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On Dihedral Covering Spaces of Knots

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With an appropriate choice of notation, the combinatorial methods of [13] may be conveniently applied to the irregular coverings studied by Reidemeister and Bankwitz [16; 1], revealing simple formulas for, and new relationships among, linking numbers between branch curves[1].

To each homomorphism of the group of a tame knot onto an abstract group $\mathscr{G}$ of finite order o there corresponds a partially ordered set of covering spaces of S^3, branched along the knot. Each distinct i-sheeted element may be associated biuniquely with a class of conjugate subgroups of index i in $\mathscr{G}$ and constructed explicitly by following the homomorphism with the associated coset representation of $\mathscr{G}$ onto a transitive permutation group of degree i. Cf. [13; 3, Ch. XII]. Where one such class is "contained in" another (in the sense of inclusion for some choice of particular subgroups) the associated covering covers the other by a covering map which may be thought of as the identification of corresponding points in various copies of S^3. The set of such coverings and covering maps forms a commutative diagram. Cf. [14, p. 49].

Since the o-sheeted covering which covers all others is regular, its covering translations restrict the linking numbers between pairs of its branch curves; for if one such pair may be moved to another, the linking numbers between them are obviously equal. Linking numbers in this *maximal* covering may then be trickled down to the lesser-sheeted ones by a simple transfer argument, formulated in the following generality by the late R. H. Fox.

Let k be a link in a closed 3-manifold M with components $k_1, \ldots, k_n$ and let $f\colon \overline{M} \to M$ be a covering of finite degree branched over k. Let $\bar{k}_{i1}, \ldots, \bar{k}_{im(i)}$ be the components of $\bar{k}_i = f^{-1}(k_i)$ in $\overline{M}$. Let c_{ia} be the branching index of f at $\bar{k}_{ia}$ and let d_{ia} be the degree of the mapping $f|\bar{k}_{ia}$. The links k_i and $\bar{k}_i$ having been so oriented that $d_{ia} > 0$, let k_i and $\bar{k}_{ia}$ denote also the 1-cycles carried by these oriented knots. If k_i is a torsion cycle, then there is a 2-chain C_i in M whose boundary ∂C_i is, say, $t_i k_i$. If k_j is also a torsion cycle, then the linking number $L(k_i, k_j)$ is defined and equal to $(1/t_i) S(C_i, k_j)$ where S denotes the intersection number. The *transfer* $\bar{C}_i$ of C_i is a 2-chain in $\overline{M}$ that is easily seen to have the

[1] Cf. [8, § 5]. These (and other) noncyclic invariants may be used to resolve the amphicheirality, primeness and bridge number problems for the 10-crossing knots tabled in [12]

following two properties:

$$\partial \bar{C}_i = t_i \sum_{h=1}^{m(i)} c_{ih} k_{ih},$$

$$S(\bar{C}_i, k_{jk}) = d_{jk} S(C_i, k_j).$$

From this it follows that

$$d_{jk} L(k_i, k_j) = \sum_{h=1}^{m(i)} c_{ih} L(k_{ih}, k_{jk})$$

provided, of course, that the k_{ih} and k_{jk} are all torsion cycles. Thus the linking numbers $L(k_i, k_j)$ are determined by the linking numbers $L(k_{ih}, k_{jk})$ whenever the latter are all defined.

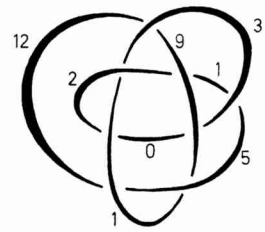
For example, the *tetrahedral* coverings obtained from a homomorphism of a knot group onto A_4 [18, p. 609] give rise to just one linking number (where defined) between pairs of branch curves in the maximal covering, multiples of which appear in the lesser-sheeted ones. The same, rather uninteresting situation arises from homomorphisms onto the metacyclic group Γ_5 considered in [8]. This paper examines the case where $\mathscr{G}$ is the dihedral group of order $2p$.

