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Counting Types of Rigid Frameworks

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1 Introduction and Statement of Results

Given a finite, connected, simplicial 1-complex K, which is assumed fixed throughout the paper, we consider maps φ: $K \to \mathbb{R}^3$ which are linear on each simplex of K. We call such maps Euclidean frameworks modeled on K, or, simply, frameworks. If K has v vertices, the totality of frameworks modeled on K may be conveniently identified with $\mathbb{R}^{3v}$ and studied by the methods of linear algebra and real-algebraic geometry. Connelly [3] and Gluck [4] have used this approach to obtain some striking results on rigidity. Here, we use the decision theory for real-closed fields [7], p. 295ff., as well as some real-algebraic geometry, to answer questions about frameworks motivated by remarks of Grünbaum [5].

We call a framework φ rigid if every edge-length-preserving deformation of φ through frameworks is of the form $\omega_t \circ \varphi$, for some path ω_t of rigid, affine motions of $\mathbb{R}^3$, $\omega_0 = \text{identity}$.

Let RIG denote the set of rigid frameworks. Following Grünbaum [5], p. 2.15, we are interested in connectedness properties of RIG; however, first we must take care to exclude certain degenerate cases. To see why, note that the contraction h_t of $\mathbb{R}^3$ given by $h_t(x) = (1-t) \cdot x$, $0 \leqq t \leqq 1$, induces a contraction of RIG to a constant framework φ (i.e., φ is the constantly 0 map). Accordingly, we exclude these degeneracies, the set Δ of constant frameworks (see § 2.2), and we partition RIG$\setminus\Delta$ into path-components, which we call rigid types. Grünbaum (*op. cit.*) asks for a bound on the number of rigid types.

Theorem A. *The sum of the Betti numbers of* RIG$\setminus\Delta$ *is* $< 2^{36^v}$.

The Betti numbers used here are the ranks of the singular homology groups, so that the stated bound is also a bound on the number of path-components, i.e., the number of rigid types.

We are able to obtain a somewhat better bound on the number of rigid types (but not on the Betti-number sum) by an argument that first produces cor-

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responding bounds for certain restricted classes of rigid types (cf. end of § 5) and then pieces these together. If $\pi_0(\mathrm{RIG}\setminus\Delta)$ denotes the set of path-components of $\mathrm{RIG}\setminus\Delta$, then the improved bound is

$$\operatorname{card}\pi_0(\mathrm{RIG}\setminus\Delta)<4^{3^{6v-9}}. \tag{1}$$

Let FLEX denote the set of flexible frameworks: that is, $\mathrm{FLEX}=\mathbb{R}^{3v}\setminus\mathrm{RIG}$. Theorem A and (1) have analogues which apply to FLEX in place of $\mathrm{RIG}\setminus\Delta$. The bounds are similar.

To avoid some trivial special cases later, we shall make the harmless assumption from now on that the number v of vertices of K is $\geqq 3$.

We now consider the set $\mathrm{RIG}^\infty\subseteq\mathbb{R}^{3v}$ of all infinitesimally rigid (IR) frameworks (see § 2.1 for a definition). The implicit function theorem implies that $\mathrm{RIG}^\infty\subseteq\mathrm{RIG}$ (cf. [4], p. 234). The inclusion is proper. In fact, $\mathrm{RIG}\setminus\mathrm{RIG}^\infty\supseteq\Delta$ (see § 2.1), but there also may be frameworks in $\mathrm{RIG}\setminus\mathrm{RIG}^\infty$ that are geometrically interesting (see Connelly [3]).

Theorem B. *The sum of the Betti numbers of* RIG^∞ *is* $\leqq(6v-11)(12v-23)^{3v-3}$.

As in the case of Theorem A, the bound in Theorem B applies to the number of path-components. However, for path-components alone, we can again improve the bound:

$$\operatorname{card}\pi_0(\mathrm{RIG}^\infty)\leqq(6v-11)(12v-23)^{3v-6}. \tag{2}$$

Under certain conditions, the bound in (2) applies also to rigid types. Moreover specifically, we have

Theorem C. *If* $\mathrm{RIG}^\infty\neq\emptyset$, *then the inclusion-induced map* $\pi_0(\mathrm{RIG}^\infty)\to\pi_0(\mathrm{RIG}\setminus\Delta)$ *is onto. Therefore, when* $\mathrm{RIG}^\infty\neq\emptyset$,

$$\operatorname{card}\pi_0(\mathrm{RIG}\setminus\Delta)\leqq(6v-11)\cdot(12v-23)^{3v-6}. \tag{3}$$

Of course the bound in (3) is much better than that in (1). However, the range of applicability of (3), while by no means uninteresting, is limited. We give examples in § 3.2.

