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The Monodromy of Reducible Plane Curves

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I. Introduction

Let $f: C^{n+1} \to C$ be a polynomial function such that $f(0)=0$ and f has an isolated singularity at 0. The local geometry of $f^{-1}(0)$ near 0 is completely described by the algebraic link L^{2n-1} of the singularity, where $L^{2n-1}=f^{-1}(0)\cap S_\varepsilon^{2n+1}$ is a smooth $(n-2)$-connected submanifold of the small ε-sphere S_ε^{2n+1} about the origin [13]. Moreover, the link complement $X=S^{2n+1}-L$ fibers over S^1 with fiber $F^{2n}\simeq S^n\vee\cdots\vee S^n$. Let $\tilde{X}$ denote the infinite cyclic covering space of X associated with the epimorphism $\Pi_1(X)\to\Pi_1(S^1)$ induced by the fibration. The action of the multiplicative generator t of $\Pi_1(S^1)$ on $H_n(F;Z)$ is called the *local algebraic monodromy transformation of* f *at* 0. Since $\tilde{X}\approx F\times R$, t can be viewed as the generator of the infinite cyclic covering translation group, and the algebraic monodromy is an automorphism $t: H_n(\tilde{X};Z)\xrightarrow{\cong} H_n(\tilde{X};Z)$. The polynomial invariants of $H_n(\tilde{X};Z)$ as a $\Lambda=Z[t,t^{-1}]$ module are therefore invariants of the monodromy. For example, the Alexander polynomial $\Delta_1^n(t)$ is the characteristic polynomial of the monodromy.

We are interested in the following problem: Given f, determine whether the algebraic monodromy is of finite or infinite order. The problem is most amenable to solution in the case $n=1$, because the geometry is completely understood [3, 9, 15]. In the classical case ($n=1$), $f^{-1}(0)$ is called a plane curve, and the algebraic link L^1 has r components, one corresponding to each of the branches of $f^{-1}(0)$ at the origin. Each branch corresponds to a distinct analytically irreducible factor of f which maps 0 to 0. Each component of L is an iterated torus knot, and both the iteration on each component and the linking among the various components is completely specified by the Puiseux developments [1, 10] corresponding to the branches.

In the case that f is analytically irreducible ($r=1$), f always has finite monodromy [10, 1, 19]. If $r\geqq 2$, the monodromy can be infinite [1, 20]. This paper

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attacks the monodromy problem by calculating the characteristic polynomial of the monodromy for any algebraic link. Our calculation provides a new proof of the result of Grothendieck [13, pp. 72] for plane curves, which states that the roots of the characteristic polynomial of the monodromy are roots of unity. It also answers the question of Milnor [13, pp. 72] for the case $n=1$ and $r \geqq 3$ by providing an algorithm for calculating the characteristic polynomial. We also attempt to calculate the minimal polynomial of the monodromy, and are completely successful for $r=1$, but only partially successful for $r \geqq 2$. For curves of one or two branches, the calculation of the characteristic polynomial of the monodromy is due to Burau [4, 5]. He calculated the Alexander polynomial of the universal abelian covering space of the complement of the algebraic link, proceeding via calculation of the fundamental group of the complement. The method employed here is largely homological in nature and computationally simpler than Burau's, and relies on a Mayer-Vietoris splitting for the universal abelian covering. We obtain complete information about the homology of the universal abelian covering for all r, recovering and generalizing Burau's results for $r \leqq 2$. From the universal abelian information can be extracted much of the desired information about the infinite cyclic covering. Moreover, our calculations apply to all iterated torus links, a class of links properly containing algebraic links. The introduction of the universal abelian covering complicates the calculation, but the details of the Mayer-Vietoris argument show that for $r \geqq 2$ there is *no* relation between infinite cyclic covers from one stage to the next in the iteration on the link in the base space. On the other hand, there is a natural geometric iterative relationship between the universal abelian covers. For $r=1$, the universal abelian and the infinite cyclic covers coincide, and this method provides a particularly easy calculation for the minimal polynomial and a proof of finite monodromy [19].

For $r=1$, the method of proof reduces the general case of calculating the invariants of an iterated torus knot to the calculation of the invariants for a sequence of ordinary torus knots. For $r \geqq 2$, the general case is iteratively reduced to a sequence of simple "model" links of either 2 or 3 components. In this context, a model link of 2 components is an ordinary torus knot linked with an unknotted core (either inner or outer), and a model link of 3 components is an ordinary torus knot linked with both unknotted cores. In other words, the perhaps surprising result is that for a link with a large number of components, the most complicated link arising in the iterative calculation is a link with only three components, two of which are unknotted!

We then compute all of the homology invariants for the models, iteratively obtaining calculations for the characteristic and minimal polynomial of the universal abelian cover of any iterated torus link. We then project the universal abelian information down to the infinite cyclic cover, applying methods similar to those in [20] to obtain monodromy information, calculating the characteristic polynomial of the monodromy and a divisor of the minimal polynomial of the monodromy.

II. Background Algebra

Let R be a commutative Noetherian UFD with a 1, and A be a finitely-generated R-module. Suppose that the $(m \times n)$ matrix M with entries in R is a relation

(presentation) matrix for A [see 17 p. 102]. The *k-th elementary ideal* $\varepsilon_k(A)$ is the ideal in R generated by the $(n-k+1)\times(n-k+1)$ minor determinants of M, with the conventions that $\varepsilon_k=R$ if $(n-k+1)<1$ and $\varepsilon_k=0$ if $(n-k+1)>m$. Let $\Delta_k(A)$ denote the generator of the smallest principal ideal in R containing $\varepsilon_k(A)$. $\Delta_k(A)$ is defined up to units in R. We call $\Delta_1(A)$ the *characteristic polynomial of* A. Since $\Delta_{k+1}(A)$ divides $\Delta_k(A)$ in R for all k, we set $\lambda_k(A)=\Delta_k(A)/\Delta_{k+1}(A)$. We call $\lambda_1(A)$ the *minimal polynomial of* A. All of the above are invariants of R-isomorphism type of A. [7, 21].

Lemma 2.1. *If* $0\to A\to B\to C\to 0$ *is a short exact sequence of finitely generated* R*-modules, then*

(i) $\Delta_1(B)=\Delta_1(A)\cdot\Delta_1(C)$.

(ii) *If the sequence splits, then* $\lambda_1(B)=$ *least common multiple of* $\lambda_1(A)$, $\lambda_1(C)$ $=$l.c.m. $[\lambda_1(A), \lambda_1(C)]$.

(iii) *If* $\Delta_1(C)=1$ *then* $\Delta_i(A)=\Delta_i(B)$ *for all* i.

Proof. (i) is proved by J. Levine [11], and we are indebted to J. Levine for pointing out the following elementary proof of (ii): Let A be presented by the $(m\times n)$-matrix M, and C be presented by the $(p\times q)$-matrix N. Then $A\oplus C$ is presented by the direct sum of M and N, an $(m+p)\times(n+q)$-matrix. Now suppose the collection $\{x_i\}$ generates $\varepsilon_1(A)$, $\{x_{ij}\}$ generates $\varepsilon_2(A)$, $\{y_k\}$ generates $\varepsilon_1(C)$, and $\{y_{kr}\}$ generates $\varepsilon_2(C)$. Then $\Delta_1(A)=$g.c.d. $\{x_i\}$ (greatest common divisor), $\Delta_2(A)$ $=$g.c.d. $\{x_{ij}\}$, etc. Now $\Delta_1(A\oplus C)=\Delta_1(A)\cdot\Delta_1(C)=$g.c.d. $\{x_j\}\cdot$g.c.d. $\{y_k\}$. Moreover, $\Delta_2(A\oplus C)=$g.c.d. $\{x_i y_{kr}, x_{ij} y_k\}$. This is because any $(n+q-1)\times(n+q-1)$ minor of $M\oplus N$ is obtained by deleting a column and taking $(n+q-1)$ rows. The minor matrix is itself a direct sum, and has determinant $=0$ unless it is the direct sum of square matrices. Hence, the only possible non-zero determinants are of the form $x_i y_{kr}$ or $x_{ij} y_k$. We have

$$\lambda_1(A\oplus C)=\frac{\Delta_1(A\oplus C)}{\Delta_2(A\oplus C)}=\frac{\Delta_1(A)\cdot\Delta_1(C)}{\text{g.c.d. }\{x_i y_{kr}, x_{ij} y_k\}}.$$

Also,

$$\begin{aligned}&\text{g.c.d. }\{x_i y_{kr}, x_{ij} y_k\}=\text{g.c.d. }\{\text{g.c.d. }\{x_i y_{kr}\}, \text{g.c.d. }\{x_{ij} y_k\}\}\\&=\text{g.c.d. }\{\text{g.c.d. }\{x_i\}\cdot\text{g.c.d. }\{y_{kr}\}, \text{g.c.d. }\{x_{ij}\}\cdot\text{g.c.d. }\{y_k\}\}\\&=\text{g.c.d. }\{\Delta_1(A)\cdot\Delta_2(C), \Delta_2(A)\cdot\Delta_1(C)\}=\Delta_2(A)\cdot\Delta_2(C)\cdot\text{g.c.d. }\{\lambda_1(A), \lambda_1(C)\}.\end{aligned}$$

Hence,

$$\begin{aligned}\lambda_1(A\oplus C)&=\frac{\Delta_1(A)\cdot\Delta_1(C)}{\Delta_2(A)\cdot\Delta_2(C)\cdot\text{g.c.d. }\{\lambda_1(A), \lambda_1(C)\}}=\frac{\lambda_1(A)\cdot\lambda_1(C)}{\text{g.c.d. }\{\lambda_1(A), \lambda_1(C)\}}\\&=\text{l.c.m. }\{\lambda_1(A), \lambda_1(C)\}.\end{aligned}$$

To prove (iii) we use results of Blanchfield [2]. By Theorem 4.2 of [2] we have that A is a submodule of B, so $\Delta_i(A)$ divides $\Delta_i(B)$ for all i. By Theorem 4.9 of [2], we have that $\Delta_i(B)$ divides $\Delta_i(A)\cdot\Delta_1(C)$ (subtract 1 from our subscripts to get Blanchfield's subscripts). Now $\Delta_1(C)=1$, so $\Delta_i(A)=\Delta_i(B)$ for all i. This completes the proof of Lemma 2.1.

III. Covering Space Preliminaries

The preliminaries through and including Lemma 3.1 are as given by Shinohara [16] and are included here for completeness. The approach used in the remainder of the section is a generalization of his approach.

Let X be a connected, locally path-connected and semilocally 1-connected space with Y a connected and locally path-connected subspace of X. Further assume there exists a homomorphism

$$\phi: \pi_1(X, y_0) \to G$$

of $\pi_1(X, y_0)$ onto an abelian group G (written multiplicatively), where y_0 is a point of Y. Then there exists a regular covering space

$$p: \tilde{X} \to X$$

associated with $\mathrm{Ker}(\phi)$. G acts on $\tilde{X}$ as the group of covering transformations, with the action described as follows:

Choose a point $\tilde{y}_0$ in $p^{-1}(y_0)$. Let g be an element of G and let γ be an element of $\pi_1(X, y_0)$ such that

$$\phi(\gamma) = g.$$

Then there is a unique path class $\tilde{\gamma}$ in $\tilde{X}$ with initial point $\tilde{y}_0$ such that $p_\#(\tilde{\gamma}) = \gamma$. The terminal point $\tilde{y}_1$ of $\tilde{\gamma}$ is independent of the choice of γ satisfying $\phi(\gamma) = g$. Thus g corresponds to a unique covering transformation which maps $\tilde{y}_0$ to $\tilde{y}_1$. This correspondence is an isomorphism of G onto the group of covering transformations and is independent of the choice of $\tilde{y}_0 \in p^{-1}(y_0)$ since G is abelian. The action is called the *ϕ-action of G on X*, and for each subgroup H of G, and ϕ-action of G induces an action of H on $\tilde{X}$. For a more detailed treatment of "ϕ-actions" see Massey, Chapter 5, Section 7 [12].

Let $Y \subset X$, and let H be the image of $\pi_1(Y, y_0)$ under the homomorphism

$$\phi \circ i_\# : \pi_1(Y, y_0) \to G$$

where i is the inclusion map of Y into X. Let $\{g_1 = 1, g_2, \ldots, g_\mu\}$ be a coset representative system of $G \bmod H$ where $\mu = [G : H]$ may be infinite (i.e., $G = g_1 H + g_2 H + \cdots + g_\mu H$). Let Y_1 be a path component of $p^{-1}(Y)$ and let $Y_i = g_i(Y_1)$, $1 \leqq i \leqq \mu$. Then we have the following well-known properties:

Lemma 3.1. *Let $\psi = \phi \circ i_\#$. Then*

(1) *$p | Y_i : Y_i \to Y$ is a covering space which belongs to $\mathrm{Ker}(\psi)$, $i = 1, 2, \ldots, \mu$.*

(2) *For $i = 1, 2, \ldots, \mu$, the ϕ-action of H on $\tilde{X}$ induces an action of H on Y_i by restricting every element of H to Y_i. Furthermore, this induced action of H on Y_i coincides with the ψ-action of H on Y_i.*

(3) *$p^{-1}(Y)$ is the disjoint union of $Y_1, Y_2, \ldots, Y_\mu$.*

Now suppose X is a finite, connected simplicial complex and Y is a connected subcomplex of X. Let G_r denote the free abelian group generated by $\{x_1, x_2, \ldots, x_r\}$. We will let Λ_r denote the integral group ring ZG_r. Now Λ_r is a Unique Factorization

Domain and is Noetherian, so the Alexander polynomials $\{\Delta_i^q\}$ defined in Section II are well-defined and unique up to units in Λ_r. Suppose that H is the free abelian subgroup of G generated by $\{x_1, \ldots, x_{i-1}, x_i, x_{i+1}^{a_{i+1}} \ldots x_r^{a_r}\}$ for some non-zero integers $\{a_j\}_{i+1}^r$. Let $p: \tilde{X} \to X$ be the abelian covering space associated with Ker ϕ where ϕ is an epimorphism

$$\phi: \pi_1(X, y_0) \to G_r.$$

Let $\tilde{Y}$ denote $p^{-1}(Y)$. Since $\tilde{Y}$ is invariant under G_r-actions and Y is a finite subcomplex of X, $H_*(\tilde{Y}; Z)$ is a finitely presented Λ_r module.

Now $[G_r: H] = \mu$ and we let $\{g_1 = 1, g_2, g_3, \ldots, g_\mu\}$ form a coset representative system of $G_r \bmod H$ (i.e., $G_r = H + g_2 H + \cdots + g_\mu H$). Let Y_1 be a path-component of $\tilde{Y}$ and $Y_j = g_j(Y_1)$ for $j = 1, 2, \ldots, \mu$. By Lemma 3.1, $p | Y_j: Y_j \to Y$ is an abelian covering space having H as the group of covering transformations. Hence $H_*(Y_j; Z)$ is a finitely presented ZH-module.

