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Autor: Friedlander, Eric M.

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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Etale *K*-theory I: Connections with Etale Cohomology and Algebraic Vector Bundles

Eric M. Friedlander*

Northwestern University, Department of Mathematics, Evanston, IL 60201, USA

We present a K-theory for algebraic varieties whose relationship to etale cohomology is similar to the relationship between complex K-theory and singular cohomology in algebraic topology. As described below, this theory is naturally related to the algebraic K-theory of algebraic vector bundles. In the forthcoming paper [11], we show how etale K-theory is also naturally related to higher algebraic K-theory of algebraic varieties; a subsequent paper will present a suitable etale K-theory for much more general schemes.

With these relationships in mind, we expect that etale K-theory will prove to be a useful tool in the hands of algebraic geometers. Experience from algebraic topology suggests that integrality properties of various K-theoretic operations can have important consequences; thus, we have established the existence and basic properties of lambda and Adams operations in etale K-theory. On the other hand, galois actions on etale cohomology provide the context for some of the deepest and most interesting questions in algebraic geometry. We briefly consider galois actions on etale K-theory, using such actions to formulate a purely K-theoretic version of the "Tate Conjecture" concerning the identification of algebraic cycles.

In Sects. 1 and 2, we consider the mod-m and l-adic K-theories of an inverse system of pairs of simplicial sets. We establish an Atiyah-Hirzebruch spectral sequence and a chern character in such a general context, even though our interest lies in the special case of the etale homotopy type of a closed immersion of algebraic varieties. Section 3 introduces the etale K-theory of a closed immersion of algebraic varieties (isomorphic to, but more intrinsically defined than, the K-theory of the etale homotopy type of the closed immersion). This etale K-theory is related to the theory of algebraic vector bundles by a natural transformation many of whose agreeable properties we establish in Sects. 3 and 4. Our K-theoretic version of the Tate Conjecture is formulated in terms of this natural transformation. An appendix provides a convergence criterion for half plane spectral sequences which we require to obtain the Atiyah-Hirzebruch spectral sequences of Sect. 1.

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1. Mod-m and l-adic K-Theory

We establish the foundations of mod-m and l-adic K-theory for inverse systems of pairs of simplicial sets. Of particular interest is the inverse system $(V_\cdot, W_\cdot)_{et}$, the etale homotopy type of a closed immersion of locally noetherian simplicial schemes (cf. Example 1.3). These K-theories for inverse systems generalize complex K-theory for a single pair (Proposition 1.2) and fit in Atiyah-Hirzebruch spectral sequences (Proposition 1.4). When specialized to the etale homotopy type of a closed immersion of algebraic varieties, these are convergent spectral sequences from etale cohomology to K-theory. When specialized even further to the etale homotopy type of a closed immersion of complex algebraic varieties, these K-theories are an algebraically defined version of the corresponding complex K-theories of the underlying topological pair (Proposition 1.6).

We let m be a positive integer and denote by $C(m)$ the mapping cone of multiplication by m on the circle. Thus,

$$S^1 \xrightarrow{m} S^1 \to C(m)$$

is a cofibre triple. The mod-m K-theory of a relative C.W. pair (T, S) is a $\mathbb{Z}/2$-graded cohomology theory defined by

$$K^0(T,S;\mathbb{Z}/m) = [C(m) \wedge T/S, BU],$$

$$K^1(T,S;\mathbb{Z}/m) = [\Sigma C(m) \wedge T/S, BU]$$

where [,] denotes pointed homotopy classes of maps, $\Sigma C(m)$ is the reduced suspension of $C(m)$, and BU is the classifying space of the infinite unitary group.

According to Araki and Toda ([2], Cor. 7.12 and 10.8), $K^*(T,S;\mathbb{Z}/m)$ is an associative, graded commutative $\mathbb{Z}/m$ algebra for $m \not\equiv 2 \pmod 4$ whose multiplication is induced by a "comultiplication" map $\Sigma^2 C(m) \to C(m) \wedge C(m)$ (see also [7], Theorem 1.7).

We shall find it convenient to work in the category of simplicial sets rather that topological spaces. We let $\mathscr{S}$ denote the category of simplicial sets, $\mathscr{S}^2$ the category of pairs (i.e., inclusions) of simplicial sets and $\mathscr{S}_0 \subset \mathscr{S}^2$ the category of pointed simplicial sets. Moreover, we let $\mathscr{H}$ (respectively, $\mathscr{H}^2$; resp., $\mathscr{H}_0$) denote the homotopy category of $\mathscr{S}$ (resp., $\mathscr{S}^2$; resp., $\mathscr{S}_0$). We employ the adjoint pair

$$(|\cdot|, \mathrm{Sin}(\cdot)): \mathscr{S} \leftrightarrow \mathrm{Top}$$

consisting of the geometric realization functor and the singular functor. We also employ the adjoint pair defined for each $n > 0$

$$(sk_n, \cos k_n): \mathscr{S} \leftrightarrow \mathscr{S}.$$

Although $\cos k_n$ is not as familiar as sk_n, it is frequently more useful because it induces a functor $\cos k_n: \mathscr{H} \to \mathscr{H}$.

We recall that the pro-category pro-$\mathscr{C}$ of a given category $\mathscr{C}$ is the category whose objects are inverse systems $\{X_i\}$ of objects of $\mathscr{C}$ (i.e., functors $I \to \mathscr{C}$ with domain a left filtering category) and whose maps are defined by

$$\mathrm{Hom}_{\text{pro-}\mathscr{C}}(\{X_i\}, \{Y_j\}) = \varprojlim_j \operatorname{colim}_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j).$$

We shall employ the Artin-Mazur completion functor [3]

$$(\ )\hat{}\colon \text{pro-}\mathscr{H}_0 \to \text{pro-}\mathscr{H}_0$$

defined with respect to a given set of primes L by sending a simplicial set $X \in \mathscr{H}_0 \subset \text{pro-}\mathscr{H}_0$ to

$$\{Y; X \to Y \in \mathscr{H}_0^2 \text{ with } \pi_i(Y) \text{ finite, } L\text{-primary for } i>0\}.$$

Another useful functor introduced by Artin and Mazur is the functor

$$\#(\cdot)\colon \text{pro-}\mathscr{H}_0 \to \text{pro-}\mathscr{H}_0$$

sending $X \in \mathscr{H}_0 \subset \text{pro-}\mathscr{H}_0$ to $\{\cos k_n X; n>0\}$. This functor converts weak equivalences in pro-$\mathscr{H}_0$ to isomorphisms [3].

In the following definition, we let $C(m)$ denote the simplicial set $\mathrm{Sin}(C(m))$. For $X \in \mathscr{S}_0$, $\Sigma^1 X = \Sigma X$ is the reduced suspension whereas $\Sigma^0 X = X$.

Definition 1.1. For any $\{X_i, Y_i\} \in \text{pro-}\mathscr{H}^2$, any $m>0$, and $\varepsilon = 0$ or 1, we define

$$K^\varepsilon(\{X_i, Y_i\}; \mathbb{Z}/m) = \mathrm{Hom}_{\text{pro-}\mathscr{H}_0}(\{\Sigma^\varepsilon C(m) \wedge X_i/Y_i\}, \#(\mathrm{Sin} BU)).$$

Moreover, if l is an (implicitly specified) prime, we define

$$\hat{K}^\varepsilon(\{X_i, Y_i\}) = \mathrm{Hom}_{\text{pro-}\mathscr{H}_0}(\{\Sigma^\varepsilon X_i/Y_i\}, \#((\mathrm{Sin} BU)\hat{}) \times \mathbb{Z}\hat{})$$

where $(\ )\hat{}$ is the Artin-Mazur completion functor with respect to l (for $X \in \mathscr{S}$, $X \times \mathbb{Z}\hat{} = \{X \times \mathbb{Z}/l^k\colon k>0\} \in \text{pro-}\mathscr{H}$).

Because $\#(\mathrm{Sin} BU)$ and $\#((\mathrm{Sin} BU)\hat{})$ are groups objects in the category pro-$\mathscr{H}_0$, $K^*(\{X_i, Y_i\}, \mathbb{Z}/m)$ and $\hat{K}^*(\{X_i, Y_i\})$ are abelian group valued functors on the homotopy category of pro-$\mathscr{H}^2$ obtained by inverting weak equivalences (i.e., $\{X_i, Y_i\} \to \{X'_j, Y'_j\}$ is a weak equivalence if the induced map $\{X_i/Y_i\} \to \{X'_j/Y'_j\}$ in pro-$\mathscr{H}$ is a disjoint union of maps of inverse systems of connected objects in pro-$\mathscr{H}_0$ each of which induces an isomorphism of pro-homotopy groups).

In the following proposition, we see that Definition 1.1 specializes to more familiar concepts.

Proposition 1.2. *Assume the notation of Definition 1.1. Then $K^*(\{X_i, Y_i\}; \mathbb{Z}/m)$ is an associative, $\mathbb{Z}/2$ graded commutative, $\mathbb{Z}/m$ algebra for $m \not\equiv 2 \pmod 4$, and $\hat{K}^*(\{X_i, Y_i\})$ is a $\mathbb{Z}/2$-graded commutative algebra over the l-adic integers $\mathbb{Z}_l$ for any prime l. Furthermore,*

i) *Bott periodicity determines an isomorphism of rings*

$$\hat{K}^*(\{X_i, Y_i\}) \simeq \varprojlim_k K^*(\{X_i, Y_i\}; \mathbb{Z}/l^k).$$

ii) *The ring structure on*

$$\hat{K}^*(\{X_i, Y_i\})_{\mathrm{red}} = \mathrm{Ker}(\hat{K}^*(\{X_i, Y_i\}) \to \varprojlim_k \varinjlim_i H^0(X_i, Y_i; \mathbb{Z}/l^k)$$
$$\simeq \mathrm{Hom}_{\text{pro-}\mathscr{H}_0}(\{\Sigma^* X_i/Y_i\}, \#((\mathrm{Sin} BU)\hat{}))$$

is induced by the Artin-Mazur completion of the ring space structure on BU.

iii) $K^*(\{X_i, Y_i\}; \mathbb{Z}/m)$ *is naturally isomorphic to*

$$\lim_{\leftarrow n} \operatorname{colim}_{\rightarrow i} \operatorname{im}[K^*(|sk_{n+1}X_i|, |sk_{n+1}Y_i|; \mathbb{Z}/m) \to K^*(|sk_nY_i|, |sk_nY_i|; \mathbb{Z}/m)].$$

Moreover, if $\{Z_j, W_j\} \in \text{pro-}\mathscr{H}^2$ *is weakly equivalent to* $\{X_i, Y_i\}$ *in* pro-$\mathscr{S}^2$, *then*

$$K^*(\{X_i, Y_i\}; \mathbb{Z}/m) \simeq \lim_{\leftarrow n} \operatorname{colim}_{\rightarrow j} K^*(|sk_nZ_j|, |sk_nW_j|; \mathbb{Z}/m).$$

iv) *If* (T, S) *is a C.W. pair such that* $(\mathrm{Sin}(T), \mathrm{Sin}(S))$ *is weakly equivalent to* $\{X_i, Y_i\}$, *then*

$$K^*(\{X_i, Y_i\}; \mathbb{Z}/m) \simeq K^*(T, S; \mathbb{Z}/m).$$

v) *If* (T, S) *is a relatively finite C.W. pair such that* $(\mathrm{Sin}(T), \mathrm{Sin}(S))$ *is weakly equivalent to* $\{X_i, Y_i\}$, *then*

$$\hat{K}^*(\{X_i, Y_i\}) \simeq K^*(T, S) \otimes \mathbb{Z}_l.$$

Proof. The ring structure on $K^*(\{X_i, Y_i\}; \mathbb{Z}/m)$ is induced by the comultiplication map $\Sigma^2 C(m) \to C(m) \wedge C(m)$ and the ring space structure on BU. Bott periodicity, of course, provides $\mathbb{Z}/2$-periodicity which enables $K^*(\{X_i, Y_i\}; \mathbb{Z}/m)$ to be a $\mathbb{Z}/2$ graded (as opposed to a $\mathbb{Z}$-graded) multiplicative theory. The associativity and graded commutativity of $K^*(\{X_i, Y_i\}; \mathbb{Z}/m)$ follow from that of $K^*(\ ; \mathbb{Z}/m)$ given in [2].

The Bott map β: $BU \times \mathbb{Z} \to \Omega^2 BU$ induces a weak equivalence

$$BU^{\wedge} \to \{\mathrm{Sin}(BU^{C(l^k)})_0; k > 0\}$$

where $(BU^{C(l^k)})_0$ is the connected component of the mapping space $BU^{C(l^k)} \simeq (BU^{C(l^k)})_0 \times \mathbb{Z}/l^k$. Therefore, $\{\#(\mathrm{Sin}BU)^{C(l^k)}\}$ is isomorphic (as pro-H-spaces in pro-$\mathscr{H}_0$) to $\#((\mathrm{Sin}BU)^{\wedge}) \times \mathbb{Z}^{\wedge}$. This implies the isomorphism of graded groups:

$$\hat{K}^*(\{X_i, Y_i\}) \simeq \lim_{\leftarrow k} K^*(\{X_i, Y_i\}; \mathbb{Z}/l^k).$$

Because the above discussion of the ring structure on $K^*(\{X_i, Y_i\}; \mathbb{Z}/m)$ applies to $m = 2^k$ for $k > 1$ as well as m odd, we may use this isomorphism to provide a ring structure on $\hat{K}^*(\{X_i, Y_i\})$ for any prime l. The fact that this ring structure on $\hat{K}^*(\{X_i, Y_i\})_{\mathrm{red}}$ is induced by that on $\#((\mathrm{Sin}BU)^{\wedge})$ also follows from Bott periodicity.

If two maps $sk_nZ \rightrightarrows B$ in $\mathscr{S}$ are homotopic, then their adjoints $Z \rightrightarrows \cos k_n B$ are also homotopic; if $Z \rightrightarrows \cos k_{n+1} B$ in $\mathscr{S}$ are homotopic, then the restriction of their adjoints $sk_nZ \rightrightarrows B$ are also homotopic. This readily implies property (1.2iii).

Because $K^*(T, S; \mathbb{Z}/m)$ is finite whenever (T, S) is a finite C.W. pair, $K^*(\cdot; \mathbb{Z}/m)$ is a compact representable functor [16]. We thus conclude that $K^*(T, S; \mathbb{Z}/m) \to \lim_{\leftarrow n} K^*(sk_nT, sk_nS; \mathbb{Z}/m)$ is an isomorphism for any C.W. pair (T, S). Together with (1.2iii), this implies property (1.2iv).

