega}$.

Proof. Let $\omega, \omega' \in \hat{\Omega}$ be such that $\mathscr{S}_p(\omega) = \mathscr{S}_p(\omega')$ for all $p \in P_+^\vee$. We must show that then $\omega' \in W \cdot \{\omega\}$. For this purpose we may assume that there exist special subsets X, $X' \subset B$ such that $\omega \in \Omega(X)$, $\omega' \in \Omega(X')$ and that $\mathrm{Im}(\omega)$ resp $\mathrm{Im}(\omega')$ is in the image of $\bar{C} \cap \mathring{I}$ under the projection π_X resp. $\pi_{X'}$. We claim that $X \subset X'$. If not, let $a \in X' - X$ and define $p \in P_+^\vee$ by $\langle \beta, p\rangle = \delta_{\alpha\beta}$ ($\beta \in B$). Then $\mathscr{S}_{kp}(\omega') = 0$ by (3.3) and $\mathscr{S}_{kp}(\omega)\exp(2\pi i\langle \omega, kp\rangle)$ tends to a positive integer by (3.5), thus contradicting the assumption that $\mathscr{S}_{kp}(\omega) = \mathscr{S}_{kp}(\omega')$.

Similarly we have $X' \subset X$ and so $X = X'$. For any $p \in P_+^\vee$ which vanishes on X, we put $u_p := \exp 2\pi i\langle \omega' - \omega, p\rangle$. By assumption we have $u_p^k \mathscr{S}_{kp}(\omega)\exp(2\pi i\langle \omega, kp\rangle) = \mathscr{S}_{kp}(\omega')\exp(2\pi i\langle \omega', kp\rangle)$. If we let k go to ∞, then it follows from the previous lemma that u_p^k tends to a positive number. This can only be if $u_p = 1$. So we have $\langle \omega' - \omega, p\rangle \in \mathbf{Z}$ for all $p \in P_+^\vee$ which vanish on X. This implies that $\omega' - \omega \in (Q + V_X)/V_X$ and in particular that $\omega' \in \tilde{W} \cdot \{\omega\}$.

We are now in a position to prove a large portion of Theorem (3.1). Let $\mathscr{O}$ denote the sheaf of continuous functions on $\hat{M}$ whose restriction to any stratum $M(X)$ is analytic. A section of $\mathscr{O}$ over a subset $\mathscr{V} \subset \hat{M}$ will be also called a holomorphic function on $\mathscr{V}$. By (3.4) every $\mathscr{S}_p$ ($p \in P_+^\vee$) defines a holomorphic function on $\hat{M}$; we will denote it by $\mathscr{S}_p$.

(3.7) **Proposition.** *The ringed space* $(\hat{M}, \mathscr{O})$ *is a normal analytic space and the partition* $\{M(X) : X \subset B$ *special*$\}$ *is an analytic stratification of* M *relative this analytic structure* (*i.e. the topological closure of each stratum is a union of strata and an analytic subvariety of* M).

Proof. We invoke the analyticity criterion of Baily-Cartan-Borel [2], Theorem 9.2. We know by (2.19) and (2.20) that $\hat{M}$ is a locally compact Hausdorff space admitting a countable basis. According to the result just cited, it then suffices to prove the following four properties.

(i) The topological boundary of each stratum $M(X)$ is a union of lower dimensional strata.

(ii) Each point of $\hat{M}$ admits a neighbourhood basis of which each member intersects M in a connected set.

(iii) The restriction of the elements of $\mathscr{O}$ to any stratum $M(X)$ induce the structure sheaf of $M(X)$.

(iv) Each point of M has a neighbourhood $\mathscr{V}$ such that $\mathscr{O}(\mathscr{V})$ separates the points of $\mathscr{V}$.

Clearly (i) and (ii) hold and (iv) follows from (4.6). To check (iii), let $Z \subset B$, let $X \subset Z$ denote the maximal special subset, $\omega \in \mathrm{Im}^{-1}(\hat{F}_Z)$ and ϕ a germ at ω of a W_Y-invariant holomorphic function on $\Omega(X)$ (where $Y := Z - X$). We must show that ϕ is the restriction of an element of $p^*\mathscr{O}$. Let $\tilde{\pi}$: $\mathrm{Im}^{-1}(St(\hat{F}_Z)) \to \mathrm{Im}^{-1}(St(\hat{F}_Z)) \cap \Omega(X)$ denote the canonical map (which coincides with $\pi_X^{V'}$ on $\mathrm{Im}^{-1}(St(\hat{F}_Z)) \cap \Omega(V')$) and let and let $\tilde{\phi}$ denote the germs of $\phi \circ \tilde{\pi}$ at

ω. Then $\tilde{\phi}$ is a $\tilde{W}_\omega$-invariant germ and hence the restriction of an element of $p^*\mathcal{O}$.

Next we want to show that $(\hat{M}, \mathcal{O})$ is a Stein space. For this purpose we need the following

(3.8) **Lemma.** *Let $K' \subset V$ and $K'' \subset \bar{C} \cap \mathring{I}$ be compact convex sets and put $K := K' + iK''$. Define $\hat{K} \subset \hat{\Omega}$ by $\hat{K} \cap \Omega(V') = \pi_{V'}(\mathscr{H}(\tilde{W} \cdot K))$. Then for any $\omega \in \hat{\Omega} - \hat{K}$ there exists a section f of $p^*\mathcal{O}$ such that $|f(\omega)| > \sup\{|f(\omega')| : \omega' \in \hat{K}\}$.*

Proof. Since $\hat{K}$ is $\tilde{W}$-invariant we may replace ω by any other element in its $\tilde{W}$-orbit and so assume that there exists a special $X \subset B$ and a $\tilde{\omega} \in \mathrm{Im}^{-1}(\bar{C} \cap \mathring{I})$ with $\pi_X(\tilde{\omega}) = \omega$. If $\tilde{\omega} \notin K + V_{B,\mathbf{C}}$, then choose $p \in F^\vee_{B,\mathbf{C}}$ such that $|\langle \tilde{\omega}, p\rangle| > \sup\{|\langle \omega', p\rangle| : p \in K\}$ (this is possible since K is convex). Since $\mathscr{H}(\tilde{W} \cdot K) \subset K + V_{B,\mathbf{C}}$ and p vanishes on $V_{B,\mathbf{C}}$ it follows that the function f on $\hat{\Omega}$ induced by p is a section of $p^*\mathcal{O}$ with the desired property. So let us assume that $\tilde{\omega} \in K + V_{B,\mathbf{C}}$. Writing $\tilde{\omega} = x + iy$, we then

Claim. $y \notin K'' + V_X + \mathbf{R}_- \cdot (B - X)$.