Let D_p be the group of symmetries of a regular, p-sided polygon for odd p. A knot group admits a homomorphism onto D_p if and only if the integers mod p are a quotient group of the first homology group of the knot's 2-fold branched cyclic covering [15, p. 715]. Such a homomorphism may be thought of as a nontrivial assignment of an integer to each segment of a knot diagram such that $x-2y+z \equiv 0 \bmod p$ at each crossing, where y is the integer assigned to the overpassing segment. Cf. [7, p. 133]. Here, of course, each integer i represents the group element ab^{2i} in D_p presented as $\{a, b\colon a^2, b^p, abab\}$.

The corresponding p-sheeted irregular covering M_p may be obtained from the representation of D_p into S_p which sends a to $0\ (1\,p-1)(2\,p-2)(3\,p-3)\ldots((p-1)/2(p+1)/2)$ and b to $(0\ 1\ 2\ 3 \ldots p-1)$. Here it is helpful to visualize the permutation assigned to each segment labelled i as a p-gon with vertices cyclically numbered from 0 to $p-1$ and pairwise connected by parallel lines, except for the vertex numbered i. It then turns out that the $(p+1)/2$ branch curves of M_p may be consistently identified as k_j^p, $j=0, 1, 2, \ldots, (p-1)/2$, where each is the one approachable from the $(i \pm j)$-th copies of S^3 along any segment labeled i.

The following example sets forth M_{15} of knot 8_{21} and the linking numbers v_{ij}^{15} between branch curves k_i^{15} and k_j^{15}.

$i=1$		0						
2		4	1					
3		2	2	−2				
4		0	−1	1	2			
5		−4	−1	3	4	−1		
6		−2	0	2	0	0	0	
7		4	1	−3	−2	1	3	2
j	=	0	1	2	3	4	5	6



Note that such linking numbers should be interpreted as $\pm$ and $\mp$ the indicated values, depending on the orientation of S^3 [16; 8, § 5], and that our preferred ordering of branch curves is not invariant under inner automorphisms of D_p.

Hilden and Montesinos have independently shown that every closed, orientable 3-manifold is some such M_3 and Cappell and Shaneson have obtained a formula for the Hirzebruch-Rochlin invariant by studying such manifolds as M_p [4].

If, say, some $n \mid p$, then the unique M_n which arises from a given homomorphism onto $D_p \supset D_n$ may be obtained from M_p by reducing the symbols i, and letters of the corresponding permutations representing ab^{2i}, to their equivalence classes mod n. Cf. [2].

The maximal ($2p$-sheeted) dihedral covering M_{2p} may be conveniently visualized as a 2-fold branched cyclic covering of $M_p - k_0^p$ by using the representation of D_p into S_{2p} which sends a to $((3p+1)/2(3p-1)/2) \dots (2p-1\, p+1)(0p)(1\, p-1) \dots ((p-1)/2(p+1)/2)$ and b to $(0\,2\,4 \dots 2p-2)(1\,3\,5 \dots 2p-1)$. Identification of letters mod p then yields an M'_p which may be obtained from the M_p described above by the inner automorphism of $D_p \subset S_p$ which replaces each letter x with the letter $2x$ mod p. Meridian permutations for M_{2p} may be thought of as parallel lines connecting pairs of vertices of a $2p$-gon at various angles.

Alternatively, we may construct an M'_{2p} by representing D_p into S_{2p} by sending a to $\dots (2p-1\, p-1)(0p)(1\, p+1)(2p+2) \dots$ and b to $(2p-2 \dots 4\,2\,0)(1\,3\,5 \dots 2p-1)$. Cf. [10, § 4]. Although M'_{2p} is just M_{2p} with copy numbers of S^3 relabeled by the inner automorphism of $D_p \subset S_{2p}$ which switches even numbered letters (except 0) according to the permutation $(2\,2p-2)(4\,2p-4) \dots (p-1\, p+1)$, it has the advantage of rendering easily perceivable the effect of covering translations upon branch curves. The meridian permutations of M'_{2p} consist of sets of chords of a $2p$-gon lying at various distances from its center, and their rotational symmetries reflect covering translations which may be used to send any one branch curve to any other.

Using the foregoing notation to keep track of branch curves, it is easy to see how linking numbers in any irregular dihedral covering are determined by those in some maximally-sheeted one and (surprisingly) how the linking number between any pair of branch curves in such a covering is determined by those between the branch curve of index 1 and the rest.