The set of infinitesimally flexible (IF) frameworks will be denoted by FLEX^∞: that is, $\mathrm{FLEX}^\infty=\mathbb{R}^{3v}\setminus\mathrm{RIG}^\infty$. We can obtain a bound in this case that is much better than those in (1)–(3): namely, it is strictly exponential in v:

Theorem D. $\operatorname{card}\pi_0(\mathrm{FLEX}^\infty\setminus\Delta)<4\cdot 6^{6v-11}$.

We do not know whether a similarly improved bound can be obtained for $\operatorname{card}\pi_0(\mathrm{FLEX})$.

The paper is organized as follows:

After some preliminaries (§ 2), we treat the case of IR and IF frameworks (§§ 3, 4), that is, Theorems B–D and (2). Our main tool is a theorem of Milnor [8] (see 3.1 this paper). The main idea of §§ 3, 4 is to give algebraic descriptions of RIG^∞ and $\mathrm{FLEX}^\infty\setminus\Delta$, or of suitable subsets of these, to which Milnor's theorem can be applied to good effect.

The case of rigid frameworks, however, leads to descriptions (§ 5) that require quantifier-elimination before Milnor's theorem can be applied. The decision procedure of Tarski and Seidenberg (see [7] or [10]) involves exactly the right kind of quantifier-elimination, but it is far too inefficient and inexplicit for our needs. Fortunately, there is a recent variant of the $T-S$ procedure, due to Collins [2], which is relatively efficient. Moreover, it has the additional advantage of directly producing the desired final bound, obviating the need for Milnor's theorem in this case and thereby saving one order of exponentiation.

The resulting bounds in Theorem A and in (1) are larger than those obtained in the infinitesimal cases, and all of these bounds are almost surely far from optimal. However, it is unlikely that significant improvements in our bounds can be obtained unless completely different techniques are used, which are more closely adapted to the specific problem.

In § 6, we describe Collins' main results and complete the proof of Theorem A.

The mere existence of the Tarski-Seidenberg decision procedure, when coupled with the results of § 5, yields a conclusion of some independent interest.

Theorem E. *There exists a finite algorithm for deciding whether a given framework modeled on K is rigid or flexible.*

Collins' more efficient and precise procedure produces an algorithm, together with a crude bound on its length (see 6.5). The bound, like that in Theorem A, is super-exponential as a function of v, and, thus, the algorithm cannot be used for actual computation.

Note that decidability questions for the infinitesimal cases are much easier, since they can be expressed in terms of determinants.

In Sect. § 7, we mention some variants on the above results, including another algorithm of some interest.

At this point I want to thank R. Connelly for his inspiring discussions of rigidity that aroused my interest and for bringing to my attention B. Grünbaum's provocative survey article [5]. I also want to thank A. Nerode for bringing Collins' work to my attention. And finally, I want to thank C.M. Wagner for some helpful conversations and especially for pointing out an error in the first draft.

2 Preliminaries

2.1 Definitions. We begin with the algebraic setting, which is similar to that described in [4].

Order the vertices of K in some arbitrary but fixed manner. A framework may be viewed then as an ordered v-tuple $p=(p_1, p_2, \ldots, p_v) \in \mathbb{R}^{3v}$ of vertices in $\mathbb{R}^3$, together with an edge structure $\mathscr{E}$. $\mathscr{E}$ is a set of ordered pairs (i,j), $1 \leqq i < j \leqq v$, with (i,j) in $\mathscr{E}$ if and only if the i^{th} and j^{th} vertices of K are joined by an edge. $\mathscr{E}$ depends only on K, which is fixed, and so $\mathscr{E}$ is usually suppressed from the notation and discussion.

The constant frameworks in this setting form a 3-dimensional vector subspace Δ (the "diagonal") $\subseteq \mathbb{R}^{3v}$, given by the equations $p_1 = p_2 = \ldots = p_v$.

For any (i,j), $1 \leqq i < j \leqq v$, we define a quadratic polynomial

$$\lambda_{ij}(p) = \|p_i - p_j\|^2,$$

where $\| \ \|$ is the usual Euclidean norm. We then define functions $\lambda: \mathbb{R}^{3v} \to \mathbb{R}^e$ and $\Lambda: \mathbb{R}^{3v} \to \mathbb{R}^E$ (e = cardinality $\mathscr{E}$, $E = \frac{1}{2}v(v-1)$) as follows: For $p \in \mathbb{R}^{3v}$, $\lambda(p)$ (resp., $\Lambda(p)$) is the e-tuple (resp., E-tuple) of all $\lambda_{ij}(p)$ such that $(i,j) \in \mathscr{E}$ (resp., such that $1 \leqq i < j \leqq v$), the $\lambda_{ij}(p)$ ordered lexicographically according to the subscripts.