Associated to any relatively finite C.W. triple (T, T', S), there is a natural map of long exact sequences

$$\begin{array}{ccccccc} \ldots \to K^*(T,T')\otimes \mathbb{Z}_l & \longrightarrow & K^*(T,S)\otimes \mathbb{Z}_l & \longrightarrow & K^*(T',S)\otimes \mathbb{Z}_l \to \ldots \\ \downarrow & & \downarrow & & \downarrow \\ \ldots \to \hat{K}^*(T,T') & \longrightarrow & \hat{K}^*(T,S) & \longrightarrow & \hat{K}^*(T',S) \to \ldots \end{array}$$

where $\hat{K}^*(\cdot) = \varprojlim K^*(\cdot; \mathbb{Z}/l^k)$ and where the bottom row is exact because it is an inverse limit of exact sequences of finite abelian groups. Using induction on the number of relative cells and the easy special case of one relative cell, we conclude property (1.2v).

We observe that for a general $\{Z_j, W_j\} \in \text{pro-}\mathscr{S}^2$, the canonical map

$$\varinjlim_j K^*(|Z_j|, |W_j|; \mathbb{Z}/m) \to \varprojlim_n \varinjlim_j K^*(|sk_n Z_j|, |sk_n W_j|; \mathbb{Z}/m)$$

is not an isomorphism (e.g., $\{Z_j, W_j\} = \#(\mathrm{Sin} BU)$). The reason we consider the inverse limit over skeleta is that this is the abutment of the Atiyah-Hirzebruch spectral sequence of Prop. 1.4. We also observe that the finiteness hypothesis in (1.2v) is necessary: $K^0(T, *) \otimes \mathbb{Z}_l$ is not isomorphic to $\varprojlim_k K^0(T, *; \mathbb{Z}/l^k)$ if T is an infinite bouquet of 2-spheres.

Example 1.3. Let i: $W. \to V.$ be a closed immersion of locally noetherian simplicial schemes. We consider the etale homotopy of $(V., W.)$,

$$(V., W.)_{\mathrm{et}} = \{\pi(\Delta U..), \pi(\Delta \psi_{W.}(U..); U.. \in HR(V.)\} \in \text{pro-}\mathscr{H}^2$$

where Δ is the functor sending a bi-simplicial set to its diagonal, where the indexing category $HR(V.)$ is the left filtering category of hypercoverings of $V.$, where $\psi_{W.}(U..) \to U..$ is the open and closed immersion defined by setting $\psi_{W.}(U..)_{s,t} \to U_{s,t}$ equal to the inclusion of those connected components of $U_{s,t}$ with nonempty intersection with W_S, and where $\pi(\Delta U..)$ is the simplicial set of connected components of the bi-simplicial scheme $U...$ So defined, $(V., W.)_{\mathrm{et}}$ is a somewhat modified version of the general definition (specialized to i: $W. \to V.$) of the etale homotopy type of a diagram of schemes which yields the same weak homotopy type (cf. [10], Chap. 11).

As shown in [10], there is a natural weak equivalence $(V., W.)_{\mathrm{et}} \to (V., W.)_{\mathrm{Ret}}$ in pro-$\mathscr{H}^2$ with $(V., W.)_{\mathrm{Ret}} \in \text{pro-}\mathscr{S}^2$ defined by considering the category $HRR(V.)$ of rigid hypercoverings of $V.$ By (1.2iii),

$$K^*((V., W.)_{\mathrm{et}}; \mathbb{Z}/m) \simeq \varprojlim_n \varinjlim_{HRR(V.)} K^*(sk_n(V., W.)_{\mathrm{Ret}}; \mathbb{Z}/m).$$

The right hand side of the above isomorphism was briefly considered in the discussion of generalized cohomology presented in [10].

We next establish Atiyah-Hirzebruch spectral sequences for $K^*(\{X_i, Y_i\}; \mathbb{Z}/m)$ and $\hat{K}^*(\{X_i, Y_i\})$. We remind the reader that $H^*(\{X_i, Y_i\}, \mathbb{Z}/m)$

is defined to be $\operatorname{colim}_{\overrightarrow{i\in I}} H^*(X_i, Y_i; \mathbb{Z}/m)$. For notational convenience, we let $K^*(X, Y)$ denote $K^*(|X|, |Y|)$ for any $(X, Y)\in\mathscr{S}^2$.

Proposition 1.4. *For any* $\{X_i, Y_i\}\in\text{pro-}\mathscr{H}^2$ *and any* $m>0$, *there is a natural spectral sequence*

$$E_2^n(\mathbb{Z}/m)=H^n(\{X_i, Y_i\}; \mathbb{Z}/m) \Rightarrow K^*(\{X_i, Y_i\}; \mathbb{Z}/m)$$

which is convergent provided that $E_2^n(\mathbb{Z}/m)=0$ *for n sufficiently large or that* $E_2^n(\mathbb{Z}/m)$ *is finite for all n. Moreover,*

$$d_r\colon\ E_2^n(\mathbb{Z}/m)\to E_2^{n+r}(\mathbb{Z}/m)$$

is the 0-map *for r even, so that* $\{E_r^n(\mathbb{Z}/m)\}$ *collapses if* $H^n(\{X_i, Y_i\}, \mathbb{Z}/m)=0$ *either for all odd n or all even n.*

Furthermore, the inverse limit of $\{E_r^n(\mathbb{Z}/l^k)\}$ *with respect to k for any prime l determines a spectral sequence*

$$E_2^n=\varprojlim_{\overleftarrow{k}} H^n(\{X_i, Y_i\}; \mathbb{Z}/l^k) \Rightarrow \hat{K}^*(\{X_i, Y_i\})$$

provided that $H^n(\{X_i, Y_i\}; \mathbb{Z}/l)$ *is finite for each n. This spectral sequence converges provided that* $\varprojlim H^n(\{X_i, Y_i\}; \mathbb{Z}/l^k)$ *either equals* 0 *for n sufficiently large or is finite for all n.*

Proof. For each i, we consider $(1.4.1)'_i$, the first derived exact couple of the following exact couple

$$(1.4.1)_i \qquad \begin{array}{ccc} \bigoplus_n K^*(sk_{n-1}X_i, sk_{n-1}Y_i; \mathbb{Z}/m) & \leftarrow & \bigoplus_n K^*(sk_nX_i, sk_nY_i; \mathbb{Z}/m) \\ & \searrow \quad \nearrow & \\ & \bigoplus_n K^*(sk_nX_i, sk_nY_i \cup sk_{n-1}X_i; \mathbb{Z}/m) & \end{array}$$

Using the fact that d_1 can be identified with the coboundary map of the cochain complex $C^*(X_i, Y_i; \mathbb{Z}/m)$ and the fact that two maps $(X_i, Y_i)\rightrightarrows(X_j, Y_j)$ in $\mathscr{S}^2$ which are equal in $\mathscr{H}^2$ determine the same map $(1.4.1)'_j\to(1.4.1)'_i$ (cf. [9], Lemma 3.2), we conclude that $\operatorname{colim}_{\overrightarrow{i}} (1.4.1)'_i$ is an exact couple with associated spectral sequence $\{E_r^n(\mathbb{Z}/m)\}$. Because $K^\varepsilon(sk_nX_i, sk_nY_i\cup sk_{n-1}X_i; \mathbb{Z}/m)\neq 0$ if and only if $n+\varepsilon$ is even, the even differentials vanish. The convergence criteria follow using (1.2iii) from Prop. A.2 of the appendix. As remarked above Example 1.3, the abutment of this colimit spectral sequence is not necessarily the colimit of the abutments.

We now assume that $H^n(\{X_i, Y_i\}; \mathbb{Z}/l)$, and hence also $H^n(\{X_i, Y_i\}; \mathbb{Z}/l^k)$, is finite for all n. Using the exact couple $\operatorname{colim}_{\overrightarrow{i}} (1.4.1)'_i$ with $m=l^k$, we conclude that

$$\operatorname{colim}_{\overrightarrow{i}} \operatorname{im}[K^*(sk_{n+1}X_i, sk_{n+1}Y_i; \mathbb{Z}/l^k)\to K^*(sk_nX_i, sk_nY_i; \mathbb{Z}/l^k)]$$

is finite for all n. Consequently, the inverse limit of $\operatorname{colim}_{\overrightarrow{i}}$ (1.4.1)$_i$ with respect to k is again an exact couple with associated spectral sequence

$$\lim_{\overleftarrow{k}} E_2^n(\mathbb{Z}/l^k) = \lim_{\overleftarrow{k}} H^n(\{X_i, Y_i\}; \mathbb{Z}/l^k) \Rightarrow \hat{K}^*(\{X_i, Y_i\})$$

in view of (1.2i). The convergence of this spectral sequence when either of the asserted convergence criteria is satisfied follows from the same general convergence criterion.

We specialize Prop. 1.4 to the algebraic situation of Example 1.3. We remind the reader that $H^*((V., W.)_{et}; \mathbb{Z}/m)$ is naturally isomorphic to the relative etale cohomology $H^*_{et}(V., W.; \mathbb{Z}/m)$ (cf. [10], Proposition 3.10 for a description of this in terms of sheaf cohomology). Our consideration of schemes of finite type over complete discrete valuation rings with separably closed residue fields is a useful generalization of the algebraic varieties with which we will be primarily interested. Namely, this generalization permits lifting arguments comparing varieties in characteristic $p \neq 0$ with varieties in characteristic 0 (see, for example, Proposition 3.6 below).

Corollary 1.5. *Let i: $W. \to V.$ be a closed immersion of simplicial schemes of finite type over a complete discrete valuation ring F with separably closed residue field. For any integer m and any prime l both of which are invertible in F, there are spectral sequences*

$$E_2^n(\mathbb{Z}/m) = H^n_{et}(V., W.; \mathbb{Z}/m) \Rightarrow K^*((V., W.)_{et}; \mathbb{Z}/m)$$
$$E_2^n = \lim_{\overleftarrow{k}} H^n_{et}(V., W.; \mathbb{Z}/l^k) \Rightarrow \hat{K}^*((V., W.)_{et})$$

natural with respect to $W. \to V.$ and $\mathbb{Z}/m$, where $H^*_{et}(\ )$ denotes etale cohomology. The spectral sequence $\{E_r^n(\mathbb{Z}/m)\}$ is always convergent, whereas $\{E_r^n\}$ is convergent provided that $\lim_{\overleftarrow{k}} H^n_{et}(V., W.; \mathbb{Z}/l^k) = 0$ for n sufficiently large.

Proof. Deligne's finiteness theorem (Theorem 1.1 of "Th. finitude" of [8]) implies that $H^n_{et}(V, \mathbb{Z}/m)$ is finite for each n whenever V is a scheme of finite type over F. As argued in [10], this easily implies that $H^n_{et}(V., W.; \mathbb{Z}/m)$ is also finite for each n provided that both $V.$ and $W.$ are of finite type over F. Because $H^n_{et}(V., W.; \mathbb{Z}/m)$ is isomorphic to $H^n((V., W.)_{et}; \mathbb{Z}/m)$, the corollary follows directly from Prop. 1.4.

If i: $W. \to V.$ of Corollary 1.5 is a closed immersion of schemes, then $H^i_{et}(V., W.; \mathbb{Z}/l) = 0$ for i greater than twice the dimension of $V.$, so that the second spectral sequence also converges.

The spectral sequences of Corollary 1.5 and the Artin-Grothendieck comparison theorem between etale and singular cohomology provide the following comparison between the K-theory of the etale homotopy type and the K-theory of (the underlying topological space of) a complex algebraic variety. In particular, we conclude that the mod-m and l-adic K-theory of a complex algebraic variety is functorial with respect to (not necessarily continuous) algebraic maps.

Proposition 1.6. *For any closed immersion $i: W \to V$ of complex algebraic varieties, any integer $m>0$, and any prime l, there are natural isomorphisms*

$$K^*((V,W)_{et}; \mathbb{Z}/m) \simeq K^*(V^{top}, W^{top}; \mathbb{Z}/m)$$

$$\hat{K}^*((V,W)_{et}) \simeq K^*(V^{top}, W^{top}) \otimes \mathbb{Z}_l$$

where V^{top} and W^{top} denote the underlying topological spaces (with the complex topology) of V and W.

Proof. Because V^{top} admits a finite triangulation, (1.2.v) implies that the second asserted isomorphism follows immediately from the first. Let $HR(V^{top})$ denote the left filtering category of local homeomorphism hypercoverings $Z. \to V^{top}$. The forgetful functor $HR(V) \to HR(V^{top})$ together with the canonical maps $\pi(Z.) \leftarrow \Delta \circ \mathrm{Sin}(Z.) \to \mathrm{Sin}(V^{top})$ induce isomorphisms in cohomology

$$\begin{aligned} H^*_{et}(V,W; \mathbb{Z}/m) &\simeq H^*(\{\pi(U.), \pi(\psi_W(U.))\}_{U. \in HR(V)}; \mathbb{Z}/m) \\ &\xleftarrow{\sim} H^*(\{\pi(Z.), \pi(\psi_W(Z.))\}_{Z. \in HR(V^{top})}; \mathbb{Z}/m) \\ &\xrightarrow{\sim} H^*(\{\Delta \circ \mathrm{Sin}(Z.), \Delta \circ \mathrm{Sin}\,\psi_W(Z.)\}; \mathbb{Z}/m) \\ &\xrightarrow{\sim} H^*(\mathrm{Sin}(V^{top}), \mathrm{Sin}(W^{top}); \mathbb{Z}/m) \end{aligned}$$

by the Artin-Grothendieck comparison theorem as presented in [3] (see also [10], Proposition 11.3 for an appropriate version of Cox's tubular neighborhood theorem). By Corollary 1.5, these isomorphisms imply the isomorphism

$$K^*((V,W)_{et}; \mathbb{Z}/m) \simeq K^*(V^{top}, W^{top}; \mathbb{Z}/m)$$

between abutments of Atiyah-Hirzebruch spectral sequences, because we are in the situation in which these spectral sequences converge.

2. Chern Character

Proposition 2.3 establishes a multiplicative chern character from l-adic K-theory to rational l-adic cohomology for $\{X_i, Y_i\} \in \text{pro-}\mathscr{H}^2$. Using the Atiyah-Hirzebruch spectral sequence of Prop. 1.4, we give in Theorem 2.4 sufficient conditions that this chern character be a rational isomorphism. As seen in Corollary 2.5, these conditions apply to the etale homotopy type of a closed immersion of algebraic varieties. In the special case of complex algebraic varieties, the chern character corresponds to the usual topological chern character.

Because the inverse limit argument used to obtain the l-adic Atiyah-Hirzebruch spectral sequence requires a finiteness condition which only applies after taking derived exact couples, our proof of Theorem 2.4 utilizes the mod-l^k Atiyah-Hirzebruch spectral sequences. Consequently, we begin with a mod-m approximation of the chern character.

