Werk

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Autor: Whitten, Wilbur

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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Algebraic and Geometric Characterizations of Knots

Wilbur Whitten (Princeton)

Many topological problems ask one to characterize and portray equivalence classes of spaces or maps. Among such problems, the exemplar, which is, incidentally, the fundamental problem of topology, asks for the classification of topological spaces. Most characterization problems, however, hypothesize equivalence relations that are more subtle than the topological distribution of spaces. For example, homotopies, isotopies, and ambient autohomeomorphisms and isotopies divide the set of imbeddings of one topological space in another into multifarious equivalence classes called relative placement types; the placement problem is to characterize the various placement types. Classical knot theory, covering the placement problem of knots in three-space, has developed steadily since its pristine form seventy-five years ago, and now contains many elaborate and practical invariants of relative placement types, especially knot types, the classes of knots ambient autohomeomorphisms produce. Although a knot's group, a most important invariant of a knot's type, generally does not determine the type [9], J. Simon [25] recently characterized the type by a group – the free product of two cable-knot groups; the cables wind about the composition of the given knot and a suitable fixed knot. More direct algebraic characterizations are possible, however. In this paper, we prove, for a given knot K, that the group of a canonical double $D(K)$ of K characterizes the (ambient) isotopy type of K. This portrayal of isotopy types yields, as a corollary, an algebraic characterization of amphicheiral knots.

Turning to geometric classifications of knots, we note the basic conjecture that the topological type of a knot's complement determines the knot's type. Numerous persons have in fact verified this for several cases: composites ([2, 11, 18]), torus ([1, 10, 12]), doubled ([2, 11]), most cables ([11, 23]), and a few others ([5, 15, 23, 24]). We observe that, for a doubled knot $D(K)$, the topological type of $S^3 - D(K)$ characterizes not only the knot type of $D(K)$, but also the isotopy type of K; see Theorem 2.1. This also follows, of course, from our main result stated in the preceding paragraph.

In Section 1, we give preliminary agreements and definitions, and we prove three lemmas needed in the sequel. We state and prove the "doubled-knot" classifications of knots in Section 2.

I am grateful to the referee for his thoughtful suggestions.

1. Preliminaries

Throughout this work, the three-sphere S^3 has a fixed orientation; all maps are piecewise linear, all submanifolds are polyhedral, and all knots and links are oriented. Let K_1 and K_2 be knots in S^3, and let $h\colon (S^3, K_1) \to (S^3, K_2)$ be a homeomorphism. We distinguish two equivalence relations on knots in S^3: (1) K_1 and K_2 *are of (or belong to) the same isotopy type*, if h preserves the orientation of S^3 and is, therefore, isotopic to the identity map on S^3 [8]; (2) for an arbitrary homeomorphism h, K_1 and K_2 are *equivalent* and *are of (or belong to) the same knot type*. The symbol $\{K\}$ denotes the isotopy type of K; K^*, the mirror image of K.

The symbols $\approx$, $\cong$, and $\simeq$ denote, respectively, the three relations, isomorphism of groups, homeomorphism of spaces, and homotopy of maps. If S is a subset of a group G, then $\langle S \rangle$ denotes the consequence (or normal closure) of S in G. The complement C of an open solid torus in S^3 is a *toral solid*; if a core K of the torus is knotted, C is a *K-knot manifold*. The complement of $\nu(>1)$ mutually disjoint, open, solid tori in S^3 is a *ν-link manifold*. For a knot K, a meridian-longitude pair (μ, λ) of simple closed curves is always *oriented with respect to K*: μ has linking number $+1$ with K, and the orientations of λ and K match; that is, λ and K are homologous in some second-regular neighborhood of K.

A surface F in a 3-manifold M is *properly imbedded*, if $F \cap \partial M = \partial F$; F is *incompressible* in M, if the inclusion-induced homomorphism $\pi_1(F) \to \pi_1(M)$ is a monomorphism and if F is neither a 2-sphere nor a projective plane. Let F_1 and F_2 be properly imbedded surfaces in M; F_1 and F_2 are *parallel*, if there is a homeomorphism $k\colon F_1 \times I \to M$ satisfying $k(F_1 \times 0) = F_1$, $k(F_1 \times 1) = F_2$, and $k((\partial F_1) \times I) \subset \partial M$. A properly imbedded surface F in M is *boundary parallel*, if there is an imbedding $k\colon F \times I \to M$ with $k(F \times 0) = F$ and $k(((\partial F) \times I) \cup (F \times 1)) \subseteq \partial M$. In particular, a closed surface F_1 is boundary parallel, if it is parallel to a component F_2 of ∂M; in such a case, we sometimes say that F_1 is *parallel to the boundary of M*. A knot K' is a *companion* of K, if K' is a core of a solid torus whose interior contains K, and whose boundary is incompressible in a K-toral solid C and is not parallel to ∂C.