Lemma 3.2. *A presentation matrix for the ZH-module $H_q(Y_1; Z)$ is simultaneously a presentation matrix for the Λ_r-module $H_q(\tilde{Y}; Z)$, by considering the ZH-entries as elements of Λ_r via the inclusion homomorphism.*

Proof. Let $F_2 \xrightarrow{M} F_1 \longrightarrow H_q(Y_1; Z)$ present $H_q(Y_1; Z)$ as a ZH-module. Assume that we have a fixed coset decomposition of G by H,

$$G = H \cup g_2 H \cup g_3 H \cup \cdots \cup g_\mu H$$

and a corresponding fixed subset D,

$$D = \{g_1 = 1, g_2, g_3, \ldots, g_\mu\}.$$

Then each element of G has a unique representation of the form $g_i h$, where $g_i \in D$, $h \in H$. Furthermore, it follows from Lemma 3.1 that as an abelian group,

$$H_q(\tilde{Y}; Z) \cong \bigoplus_{i=1}^{\mu} H_q(Y_i; Z).$$

Since $g_i: Y_1 \to Y_i$ is a homeomorphism for each i and the group of covering transformations is abelian – in particular, $g_i h = h g_i$ for each $h \in H$ – it follows that $g_{i*}: H_q(Y_1; Z) \to H_q(Y_i; Z)$ is a ZH isomorphism, $1 \leqq i \leqq \mu$. Furthermore,

$$H_q(\tilde{Y}; Z) \cong \bigoplus_{i=1}^{\mu} H_q(Y_i; Z) \quad \text{as} \quad ZH\text{-modules.}$$

Here we are abusing notation by identifying G_r with the group of covering transformations. Now, if α is an element of $H_q(\tilde{Y}; Z)$, then

$$\alpha = \sum_{j=1}^{m} \alpha_{i_j} \quad \text{where } i_j \in \{i\}_1^\mu \text{ and } \alpha_{i_j} \in H_q(Y_{i_j}; Z).$$

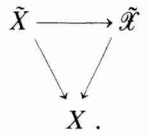
Also, $\alpha_{i_j} = g_{i_j *}(\beta_{i_j})$ for some $\beta_{i_j} \in H_q(Y_1; Z)$. Now, since

$$F_2 \otimes_{ZH} \Lambda_r \xrightarrow{M \otimes 1} F_1 \otimes_{ZH} \Lambda_r \longrightarrow H_q(Y_1; Z) \otimes_{ZH} \Lambda_r \longrightarrow 0$$

is exact, it suffices to show that $H_q(Y_1;Z)\otimes_{ZH}\Lambda_r \cong H_q(\tilde{Y};Z)$ as Λ_r-modules. Using the standard homomorphism defined by setting $\tau(y\otimes g)=g_*(y)$ for $g\in G_r$, $y\in H_q(Y_1;Z)$ and extending linearly, the lemma follows immediately (modulo notational complications) from the information given above.

Let $L=L_1\cup\cdots\cup L_r$ be an oriented n-link in S^{n+2}. That is, L is the disjoint union of oriented submanifolds, each homeomorphic to S^n. Let $X=S^{n+2}-L$; then $H_1(X;Z)$ is free abelian multiplicative group on the symbols $\{t_i\}_{i=1}^r$, where each t_i is geometrically carried by a meridian circle with homological linking number δ_{ij} with L_j. The Hurewicz homomorphism $\phi\colon \pi_1(X)\to H_1(X;Z)$ takes the form $\phi(g)=\prod_{i=1}^{r} t_i^{\langle g, L_i\rangle}$, where $\langle\alpha,\beta\rangle$ denotes the homological linking number of α and β. Let J denote the infinite cyclic multiplicative group on the generator t. By sending each of the t_i to t and extending linearly, we obtain the epimorphism $\Phi\colon \Pi_1(X)\to J$.

The universal abelian covering $\tilde{\tilde{X}}$ of X is the regular covering associated with Ker ϕ; the infinite cyclic covering $\tilde{\tilde{X}}$ of X is the regular covering associated with Ker Φ. We have the commutative diagram of covering spaces



We also have that $H_q(\tilde{X};Z)$ is a Λ_r-module, and as such has a characteristic polynomial $\Delta_1^q(t_1,\ldots,t_r)$ and a minimal polynomial $\lambda_1^q(t_1,\ldots,t_r)$. Likewise $H_q(\tilde{\tilde{X}};Z)$ is a Λ-module, and has characteristic polynomial $\Delta_1^q(t)$ and minimal polynomial $\lambda_1^q(t)$. We are confronted with the following problem, unsolved in general: What is the relationship (if any) between $\Delta_1^q(t,\ldots,t)$ and $\Delta_1^q(t)$? Or, more important for algebraic links, how are $\lambda_1^q(t,\ldots,t)$ and $\lambda_1^q(t)$ related?

Milnor answers the first question for the case $q=1$ [13, pp. 95].

Theorem 3.3 (Milnor).

$$(t-1)\,\Delta_1^1(t,\ldots,t)=\Delta_1^1(t).$$

By reasoning similar to that used in the proof of the above theorem, it is clear that $\lambda_1^1(t,\ldots,t)$ divides $\lambda_1^1(t)$. We are unable, however, to say anything more in general about their relationship.

Throughout the rest of the paper, we fix the following notation:

$$\Delta_1^1(t_1,\ldots,t_r)=\Delta(t_1,\ldots,t_r),\qquad \lambda_1^1(t_1,\ldots,t_r)=\lambda(t_1,\ldots,t_r),\qquad \Delta_1^1(t)=\Delta(t)$$

and $\lambda_1^1(t)=\lambda(t)$.

IV. Geometric Structure of Algebraic Links

In this section we describe how each algebraic link can be constructed by an iterative procedure. This construction is completely determined by the charac-

teristic Puiseux developments associated with the link. The Puiseux numbers determine when branching occurs, how the branches are linked, and also determine the structure of the branches. Each branch is an *iterated tube knot*, which is defined below.

Definition. Let K be an oriented knot in S^3 and let V be a tubular neighborhood of K. The longitude of V is the curve (unique up to isotopy) on ∂V which lies on a Seifert surface spanning K. Let V' be a tubular neighborhood of the trivial knot K', and let $f\colon V' \to V$ be an orientation preserving diffeomorphism of V' onto V which carries longitude to longitude. Let L' be a knot contained in the interior of V'. Then L' is homologous to $\gamma K'(L' \sim \gamma K')$ inside V' for some integer γ. For example, if L' is a torus knot of type (p, q), then $\gamma = q$. Note that we write the meridianal winding number (p) as the first coordinate of the torus knot (p, q), and the longitudinal winding number (q) as the second coordinate. This notation agrees with that of Lê [10] and A'Campo [1], but is opposite to that of Milnor [13]. Moreover, if $L = f(L')$ then L is a knot in the interior of V and $L \sim \gamma K$ in V. L will be called a *tube knot of type* $\{K, L'\}$. An *iterated tube knot* L_r *of type* $\{K_1, K_2, \ldots, K_r\}$ is the knot L_r where L_i is inductively defined to be a tube knot of type $\{L_{i-1}, K_i\}$ $2 \leqq i \leqq r$ and $L_1 = K_1$.

A beautiful classical result [3, 9, 15] relates the geometry of polynomial functions of two complex variables to algebra, proving that algebraic knots and each branch of an algebraic link are iterated tube knots, where each iteration is via an ordinary torus knot. More specifically, the structure of each branch and the linking among the various branches is completely specified by the characteristic Puiseux expansion [1, 10], which, for a fixed branch, expresses y as a fractional power series in x.

Let K be the knot associated to a branch of a plane algebraic curve by taking the intersection with a small S^3. The associated Puiseux expansion for that branch has the form of a fractional power series, truncated to only finitely many geometrically meaningful terms:

$$y = a_1 x^{\frac{m_1}{n_1}} + a_2 x^{\frac{m_2}{n_1 n_2}} + \cdots + a_s x^{\frac{m_s}{n_1 \cdots n_s}} \tag{1}$$

where $(m_i, n_i) = 1$, $1 \leqq i \leqq s$ and $m_1 > n_1$, $m_2 > n_2 m_1, \ldots, m_s > n_s m_{s-1}$. Recursively define the numbers w_i, $1 \leqq i \leqq s$ as follows: $w_1 = m_1$, and

$$w_i = m_i - m_{i-1} n_i + w_{i-1} n_{i-1} n_i$$

for $2 \leqq i \leqq s$. Then K is the iterated tube knot of type $\{K_1, K_2, \ldots, K_s\}$ where K_i is the torus knot of type (w_i, n_i), $1 \leqq i \leqq s$. Note that $(w_i, n_i) = 1$, and $w_i > n_i n_{i-1} w_{i-1}$, $2 \leqq i \leqq s$. This last inequality means that in the algebraic situation the meridianal winding numbers and the degree of the characteristic polynomial of the monodromy are getting very large very fast.

The Puiseux expansions dictate the linking among the various branches. For example, consider the case of an algebraic link with 2 branches, say K and K'. Assume that the Puiseux expansions for K and K' respectively are:

$$y = a_1 x^{\frac{m_1}{n_1}} + \cdots + a_s x^{\frac{m_s}{n_1 \cdots n_s}} \tag{2}$$

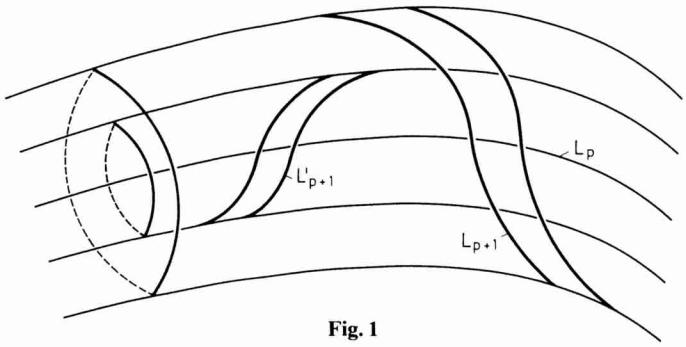


Fig. 1

and

$$y = a_1' x^{\frac{m_1}{n_1'}} + \cdots + a_t' x^{\frac{m_t}{n_1' \cdots n_t'}}. \tag{3}$$

Defining the $\{w_j\}$ and $\{w_j'\}$ as before, let K_j denote the torus knot of type (w_j, n_j) $1 \leqq j \leqq s$, and K_j' denote the torus knot of type (w_j', n_j'), $1 \leqq j \leqq t$. Without loss of generality, assume that the branching takes place at the $(p+1)$-st stage of the iteration: that is, $a_i = a_i'$ and $\frac{m_i}{n_i} = \frac{m_i'}{n_i'}$ for $1 \leqq i \leqq p$ but not both equalities holding for $i = (p+1)$. So, at the p-th stage, the algebraic link is the single iterated torus knot $L_p = \{K_1, \ldots, K_p\} = \{K_1', \ldots, K_p'\}$. The knot L_p splits into 2 knots L_{p+1} and L_{p+1}' as follows: Suppose $\frac{m_{p+1}'}{n_{p+1}} > \frac{m_{p+1}}{n_{p+2}}$. Then extend L_p to $L_{p+1} = \{L_p, K_{p+1}\}$. In a smaller tubular neighborhood of L_p, extend L_p to $L_{p+1}' = \{L_p, K_{p+1}'\}$.

Then proceed with iteration normally on L_{p+1} to obtain K, and L_{p+1}' to obtain K'. If $\frac{m_{p+1}}{n_{p+1}} > \frac{m_{p+1}'}{n_{p+1}'}$, the roles of the primed and unprimed components in the construction are interchanged. If $\frac{m_{p+1}}{n_{p+1}} = \frac{m_{p+1}'}{n_{p+1}'}$, then L_{p+1} and L_{p+1}' can be constructed by either method, and are "parallel" knots. The construction for more than two branches proceeds analogously.

One should note that algebraic knots and links are a proper subset of the set of iterated torus links, because of inequality restrictions like $w_i > n_i$ on a torus knot iterate of type (w_i, n_i).

V. The Iterative Homology Calculation

Two types of iterations occur in the geometry of an algebraic link: (i) iteration on one branch, and (ii) the addition of a new branch. We fix the following notation for the remainder of this section: let $L = L_1 \cup \cdots \cup L_\mu$ be an oriented link of μ components in S^3. A tubular neighborhood of L will be $\bigcup_{i=1}^{\mu} \{V_i\}$ where each V_i is a tubular neighborhood of L_i, and the V_i's are pairwise disjoint. We assume

(without loss of generality) that iteration is occurring on the last branch L_μ. Let V' be a tubular neighborhood of the unknot, and K' a knot contained in V'. Let U' be a tubular neighborhood of K' which is contained in the interior of V' and which misses the unknot. Let V^* be a small tubular neighborhood of the unknot which is contained in V' and misses U'. Let $f: V' \to V_\mu$ be an orientation preserving onto diffeomorphism taking longitude to longitude. Assuming that K' is homologous to γ times the unknot, we have that $K = f(K') \sim \gamma L_\mu$. Since we are discussing algebraic links, we also may assume that $\gamma \neq 0$. We call γ the *winding number* of K'.

There are two types of links arising from iteration about L_μ. Namely, (i) $L' = L_1 \cup \cdots \cup L_{\mu-1} \cup K$ where L_μ is replaced by K, and (ii) $\hat{L} = L_1 \cup \cdots \cup L_\mu \cup K$ where K is added to L to give a new link of $(\mu+1)$ components. The development of an algebraic link is obtained from the unknot via a finite sequence of operations of types (i) and (ii) with K' a torus knot. The purpose of the iterative homology calculation is to calculate the invariants of L' and $\hat{L}$ in terms of those of L and K'.

Let $U = f(U')$. We denote X by the bounded complement of the new link: that is, either (i) $X = S^3 - \mathrm{Int}(V_1 \cup \cdots \cup V_{\mu-1} \cup U)$ or (ii)

$$X = S^3 - \mathrm{Int}(V_1 \cup \cdots \cup V_{\mu-1} \cup U \cup V_\mu^*, \qquad V_\mu^* = f(V^*).$$

Let Y denote the bounded complement of the old link:

$$Y = S^3 - \mathrm{Int}(V_1 \cup \cdots \cup V_\mu).$$

Let

$$W = \begin{cases} V_\mu - \mathrm{Int}\, U & \text{(i)} \\ V_\mu - \mathrm{Int}(U \cup V_\mu^*) & \text{(ii)}. \end{cases}$$

Figure 2 shows W for a Type (ii) iteration. Let $T = \partial V_\mu$. We then obtain a very nice Mayer-Vietoris splitting for X: That is, $X = Y \underset{T}{\cup} W$. This splitting induces a splitting on the abelian covering spaces of X, which we exploit in the calculation. The main point is that W is geometrically very nice. It is an unknotted solid torus minus either a torus knot or a torus knot plus the core of the solid torus. We denote by $\tilde{X}$ the universal abelian covering of X, and let (i) ${}_{L'}\Delta(t_1, \ldots, t_\mu)$ or (ii) ${}_{\hat{L}}\Delta(t_1, \ldots, t_{\mu+1})$ denote the characteristic polynomial of $H_1(\tilde{X}; Z)$. We likewise denote the minimal polynomial by (i) ${}_{L'}\lambda(t_1, \ldots, t_\mu)$ or (ii) ${}_{\hat{L}}\lambda(t_1, \ldots, t_{\mu+1})$. In the above notation, as in § 3, the $\{t_i\}$ denote the meridian circles of the link.

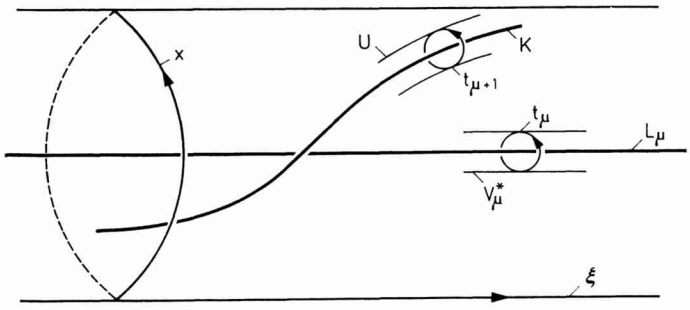


Fig. 2

Because of its elegant nature and because it most clearly illustrates the nature of the iterative method, we give a more complete version of the proof for $r=1$ found in [19].

Case 1. $\mu=1$, Type (i) iteration. (See also [16].)