Definition 2.1. For $\{X_i, Y_i\} \in \text{pro-}\mathscr{H}^2$, $m>0$, $j \geqq 0$, and $\varepsilon = 0$ or 1, we define

$$(j+1) \cdot \widetilde{ch}_j: \; K^\varepsilon(\{X_i, Y_i\}; \mathbb{Z}/m) \to H^{2j-\varepsilon}(\{X_i, Y_i\}; \mathbb{Z}/m)$$

by sending α: $\{\Sigma^\varepsilon C(m)\wedge X_i/Y_i\}\to \#(\mathrm{Sin}BU)$ to $\Sigma^{-2-\varepsilon}\circ\alpha^*(\widetilde{ch}_{j+1})$, where $\widetilde{ch}_{j+1}\in H^{2j+2}(BU,\mathbb{Z})$ is the $(j+1)$-st Newton polynomial in the universal chern classes and $\Sigma^{-2-\varepsilon}$ is the de-suspension isomorphism.

Because the Bott map b: $\Sigma^2 BU\to BU$ satisfies $b^*(\widetilde{ch}_{j+1})=(j+1)\circ\widetilde{ch}_j$, the composite $\hat{K}^\varepsilon(\{X_i,Y_i\}_{\mathrm{red}}\to K^\varepsilon(\{X_i,Y_i\};\mathbb{Z}/l^k)\to H^{2j-\varepsilon}(\{X_i,Y_i\};\mathbb{Z}/l^k)$ sends a class β: $\{\Sigma^\varepsilon X_i/Y_i\}\to\#((\mathrm{Sin}BU)^\wedge)$ to the image under β^* of $(j+1)\circ\widetilde{ch}_j\in H^{2j}(BU,\mathbb{Z}/l^k)$ for any prime l.

So defined, $(j+1)\widetilde{ch}_j$ is the $(j+1)!$ multiple of the appropriate component of the chern character. One must divide by such factorials before obtaining a natural transformation of cohomology theories. As described in the following lemma, such division is the only obstruction to constructing a natural transformation of multiplicative cohomology theories.

Lemma 2.2. *For any* $\{X_i,Y_i\}\in\text{pro-}\mathscr{H}^2$ *and any integers* $m>0$ *and* $k\geqq 0$, *the following squares commute:*

$$\begin{array}{ccc} K^0(\{Y_i,\phi\};\mathbb{Z}/m) & \xrightarrow{\delta} & K^1(\{X_i,Y_i\};\mathbb{Z}/m) \\ {\scriptstyle (j+2)}\downarrow{\scriptstyle (j+1)\widetilde{ch}_j} & & \downarrow{\scriptstyle (j+2)\widetilde{ch}_{j+1}} \\ H^{2j}(\{Y_i\};\mathbb{Z}/m) & \xrightarrow{\delta} & H^{2j+1}(\{X_i,Y_i\};\mathbb{Z}/m), \end{array}$$

$$\begin{array}{ccc} K^1(\{Y_i,\phi\};\mathbb{Z}/m) & \xrightarrow{\delta} & K^0(\{X_i,Y_i\};\mathbb{Z}/m) \\ \downarrow{\scriptstyle (j+1)\widetilde{ch}_j} & & \downarrow{\scriptstyle (j+1)\widetilde{ch}_j} \\ H^{2j-1}(\{Y_i\};\mathbb{Z}/m) & \xrightarrow{\delta} & H^{2j}(\{X_i,Y_i\};\mathbb{Z}/m). \end{array}$$

Furthermore, if $H^n(\{X_i,Y_i\};\mathbb{Z}/m)=0$ *for* $n\geqq 2(N-1)$ *and if* m *is not congruent to* 2 mod 4, *then*

$$N!ch=\bigoplus_{j\geqq 0}N!\cdot(j+1)\cdot\widetilde{ch}_j/(j+1)!\colon\ K^*(\{X_i,Y_i\};\mathbb{Z}/m)\to H^*(\{X_i,Y_i\};\mathbb{Z}/m)$$

is an additive homomorphism between $\mathbb{Z}/2$*-graded commutative rings with the following multiplicative property:*

$$N!(N!ch)(\alpha\cdot\beta)=(N!ch)(\alpha)\cdot(N!ch)(\beta).$$

Proof. For any $i\in I$, the map $K^0(Y_i,\phi;\mathbb{Z}/m)\to K^1(X_i,Y_i;\mathbb{Z}/m)$ sends α: $C(m)\wedge|Y_i|^+\to BU$ to

$$\delta\alpha=b\circ\Sigma^2\alpha\circ\gamma\colon\ \Sigma C(m)\wedge|X_i/Y_i|\to\Sigma^2 C(m)\wedge|Y_i|^+\to\Sigma^2 BU\to BU.$$

Therefore,

$$\begin{aligned}(j+2)\circ\widetilde{ch}_{j+1}(\delta\alpha)&=\Sigma^{-3}(\delta\alpha)^*(\widetilde{ch}_{j+2})=\Sigma^{-1}(\Sigma^2\alpha\circ\gamma)^*((j+2)\widetilde{ch}_{j+1})\\&=(j+2)\gamma^*(\Sigma(j+1)\widetilde{ch}_j(\alpha))=(j+2)\delta((j+1)\widetilde{ch}_j(\alpha));\end{aligned}$$

thus, the first square commutes. Because $K^1(Y_i,\phi;\mathbb{Z}/m)\to K^0(X_i,Y_i;\mathbb{Z}/m)$ sends β: $\Sigma C(m)\wedge|Y_i|^+\to BU$ to

$$\delta\beta=\beta\circ\gamma\colon\ C(m)\wedge|X_i/Y_i|\to\Sigma C(m)\wedge|Y_i|^+\to BU,$$

the commutativity of the second square follows immediately from Def. 2.1.

Because the topological chern character is multiplicative, the map $\mu: BU \wedge BU \to BU$ determined by tensor product has the following cohomological behavior

$$N!\mu^*(N!ch_j) = \bigoplus_{\substack{s+t=j \\ 0 \neq s \neq j}} (N!ch_s) \otimes (N!ch_t) \in H^{2j}(BU \wedge BU, \mathbb{Z}).$$

Therefore, the asserted multiplicative behavior for $N!ch$ follows from the fact that multiplication in $K^*(\ ; \mathbb{Z}/m)$ is induced by $\mu: BU \wedge BU \to BU$ and the comultiplication map $\Sigma^2 C(m) \to C(m) \wedge C(m)$ (which induces an injection in homology).

Lemma 2.2 immediately implies the following proposition establishing the l-adic chern character. One could easily prove this proposition without using the mod-m approximations discussed in Lemma 2.2, but it is Lemma 2.2 which we shall use to prove Theorem 2.4.

Proposition 2.3. *For any prime l and any inverse system $\{X_i, Y_i\} \in \text{pro-}\mathscr{H}^2$ with the property that $H^n(\{X_i, Y_i\}; \mathbb{Z}/l) = 0$ for $n \geqq 2(N-1)$, the map ch defined by*

$$ch = (N!ch) \otimes (1/N!): \ \hat{K}^*(\{X_i, Y_i\}) \to \varprojlim_k H^*(\{X_i, Y_i\}; \mathbb{Z}/l^k) \otimes Q$$

is a natural transformation of $\mathbb{Z}/2$ graded, multiplicative cohomology theories (where $N!ch$ is the map defined in Lemma 2.2).

The reader should recognize the original argument of Atiyah and Hirzebruch [4] in the following proof.

Theorem 2.4. *Let l be a prime and $\{X_i, Y_i\} \in \text{pro-}\mathscr{H}^2$ be an inverse system of pairs of simplicial sets which satisfies the conditions that $H^n(\{X_1, Y_i\}; \mathbb{Z}/l)$ is finite for all n and $H^n(\{X_i, Y_i\}; \mathbb{Z}/l) = 0$ for n sufficiently large. Then the map*

$$ch \otimes Q: \ \hat{K}^*(\{X_i, Y_i\}) \otimes Q \to \varprojlim_k H^*(\{X_i, Y_i\}; \mathbb{Z}/l^k) \otimes Q$$

is an isomorphism. Furthermore, there is a natural isomorphism

$$Gr(\hat{K}^*(\{X_i, Y_i\})) \simeq \varprojlim_k H^*(\{X_i, Y_i\}; \mathbb{Z}/l^k),$$

where $Gr(\hat{K}^*(\{X_i, Y_i\})$ is the graded group associated to the skeletal filtration, provided that either $\varprojlim_k H^*(\{X_i, Y_i\}; \mathbb{Z}/l^k)$ has no torsion, $\varprojlim_k H^{ev}(\{X_i, Y_i\}; \mathbb{Z}/l^k) = 0$, or $\varprojlim_k H^{od}(\{X_i, Y_i\}; \mathbb{Z}/l^k) = 0$.

Proof. By Lemma 2.2, $N!ch$ maps the terms of the exact couple $(1.4.1)_i$ for $m = l^k$ to the terms of the following exact couple

$$(2.4.1)_i \qquad \bigoplus_n H^*(sk_{n-1}X_i, sk_{n-1}Y_i; \mathbb{Z}/l^k) \leftarrow \bigoplus_n H^*(sk_nX_i, sk_nY_i; \mathbb{Z}/l^k)$$

$$\bigoplus_n H^*(sk_nX_i, sk_nY_i \cup sk_{n-1}X_i; \mathbb{Z}/l^k)$$

in such a way that failure of commutativity is multiplication by an integer depending only on cohomology degree and not on either i or k. Consequently, $N!ch$ provides a similar map from the terms of $\varinjlim_i (1.4.1)'_i$ to the terms of $\varinjlim_i (2.4.1)'_i$ for each k, and thus a similar map from the terms of $\varprojlim_k \varinjlim_i (1.4.1)'_i$ to $\varprojlim_k \varinjlim_i (2.4.1)'_i$.

Consequently, $N!ch$ determines a map of exact couples

$$(N!ch)\otimes(1/N!)\colon \varprojlim_k \varinjlim_i (1.4.1)'_i \to \varprojlim_k \varinjlim_i (2.4.1)'_i \otimes Q$$

which determines the following map of spectral sequences:

$$E_2^n = \varprojlim_k H^n(\{X_i, Y_i\}; \mathbb{Z}/l^k) \Rightarrow \hat{K}^*(\{X_i, Y_i\}) \tag{2.4.2}$$

$$\downarrow$$

$$'E_2^n = \varprojlim_k H^n(\{X_i, Y_i\}; \mathbb{Z}/l^k)\otimes Q \Rightarrow \varprojlim_k H^*(\{X_i, Y_i\}; \mathbb{Z}/l^k)\otimes Q.$$

Because the map $N!ch$ of Lemma 2.2 determines $N!$ times the coefficient isomorphism

$$K^*(sk_n X_i, sk_n Y_i \cup sk_{n-1} X_i; \mathbb{Z}/m) \to H^*(sk_n X_i, sk_n Y_i \cup sk_{n-1} X_i; \mathbb{Z}/m)$$

for any $i\in I$, $m>0$, $n\geqq 0$, we conclude that $ch\colon E_2^n \to {}'E_2^n$ is isomorphic to $\otimes Q\colon E_2^n \to E_2^n \otimes Q$. Consequently, $ch\otimes Q\colon \{E_r^n\}\otimes Q \to \{'E_r^n\}$ is an isomorphism.

The convergence of $\{E_r^n\}$ proved in Prop. 1.4 now implies that

$$ch\otimes Q\colon\ K^*(\{X_i, Y_i\})\otimes Q \to \varprojlim H^*(\{X_i, Y_i\}; \mathbb{Z}/l^k)\otimes Q$$

is an isomorphism. Because the even differentials of $\{E_r^n\}$ are 0, $\{E_r^n\}$ collapses (i.e., $E_2^n = E_\infty^n$ for all n) whenever either

$$\varprojlim H^{ev}(\{X_i, Y_i\}; \mathbb{Z}/l^k)=0 \quad \text{or} \quad \varprojlim H^{od}(\{X_i, Y_i\}; \mathbb{Z}/l^k)=0.$$

Because $\{'E_r^n\}$ collapses, and because $E_2^n \to {}'E_2^n$ is injective whenever E_2^n has no torsion, we conclude that $\{E_r^n\}$ also collapses whenever $\varprojlim H^*(\{X_i, Y_i\}; \mathbb{Z}/l^k)$ has no torsion.

As argued in the proof of Corollary 1.5, Theorem 2.4 specializes to the case of the etale homotopy type of a closed immersion of schemes of finite type over a complete discrete valuation ring with separably closed residue field. We state this special case of Theorem 2.4 as a separate corollary.

Corollary 2.5. *Let $i\colon W \to V$ be a closed immersion of schemes of finite type over a complete discrete valuation ring F with separably closed residue field and let l be a prime invertible in F. Then there exists a natural isomorphism of $\mathbb{Z}/2$ graded algebras*

$$ch\otimes Q\colon\ \hat{K}^*((V, W)_{et})\otimes Q \xrightarrow{\sim} \varprojlim_k H^*_{et}(V, W; \mathbb{Z}/l^k)\otimes Q.$$

Moreover, $\varprojlim_k H^*_{et}(V, W; \mathbb{Z}/l^k)$ *is an associated graded group of* $\hat{K}^*((V, W)_{et})$ *provided that either* $\varprojlim_k H^{ev}_{et}(V, W; \mathbb{Z}/l^k) = 0$, $\varprojlim_k H^{od}_{et}(V, W; \mathbb{Z}/l^k) = 0$, *or* $\varprojlim_k H^*_{et}(V, W; \mathbb{Z}/l^k)$ *is torsion-free.*

For a closed immersion $W \to V$ of complex algebraic varieties, $(V, W)_{et}$ is related to (V^{top}, W^{top}) by a chain of maps in pro-$\mathscr{H}^2$ which induce isomorphisms in $\mathbb{Z}/l$ cohomology (as seen in the proof of Proposition 1.6). We recall that the usual chern character in algebraic topology can be defined by sending a classifying map $\alpha: \Sigma^\varepsilon T/S \to BU$ in $K^\varepsilon(T, S)$ to $\Sigma^{-\varepsilon} \circ \alpha^*(ch) \in H^*(T, S; Q)$. Thus, the construction of ch as presented in Proposition 2.3 has the following immediate corollary.

Corollary 2.6. *Let l be a prime and let $W \to V$ be a closed immersion of complex algebraic varieties. Then the following square commutes*

$$\begin{array}{ccc} \hat{K}^*((V, W)_{et}) & \longrightarrow & K^*(V^{top}, W^{top}) \otimes \mathbb{Z}/_l \\ \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch^{top} \otimes \mathbb{Z}_l} \\ \varprojlim H^*_{et}(V, W; \mathbb{Z}/l^k) \otimes Q & \xrightarrow{\sim} & H^*(V^{top}, W^{top}; Q_l) \end{array}$$

where the upper horizontal arrow is the isomorphism of Prop. 1.6 *and the lower horizontal arrow is that given by the Artin-Grothendieck comparison theorem (as described in the proof of Prop.* 1.6).