If nontrivial surgery on K never produces a homotopy 3-sphere, we shall call K a sterile knot, thereby eschewing the emetic phrase "a knot with Property P" currently reserved for such knots. One can easily verify that the topological type of S^3 minus any sterile knot determines

the knot's type. If a knot is not sterile, we shall call it fertile. There are no known examples of nontrivial fertile knots.

Let U, W, X, and Y be spaces. We write $U = X \cup_W Y$ to denote $U = X \cup Y$ when $X \cap Y = W$. The symbol Int X denotes the interior of X.

Now to doubled knots. Let ρ and η be fixed integers with ρ arbitrary and $\eta = \pm 2$. A knot $D(K; \rho, \eta) \subset S^3$ is the *double of K with twist ρ and self-intersection number η*, if there is a solid torus W with core K and with $D(K; \rho, \eta) \subset \text{Int } W$, and if there is a homeomorphism taking $V^*_{(\rho,\eta)}$, exactly one of the four solid tori shown in Fig. 1, onto W and preserving the orientation of $V^*_{(\rho,\eta)}$ – the orientations of both $V^*_{(\rho,\eta)}$ and W are inherited from that of S^3 – taking a longitude of $V^*_{(\rho,\eta)}$ onto a longitude of W, and taking K_0 (in Fig. 1) onto $D(K; \rho, \eta)$. If a homeomorphism h between oriented solid tori preserves the orientations (inherited from that of S^3) and maps a longitude to a longitude, then h is *faithful*. The number η is the intersection number of the knot K_0 and a disk having one clasping singularity, lying in Int $V^*_{(\rho,\eta)}$, and spanning K_0; the orientation of the disk induces the orientation of K_0, and the orientation of S^3 is fixed so that η is positive or negative as stated in the four parts of Fig. 1. The twist ρ equals the "twisting number" v of $D(K; \rho, \eta)$ [19, p. 234], as one can easily show. The knot K is the *diagonal* of $D(K; \rho, \eta)$, and the system $(\{K\}, \rho, \eta)$ characterizes $\{D(K; \rho, \eta)\}$. On the other hand, $\{D(K; \rho, \eta)\}$ determines the system $(\{K\}, \rho, \eta)$ except in two cases when K is unknotted [19, p. 234]. Furthermore, the diagonal K is the unique maximal companion of $D(K; \rho, \eta)$ [19, Remark 1, p. 242] when K is knotted; when K is unknotted, $D(K; \rho, \eta)$ has no companions [19, Theorem 4, p. 242].

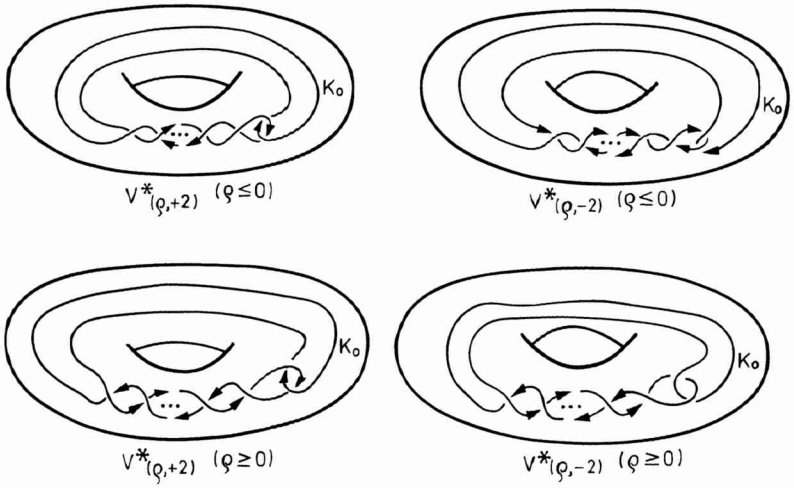

Fig. 1. (ρ, η)-twist knots in solid tori

Let (μ, λ) be a meridian-longitude pair on the boundary of a solid torus V in S^3, let K be a core of V, and let (α, β) be a pair of relatively prime integers with $\alpha \geqq 2$. A simple closed curve on ∂V homologous to $\beta\mu + \alpha\lambda$ is an (α, β)-*cable knot with core* K.

Lemma 1.1. *Let C be a knot manifold in S^3, and let A be a properly imbedded annulus in C with the following properties:*

(a) *the components of ∂A do not bound disks in ∂C;*

(b) *A is not boundary parallel in C.*

Then a core of $S^3 - \operatorname{Int} C$ is either a composite knot or a cable knot.