In this case, the infinite cyclic cover is the universal abelian cover. In the iteration, $\mu=1$ throughout and the μ subscript is suppressed. We have the covering projection $p\colon \tilde{X}\to X$, and let $\tilde{Y}=p^{-1}(Y)$, $\tilde{W}=p^{-1}(W)$, and $\tilde{T}=p^{-1}(T)$. In order to identify $\tilde{Y}$, $\tilde{W}$, and $\tilde{T}$, it suffices to study the inclusion homomorphism on H_1. Unless otherwise specified in § V, we take homology with Z coefficients. If $A\subset X$, and $\alpha\in H_1(A)$ then $i_*(\alpha)=t^{\langle\alpha, L\rangle}$ where $i_*\colon H_1(A)\to H_1(X)$ is the inclusion map and $\langle\ ,\ \rangle$ denotes homological linking number. We have that $H_1(Y)\cong Z$ generated by the meridian curve x on $T=\partial V$. Since $K\sim\gamma L$, and $H_1(X)\cong Z$ generated by the meridian curve t on ∂U, we have $i_*(x)=t^{\gamma}$. So $i_*\colon H_1(Y)\to H_1(X)$ is injective, and $\tilde{Y}$ is the union of γ copies of the infinite cyclic cover of Y (see § 3). Also,

$$H_1(W)\cong Z\oplus Z,$$

generated by t and by ξ, where ξ is the longitude of V. Since $\langle\xi, L\rangle=0$, we have that $i_*(t)=t$ and $i_*(\xi)=1$. So $i_*\colon H_1(W)\to H_1(X)$ is onto, and $\tilde{W}$ is connected. Moreover, $H_1(T)\cong Z\oplus Z$ generated by x and ξ, so $\tilde{T}$ is the union of γ copies of $R\times S^1$. As a $\Lambda=Z[t, t^{-1}]$-module, $H_1(\tilde{T})\underset{\Lambda}{\cong} H_0(\tilde{T})\underset{\Lambda}{\cong}\Lambda/(t^{\gamma}-1)$ where $\Lambda/(t^{\gamma}-1)$ denotes the cyclic Λ-module of order $=(t^{\gamma}-1)$. See also [16]. Moreover, if $W'=V'-\mathrm{Int}(U')$ and $T'=\partial V'$, then $W'\underset{T'}{\cup}(S^1\times D^2)=S^3-\mathrm{Int}\ U'=Z$ is the bounded complement of K', an ordinary torus knot. Moreover, the diffeomorphism $f\colon W'\to W$ induces a diffeomorphism of infinite cyclic covers $\tilde{f}\colon\tilde{W}'\to\tilde{W}$. But an analysis similar to that just performed for $\tilde{X}$ shows that $\tilde{Z}=\tilde{W}'\underset{\tilde{T}'}{\cup}\gamma(R\times D^2)$, where $\tilde{T}'=\gamma(R\times S^1)$ and $\gamma(R\times D^2)$ denotes the disjoint union of γ copies of $(R\times D^2)$. The $M-V$ sequence for $\tilde{Z}$ yields the short exact sequence

$$0\to H_1(\tilde{T}')\xrightarrow{i^*} H_1(\tilde{W}')\to H_1(\tilde{Z})\to 0,$$

because

$$H_0(\gamma(R\times S^1))\xrightarrow{i_*} H_0(\tilde{W}')\otimes H_0(\gamma(R\times D^2))$$

is injective. Hence we obtain

$$H_1(\tilde{Z})\underset{\Lambda}{\cong} H_1(\tilde{W}')/H_1(\tilde{T}').$$

Repeating the $M-V$ sequence for $\tilde{X}$, we find that

$$0\to H_1(\tilde{T})\to H_1(\tilde{W})\oplus H_1(\tilde{Y})\to H_1(\tilde{X})\to 0$$

is exact. Since $i_*\colon H_1(\tilde{T})\to H_1(\tilde{Y})$ is the zero map (the longitude bounds a Seifert surface for the knot which lifts to $\tilde{Y}$), we have that

$$H_1(\tilde{X})\underset{\Lambda}{\cong} H_1(\tilde{W})/H_1(\tilde{T})\oplus H_1(\tilde{Y})\underset{\Lambda}{\cong} H_1(\tilde{Z})\oplus H_1(\tilde{Y}).$$

By Lemma 3.2 a presentation matrix for $H_1(\tilde{Y})$ over $\Lambda = Z[t, t^{-1}]$ is obtained from a presentation matrix for H_1 of one of the components over $Z[x, x^{-1}]$ via replacement of x by t^γ. Hence, using Lemma 2.1, we have proved the following

Theorem 5.1. *Let L be a knot, and L' be the knot obtained via iteration of type* (i) *about L via the knot K' with winding number $\gamma \neq 0$. Then*

(a) ${}_{L'}\Delta(t) = {}_L\Delta(t^\gamma) \cdot {}_{K'}\Delta(t)$,

(b) ${}_{L'}\lambda(t) = \text{l.c.m.}\,[{}_L\lambda(t^\gamma), {}_{K'}\lambda(t)]$.

Part (a) of Theorem 5.1 was first proved by Seifert [18].

Case 2. $\mu \geqq 2$, Type (i) iteration.

We prove the following:

Theorem 5.2. *Let L be a link of $\mu \geqq 2$ components, and let L' be the link obtained from an iteration of type* (i) *about L_μ via the knot K' with the winding number $\gamma \neq 0$. Then*

(a) ${}_{L'}\Delta(t_1, \ldots, t_\mu) = {}_L\Delta(t_1, \ldots, t_\mu^\gamma) \cdot {}_M\Delta(t_\mu, y)$,

(b) ${}_{L'}\lambda(t_1, \ldots, t_\mu) = \text{l.c.m.}\,[{}_L\lambda(t_1, \ldots, t_\mu^\gamma), {}_M\lambda(t_\mu, y)]$

where M denotes the model link of two components formed by K' and the unknotted meridian curve on the boundary torus containing K', and

$$y = \prod_{i=1}^{\mu-1} t_i^{\langle L_i, L_\mu \rangle}.$$

Proof. Let $\tilde{X} \xrightarrow{p} X$ denote the universal abelian cover of the bounded complement of L', and $\tilde{Y} = p^{-1}(Y)$, $\tilde{W} = p^{-1}(W)$ and $\tilde{T} = p^{-1}(T)$ as before. We compute the inclusion map on H_1 to identify $\tilde{Y}$, $\tilde{W}$, and $\tilde{T}$. If $A \subset X$, and $\alpha \in H_1(A)$, then $i_*(\alpha) = \prod_{i=1}^{\mu} t_i^{\langle \alpha, L_i \rangle} \in H_1(X) \cong \mu Z$ generated by meridians $\{t_i\}_{i=1}^{\mu}$. Now $H_1(Y) \cong \mu Z$ is generated by $\{t_i\}_{i=1}^{\mu}$ and by x, the meridian curve on T. We have $i_*(t_i) = t_i$ $1 \leqq i \leqq \mu - 1$ and $i_*(x) = t_\mu^\gamma$. Hence $i_*: H_1(Y) \to H_1(X)$ is injective, and $\tilde{Y}$ has γ components, each of which is a copy of the universal abelian cover of Y. Likewise, $H_1(W) \cong Z \oplus Z$ generated by t_μ and ξ, where ξ is the longitude of T. Since

$$\langle \xi, L_i \rangle = \langle L_\mu, L_i \rangle \, i \neq \mu \quad \text{and} \quad \langle \xi, L_\mu \rangle = 0,$$

we have $i_*(t_\mu) = t_\mu$ and $i_*(\xi) = y = \prod_{i=1}^{\mu-1} t_i^{\langle L_i, L_\mu \rangle}$. So $i_*: H_1(W) \to H_1(X)$ is also injective, and each component of $\tilde{W}$ is a copy of the universal abelian cover of W. Similarly, each component of $\tilde{T}$ is a copy of R^2, the universal abelian cover of T.

Applying the $M - V$ sequence to $\tilde{X} = \tilde{Y} \underset{\tilde{T}}{\cup} \tilde{W}$ we obtain the following exact sequence of Λ_μ-modules:

$$0 \to H_1(\tilde{Y}) \oplus H_1(\tilde{W}) \to H_1(\tilde{X}) \to H_0(\tilde{T}) \xrightarrow{i_*} H_0(\tilde{Y}) \oplus H_0(\tilde{W}) \to H_0(\tilde{X}) \to 0. \quad (1)$$

Exact sequence (1) reduces to the following short exact sequence of Λ_μ-modules:

$$0 \to H_1(\tilde{Y}) \oplus H_1(\tilde{W}) \to H_1(\tilde{X}) \to \operatorname{Ker} i_* \to 0. \tag{2}$$

Exact sequence (2) is in fact enough to prove part (a) of Theorem 5.2, because it is fairly easily shown that the characteristic polynomial of $H_0(\tilde{T})$, hence of $\operatorname{Ker} i_*$, is the identity. However, part (b) is more delicate, and requires calculation of $\operatorname{Ker} i_*$. We show, as in the proof of Theorem 5.1, that i_* is injective.

$\tilde{W}$ has, in general, infinitely many components, each diffeomorphic to the universal abelian covering of W. Let A be the indexing set for the components of $\tilde{W}$, so $\tilde{W} = \bigcup_{\alpha \in A} \tilde{W}_\alpha$. Fix attention on a component, say $\tilde{W}_\alpha$. By examination of the inclusion map $H_1(T) \to H_1(W)$, we see that $\partial \tilde{W}_\alpha = \gamma$ copies of the universal abelian covering of $T = \bigcup_{j=1}^{\gamma} \tilde{T}_{\alpha j}$. Moreover, each $\tilde{T}_{\alpha j}$ is incident to exactly one of the components of $\tilde{Y}$, labeled $\tilde{Y}_j$. So we have $0 \to H_0\left(\bigcup_{j=1}^{\gamma} T_{\alpha j}\right) \xrightarrow{i_*} H_0\left(\bigcup_{j=1}^{\gamma} \tilde{Y}_j\right) \oplus H_0(\tilde{W}_\alpha)$ is exact. Since $H_0(\tilde{T}) \underset{Z}{\cong} \bigoplus_{\alpha \in A} H_0\left(\bigcup_{j=1}^{\gamma} \tilde{T}_{\alpha j}\right)$, we have that $i_*: H_0(\tilde{T}) \to H_0(\tilde{Y}) \oplus H_0(\tilde{W})$ is injective. Hence, $H_1(\tilde{X}) \underset{\Lambda_2}{\cong} H_1(\tilde{Y}) \oplus H_1(\tilde{W})$ and Theorem 5.2 follows.

Case 3. $\mu = 1$, Type (ii) iteration.

We prove the following

Theorem 5.3. *Let L be a knot, and $\hat{L}$ be the link of 2 components obtained via iteration of type* (ii) *about L via the knot K' with winding number $\gamma \neq 0$. Then*

(a) ${}_{\hat{L}}\Delta(t_1, t_2) = {}_L\Delta(t_1 t_2^{\gamma}) \cdot {}_N\Delta(t_1, t_2)$,

(b) ${}_{\hat{L}}\lambda(t_1, t_2) = \text{l.c.m.}\,[{}_L\lambda(t_1 t_2^{\gamma}),\ {}_N\lambda(t_1, t_2)]$

where N denotes the model link of two components formed by K' and the unknotted core of the torus containing K'.

Proof. We have that $H_1(X) = Z \oplus Z$ generated by t_1 and t_2, $H_1(Y) \cong Z$ generated by x, the meridian of T, $H_1(W) \cong Z \oplus Z \oplus Z$ generated by t_1, t_2, and ξ, the longitude of T, and $H_1(T) \cong Z \oplus Z$ generated by x and ξ. Moreover, $i_*(x) = t_1 t_2^{\gamma}$ and $i_*(\xi) = 1$. As in the proof of 5.1, we have that each component of $\tilde{T}$ is a copy of $R \times S^1$. Now, exactly as in the proof of 5.1, let $W' = V' - \operatorname{Int}(U' \cup V^*)$. We have the diffeomorphism $f: W' \to W$, and moreover, $W' \underset{T'}{\cup} (S^1 \times D^2) = S^3 - \operatorname{Int}(U' \cup V^*) = Z$ is the bounded complement of the model link N of two components formed by the knot K' and the unknotted core of V. Proceeding as in the proof of 5.1, one shows that

$$H_1(\tilde{Z}) \underset{\Lambda_2}{\cong} H_1(\tilde{W}')/H_1(\tilde{T}')$$

where $\tilde{Z}$ is the universal abelian cover of Z. One then proves that $H_1(\tilde{X}) \underset{\Lambda_2}{\cong} H_1(\tilde{Z}) \oplus H_1(\tilde{Y})$, completing the proof of 5.3.

Case 4. $\mu \geqq 2$, Type (ii) iteration.

We prove the following

Theorem 5.4. *Let L be a link $\mu \geqq 2$ components, and let $\hat{L}$ be the link obtained from an iteration of Type* (ii) *about L_μ via the knot K' with winding number $\gamma \neq 0$. Then*

(a) $_{\hat{L}}\Delta(t_1, \ldots, t_{\mu+1}) = {}_L\Delta(t_1, \ldots, t_\mu t_{\mu+1}^\gamma) \cdot {}_P\Delta(t_\mu, t_{\mu+1}, y)$,

(b) $_{\hat{L}}\hat{\lambda}(t_1, \ldots, t_{\mu+1}) = \text{l.c.m.}\,[{}_L\hat{\lambda}(t_1, \ldots, t_\mu t_{\mu+1}^\gamma), {}_P\hat{\lambda}(t_\mu, t_{\mu+1}, y)]$

where P denotes the model link of 3 components formed by K', the unknotted meridian curve on the boundary of the torus containing K', and the unknotted core of the torus containing K', and $y = \prod_{i=1}^{\mu-1} t_i^{\langle L_i, L_\mu \rangle}$.

Proof. The proof proceeds exactly as in the proof of Theorem 5.2. With the notations of that proof, we have that $H_1(X) \cong (\mu+1)Z$ generated by meridians $\{t_i\}_{i=1}^{\mu+1}$. $H_1(Y) \cong \mu Z$ generated by $\{t_i\}_{i=1}^{\mu-1}$ and x, the meridian curve on T. We have $i_*(t_i) = t_i$ $1 \leqq i \leqq \mu - 1$, $i_*(x) = t_\mu t_{\mu+1}^\gamma$. $H_1(W) \cong Z \oplus Z \oplus Z$ generated by t_μ, $t_{\mu+1}$ and ξ, the longitude of T. We have that $i_*(\xi) = y = \prod_{i=1}^{\mu-1} t_i^{\langle L_i, L_\mu \rangle}$. Exactly the same reasoning as in 5.2 shows that each component of $\tilde{Y}$, $\tilde{W}$ and $\tilde{T}$ is a copy of a universal abelian cover, and a similar argument to that in 5.2 shows that $H_0(\tilde{T}) \xrightarrow{i^*} H_0(\tilde{Y}) \oplus H_0(\tilde{W})$ is injective. Hence

$$H_i(\tilde{X}) \cong H_1(\tilde{Y}) \oplus H_1(\tilde{W})$$

and Theorem 5.4 follows.

We note that in the proofs of Theorems 5.1–5.4, the only information about K' that was used was homological – we assumed the winding number $\gamma \neq 0$. Hence all of these theorems hold for general iterated tube links, with nonzero winding numbers at each stage in the iteration. The case of $\gamma = 0$ can be similarly analyzed [16], and will be done elsewhere. One can also obtain similar results for homology invariants in all dimensions for higher-dimensional iterated tube links.

We now concentrate on the situation of iterated tube links for which all of the iterates K' are torus knots. (In fact, the following proof only depends on the fact that each iterate is a closed braid.) This class of links contains all algebraic links, and it is possible to calculate all of the other homology groups of the universal abelian cover $\tilde{X}$ in this situation, and to give more information about $H_1(\tilde{X})$.