3. Relationship Between $K^0_{alg}(\ )$ and $K^0_{et}(\ )$

Definition 3.1 introduces the l-adic K-theory of a closed immersion of simplicial schemes over a strict hensel local ring. So defined, $\hat{K}^*_{et}(V., W.)$ is isomorphic to $\hat{K}^*((V., W.)_{et})$. As seen in Theorem 3.4, $\hat{K}^0_{et}(\ )$ is related to the Grothendieck group of algebraic vector bundles by a natural transformation $\hat{\rho}: K^0_{alg}(\ ) \to \hat{K}^0_{et}(\ )$. As seen in Propositions 3.5, 3.6 and 3.7, this natural transformation is closely related to the "forgetful functor" in characteristic 0 and factors the Grothendieck chern character.

As defined below, $\hat{K}^*_{et}(V., W.)$ is more intrinsic than $\hat{K}^*((V., W.)_{et})$. Such a definition is useful in defining $\hat{\rho}: K^0_{alg}(\ ) \to \hat{K}^0_{et}(\ )$ and necessary when considering galois actions.

Definition 3.1. Let l be prime and let $W. \to V.$ be a closed immersion of simplicial schemes locally of finite type over a strict hensel, noetherian local ring F of residue characteristic $p \neq l$. We define

$$\hat{K}^*_{et}(V., W.) = \hat{K}^*_{et}(V., W.)_{red} \times \varprojlim_k H^0_{et}(V., W.; \mathbb{Z}/l^k), \hat{K}^\varepsilon_{et}(V., W.)_{red}$$

$$\hat{K}^\varepsilon_{et}(V., W.)_{red} = \varprojlim_r \operatorname*{colim}_{m,n} \mathrm{Hom}_{\text{pro-}\mathscr{H}_0}(\Sigma^\varepsilon(V./W.)_{et}, \cos k_r((\mathrm{Grass}_{m+n,n})^\wedge_{et}))$$

where $\varepsilon=0$ or 1, $\mathrm{Grass}_{m+n,n}$ is the grassmann variety over F of m planes in $m+n$ space, and

$$(V./W.)_{\mathrm{et}}=\{\pi(\Delta U..)/\pi(\Delta\psi_{W.}(U..));\ U..\in HR(V.)\}.$$

The following isomorphism between $\hat{K}^*_{\mathrm{et}}(V.,W.)$ and $\hat{K}^*((V.,W.)_{\mathrm{et}}$ is natural only with respect to maps over F. This is the reason why $K^*((V.,W.)_{\mathrm{et}})$ is not a suitable theory when one considers galois actions.

Proposition 3.2. *Assume the hypotheses and notation of Def.* 3.1. *There exists an isomorphism of rings*

$$\eta\colon\ \hat{K}^*_{\mathrm{et}}(V.,W.)\xrightarrow{\sim}\hat{K}^*((V.,W.)_{\mathrm{et}})$$

natural with respect to maps over F, where the ring structure on $K^*_{\mathrm{et}}(V.,W.)$ is that induced by external direct sum and tensor product maps on grassmann varieties.

Proof. Because $ch_0\colon \hat{K}^0((V.,W.)_{\mathrm{et}})\to\varprojlim H^0_{\mathrm{et}}(V.,W.;\mathbb{Z}/l^k)$ is split surjective, it suffices to exhibit a suitable isomorphism

$$\hat{K}^*_{\mathrm{et}}(V.,W.)_{\mathrm{red}}\xrightarrow{\sim}\hat{K}^*((V.,W.)_{\mathrm{et}})_{\mathrm{red}}.$$

Because the natural map $\mathrm{Grass}_{m+n,n}(\mathbb{C})\to BU$ is an r-equivalence for $r=2(\min(m,n)-1)$, $\hat{K}^{\varepsilon}((V.,W.)_{\mathrm{et}})$ is naturally isomorphic to

$$\varprojlim_{r}\ \varinjlim_{m,n}\ \mathrm{Hom}_{\text{pro-}\mathscr{H}_0}(\Sigma^{\varepsilon}(V./W.)_{\mathrm{et}},\ \mathrm{cos}k_r(\mathrm{Grass}_{m+n,n}(\mathbb{C})^{\wedge})). \tag{3.2.1}$$

If F has positive characteristic, let R denote the Witt vectors of the residue field $\bar{F}$ of F; otherwise, let $R=F$. Choose a common embedding of R and $\mathbb{C}$ in an algebraically closed field. Such a choice determines weak equivalences in pro-$\mathscr{H}_0$

$$\begin{aligned}\mathrm{Sin}(\mathrm{Grass}_{m+n,n}(\mathbb{C}))^{\wedge}\to(\mathrm{Grass}_{m+n,n;\mathbb{C}})^{\wedge}_{\mathrm{et}}\to(\mathrm{Grass}_{m+n,n;R})^{\wedge}_{\mathrm{et}}\\ \leftarrow(\mathrm{Grass}_{m+n,n;\bar{F}})^{\wedge}_{\mathrm{et}}\to(\mathrm{Grass}_{m+n,n})^{\wedge}_{\mathrm{et}}.\end{aligned} \tag{3.2.2}$$

These weak equivalences induce an isomorphism between (3.2.1) and $K^{\varepsilon}_{\mathrm{et}}(V.,W.)_{\mathrm{red}}$.

The group structure on $\hat{K}^*((V.,W.)_{\mathrm{et}})$ is determined by the external direct sum maps on complex grassmannians:

$$\mathrm{Grass}_{m+n,n}(\mathbb{C})\times\mathrm{Grass}_{n'+m',n'}(\mathbb{C})\to\mathrm{Grass}_{m+m'+n+n',n+n'}(\mathbb{C}).$$

The group structure on $\hat{K}^*_{\mathrm{et}}(V.,W.)_{\mathrm{red}}$ is defined by applying $(\ )^{\wedge}_{\mathrm{et}}$ to the external direct sum maps of grassmann varieties over F:

$$\mathrm{Grass}_{m+n,n}\times\mathrm{Grass}_{m'+n',n'}\to\mathrm{Grass}_{m+m'+n+n',n+n'}.$$

Because the weak equivalences of (3.2.2) fit in commutative squares relating these external direct sum maps, η is a group homomorphism. A similar

argument shows that η is multiplicative, where the multiplication on $K^*_{et}(V., W.)_{red}$ is defined by applying $(\ )^\wedge_{et}$ to the external tensor product maps of grassmann varieties over F,

$$\mathrm{Grass}_{m+n,n} \times \mathrm{Grass}_{m'+n',n'} \to \mathrm{Grass}_{(m+n)(m'+n'),nn'},$$

and using the additive structure described above to adjust this multiplication to correspond to that on the zero component of $BU \times \mathbb{Z}$.

We next introduce the classifying map of a projective module over an affine variety trivialized on a closed subvariety.

Lemma 3.3. *Let $V = \mathrm{Spec}\, A$ be finite type over a noetherian ring R, $i: W = \mathrm{Spec}\, A/I \to V$ be a closed immersion, P be a rank n projective A module, and $\alpha: (A/I)^{\oplus n} \to P \otimes A/I$ be a trivialization of $P \otimes A/I$. For m sufficiently large there exists an algebraic map*

$$f_p: V \to \mathrm{Grass}_{m+n,n;R}$$

such that $f_p \circ i: W \to \mathrm{Grass}_{m+n,n;R}$ is the "point map" (i.e., $f_p \circ i$ factors through the standard section $\mathrm{Spec}\, R \to \mathrm{Grass}_{m+n,n;R}$) and $f^*(U) = P$, where U is the universal locally free, rank n quotient sheaf generated by $m+n$ global sections.

Furthermore, if $\dim V < m$ and if $f_p, f'_p: V \to \mathrm{Grass}_{m+n,n;R}$ are two such "classifying maps for P over V trivialized on W'', then there exists an algebraic map

$$H: V \underset{\mathrm{Spec}\, R}{\times} A^1_R \to \mathrm{Grass}_{2m+2n,n;R}$$

such that $H \circ j_0 = f_p$ and $H \circ j_1 = f'_p$ (where $j_i: V \to V \underset{\mathrm{Spec}\, R}{\times} A^1_R$ are the specialization maps $A \underset{R}{\otimes} R[t] \to A$ sending t to $i \in A$, and where $\mathrm{Grass}_{m+n,n;R}$ is identified with its image in $\mathrm{Grass}_{2m+2n,n;R}$ under the natural embedding) and such that $H \circ (i \times 1)$: $W \underset{\mathrm{Spec}\, R}{\times} A^1_R \to \mathrm{Grass}_{2m+2n,n;R}$ factors through the projection $W \underset{\mathrm{Spec}\, R}{\times} A^1_R \to A^1_R$.

Proof. Choose a set of generators $(p_1, \ldots, p_{n+m})$ for P over A such that $(\bar{p}_1, \ldots, \bar{p}_n)$ is the basis for $\bar{P} = P \otimes A/I$ over $\bar{A} = A/I$ specified by α and $\bar{p}_{n+1} = \ldots = \bar{p}_{n+m} = 0$ in $\bar{P}$. The isomorphism class of such a generating map $\rho: A^{\oplus n+m} \to P$ sending the basis vector e_i to p_i is in natural one-to-one correspondence with a classifying map $f_p: V \to \mathrm{Grass}_{m+n,n;R}$ [14]. Because the induced generating map $\bar{\rho}: \bar{A}^{\oplus n+m} \to \bar{P}$ is the projection onto the first n factors, $f_p \circ i: W \to \mathrm{Grass}_{m+n,n;R}$ is the point map.

Let ρ, ρ' be two such generating maps for P whose restrictions to $P \otimes A/I$ are specified by α; let ρ^{-1}, ρ'^{-1} denote left inverses of ρ, ρ'; and let $i: Q \to A^{\oplus n+m}$, $i': Q' \to A^{\oplus n+m}$ denote the kernels of ρ, ρ'. Then

$$\psi = (\rho'^{-1} \oplus i')^{-1}(\rho^{-1} \oplus i): P \oplus Q \xrightarrow{\sim} A^{\oplus n+m} \xrightarrow{\sim} P \oplus Q'$$

has the property that $\psi \otimes A/I$ restricts to an isomorphism $\bar{\psi}_1$: $Q \otimes A/I \to Q' \otimes A/I$ commuting with the inclusions $i \otimes A/I$, $i' \otimes A/I$. Provided that $\dim(V) < m$, we may use Theorem IV.3.4 of [5] to conclude the existence of an isomorphism ψ_1: $Q \to Q'$ with $\psi_1 \otimes A/I = \bar{\psi}_1$. Namely, we use the isomorphism

$$A^{\oplus n+m} \oplus Q \simeq Q \oplus P \oplus Q \simeq Q \oplus P \oplus Q' \simeq A^{\oplus n+m} \oplus Q'$$

the above mentioned theorems, and induction on $n+m$ to obtain ψ_1. Let ψ denote the map

$$\psi = (\rho'^{-1} \oplus i') \circ (1 \oplus \psi_1) \circ (\rho^{-1} \oplus i)^{-1} \colon A^{\oplus n+m} \to A^{\oplus n+m}$$

so that $\psi \otimes A/I$ is the identity on $i(Q) \otimes A/I$ and $\rho \circ \psi$, ρ' are the given generating maps.

We define H: $V \underset{\operatorname{Spec} R}{\times} A^1_R \to \operatorname{Grass}_{2m+2n,n;R}$ to be the composite of the homotopy $V \underset{\operatorname{Spec} R}{\times} A^1_R \to \operatorname{Grass}_{2m+2n,n;R}$ given by the generating map $A[t]^{\oplus 2n+2m} \to P \underset{A}{\otimes} A[t]$ which sends $x_1, \ldots, x_{n+m}$; $y_1, \ldots, y_{n+m}$ to $(1-t)\rho(x_1, \ldots, x_{n+m}) + t\rho \circ \psi(y_1, \ldots, y_{n+m})$ and the twisting homotopy

$$\operatorname{Grass}_{2m+2n,n;R} \underset{\operatorname{Spec} R}{\times} A^1_R \to \operatorname{Grass}_{2m+2n,n;R}$$

given by the action of the 1-parameter subgroup of $G1_{2m+2n,R}$ given by

$$\begin{pmatrix} (1-t^2)I_{n+m} & tI_{n+m} \\ (t^3-2t)I_{n+m} & (1-t^2)I_{n+m} \end{pmatrix}.$$

So constructed, $H \circ j_0 = f_p$, $H \circ j_1 = f'_p$, and $H \circ (i \times 1)$ factors through the projection $W \underset{\operatorname{Spec} R}{\times} A^1_R \to A^1_R$.

Using Lemma 3.3, we construct a natural transformation $\hat{\rho}$: $K^0_{\text{alg}}(\ ) \to \hat{K}^0_{\text{et}}(\ )$. We remind the reader that $K^0_{\text{alg}}(V, W)$ is the Grothendieck group of locally free, finite rank $\mathcal{O}_V$-modules P with a given trivialization of $P \underset{\mathcal{O}_V}{\otimes} \mathcal{O}_W$.

Theorem 3.4. *Let l be a prime and let F be a strict hensel, noetherian local ring of residue characteristic $p \neq l$. For any affine scheme V and any closed subscheme $W \to V$ of finite type over F, we define*

$$\hat{\rho}\colon\ K^0_{\text{alg}}(V, W) \to \hat{K}^0_{\text{et}}(V, W)$$

by sending a projective module P to $((f_p)_{\text{et}}, rk(P))$. So defined, extends to a natural transformation of ring valued functors defined on the category of closed immersions of quasi-projective schemes over F.

Proof. If f_p: $V \to \operatorname{Grass}_{m+n,n}$ is a classifying map for P over V trivialized on W, then $(f_p)_{\text{et}}$: $V_{\text{et}} \to (\operatorname{Grass}_{m+n,n})_{\text{et}}$ factors through the projection $V_{\text{et}} \to (V/W)_{\text{et}}$ because $f_p \circ i$ is the point map. If f'_p: $V \to \operatorname{Grass}_{m'+n,n}$ is another classifying map for P over V trivialized on W with $m, m' > \dim(V)$ and if

$$H\colon V \underset{\operatorname{Spec} F}{\times} A^1_F \to \operatorname{Grass}_{2m+2n,n;F}$$

relates f_p and f'_p as in Lemma 3.3, then H determines a map

$$H_{et}: (V \times A^1, W \times A^1)_{et} \to (\mathrm{Grass}_{2m+2n,n}, A^1)_{et}.$$

Because $(j_0)^\wedge_{et}$, $(j_1)^\wedge_{et}: (V/W)_{et} \to (V \times A^1/W \times A^1)^\wedge_{et}$ are weak equivalences and because the "loop" $(A^1)_{et} \to (\mathrm{Grass}_{2m+2n,n})^\wedge_{et}$ is contractible (as can be seen by lifting to characteristic 0), the existence of H_{et} implies that $(f_p)^\wedge_{et} = (f_p)^\wedge_{et}$ in $\hat{K}^0_{et}(V, W)_{red}$.