Proof. Suppose not: $y \in K'' + V_X + \mathbf{R}_- \cdot (B - X)$. Choose a positive X-root $z = \sum_{\alpha \in X} z_\alpha \alpha$ and put $y_t = y - tz$. For $t \geqq 0$ we have $y_t \in \bar{C} \cap \mathring{I}$. Taking t sufficiently large, we can arrange that $y_t \in K'' + \mathbf{R}_- \cdot B$. But (2.5) then implies that $y_t \in \mathscr{H}(W \cdot K'')$. Since $x \in K' + V_B = \mathscr{H}(Q \cdot K')$, it would follow that $x + iy_t \in \mathscr{H}(\tilde{W} \cdot K)$ which contradicts the assumption that $\pi_X(x + iy_t) = \pi_X(x + iy) = \omega$ is not a member of $\pi_X(\mathscr{H}(\tilde{W} \cdot K))$.

Since y is not in the closed convex set $K'' + V_X + \mathbf{R}_- \cdot (B - X)$, there exists a $p \in V^*$ such that $\langle y, p\rangle > \sup\{\langle y', p\rangle : y' \in K'' + V_X + \mathbf{R}_- \cdot (B - X)\}$. Then necessarily p vanishes on V_X and is $\geqq 0$ on $B - X$, in other words $p \in \bar{F}^\vee_X$. Since such p form an open cone in $\bar{F}^\vee_X$ (K'' is compact!), we may choose p to be a dual weight in $F^\vee_X$. Following (3.5),

$$\mathscr{S}_{kp}(\omega) \exp(2\pi i k \langle \omega, p\rangle) = \mathscr{S}_{kp}(\tilde{\omega}) \exp(2\pi i k \langle \tilde{\omega}, p\rangle)$$

tends to a positive integer as $k \to \infty$. On the other hand, if $\omega \in \mathscr{H}(\tilde{W} \cdot K) \cap \mathrm{Im}^{-1}(\bar{C})$ then

$$|\exp(2\pi i k \langle \tilde{\omega} - \omega', p\rangle)| = \exp(-2\pi k \langle \mathrm{Im}(\tilde{\omega} - \omega'), p\rangle) \to 0 \quad \text{as } k \to \infty$$

(uniformly on $\mathscr{H}(\tilde{W} \cdot K) \cap \mathrm{Im}^{-1}(\bar{C})$) by (3.5). So $\mathscr{S}_{kp}(\omega') \exp(2\pi i k \langle \omega, p\rangle) \to 0$ as $k \to \infty$ (uniformly on $\mathscr{H}(\tilde{W} \cdot K) \cap \mathrm{Im}^{-1}(\bar{C})$). This implies that for k sufficiently large $f := \mathscr{S}_{kp}$ is as desired.

(3.9) **Corollary.** *$(\hat{M}, \mathcal{O})$ is a Stein space.*

Proof. We know laready that the global holomorphic functions separate the points of $\hat{M}$. Since any compact subset of M is contained in $p(\hat{K})$ for some K as in the lemma, it follows that $\hat{M}$ is holomorphically convex. So $\hat{M}$ is a Stein space.

4. Invariant Theory of Exponential Type

In this section we investigate the algebra of functions on $\hat{\Omega}$ generated by the $\{\mathscr{S}_p: p\in P_+^\vee\}$ from a formal-algebraic point of view. Our results are expressed in an invariant theory of exponential type which in the case of finite Weyl group is just the well-known theory of exponential invariants as for instance expounded in [4], Chap. VI, §3. In the affine case it is equivalent to the theory of theta invariants of [8].

For the purpose of this section we always assume a total order on every root basis (V,B): so $B=\{\alpha_1,\ldots,\alpha_l\}$ with $l=|B|$.

(4.1) Let $(V,B=\{\alpha_1,\ldots,\alpha_l\})$ be a root basis of rank 1. We introduce a partial ordering $\leqq$ on $P^\vee$ by the prescription that $p'\leqq p$ if and only if $p-p'\in\mathbf{Z}_+\cdot B^\vee$. It follows from (1.11) that any $p\in P_+^\vee$ $(=P^\vee\cap\bar{C}^\vee)$ is the unique maximal element of its W-orbit $W\cdot\{p\}$. If D, $D'\subset P^\vee$, then we say that D' *dominates* D (and we write $D\leqq D'$) if for any $p\in D$, there exists a $p'\in D'$ with $p\leqq p'$. We say that a subset D of $P^\vee$ is *finitely dominated* if there exists a finite subset D' of $P_+^\vee$ such that $D\leqq D'$. This implies that $w(D)\leqq D'$ for every $w\in W$.

If χ is an integral-valued function on $P^\vee$, then the *support* of χ (notation: *supp*(χ)) is by definition the set of $p\in P^\vee$ with $\chi(p)\neq 0$. We say that $\chi\colon P^\vee\to\mathbf{Z}$ is *of finite type* if supp(χ) is finitely dominated. The set of all $\chi\colon P^\vee\to\mathbf{Z}$ of finite type forms a $\mathbf{Z}$-module which we denote by A. If χ_1, $\chi_2\in A$ and supp(χ_i) is dominated by the finite set $D_i\subset P_+^\vee$ $(i=1,2)$, then supp$(\chi_i)\subset D_i+\mathbf{Z}_-\cdot B^\vee$, and so for any $p\in P^\vee$ the set of pairs $(p_1,p_2)\in\text{supp}(\chi_1)\times\text{supp}(\chi_2)$ with $p_1+p_2=p$ is finite. So we can define a multiplication law on A by

$$\chi_1\chi_2(p)=\sum_{p'\in P^\vee}\chi_1(p')\chi_2(p-p').$$

The product is certainly in A, since supp$(\chi_1\chi_2)$ is dominated by D_1+D_2. Thus A becomes a commutative ring. It contains the subring A_0 consisting of the $\chi\in A$ with supp$(\chi)\subset F_B^\vee$ and this gives A the structure of a commutative A_0-algebra. Clearly, W acts on A as a group of A_0-algebra automorphisms. We call the A_0-algebra A, endowed with the action of W, the *dual-weight algebra* of (V,B). It is easily seen that the dual-weight algebra of a direct sum of root bases is canonically isomorphic to the tensor product (over $\mathbf{Z}$) of the dual-weight algebras of the factors.

In case (V,B) is irreducible and W is finite, we have $F_B^\vee=\{0\}$ and $I^\vee=V^*$. So $P^\vee$ is then the lattice dual to Q. If $D\subset P^\vee$ is dominated by a finite subset E of $P_+^\vee$, then $(W\cdot D)\cap\bar{C}^\vee\subset(E+\mathbf{Z}_-\cdot B)\cap\bar{C}^\vee$. Since $(\mathbf{R}_-\cdot B)\cap\bar{C}^\vee=\{0\}$ by (1.17), we have that $(E+\mathbf{Z}_-\cdot B^\vee)\cap\bar{C}^\vee$ is finite. It follows that D is finite. This implies that $\chi\in A$ iff supp(χ) is finite.