Proof. Hypothesis (a) implies that the components of ∂A bound annuli in ∂C. Hence, there are toral solids X_1 and X_2 with $C = X_1 \cup_A X_2$, since C is a knot manifold.

If each of X_1 and X_2 is a knot manifold, hypothesis (a) and [25, proof of Lemma 2.1, p. 4] imply that the components of ∂A bound meridional disks in $S^3 - \operatorname{Int} C$. Therefore, a core of $S^3 - \operatorname{Int} C$ is, by definition, a composite knot.

Suppose now that X_2, say, is a solid torus. There is an annulus $B \subset \partial C$ satisfying $B \cup A = \partial X_2$. If the inclusion-induced homomorphism $\pi_1(A) \to \pi_1(X_2)$ is surjective, then the inclusion $A \to X_2$ is a homotopy equivalence [29], and A, therefore, is boundary parallel (to B) in C [21, Theorem 3.1, p. 168], contradicting hypothesis (b). The result [25, Lemma 2.2, p. 4] then implies that a core of $S^3 - \operatorname{Int} C$ is a cable knot. This concludes the lemma's proof.

Remark. H.C. Lyon has stated a lemma similar to Lemma 1.1; see [14, Lemma 2].

Lemma 1.2. *Let $M_1 \subset M_2 \subset M_3$ be a sequence of knot manifolds satisfying the following conditions*: (1) $\partial M_i \cap \partial M_j = \emptyset$ $(i \neq j)$, (2) *∂M_i is incompressible in M_{i+1} $(i = 1, 2)$, and* (3) *∂M_1 is not parallel to ∂M_2. Then ∂M_1 is incompressible in M_3 and is not parallel to ∂M_3.*

Proof. Suppose ∂M_1 is compressible in M_3. By Kneser's lemma [13], there is a nonsingular disk $D \subset M_3$ such that $D \cap \partial M_1 = \partial D$ and such that ∂D is not contractible on ∂M_1. Since M_1 is a knot manifold, $\operatorname{Int} D \subset M_3 - M_1$. Since ∂M_1 is incompressible in M_2, $D \cap \partial M_2 \neq \emptyset$. Put D in general position with respect to ∂M_2. Then $D \cap \partial M_2$ is a collection of simple closed curves in $\operatorname{Int} D$; assume that the number of curves in $D \cap \partial M_2$ is minimal. One of these curves must be noncontractible on ∂M_2 and bound a disk (in D) whose interior is in $M_3 - M_2$. But this is impossible, since ∂M_2 is incompressible in M_3; therefore, ∂M_1 is incompressible in M_3.

Suppose ∂M_1 is parallel to ∂M_3. Then $M = M_3 - \operatorname{Int} M_1 \cong \partial M_1 \times I$, and ∂M_2 is incompressible in M, since it is incompressible in M_3. Hence, ∂M_2 is parallel to ∂M_1 [21, Theorem 3.2, p. 169], and the lemma is proved.

Using Lemma 1.2 and induction, one can easily construct a proof of the following lemma; we shall omit details of the proof.

Lemma 1.3. *Let $M_1 \subset \cdots \subset M_n$ $(n \geqq 2)$ be a sequence of knot manifolds satisfying the following conditions*: (1) $\partial M_i \cap \partial M_j = \emptyset$ $(i \neq j)$, (2) ∂M_i *is incompressible in M_{i+1} $(i = 1, \ldots, n-1)$, and* (3) ∂M_1 *is not parallel to ∂M_2. Then ∂M_1 is incompressible in each of M_2, $M_3, \ldots,$ and M_n, and ∂M_1 is not parallel to any torus ∂M_j $(j = 2, \ldots, n)$.*

2. The Characterizations

Let V be a second-regular neighborhood of a doubled knot $D(K; \rho, \eta)$, and set $C^3(K; \rho, \eta) = S^3 - \operatorname{Int} V$. Our geometric classification is contained in the following theorem.

Theorem 2.1. *Let K_1 and K_2 be knots in S^3, and let ρ and η be fixed integers; ρ, arbitrary; $\eta = \pm 2$. Then K_1 and K_2 belong to the same isotopy type if and only if $C^3(K_1; \rho, \eta) \cong C^3(K_2; \rho, \eta)$.*

Proof. If K_1 and K_2 are of the same isotopy type, then $D(K_1; \rho, \eta)$ and $D(K_2; \rho, \eta)$ are also of the same isotopy type [19, p. 234]. Consequently, $C^3(K_1; \rho, \eta) \cong C^3(K_2; \rho, \eta)$.