Let v_{ij}^p and v_{ij}^n be the linking numbers between branch curves of M_p and M_n ordered consistently as above.

Theorem 1. *For any* $h \equiv \pm i \bmod n$, $v_{i0}^n = -v_{h0}^p + 2\sum v_{hk}^p$ *over all* $k \equiv 0 \bmod n$ *and, for* $j \neq 0$, $v_{ij}^n = \sum v_{hk}^p$ *over all* $k \equiv \pm j \bmod n$.

Proof is by examination of the (p/n)-sheeted covering of M_n by M_p obtained by mapping each point in any i-th copy of S^3 (for M_p) to the same point in the j-th copy of S^3 (for M_n) for all $i \equiv j \bmod n$. Note that k_0^n is covered by $k_0^p + 2\sum k_k^p$ over all $0 \neq k \equiv 0 \bmod n$, while any other k_j^n is covered by $\sum k_k^p$ over all $k \equiv \pm j \bmod n$. Clearly v_{ij}^n equals any *one* covering curve of k_i^n linked with *all* covering curves of k_j^n by the transfer argument set forth above.

Theorem 2. *For any knot, any p, and any* $i>j>0$, $2v_{ij}^p=v_{h0}^p+v_{k0}^p$ *where* $h=i-j$ *and k is the smallest value of* $|\pm(i+j) \bmod p|$.

Proof is by examination of the covering translations of M'_{2p} and the covering map from M_{2p} to M'_p, tracing branch curves through from M_p. Recall that in M_p we have named the branch curves $k_0^p, k_1^p, k_2^p, \ldots, k_{(p-1)/2}^p$ in accordance with their "distance" from the letter which remains fixed by a meridian permutation. The relabeling of letters which yields M'_p shuffles these curves to the following order, as one moves away from the letter which remains fixed: k_0^p, $k_{(p-1)/2}^p$, k_1^p, $k_{(p-3)/2}^p$, k_2^p, etc. In M_{2p}, k_0^p is covered by a curve k_0^{2p} and each other k_i^p is covered by two curves which we may call k_i^{2p} and k_{-i}^{2p} in some consistent fashion so that all negative subscripts represent cyclically adjacent chords of the $2p$-gons which represent meridian permutations of M'_{2p}. It is not *too* hard to see that these chords are then distributed cyclically around these $2p$-gons as follows: $\ldots, k_{-2}^{2p}, k_{-1}^{2p}, k_0^{2p}, k_1^{2p}, k_2^{2p}, \ldots$ The rotation which moves each vertex numbered i to that numbered $i+2x \bmod 2p$ evidences a covering translation which moves each k_i^{2p} to k_{i+x}^{2p} (in M'_{2p}). From this it is clear that $v_{0i}^{2p}=v_{0-i}^{2p}$ and $v_{ij}^{2p}+v_{i-j}^{2p}=v_{0(i-j)}^{2p}+v_{0(i+j)}^{2p}$ where $i>j$ and by the subscript $(i+j)$ we mean the shortest distance from k_i^{2p} to k_{-j}^{2p} which, for $i+j>(p-1)/2$, is just $|-(i+j) \bmod p|$. By examination of the covering of M'_p by M_{2p} it is also clear that $v_{i0}^p=2v_{i0}^{2p}$ and $v_{ij}^p=v_{ij}^{2p}+v_{i-j}^{2p}$. This proves the theorem.

Corollary. $\sum v_{ij}^p=\dfrac{p+1}{4}\sum v_{i0}^p$.

We call this latter sum v_p and observe that it is a handy invariant of the placement of the branch link in oriented M_p.

Theorem 3. *If any one* v_{ij}^p *is defined, then they all are.*

Let, say, $j\neq 0$. Then k_j^p (perhaps weakly) bounds some 2-chain C which lifts to C' bounding $k_j^{2p}+k_{-j}^{2p}$. By covering translations, C' may be translated to other 2-chains which bound $k_{x+j}^{2p}+k_{x-j}^{2p}$ for any x. Let $x=0, 2j, 4j, \ldots, (2p-2)j$ and sum up the resulting 2-chains with coefficients alternatingly multiplied by ± 1 to get a 2-chain which bounds $2k_j^{2p}$. Now translate this to one bounding $2k_0^{2p}$ and project it to one bounding $2k_0^p$ in M_p; for any $i\neq 0$, translate to $2k_i^{2p}$ or $2k_{-i}^{2p}$ and project to bound $2k_i^p$.