In case (V,B) is of affine type, then recall that there exist $n=n_1\alpha_1+\ldots+n_l\alpha_l$ resp. $n^\vee=n_1^\vee\alpha_1^\vee+\ldots+n_l^\vee\alpha_l^\vee$ (with $n_\lambda, n_\lambda^\vee\in\mathbf{N}$) generating F_B resp. $F_B^\vee$. We make n and $n^\vee$ unique by requiring that n and $n^\vee$ are indivisible as elements of Q resp. $Q^\vee$ (or equivalently, that $(n_1,\ldots,n_l)$ and $(n_1^\vee,\ldots,n_l^\vee)$ are relatively prime). Then A_0 consists of the $\chi\colon P^\vee\to\mathbf{Z}$ whose support is contained in $D+\mathbf{Z}_+\cdot\{n\}$ for some finite subset D of $F_B^\vee$, because $\bar{C}^\vee\cap(\mathbf{R}\cdot B^\vee)=\mathbf{R}\cdot\{n\}$ by (1.17). Any such nonzero

χ is invertible in A_0, with $\mathrm{supp}(\chi^{-1}) \subset (-D) + \mathbf{Z}_+ \cdot \{n\}$, so A_0 is a field. Note that A_0 is the degree zero part of a natural (W-invariant) grading on A: if we put $A_k := \{\chi \in A: p \in \mathrm{supp}(\chi) \text{ implies } \langle n, p\rangle = k\}$, then $A = \bigoplus_{k=0}^{\infty} A_k$. For n takes nonnegative integral values on $P^\vee$ and if $\chi \in A$ is dominated by the finite set $D \subset P_+^\vee$, then its component χ_k in A_k is also dominated by D. Given $p \in P^\vee$, the conditions $\langle n, p\rangle = 0$ and $p \in F_B^\vee$ are equivalent, so the two definitions for A_0 agree.

If (V, B) has no irreducible component of affine type, then we have $F_B^\vee \cap (\mathbf{R}_+ \cdot B^\vee) = \{0\}$, by (1.16) and so for any finite subset E of $P_+^\vee$, $E + (\mathbf{Z}_- \cdot B^\vee)$ intersects $F_B^\vee$ in a finite set. This implies that $\chi \in A_0$ is equivalent to $\mathrm{supp}(\chi)$ is finite and contained in $F_B^\vee$.

The set of W-invariant elements of A, denoted by A^W, is obviously an A_0-subalgebra of A. The nontrivial character $\det: W \to \{\pm 1\}$ defines $A^{-W} := \{\chi \in A: w(\chi) = \det(w)\chi\}$. We have $A^{-W} \cdot A^W \subset A^{-W}$ and $A^{-W} \cdot A^{-W} \subset A^W$, so A^{-W} is an A^W-module and $A^W \oplus A^{-W}$ is a subalgebra of A (in the affine case, insert the adjective *graded*). We may obtain A_0-bases for A^W and A^{-W} in the following manner. Write e^p for the characteristic function of $p \in P^\vee$ (so that $e^p \cdot e^{p'} = e^{p+p'}$) and put

$$S_p := \sum_{p' \in W \cdot \{p\}} e^{p'} \quad \text{and} \quad J_p := \sum_{w \in W} \det(w)\, e^{w(p)}.$$

Obviously, $S_p \in A^W$ and $J_p \in A^{-W}$. Now, let $p_1, \ldots, p_l \in P^\vee$ be such that $\langle \alpha_\lambda, p_\mu\rangle = \delta_{\lambda\mu}$. Then $\bar{C}^\vee = F_B + \mathbf{R}_+ \cdot \{p_1, \ldots, p_l\}$, $P_+^\vee = F_B + \mathbf{Z}_+ \cdot \{p_1, \ldots, p_l\}$ and $P_{++}^\vee = F_B + \mathbf{N} \cdot \{p_1, \ldots, p_l\}$. We claim that $\{S_p: p \in \mathbf{Z}_+ \cdot \{p_1, \ldots, p_l\}\}$ resp. $\{J_p: p \in \mathbf{N} \cdot \{p_1, \ldots, p_l\}\}$ constitutes an A_0-basis of A^W resp. A^{-W}. The proof of this fact is simple and therefore left to the reader.

In order to state the main result of this section, we need two more notions. If $\chi \in A$, we call the set of maximal elements the *maximal support* of χ. The (unique) function on $P^\vee$ which coincides with χ on its maximal support and vanishes elsewhere is called the *initial term* of χ and denoted by $in(\chi)$. If $\chi \in A^W$ (resp. A^{-W}) then the maximal support of χ is contained in $P_+^\vee$ (resp. $P_{++}^\vee$). According to [4], Chap. VI, §3.2, we have $in(\chi\chi') = in(\chi)e^p$ if $in(\chi') = e^p$. We further note that $in(S_p) = e^p (p \in P_+^\vee)$ and $in(J_p) = e^p (p \in P_{++}^\vee)$.

(4.2) **Theorem.** *Let $(V, B = \{\alpha_1, \ldots, \alpha_l\})$ be a root basis and let $p_1, \ldots, p_l \in P^\vee$ satisfy $\langle \alpha_\lambda, p_\mu\rangle = \delta_{\lambda\mu}$. Let $S_1, \ldots, S_l \in A^W$ and $J \in A^{-W}$ be such that $in(S_\lambda) = e^{p_\lambda}$ and $in(J) = e^{p_1 + \ldots + p_l}$ (and in the affine case also homogeneous).*

(i[a]) *If W is finite, then the canonical map $A_0[X_1, \ldots, X_l] \to A$, induced by $X_\lambda \mapsto S_\lambda$, is an isomorphism of A_0-algebras.*

(i[b]) *If (V, B) is of affine type, the canonical map $A_0[X_1, \ldots, X_l] \to A$, $X_\lambda \mapsto S_\lambda$, is an isomorphism of graded A_0-algebras, if we put $\deg(\chi_\lambda) = n_\lambda$.*

(i[c]) *If (V, B) is irreducible with W infinite, then $m := \{\chi \in A: \mathrm{supp}(\chi) \subset P^\vee - F_B^\vee\}$ is an ideal of A and the canonical map $A_0[X_1, \ldots, X_l] \to A^W$, $X_\lambda \mapsto S_\lambda$ induces an isomorphism of A_0-algebras from $A_0[[X_1, \ldots, X_l]]$ onto the m^W-adic completion $(A^W, m^W)^\wedge$ of A^W.*

(ii) *The A^W-module A^{-W} is freely generated by J.*

Proof. Any $p\in P_+^\vee$ is uniquely written as $p'+\nu_1 p_1+\ldots+\nu_l p_l$ with $\nu_\lambda\in\mathbf{Z}_+$ and $p'\in F_B^\vee$. We then put $S_{[p]}=e^{p'}S_1^{\nu_1}\ldots S_l^{\nu_l}$; this is clearly an element of $A_0[A_1,\ldots,S_l]$ and $in(S_{[p]})=e^p$.