Theorem 5.5. *Under the above hypotheses, we have $H_i(\tilde{X}) = 0$ $i \geqq 2$, and $H_1(\tilde{X})$ is free abelian.*

Proof. We show that $\tilde{X}$ has the homology of a 1-complex. Via the $M - V$ sequence, the proof iteratively reduces to proving that $\tilde{W}$ has the homology of a 1-complex, for $\tilde{W}$ the universal abelian cover of W obtained in iterations of types (i) and (ii). We prove, in fact, that $\tilde{W}$ is homotopy equivalent to a 1-complex. In both types of iteration, W is a fiber bundle over S^1 (the longitude circle ξ of T), with fiber

F a 2-disc punctured by K (and possibly L_μ). We build $\tilde{W}$ in two stages. First, we have $Y \approx F \times R$, an infinite cyclic cover obtained by unwrapping ξ. Let $\tilde{F}$ be one of the components of $p^{-1}(F)$, where $\tilde{W} \xrightarrow{p} W$ is the universal abelian cover. Then $\tilde{W} \approx \tilde{F} \times R$, and $\tilde{F}$ is homotopy equivalent to a 1-complex, since F is.

Corollary 5.6. *With the hypotheses of Theorem* 5.5, *let* G' *be the commutator subgroup of* $G = \Pi_1(X)$. *Then* $H_i(G') = 0$, $i \geqq 2$, *and* $H_1(G')$ *is free abelian.*

Proof. Since $\Pi_i(\tilde{X}) = 0$ $i \geqq 2$ [14], $\tilde{X}$ is the Eilenberg-MacLane space $K(G', 1)$.

VI. Homology Invariants of the Model Links

In Section V we have seen that the characteristic and minimal polynomials of the universal abelian covers of the model links of 1, 2, and 3 components play a central role in the iterative calculation. In this section we calculate these invariants for all possible model links.

The model link of one component is an ordinary torus knot of type (m, n), and we have the following well-known [4, pp. 131]

Theorem 6.1. *For the torus knot of type* (m, n),

$$\Delta(t) = \lambda(t) = \frac{(t^{mn} - 1)(t - 1)}{(t^m - 1)(t^n - 1)}.$$

We now consider the case of the model link of two components, a torus knot K of type (m, n) linked with an exterior unknotted core, say L. This is precisely the situation encountered in Theorem 5.2, with winding number $\gamma = n$. Figure 3

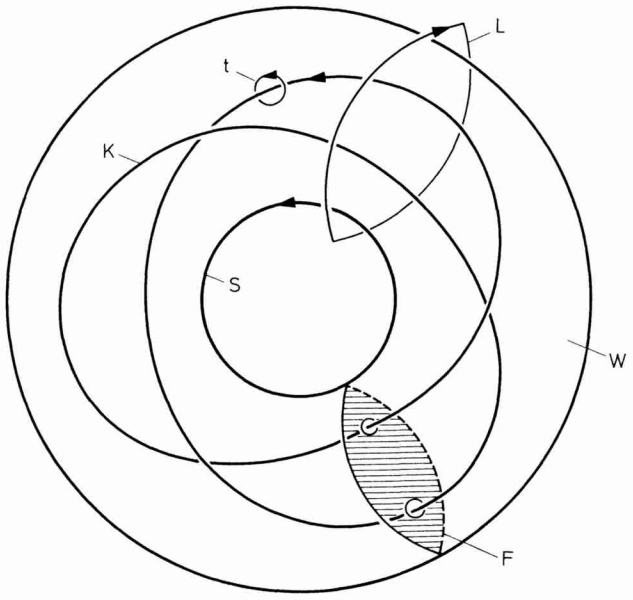


Fig. 3

shows the model link of the torus knot (3, 2) (labelled K) linked with its exterior unknotted core (labelled L).

The complement $S^3-(K\cup L)$ deformation retracts to W, a solid torus with a hole modeled on the torus knot K drilled out of it. We have that $H_1(W)\cong \underset{s}{Z}\oplus \underset{t}{Z}$ where s and t are the meridians of L and K, respectively. Now W fibers over the meridian circle of L with fiber F, a 2-disc punctured n times. We will consider the following 2-stage construction of the universal abelian cover $\tilde{W}$ of W (see proof of Theorem 5.5): First unwrap the homology generator s to obtain the infinite cyclic cover $F\times R\approx \tilde{Y}\to W$ associated with the epimorphism $\pi_1(W)\to H_1(W)\to \underset{s}{Z}$ obtained by projecting $H_1(W)$ onto the cyclic summand generated by s. Now $\tilde{Y}\simeq F$, and we consider the infinite cyclic covering space $\tilde{F}\to F$ associated with the epimorphism $\pi_1(F)\xrightarrow{i_*}\pi_1(W)\to H_1(W)\to \underset{t}{Z}$ which sends each generator $\{x_i\}_{i=1}^n$ of the free group $\pi_1(F)$ onto t. This induces an infinite cyclic covering $\tilde{W}\to \tilde{Y}$, and $\tilde{W}\approx \tilde{F}\times R$. Moreover $\tilde{W}$ is the universal abelian cover of W. Consider now the structure of $H_1(\tilde{F})$ as a $\Lambda_t=Z[t,t^{-1}]$ module which is induced by the infinite cyclic cover $\tilde{F}\to F$. We have that $\tilde{F}$ is homotopy equivalent to a 1-complex (Figure 4), so from the augmented chain complex of $\tilde{F}$ we extract the following split short exact sequence of Λ_t-modules:

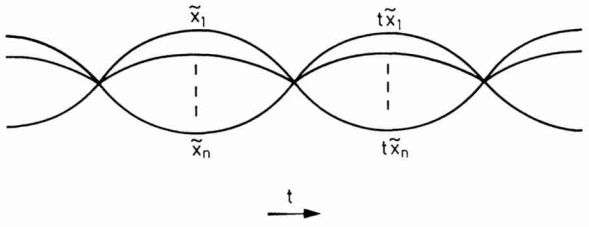
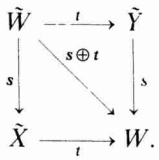


Fig. 4. Spine of $\tilde{F}$

$$0\to H_1(\tilde{F})\to C_1(\tilde{F})\xrightarrow{\varepsilon} I\to 0 \tag{1}$$

where $I\cong\Lambda_t$ is the augmentation ideal of Λ_t. $C_1(\tilde{F})$ is the free Λ_t-module on generators $\{\tilde{x}_i\}_{i=1}^n$, $\tilde{x}_i$ being a lift of x_i. Hence $H_1(\tilde{F})$ is the free Λ_t-module of rank $(n-1)$ on the generators $\{\alpha_i=\tilde{x}_i-\tilde{x}_{i+1}\}_{i=1}^{n-1}$. Now consider the structure of $H_1(\tilde{W})$ as a module over $\Lambda_{s,t}=Z[t,t^{-1},s,s^{-1}]\cong(Z[t,t^{-1}])[s,s^{-1}]$. Since $\tilde{W}\cong\tilde{F}\times R$, then $H_1(\tilde{W})$ is a free Λ_t-module, with the infinite cyclic group generated by s acting on it as a Λ_t-automorphism. This is due to the fact that $\tilde{W}$ can be viewed as an infinite cyclic cover of $\tilde{X}$, where $\tilde{X}$ is the infinite cyclic cover of W associated with the epimorphism $\pi_1(W)\to H_1(W)\to \underset{(t)}{Z}$. We have the commutative diagram of covering spaces and generators of free abelian covering translation groups:



In the above diagram, $\tilde{W} \xrightarrow{t} \tilde{Y}$ denotes the infinite cyclic covering with t as the generator of the infinite cyclic covering translation group.

It is therefore clear that the characteristic and minimal polynomial of $H_1(\tilde{W})$ as a $\Lambda_{s,t}$-module are identical to the characteristic and minimal polynomial of the Λ_t-automorphism of $H_1(\tilde{F})$ induced by covering translation s in the infinite cyclic cover $\tilde{F} \times R \approx \tilde{W} \xrightarrow{s} \tilde{X}$. In fact, if we choose the Λ_t-basis $\{\alpha_i\}_{i=1}^{n-1}$ for $H_1(\tilde{W})$ and represent the automorphism s by the square $(n-1)\times(n-1)$ matrix A, then $A - sI$ is a presentation matrix for $H_1(\tilde{W})$ as a $\Lambda_{s,t}$ module (I denotes the $(n-1)\times(n-1)$ identity matrix), and its determinant is the characteristic polynomial of both the Λ_t-linear transformation s and the $\Lambda_{s,t}$-module $H_1(\tilde{W})$.

In order to compute the matrix A, it is necessary to study the action of s on $\pi_1(F)$, the twisting taking place in the bundle $F \to W$. We can regard F as a tele-

$$\downarrow$$

$$S^1_{(s)}$$

phone dial with n holes in it, and the action of s can be regarded as a clockwise dial (regard $m>0$) through one click, iterated m times. That is, a dial corresponds to rotating F through $\frac{2\pi}{n}$, and the mth iteration of the dial is the glueing diffeomorphism of the fibration. Hence, if we represent the dial automorphism on $H_1(\tilde{F})$ by the $(n-1)\times(n-1)$ matrix D over Λ_t, then $A = D^m$.

In Figure 5, we have the fiber F with basepoint as shown, and paths connecting the basepoint to the holes $\{x_i\}_{i=1}^n$. The generator x_i of $\pi_1(F)$ consists of going out the path from the basepoint to the hole, once around the hole, and back to the basepoint. For ease in calculation, we also have the radial reference lines going from the outside edge of the holes to the edge of the disc. The reference lines allow us to easily calculate the homotopy class of any based loop, by simply writing down the sequence of signed intersections with the reference lines.

In Figure 6, we have the segment of the bundle twisted through $\frac{2\pi}{n}$ (one dial), and a diffeomorphism which straightens the picture out to a product (the bottom $\times I$).

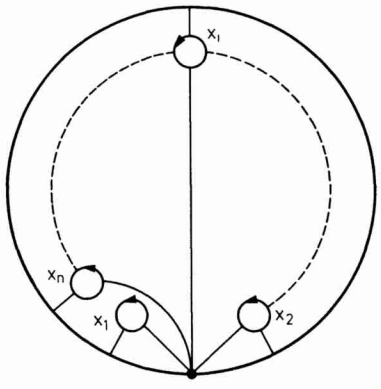


Fig. 5

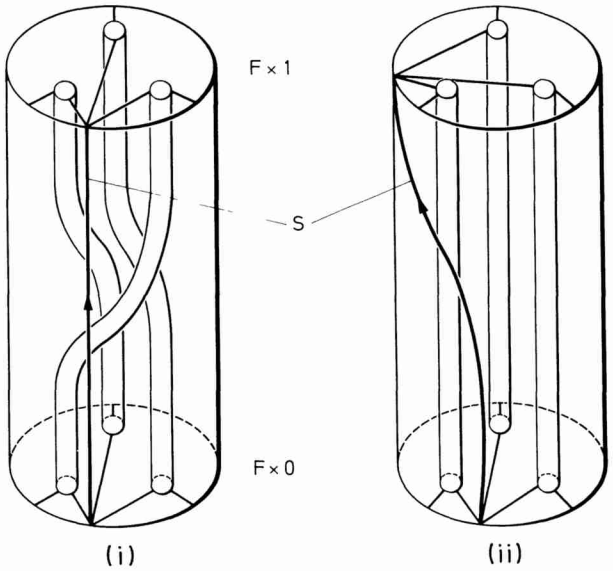


Fig. 6

Untwisting the tubes in Figure 6(i) twists the path s as shown in Figure 6(ii). The action of the generator s on the loop x_i is computed by traveling up s, doing x_i on the copy of $F \times 1$ in Figure 6(ii) then traveling back down s. Note that the diffeomorphism in Figure 6 taking the twisted picture to the straight picture, when restricted to $F \times 1$, gives a dial diffeomorphism $d: F \to F$ which is rotation through $\frac{-2\pi}{n}$. If ξ represents the basepoint in F, chosen to lie on ∂F between x_1 and x_2, then we choose the path traced out by ξ during rotation, which connects ξ to $d(\xi)$, to compute the effect of $d_*: \pi_1(F) \xrightarrow{\cong} \pi_1(F)$. We have that the path $s\, x_i\, s^{-1}$ (read left to right) pulls off $F \times 1$ and projects straight down to $d_*(x_i)$ in $F \times 0$ (keeping $\xi \times 0$ fixed). So we have

$$
\begin{aligned}
&s\, x_1\, s^{-1} = d_*(x_1) = x_1^{-1}\, x_n\, x_1 \\
&s\, x_2\, s^{-1} = d_*(x_2) = x_1^{-1}\, x_1\, x_1 = x_1 \\
&\quad\vdots \\
&s\, x_n\, s^{-1} = d_*(x_m) = x_1^{-1}\, x_{n-1}\, x_1 .
\end{aligned}
$$

The diffeomorphism $d: F \to F$ lifts to $\tilde{d}: \tilde{F} \to \tilde{F}$, and we wish to compute D, the matrix representing the Λ_t-automorphism $\tilde{d}_*: H_1(\tilde{F}) \to H_1(\tilde{F})$ with respect to the basis $\{\alpha_i\}_{i=1}^{n-1}$. D can be computed by computing the chain automorphism $\tilde{d}_\#: C_1(\tilde{F}) \to C_1(\tilde{F})$. We have $\tilde{d}_\#(\tilde{x}_i) =$ the lift of the path $x_1\, x_{i-1}\, x_1^{-1}$ to $\tilde{F} = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}(d_*(x_i))\right)^{\psi} \tilde{x}_j$,

where $\dfrac{\partial}{\partial x_j}$ denotes the Fox Free Derivative [8] and ψ denotes the linear extension of the abelianizing homomorphism

$$\psi: \pi_1(F) \to Z,$$
$$x_i | \to t.$$

In other words,

$$\tilde{d}_{\#}(\tilde{x}_1) = (1-t^{-1})\,\tilde{x}_1 + t^{-1}\,\tilde{x}_n$$
$$\tilde{d}_{\#}(\tilde{x}_2) = \tilde{x}_1$$
$$\vdots$$
$$\tilde{d}_{\#}(\tilde{x}_n) = (1-t^{-1})\,\tilde{x}_1 + t^{-1}\,\tilde{x}_{n-1}.$$

So $\tilde{d}_*(\alpha_1) = \tilde{d}_{\#}(\tilde{x}_1 - \tilde{x}_2) = t^{-1}(\tilde{x}_n - \tilde{x}_1)$
$$= -t^{-1}(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1})$$
$$\tilde{d}_*(\alpha_2) = t^{-1}(\alpha_1)$$
$$\vdots$$
$$\tilde{d}_*(\alpha_{n-1}) = t^{-1}(\alpha_{n-2}).$$

Hence we have the $(n-1)\times(n-1)$ matrix

$$D = t^{-1}\begin{pmatrix} -1 & \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & & & & \cdot & 0 \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & & \cdot & 0 & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix}.$$

Moreover, tD is the companion matrix to the polynomial $1+\xi+\xi^2+\cdots+\xi^{n-1}$, so since t is invertible in Λ_t we have that $(tD)^n = I$ so (tD) generates a cyclic subgroup of order n in $GL(n-1, Z)$. (This is no surprise because D is induced by a dial of F which obviously has order n). But since n is coprime to m, $t^m A = (tD)^m$ generates same cyclic subgroup of $GL(n-1, Z)$, so $t^m A$ has the same characteristic polynomial, corresponding to the sum of all the elements in the cyclic subgroup. So

$$t^{m(n-1)}\det(A - sI) = \det(t^m A - t^m sI) = 1 + (st^m) + \cdots + (st^m)^{n-1} = \frac{(st^m)^n - 1}{(st^m) - 1}.$$

Moreover, since A is the companion matrix to its characteristic polynomial, its characteristic polynomial is also its minimal polynomial. The above arguments provide an essentially homological proof of the following generalization of the classical results of Burau:

Theorem 6.2. *Let M (as in Theorem 5.2) denote the model link of a torus knot K of type (m, n) linked with its unknotted exterior core L. Let t denote the homology*

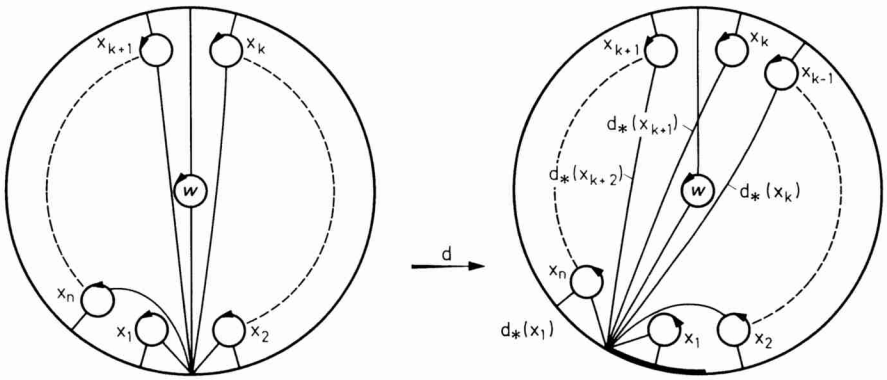


Fig. 7

generator coming from K, and s denote the homology generator coming from L. Then

$$_M\Delta(s,t) = {}_M\Lambda(s,t) = \frac{(s\,t^m)^n - 1}{(s\,t^m) - 1}.$$

We note here the Theorem 6.2 covers the model links arising in Theorem 5.3 as well, by reversing the roles of m and n.

