Werk

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using homology surgery theorem [CS2] that even when $H \subset G$ is not a square-root closed subgroup, if $\Phi(\tau(f))=0$, $n=2k>4$, there is a "homology h-cobordism" of W to a split "homology-equivalent manifold", where homology is taken using local coefficients in $Z[\frac{1}{2}][\pi_1(Y)]$. This can be directly applied to compute the Wall groups of the ring $R[G_1 *_H G_2]$, $Z[\frac{1}{2}] \subset R \subset Q$, even when H is not square-root closed in G_1 and G_2.

3) If $n \geqq 6$ and $H=G_1$, even if H is not square-root closed in G, and in fact even if $H \to G_2$ is not injective, any homotopy equivalence $f\colon W \to Y$ is splittable. For f a simple homotopy equivalence this is Theorem 12.1 of [W2]. Essentially the same proof, but using the surgery theory associated to the homotopy equivalence problem in place of that for simple homotopy equivalences, shows that W is h-cobordant to a split manifold. But if $H=G_1$, then $G=G_1 *_H G_2=G_2$ and in particular, $\mathrm{Wh}(G_2) \to \mathrm{Wh}(G)$ is surjective. From this, the h-cobordism to a split homotopy equivalence is easily replaced by an s-cobordism to a split homotopy equivalence.

Chapter III: The Even-Dimension Case

§ 1. Splitting in a Covering Space

The present chapter completes the proofs of Theorems 1 and 2 for $n=2k-1$, that is for dimension Y even. The results of Chapters I and II are heavily used below.

In many cases, for example when $\tilde{K}_0(H)$ is zero, the results of the present chapter could be derived quite easily from those of Chapter II. However, in the general case considered here considerable effort is needed, when $n=2k-1$, to establish the relationship between the projective modules arising in the proof and the initial data on the Whitehead torsion of f. This is accomplished in Lemma III.4.

We briefly outline the argument of the present chapter. In § 1, we consider the problem of splitting the map of covering spaces $\hat{f}\colon \hat{W} \to \hat{Y}$ along $X \subset \hat{Y}$. Lemma III.2 shows that the only obstruction is an element of $\tilde{K}_0(H)$. In § 2, this obstruction is shown to depend only on $\tau(f)$. Using the results of § 1, we carefully construct a submanifold $V \subset W \times S^1$, with V a transverse inverse image of $X \times S^1 \subset Y \times S^1$ and with $V \to X \times S^1$ a k connected map. This is the situation which was studied in Lemma II.1 and we perform the nilpotent normal cobordism construction of Chapter II. The main result of § 2, Lemma III.6 shows that when $\Phi(\tau(f))=0$ this produces a nilpotent normal cobordism of $W \times S^1$ to a manifold U, with $\mathring{W} \subset U$ and $\mathring{W} \to Y$ a split homotopy equivalence. In § 3 results of Chapter II are used to replace this normal cobordism by an s-cobordism of $W \times S^1$ to U', with $\mathring{W} \subset U'$. Hence $\mathring{W} \subset W \times S^1$ and we complete the proof of the h-splitting theorem by showing that this implies W and $\mathring{W}$ are h-cobordant. The s-splitting result for $n=2k-1$ is derived from the h-splitting result.

The assumption that H is square-root closed in G is not used in § 1 and § 2. It is used in § 3 in replacing a nilpotent normal cobordism by an s-cobordism. In particular, even if H is not spare-root closed in G the results of § 2 show that if

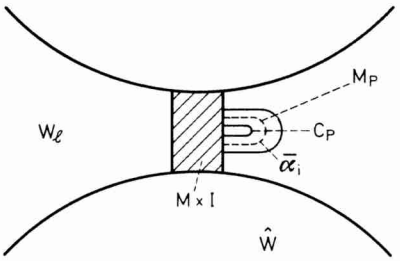
$\bar{\Phi}(\tau(f))=0$, (W,f) is normally cobordant to a split homotopy equivalence, $n=2k-1$. For $n=2k$ this was proved in Chapter II.