Two frameworks $p, q \in \mathbb{R}^{3v}$ are called isometric if $\lambda(p) = \lambda(q)$ and congruent if $\Lambda(p) = \Lambda(q)$, [4]. The isometry (resp., congruence) class of p is just the fibre $\lambda^{-1}\lambda(p)$ (resp., $\Lambda^{-1}\Delta(p)$). Clearly, $\Lambda^{-1}\Lambda(p) \subseteq \lambda^{-1}\lambda(p)$, and both are algebraic sets.

Let $E(3)$ denote the group of rigid, affine motions of $\mathbb{R}^3$ (i.e., the group generated by rotations, reflections, and translations), and let $E^+(3)$ denote its identity component. $E(3)$ acts on $\mathbb{R}^{3v}$ via the diagonal action on $\mathbb{R}^3 \times \mathbb{R}^3 \times \ldots \times \mathbb{R}^3$, and the fibres $\lambda^{-1}\lambda(p)$ and $\Lambda^{-1}\Lambda(p)$ are invariant under this action. In fact, it is not hard to verify that $\Lambda^{-1}\Lambda(p) = E(3) \cdot p$, [4]. If $C(p)$ denotes the path-component of p in $\lambda^{-1}\lambda(p)$, then, of course, $E^+(3) \cdot p \subseteq C(p)$.

The definition of rigidity asserts that p is rigid if and only if $C(p) \subseteq E^+(3) \cdot p$.

Since the constant frameworks Δ are transformed transitively by $E^+(3)$, we have, for $p \in \Delta$, $\Delta = C(p) = E^+(3) \cdot p$, and so the constant frameworks are rigid. That is, recalling the notation introduced in § 1, $\Delta \subseteq \mathrm{RIG}$.

Both Δ and RIG are $E(3)$-invariant subsets of $\mathbb{R}^{3v}$.

Next we introduce the notion of infinitesimal rigidity, recalling that we have assumed $v \geqq 3$. We say that a framework p is infinitesimally rigid (IR) if and only if the differential $d\lambda(p)$: $\mathbb{R}^{3v} \to \mathbb{R}^e$ has rank $3v-6$, which is the maximum possible (see [4], p. 238 ff.). As mentioned in § 1, the implicit function theorem shows that $\mathrm{RIG}^\infty \subseteq \mathrm{RIG}$, [4]. Clearly $\operatorname{rank} d\lambda(p) = 0$ when $p \in \Delta$, so that $\Delta \subseteq \mathrm{FLEX}^\infty$.

One verifies easily that RIG^∞ is $E(3)$-invariant.

Now let X denote any one of the $E(3)$-invariant sets RIG, FLEX, RIG^∞, or FLEX^∞. We show how X decomposes with respect to Δ.

2.2 Lemma. *Let V denote any vector subspace of $\mathbb{R}^{3v}$ complementary to Δ, i.e., we have an internal direct sum $\mathbb{R}^{3v} = \Delta \oplus V$. Then,*

$$X = \Delta \oplus (X \cap V).$$

Proof. Let $\pi: \mathbb{R}^{3v} \to V$ be the projection along Δ. For any $p \in V$, $\pi^{-1}(p) \subseteq E(3) \cdot p$. It follows that $X = \pi^{-1}(X \cap V)$, from which the result is immediate. □

Since Δ is contractible, 2.2 shows that X and $X \cap V$ have the same homotopy type.

3 Counting IR Types

The bounds we obtain here derive from the following theorem of Milnor.

3.1 Theorem [8]. *Let A be an algebraic subset of $\mathbb{R}^m$ defined by a finite number of polynomials each having degree $\leqq k$. Then the sum of the Betti numbers*[1] *of A is $\leqq k(2k-1)^{m-1}$.*

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[1] Milnor uses Čech Betti numbers, but since A has the homotopy type of a finite CW complex (see 6.2(d)), these coincide with the usual singular Betti numbers. The same holds for all the sets we consider. In particular, β_0 equals the number of path-components

Proof of Theorem B. Choose a vector subspace $V \subseteq \mathbb{R}^{3v}$ complementary to Δ (cf. 2.2), and identify it with $\mathbb{R}^{3v-3}$. For $p \in V$, let $D(p)$ denote the sum of the squares of all $(3v-6) \times (3v-6)$ subdeterminants of $d\lambda(p)$. An easy calculation shows that $D(p)$ is a polynomial in $3v-3$ variables of degree $6v-12$. Clearly, $p \in \mathrm{RIG}^\infty \cap V \Leftrightarrow D(p) \neq 0$. Thus, $\mathrm{RIG}^\infty \cap V$ may be obtained from

$$A = \{(p,t) \in V \times \mathbb{R} = \mathbb{R}^{3v-2} \mid t \cdot D(p) - 1 = 0\}$$

by projecting $(p,t) \mapsto p$. In fact, an inverse $\mathrm{RIG}^\infty \cap V \to A$ exists, given by $p \mapsto (p, D(p)^{-1})$, showing that these two sets are homeomorphic.