Since $in(S_1^{\nu_1}\ldots S_l^{\nu_l})=e^{\nu_1 p_1+\cdots+\nu_l p_l}$, the elements $S_1,\ldots,S_l$ are algebraically independent. Now, let $\chi\in A^W$. Then $\mathrm{supp}(\chi)$ is dominated by a finite subset D of $P_+^\vee$. Let $D_\nu(\nu=0,1,2,\ldots)$ denote the set of $p\in P_+^\vee$ of the form $p'-\nu_1\alpha_1^\vee-\ldots-\nu_l\alpha_l^\vee$ with $p'\in D$, $\nu_\lambda\in\mathbf{Z}_+$ and $\nu_1+\ldots+\nu_l\geqq\nu$. We construct inductively a sequence $X_\nu\in A_0[S_1,\ldots,S_l]$ such that $\mathrm{supp}(\chi_\nu)$ is dominated by D and $\mathrm{supp}(\chi-\chi_\nu)$ is dominated by D_ν: we take $\chi_0=0$ and define $\chi_{\nu+1}$ by

$$(*)\qquad \chi_{\nu+1}:=\chi_\nu+\sum_{p\in D_\nu-D_{\nu+1}}(\chi-\chi_\nu)(p)S_{[p]}.$$

Since $in(S_{[p]})=e^p$, $\mathrm{supp}(\chi_{\nu+1})$ is dominated by D. Clearly,

$$\mathrm{supp}(\chi-\chi_{\nu+1})\cap\bar{C}^\vee\subset(\mathrm{supp}(\chi-\chi_\nu)\cap\bar{C}^\vee)\cup D_\nu=D_\nu.$$

On the other hand, $\mathrm{supp}(\chi-\chi_{\nu+1})\cap(D_\nu-D_{\nu+1})=\emptyset$, so $\mathrm{supp}(\chi-\chi_{\nu+1})\cap\bar{C}^\vee\subset D_{\nu+1}$. Since $\{D_\nu\}_\nu$ is a decreasing sequence of subsets of $P_+^\vee$ with $\cap\{D_\nu:\nu\in\mathbf{Z}_+\mathrm{K}=\emptyset$, it follows that $\lim_{\nu\to\infty}\chi_\nu(p)=\chi(p)$ for all $p\in P^\vee$.

First assume that W is finite. Then we can choose a $\rho=\sum_{\lambda=1}\rho_\lambda\alpha_\lambda\in V_B$ with $\langle\rho,\alpha_\lambda^\vee\rangle=1$ $(\lambda=1,\ldots,l)$. We then have $\rho_\lambda>0$ for all λ, so x is $\geqq 0$ on $\bar{C}^\vee$. If $p\in D_\nu$, then $p=p'-\nu_1\alpha_1^\vee+\ldots+\nu_l\alpha_l^\vee$ for some $p'\in D$ and $\nu_\lambda\in\mathbf{Z}_+$ with $\nu_1+\ldots+\nu_l\geqq\nu$, so $\langle x,p\rangle\leqq\langle x,p'\rangle-\nu$. It follows that $D_\nu=\emptyset$ if $\nu\geqq\nu_0:=\max\{\langle x,p'\rangle:p'\in D\}$. So $\chi_\nu=\chi_{\nu_0}$ for $\nu\geqq\nu_0$. This proves that $\chi=\chi_{\nu_0}\in A_0[S_1,\ldots,S_l]$.

Next, assume that (V,B) is of affine type. We choose a $\rho\in V$ such that $\langle\rho,\alpha_\lambda^\vee\rangle=1$ for all λ. Then $\langle\rho,n^\vee\rangle=n_1^\vee+\ldots+n_l^\vee=:h^\vee$. For the proof of (i^b), we may without loss of generality replace p_λ by $p_\lambda-\frac{1}{h^\vee}\langle\rho,p_\lambda\rangle n$ (for $e^{(h^\vee)^{-1}n^\vee}\in A_0$) and therefore suppose that $\langle\rho,p_\lambda\rangle=1$ for all λ. Now fix a $k\in\mathbf{Z}_+$ and let D denote the (finite) set of $\nu_1 p_1+\ldots+\nu_l p_l$ with $\nu_\lambda\in\mathbf{Z}_+$ and $n_1\nu_1+\ldots+n_l\nu_l=k$. Then $\{S_p:p\in D\}$ is an A_0-basis of A_k. If D_ν is defined as above, then any $p\in D_\nu$ can be written as $p=p'-\nu_1\alpha_1^\vee-\ldots-\nu_l\alpha_l^\vee$ with $\nu_\lambda\in\mathbf{Z}_+$ and $\nu_1+\ldots+\nu_l\geqq\nu$ and $p'\in D$ and also as $p=p''-\lambda n^\vee$ with $p''\in D$, $\lambda\in\mathbf{R}$. We evaluate ρ on these equalities. Comparing the answers shows that $\lambda h^\vee=\nu_1+\ldots+\nu_l\geqq\nu$. So if we take $\chi=S_p$ for any $p\in D$, and define the sequence $\{\chi_\nu\}$ by (*) then

$$\chi_{\nu+1}-\chi_\nu\in e^{-\nu(h^\vee)^{-1}n^\vee}\mathbf{Z}[e^{-(h^\vee)^{-1}n^\vee}][S_1,\ldots,S_l].$$

Hence $\{\chi_\nu\}$ converges (in the $(e^{-(h^\vee)^{-1}n^\vee})$-adic topology) to an element of $\mathbf{Z}[[e^{-(h^\vee)^{-1}n^\vee}]][S_1,\ldots,S_l]$, in particular $\chi\in A_0[S_1,\ldots,S_l]$. If follows that $A_0[S_1,\ldots,S_l]=A^W$.

For the proofs of (i^c) and (ii) we need a lemma.