We are left with the most complicated situation – a model link P as in Theorem 5.4. In this case P consists of the torus knot K of type (m, n) linked with both its exterior unknotted core L (a meridian on the boundary of the solid torus containing K) and its interior unknotted core C (the core of the solid torus). Let t, s and w denote the homology generators of $H_1(S^3 - P) \cong Z \oplus Z \oplus Z$ corresponding respectively to K, L and C. As before we consider the infinite cyclic covering space $\tilde{Y} \to W = S^3 - P$ corresponding to projection on the infinite cyclic factor $H_1(W) \to \underset{s}{Z}$. We have that W fibers over S^1 with fiber F, a telephone dial with n holes around the edge (corresponding to K), and a hole in the middle (corresponding to C). Figure 7 shows F in this situation.

We choose the reference line for w to go between x_k and x_{k+1} if $m = 2k$ or $m = 2k+1$.

As before we have that $\tilde{Y} \approx F \times R$, and we consider the abelian covering $\tilde{W}$ of $\tilde{Y}$ induced by the epimorphism $\pi_1(F) \xrightarrow{i_*} \pi_1(W) \to H_1(W) \to \underset{t}{Z} \oplus \underset{w}{Z}$. Now $\tilde{W} \approx \tilde{F} \times R$ where $\tilde{F}$ is the abelian cover of F induced by sending all the x_i's to t and w to w. Moreover, $\tilde{W}$ is the universal abelian covering of W. We now desire to study the action of covering translation s on the fiber $\tilde{F}$. We decompose s into the composition of m twists, or "dials" as before, and are reduced to studying the dial d: $F \to F$ which rotates F through $\dfrac{-2\pi}{n}$. The situation is unfortunately more complicated than the previous one, because $H_1(\tilde{F})$ is no longer free as a $\Lambda_{t,w} = Z[t, t^{-1}, w, w^{-1}]$-module. We have that F is homotopy equivalent to a bouquet of $(n+1)$ copies of S^1, so the augmented chain complex of $\tilde{F}$ becomes the exact sequence

$$0 \to H_1(\tilde{F}) \to C_1(\tilde{F}) \xrightarrow{\partial} I \to 0$$

where I is the augmentation ideal of $\Lambda_{t,w}$, generated by $(t-1)$ and $(w-1)$. Sadly, I is not free and the sequence does not split, so $H_1(\tilde{F})$ is a submodule of a free $\Lambda_{t,w}$-module, but not itself free.

We can as before proceed with an analysis of the twisting on $\pi_1(F)$:

$$
\begin{aligned}
s\,x_1\,s^{-1} &= d_*(x_1) = x_1^{-1}\,x_n\,x_1\\
s\,x_2\,s^{-1} &= d_*(x_2) = x_1^{-1}\,x_1\,x_1 = x_1\\
&\vdots\\
s\,x_k\,s^{-1} &= d_*(x_k) = x_1^{-1}\,x_{k-1}\,x_1\\
s\,x_{k+1}\,s^{-1} &= d_*(x_{k+1}) = x_1^{-1}\,w^{-1}\,x_k\,w\,x_1\\
s\,x_{k+2}\,s^{-1} &= d_*(x_{k+2}) = x_1^{-1}\,x_{k+1}\,x_1\\
&\vdots\\
s\,x_n\,s^{-1} &= d_*(x_m) = x_1^{-1}\,x_{n-1}\,x_1\\
s\,w\,s^{-1} &= d_*(w) = x_1^{-1}\,w\,x_1 .
\end{aligned}
$$

One notes that the only significant difference between this and the previous case is the introduction of the middle generator w in $d_*(x_{k+1})$ and $d_*(w)$.

As before, $d\colon F \to F$ lifts to $\tilde{d}\colon \tilde{F} \to \tilde{F}$, and the $\Lambda_{t,w}$-chain isomorphism

$$\tilde{d}_{\#}\colon C_1(\tilde{F}) \to C_1(\tilde{F})$$

is given by $\tilde{d}_{\#}(\tilde{\sigma}) = \sum_{\tau \in G} \left(\frac{\partial(d_*(\sigma))}{\partial \tau}\right)^{\psi} \tilde{\tau}, \forall\, \tilde{\sigma} \in \tilde{G}$ where $G = \{x_1, \ldots, x_n, w\}$ is the set of free generators of $\pi_1(F)$. $\tilde{G} = \{\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{w}\}$ is the set of $\Lambda_{t,w}$-generators of $C_1(\tilde{F})$, and ψ is the abelianizing homomorphism $\psi\colon \pi_1(F) \to \underset{t}{Z} \oplus \underset{w}{Z}$.

We have the following generators for $H_1(\tilde{F})$ over $\Lambda_{t,w}$ (see Fig. 8):

$$\{\alpha_i = (\tilde{x}_i - \tilde{x}_{i+1})\}_{i=1}^{n-1} \quad \text{and} \quad \{\beta_j = (w-1)\,\tilde{x}_j + (1 - t\,\tilde{w}\}_{j=1}^{n} .$$

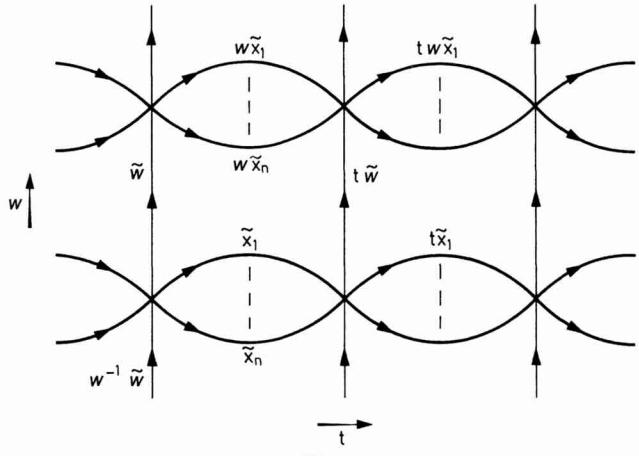


Fig. 8

In this case, we regard $\{\beta_j\}$ as the "essential" generators and $\{\alpha_i\}$ as the "inessential" generators, because of the relations

$$(w-1)\,\alpha_i=\beta_i-\beta_{i+1} \qquad 1\leqq i\leqq n-1.$$

Let H' be the free $\Lambda_{t,w}$ submodule of $H_1(\tilde{F}) \underset{\Lambda_{t,w}}{\cong} H_1(\tilde{W})$ which is generated by $\{\beta_j\}_{j=1}^n$. We have

$$\begin{aligned}
\tilde{d}_{\#}(\tilde{x}_1)&=(1-t^{-1})\,\tilde{x}_1+t^{-1}\,\tilde{x}_n\\
\tilde{d}_{\#}(\tilde{x}_2)&=\tilde{x}_1\\
&\vdots\\
\tilde{d}_{\#}(\tilde{x}_k)&=(1-t^{-1})\,\tilde{x}_1+t^{-1}\,\tilde{x}_n\\
&\vdots\\
\tilde{d}_{\#}(\tilde{x}_{k+1})&=(1-t^{-1})\,\tilde{x}_1+(w^{-1}-t^{-1}\,w^{-1})\,\tilde{w}+(t^{-1}\,w^{-1})\,\tilde{x}_k\\
&=(1-t^{-1})\,\tilde{x}_1+w^{-1}(1-t^{-1})\,\tilde{w}+(t^{-1}\,w^{-1})\,\tilde{x}_k\\
\tilde{d}_{\#}(\tilde{x}_{k+2})&=(1-t^{-1})\,\tilde{x}_1+t^{-1}\,\tilde{x}_{k+1}\\
&\vdots\\
\tilde{d}_{\#}(\tilde{x}_n)&=(1-t^{-1})\,\tilde{x}_1+t^{-1}\,\tilde{x}_{n-1}\\
\tilde{d}_{\#}(\tilde{w})&=t^{-1}(w-1)\,\tilde{x}_1+t^{-1}\,\tilde{w}.
\end{aligned}$$

Hence

$$\tilde{d}_*(\beta_j)=\begin{cases}t^{-1}\,\beta_n & j=1\\ t^{-1}\,\beta_{j-1} & j\neq 1,\quad k+1\\ t^{-1}\,w^{-1}\,\beta_k & j=k+1.\end{cases}$$

Perform the following change of basis:

$$\beta_i^*=\begin{cases}w^{-1}\,\beta_i & 1\leqq i\leqq k\\ \\ \beta_i & k+1\leqq i\leqq n.\end{cases}$$

Then with respect to this new basis,

$$\tilde{d}_*(\beta_j^*)=\begin{cases}t^{-1}\,w^{-1}\,\beta_n^* & j=1\\ t^{-1}\,\beta_{j-1}^* & 2\leqq j\leqq n\end{cases}$$

and the matrix D of the automorphism $\tilde{d}_*\colon H'\to H'$ with respect to the basis $\{\beta_j^*\}$ is

$$D=\left(\begin{array}{ccc|c}0 & \cdots & 0 & t^{-1}\,w^{-1}\\ \hline t^{-1} & \ddots & & 0\\ & \ddots & 0 & \vdots\\ & & & 0\\ 0 & & t^{-1} & \end{array}\right)=t^{-1}\left(\begin{array}{ccc|c}0 & \cdots & 0 & w^{-1}\\ \hline 1 & \ddots & & 0\\ & \ddots & 0 & \vdots\\ & & & 0\\ 0 & & 1 & \end{array}\right).$$

Moreover, tD is the companion matrix to the polynomial ξ^n-w^{-1}, so $w(tD)^n=I$. This means that $(tD)^m$ satisfies the polynomial ξ^n-w^{-m}. Furthermore, no poly-

nomial of degree less than n annihilates $(t\,D)^m$, provided m and n are coprime. This is because both $(t\,D)$ and $(t\,D)^m$ are permutations of order n (modulo units in $\Lambda_{t,w}$) on the basis elements $\{\beta_i\}_{i=1}^n$. In other words, as before, the minimal polynomial of $(t\,D)^m$ is also $\xi^n - w^{-m}$. Therefore, if $A = D^m$, then

$$t^{mn}\det(A - s\,I) = t^{mn}\det(D^m - s\,I) = \det((t\,D)^m - t^m\,s\,I) = (t^m\,s)^n - w^{-m}.$$

Therefore, both the minimal and characteristic polynomial of the $\Lambda_{s,t,w}$-submodule H' of $H = {}_1(\tilde{W})$ generated by $\{\beta_j\}_{j=1}^m$ is $[(t^m\,s)^n\,w^m - 1]$. In order to prove that the polynomial is in fact the minimal and characteristic polynomial of the $\Lambda_{s,t,w}$-module H, we need to get rid of the relations coming from the "inessential" generators $\{\alpha_i\}_{i=1}^{n-1}$.

Consider the short exact sequence of $\Lambda_{s,t,w}$-modules:

$$0 \to H' \to H \xrightarrow{\pi} H/H' \to 0.$$

We have that H/H' is generated by $\{\pi(\alpha_i)\}_{i=1}^{n-1}$, and we also have that

$$(w-1)\,\pi(\alpha_i) = 0 \qquad 1 \leqq i \leqq n-1,$$

so $\Delta(H/H')$ divides $(w-1)^{n-1}$, or $\Delta(H/H') = (w-1)^b$ for some integer $0 \leqq b \leqq n-1$. Now

$${}_p\Delta(s,t,w) = \Delta(H) = \Delta(H')\,\Delta(H/H') = [(t^m\,s)^n\,w^m - 1]\,(w-1)^b$$

by Lemma 2.1 (i). Hence ${}_p\Delta(t,t,t) = (t^{mn+m+n} - 1)\,(t-1)^b$, and by Theorem 3.3 $\bar{\Delta}(t) = (t-1)\,(t^{mn+m+n} - 1)\,(t-1)^b$. We now observe that, although we may be working on an iterated tube link which may not be an algebraic link because of violation of the inequalities in Section IV, our model link P of 3 components is itself an algebraic link. We are therefore in a position to use the theorem of Durfee [6, Prop. 3], says that $\nabla(1) \neq 0$ for an algebraic link, where $\dfrac{\bar{\Delta}(t)}{(t-1)^2}$ is the Hosokawa polynomial. In the case of our model algebraic link,

$$\nabla(t) = (1 + t + \cdots + t^{mn+m+n-1})\,(t-1)^b$$

and $\nabla(1) \neq 0$ so $b = 0$. This of course means that $\Delta(H/H') = 1$, and by Lemma 2.1 that the $\Lambda_{s,t,w}$-invariants of H' and H are identical. We have proved the following:

Theorem 6.3. *Let P (as in Theorem 5.3) denote the model link of a torus knot K of type (m, n) linked with both its exterior unknotted core L and its interior unknotted core C. Let t, s and w denote the homology generators arising from K, L and C, respectively. Then*

$${}_p\Delta(s,t,w) = {}_p\lambda(s,t,w) = (t^m\,s)^n\,w^m - 1.$$

VII. Applications to Monodromy

Let L be an algebraic link. Milnor [13] shows that $S^3 - L = X$ is a smooth fiber bundle over S^1, with projection mapping

$$\psi(z) = f(z)/|f(z)|, \qquad z \in C^2.$$

Each fiber F_θ has the homotopy type of a bouquet $S^1 \vee S^1 \vee \cdots \vee S^1$ of μ copies of S^1. The *local monodromy map h*, the natural action of a generator of $\pi_1(S^1)$ on the homology of the fiber, is described by

$$h: H_*(F_0) \to H_*(F_0).$$

Here h is the map induced by the characteristic homeomorphism $\tilde{h}$ of the fiber $F_0 = \psi^{-1}(1)$. The map $\tilde{h}$ is obtained via the Covering Homotopy Theorem by choosing a continuous family of homeomorphisms $h_t: F_0 \to F_t$ for $0 \leqq t \leqq 2\pi$, where h_0 is the identity and $\tilde{h} = h_{2\pi}$ is the required characteristic homeomorphism of the fiber [13].

It follows that the infinite cyclic covering space $\tilde{X}$ is homeomorphic to $F_0 \times R$ and $\tilde{h}$ can be considered as the generator of the group of covering transformations of $\tilde{X}$ [13].