By 3.1, the sum of the Betti numbers of A (and, hence, of $\mathrm{RIG}^\infty \cap V$) is $\leqq (6v-11)(12v-23)^{3v-3}$. By 2.2, this also applies to RIG^∞. □

If we are interested only in a bound on card $\pi_0(\mathrm{RIG}^\infty)$, we can improve on the above by restricting to smaller subspaces of $\mathbb{R}^{3v}$. In particular, if $W \subseteq \mathbb{R}^{3v}$ is a subspace that meets every $E^+(3)$-orbit, then the inclusion-induced map

$$\pi_0(\mathrm{RIG}^\infty \cap W) \to \pi_0(\mathrm{RIG}^\infty)$$

is onto, and so it suffices to obtain a bound on card $\pi_0(\mathrm{RIG}^\infty \cap W)$. We choose a subspace W of dimension $3v-6$ (below), and then we argue as in the proof of Theorem B. The corresponding polynomial D involves $3v-6$ variables and has degree $6v-12$. Therefore, the sum of the Betti numbers of $\mathrm{RIG}^\infty \cap W$ is $\leqq (6v-11)(12v-23)^{3v-6}$, which yields the desired bound on card $\pi_0(\mathrm{RIG}^\infty)$ (thus proving (2) of § 1).

To define W, we assume without loss of generality that $(1,2)$ and $(1,3) \in \mathscr{E}$. Then, W is given by the conditions $p_1 = 0$, $p_2 \in \mathbb{R} \times 0 \times 0$, and $p_3 \in \mathbb{R} \times \mathbb{R} \times 0$. Clearly, W has dimension $3v-6$ and meets every $E^+(3)$-orbit. Henceforth, we reserve the symbol "W" to denote this particular subspace of $\mathbb{R}^{3v}$.

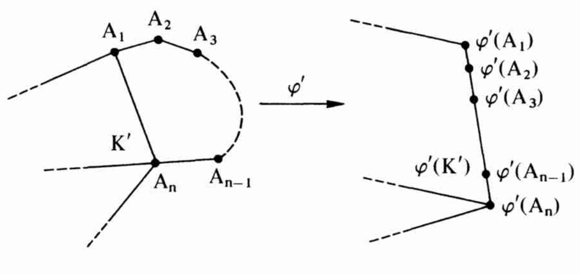
Proof of Theorem C. We must show that if $\mathrm{RIG}^\infty \neq \emptyset$, then $\pi_0(\mathrm{RIG}^\infty) \to \pi_0(\mathrm{RIG} \setminus \Delta)$ is onto.

Recall that $\mathrm{RIG}^\infty \subseteq \mathrm{RIG} \setminus \Delta$. Since RIG^∞ is the complement of an algebraic set and non-empty, it must be dense in $\mathbb{R}^{3v}$, hence dense in $\mathrm{RIG} \setminus \Delta$. But this latter set is semi-algebraic (see § 6): that is, it is a finite union of solutions of systems of polynomial equalities and inequalities. A semi-algebraic set is locally-path connected [6] (also see § 6, esp. 6.2(*d*)), so that its path-components are open. Thus, each component of $\mathrm{RIG} \setminus \Delta$ meets RIG^∞. □

3.2 *Remarks. a*) When $\mathrm{RIG}^\infty \neq \emptyset$, every framework $p \in \mathbb{R}^{3v}$ admits arbitrarily small analytic perturbations $p(t)$, $0 \leqq t < \varepsilon$, with $p = p(0)$ and $p(t) \in \mathrm{RIG}^\infty$, $0 < t < \varepsilon$. This uses Milnor's Curve Selection Lemma [9] or a similar result in Wallace [11], and it provides an alternate, somewhat less elementary, proof of Theorem C.

b) By a result in Gluck [4], every 1-skeleton of a triangulation of S^2 admits imbeddings into $\mathbb{R}^3$ which are IR frameworks. Therefore, for every such K, $\mathrm{RIG}^\infty \neq \emptyset$, and so Theorem C applies.

There are, however, simple modifications of these K for which $\mathrm{RIG}^\infty = \emptyset$. Let a K as above be given, and, to an edge labeled $A_1 A_n$, attach edges $A_1 A_2, A_2 A_3, \ldots, A_{n-1} A_n$, for every $n \geqq 4$. Call the result K'.