To begin the argument, note that from Lemma I.1, when $n=2k-1$ we may assume that the homotopy equivalence $f\colon W\to Y$ is transverse to $X\subset Y$ with, setting as usual $M=f^{-1}(X)$, $f!\colon M\to X$ inducing isomorphisms of fundamental groups and $K_i(M)=0$, $i\neq k-1, k$. Unfortunatelly, unlike the situation studied in Chapter II, $K_{k-1}(M)$ and $K_k(M)$ will not, in general, be free or even projective modules[15] over $Z[H]$.

Recall the decomposition defined in I §4, $K_{k-1}(M)=P\oplus Q$ where $P\cong K_k(W_r, M)$ and $Q\cong K_{k-1}(W_l, M)$ are by Lemma I.6 finitely generated $Z[H]$ modules. Let $\alpha_1,\ldots,\alpha_r$ be a $Z[H]$ module generating set for P and let $\beta_1,\ldots,\beta_s$ be a $Z[H]$ module generating set for Q. These elements are represented by disjoint embedded $(k-1)$ dimensional spheres in M^{2k-1}. In the middle-dimension under consideration here, we cannot apply Lemma I.3 to perform handle exchange on $M\subset W$. However, as we shall see below handle-exchanges, on discs with boundaries the spheres $\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s$, can be performed on $M\subset\hat{W}$.

Let $M\times I$, $I=[-1,1]$ be a neighborhood of $M\times 0=M$ in W, lifting to a neighborhood $M\times I$ of M in $\hat{W}$ with $M\times 1\subset W_r$, $M\times -1\subset W_l$. Recall from Chapter I that $K_{i+1}(W_r, M)\oplus K_{i+1}(W_l, M)\cong K_i(M)$ and hence $K_i(W_r, M)=0$ for $i<k$. Hence as $\pi_1(M)=\pi_1(W_r)=\pi_1(W_l)$ by the relative Hurewicz theorem each class of $K_k(W_r, M)$ is represented by an immersed disc $(D^k, S^{k-1})\to(W_r, M)$. In particular, let $\bar{\alpha}_i\colon (D^k, S^{k-1})\to(W_r, M)$ be an immersed disc with $\partial\bar{\alpha}_i=\alpha_i$. Similarly there are immersions $\bar{\beta}_i\colon (D^k, S^{k-1})\to(W_l, M)$ with $\partial\bar{\beta}_i=\beta_i$. As $\pi_1(M)=\pi_1(W_r)$ a standard piping argument (see [Z, Lemma 48], [M2, p. 71], [W2, p. 39]) shows that the immersions $\{\bar{\alpha}_i\}$ $1\leqq i\leqq r$ are regularly homotopic to disjoint embeddings in (W_r, M). Similarly, the immersions $\{\bar{\beta}_i\}$ $1\leqq i\leqq s$ are regularly homotopic to disjoint embeddings in (W_l, M).

Thus, we may assume that $\{\bar{\alpha}_i\}$ $1\leqq i\leqq r$, $\{\beta_j\}$ $1\leqq j\leqq s$ are disjointly embedded discs in $(\hat{W}, M)$. Thickening M and the discs $\bar{\alpha}_i\subset W_r$, $1\leqq i\leqq r$, we obtain a cobordism C_P of M to a manifold $M_P\subset W_r$.



Similarly, thickening up the discs $\bar{\beta}_i$, $1\leqq i\leqq s$, produces a cobordism C_Q in W_l of M to $M_Q\subset W_l$. Then $C=C_P\cup_M C_Q\subset\hat{W}$ is a cobordism of M_P to M_Q.

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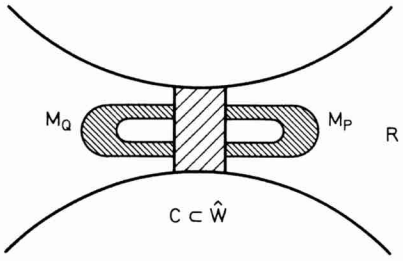
[15] In fact, if $K_{k-1}(M)$ is a projective module and $\tilde{K}_0(H)=0$ it is not too difficult to construct, without using the square-root closed condition or any condition on elements of order 2 in $\pi_1(Y)$, an h-cobordism of W to a split homotopy equivalent manifold. See in this connection the discussion on Farrell-Hsiang splitting in Chapter VI



The map $\hat{f}\colon \hat{W} \to \hat{Y}$ restricts, after being varied by a homotopy, to $\hat{f}!\colon C \to X \times I$, $X \times I$ a neighborhood of X in $\hat{Y}$. Let $R =$ closure of $W_r - C_P$, $L =$ closure of $W_l - C_Q$ in $\hat{W}$.