(4.3) **Lemma.** *Any finitely dominated subset of $P_{+}^\vee{}_+$ (resp. $P_+^\vee-F_B^\vee$) is dominated by a finite subset of $P_{+}^\vee{}_+$ (resp. $P_+^\vee-F_B^\vee$).*

Proof. Let $p\in P_+^\vee$ and let L denote the set of $\xi\in p+(\mathbf{R}_-\cdot B^\vee)$ satisfying $\langle\alpha,\xi\rangle\geqq 1$ $(\lambda=1,\ldots,l)$. Since L is a closed set defined by a finite number of linear

inequalities with integral coefficients, there exists a compact subset K of L such that $L=K+((\mathbf{Z}_-\cdot B^\vee)\cap \bar{C}^\vee)$. So the finite set $K\cap(p+Q^\vee)$ dominates $(p+(\mathbf{R}_-\cdot B^\vee))\cap P_{++}^\vee$, in other words, any subset of $P_{++}^\vee$ dominated by p is dominated by $K\cap(p+Q^\vee)$. Since $K\cap(p+Q^\vee)$ is a finite subset of $P_{++}^\vee$, it follows that any finitely dominated subset of $P_{++}^\vee$ is dominated by a finite subset of $P_{++}^\vee$. The case $P_+^\vee - F_B^\vee$ is treated in the same way (with the inequalities $\langle\alpha_\lambda,\xi\rangle\geqq 1$ replaced by $\langle\alpha_1+\ldots+\alpha_l,\xi\rangle\geqq 1$, $\langle\alpha_\lambda,\xi\rangle\geqq 0\,(\lambda=1,\ldots,l)$).

Continuation of the Proof of (4.2). Assume that (V,B) is irreducible and has infinite Weyl group. Choose a positive B-root x. By (2.2) we have $\langle x,p\rangle\geqq 0$ and equality iff $p\in F_B^\vee$. So if $p\in P_+^\vee - F_B^\vee$, $p'\in P_+^\vee$, then $\langle x,p+p'\rangle\geqq\langle x,p\rangle>0$ and hence $p+p'\in P_+^\vee - F_B^\vee$. This implies that m is an ideal of A. Since $S_1,\ldots,S_l\in m$, the assignment $X_\lambda\mapsto S_\lambda$ induces a homomorphism $A_0[[X_1,\ldots,X_l]]\to(A^W,m^W)^\wedge$ which is injective, since $S_1,\ldots,S_l$ are algebraically independent. Surjectivity will follow if we show that m^W is generated by $S_1,\ldots,S_l$. If $\chi\in m^W$, then $\mathrm{supp}(\chi)$ is dominated by a finite subset D of $P_+^\vee - F_B^\vee$. Choose finite sets $E_\lambda\subset P_+^\vee$ $(\lambda=1,\ldots,l)$ such that $D\subset\bigcup_{\lambda=1}^{l}(p_\lambda+E_\lambda)$ and let for any $p\in P_+^\vee$ dominated by D, $\lambda(p)\in\{1,\ldots,l\}$ be such that p is dominated by $p_{\lambda(p)}+E_{\lambda(p)}$. Recall from the beginning of the proof that we have $\chi=\sum_{p\in P_+^\vee} c_p S^{[p]}$ for certain $c_p\in\mathbf{Z}$, (with $c_p=0$ unless p is dominated by D). We put $\chi^\lambda:=\sum_{p\in P_+^\vee,\,\lambda(p)=p} c_p S^{[p-p_\lambda]}$. Then $\mathrm{supp}(\chi^\lambda)$ is dominated by E_λ and so $\chi^\lambda\in A^W$. Moreover, $\chi=\sum_{\lambda=1}^{l}\chi_\lambda S_\lambda$.

It remains to prove (ii). Let $\chi\in A^{-W}$. Since $\mathrm{supp}(\chi)\cap\bar{C}^\vee\subset P_{++}^\vee$, it follows from the lemma that $\mathrm{supp}(\chi)$ is dominated by a finite subset D of $P_{++}^\vee$. So if we put $\rho^\vee=p_1+\ldots+p_l$, then $D-\rho^\vee$ is a subset of $P_+^\vee$. We define the sequence $\{D_\nu:\nu\in\mathbf{Z}_+\}$ as before and construct inductively a sequence $\chi_\nu\in A_0[S_1,\ldots,S_l]$ such that $\mathrm{supp}(\chi_\nu)$ is dominated by $D-\rho^\vee$ and $\mathrm{supp}(\chi-\chi_\nu J)$ is dominated by D_ν. We put $\chi_0=0$ and define $\chi_{\nu+1}$ by

$$\chi_{\nu+1}:=\chi_\nu+\sum_{p\in D_\nu - D_{\nu+1}}(\chi-J\chi_\nu)(p)S^{[p]}.$$

This sequence clearly possesses the claimed properties. Since $\cap\{D_\nu:\nu\in\mathbf{Z}_+\}=\emptyset$, we can define $\chi':=\lim_{\nu\to\infty}\chi_\nu$. Then χ' is a W-invariant function on $P^\vee$ whose support is dominated by $D-\rho^\vee$ (so $\chi'\in A^W$) and we have $\chi=J\chi'$.

(4.4) In the theorem we could take for S_λ S_{p_λ}; by the second clause, an alternative choice for S_λ is $J_{p_\lambda+\rho^\vee}J_{\rho^\vee}^{-1}$. Following Kac [6], the last expression describes the character of a 'fundamental representation' of the Lie algebra associated to (V,B).

The theorem shows that the ideal generated by $(A^{-W})^2$ in A^W is principal; it is called the *discriminant ideal* (in the affine case, the discriminant ideal is generated by a homogeneous element of degree $2(n_1+\ldots+n_l)$). In particular, there exists a unique $F\in A_0[[X_1,\ldots,X_l]]$ (in the affine or finite case, we have $F\in A_0[X_1,\ldots,X_l]$) such that $J_{\rho^\vee}^2=F(S_{p_1},\ldots,S_{p_l})$. It is not difficult to show that F is independent of the particular choices of $p_1,\ldots,p_l$ and each coefficient of F has

its support contained in $F_B^\vee \cap (\mathbf{Z}_- \cdot B^\vee)$ (so the affine case $F \in \mathbf{Z}[[e^{-n^\vee}]][X_1, \ldots, X_l]$; the coefficients appear to be the formal q-expansions of certain modular forms, see (5.2)).

(4.5) *Example.* Consider the rank two case with infinite Weyl group. So the Cartan matrix is given by $\begin{pmatrix} 2 & -u \\ -v & 2 \end{pmatrix}$ with $u, v \in \mathbf{N}$ and $uv \geqq 4$. If $uv=4$ (i.e. if $\{u, v\}$ equals $\{2, 2\}$ or $\{1, 4\}$), then the root basis is of affine type (as we've seen in (1.2)), while for $uv>4$, this is not the case. If $\{u, v\}=\{2, 2\}$ (resp. $\{1,4\}$), then $n = \alpha_1+\alpha_2$ (resp. $\alpha_1+2\alpha_2$) and $n^\vee = \alpha_1^\vee + \alpha_2^\vee$ (resp. $2\alpha_1^\vee + \alpha_2^\vee$), so $\deg(X_1)=1$ (resp. 1), $\deg(X_2)=1$ (resp. 2) and $F \in \mathbf{Z}[[e^{-n^\vee}]][X_1, X_2]$ is homogeneous of degree 4 (resp. 6).