By a standard counting argument, K satisfies $e=3v-6$, whereas K' satisfies $e'=e+n-1$, $v'=v+n-2$. Therefore, $e'<3v'-6$ so that K' has no IR frameworks.

On the other hand, K' does have rigid frameworks. For an example, let φ: $K\to\mathbb{R}^3$ be an IR imbedding and extend it to a framework φ': $K'\to\mathbb{R}^3$ by mapping the vertices $A_2, A_3, \ldots, A_{n-1}$ in order onto the edge $\varphi(A_1 A_n)$, as pictured above. φ' is rigid.

4 Counting IF Types

The proof of Theorem B in § 3 can easily be modified to give a similar bound for card $\pi_0(\mathrm{FLEX}^\infty\setminus\Delta)$. However, a much better (strictly exponential) bound can be obtained by a less direct argument.

We use the notation and conventions of § 3. In particular, we recall that W denotes the subspace of $\mathbb{R}^{3v}$ given by $p_1=0$, $p_2\in\mathbb{R}\times 0\times 0$, and $p_3\in\mathbb{R}\times\mathbb{R}\times 0$, where we assume $(1,2)$, $(1,3)\in\mathscr{E}$.

Certain kinds of degenerate frameworks must be avoided in our arguments: namely, the frameworks p for which p_1, p_2, p_3 are collinear. We call all remaining frameworks generic, and we denote the set of generic frameworks by GEN.

4.1 Lemma. *The $E(3)$-orbit of a generic framework is a smooth 6-manifold transverse to W.*

Proof. Suppose p is generic. The first assertion follows immediately from standard facts about smooth actions, together with the fact that the isotropy subgroup of p is finite. It remains to verify transversality.

We may assume that $p\in W$. The tangent directions of $E(3)\cdot p$ at $p=(0, p_2, p_3, \ldots, p_v)\in W$ consist of all vectors of the form

$$(*)\qquad (0, A p_2, A p_3, \ldots, A p_v)+(b, b, \ldots, b),$$

where A ranges over all 3×3 skew-symmetric matrices and b ranges over $\mathbb{R}^3$. Given any $q_1, q_2, q_3\in\mathbb{R}^3$, it is not hard to find A and b, together with $p'_2\in\mathbb{R}\times 0\times 0$ and $p'_3\in\mathbb{R}\times\mathbb{R}\times 0$, such that

$$(0, A p_2, A p_3)+(b, b, b)+(0, p'_2, p'_3)=(q_1, q_2, q_3).$$

It follows that the vectors $(*)$, together with W, span $\mathbb{R}^{3v}$, which is just the assertion of transversality. □

4.2 Lemma. *A framework* $p \in W \cap \mathrm{GEN}$ *is* $\mathrm{IR} \Leftrightarrow d(\lambda|W)(p)$: $W \to \mathbb{R}^e$ *has* rank $3v-6$.

Proof. In one direction the implication is trivial. Suppose then that $p \in W \cap \mathrm{GEN} \cap \mathrm{RIG}^\infty$, so that $d\lambda(p)$: $\mathbb{R}^{3v} \to \mathbb{R}^e$ has (maximum) rank $3v-6$. Since rigid motions do not change edge-length, the tangent space of $E(3) \cdot p$ at p is contained in $\ker d\lambda(p)$. By 4.1, this tangent space has dimension six and is complementary to W. Therefore, it equals $\ker d\lambda(p)$, and $\ker d(\lambda|W)(p) = \ker d\lambda(p) \cap W = \{0\}$, as desired. □

4.3 Corollary. *A framework* $p \in W \cap \mathrm{GEN}$ *is* $\mathrm{IF} \Leftrightarrow \exists q \in W$ *such that* $q \neq 0$ *and* $d\lambda(p)(q) = 0$. □

Now choose any $p \in W$, so that $p_1 = 0$, $p_2 = (x,0,0)$, and $p_3 = (y,z,0)$, and define $r(p) = xz$. Clearly $p \in W \cap \mathrm{GEN} \Leftrightarrow r(p) \neq 0$. It follows from this and 4.3 that $W \cap \mathrm{GEN} \cap \mathrm{FLEX}^\infty$ consists of all $p \in W$ such that, for some $q \in W$,

$$d\lambda(p)(q) = 0, \qquad q \neq 0, \qquad r(p) \neq 0.$$

4.4 Proposition. $\operatorname{card} \pi_0(W \cap \mathrm{GEN} \cap \mathrm{FLEX}) \leqq 3 \cdot 5^{6v-11}$.

Proof. The equation $d\lambda(p)(q) = 0$ above is equivalent to the simultaneous system of equations