The following result is proved by a method similar to that used in proving Lemma 4.1 of [FH1].

Lemma III.1. *$K_i(M_P)=0$, $K_i(M_Q)=0$ for $i \neq k-1$, k. $K_{k-1}(M_P)$, $K_{k-1}(M_Q)$, $K_k(M_P)$ and $K_k(M_Q)$ are finitely generated projective $Z[H]$ modules. $K_i(C)=0$ for $i \neq k$ and $K_k(M_P) \oplus K_k(M_Q) \to K_k(C)$ is an isomorphism of stably free $Z[H]$ modules. $K_i(R)=0$, $K_i(L)=0$ for $i \neq k-1$ and $K_{k-1}(M_P) \to K_{k-1}(R)$, $K_{k-1}(M_Q) \to K_{k-1}(L)$ are isomorphisms of $Z[H]$ modules. Also $K_k(M_Q) \to K_k(W_r \cup_M C_Q)$ and $K_k(M_P) \to K_k(W_l \cup_M C_P)$ are isomorphisms, and $K_i(W_r \cup_M C_Q)=0$, $K_i(W_l \cup_M C_P)=0$ for $i \neq k$.*

Proof. We will need to employ in this proof the cohomology cokernel groups $K^i(C)$ and $K^i(C, \partial C)$ associated to the maps $C \to X \times I$. Such groups are discussed in a more general setting in [W2; Chapter II].

First note that up to homotopy, C was produced by attaching k-dimensional cells to a generating set for $P \oplus Q = K_{k-1}(M)$. Hence, $K_j(C)=0$ for $j \neq k$. Moreover, we claim that $K^{k+1}(C; B)=0$, B any $Z[H]$ module, and hence by a standard argument [W2; p. 26] $K_k(C)$ is a stably free $Z[H]$ module. To show the vanishing of $K^{k+1}(C; B)$ recall that by Poincaré duality, $K^{k+1}(C; B) \cong K_{k-1}(C, \partial C; B)$ and we have the exact sequence

$$K_{k-1}(C; B) \to K_{k-1}(C, \partial C; B) \to K_{k-2}(\partial C; B).$$

Moreover, $\partial C = M_P \cup M_Q$ with M_P and M_Q obtained from M by surgeries on $(k-1)$-dimensional spheres and hence as $K_j(M; B)=0$ $j<k-1$, we get $K_j(\partial C; B) = K_j(M_P; B) \oplus K_j(M_Q; B)=0$ for $j<k-1$. Thus, $K_k(C)$ is a stably free finitely generated $Z[H]$ module.

We proceed to relate $K_k(C)$ to $K_k(M_P) \oplus K_k(M_Q) = K_k(\partial C)$. In the exact sequence

$$K_{k+1}(C, \partial C) \to K_k(\partial C) \to K_k(C)$$

by Poincaré duality $K_{k+1}(C, \partial C) \cong K^{k-1}(C)=0$. Hence

$$K_k(\partial C) = K_k(M_P) \oplus K_k(M_Q) \to K_k(C)$$

is injective. To see that this map is also surjective, use the Mayer-Vietories sequence of $\hat{W}=L\cup_{M_Q} C\cup_{M_P} R$; this reduces, as $\hat{W}\to\hat{Y}$ is a homotopy equivalence, to the isomorphisms

(1) $K_i(M_P)\oplus K_i(M_Q)\to K_i(L)\oplus K_i(C)\oplus K_i(R)$.

In particular, $K_k(M_P)\oplus K_k(M_Q)\to K_k(C)$ is also surjective and hence is an isomorphism of $Z[H]$ modules.

To compute $K_i(R)$ and $K_i(L)$, from (1) and $K_i(C)=0$, $i\neq k$, $K_k(C)=K_k(M_P)\oplus K_k(M_Q)$ conclude that $K_i(R)=0$, $K_i(L)=0$ $i\neq k-1$, and $K_{k-1}(M_P)\to K_{k-1}(R)$, $K_{k-1}(M_Q)\to K_{k-1}(L)$ are isomorphisms.