To verify that sending P to $((f_p)^\wedge_{et}, rk(P))$ (where $rk(P)$ in $\varprojlim H^0_{et}(V, W; \mathbb{Z}/l^k)$ sends each component of V not meeting W to the rank of P on that component) determines a ring homomorphism, we observe that

$$(f_{P \oplus Q})^\wedge_{et} = (f_p)^\wedge_{et} + (f_q)^\wedge_{et}, \quad (f_{P \otimes Q})^\wedge_{et} = (f_p)^\wedge_{et} \cdot (f_q)^\wedge_{et}$$

because a classifying map for $P \oplus Q$ (respectively, $P \otimes Q$) trivialized on W can be obtained by composing the product of f_p and f_q with the external direct sum (resp., tensor product) map on grassmann varieties.

To extend $\hat{\rho}$, we employ "affine resolutions": a map of finite type $\alpha: \tilde{V} \to V$ is said to be an affine resolution if $\tilde{V}$ is affine and if α is locally in the etale topology a product projection with affine spaces as fibre. As shown by Jouanolou in [13], any quasi-projective algebraic variety admits such an affine resolution. Using the Leray spectral sequence for an affine resolution $\alpha: \tilde{V} \to V$, we conclude that $\alpha_{et}: \tilde{V}_{et} \to V_{et}$ is a $\mathbb{Z}/l$-equivalence.

For any closed immersion $W \to V$ of quasi-projective schemes over F, we define $\hat{\rho}$ to be the composition

$$(\alpha_{et})^{*-1} \circ \hat{\rho} \circ \alpha^*: K^0_{alg}(V, W) \to K^0_{alg}(\tilde{V}, \tilde{W}) \to \hat{K}^0_{et}(\tilde{V}, \tilde{W}) \to \hat{K}^0_{et}(V, W)$$

where $\alpha: \tilde{V} \to V$ is an affine resolution with restriction $\tilde{W} \to W$ and $(\alpha_{et})^*: \hat{K}^0_{et}(V, W) \to \hat{K}^0_{et}(\tilde{V}, \tilde{W})$ is induced by the $\mathbb{Z}/l$-equivalence $(\tilde{V}/\tilde{W})_{et} \to (V/W)_{et}$. Because any two affine resolutions of V are dominated by a third, the naturality of $\hat{\rho}$ for affine varieties implies that $\hat{\rho}$ as defined above is independent of α. The naturality of $\hat{\rho}$ so extended is easily verified.

In Prop. 3.5, we show that $\hat{\rho}$ for closed immersion of complex algebraic varieties corresponds under the comparison isomorphism of Prop. 1.6 to the "forgetful functor."

Proposition 3.5. *Let l be a prime and let $i: W \to V$ be a closed immersion of complex algebraic varieties. Let*

$$\rho: K^0_{alg}(V, W) \to K^0(V^{top}, W^{top})$$

be the "forgetful functor" sending a finitely generated, locally free sheaf P on V with a given trivialization on W to the complex vector bundle $P^{top} = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_V}(P^*))^{top} \to V^{top}$ with the associated trivialization on W^{top}, where P^*

is the dual of P. Then the following square commutes

$$\begin{array}{ccc} K^0_{\mathrm{alg}}(V, W) & \xrightarrow{\rho \otimes \mathbb{Z}_l} & K^0(V^{\mathrm{top}}, W^{\mathrm{top}}) \otimes \mathbb{Z}_l \\ \downarrow{\scriptstyle \hat{\rho}} & & \downarrow \\ \hat{K}^0_{\mathrm{et}}(V, W) & \xrightarrow{\eta} & K^0((V, W)_{\mathrm{et}}) \end{array}$$

where the right vertical arrow is the isomorphism of Prop. 1.6.

Proof. If P is induced by the classifying map $f_P\colon V \to \mathrm{Grass}_{m+n,n;\mathbb{C}}$ trivialized on W, then $f_P^{\mathrm{top}}\colon V^{\mathrm{top}} \to \mathrm{Grass}_{m+n,n}(\mathbb{C})$ induces $P^{\mathrm{top}} \to V^{\mathrm{top}}$. Therefore, $(\rho \otimes \mathbb{Z}_l)(P)$ is determined by

$$i^\wedge \circ f_P^{\mathrm{top}}\colon\ V^{\mathrm{top}} \to \mathrm{Grass}_{m+n,n}(\mathbb{C}) \to BU^\wedge \times \mathbb{Z}^\wedge.$$

This map is sent to the composite map in pro-$\mathscr{H}_0$ determined by $(f_P)_{\mathrm{et}}$ and the comparison maps

$$V_{\mathrm{et}} \to (\mathrm{Grass}_{m+n,n;\mathbb{C}})_{\mathrm{et}} \to \#(\mathrm{Grass}_{m+n,n}(\mathbb{C})^\wedge) \to \#((\mathrm{Sin}\, BU)^\wedge) \times \mathbb{Z}^\wedge$$

by the isomorphism of Prop. 1.6. This composite map represents the image in $K^0((V, W)_{\mathrm{et}})$ of $((f_P)^\wedge_{\mathrm{et}}, rk(P)) \in K^0_{\mathrm{et}}(V, W)$ under η. Consequently, we have proved the proposition for V affine (in which case every P has a classifying map). The general case follows by using an affine resolution $\alpha\colon \tilde{V} \to V$ and the naturality of each of the maps $\hat{\rho}$, $\rho \otimes \mathbb{Z}_l$, and the isomorphisms of Propositions 1.6 and 3.2.

In conjunction with Proposition 3.5, the following proposition serves to partially identify $\hat{\rho}$ for positive residue characteristic varieties which are liftable to characteristic 0.

Proposition 3.6. *Let R be a strict hensel, noetherian local ring with residue field F and let K be a separably closed field containing R. Let l be a prime not equal to the characteristic of F. Let $i_R\colon W_R \to V_R$ be a closed immersion of schemes quasiprojective over R, let $j_0\colon (V_F, W_F) \to (V_R, W_R)$ and $j_1\colon (V_K, W_K) \to (V_R, W_R)$ be induced by base change (so that $V_F = V_R \underset{\mathrm{Spec}\, R}{\times} \mathrm{Spec}\, F$, etc.). Then the following diagram commutes*

$$\begin{array}{ccccc} K^0_{\mathrm{alg}}(V_F, W_F) & \xleftarrow{j_0^*} & K^0_{\mathrm{alg}}(V_R, W_R) & \xrightarrow{j_1^*} & K^0_{\mathrm{alg}}(V_K, W_K) \\ \downarrow{\scriptstyle \hat{\rho}} & & \downarrow{\scriptstyle \hat{\rho}} & & \downarrow{\scriptstyle \hat{\rho}} \\ \hat{K}^0_{\mathrm{et}}(V_F, W_F) & \xleftarrow{(j_0)^*_{\mathrm{et}}} & \hat{K}^0_{\mathrm{et}}(V_R, W_R) & \xrightarrow{(j_1)^*_{\mathrm{et}}} & \hat{K}^0_{\mathrm{et}}(V_K, W_K). \end{array}$$

Moreover, if V_R and W_R are projective and smooth over Spec$R$, *then $(j_0)^*_{\mathrm{et}}$ and $(j_1)^*_{\mathrm{et}}$ are isomorphisms.*

Proof. The map $(j_0)^*_{\mathrm{et}}$ is defined by composition on the left with the "inverse" of the weak equivalence $(\mathrm{Grass}_{m+n,n;F})^\wedge_{\mathrm{et}} \to (\mathrm{Grass}_{m+n,n;R})^\wedge_{\mathrm{et}}$ and on the right with $(V_F/W_F)_{\mathrm{et}} \to (V_R/W_R)_{\mathrm{et}}$; the map $(j_1)^*_{\mathrm{et}}$ is defined similarly. These maps are isomorphisms whenever both V_R and W_R are projective and smooth over Spec$R$, because $(V_F/W_F)_{\mathrm{et}} \to (V_R/W_R)_{\mathrm{et}}$ and $(V_K, W_K)_{\mathrm{et}} \to (V_K, W_R)_{\mathrm{et}}$ are then $\mathbb{Z}/l$-equivalences [10]. The commutativity of the left hand square follows from the fact that

if P is a locally free sheaf of rank n on V_R generated by $m+n$ global sections and provided with a trivialization when restricted to W_R and if $f_P\colon V_R \to \mathrm{Grass}_{m+n,n;R}$ is a classifying map for P over V_R trivialized on W_R, then a classifying map for $j_0^*(P)$ over V_F trivialized on W_F is given by $f_P \underset{\mathrm{Spec}\,R}{\times} \mathrm{Spec}\,F\colon V_F \to \mathrm{Grass}_{m+n,n;F}$. The commutativity of the right hand square is proved similarly.

Propositions 3.5 and 3.6 enable us to identify $ch \circ \hat{\rho}$ with Grothendieck's chern character from $K^0_{\mathrm{alg}}(\ )$ to $\varprojlim_k H^{\mathrm{ev}}_{\mathrm{et}}(\ ; \mathbb{Z}/l^k) \otimes Q$ [12].

Proposition 3.7. *Let l be a prime and let V be a quasi-projective algebraic variety over a separably closed field F of characteristic different from l. Then the composition*

$$ch \circ \eta \circ \hat{\rho}\colon\ K^0_{\mathrm{alg}}(V) \to \hat{K}^0_{\mathrm{et}}(V, \emptyset) \to \hat{K}^0((V,\emptyset)_{\mathrm{et}}) \to \varprojlim_k H^{\mathrm{ev}}_{\mathrm{et}}(V; \mathbb{Z}/l^k) \otimes Q$$

is the chern character of Grothendieck (see Prop. 5.5 for a Galois equivariant refinement).

Proof. Let $ch'\colon K^0_{\mathrm{alg}}(V) \to \varprojlim H^{\mathrm{ev}}_{\mathrm{et}}(V; \mathbb{Z}/l^k) \otimes Q$ denote Grothendieck's chern character. If $F = \mathbb{C}$, the following square commutes

$$\begin{array}{ccc} K^0_{\mathrm{alg}}(V, \emptyset) & \xrightarrow{\rho} & K^0(V^{\mathrm{top}}, \emptyset) \\ \downarrow{\scriptstyle ch'} & & \downarrow{\scriptstyle ch^{\mathrm{top}} \otimes \mathbb{Z}_l} \\ \varprojlim H^{\mathrm{ev}}_{\mathrm{et}}(V; \mathbb{Z}/l^k) \otimes Q & \xrightarrow{\sim} & H^{\mathrm{ev}}(V^{\mathrm{top}}, Q_l) \end{array}$$

where the bottom isomorphism is given by the Artin-Grothendieck comparison theorem. We conclude that $ch' = ch \circ \eta \circ \hat{\rho}$ by using Corollary 2.6 and Prop. 3.5 for complex quasi-projective varieties.

More generally, Prop. 3.6 and the naturality of $ch \circ \eta \circ \hat{\rho}$ and ch' enable us to conclude that $ch'(U) = ch \circ \eta \circ \hat{\rho}(U)$ where U is the universal locally free rank n quotient sheaf on $\mathrm{Grass}_{m+n,n;F}$ (which is the restriction of U_R over $\mathrm{Grass}_{m+n,n;R}$) for any $m, n > 0$. Consequently, $ch \circ \eta \circ \rho = ch'$ whenever V is affine because any finitely generated projective P is obtained from U over some $\mathrm{Grass}_{m+n,n;F}$ by pull-back via a classifying map for P. For an arbitrary quasi-projective V over F, we employ an affine resolution $\alpha\colon \tilde{V} \to V$: naturality of ch' and $ch \circ \eta \circ \hat{\rho}$ and the fact that

$$\alpha^*\colon \varprojlim H^{\mathrm{ev}}_{\mathrm{et}}(V; \mathbb{Z}/l^k) \to \varprojlim H^{\mathrm{ev}}_{\mathrm{et}}(\tilde{V}; \mathbb{Z}/l^k)$$

is an isomorphism imply that

$$ch \circ \eta \circ \hat{\rho} = ch'\colon\ K^0_{\mathrm{alg}}(V) \to \varprojlim H^{\mathrm{ev}}_{\mathrm{et}}(V; \mathbb{Z}/l^k) \otimes Q.$$

The preceeding proof, slightly modified, shows that Props. 3.5 and 3.6 characterize $\hat{\rho}$.

4. Operations in K-Theory

In Prop. 4.2, we exhibit a semi-ring $\mathrm{Vect}^{\wedge}(\{X_i, Y_i\})$ whose group completion is isomorphic to $\hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}} \times \mathbb{Z}$ and which admits the structure of a λ-semi-ring. These λ-operations determine Adams operations on $\hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}}$. In a similar manner, we obtain Adams operations on $\hat{K}^0_{\mathrm{et}}(V., W.)_{\mathrm{red}}$ arising from a λ-semi-ring structure on $\mathrm{Vect}^{\wedge}_{\mathrm{et}}(V., W.)$. As seen in Prop. 4.8, the natural transformation $\hat{\rho}$: $K^0_{\mathrm{alg}}(\cdot) \to \hat{K}^0_{\mathrm{et}}(\ )$ commutes with Adams operations. The reader will find several other propositions and corollaries in this section which help provide a computational description of the Adams operations.

In order to exhibit our "unstable" description of $\hat{K}^0(\{X_i, Y_i\})$ in Prop. 4.2, we require the following lemma (whose straightforward obstruction theoretic proof we omit).

Lemma 4.1. *Let l be a prime and let $\{X_i, Y_i\} \in \mathrm{pro}\text{-}\mathscr{H}^2$ be such that $H^n(\{X_i, Y_i\}; \mathbb{Z}/l) = 0$ for all $n \geqq N$, some given integer N. Let $\{B_j\} \in \mathrm{pro}\text{-}\mathscr{H}_0$ be such that each B_j is simply connected with only finitely many non-zero homotopy groups all of which above dimension $N-1$ are l-primary. Then f induces a bijection*

$$\mathrm{Hom}_{\mathrm{pro}\text{-}\mathscr{H}_0}(\{X_i/Y_i\}, \{B_j\}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{pro}\text{-}\mathscr{H}_0}(\{X_i/Y_i\}, \cos k_N\{B_j\}).$$

In the proof of the following proposition, we apply Lemma 4.1 to $\#((\mathrm{Sin}(BU))^{\wedge})$, using the fact that $\cos k_N(\#((\mathrm{Sin}(BU))^{\wedge}))$ is isomorphic to $\cos k_N(\#((\mathrm{Sin}(\mathrm{Grass}_{m+n,n}(C))^{\wedge})))$ for $N \leqq 2\min(m, n)$.