$$a) \ (p_i - p_j) \cdot (q_i - q_j) = 0, \qquad (i,j) \in \mathscr{E}.$$

The two inequalities above may be changed to equalities (cf. §3, proof of Theorem B) by introducing two slack variables t_1, t_2:

$$b) \ t_1 r(p) - 1 = 0.$$

$$c) \ t_2 \|q\|^2 - 1 = 0.$$

The simultaneous solutions (t_1, t_2, p, q) to (a)-(c) form algebraic subset C of $\mathbb{R}^2 \times W^2 = \mathbb{R}^{6v-10}$ whose projection onto the p-coordinate is $W \cap \mathrm{GEN} \cap \mathrm{FLEX}^\infty$. All the polynomials in (a)-(c) have degree $\leqq 3$. Therefore, by 3.1, the sum of the Betti numbers of C, hence $\operatorname{card} \pi_0(C)$, is $\leqq 3 \cdot 5^{6v-11}$. Since projection cannot increase the number of path-components, the desired result follows. □

Since W meets every $E^+(3)$-orbit and GEN is $E(3)^+$-invariant, we conclude that the natural map

$$\pi_0(W \cap \mathrm{GEN} \cap \mathrm{FLEX}^\infty) \to \pi_0(\mathrm{GEN} \cap \mathrm{FLEX}^\infty)$$

is onto. Therefore:

4.5 Corollary. $\operatorname{card} \pi_0(\mathrm{GEN} \cap \mathrm{FLEX}^\infty) \leqq 3 \cdot 5^{6v-11}$.
We now sketch the rest of the proof of Theorem D.

Recall that the definition of GEN distinguishes the vertices p_1, p_2, p_3, which corresponds to our assumption that $(1,2), (1,3) \in \mathscr{E}$ (i.e., the stated vertices belong to adjacent edges). Any other choice of two adjacent edges determines a similar set GEN′ to which the foregoing arguments and results apply *mutatis mutandis*. By 4.5, each of the sets GEN′ contributes at most $3 \cdot 5^{6v-11}$ to the number of path-components of $\mathrm{FLEX}^\infty \backslash \Delta$.

Frameworks that are not generic with respect to *any* choice of two adjacent edges are characterized by the property that all their vertices are collinear. It follows from this that the non-constant, non-generic frameworks contribute at most one path-component to $\mathrm{FLEX}^\infty \setminus \Delta$.

Combining the two preceding paragraphs, we obtain

4.6 Corollary. $\operatorname{card} \pi_0(\mathrm{FLEX}^\infty \setminus \Delta) \leqq k \cdot 3 \cdot 5^{6v-11} + 1$, *where* k *is the number of sets* GEN′.

Finally, we observe that not all the sets GEN′ are necessary, since there may be much overlap. In fact, a short combinatorial argument shows that at most $4(v-1)$ such sets are needed to exhaust the non-collinear frameworks, and so we may take $k = 4(v-1)$ in 4.6.

Theorem D is now an immediate consequence of the emended 4.6.

5 Characterizing Rigid Frameworks

Recall that $p \in \mathbb{R}^{3v}$ is rigid if and only if the path-component $C(p)$ of p in $\lambda^{-1}\lambda(p)$ is contained in $E^+(3) \cdot p$.

5.1 Lemma. *p is rigid $\Leftrightarrow$ p is interior to $\Lambda^{-1}\Lambda(p)$ in $\lambda^{-1}\lambda(p)$.*

This is equivalent to the characterization in [4] of rigidity as ε-rigidity. We give here an independent proof which avoids the use of the Curve Selection Lemma [9] (cf. 3.7(a) above).

Proof of 5.1. We use two facts: (a) Algebraic sets are locally-path-connected. (b) $E^+(3)$-orbits in $\mathbb{R}^{3v}$ are closed in $\mathbb{R}^{3v}$. The first fact is well known (cf. 6.2 (d)), and the second may easily be verified directly.

Fact (a) implies that $C(p)$ is open in $\lambda^{-1}\lambda(p)$. Thus, if p is rigid, then $p \in C(p) \subseteq E^+(3) \cdot p \subseteq \Lambda^{-1}\Lambda(p)$, so that $p \in \operatorname{int} \Delta^{-1}\Delta(p)$.