To check that $K_{k-1}(M_P)$ is projective, we show that $K_{k-1}(M_P)\oplus K_{k-1}(M_Q)=K_{k-1}(\partial C)$ is a stably free $Z[H]$ module. As $K_k(\partial C)\to K_k(C)$ was seen above to be an isomorphism and as $K_{k-1}(C)=0$, we have $K_k(C,\partial C)\cong K_{k-1}(\partial C)$ and $K_i(C,\partial C)=0$ for $i\neq k$. Hence by an argument used above, to show that $K_k(C,\partial C)$ is a stably free $Z[H]$ module, it suffices to check that $K^{k+1}(C,\partial C\colon B)=0$ for any $Z[H]$ module B. But, $K^{k+1}(C,\partial C\colon B)\cong K_{k-1}(C\colon B)=0$ by Poincaré duality [W 2].

Lastly, these computations and the Mayer-Vietoris sequence of $\hat{W}=L\cup_{M_P}(C_P\cup_M W_l)$

$$\cdots\to K_i(M_P)\to K_i(L)\oplus K_i(C_P\cup_M W_l)\to K_i(\hat{W})\to\cdots$$

show immediately that $K_i(C_P\cup_M W_l)=0$ for $i\neq k$, $K_k(C_P\cup_M W_l)=K_k(M_P)$. Similar remarks apply to $C_Q\cup_M W_r$.

After variation by a homotopy, the map $\hat{f}\colon \hat{W}\to\hat{Y}$ restricts to $f!\colon C\to X\times I$, $I=[-1,1]$, $X\times I$ a neighborhood of $X\times 0=X\subset Y$, with $\hat{f}(M_P)=X\times 1$, $\hat{f}(M_Q)=X\times -1$.

Lemma III.2. *If $[K_k(M_P)]\in\tilde{K}_0(H)$ is zero, then, after performing further trivial ambient surgeries on $(k-1)$ spheres of M_Q and still calling the resulting manifold M_P, $K_{k-1}(M_P)$ and $K_k(M_P)$ will be free $Z[H]$ modules. If $K_{k-1}(M_P)$ is a free $Z[H]$ module, then $\hat{f}!\colon C\to X\times I$ is homotopy by a homotopy fixed on ∂C, to a map, which we continue to denote $\hat{f}$, transverse to X with $\hat{f}^{-1}(X)\to X$ a homotopy equivalence.*

Proof. As $K_{k-1}(M_P)$ and $K_k(M_P)$ are projective and are the only nonzero homology groups of a free $Z[H]$ complex standard arguments show that $[K_{k-1}(M_P)]-[K_k(M_P)]=0$ in $\tilde{K}_0(H)$. As performing trivial surgeries stabilizes $K_{k-1}(M_P)$ and $K_k(M_P)$, the first statement of the lemma is obvious. Note that performing trivial surgeries corresponds to enlarging the generating set $(\alpha_1,\ldots,\alpha_r)$ of P with copies of the zero element.

To see the second statement of the lemma, let $e_1,\ldots,e_v$ be a basis for the free $Z[H]$ module $K_{k-1}(M_P)$. As $K_{k-1}(C)=0$, $K_k(C,M_P)\to K_{k-1}(M_P)$ is surjective. Moreover, as $K_i(C,M_P)=0$ for $i<k$, by the relative Hurewicz theorem there are immersed discs $d_i\colon (D^k,S^{k-1})\to(C,M_P)$ with ∂d_i representing e_i, $1\leqq i\leqq v$. As $\pi_1(C)=\pi_1(M_P)$ a standard piping argument, see [W 2; p. 39], used above shows that after variation by regular homotopies the discs $\{d_i\}$ $1\leqq i\leqq v$ may be taken to be disjoint embeddings. Performing handle exchanges on M_P

in C using these embedded discs, we obtain a cobordism C'_P of M_P to M' with $C'_P \subset C$.