Clearly $H_1(\tilde{X}) \cong H_1(F_0 \times R) \cong \mu Z$ as an abelian group, and $h: H_1(\tilde{X}) \to H_1(\tilde{X})$ is an automorphism.

The following lemma enables us to show that the minimal polynomial of the monodromy map h is $\lambda(t)$, the polynomial invariant defined in Section II.

Lemma 7.1. *If M is the integral matrix of h as an automorphism of a free abelian group, then $M - hI$ is a presentation matrix for $H_1(X)$ as a $\Lambda \cong Z[t, t^{-1}]$ module.*

The proof of the lemma is a straight-forward algebraic argument [20] and is omitted. By a theorem of Frobenius applied to the above result, the minimal polynomial of h is $\lambda(t) = \Delta_1(t)/\Delta_2(t)$, where $\Delta_1(t)$ and $\Delta_2(t)$ are the polynomial invariants defined in Section II for the link L.

Applying the results of Sections V and VI enables us to give a simple proof of Grothendieck's result and answer Milnor's question in the classical case [13, page 72]. The detailed algorithm for calculating $\Delta(t)$ is given later in this section.

Theorem 7.2. *For iterated torus links, all roots of $\Delta(t)$ are roots of unity.*

Proof. The immediate result of Sections V and VI is that each factor of $\Delta(t)$ has one of the following forms:

(i) $\dfrac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)}$,

(ii) $\dfrac{t^{ab}-1}{t^a-1}$,

(iii) $t^k - 1$.

Thus, each root is a root of unity.

We are interested in the question of whether or not h has finite order. An equivalent question is whether or not there exists a positive integer d such that $h^d = 1$. But $h^d = 1$ if and only if the minimal polynomial of h, $\lambda(t)$, divides $t^d - 1$. Since it is known (Grothendieck, or Theorem 7.2 above) that the roots of $\lambda(t)$ are roots of unity, the following lemma is proved.

Lemma 7.3. *The monodromy map h has finite order iff $\lambda(t)$ has distinct roots.*

For algebraic knots, we note that the polynomial factors involved in the recursive formula for $\Delta(t)$ are of the form in 7.2 (i) above and thus have distinct roots. Since $\lambda(t)$ divides a polynomial developed sequentially in terms of least common multiples of such factors, and repeated iterations do not introduce common roots in previously relatively prime factors, $\lambda(t)$ has distinct roots. Applying Lemma 7.3 we have a simple proof of the following Theorem [1, 10, 19].

Theorem 7.4. *If L is an algebraic knot, then h has finite order.*

Note. Theorem 7.4 is true for the much larger class of iterated torus knots, and the proof is the same [19].

It is well known that for an algebraic link with $r \geqq 2$ components, the monodromy map h may have either finite or infinite order [1, 20]. Our algorithm for calculating the characteristic polynomial and the following theorem provide good tools for studying specific links and/or developing general criteria for finite (infinite) monodromy. It generalizes a theorem of Durfee [6].

Theorem 7.5. *If h has finite order and ξ is a p^m-th root of unity for any prime p and any positive integer m, then the multiplicity of ξ as a root of $\Delta(t)$ is at most $r-1$, where r is the number of components of the algebraic link L* [20].

Note. It is known that $\Delta(t)$ has exactly $r-1$ factors of $t-1$ [6].

Proof. Recalling the results of Milnor mentioned above, we argue as in [17], Theorem 2.4. Consider the short exact sequence of chain complexes with integral coefficients

$$0 \longrightarrow C_*(\tilde{X}) \xrightarrow{h-1} C_*(\tilde{X}) \longrightarrow C_*(X) \longrightarrow 0.$$

This induces the long exact sequence of homology

$$\cdots \to H_i(\tilde{X}) \longrightarrow H_i(\tilde{X}) \longrightarrow H_i(X) \xrightarrow{\partial} H_{i-1}(\tilde{X}) \to \cdots.$$

Since $n=1$, we have

$$0 \longrightarrow H_2(X) \xrightarrow{\partial} H_1(\tilde{X}) \xrightarrow{h_*-I} H_1(\tilde{X}) \longrightarrow H_1(X) \longrightarrow H_0(\tilde{X}) \longrightarrow H_0(\tilde{X}) \longrightarrow H_0(X) \longrightarrow 0.$$

Also, from Alexander duality, we have [17]

$$H_2(X) \cong (r-1)\,Z$$

and

$$H_1(X) \cong r\,Z.$$

Now $H_0(X) \cong H_0(\tilde{X}) \cong Z$, so the above sequence is

$$0 \longrightarrow (r-1)\,Z \longrightarrow H_1(\tilde{X}) \xrightarrow{h_*-I} H_1(\tilde{X}) \longrightarrow r\,Z \longrightarrow Z \longrightarrow Z \longrightarrow Z \longrightarrow 0,$$

which reduces to

$$0 \longrightarrow (r-1)\,Z \longrightarrow H_1(\tilde{X}) \xrightarrow{h_*-I} H_1(\tilde{X}) \longrightarrow (r-1)\,Z \longrightarrow 0.$$

From the argument of [17, pp. 104], it follows that

$$\Delta_i(1)=\begin{cases}0, & 1\leqq i\leqq r-1\\ 1, & i\geqq r.\end{cases}$$

Proceeding to argue by contradiction, we assume h has finite order and the multiplicity of ξ as a root of $\Delta(t)$ is greater than $r-1$. Now by Lemma 7.3, $\lambda(t)$ has distinct roots, so $\Delta_i(t)/\Delta_{i+1}(t)$ has distinct roots for each $i\geqq 1$. Hence ξ is a root of $\Delta_r(t)$. Since $\Delta_r(t)$ is a product of cyclotomic polynomials, it follows that $\phi_{p^m}(t)$ divides $\Delta_r(t)$, where $\phi_{p^m}(t)$ denotes the p^m-th cyclotomic polynomial. Thus $\Delta_r(1)$ is a multiple of $\phi_{p^m}(1)$, but

$$\phi_{p^m}(1)=\begin{cases}0, & m=0\\ p, & m>0,\end{cases}$$

contradicting 1) above.

In order to illustrate the utility of the theorems proved in Sections V and VI, we calculate $\Delta(t)$ for general algebraic links of 2 and 3 components. The calculation for $r=2$ is a simpler proof of an earlier result of Burau [5]. The calculation for $r=3$ is a new result and could be done for any $r\geqq 3$. The iterative development is similar for any $r\geqq 3$, but the number of factors and the size of the exponents in the polynomial become large very rapidly. However, only three basic types of factors are involved, and these will come from the three model links discussed in Section VI – a torus knot, a torus knot linked with its exterior core, and a torus knot linked with both cores. The corresponding polynomials will be called $P(m, n, x)$, $Q(m, n, x, y)$, and $B(m, n, x, y, z)$ respectively for a torus knot of type (m, n), where x, y, and z are the homology generators arising from the torus knot, exterior core, and interior core respectively. From Section VI we have:

$$P(m, n, x)=\frac{x^{mn}-1}{x^m-1}\cdot\frac{x-1}{x^n-1}, \tag{1}$$

$$Q(m, n, x, y)=\frac{(y\,x^m)^n-1}{y\,x^m-1}, \tag{2}$$

$$B(m, n, x, y, z)=(y\,x^m)^n\,z^m-1. \tag{3}$$

We will first give the geometric and polynomial development for an algebraic link with two components. Then that development is simply extended to obtain a link with three components.

Assume the two characteristic Puiseux developments are

$$y_1=a_{11}\,x_1^{m_{11}/n_{11}}+a_{12}\,x_1^{m_{12}/n_{11}\,n_{12}}+\cdots+a_{1,s_1}\,x_1^{m_{1,s_1}/n_{11}\,n_{12}\cdots n_{1,s_1}}$$

and

$$y_2=a_{21}\,x_2^{m_{21}/n_{21}}+a_{22}\,x_2^{m_{22}/n_{21}\,n_{22}}+\cdots+a_{2,s_2}\,x_2^{m_{2,s_2}/n_{21}\,n_{22}\cdots n_{2,s_2}},$$

which are the same through the first $i \geqq 0$ terms, but differ at the $(i+1)^{st}$ terms. Either $\frac{m_{1,i+1}}{n_{1,i+1}} \neq \frac{m_{2,i+1}}{n_{2,i+1}}$ or $a_{1,i+1} \neq a_{2,i+1}$. Without loss of generality, assume $\frac{m_{1,i+1}}{n_{1,i+1}} > \frac{m_{2,i+1}}{n_{2,i+1}}$ (see Section V or [3, 5, 9]). Recall that $w_{k1} = m_{k1}$ and $w_{kj} = m_{kj} - n_{kj} m_{k,j-1} + w_{k,j-1} n_{kj} n_{k,j-1}$ for $2 \leqq j \leqq s_k$. Then the geometric development is as follows [5]: start with the unknot; iterate by a torus knot of type (m_{11}, n_{11}); iterate by a torus knot of type (w_{12}, n_{12}); ... ; iterate by a torus knot of type (w_{1i}, n_{1i}) to get L_1^i; iterate by a torus knot of type $(w_{2,i+1}, n_{2,i+1})$ to get L_2^{i+1}. Now L_1^i and L_2^{i+1} taken together give a link with two components. We note that L_1^i is the inner branch and L_2^{i+1} is the outer branch. Iterate on the inner branch by a torus knot of type $(w_{1,i+1}, n_{1,i+1})$; then by $(w_{1,i+2}, n_{1,i+2})$; ... ; (w_{1,s_1}, n_{1,s_1}) to complete the inner branch, $L_1^{s_1}$. The iterate on the outer branch successively by torus knots of types $(w_{2,i+2}, n_{2,i+2})$, $(w_{2,i+3}, n_{2,i+3})$, ..., (w_{2,s_2}, n_{2,s_2}) to complete the outer component, $L_2^{s_2}$.

Applying the theorems of Sections V and VI we get the corresponding iterative development for the Alexander polynomial of the universal abelian cover. Since products of the n's appear regularly, it is convenient to use the notation $b_{k,l,m} \equiv \prod_{j=l}^{m} n_{k,j}$, where $b_{k,l,m} = 1$ if $l > m$. The form of the polynomial at each iteration is shown, with the ordinal number of the iteration given as a left subscript of Δ.

$$
\begin{aligned}
{}_1\Delta(t) &= P(w_{11}, n_{11}, t) = \frac{t^{w_{11} n_{11}} - 1}{t^{w_{11}} - 1} \cdot \frac{t-1}{t^{n_{11}} - 1}. \\
{}_2\Delta(t) &= {}_1\Delta(t^{n_{12}})\, P(w_{12}, n_{12}, t) = P(w_{11}, n_{11}, t^{n_{12}})\, P(w_{12}, n_{12}, t). \\
&\vdots \\
{}_i\Delta(t) &= {}_{i-1}\Delta(t^{n_{1i}})\, P(w_{1i}, n_{1i}, t) = \prod_{j=1}^{i} P(w_{1j}, n_{1j}, t^{b_{1,j+1,i}}).
\end{aligned}
$$

Now the branching occurs. In the following, t_1 and t_2 will be related to the inner and outer components respectively. Applying Theorem 5.3 we have

$$
\begin{aligned}
{}_{i+1}\Delta(t_1, t_2) &= {}_i\Delta(t_1 t_2^{n_{2,i+1}})\, Q(n_{2,i+1}, w_{2,i+1}, t_2, t_1) \\
&= \left[\prod_{j=1}^{i} P(w_{1j}, n_{ij}, [t_1 t_2^{n_{2,i+1}}]^{b_{1,j+1,i}}) \right] \frac{(t_1 t_2^{n_{2,i+1}})^{w_{2,i+1}} - 1}{t_1 t_2^{n_{2,i+1}} - 1}.
\end{aligned}
$$

Note. For a torus knot of type (m, n) linked with its inner core, the Alexander polynomial is $\frac{(x^n y)^m - 1}{x^n y - 1}$ (compare with (2) above). Applying Theorem 5.2 and Theorem 6.2, the next $s_1 - i$ iterations each contribute a factor of the form $Q(w_{1,j}, n_{1,j}, t_1, t_2^{\alpha})$ and alter the preceding polynomial by substituting $t_1^{n_{1j}}$ for t_1. Here, α is the linking number of the two components prior to the iteration. Specifically,

$$
{}_{i+2}\Delta(t_1, t_2) = {}_{i+1}\Delta(t_1^{n_{1,i+1}}, t_2)\, Q(w_{1,i+1}, n_{1,i+1}, t_1, y_{i+1}),
$$

where

$$y_{i+1} = t_2^{\langle L_1^i, L_2^{i+1}\rangle} \quad \text{(see Theorem 5.2)}$$
$$= t_2^{w_{2,i+1}} \quad \text{(recall how } L_2^{i+1} \text{ was obtained).}$$

After the next iteration (by the torus knot of type $(w_{1,i+2}, n_{1,i+2})$ on the inner branch), we have

$$_{i+3}\Delta(t_1, t_2) = {}_{i+2}\Delta(t_1^{n_{1,i+2}}, t_2)\, Q(w_{1,i+2}, n_{1,i+2}, t_1, y_{i+2})$$

where

$$y_{i+2} = t_2^{\langle L_1^{i+1}, L_2^{i+1}\rangle} = t_2^{\langle L_1^i, L_2^{i+1}\rangle n_{1,i+1}} = t_2^{w_{2,i+1} n_{1,i+1}}.$$

Note. For $i+1 \leqq j \leqq s_2$, $\langle L_1^j, L_2^{i+1}\rangle = \langle L_1^{j-1}, L_2^{i+1}\rangle \cdot n_{1,j}$ since L_1^j is obtained from L_1^{j-1} by an iteration of type $(w_{1,j}, n_{1,j})$ in a tubular neighborhood of L_1^{j-1}, disjoint from L_2^{i+1}. Thus the linking number is changed directly by the longitudinal number, $n_{1,j}$.

The iterations continue in this manner until the inner component is completely developed. At that stage we have

$$_{s_1+1}\Delta(t_1, t_2) = {}_{s_1}\Delta(t_1^{n_{1,s_1}}, t_2)\, Q(w_{1,s_1}, n_{1,s_1}, t_1, y_{s_1})$$

where

$$y_{s_1} = t_2^{\langle L_1^{s_1-1}, L_2^{i+1}\rangle} = t_2^{w_{2,i+1} b_{1,i+1,s_1-1}}.$$

Summarizing the effects of the iterations to this stage, we have

$$_{s_1+1}\Delta(t_1, t_2) = \left(\prod_{j=1}^{i} P(w_{1,j}, n_{1,j}, t_1^{b_{1,j+1,s_1}}, t_2^{b_{2,j+1,i+1}})\right) Q(n_{2,i+1}, w_{1,i+1},$$

$$t_2, t_1^{b_{1,i+1,s_1}}) \left(\prod_{j=i+1}^{s_1} Q(w_{1,j}, n_{1,j}, t_1^{b_{1,j+1,s_1}}, t_2^{w_{2,i+1} b_{1,i+1,j-1}})\right).$$

Finally, the outer branch is developed, beginning with an iteration of type $(w_{2,i+2}, n_{2,i+2})$ on L_2^{i+1} and we have

$$_{s_1+2}\Delta(t_1, t_2) = {}_{s_1+1}\Delta(t_1, t_2^{n_{2,i+2}})\, Q(w_{2,i+2}, n_{2,i+2}, t_2, v_{i+2})$$

where

$$v_{i+2} = t_1^{\langle L_1^{s_1}, L_2^{i+1}\rangle} = t_1^{w_{2,i+1} b_{1,i+1,s_1}}.$$

Continuing,

$$_{s_1+3}\Delta(t_1, t_2) = {}_{s_1+2}\Delta(t_1, t_2^{n_{2,i+3}})\, Q(w_{2,i+3}, n_{2,i+3}, t_2, v_{i+3})$$

where

$$v_{i+3} = t_1^{\langle L_1^{s_1}, L_2^{i+2}\rangle} = t_1^{\langle L_1^{s_1}, L_2^{i+1}\rangle n_{2,i+2}} = (v_{i+2})^{n_{2,i+2}} = t_1^{w_{2,i+1} b_{1,i+1,s_1} b_{2,i+2,i+2}}.$$

These iterations and corresponding changes in the polynomial continue until the outer branch is completely developed. The final polynomial is

$$_{s_1+s_2-i}\Delta(t_1, t_2) = {}_{s_1+s_2-i-1}\Delta(t_1, t_2^{n_{2,s_2}, s_2})\, Q(w_{2,s_2}, n_{2,s_2}, t_2, v_{s_2})$$

where

$$v_{s_2} = (v_{s_2-1})^{n_{2,s_2-1}} = t_1^{w_{2,i+1} b_{1,i+1,s_1} b_{2,i+2,s_2-1}}.$$

Summarizing, the complete Alexander polynomial is

$$\Delta(t_1, t_2) = \prod_{j=1}^{i} P(w_{1,j}, n_{1,j}, t_1^{b_{1,j+1,s_1}} \cdot t_2^{b_{2,j+1,s_2}})\, Q(n_{2,i+1}, w_{2,i+1}, t_2^{b_{2,i+2,s_2}},$$

$$t_1^{b_{1,i+1,s_1}}) \prod_{j=i+1}^{s_1} Q(w_{1,j}, n_{1,j}, t_1^{b_{1,j+1,s_1}}, t_2^{w_{2,i+1} b_{1,i+1,j-1} b_{2,i+2,s_2}})$$

$$\prod_{j=i+2}^{s_2} Q(w_{2,j}, n_{2,j}, t_2^{b_{2,j+1,s_2}}, t_1^{w_{2,i+1} b_{1,i+1,s_1} b_{2,i+2,j-1}}). \tag{4}$$

This completes a new proof of an earlier result of Burau [5] (see also [20]).