Proposition 4.2. *Let l be a prime and let $\{X_i, Y_i\} \in \mathrm{pro}\text{-}\mathscr{H}^2$ be such that $H^n(\{X_i, Y_i\}; \mathbb{Z}/l) = 0$ for all $n \geqq N$, some N sufficiently large. Let $\mathrm{Vect}^{\wedge}(\{X_i, Y_i\}) = \coprod_{n \geqq 0} \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})_n$ denote the monoid with*

$$\mathrm{Vect}^{\wedge}(\{X_i, Y_i\})_n = \operatorname*{colim}_m \mathrm{Hom}_{\mathrm{pro}\text{-}\mathscr{H}_0}(\{X_i/Y_i\}, \#((\mathrm{Sin}(\mathrm{Grass}_{m+n,n}(\mathbb{C}))^{\wedge})$$

whose additive structure is determined by the external direct sum maps

$$\mathrm{Grass}_{m+n,n}(\mathbb{C}) \times \mathrm{Grass}_{m'+n',n'}(\mathbb{C}) \to \mathrm{Grass}_{m+m'+n+n',n+n'}(\mathbb{C}).$$

Let ξ: $\mathrm{Vect}^{\wedge}(\{X_i, Y_i\}) \to \hat{K}^0(\{X_i, Y_i\})$ be the homomorphism sending $\alpha \in \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})_n$ to $(i^{\wedge} \circ \alpha, n) \in \hat{K}^0(\{X_i, Y_i\})$, where i: $\mathrm{Grass}_{m+n,n}(\mathbb{C}) \to BU$ is the natural inclusion. Then the group completion of ξ,

$$\xi^+ \colon \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})^+ \to \hat{K}^0(\{X_i, Y_i\})$$

is a ring homomorphism provided that $\mathrm{Vect}^{\wedge}(\{X_i, Y_i\})$ is given the multiplicative structure determined by the external tensor product maps on grassmannians. Furthermore, $\mathrm{Vect}^{\wedge}(\{X_i, Y_i\})^+$ can be identified (as groups) with $\hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}} \times \mathbb{Z}$ in such a way that ξ^+ restricts to the projection

$$\hat{K}(\{X_i, Y_i\})_{\mathrm{red}} \times [0] \xrightarrow{\sim} \hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}}.$$

Proof. Because the ring structure on $\hat{K}^0(\{X_i, Y_i\})$ is induced by the ring structure on $BU \times \mathbb{Z}$, which in turn is induced by the external direct sum and tensor product maps on complex grassmanians, ξ is a map of bi-monoids. Thus, ξ^+ is a ring homomorphism.

By Lemma 4.1, any element $\gamma\colon \{X_i/Y_i\} \to \#((\mathrm{Sin}\, BU)^\wedge)$ in $\hat{K}^0(X_i, Y_i\})_{\mathrm{red}}$ is of the form $\hat{i} \circ \alpha$ for some uniquely determined $\alpha \in \mathrm{Vect}^\wedge(V., W.)_n$ provided that $n \geqq N/2$. Moreover, if $n < N/2$ and if $\hat{i} \circ \alpha = \hat{i} \circ \alpha'$ with $\alpha, \alpha' \in \mathrm{Vect}^\wedge(\{X_i, Y_i\})_n$, then for any $\beta \in \mathrm{Vect}(\{X_i, Y_i\})_{n'}$

$$\alpha + \beta = \alpha' + \beta \in \mathrm{Vect}^\wedge(\{X_i, Y_i\})_n \quad \text{if } n + n' \geqq N/2.$$

This readily implies that $\mathrm{Vect}^\wedge(\{X_i, Y_i\})^+$ is isomorphic to $K^0(\{X_i, Y_i\})_{\mathrm{red}} \times \mathbb{Z}$ and that $\xi^+\colon \mathrm{Vect}^\wedge(\{X_i, Y_i\})^+ \to \hat{K}^0(\{X_i, Y_i\})$ restricts to the projection

$$\hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}} \times [0] \xrightarrow{\sim} \hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}}.$$

Because the weak equivalences of (3.2.2) are compatible with external direct sum and tensor product maps, we immediately conclude the following more natural form of Proposition 4.2.

Corollary 4.3. *Let l be a prime and let $W. \to V.$ be a closed immersion of simplicial schemes locally of finite type over a strict hensel, noetherian local ring F of residue characteristic $p \neq l$. Assume that $H^n_{\mathrm{et}}(V., W.; \mathbb{Z}/l) = 0$ for n sufficiently large. Let $\mathrm{Vect}^\wedge_{\mathrm{et}}(V., W.) = \coprod_{n \geqq 0} \mathrm{Vect}^\wedge_{\mathrm{et}}(V., W.)_n$ denote the semi-ring with*

$$\mathrm{Vect}^\wedge_{\mathrm{et}}(V., W.)_n = \varinjlim_m \mathrm{Hom}_{\text{pro-}\mathscr{H}_0}((V./W.)_{\mathrm{et}}, \#((\mathrm{Grass}_{m+n,n})^\wedge_{\mathrm{et}}))$$

with additive and multiplicative structures given by external direct sum and external tensor product maps on grassmannians over F. Then there exists a natural ring homomorphism

$$\xi^+\colon \mathrm{Vect}^\wedge_{\mathrm{et}}(V., W.)^+ \to \hat{K}^0_{\mathrm{et}}(V., W.)$$

and an identification of $\mathrm{Vect}^\wedge_{\mathrm{et}}(V., W.)^+$ with $\hat{K}^0_{\mathrm{et}}(V., W.)_{\mathrm{red}} \times \mathbb{Z}$ such that ξ^+ restricts to an isomorphism $\hat{K}^0_{\mathrm{et}}(V., W.)_{\mathrm{red}} \times [0] \xrightarrow{\sim} \hat{K}^0_{\mathrm{et}}(V., W.)_{\mathrm{red}}$.

Using the "unstable" monoid $\mathrm{Vect}^\wedge(\{X_i, Y_i\})$, we introduce λ-operations in the following proposition.

Proposition 4.4. *Let l be a prime and let $\{X_i, Y_i\} \in \text{pro-}\mathscr{H}^2$ be such that $H^n(\{X_i, Y_i\}; \mathbb{Z}/l) = 0$ for n sufficiently large. For any $n \geqq 0$ and any $k > 0$, let N denote the binomial coefficient $\binom{n}{k}$ and define*

$$\lambda^k\colon \mathrm{Vect}^\wedge(\{X_i, Y_i\})_n \to \mathrm{Vect}^\wedge(\{X_i, Y_i\})_N$$

to be the map induced by composition on the left with the weak completion of the maps

$$\lambda^k\colon \mathrm{Grass}_{m+n,n}(\mathbb{C}) \to \mathrm{Grass}_{M+N,N}(\mathbb{C})$$

which classify the k-th *exterior power of the universal rank* n *quotient bundle on* $\mathrm{Grass}_{m+n,n}(\mathbb{C})$ $\left(\text{so that } M=\binom{m+n}{k}-N\right)$. *So defined, the* λ^k's *determine a* λ*-ring structure on* $\mathrm{Vect}^{\wedge}(\{X_i, Y_i\})^+$. *In particular, the* λ^k's *determine Adams operations*

$$\psi^k\colon \hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}} \to \hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}}.$$

Proof. The compatibility of λ^k on $\mathrm{Grass}_{m+n,n}(\mathbb{C})$ as m varies and n is held fixed implies that λ^k is well defined on $\mathrm{Vect}^{\wedge}(\{X_i, Y_i\})$. Because $\lambda^0\colon \mathrm{Grass}_{m+n,n}(\mathbb{C}) \to \mathrm{Grass}_{1,1}(\mathbb{C}) = \{pt\}$ is the point map, $\lambda^0(\alpha)$ is the multiplicative unit of $\mathrm{Vect}^{\wedge}(\{X_i, Y_i\})$ for any $\alpha \in \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})$; because $\lambda_1\colon \mathrm{Grass}_{m+n,n}(\mathbb{C}) \to \mathrm{Grass}_{m+n,n}(\mathbb{C})$ is the identity map, $\lambda^1(\alpha)=\alpha$ for any $\alpha \in \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})$.

Following the usual construction of a λ-ring structure on K-theory, we define $\lambda_t\colon \mathrm{Vect}^{\wedge}(\{X_i, Y_i\}) \to \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})[[t]]$ by sending $\alpha \in \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})$ to $\lambda_t(\alpha) = \sum_{k \geqq 0} \lambda^k(\alpha) t^k$. Because

$$\lambda^k \circ \oplus = \oplus \circ \Big(\times_{s+t=k} \lambda^s \circ pr_1 \otimes \lambda^t \circ pr_2\Big)\colon \mathrm{Grass}_{m+n,n}(\mathbb{C}) \times \mathrm{Grass}_{m'+n',n'}(\mathbb{C}) \to \mathrm{Grass}_{m''+n'',n''}(\mathbb{C}),$$

we conclude that λ_t extends to a homomorphism

$$\lambda_t\colon \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})^+ \to (\mathrm{Vect}^{\wedge}(\{X_i, Y_i\})^+[[t]])^{\cdot}$$

where the right hand side is the multiplicative group of power series with constant term equal to 1. We define

$$\lambda^k\colon \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})^+ \to \mathrm{Vect}^{\wedge}(\{X_i, Y_i\})^+$$

to be the k-th coefficient of λ_t. Clearly, $\lambda^k(\alpha+\beta) = \bigoplus_{s+t=k} \lambda^s(\alpha)\lambda^t(\beta)$.

The Adams operation ψ^k on $\mathrm{Vect}^{\wedge}(\{X_i, Y_i\})^+$ is defined to be the k-th Newton polynomial in $\lambda^1, \ldots, \lambda^k$. Because ψ^k commutes with the projection $\mathrm{Vect}^{\wedge}(\{X_i\ Y_i\})^+ \simeq \hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}} \times \mathbb{Z} \to \mathbb{Z}$, we conclude that ψ^k induces a map $\psi^k\colon \hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}} \to \hat{K}^0(\{X_i, Y_i\})_{\mathrm{red}}$.

The following is a corollary of the proof of Prop. 4.4.

Corollary 4.5. *Let* l *be a prime and let* $W. \to V.$ *be a closed immersion of simplicial schemes locally of finite type over a strict hensel, noetherian local ring* F *of residue characteristic* $p \neq l$. *Assume that* $H^n_{\mathrm{et}}(V., W.; \mathbb{Z}/l)=0$ *for* n *sufficiently large. Then the exterior power maps on grassmannians over* F,

$$\lambda^k\colon \mathrm{Grass}_{m+n,n} \to \mathrm{Grass}_{M+N,N}; \quad N=\binom{n}{k},\ M=\binom{m+n}{k}-N,$$

determine a λ*-ring structure on* $\mathrm{Vect}^{\wedge}_{\mathrm{et}}(V., W.)^+$ *and Adams operations on* $\hat{K}^0_{\mathrm{et}}(V., W.)_{\mathrm{red}}$.

For computational purposes, we provide another description of the Adams operations on $\hat{K}^0(\{X_i, Y_i\})_{\text{red}}$ *and thus on* $\hat{K}^0_{\text{et}}(V., W.)_{\text{red}}$ *via the isomorphism*

$$\eta\colon \hat{K}^0_{\text{et}}(V., W.)_{\text{red}} \xrightarrow{\sim} \hat{K}^0((V., W.)_{\text{et}})_{\text{red}}.$$

Proposition 4.6. *Assume the hypotheses and notation of Prop.* 4.4. *Then the Adams operation*

$$\psi^k\colon\ \hat{K}^0(\{X_i, Y_i\})_{\text{red}} \to \hat{K}^0(\{X_i, Y_i\})_{\text{red}}$$

is induced by composition on the left with the weak completion of the map ψ^k: $BU \to BU$ *representing the k-th Adams operation in complex K-theory. Moreover, for any* $j>0$,

$$ch_j \circ \psi^k = k^j \circ ch_j\colon\ \hat{K}^0(\{X_i, Y_i\})_{\text{red}} \to \varprojlim_t H^{2j}(\{X_i, Y_i\}; \mathbb{Z}/l^t) \otimes Q.$$

Proof. Both $\psi^k\colon BU \times \mathbb{Z} \to BU \times \mathbb{Z}$ and $\psi^k\colon \text{Vect}^\wedge(\{X_i, Y_i\})^+ \to \text{Vect}^\wedge(\{X_i, Y_i\})^+$ are induced by the maps

$$S_k \circ \times i_r \circ \times \lambda^r\colon\ \text{Grass}_{m+n,n}(\mathbb{C}) \to \Pi\, \text{Grass}_{M_r+N_r,N_r}(\mathbb{C}) \to \Pi \text{BU} \times \mathbb{Z} \to \text{BU} \times \mathbb{Z}$$

where the products are indexed by r, $1 \leqq r \leqq k$, where $N_r = \binom{n}{r}$ and $M_r = \binom{m+n}{r} - N_r$, and where S_k is the k-th Newton polynomial. This implies that composition on the left with $\psi^{k\wedge}$ on $\#((\text{Sin}BU)^\wedge)$ determines ψ^k on $\hat{K}^0(\{X_i, Y_i\})_{\text{red}}$.

To prove that $ch_j \circ \psi^k = k^j \circ ch_j$, it suffices to prove for all $t>0$ that

$$(j+1)\widetilde{ch}_j \circ \psi^k = k^j \cdot (j+1)\widetilde{ch}_j\colon\ \hat{K}^0(\{X_i, Y_i\}) \to H^{2j}(\{X_i, Y_i\}; \mathbb{Z}/l^t)$$

where $(j+1)\widetilde{ch}_j$ is defined in Def. 2.1. Because $(j+1)\widetilde{ch}_j$ is the pull-back of $(j+1)!ch \in H^{2j}(BU, \mathbb{Z}/l^t)$, the above equality follows from the well known fact that $\psi^k(ch_j) = k^j ch_j \in H^{2j}(BU, Q)$.

The following is an essentially immediate corollary of Prop. 4.6.