When $uv>4$, then the expansion of $F \in \mathbf{Z}[[X_1, X_2]]$ begins with $X_1^{u+2} + X_1^2 X_2^2 + X_2^{v+2}$ (the remaining terms are of order $>\max(u+2, v+2)$).

5. Proof that $\hat{\Omega}/\hat{W}$ is Nonsingular

The only assertion of (3.1) which remains to prove is the nonsingularity of $\hat{M}$. For this purpose we observe that if (V, B) decomposes into $(V_1, B_1) \oplus (V_2, B_2)$, then M decomposes accordingly into the corresponding factors M_1 and M_2. Therefore it suffices to treat the irreducible case. When W is finite we have $\hat{M} = M$ and so there is nothing to prove.

(5.1) So let us suppose that (V, B) is of affine type. The main goal of the following discussion is to show that this case is covered in the literature.

Recall that $n = \sum_{\lambda=1}^{l} n_\lambda \alpha_\lambda$ (resp. $n^\vee = \sum_{\lambda=1}^{l} n_\lambda^\vee \alpha_\lambda^\vee$) are the unique indivisible elements of Q (resp. $Q^\vee$) orthogonal to $Q^\vee$ (resp. Q) with at least one (and therefore every) coefficient positive. We put $N := \mathbf{Z}n$ and $N^\vee := \mathbf{Z}n^\vee$. Since W leaves N (pontwise) fixed, W acts on the lattice $Q' := Q/N$. According to (1.16) W leaves invariant a positive definite quadratic form on Q' and therefore acts through a *finite* reflection group W'. Let $T \subset W$ denote the kernel of this action and put $V' := V_B/N_{\mathbf{R}}\ (= Q'_{\mathbf{R}})$. Since T acts faithfully on $V/N_{\mathbf{R}}$ by (1.19) and trivially on the subspace V' of $V/N_{\mathbf{R}}$, there exists for each $t \in T$ a unique $t' \in V'$ such that $t(x) \equiv x + \langle x, n^\vee \rangle t' \pmod{N_{\mathbf{R}}}$, which is $\neq 0$ if $t \neq 1$. We thus identify T with a lattice in V'. This identification clearly respects the W'-actions. Since V' is a simple W'-module by (1.19), the lattice T must generate V'. We further claim that $T \subset Q'$. For if we identify $\mathrm{End}(V)$ with $V \oplus V^*$, then W is contained in $\mathbf{1} + Q \otimes Q^\vee$ (for this subset is closed under composition and contains $s_{\alpha_1}, \ldots, s_{\alpha_l}$). As $n^\vee$ is indivisible in $Q^\vee$, this implies that $t' \in Q'$. The situation gives rise to two exact sequences

$(5.1)_1$ $$0 \to T \cdot Q \to \tilde{W} \to W' \to 1$$

and

$(5.1)_2$ $$0 \to N \to T \cdot Q \to T \times Q' \to 0.$$

The first sequence shows that $T\cdot Q$ is the unipotent radical of $\tilde{W}$, while the second implies that $T\cdot Q$ is at most 2-step unipotent. The sequence also defines a W'-invariant bilinear form $(|)$: $Q'\otimes T\to\mathbf{Z}$ by

$$t(x+q)=t(x)+q-(q'|t)n \tag{5.1}_3$$

where $t\in T$, $q'\in Q'$ and $q\in Q$ is a representative of q', which extends to one on V' (also denoted by $(|)$). Since V' is a simple W'-module, $(|)$ must be a scalar multiple of a positive definite symmetric form (compare [4], exerc. 3 of Chap. V, §2).

We shall determine T and $(|)$ in a more precise manner although this is not essential for the sequel. The set R projects to a finite root system R' in V' of which W' is the Weyl group. Choose $\alpha\in B$ such that $B-\{\alpha\}$ maps to a root basis of R'. It is convenient here to relabel the elements by $\{0,\dots,l'\}$ (with $l'=l-1$) such that $\alpha=\alpha_0$. We further let V_0, resp. W_0, resp. R_0 denote the subspace of V spanned by $B-\{\alpha_0\}$, resp. the subgroup $W_{B-\{\alpha_0\}}$ of W, resp. the subroot system $W_0\cdot(B-\{\alpha_0\})$ of R. So there exist a $\beta_0\in R_0$ and a $u\in\mathbf{N}$ such that $\alpha_0+u\beta_0\in N$. Since we have $\langle\beta_0,\alpha_0^\vee\rangle=-\frac{1}{u}\langle\alpha_0,\alpha_0^\vee\rangle=-\frac{2}{u}$, it follows that $\beta_0\in R_+$ and that $u\in\{1,2\}$ (in fact, $u=2$ only occurs for the types which are usually denoted by BC_l, $l\geqq 1$). Since $\alpha_0+u\beta_0$ is indivisible, we must have $\alpha_0+u\beta_0=n$. Now s_{α_0} and s_{β_0} induce the same reflection in V' and so $u\alpha_0^\vee+\beta_0^\vee\in N^\vee$. Since $\beta_0^\vee$ is indivisible, it follows that $u\alpha_0^\vee+\beta_0^\vee=n^\vee$. A simple calculation shows that

$$s_{\alpha_0}s_{\beta_0}(x)=x+\langle x,n^\vee\rangle\beta_0-\frac{1}{u}\langle x,\beta_0^\vee\rangle n-\frac{1}{u}\langle x,n^\vee\rangle n, \tag{5.1}_4$$

from which it follows that $s_{\alpha_0}s_{\beta_0}$ is the element of T corresponding to $\beta_0'\in Q'$. Since W is generated by W_0 and $s_{\alpha_0}s_{\beta_0}$, this implies that T is the sublattice of Q' spanned by the W'-orbit of β_0'. It also follows from $(5.1)_4$ that $(q'|\beta_0')=\frac{1}{u}\langle q,\beta_0^\vee\rangle$. In particular $(\beta_0'|\beta_0')=\frac{2}{u}$, which shows that $(|)$ is positive definite. Since $(|)$ is not identically zero, the exact sequence $(5.1)_2$ doesn't split. So $T\cdot Q$ is two-step unipotent and N is the centre of $\tilde{W}$.