Theorem 7.6. *The characteristic polynomial $\Delta(t)$ of the monodromy map h for an* algebraic link of two components is

$$\Delta(t) = (t-1)\,\Delta(t, t)$$

$$= (t-1) \prod_{j=1}^{i} \left(\frac{t^{w_{1,j} e_{0,j}} - 1}{t^{w_{1,j} e_{0,j+1}} - 1} \cdot \frac{t^{e_{0,j+1}} - 1}{t^{e_{0,j}} - 1} \right)$$

$$\cdot \left\{ \frac{t^{w_{2,i+1} e_{0,i+1}} - 1}{t^{e_{0,i+1}} - 1} \cdot \prod_{k=1}^{2} \prod_{j=i+k}^{s_k} \left(\frac{t^{n_{k,j} e_{k,j}} - 1}{t^{e_{k,j}} - 1} \right) \right\}$$

where

$$e_{q,j} = \begin{cases} b_{1,j,s_1} + b_{2,j,s_2} & q=0 \\ w_{2,i+1}\, b_{2,i+2,s_2}\, b_{1,i+1,j-1} + w_{1,j}\, b_{1,j+1,s_1} & q=1 \\ w_{2,i+1}\, b_{1,i+1,s_1}\, b_{2,i+2,j-1} + w_{2,j}\, b_{2,j+1,s_2} & q=2. \end{cases}$$

It is now possible to give simple numerical criteria for h to have finite or infinite order. These criteria are based on the characteristic Puiseux numbers, and the criteria eliminate the need to calculate $\Delta(t)$ and $\lambda(t)$ for many algebraic links with two components. The following is a sharpening of Theorem 3.1 of [20].

Theorem 7.7. *For an algebraic link L of 2 components which branches at the $i+1$ iteration (in effect, i is as defined above), h has*

(i) *infinite order if the elements of the set*

$$C = \{w_{2,i+1}, n_{2,i+2}, n_{2,i+3}, \ldots, n_{2,s_2}, n_{1,i+1}, n_{1,i+2}, \ldots, n_{1,s_1}\}$$

are not pair-wise relatively prime.

(ii) *finite order if the elements of C are pair-wise relatively prime, and for each element $c \in C$, c is relatively prime to $n_{ij} w_{ij}$, $1 \leqq j \leqq i$.*

Proof (see also [20]). (i) By consideration of cases it is fairly easy to show that if the elements of C are not pairwise relatively prime, then $\Delta(t)$ has repeated p-th roots for some prime p. Then Theorem 7.5 is applied to show infinite monodromy since $r-1=2-1=1$. The key observation needed in the proof is that each factor of $\Delta(t)$ within the braces "{ }" has the form $(t^{ck}-1)/(t^k-1)$ for some $c\in C$ (refer to Theorem 7.6).

(ii) Under these conditions it is clear that each factor of $\Delta(t)$ has roots distinct from those of the other factors. Thus $\Delta(t)$ has distinct roots and $\Delta(t)=\lambda(t)$. By Lemma 7.3, h has finite order.

These criteria identify large classes of algebraic links for each category; (i) infinite monodromy, and (ii) finite monodromy. However, the criteria are not exhaustive and it is possible (see Example 3 below) to find algebraic links with two components such that neither criterion is satisfied.

Example 1. $f(x, y)=(x^n+y^m)(x^{m'}+y^{n'})$ where $m'/n'>m/n>1$ and $(n, m)=1=(n', m')$, but $(m, n')\neq 1$. The set C associated with criterion (i) is $\{m, n'\}$ and by assumption the elements are not pair-wise relatively prime. Thus h has infinite order.

Example 2. $f(x, y)=(x^n+y^m)(a\,x^n+y^m)$, $a\neq 1$, where $m/n>1$ and $(n, m)=1$. Then the set associated with criterion (ii) is $\{m, n\}$ and the elements are relatively prime by hypothesis. Hence h has finite order.

Example 3. Let L be the algebraic link with two components with characteristic Puiseux pairs $\{(m_{11}, n_{11})\}=\{(3, 2)\}$ and $\{(m_{21}, n_{21}), (m_{22}, n_{22})\}$ where $m_{21}=3$, $n_{21}=2$, $n_{22}=2$, and $m_{22}=9$. Then the set C in Theorem 7.7 (i) is $C=\{w_{22}\}=\{15\}$ so criterion (ii) is not met. Since neither criteria is met, we consider $\Delta(t)$, and by Theorem 7.6,

$$\Delta(t)=(t-1)\frac{t^{18}-1}{t^9-1}\cdot\frac{t^3-1}{t^6-1}\cdot\frac{t^{45}-1}{t^3-1},$$

which has distinct roots, so h has finite order.

Note. We have assumed in applying Theorem 7.7 and 7.6 to this example that the two Puiseux developments are identical through the first term.

Example 4. Let L be determined by the same Puiseux pairs as in Example 3, but assume that the Puiseux developments differ in the first term (i.e., the first coefficients a_{11} and a_{21} differ). Then the branching is immediate, $i=0$, and

$$C=\{w_{21}, n_{22}, n_{11}\}=\{3, 2, 2\},$$

so by Theorem 7.7 (i), h has infinite order. Or alternately, by Theorem 7.6

$$\begin{aligned}\Delta(t)&=(t-1)\frac{t^{w_{21}e_{01}}-1}{t^{e_{01}}-1}\cdot\frac{t^{n_{11}e_{11}}-1}{t^{e_{11}}-1}\cdot\frac{t^{n_{22}e_{22}}-1}{t^{e_{22}}-1}\\&=(t-1)\frac{t^{18}-1}{t^6-1}\cdot\frac{t^{18}-1}{t^9-1}\cdot\frac{t^{42}-1}{t^{21}-1}.\end{aligned}$$

Clearly, the third and fourth factors give repeated 2-nd roots, -1, and by Theorem 7.5, h has infinite order.

Remark. It is interesting to note that the links in Examples 3 and 4 are identical except for the manner in which the two components are linked to each other – the components themselves are the same in Example 4 as in Example 3. Yet in Example 3, h has finite order, while in Example 4, h has infinite order.

We now consider a polynomial of three components and without loss of generality assume the corresponding Puiseux developments are

$$y_1 = a_{11}\, x_1^{m_{11}/n_{11}} + a_{12}\, x_1^{m_{12}/n_{11}\, n_{12}} + \cdots + a_{1,s_1}\, x_1^{m_{1,s_1}/b_{1,1,s_1}},$$
$$y_2 = a_{21}\, x_2^{m_{21}/n_{21}} + \cdots + a_{2,s}\, x_2^{m_{2,s_2}/b_{2,1,s_2}},$$
$$y_3 = a_{31}\, x_3^{m_{31}/n_{31}} + \cdots + a_{3,s_3}\, x_3^{m_{3,s_3}/b_{3,1,s_3}}$$

where all three developments agree through the first $i \geqq 0$ terms, but

$$m_{2,i+1}/n_{2,i+1} < m_{1,i+1}/n_{1,i+1}.$$

Further assume that the second and third developments agree through the $i+m$ term ($m>0$), but $m_{3,i+m+1}/n_{3,i+m+1} < m_{2,i+m+1}/n_{2,i+m+1}$. Then in the final link, the first, second, and third developments correspond to the inner, middle, and outer branches. The development through the first s_1+m iterations is the same as for the link of two components given above, with resulting link $L_1^{s_1} \cup L_2^{i+m}$ and polynomial ${}_{s_1+m}\Delta(t_1, t_2)$ given by the final polynomial formula for two components (using $i+m$ instead of s_2).

The development is continued with an iteration of type $(w_{3,m+i+1}, n_{3,m+i+1})$ about L_2^{i+m} to obtain L_3^{i+m+1}. Consider the link $L_1^{s_1} \cup L_2^{i+m} \cup L_3^{i+m+1}$, with inner, middle, and outer components corresponding to subscripts 1, 2, and 3 respectively. The link will be completed by successive iterations on the middle and outer branches. The current polynomial is (Thm. 5.4 and Thm. 6.3)

$${}_{s_1+m+1}\Delta(t_1, t_2, t_3) = {}_{s_1+m}\Delta(t_1, t_2\, t_3^{n_{3,i+m+1}}) \cdot B(w_{3,i+m+1}, n_{3,i+m+1}, t_3, t_1^{\alpha}, t_2)$$

where

$$\alpha = \langle L_1^{s_1}, L_2^{i+m} \rangle = w_{2,i+1}\, b_{1,i+1,s_1}\, b_{2,i+2,i+m},$$

with

$$B(\ldots) = (t_1^{\alpha}\, t_3^{w_{3,i+m+1}})^{n_{3,i+m+1}}\, t_2^{w_{3,i+m+1}} - 1 \qquad \text{[see (3) above]}.$$

Also, ${}_{s_1+m}\Delta(x, y)$ has the form of (4) above with s_2 replaced by $i+m$.

Next, the middle branch is developed by successive iterations of types

$$(w_{2,i+m+j}, n_{2,i+m+j}) \qquad 1 \leqq j \leqq s_2 - i - m.$$

The corresponding development of the characteristic polynomial is:

$${}_{s_1+m+2}\Delta(t_1, t_2, t_3) = {}_{s_1+m+1}\Delta(t_1, t_2^{n_{2,i+m+1}}, t_3) \cdot Q(w_{2,i+m+1}, n_{2,i+m+1}, t_2, v_1),$$

where

$$v_1 = t_1^{\langle L_1^{s_1}, L_2^{i+m} \rangle}\, t_3^{\langle L_3^{i+m+1}, L_2^{i+m} \rangle} = t_1^{w_{2,i+1}\, b_{1,i+1,s_1}\, b_{2,i+2,i+m}}\, t_3^{w_{3,i+m+1}}.$$

Continuing,

$${}_{s_1+m+3}\Delta(t_1, t_2, t_3) = {}_{s_1+m+2}\Delta(t_1, t_2^{n_{2,i+m+2}}, t_3)\, Q(w_{2,i+m+2}, t_2, v_2)$$

where

$$v_2 = t_1^{\langle L_1^{s_1},\, L_2^{i+m+1}\rangle}\, t_3^{\langle L_3^{i+m+1},\, L_2^{i+m+1}\rangle}$$
$$= t_1^{\langle L_1^{s_1},\, L_2^{i+m}\rangle^{n_{2,\,i+m+1}}}\, t_2^{\langle L_3^{i+m+1},\, L_2^{i+m}\rangle^{n_{2,\,i+m+1}}} = (v_1)^{n_{2,\,i+m+1}}.$$

$$_{s_1+m+4}\Delta(t_1, t_2, t_3) = {}_{s_1+m+3}\Delta(t_1, t_2^{n_{2,\,i+m+3}}, t_3)\, Q(w_{2,\,i+m+3}, n_{2,\,i+m+3}, t_2, v_3),$$

where

$$v_3 = (v_2)^{n_{2,\,i+m+2}} = (v_1)^{b_{2,\,i+m+1,\,i+m+2}}.$$

The form of the polynomial at the completion of the middle branch is:

$$_{s_1+s_2-i+1}\Delta(t_1, t_2, t_3) = {}_{s_1+s_2-i}\Delta(t_1, t_2^{n_{2,\,s_2}}, t_3)\cdot Q(w_{2,\,s_2}, n_{2,\,s_2}, t_2, v_{s_2-i-m}),$$

where

$$v_{s_2-i-m} = (v_{s_2-i-m-1})^{n_{2,\,s_2-1}} = v_1^{b_{2,\,i+1,\,s_2-1}}.$$

We now do iterations of types $(w_{3,\,i+m+j}, n_{3,\,i+m+j})$, $2 \leqq j \leqq s_3 - i - m$, on the outer branch to complete the link. The corresponding development of the Alexander polynomial is:

$$_{s_1+s_2-i+2}\Delta(t_1, t_2, t_3) = {}_{s_1+s_2-i+1}\Delta(t_1, t_2, t_3^{n_{3,\,i+m+2}})\, Q(w_{3,\,i+m+2}, n_{3,\,i+m+2}, t_3, u_1)$$

where

$$u_1 = t_2^{\langle L_2^{s_2},\, L_3^{i+m+1}\rangle}\, t_1^{\langle L_1^{s_1},\, L_3^{i+m+1}\rangle} = t_2^{\langle L_2^{s_2-1},\, L_3^{i+m+1}\rangle^{n_{2,\,s_2}}}\, t_1^{\langle L_1^{s_1},\, L_2^{i+m}\rangle^{n_{3,\,i+m+1}}}$$
$$= t_2^{w_{3,\,i+m+1}\, b_{2,\,i+m+1,\,s_2}}\, t_1^{w_{2,\,i+1}\, b_{1,\,i+1,\,s_1}\, b_{2,\,i+2,\,i+m}\, n_{3,\,i+m+1}}$$
$$= t_2^{w_{3,\,i+m+1}\, b_{2,\,i+m+1,\,s_2}}\, t_1^{w_{2,\,i+1}\, b_{1,\,i+1,\,s_1}\, b_{3,\,i+2,\,i+m+1}}.$$

Note. We found u_1 by referring back to v_{s_2-i-m} and v_1.