Corollary 4.7. *Let* l *be a prime and let* $W \to V$ *be a closed immersion of complex algebraic varieties. Then* ψ^k *on* $\hat{K}^0_{\text{et}}(V, W)_{\text{red}}$ *corresponds to* $\psi^k \otimes 1$ *on* $K^0(V^{\text{top}}, W^{\text{top}})_{\text{red}} \otimes \mathbb{Z}_l$ *under the isomorphism* $\hat{K}^0_{\text{et}}(V, W) \simeq K^0(V^{\text{top}}, W^{\text{top}}) \otimes \mathbb{Z}_l$ *given by Prop.* 1.6 *and* 3.2.

We next compare ψ^k on $\hat{K}^0_{\text{et}}(V, W)$ to ψ^k on $K^0_{\text{alg}}(V, W)$.

Proposition 4.8. *Let* l *be a prime and let* $W \to V$ *be a closed immersion of quasi-projective schemes over a strict hensel, noetherian local ring of residue characteristic different from* l. *Then the homomorphism* $\hat{\rho}$ *of Theorem* 3.4 *factors as the following composite*

$$\hat{\rho} = \xi^+ \circ \rho'\colon\ K^0_{\text{alg}}(V, W) \to \text{Vect}^\wedge_{\text{et}}(V, W)^+ \to \hat{K}^0_{\text{et}}(V, W)$$

with ρ' *a homomorphism of* λ*-rings.*

Proof. Using an affine resolution $\alpha\colon \tilde{V} \to V$ of V, we may dominate (V, W) by the affine pair $(\tilde{V}, \tilde{W})$. Because $\alpha^*_{\text{et}}\colon \text{Vect}^\wedge_{\text{et}}(V, W) \to \text{Vect}^\wedge_{\text{et}}(\tilde{V}, \tilde{W})$ is an isomorphism, we

may assume that V is affine: $V=\operatorname{Spec}A$ and $W=\operatorname{Spec}A/I$. If P is a rank n projective A-module provided with α: $P\otimes A/I\simeq(A/I)^{\oplus n}$ and if f_p: $V\to \operatorname{Grass}_{m+n,n}$ is a classifying map for P on V trivialized on W, then $\lambda^j\circ f_p$: $V\to\operatorname{Grass}_{m+n,n}\to\operatorname{Grass}_{M+N,N}$ is a classifying map for $\lambda^j(P)$ on V trivialized on W for any $j\geqq 0$. Therefore, sending P to $(f_p)^{\wedge}_{\text{et}}$ determines a map of λ-semi-rings which induces ρ': $K^0(V,W)\to\operatorname{Vect}^{\wedge}_{\text{et}}(V,W)^+$. The equality $\hat{\rho}=\xi^+\circ\rho'$ follows immediately from the definitions of $\hat{\rho}$, ξ^+, and ρ'.

We conclude this section by giving another description of ψ^p for simplicial schemes in characteristic $p>0$.

Proposition 4.9. *Let l be a prime and let $W.\to V.$ be a closed immersion of simplicial schemes locally of finite type over a separably closed field of characteristic p, $0\neq p\neq l$. Assume that $H^n_{\text{et}}(V.,W.;\mathbb{Z}/l)=0$ for n sufficiently large. For any m, $n>0$, let φ^p: $\operatorname{Grass}_{m+n,n}\to\operatorname{Grass}_{m+n,n}$ be the geometric frobenius map. Then the following diagram commutes for each $n\geqq 0$*

$$\begin{array}{ccc}
\operatorname{Vect}^{\wedge}_{\text{et}}(V.,W.)_n & \xrightarrow{\varphi^{p\,\wedge}} & \operatorname{Vect}^{\wedge}_{\text{et}}(V.,W.)_n \\
\downarrow & & \downarrow \\
\operatorname{Vect}^{\wedge}_{\text{et}}(V.,W.)^+ & \xrightarrow{\psi^p} & \operatorname{Vect}^{\wedge}_{\text{et}}(V.,W.)^+
\end{array}$$

where the vertical maps are the canonical ones.

Proof. As discussed in the proof of Prop. 4.6, it suffices to prove that the following square commutes in pro-$\mathscr{H}_0$:

$$\begin{array}{ccc}
\#((\operatorname{Grass}_{m+n,n})^{\wedge}_{\text{et}}) & \xrightarrow{\varphi^{p\,\wedge}} & \#((\operatorname{Grass}_{m+n,n})^{\wedge}_{\text{et}}) \\
\downarrow{\scriptstyle \theta_{m,n}} & & \downarrow{\scriptstyle \hat{i}_{m,n}\circ\theta_{m,n}} \\
\#((\operatorname{Sin}\operatorname{Grass}_{m+n,n}(\mathbb{C})^{\wedge}) & \xrightarrow{(S_k\circ\times i_r\circ\lambda^r)^{\wedge}} & \#((\operatorname{Sin}BU\times\mathbb{Z})^{\wedge})
\end{array}$$

where $\theta_{m,n}$ is induced by the weak equivalences of (3.2.2). Because the map $\hat{i}_{m,n}\circ\theta_{m,n}$ represents $\eta\circ\hat{\rho}(U)\in\hat{K}^0((\operatorname{Grass}_{m+n,n})_{\text{et}})$, where U is the universal locally free rank n quotient sheaf on $\operatorname{Grass}_{m+n,n}$, the naturality of $\eta\circ\hat{\rho}$ implies that $\hat{i}_{m,n}\circ\theta_{m,n}\circ\varphi^{p\,\wedge}$ represents $\eta\circ\hat{\rho}(\varphi^{p*}(U))$. Because U is defined over $\mathbb{F}_p$, $\psi^p(U)=\varphi^{p*}(U)\in K_{\text{alg}}(\operatorname{Grass}_{m+n,n})$ (cf. [15]). By Prop. 4.8, $\eta\circ\hat{\rho}(\psi^p(U))=\psi^p\circ\eta\circ\hat{\rho}(U)$, so that the commutativity of the square follows from the observation in the proof of Prop. 4.6 that $(S_k\circ\times i_r\circ\lambda^r)^{\wedge}\circ\theta_{m,n}$ represents $\psi^p\circ\hat{i}_{m,n}\circ\theta_{m,n}=\psi^p\circ\eta\circ\hat{\rho}(U)$.

5. Galois Actions

In this section, we consider $\hat{K}^0_{\text{et}}(V.,W.)$ as a galois module. As seen in Prop. 5.3, the image of $\hat{\rho}$: $K^0_{\text{alg}}(V,W)\to\hat{K}^0_{\text{et}}(V,W)$ lies in the "discrete" submodule of $\hat{K}^0_{\text{et}}(V,W)$. Proposition 5.5 verifies that the chern character can be defined to be equivariant with respect to the galois action. This leads to Theorem 5.6 which asserts that the Tate Conjecture concerning algebraic cycles admits a K-theoretic formulation.

The following definition introduces the galois action. Throughout this section, we consider a given field F finitely generated over the prime field of characteristic $p \geqq 0$ and some algebraic closure $\bar{F}$ of F. We consider a fixed prime $l \neq p$, so that $(\ )^\wedge$ refers to l-adic completion.

Definition 5.1. Let $W. \to V.$ be a closed immersion of simplicial schemes locally of finite type over $\bar{F}$ defined over some finite extension L of F (i.e., obtained by base change from a map of simplicial schemes over L). The galois action of $\mathrm{Gal}(\bar{F}, L)$ on $\hat{K}^*_{\mathrm{et}}(V., W.)_{\mathrm{red}}$,

$$\mu\colon\ \mathrm{Gal}(\bar{F}, L) \times \hat{K}^\varepsilon_{\mathrm{et}}(V., W.)_{\mathrm{red}} \to \hat{K}^\varepsilon_{\mathrm{et}}(V., W.)_{\mathrm{red}},$$

is defined for $\varepsilon = 0$ or 1 by sending $(\sigma, \{\alpha_r\})$ to

$$\{(1 \otimes \sigma^{-1})^\wedge_{\mathrm{et}} \circ \alpha_r \circ (1 \otimes \sigma)_{\mathrm{et}}\colon\ \Sigma^\varepsilon(V./W.)_{\mathrm{et}} \to \mathrm{cos}\, k_r((\mathrm{Grass}_{m+n,n})^\wedge_{\mathrm{et}}\}.$$

We extend this action to all of $\hat{K}^*_{\mathrm{et}}(V., W.)$ by defining the action to be trivial on $\varprojlim_k H^0_{\mathrm{et}}(V., W.; \mathbb{Z}/l^k)$. We define the discrete submodule

$$\hat{K}^*_{\mathrm{et}}(V., W.)^G \subset \hat{K}^*_{\mathrm{et}}(V., W.)$$

to consist of those elements of $\hat{K}^*_{\mathrm{et}}(V., W.)$ left fixed by some subgroup of $\mathrm{Gal}(\bar{F}, L)$ of finite index.

In the above definition,

$$1 \otimes \sigma\colon\ \mathrm{Grass}_{m+n,n;F} = \mathrm{Grass}_{m+n,n;L} \underset{L}{\otimes} \bar{F} \to \mathrm{Grass}_{m+n,n;F}$$

is defined to be the map induced from $\sigma\colon \mathrm{Spec}\,\bar{F} \to \mathrm{Spec}\,\bar{F}$ by base change $\mathrm{Grass}_{m+n,n;F} \to \mathrm{Spec}\,\bar{F}$. The map $1 \otimes \sigma^{-1}\colon (V., W.) \to (V., W.)$ is similarly defined.

If $f\colon (V, W) \to (V', W')$ is a map between closed immersions of schemes of finite type over $\bar{F}$, then f is of the form

$$f_L \otimes 1\colon\ (V_L \underset{L}{\otimes} \bar{F}, W_L \underset{L}{\otimes} \bar{F}) \to (V'_L \underset{L}{\otimes} \bar{F}, W'_L \underset{L}{\otimes} \bar{F})$$

for some finite extension L of F. Consequently, $f^*\colon \hat{K}^0_{\mathrm{et}}(V', W') \to \hat{K}^0_{\mathrm{et}}(V, W)$ is $\mathrm{Gal}(\bar{F}, L)$-equivariant. In particular, f^* restricts to

$$f^*\colon\ \hat{K}^0_{\mathrm{et}}(V', W')^G \to \hat{K}^0_{\mathrm{et}}(V, W)^G.$$

We summarize these observations in the following proposition.

Proposition 5.2. *A map $f\colon (V, W) \to (V', W')$ between closed immersions of schemes of finite type over $\bar{F}$ induces a map $f^*\colon \hat{K}^0_{\mathrm{et}}(V', W) \to \hat{K}^0_{\mathrm{et}}(V, W)$ which is $\mathrm{Gal}(\bar{F}, L)$ equivariant for some finite extension L of F. Consequently, $\hat{K}^0_{\mathrm{et}}(V, W)^G$ determines a functor*

$$\hat{K}^0_{\mathrm{et}}(\ ,\ )^G\colon\ (\textit{closed immersion of finite type}/\bar{F}) \to (\textit{Rings}).$$

The fact that $\hat{K}^0_{\text{et}}(V,W)^G$ *is a ring is a consequence of the fact that addition and multiplication in* $\hat{K}^0_{\text{et}}(V,W)^G$ *are induced by maps*

$$\text{Grass}_{m+n,n} \times \text{Grass}_{m'+n',n'} \to \text{Grass}_{m+m'+n+n',n+n'}$$

and

$$\text{Grass}_{m+n,n} \times \text{Grass}_{m'+n',n'} \to \text{Grass}_{(m+n)(m'+n'),nn'}$$

which are defined over the prime field.

The importance of the subfunctor $\hat{K}^0_{\text{et}}(\ ,\ )^G$ *of* $\hat{K}^0_{\text{et}}(\ ,\ )$ *is revealed in the following proposition.*

Proposition 5.3. *Let* $W \to V$ *be a closed immersion of quasi-projective algebraic varieties over* $\bar{F}$. *Then the image of* $\hat{\rho}$: $K^0_{\text{alg}}(V,W) \to \hat{K}^0_{\text{et}}(V,W)$ *lies in* $\hat{K}^0_{\text{et}}(V,W)^G$.

Proof. By replacing (V,W) by $(\tilde{V},\tilde{W})$ for some affine resolution α: $\tilde{V} \to V$, we may assume (V,W) is an affine pair. Let $V = \text{Spec}\, A$, $W = \text{Spec}\, A/I$, P be a finitely generated projective A-module with a given trivialization of $P \otimes A/I$, and f_p: $V \to \text{Grass}_{m+n,n}$ be a classifying map for P over V trivialized on W. Because f_p is defined over some finite extension L of F, $f_p = (1 \otimes \sigma^{-1}) \circ f_p \circ (1 \otimes \sigma)$ for any σ in $\text{Gal}(\bar{F}, L)$. Therefore, $\hat{\rho}(P)$ in $\hat{K}^0_{\text{et}}(V,W)$ is invariant under $\text{Gal}(\bar{F}, L)$, a subgroup of finite index in $\text{Gal}(\bar{F}, F)$.

Following Tate in [17], we consider the $\text{Gal}(\bar{F}, F)$ module

$$Q_l(i) = Q \underset{\mathbb{Z}}{\otimes} (\varprojlim_n \mu_{l^n})^{\otimes i} \qquad i \geqq 0$$

where $\mu_{l^n} \subset \bar{F}^*$ denotes the multiplicative group of l^n-th roots of unity. A choice of isomorphism $\varprojlim_n \mu_{l^n} \simeq \mathbb{Z}_l$ identifies $Q_l(i)$ with Q_l provided with a specified $\text{Gal}(\bar{F}, F)$ module structure.

Definition 5.4. Let $W. \to V.$ be a closed immersion of simplicial schemes locally of finite type over $\bar{F}$ defined over a finite field extension L of F. We define $H^*(V., W.)(*)$ to be the following $\text{Gal}(\bar{F}, L)$ module

$$H^*(V., W.)(*) = \bigoplus_{j \geqq 0} (\varprojlim_k H^j(V., W.; \mathbb{Z}/l^k)) \underset{\mathbb{Z}_l}{\otimes} Q_l([(j+1)/2])$$

where $[i]$ is the greatest integer in i. We define the discrete submodule

$$H^*_l(V., W.)(*)^G \subset H^*_l(V., W.)(*)$$

to consist of those elements left fixed by some subgroup of $\text{Gal}(\bar{F}, L)$ of finite index.

We now proceed to refine the chern character of Prop. 2.3 so that it becomes galois equivariant on $\hat{K}^*_{\text{et}}(V., W.)$. This is done by limiting the un-naturality involved to the choice of values on the universal classes $\hat{\rho}(U_{m,n})$ (which has the effect of choosing an embedding of $\varprojlim_n \mu_{l^n}$ into $\mathbb{C}$).