The sublattices Q_0 resp. T_0 of Q spanned by $B-\{\alpha_0\}$ resp. $W_0\cdot\left\{\frac{1}{u}\beta_0\right\}$ map isomorphically onto Q' resp. T. If we choose $x_0\in V$ such that $\langle x_0,\alpha_\lambda^\vee\rangle=\frac{1}{u}\delta_{0\lambda}$, then x_0 is a fixed point of W_0 satisfying $\langle x_0,n^\vee\rangle=1$. Since

$$s_{\alpha_0}s_{\beta_0}(x_0)=x_0+\beta_0-\frac{1}{u}n=x_0+\beta_0-\tfrac{1}{2}(\beta_0'|\beta_0')n,$$

it is not difficult to verify that

$$t(x_0)=x_0+t-\tfrac{1}{2}(t'|t')n \quad \text{for all } t\in T_0. \tag{5.1}$$

(5.2) **Lemma.** *The cone $\mathring{I}$ is the half space of V defined by $n^\vee > 0$.*

Proof. Clearly $n^\vee$ is positive on $\mathring{I}$. If $\xi \in V^*$ is positive on $\mathring{I}$, then ξ vanishes on $F_B^\vee$ and so $\langle n, \xi\rangle = 0$. If ξ is not a multiple of $n^\vee$, then $\langle \alpha, \xi\rangle \neq 0$ for some $\alpha \in B$ and hence $\langle t, \xi\rangle \neq 0$ for some $t \in T$ (for T is of finite index in Q'). Since $t^k(\xi) = \xi + k\langle t, \xi\rangle n^\vee$ ($k \in \mathbf{Z}$) is also positive on $\mathring{I}$, it would follow that $n^\vee$ vanishes on I which is clearly a contradiction. So the dual cone of $\mathring{I}$ is $\mathbf{R}_+ \cdot n^\vee$. Since $\mathring{I}$ is an open convex cone, it follows that $\mathring{I}$ is defined by $n^\vee > 0$.

(5.3) By this lemma we have a $\tilde{W}$-equivariant fibration $n^\vee : \Omega \to \mathscr{H}_+$, where $\mathscr{H}_+$ denotes the Poincaré upper half plane. We get corresponding projection $M \to \mathscr{H}_+$ and $\hat{M} \to \mathscr{H}_+$. We fix a $\tau \in \mathscr{H}_+$ and let Ω_τ, M_τ and $\hat{M}_\tau$ denote the corresponding fibres over τ. If we identify Ω_τ with $V_{B,\mathbf{C}}$ through the assignment $\omega \in V_{B,\mathbf{C}} \mapsto \tau x_0 + \omega \in \Omega_\tau$ then it follows from $(5.1)_3$ and $(5.1)_5$ that the corresponding action of $t \in T_0$ on $V_{B,\mathbf{C}}$ is given by

$$t(\omega) = \omega + \tau t - (\omega' | t')n - \tfrac{1}{2}(t' | t')n \qquad (5.3)_1$$

which is precisely the action we studied in [8]. To bring out the relationship more clearly, consider the $\mathbf{C}^*$-bundle

$$N \backslash N_{\mathbf{C}} \to N \backslash \Omega_\tau \to N_{\mathbf{C}} \backslash \Omega_\tau$$

which we orient by means of the identification $\mathbf{C}/\mathbf{Z} \to N \backslash N_{\mathbf{C}}$, $\lambda \mapsto -\lambda n$. Notice that $(T \cdot Q)/N \cong T \times Q'$ acts on $N_{\mathbf{C}} \backslash \Omega_\tau$ as a translation lattice: (t, q') sends $\omega + N_{\mathbf{C}}$ to $\omega + \tau t + q' + N_{\mathbf{C}}$. So if we divide out by $T \cdot Q$ we get a $\mathbf{C}^*$-bundle over a complex torus:

$$\mathscr{L}^* : \; N \backslash N_{\mathbf{C}} \to (T \cdot Q) \backslash \Omega_\tau \to (T \cdot Q + N_{\mathbf{C}}) \backslash \Omega_\tau .$$

The complex torus is isomorphic (as a W'-variety) to the quotient of $V'_{\mathbf{C}}$ by the lattice $Q' + \tau T$. The latter is isogenous to the l'-fold product of the elliptic curve $\mathbf{C}/(\mathbf{Z} + \tau\,\mathbf{Z})$ and hence an abelian variety. Filling in the zero section of $\mathscr{L}^*$ (relative the orientation defined above) yields a line bundle $\mathscr{L}$ over $(T \cdot Q + N_{\mathbf{C}}) \backslash \Omega_\tau$ for which $-(\,|\,)$ is a Riemann form. Since the Riemann form is negative definite, we may collapse the zero section to a point and obtain a normal variety $\tilde{M}_\tau$. It is not difficult to see that $\tilde{M}_\tau$ is naturally isomorphic to $\hat{M}_\tau$. In [8], we claimed to prove that $\tilde{M}_\tau$ is nonsingular, but as Bernstein and Schwartzman have pointed out, the proof presented there is incomplete (the statement is only proved for generic τ but the general case then follows using Thm. 2.1 of R. Ephraim: C^∞ and analytic equivalence of singularities, in the Proc. of a Conf. on Complex Analysis Rice University, 11–32 (1972).) Bernstein and Schwartzman also give a correct proof in [3].

(5.4) *Remark.* Let v be the smallest natural number such that $v\Omega' \subset T$ (it can be checked that we always have $v \in \{1, 2, 3\}$) and let Γ denote the group of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ with $c \in \mathbf{Z}uv$ and $b \in \mathbf{Z}u$ (if $u = 2$, then $v \neq 3$). We let Γ act on the sequence $(5.1)_1$ by letting the action on W' be trivial and putting

$$(t, q) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (at + cq, bt + dq + (\tfrac{1}{2}ab(t|t) + bc(q|t) + \tfrac{1}{2}cd(q|q))n).$$

We can extend the action of $\tilde{W}$ on Ω to one of $\tilde{W}\cdot\Gamma$: if we identify $\mathscr{H}_+ \times V_{0,\mathbf{C}} \times \mathbf{C}$ with Ω through the assignment $(\tau,\omega,\lambda)\mapsto \tau x_0+\omega+\lambda n$, then we let Γ act on Ω by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau,\omega,\lambda)=\left(\frac{a\tau+b}{c\tau+d},\frac{\omega}{c\tau+d},\lambda+\frac{1}{2}c\frac{(\omega|\omega)}{c\tau+d}\right).$$

A straight forward computation shows that this defines an action of $\tilde{W}\cdot\Gamma$ on Ω. As a consequence, Γ acts on the space of $\tilde{W}$-invariant holomorphic functions on Ω. This explains the occurrence of modular forms in the exponential invariant theory for affine root systems (see for instance Macdonald [10], Van Asch [1], Looijenga [8], Kac [7]).