Similarly,

$$_{s_1+s_2-i+3}\Delta(t_1, t_2, t_3) = {}_{s_1+s_2-i+2}\Delta(t_1, t_2, t_3^{n_{3,\,i+m+3}})\, Q(w_{3,\,i+m+3}, n_{3,\,i+m+3}, t_3, u_2),$$

where

$$u_2 = (u_1)^{n_{3,\,i+m+2}}.$$

After the final iteration, the polynomial is:

$$_{s_1+s_2+s_3-2i-m}\Delta(t_1, t_2, t_3) = {}_{s_1+s_2+s_3-2i-m-1}\Delta(t_1, t_2, t_3^{n_{3,\,s_3}})$$
$$\cdot Q(w_{3,\,s_3}, n_{3,\,s_3}, t_3, u_{s_3-i-m-1})$$

where

$$u_{s_3-i-m-1} = (u_{s_3-i-m-2})^{n_{3,\,s_3-1}} = u_1^{b_{3,\,i+m+2,\,s_3-1}}.$$

Summarizing the above recursive information, the final polynomial is:

$$\begin{aligned}
\Delta(t_1, t_2, t_3) = & \prod_{j=1}^{i} P(w_{ij}, n_{ij}, t_1^{b_{1,j+1,s_1}} \cdot t_2^{b_{2,j+1,s_2}} \cdot t_3^{b_{3,j+1,s_3}}) \\
& \cdot Q(n_{2,i+1}, w_{2,i+1}, t_2^{b_{2,i+2,s_2}}, t_3^{b_{3,i+2,s_3}}, t_1^{b_{1,i+1,s_1}}) \\
& \cdot \prod_{j=i+1}^{s_1} Q(w_{ij}, n_{ij}, t_1^{b_{1,j+1,s_1}}, (t_2^{b_{2,i+2,s_2}} t_3^{b_{3,i+2,s_3}})^{w_{2,i+1} b_{1,i+1,j-1}}) \\
& \cdot \prod_{j=i+2}^{i+m} Q(w_{2j}, n_{2j}, t_2^{b_{2,j+1,s_2}} t_3^{b_{3,j+1,s_3}}, t_1^{w_{2,i+1} b_{1,i+1,s_1} b_{2,i+2,j-1}}) \\
& \cdot [(t_1^{w_{2,i+1} b_{1,i+1,s_1} b_{2,i+2,i+m}} t_3^{w_{3,i+m+1} b_{3,i+m+2,s_3}})^{n_{3,i+m+1}} \\
& \cdot t_2^{w_{3,i+m+1} b_{2,i+m+1,s_2}} - 1] \\
& \cdot \prod_{j=i+m+1}^{s_2} Q(w_{2j}, n_{2j}, t_2^{b_{2,j+1,s_2}}, t_1^{w_{2,i+1} b_{1,i+1,s_1} b_{2,i+2,j-1}} \\
& \cdot t_3^{w_{3,i+m+1} b_{3,i+m+2,s_3} b_{2,i+m+1,j-1}}) \\
& \cdot \prod_{j=i+m+2}^{s_3} Q(w_{3j}, n_{3j}, t_3^{b_{3,j+1,s_3}}, t_1^{w_{2,i+1} b_{1,i+1,s_1} b_{3,i+2,j-1}} \\
& \cdot t_2^{w_{3,i+m+1} b_{2,i+m+1,s_2} b_{3,i+m+2,j-1}}).
\end{aligned}$$

In a similar manner, it is possible to write out the general form for an algebraic link with any number of branches and to give the specific polynomial for a given link. The algorithmic calculation actually seems as easy to follow through in the general case as for a specific link. Given the length and tedious nature of the calculations for a large number of branches, it clearly is a good problem to assign to computer calculation.

In summary, the algorithmic calculation for the Alexander polynomial for an algebraic link of three or more components is as follows: (the branches are numbered in order from innermost to outer, with correspondingly subscripted variable t_j for the j-th branch).

For each iteration, $2 \leqq j \leqq i$, up to the first branching (at stage $i+1$), the previous polynomial is evaluated at $t_1^{n_{ij}}$ and multiplied by $P(w_{ij}, n_{ij}, t_1)$. At the branching the previous polynomial is evaluated at $t_1 t_2^{n_{2,i+1}}$ and multiplied by

$$Q(n_{2,i+1}, w_{2,i+1}, t_2, t_1).$$

From then on, an iteration on the k-th branch by the torus knot of type (w_{kj}, n_{kj}) gives the previous polynomial evaluated at $t_k^{n_{kj}}$ and multiplied by

$$Q(w_{k,j}, n_{k,j}, t_k, \prod_{l \neq k} t^{\langle L_l^j, L_k^{j-1} \rangle})$$

where L_p^q denotes the p-th branch after the q-th iteration. The only exception is at a branching, where the previous polynomial is evaluated at $t_{k-1} t_k^{n_{k,j}}$ and multiplied by $B(w_{k,j}, n_{k,j}, t_k \prod_{l<k-1} t_l^{\langle L_l^j, L_{k-1}^j \rangle}, t_{k-1})$ where the link changes from $k-1$ branches to k branches.

Remark. The above development assumes for simplicity of notation that the branching always occurs on the outer component. In other cases, the development of the Alexander polynomial would be analogous.

To simplify the final notation and to emphasize the structure of the polynomial as related to the geometric development of the link, we define

$$e_{q,j}=\begin{cases}\sum_{k=1}^{3} b_{k,j,s_k} & q=0\\ w_{2,i+1}\,b_{1,i+1,j-1}\left(\sum_{k=2}^{3} b_{k,i+2,s_k}\right)+w_{1,j}\,b_{1,j+1,s_1} & q=1\\ w_{2,i+1}\,b_{2,i+2,j-1}\,b_{1,i+1,s_1}+w_{2,j}\left(\sum_{k=2}^{3} b_{k,j+1,s_k}\right) & q=2\\ b_{2,i+m+1,j-1}(w_{3,i+m+1}\,b_{3,i+m+2,s_3}+w_{2,i+1}\,b_{2,i+2,i+m}\,b_{1,i+1,s_1})\\ \quad +w_{2,j}\,b_{2,j+1,s_2} & q=3\\ b_{3,i+m+2,j-1}(w_{3,i+m+1}\,b_{2,i+m+1,s_2}+w_{2,i+1}\,b_{3,i+2,i+m+1}\,b_{1,i+1,s_1}\\ \quad +w_{3,j}\,b_{3,j+1,s_3} & q=4.\end{cases}$$

The values of q are related to the geometric structure of the link as follows: (0) the exponents involving $q=0$ are derived from the iterated torus knot up to the first branching, (1) for $q=1$, the completion of the inner component, (2) for $q=2$, the development of the middle branch up to the second branching, (3) for $q=3$, the completion of the middle component, and (4) for $q=4$, the completion of the outer branch.

Theorem 7.8. The characteristic polynomial of an algebraic link with three components (as defined above) is:

$$(t-1)\prod_{j=1}^{i}\left[\frac{t^{n_{1,j}e_{0,j}}-1}{t^{w_{1,j}e_{0,j+1}}-1}\cdot\frac{t^{e_{0,j+1}}-1}{t^{e_{0,j}}-1}\right]\cdot\left(\frac{t^{w_{2,i+1}e_{0,i+1}}-1}{t^{e_{0,i+1}}-1}\right)\cdot\prod_{j=i+1}^{s_1}\left[\frac{t^{n_{1,j}e_{1,j}}-1}{t^{e_{1,j}}-1}\right]$$
$$\cdot\prod_{j=i+2}^{i+m}\left[\frac{t^{n_{2,j}e_{2,j}}-1}{t^{e_{2,j}}-1}\right]\cdot(t^{w_{3,i+m+1}\left(\sum_{k=2}^{3} b_{k,i+m+1,s_k}\right)+w_{2,i+1}\,b_{1,i+1,s_1}\,b_{3,i+2,i+m+1}}-1)$$
$$\cdot\prod_{j=i+m+1}^{s_2}\left[\frac{t^{n_{2,j}e_{3,j}}-1}{t^{e_{3,j}}-1}\right]\cdot\prod_{j=i+m+2}^{s_3}\left[\frac{t^{n_{3,j}e_{4,j}}-1}{t^{e_{4,j}}-1}\right].$$

Using the polynomial, it is possible to give examples of links with 3 components having either finite or infinite monodromy. Perhaps the most interesting type of example is the following. We start with an algebraic link of two components which has infinite monodromy and by adding a third component produce an algebraic link which has finite monodromy. The sets of characteristic Puiseux pairs are {(11,5), (223,20)}, {(17, 13), (331,19)}, and {(31, 30), (1277, 41)}. Since $11/5>17/13>31/30$, the branching is immediate and the branches are inner, middle, and outer respectively. Considering first the algebraic link determined by the inner and middle branches, the key exponent values are

$$e_{0,1}=n_{11}\,n_{12}+n_{21}\,n_{22}=347,$$
$$e_{1,1}=n_{22}\,m_{21}+n_{12}\,m_{11}=553=7\times 79,$$
$$e_{1,2}=n_{11}\,n_{22}\,m_{21}+w_{12}=2718=3^2\times 302,$$
$$e_{2,2}=w_{22}+n_{11}\,n_{12}\,m_{21}=5907=3\times 11\times 179.$$

The exponents are given in prime factored form to simplify recognition of the various roots of the polynomial. The characteristic polynomial is

$$\frac{t^{17\times 347}-1}{t^{347}-1}\cdot\frac{t^{5\times 7\times 79}-1}{t^{7\times 79}-1}\cdot\frac{t^{5\times 2^2\times 3^2\times 302}-1}{t^{3^2\times 302}-1}\cdot\frac{t^{19\times 3\times 11\times 179}-1}{t^{3\times 11\times 179}-1},$$

which clearly has 5th roots in the second and third factors and hence by Theorem 6.3 the monodromy is infinite. Alternately, one could simply note that the monodromy is infinite by applying Theorem 7.5 (i) to $C=\{w_{21}, n_{22}, n_{11}, n_{12}\}=\{17, 19, 5, 20\}$.

Considering the algebraic link formed by all three branches, we find a link with finite monodromy since the characteristic polynomial (and thus the minimal polynomial) has distinct roots – see Lemma 7.3.

Note. In this example $m=0$, while in the development we required $m>0$. The only change is in the subscripting. The iterations on the 2nd branch must begin with the torus knot of type $(w_{2,l}, n_{2,l})$ where l is $\max\{i+2, i+m+1\}$ since the knot of type $(w_{2,i+1}, n_{2,i+1})$ is used for branching. We have $l=i+m+1$ except when $m=0$. When $m=0$, the derived polynomial is modified by replacing $i+m+1$ by $i+2$ in any $b_{k,i+m+1,j}$, for $2\leqq k\leqq 3$.

$$\begin{aligned}
&e_{0,1}=n_{11}n_{12}+n_{21}n_{22}+n_{31}n_{32}=1577=19\cdot 83,\\
&e_{1,1}=(n_{22}+n_{32})m_{21}+n_{12}m_{11}=1240=5\cdot 2^3\cdot 31,\\
&e_{1,2}=n_{11}(n_{22}+n_{32})m_{21}+w_{12}=6203\ \text{(prime)},\\
&(n_{22}+n_{31}n_{32})m_{31}+n_{11}n_{12}m_{21}=40419=3^4\cdot 499,\\
&e_{3,2}=w_{22}+(n_{32}m_{31}+n_{11}n_{12}m_{21})=2\cdot 37\cdot 97,\\
&e_{4,2}=w_{32}+n_{22}m_{31}+n_{11}n_{12}m_{21}=3\cdot 5^2\cdot 7^2\cdot 11.
\end{aligned}$$

The characteristic polynomial is

$$\left(\frac{t^{17\times 19\times 83}-1}{t^{19\times 83}-1}\right)\cdot\left(\frac{t^{5^2\times 2^3\times 31}-1}{t^{5\times 2^3\times 31}-1}\,\frac{t^{5\times 2^2\times 6203}-1}{t^{6203}-1}\right)\cdot(t^{3^4\cdot 499}-1)$$
$$\cdot\left(\frac{t^{19\cdot 2\cdot 37\cdot 97}-1}{t^{2\cdot 37\cdot 97}-1}\right)\cdot\left(\frac{t^{41\cdot 3\cdot 5^2\cdot 7^2\cdot 11}-1}{t^{3\cdot 5^2\cdot 7^2\cdot 11}-1}\right),$$

which clearly has distinct roots.

Note. In comparing the above polynomial to the general polynomial, please observe that the general polynomial has seven major parts, only five of which appear above. This is due to the immediate branching ($i=m=0$), which causes the first and fourth products in the general form to not appear.

In summary, it is now possible, given the Puiseux development, to write out the characteristic polynomial for any specific algebraic link and for any general type of algebraic link (any fixed number and pattern of branchings). This can be used to study monodromy properties of specific links, or to develop criteria for finite or infinite monodromy for classes of links. It should also be useful in answering other questions about algebraic links. The algorithm is clear, but the tedium becomes quickly overwhelming with a large number of components and/or iterations, and is best consigned to an electronic computer.

Questions

1. We have that $\lambda(t)=f(t)\,\lambda(t,\ldots,t)$. What is $f(t)$ in the case of algebraic links?

2. Does the Alexander Polynomial of the universal abelian cover classify the isotopy type or the cobordism type of an algebraic (or iterated torus) link? Burau [4, 5] proved that for algebraic links of one or two branches, the Alexander Polynomial determines the isotopytype of the link, and Lê [10] proved that for an irreducible plane curve, the Alexander Polynomial determines the cobordism type of the knot.

3. Do complements of arbitrary iterated torus knots or links fiber over S^1?[1]

References

1. A'Campo, N.: Sur la monodromie des singularités isolées d'hypersurfaces complexes. Inventiones math. **20**, 147–169 (1973)
2. Blanchfield, R.C.: Intersection theory of manifolds with operators with applications to knot theory. Ann. Math. **65**, 340–356, (1957)
3. Brauner, K.: Zur Geometrie der Funktionen zweier komplexer Veränderlichen. Abh. Math. Sem. Hamburg **6**, 1–54 (1928)
4. Burau, W.: Kennzeichnung der Schlauchknoten. Abh. Math. Sem. Hamburg **9**, 125–133 (1932)
5. Burau, W.: Kennzeichnung der Schlauchverkettungen. Abh. Math. Sem. Hamburg **10**, 285–297 (1934)
6. Durfee, A.H.: The Characteristic Polynomial of the Monodromy. Pac. J. Math. **59**, 21–26 (1975)
7. Fitting, H.: Die Determinantenideale eines Moduls. Jahresbericht d. Deutschen Mathematiker Vereinigung **46**, 195–228 (1936)
8. Fox, R.H.: A quick trip through knot theory. Topology of 3-manifolds and related topics, [pp. 120–167.] Englewood Cliffs, N.J.: Prentice-Hall 1961
9. Kahler, K.: Über die Verzweigung einer algebraischen Funktion Zweier Veränderlichen in der Umgebung einer singularen Stelle. Math. Z. **30**, 188–204 (1929)
10. Lê, Dũng Tráng: Sur les noeuds algebriques. Comp. Math. **25**, 281–321 (1972)
11. Levine, J.: A method for generating link polynomials. Am. J. Math. **89**, 69–84 (1967)
12. Massey, W.S.: Algebraic topology: An introduction. New York, N.Y.: Harcourt Brace & World 1967
13. Milnor, J.: Singular points of complex hypersurfaces. Ann. of Math. Study 61, Princeton Univ. Press (1968)
14. Papakyriakopolous, C.D.: On Dehn's lemma and the asphericity of knots. Ann. of Math. **66** (2) 1–26 (1957)
15. Reeve, J.E.: A summary of results in the topological classification of plane algebroid singularities. Rendiconti del Seminario Matematico (Torino) **14°**, 159–187 (1954–55)
16. Shinohara, Y.: Higher dimensional knots in tubes. Trans. A.M.S. **161**, 35–49 (1971)
17. Shinohara, Y., Sumners, D.W.: Homology invariants of cyclic coverings with applications to links. Trans. A.M.S. **163**, 101–121 (1972)
18. Seifert, H.: On the homology invariants of knots. Quart. J. Math. Oxford Ser. **1** (2), 23–32 (1950)
19. Sumners, D.W., Woods, J.M.: A short proof of finite monodromy for analytically irreducible plane curves. Comp. Math. **28**, 213–216 (1974)
20. Woods, J.M.: Some criteria for finite and infinite monodromy of plane algebraic curves. Inventiones math. **26**, 179–185 (1974)
21. Zassenhaus, H.: The theory of groups. New York: Chelsea 1958

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[1] **Added in Proof.** The first author recently answered this question in the affirmative