We claim that $M' \to X$ is a homotopy equivalence. Actually, it is a standard fact that surgery on a free $Z[H]$ module basis for $K_{k-1}(M_P^{2k-1})$ produces a homotopy equivalence, but for completeness we give in the present case a direct argument. Clearly $\pi_1 M' = \pi_1 X$. Moreover, $C'_P = \partial(R \cup_{M_P} C'_P)$ and as $K_i(R)=0$, $i \neq k-1$ and C'_P is up to homotopy, produced by attaching cells to a basis of $K_{k-1}(M_P)$, $K^j(C'_P) \to K^j(M_P)$ is an isomorphism for $j \neq k-1$. But this map factors through $K^j(C'_P) \to K^j(C'_P)$ which is onto for $j \neq k-1$ as

$$K^{j+1}(C'_P, \partial C'_P) \cong K_{2k-j-1}(C'_P) = 0 \quad \text{for } j \neq k-1.$$

Hence as $\partial C = M_P \cup M'$, $K^j(M')=0$ for $j \neq k-1$ and using Poincaré Duality, we conclude that $K_i(M')=0$ for all i.

We will show below in Lemma III.4 that $\Phi(\tau(f)) = [K_k(M_P)] \in \tilde{K}_0(H)$ and hence from Lemma III.2 if $\Phi(\tau(f))=0$, $\hat{f}$ is homotopic to a map split along $X \subset \hat{Y}$.

To proceed with the proof in § 2 we will need to be more careful in our choice of a basis for P and Q and correspondingly in the construction of C_P and C_Q. Using Lemma I.8, we may choose finite $Z[H]$ module generating sets $\alpha_1, \ldots, \alpha_u$ for P, $\beta_1, \ldots, \beta_u$ for Q with $\rho_1(\alpha_1)=0$, $\rho_2(\beta_1)=0$, $\rho(\alpha_i)$ and $\rho(\beta_i)$ elements of the $Z[H]$ submodule of $(P \oplus Q) \otimes_{ZH} ZG$ generated by $\alpha_1, \ldots, \alpha_{i-1}$, $\beta_1, \ldots, \beta_{i-1}$, $1 < i \leqq u$. The corresponding construction of C_P and C_Q given above constructed embedded discs $\bar{\alpha}_i$, $\bar{\beta}_i$, $1 \leqq i \leqq u$ with $\partial\bar{\alpha}_i$, $\partial\bar{\beta}_i$ being spheres in M representing α_i and β_i respectively. Note that these embedded discs $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_i, \bar{\beta}_i, \ldots, \bar{\beta}_i\}$ can be chosen so that their images in W are in general position and thus, as dimension $W = 2k$, have only isolated points of self-intersection. (For example, given an embedding $\bar{\alpha}_i : (D^k, S^{k-1}) \to (W_r, M)$ by varing the image of $\bar{\alpha}$ in W by a small regular homotopy fixed on S^{k-1} this image may be put in general position. The lift of this homotopy is a regular homotopy of $\bar{\alpha}$ to a map, which if the regular homotopy was chosen sufficiently small, will still be an embedding.)

We may impose one further restraint on the choice of the discs $\bar{\alpha}_i$, $\bar{\beta}_j$. Filter the space C_P by setting

$$C_{P,i} = M \times [0,1] \cup \text{handles obtained by thickening } \bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_i$$

so that $C_P = C_{P,u} \supset C_{P,u-1}, \ldots, C_{P,0} = M \times [0,1]$. Similarly, filter C_Q by setting

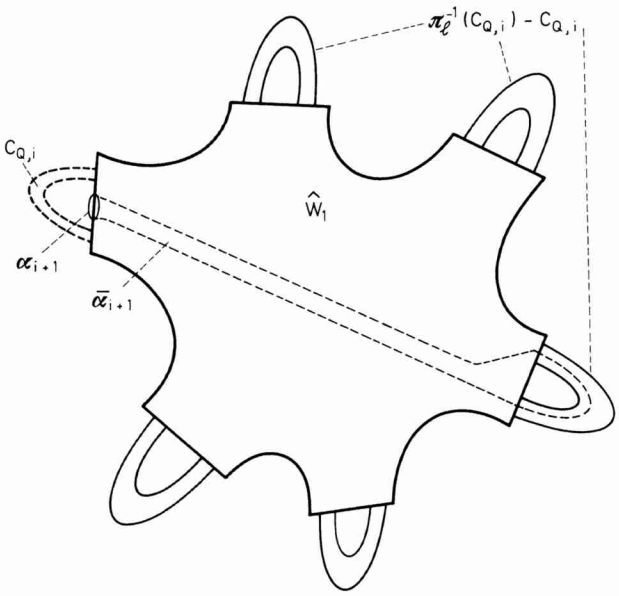
$$C_{Q,i} = M \times [0,-1] \cup \text{handles obtained by thickening } \beta_1, \ldots, \beta_i.$$