We are now in a position to complete the proof of (3.1):

(5.5) **Proposition.** *The analytic space $(\hat{M},\mathscr{O})$ is nonsingular and so is each stratum $M(X)$.*

Proof. We found already in (2.18) that $M(X)$ is an analytic manifold. Since $\mathscr{O}$ induces the structure sheaf on $M(X)$, $M(X)$ is locally analytic in $\hat{M}$. We now prove with induction on $|B|$ that M is nonsingular. In view of the discussion above, we may assume that (V,B) is an irreducible root basis, not of affine type, with W infinite. Then $M(B)$ is the stratum of lowest dimension. It is an open subset of $V_{\mathbf{C}}/V_{B,\mathbf{C}}$. Our induction hypothesis implies that $\hat{M}-M(B)$ is nonsingular.

To prove that $\hat{M}$ is nonsingular at $\omega\in M(B)$, let $\mathscr{O}'_\omega$ denote the subring of Q_ω generated by the elements $\mathscr{S}_p$ $(p\in P_+^\vee)$. Since these elements separate the points of $\hat{M}$ near ω, it suffices to show that $\mathscr{O}'_\omega$ is regular. As (V,B) is not of affine type, each element of A_0 has finite support and so the assignment $S_p\mapsto\mathscr{S}_p$ maps A_0 (isomorphically) onto the subalgebra $\mathscr{A}_0$ of $\Gamma(M,\mathscr{O})$ generated by the elements $\{\mathscr{S}_p\colon p\in F_B^\vee\}$. We may also view $\mathscr{A}_0$ as an algebra of functions on $V_{\mathbf{C}}/V_{B,\mathbf{C}}$ and as such it generates the local ring of ω in $V_{\mathbf{C}}/V_{B,\mathbf{C}}$. Let $\mathfrak{m}_0\subset\mathscr{A}_0$ denote the maximal ideal of $\mathscr{A}_0$ defined by ω. Then $\mathfrak{m}_0$ and the elements $\{\mathscr{S}_p\colon p\in P_+^\vee-F_B^\vee\}$ generate the maximal ideal of $\mathscr{O}'_\omega$. It follows from (4.2) that $\hat{\mathscr{O}}'_\omega\cong\hat{\mathscr{A}}_0[[X_1,\ldots,X_l]]$, where $\hat{\mathscr{A}}_0$ denotes the $\mathfrak{m}_0$-adic completion of $\mathscr{A}_0$. Since $\hat{\mathscr{A}}_0$ is regular, so is $\mathscr{O}'_\omega$.

(5.6) Let $W_+\subset W$ denote the kernel of the epimorphism $\det\colon W\to\{\pm 1\}$ and put $\tilde{W}_+:=W_+\cdot Q$ and $M_+:=\Omega/\tilde{W}_+$. We have a canonical projection $\pi_+\colon M_+\to M$ which realizes M_+ as a two-fold branched covering over M. Since $\hat{M}-M$ is an analytic subvariety of M of codimension $\geqq 2$, there is a canonical branched covering $\hat{M}_+\to\hat{M}$ extending π_+. Let $\hat{D}$ denote the discriminant of this covering.

If $\omega\in\Omega$ has a nontrivial stabilizer, then it follows from (2.17) that there exists a $\alpha\in R$ such that $\langle\omega,\alpha^\vee\rangle\in\mathbf{Z}$. Let $p^\vee\in V^*$ be such that $\langle\alpha,p^\vee\rangle=1$ for all B and consider the infinite product

$$\mathscr{J}(\omega):=\exp(-2\pi i\langle\omega,p^\vee\rangle)\prod_{\alpha\in R_+}(1-\exp(2\pi i\langle\omega,\alpha^\vee\rangle)).$$

This product vanishes if ω has a nontrivial stabilizer and converges elsewhere, since for $\omega\in\mathrm{Im}^{-1}(C)$ we have

$$\sum_{\alpha\in R_+} |\exp(2\pi i\langle\omega,\alpha^\vee\rangle)|$$
$$\leqq \sum_{(m_1,\ldots,m_l)\in \mathbf{Z}_+^l} \exp(-2\pi\langle \operatorname{Im}\omega, m_1\alpha_1^\vee+\ldots+m_1\alpha_l^\vee\rangle)$$
$$\leqq \prod_{\lambda=1}^{l}(1-\exp(-2\pi\langle\operatorname{Im}\omega,\alpha_\lambda\rangle))^{-1}<\infty.$$

It is easily verified that $\mathcal{J}$ is $\tilde{W}$-anti-invariant (and hence $\tilde{W}_+$-invariant). Since $\mathcal{J}$ separates the leaves of the projection map $\hat{M}_+\to\hat{M}$ it follows that the points of $\hat{M}_+$ are separated by the global holomorphic functions. Moreover, $\hat{M}_+$ is holomorphically convex (for $\hat{M}$ is) and so $\hat{M}_+$ is also a Stein space. It also follows that D is defined by $\mathcal{J}^2$, viewed as an element of $\Gamma(M,\mathcal{O})$.

(5.7) *Example.* Let (V,B) be an irreducible root basis with Cartan matrix $\begin{pmatrix}2 & -u\\ -v & 2\end{pmatrix}$, $uv\geqq 4$.

If $uv=4$, then it follows from (4.5) and (5.6) that near $M(B)\cong\mathcal{H}_+$, M_+ is defined by an equation of the form $Z^2=F(X_1,X_2;\tau)\,(\tau\in\mathcal{H}_+)$ with F is weighted homogeneous in X_1 and X_2.

This defines a family of so-called simply-elliptic singularities. More precisely, blowing up the singular locus (which is defined by $X_1=X_2=Z=0$) desingularizes $\hat{M}_+$ and the exceptional set is a bundle of elliptic curves over $\mathcal{H}_+$ (the curve over τ being isomorphic to $\mathbf{C}/(\mathbf{Z}+\tau\,\mathbf{Z})$). If we fix $\tau\in\mathcal{H}_+$, then the *degree* of the singularity is 2 resp. 1 if $\{u,v\}=\{2,2\}$ resp. $\{1,4\}$, see Saito [14].

If $uv>4$, then near $M(B)=\{*\}$, $\hat{M}_+$ is given by an equation of the form

$$Z^2=X_1^{u+2}+X_1^2X_2^2+X_2^{v+2}$$

modulo terms of order $>\max(u+2,v+2)$. By choosing new coordinates we can get rid of these remaining terms. Then the equation defines a so-called cusp singularity of type $T_{2,u+2,v+2}$.

This last case had been obtained earlier by Moody and Lepowski [12].

In a subsequent paper we shall show that the discriminant variety of these singularities can be described with the help of a similar construction.

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Note Added in Proof

V. Kac drew my attention to a paper by E. Vinberg: Discrete linear groups generated by reflections. Izv. Akad. Nauk SSSR **35**, n° 5 (1971), in which statements equivalent to (1.16), (1.17) and (1.18) are proved.