Conversely, if p is interior to $\Lambda^{-1}\Lambda(p) = E(3) \cdot p$ in $\lambda^{-1}\lambda(p)$, then $E(3) \cdot p$ is open in $\lambda^{-1}\lambda(p)$. By fact (b), $E^+(3) \cdot p$ is then both open and closed in $\lambda^{-1}\lambda(p)$, so that $E^+(3) \cdot p = C(p)$. □

The characterization given in 5.1 may be expressed as a first-order real-algebraic formula: namely,

5.2 p is rigid $\Leftrightarrow$

$$\exists \varepsilon \forall q [\varepsilon > 0 \,\&\, \{(\|p-q\|^2 < \varepsilon \,\&\, \lambda(p) = \lambda(q)) \Rightarrow \Lambda(p) = \Lambda(q)\}].$$

The additional condition that we require, namely that p not be constant, may be most conveniently expressed by the following formula:

5.3 p is non-constant $\Leftrightarrow$

$$\sum_{(i,j) \in \mathscr{E}} \lambda_{ij}(p) \neq 0,$$

where the λ_{ij} are the quadratic functions defined in 2.1.

These formulas together involve $6v+1$ variables. If we restrict to $(\mathrm{RIG}\setminus\Delta)\cap V$, V complementary to Δ (cf. § 2.2), the number is reduced to $6v-2$.

For the purposes of Theorem A, our descriptions are complete.

An improved bound on card $\pi_0(\mathrm{RIG}\setminus\Delta)$ ((1) of § 1) may be obtained using the ideas of § 4. The key is a formula in $6v-11$ variables that describes $W\cap\mathrm{GEN}\cap(\mathrm{RIG}\setminus\Delta)$. We shall not give further details.

6 Eliminating Quantifiers and Counting Types of Rigid Frameworks

This section briefly describes the main results of Collins [2] and concludes by applying these to prove Theorem A.

Consider formulas

$$(*)\quad \Psi=(Q\,x_{k+1})\dots(Q\,x_r)\,\Phi(x_1,\dots,x_r),$$

where $0\leqq k\leqq r$, $Q\,x_i$ is $\forall x_i$ or $\exists x_i$ and $\Phi(x_1,\dots,x_r)$ is a finite disjunction of simultaneous systems of equalities and inequalities involving polynomials in $\mathbb{Z}[x_1,\dots,x_r]$. Assertions 5.2 and 5.3 are examples of such formulas (modulo some elementary manipulations of first-order predicate calculus).

The truth or falsity of Ψ may be considered as a function T_Ψ of the unquantified variables $x_1,x_2,\dots,x_k$, with value 0 representing falsity and 1 representing truth. When all the variables are quantified, T_Ψ reduces to a constant.

Given a subset $S\subseteq\mathbb{R}^k$, we say that Ψ is invariant on S, or S is Ψ-invariant, provided that T_Ψ is constant on S. For example, both the sets $T_\Psi^{-1}(1)$ and $T_\Psi^{-1}(0)$ are Ψ-invariant. The former is called the truth set of Ψ. In the case that Ψ is the formula of 5.2, the truth set of Ψ is RIG. If Ψ is the conjunction of 5.2 and 5.3, the truth set is $\mathrm{RIG}\setminus\Delta$.

The basic contribution of quantifier-elimination in this context is to show that the truth sets of formulas Ψ coincide with the semi-algebraic (s.a.) sets, whose definition we now recall. The simplest examples of s.a. sets are ones of the form $\{x\in\mathbb{R}^k\,|\,f(x)>0\}$, for some $f\in\mathbb{Z}[x_1,\dots,x_k]$ and some $k\geqq 1$. The s.a. subsets of $\mathbb{R}^k$, k fixed, are closed under finite union and complementation, and the s.a. sets, in general, are closed under projections $\mathbb{R}^{k+1}\to\mathbb{R}^k$, $k\geqq 1$. The collection of all s.a. sets is the smallest family of sets satisfying these conditions.

Collins' method, which we cannot describe here, is closely related to the procedure for triangulating semi-algebraic sets given by Hironaka in [6].

We now state the main results of Collins [2].

6.1 Theorem. *Let Ψ be as in $(*)$ above. Then there exists a partition of $\mathbb{R}^k$ into finitely many non-empty sets S_i with the following properties: (a) Each S_i is Ψ-invariant. (b) Each S_i is homeomorphic to an open j-cell, for some $j\leqq k$, j depending on i. (c) Each S_i is semi-algebraic. (d) There is a finite algorithm which produces all the defining equalities and inequalities for each S_i.* □

6.2 *Remarks. a)* It follows from 6.1 that the sum of the Betti numbers (and the number of path-components) of the truth set of Ψ is bounded by the number of sets S_i.

b) It also follows from 6.1 that there is a finite algorithm for deciding whether a given $(x_1, \ldots, x_k) \in \mathbb{R}^k$ belongs to the truth set (cf. Theorem *E*) – this requires an induction on the number of quantifiers $r-k$.

c) Note that 6.1 implies that the truth set of Ψ is a finite union of semi-algebraic sets, hence it is semi-algebraic. This is the desired quantifier-elimination.

d) It is not difficult to prove inductively that the partition $\{S_i\}$ admits a refinement as a locally-finite simplicial complex (cf. Hironaka [6]). The truth sets are not subcomplexes, in general, unless they are closed sets, but they are locally-finite unions of simplexes and they have the homotopy type of finite complexes.