Schubert (and Seifert) examined the class of 2-bridged knots (previously studied under the name of "Viergeflechte" [1]) by reference to the lens spaces which constitute their 2-fold branched cyclic coverings [19, p. 146]. Conway has subsequently introduced a nice notation for a regular knot diagram of such a knot, from which the parameters of those lens spaces may be easily obtained as the numerator and denominator of a continued fraction [6, pp. 332, 336]. We show, by direct calculation, how the dihedral linking numbers of these knots may be easily reckoned from their normal forms.

Recall that a 2-bridged knot has a unique $\Delta(-1)$ fold dihedral covering, where $\Delta(t)$ is its Alexander polynomial. The fact that each $v_{i0}^{\Delta(-1)} = \pm 2$ was known to Bankwitz and Schumann [16; 1, § 5], but they seem not to have proven the general case. Their related result that each $v_{ij}^{\Delta(-1)} = \pm 2$ or 0 now follows from Theorems 4 and 2.

Theorem 4. *For a 2-bridged knot with normal form* (α, β) *and* $\beta > 0$, *each* $v_{i0}^{\Delta(-1)} = 2(-1)^j$ *where* $j\alpha < i\beta < (j+1)\alpha$.

We work from a prototypical knot diagram associated with a given (non-unique) normal form [19, Figs. 3–5]. Assigning the integers 0 and 1 to the two overpassing segments we may construct a 2-chain bounding $k_0^{\Delta(-1)}$ by assigning the coefficient 1 to the 2-cells lying beneath these segments in copies 0 and 1, respectively. As discussed in [13, § 2] these will then work themselves out to bound the rest of $k_0^{\Delta(-1)}$ and, in this case, the way in which they do so may be ignored, as there are no intersections under other segments. Now follow along the knot, starting at an overpassing segment, and see what happens. Starting at, say, the segment to which 1 is assigned, we pass alternatingly under the two overpassing segments 0, 1, 0, 1, etc. As we pass thereunder, the integers assigned to the segments of the diagram which we thus encounter change from 1 to -1 to 3 to -3 to 5 to -5, etc., until, after $p-1$ underpasses, we reach the segment to which 0 is assigned (after which, everything behaves in an entirely symmetric fashion for the "second half" of the knot). Accordingly, a representative curve for $k_i^{\Delta(-1)}$ lies in copies $1-i$, $i-1$, $3-i$, $i-3$, $5-i$, $i-5$ (mod p), etc., and clearly intersects the 2-chain bounding $k_0^{\Delta(-1)}$ only (during this first half-circuit) at the i-th crossing thus successively encountered. Clearly the sign of this intersection is $(-1)^j$ where $j\alpha < i\beta < (j+1)\alpha$, for the signs of the underpasses are (from an inspection of the prototypical diagram) reversed each time that a multiple of β surpasses a multiple of α. This completes the proof.

Observe that the algebraic sum of the signs of a projection in normal form is $2v_{\Delta(-1)}$. We understand (from K. Murasugi) that it has been proven that half this sum is equal to the signature [11] of the knot. So we have at least the following.

Conjecture. *The signature of a 2-bridged knot* $= v_{\Delta(-1)}$.

Compare Theorem 4 and the conjecture with Recipe 7.10 of [9, p. 51] and Theorem 5 of [5, p. 170].

From Theorems 4 and 1 we obtain:

Corollary. *For a 2-bridged knot, any* $v_{i0}^p \equiv 2 \bmod 4$.

For any knot, it is known that the numerator of each v_{i0}^p is even, where defined [17]. Cappell and Shaneson have conjectured, and proven for $p=3$ among other values, that each is $\equiv 2 \bmod 4$ for M_p a Z_2-homology sphere [4]. Counterexamples to the converse are furnished by knots 10_{82}, 10_{128} and 10_{152} of [12].

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