We state the condition first in case A, that is $Y - X$ has 2 components. Note that $W_l \cup_M W_1$ is the covering space of W corresponding to the subgroup $\pi_1(W_1) \subset \pi_1(W)$ and similarly $W_2 \cup_M W_r$ is the covering space of W corresponding to $\pi_1(W_2) \subset \pi_1(W)$. Let $\pi_l : \hat{W} \to W_l \cup_M W_1$, $\pi_r : \hat{W} \to W_2 \cup_M W_r$ denote the obvious covering space maps corresponding to $H \subset G_1$ and to $H \subset G_2$. Now we claim that the discs $\bar{\alpha}_i$, $\bar{\beta}_i$ can be chosen inductively so that

(2) $\bar{\alpha}_{i+1} \subset \text{closure}\,(\pi_l^{-1}(W_1 \cup_M C_{Q,i}) - C_{Q,i})$ in $\hat{W}$,

(2) $\bar{\beta}_{i+1} \subset \text{closure}\,(\pi_r^{-1}(W_2 \cup_M C_{P,i}) - C_{P,i})$ in $\hat{W}$.

Also, the disc $\bar{\alpha}_{i+1}$ (resp; $\bar{\beta}_{i+1}$) is to be chosen so that its image represents the zero element of $\pi_k(\hat{Y}_1, X)$ (resp; $\pi_k(\hat{Y}_2, X)$).



Essentially the same arguments as those employed in I § 5 show that the discs $\bar{\alpha}_i$, $\bar{\beta}_i$ may be inductively chosen to satisfy this condition. The point is again, see Lemma I.7, α_i and $\rho(\alpha_i) \in Q \otimes_{ZH} ZH_1$ represent essentially from the definition of ρ the same element of $K_{k-1}(W_1)$. But from our choice of generating sets, $\rho(\alpha_i) \in \{\beta_1, \ldots, \beta_{i-1}\} \otimes_{ZH} \widetilde{ZG}_1$ where $\{\beta_1, \ldots, \beta_{i-1}\}$ is the ZH submodule of Q generated by $\beta_1, \ldots, \beta_{i-1}$ and

$$K_{k-1}(\pi_r^{-1}(W_1 \cup_M C_{Q,i}) - C_{Q,i}) = K_{k-1}(W_1)/\{\beta_1, \ldots, \beta_{i-1}\} \otimes_{ZH} \widetilde{ZG}_1.$$

Hence as in the arguments of Lemma II.6 α_i bounds a disc in $(\pi_r^{-1}(W_1 \cup_M C_{Q,i}) - C_{Q,i})$ and as observed above this disc $\bar{\alpha}_i$ may be taken to be embedded and with its image in W having only point self-intersections.

Note that as $C_{P,i+1}$ is obtained from $C_{P,i}$ by attaching a handle obtained by thickening $\bar{\alpha}_{i+1}$, we may inductively construct $C_{P,i+1}$ so that

$$C_{P,i+1} \subset \text{closure}\,(\pi_l^{-1}(W_1 \cup_M C_{Q,i}) - C_{Q,i}) \quad \text{in } \hat{W}.$$

Similarly, we may assume that

$$C_{Q,i+1} \subset \text{closure}\,(\pi_r^{-1}(W_2 \cup_M C_{P,i}) - C_{P,i}) \quad \text{in } \hat{W}.$$

We now briefly state the corresponding condition which we may impose in case B on the choice of the discs $\bar{\alpha}_i$ and $\bar{\beta}_i$. Note that $W_l \cup_{M_1} W' \cup_{M_2} W_r$, with basepoint the basepoint of M_1, is the covering space of W corresponding to $J \subset J *_H \{t\}$. If we pick as the basepoint of $W_l \cup_{M_1} W' \cup_{M_2} W_r$ the basepoint of M_2,