Various other facts about the simplicial or cellular structure of semi-algebraic sets may be deduced, but we do not do so here.

We turn next to Collins' analysis of the relevant bounds.

6.3 Theorem. *Suppose that* Ψ *contains m polynomials in the variables* $x_1, x_2, \ldots, x_r$, *so that no polynomial has degree (in any single variable) exceeding n. Then:*

a) The number of S_i *described in 6.1 is* $<(2n)^{3^{r+1}} m^{2^r}$.

b) The computing time for the algorithm of 6.1 is dominated by $(2n)^{2^{2r+8}} \cdot m^{2^{r+6}} \cdot b$, *where b is a certain function of the number of polynomial equalities and inequalities appearing in* Ψ *and on the particular integers appearing as coefficients.* □

Here (according to Collins [1]), we say that $f(x,y,z,\ldots)$ is dominated by $g(x,y,z,\ldots)$ if, for some constant $c>0$, $f(x,y,z,\ldots)<cg(x,y,z,\ldots)$, for all $x,y,z,\ldots$. The constant c depends on the choice of scale, choice of computer, etc.

6.4 Corollary to 6.3 (Theorem A). *The sum of the Betti numbers of* $\mathrm{RIG}\setminus\Delta$ *is less than* $2^{3^{6v}}$.

Proof. As stated in § 5, the conjunction of formulas 5.2 and 5.3 involves $6v+1$ variables, which may be reduced to $6v-2$ by restricting to a complement of Δ. The number of polynomials is $e+\frac{1}{2}v(v-1)+3$, where $e=\operatorname{card}\mathscr{E}$, and none has degree exceeding two. Theorem 6.3 then implies that the number of S_i (and hence the desired sum of Betti numbers) is $<4^{3^{6v-1}}(e+3+\frac{1}{2}v(v-1))^{2^{6v-2}}$, which one easily shows is $<2^{3^{6v}}$. □

6.5 *Remark.* The length of the algorithm in Theorem E has a bound given by 6.3(b). In this case, $n=2$, $r=6v-2$, $m=e+\frac{1}{2}v(v-1)+3$ (cf. 6.4) and b is essentially negligible. Thus, for example, $4^{2^{12v+5}}$ is a bound.

7 Extensions to Other Cases

We now briefly describe two kinds of variations on the notion of framework. Questions about rigid types and decidability questions arise just as before, and they may be answered by the foregoing methods because all of the relevant characterizations are first-order real-algebraic.

7.1 *Restricted Frameworks*

Our notion of frameworks is more general than the one often used. For example, often frameworks are required to be imbedded (i.e., not two distinct vertices meet nor edges cross), or to be 1-skeleta of imbedded polyhedral surfaces (which we shall call polyhedral frameworks). Or, for another example, various weaker kinds of non-degeneracy may be required (as in our restriction to non-constant frameworks or to generic frameworks).

Any restriction on the general notion of framework that can be expressed by a first-order formula admits treatment by the foregoing techniques. The restriction that a framework p be imbedded, for example, can be expressed via linear conditions on $p_1, \ldots, p_v$. More complicated expressions are needed to characterize polyhedral frameworks.

One interesting application to this last case is the following: Let K be the 1-skeleton of a polyhedral surface. Then there is a finite algorithm for deciding whether or not all polyhedral frameworks modeled on K are rigid.

7.2 *Cabled Frameworks* (see [3])

Such frameworks p, in addition to being equipped with an edge structure $\mathscr{E}$, also have certain pairs of vertices (call their totality $\mathscr{C}$) joined by "cables". That is, the admissible deformations now are those that preserve edge length and do not increase cable length.

In place of the set $\lambda^{-1}\lambda(p)$ considered in §2, we now consider

$$L(p)=\{q \mid \lambda_{ij}(q)=\lambda_{ij}(p), \quad \text{for } (i,j)\in\mathscr{E}, \quad \text{and}$$
$$\lambda_{ij}(q)\leqq\lambda_{ij}(p), \quad \text{for } (i,j)\in\mathscr{C}\}.$$

This is a semi-algebraic set (hence, locally path-connected) and is $E(3)$-invariant, so that Lemma 5.1 still applies, with $L(p)$ replacing $\lambda^{-1}\lambda(p)$. We obtain, thus, the desired firstorder, real-algebraic characterization of rigid, cabled frameworks.

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