Proposition 5.5. *Choose an embedding of the Witt vectors of* F *into* $\mathbb{C}$. *Let* $W. \to V.$ *be a closed immersion of simplicial schemes locally of finite type over* $\bar{F}$

defined over some finite extension L of F. Assume that $H^j_{et}(V.,W.;\mathbb{Z}/l)=0$ *for* $j\geqq N$. *Then composition of the chern character of Prop.* 2.3 *and the isomorphism* η *of Prop.* 3.2 *(determined by this choice of embedding)*

$$ch\circ\eta\colon\ \hat{K}^*_{et}(V.,W.)\to\varprojlim_k H^*((V.,W.)_{et};\mathbb{Z}/l^k)\otimes Q$$

can be defined as the map sending $\{\alpha_r\colon \Sigma^\varepsilon(V./W.)_{et}\to\cos k_r((\mathrm{Grass}_{m+n,n})^\wedge_{et})\}$ *in* $K_{et}(V.,W.)$ *to* $\Sigma^{-\varepsilon}\alpha_r^*(ch\circ\eta\circ\hat{\rho}(U_{m,n}))$ *for any* $r>N$, *where* $U_{m,n}$ *is the canonical locally free rank n quotient sheaf over* $\mathrm{Grass}_{m+n,n}$. *So defined, this chern character determines a* $\mathrm{Gal}(\bar{F},L)$*-equivariant homomorphism*

$$ch\colon\ \hat{K}^*_{et}(V.,W.)\to H^*_l(V.,W.)(*).$$

Proof. Because the natural inclusion $i\colon \mathrm{Grass}_{m+n,n}\to\mathrm{Grass}_{m+a+n+b,n+b}$ has the property that

$$i^*(ch\circ\hat{\rho}(U_{m+a,n+b}))=ch\circ\hat{\rho}(U_{m,n})\in\varprojlim_k \tilde{H}^*_{et}(\mathrm{Grass}_{m+n,n};\mathbb{Z}/l^k)\otimes Q$$

$\alpha_r^*(ch\circ\hat{\rho}(U_{m,n}))$ depends only on $\alpha=\{\alpha_r\}\in\hat{K}^*_{et}(V.,W.)_{red}$ provided that $r>N$. If $\alpha=\hat{\rho}(U_{m,n})\in\hat{K}^0_{et}(\mathrm{Grass}_{m+n,n})_{red}$, then clearly $\alpha_r^*(ch\circ\eta\circ\hat{\rho}(U_{m,n}))=ch\circ\eta\circ\hat{\rho}(U_{m,n})$ because α_r can be taken to be the canonical map

$$(\mathrm{Grass}_{m+n})_{et}\to\cos k_r((\mathrm{Grass}_{m+n,n})^\wedge_{et}).$$

Naturality of the definition of ch in Prop. 2.3 implies in general that

$$\Sigma^{-\varepsilon}\alpha_r^*(ch\circ\eta\circ\hat{\rho}(U_{m,n}))$$

equals ch applied to the class in $\hat{K}^\varepsilon((V.,W.)_{et})_{red}$ corresponding to

$$\{\alpha_r\}\in\hat{K}^\varepsilon_{et}(V,W.).$$

For any $\sigma\in\mathrm{Gal}(\bar{F},L)$,

$$(1\otimes\sigma^{-1})^*\colon\ \varprojlim_k \mathrm{H}^{2j}_{et}(\mathrm{Grass}_{m+n,n};\mathbb{Z}/l^k)\otimes Q$$
$$\to\varprojlim_k \mathrm{H}^{2j}_{et}(\mathrm{Grass}_{m+n,n};\mathbb{Z}/l^k)\otimes Q$$

sends $ch_j(\hat{\rho}(U_{m,n}))$ to $\sigma(1)^j\cdot ch_j(\hat{\rho}(U_{m,n}))$, where ch_j denotes $ch_j\circ\eta$ and $\sigma(1)^j$ is the eigenvalue for $\sigma\colon Q_l(j)\to Q_l(j)$ (this follows from the fact that $ch_j(\hat{\rho}(U_{m,n}))$ is an algebraic cohomology class). Consequently,

$$ch_j(\mu(\sigma,\alpha))=\Sigma^{-\varepsilon}(1\otimes\sigma)^*\circ\alpha_r^*(\sigma(1)^j\cdot ch_j(\hat{\rho}(U_{m,n})))$$
$$=\sigma(1)^j\cdot(1\otimes\sigma)^*\,ch_j(\alpha)$$

where μ is the Galois action of Def. 5.1.

This is precisely the image under the action of σ of $ch_j(\alpha)$ when viewed in $H^{2j-\varepsilon}_l(V.,W.)(j)$.

We conclude with our K-theoretic formulation of Tate's conjecture which asserts that the span (as an l-adic vector space) of the algebraic cycles in the l-adic cohomology of a projective, smooth, irreducible variety over $\bar{F}$ is precisely $H_l^{ev}(V)(*)^G$ [17].

Theorem 5.6. *Let V be a projective, smooth, irreducible variety over $\bar{F}$. Then Tate's conjecture is true if and only if*

$$\hat{\rho}\otimes Q_l\colon\ K^0_{alg}(V)\otimes Q_l\to(\hat{K}^0_{et}(V)\otimes Q)^G$$

is surjective.

Proof. Let $A^*(V)$ denote the Chow ring of cycles (with integer coefficients) modulo rational equivalence. Let

$$c\colon\ A^*(V)\to H_l^{ev}(V)(*)^G$$

be the cycle map sending a cycle to its cohomology class and let

$$ch^A\colon\ K^0_{alg}(V)\to A^*(V)\otimes Q$$

denote Grothendieck's chern character. Then the following square

$$\begin{array}{ccc} K^0_{alg}(V)\otimes Q_l & \xrightarrow{ch^A\otimes Q_l} & A^*(V)\otimes Q_l \\ \downarrow{\scriptstyle \hat{\rho}\otimes Q_l} & & \downarrow{\scriptstyle c\otimes Q_l} \\ (\hat{K}^0_{et}(V)\otimes Q)^G & \xrightarrow{ch\circ\eta} & (H_l^{ev}(V)(*))^G \end{array}$$

commutes by Props. 3.7 and 5.5. Corollary 2.5 and Prop. 5.5 imply that $ch\circ\eta\colon (\hat{K}^0_{et}(V)\otimes Q)^G\to(H^{ev}(V)(*))^G$ is an isomorphism. The surjectivity of $ch^A\otimes Q_l\colon K^0_{alg}(V)\otimes Q_l\to A^*(V)\otimes Q_l$ is proved in [14]. The theorem now follows immediately.

Appendix – A Convergence Criterion for Exact Couples

Our favorite reference for a convergence criterion for half-plane spectral sequences is that given by Adams in Theorem 8.2 of [1]. This is based on the work of Boardman (cf. [6], Appendix B). Because there appears to be no readily accessible proof of such convergence criteria published in the literature, we provide in Prop. A.2 a statement and proof of our interpretation of the Adams-Boardman convergence criterion. The greater generality of Prop. A.2 in comparison with Adams' Theorem 8.2 is required in our applications in Sect. 1.

We thank Henry Cejtin for several useful discussions.

We begin with an elementary lemma required in the proof of Prop. A.2. We employ $\varprojlim$ and $\varprojlim^1$ to denote the inverse limit functor and its first right derived functor (cf. [1], Sect. II.8).

Lemma A.1. Let $\mathbb{N}\times\mathbb{N}$ be the partially ordered set of pairs on non-negative integers in which the pair (m,n) maps to the pair (m',n') if and only if $m\geqq m'$ and

$n \geqq n'$. Let $A^{\cdot\cdot}\colon \mathbb{N}\times\mathbb{N} \to Ab$ be a functor from $\mathbb{N}\times\mathbb{N}$ to the category of abelian groups with the property that

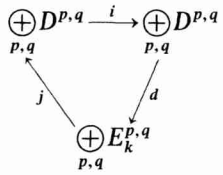
$$\varprojlim^1_m \varprojlim_n A^{\cdot\cdot} = 0 = \varprojlim_m \varprojlim^1_n A^{\cdot\cdot}.$$

Then

$$\varprojlim^1_n \varprojlim_m A^{\cdot\cdot} = \varprojlim_n \varprojlim^1_m A^{\cdot\cdot} = \varprojlim^1_{m,n} A^{\cdot\cdot} = 0.$$

Proof. We observe that $\varprojlim_m \varprojlim_n = \varprojlim_n \varprojlim_m = \varprojlim_{m.n}$ as functors from $Ab^{\mathbb{N}\times\mathbb{N}}$ to Ab. We recall that $Ab^{\mathbb{N}\times\mathbb{N}}$ has enough injectives which are a product of injectives of the form $\underline{I}^{\cdot\cdot}_{r,s}$, where I is an injective abelian group, $\underline{I}^{m,n}_{r,s} = I$ if $m \geqq r$ and $n \geqq s$, and $\underline{I}^{m,n}_{r,s} = 0$ otherwise. Moreover, $\varprojlim_m$ applied to such an injective of $Ab^{\mathbb{N}\times\mathbb{N}}$ is an injective of $Ab^{\mathbb{N}}$ because $\varprojlim_m$ commutes with products and $\varprojlim_m \underline{I}^{\cdot\cdot}_{r,s} = \underline{I}^{\cdot}_{s}$. Thus, we may use the Groethendieck spectral sequence for the right derived functors of the composite functor $\varprojlim_m \varprojlim_n$ to conclude that our hypotheses imply that $\varprojlim^1_{m,n} A^{\cdot\cdot} = 0$. Using the Grothendieck spectral sequence for the composite functor $\varprojlim_n \varprojlim_m$ plus the fact that $\varprojlim^2_m = 0$, we conclude that $\varprojlim^1_n \varprojlim_m A^{\cdot\cdot} = 0 = \varprojlim_n \varprojlim^1_m A^{\cdot\cdot}$.

Proposition A.2. Consider the following exact couple of bigraded abelian groups for some positive integer k:



where $-\infty < q < \infty$ and $0 \leqq p$, and where the bi-degrees of j, i, and d are $(0,0)$, $(-1,1)$ and $(k, 1-k)$ respectively. Let $E_r^{p,q}$ denote the $(r-k)^{\text{th}}$ derived group of $E_k^{p,q}$. Let $D^n = \varprojlim_p D^{p,n-p}$, and let $F^p D^n = \operatorname{im}(D^n \to D^{p,n-p})$.

If $\varprojlim^1_r E_r^{p,q} = 0$ for all p and q, then the exact couple converges in the following sense:

a) For any p and $q = n-p$, there is a short exact sequence

$$0 \to \varprojlim_r E_r^{p,q} \to F^p D^n \to F^{p-1} D^n \to 0.$$

b) For any p and any $q = n-p$, there is an isomorphism $D^n \xrightarrow{\sim} \varprojlim_p F^p D^n$.

Proof. The map $j\colon E_k^{p,q} \to D^{p,q}$ determines a non-zero map on those classes not in the image of $d_r\colon E_r^{p-r,q+r-1} \to E_r^{p,q}$ for any r with $k \leqq r \leqq p$. Thus, j determines the following isomorphisms for any $r > p+1$:

$$E_r^{p,q} \xrightarrow{\sim} \operatorname{im}[\ker(D^{p+r-1,q-r+1} \to D^{p-1,q+1}) \to \ker(D^{p,q} \to D^{p-1,q+1})]$$
$$\xrightarrow{\sim} \ker[\operatorname{im}(D^{p+r-1,q-r+1} \to D^{p,q}) \to \operatorname{im}(D^{p+r-1,q-r+1} \to D^{p-1,q+1})].$$

Because $\varprojlim^1 E_r^{p,q}=0$, we conclude the following short exact sequence

$$(*) \qquad 0\to \varprojlim E_r^{p,q}\to \varprojlim \operatorname{im}(D^{p+r-1,q-r+1}\to D^{p,q})$$
$$\to \varprojlim \operatorname{im}(D^{p+r-1,q-r+1}\to D^{p-1,q+1})\to 0.$$

Fix p and q. We define $A^{r,s}=\ker(D^{p+r-1,q-r+1}\to D^{p+s,q-s})$ for $s\leqq r-2$ and we define $A^{r,s}=0$ for $s>r-2$. Then $A^{r,s}$ determines a functor $A^{\cdot\cdot}\colon \mathbb{N}\times\mathbb{N}\to Ab$. We proceed to prove that

$$\varprojlim{}^1 A^{r,0}=\varprojlim{}^1 \ker(D^{p+r-1,q-r+1}\to D^{p,q})=0.$$

The serpent lemma implies the following short exact sequences whenever $r-s>p+s+1$

$$0\to A^{r,s}\to A^{r,s-1}\to E_{r-s}^{p+s,q-s}\to 0.$$

Because $\varprojlim^1 E_r^{p+s,q-s}=0$, we conclude that $\varprojlim^1 A^{r,s}\to\varprojlim^1 A^{r,s-1}$ and thus $\varprojlim_s \varprojlim^1 A^{r,s}\to\varprojlim^1 A^{r,0}$ are surjective. Therefore, to prove that $\varprojlim^1 A^{r,0}=0$, it suffices to prove that $\varprojlim_s \varprojlim^1 A^{r,s}=0$. By Lemma A.1, it suffices to show that $\varprojlim_s A^{r,s}=0=\varprojlim^1_s A^{r,s}$ for each r. This follows from the fact that $A^{r,s}=0$ for $s>r-2$.

Consequently, $\varprojlim \operatorname{im}(D^{p+r-1,q-r+1}\to D^{p,q})\xrightarrow{\sim} F^pD^{p+q}$, and similarly

$$\varprojlim \operatorname{im}(D^{p+r-1,q-r+1}\to D^{p-1,q-1})\xrightarrow{\sim} F^{p-1}D^{p+q}.$$

The short exact sequence of a.) now follows from the short exact sequence (*).

To prove the isomorphism of b), we consider the short exact sequences

$$0\to \ker(D^{p+r,n-p+r}\to D^{p,n-p})\to D^{p+r,n-p+r}$$
$$\to \operatorname{im}(D^{p+r,n-p+r}\to D^{p,n-p})\to 0.$$

Let $B^{p,r}=\ker(D^{p+r,n-p+r}\to D^{p,n-p})$. Applying $\varprojlim_p\varprojlim_r$ to the above short exact sequences and using the isomorphism $\varprojlim_r \operatorname{im}(D^{p+r,n-p+r}\to D^{p,n-p})\xrightarrow{\sim} F^pD^n$, we conclude that it suffices to prove that

$$\varprojlim_{p,r} B^{p,r}=0=\varprojlim^1_{p,r} B^{p,r}.$$

These equalities follow from the fact that $B^{p',r'}\to B^{p,r}$ is the zero map whenever $p'\geqq p+r$ and $r'\geqq r$ (so that $B^{\cdot\cdot}$ satisfies the Mittag-Leffler condition).

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