Conversely, suppose $C^3(K_1; \rho, \eta) \cong C^3(K_2; \rho, \eta)$. Application of Theorem 2.2 (below) immediately concludes the proof; another approach is feasible, however, and goes as follows. If $C^3(K_1; \rho, \eta)$ and $C^3(K_2; \rho, \eta)$ are solid tori, then $D(K_1; \rho, \eta)$ and $D(K_2; \rho, \eta)$ are trivial knots. This means that K_i $(i = 1, 2)$ must also be trivial, since $D(K_i; \rho, \eta)$, as a trivial knot, has no companions [19, p. 193]; in this case, we have $\{K_1\} = \{K_2\}$. If $C^3(K_1; \rho, \eta)$ and $C^3(K_2; \rho, \eta)$ are knot manifolds, there is an autohomeomorphism h of S^3 such that $h(D(K_1; \rho, \eta)) = D(K_2; \rho, \eta)$, since nontrivial doubled knots are sterile ([2, 11]). If h preserves the orientation of S^3, then $\{K_1\} = \{K_2\}$, since the isotopy type of a doubled knot uniquely determines the isotopy type of its diagonal [19, p. 234].

Now note that $D(K_1^*; -\rho, -\eta)$ is the mirror image of $D(K_1; \rho, \eta)$. If h reverses the orientation of S^3, then $\{D(K_1^*; -\rho, -\eta)\} = \{D(K_2; \rho, \eta)\}$. Thus, $\{K_1^*\} = \{K_2\}$. But if K_1^* and K_2 are nontrivial, then $-\eta = \eta$, since a doubled knot with nontrivial diagonal uniquely determines its self-intersection number. Since $\eta = \pm 2$, we have a contradiction; hence, K_1 and K_2 are trivial knots. Therefore, $\{K_1\} = \{K_1^*\} = \{K_2\}$, and the theorem is proved.

We now state the main characterization theorem.

Theorem 2.2. *Let K_1 and K_2 be knots in S^3, and let ρ and η be fixed integers; ρ, arbitrary; $\eta = \pm 2$. Then K_1 and K_2 are of the same isotopy type if and only if $\pi_1(C^3(K_1; \rho, \eta)) \approx \pi_1(C^3(K_2; \rho, \eta))$.*

Corollary 2.3. *Let ρ and η be fixed integers with ρ arbitrary and $\eta = \pm 2$. A knot K in S^3 is amphicheiral if and only if*

$$\pi_1(C^3(K; \rho, \eta)) \approx \pi_1(C^3(K^*; \rho, \eta)).$$

Furthermore, K is amphicheiral if and only if $C^3(K; \rho, \eta) \cong C^3(K^*; \rho, \eta)$.

Corollary 2.3, whose proof follows immediately from Theorems 2.1 and 2.2, presents both algebraic and geometric characterizations of amphicheiral knots.

Proof of Theorem 2.2. The necessity is trivial. If $\{K_1\} = \{K_2\}$, then, as in the proof of Theorem 2.1, $C^3(K_1; \rho, \eta) \cong C^3(K_2; \rho, \eta)$; therefore, $\pi_1(C^3(K_1; \rho, \eta)) \approx \pi_1(C^3(K_2; \rho, \eta))$.

For the sufficiency, we suppose, henceforth, that $\pi_1(C^3(K_1; \rho, \eta)) \approx \pi_1(C^3(K_2; \rho, \eta))$. Let V_i denote a second-regular neighborhood of $D(K_i; \rho, \eta)$ $(i = 1, 2)$, and set $C_i = C^3(K_i; \rho, \eta) = S^3 - \operatorname{Int} V_i$. The diagonal K_2 of $D(K_2; \rho, \eta)$ is a core of a solid torus $\mathscr{V}$ containing V_2 in its interior; furthermore, with the orientation of $\mathscr{V}$ inherited from the orientation of S^3, we shall assume that there is a faithful homeomorphism taking exactly one of the solid tori of Fig. 1 onto $\mathscr{V}$ and taking K_0 onto $D(K_2; \rho, \eta)$.

If each of K_1 and K_2 is a trivial knot, the conclusion follows. Otherwise, by exchanging indices if necessary, we can assume that K_2 is not trivial. Consequently, $T^2 = \partial \mathscr{V}$ is an incompressible torus in $\operatorname{Int} C_2$ not parallel to ∂C_2. Set $K^3 = S^3 - \operatorname{Int} \mathscr{V}$ and $\Lambda^3 = \mathscr{V} - \operatorname{Int} V_2$. Then $C_2 = K^3 \cup_{T^2} \Lambda^3$. The group $\pi_1(C_2) = \pi_1(K^3) *_{\pi_1(T^2)} \pi_1(\Lambda^3)$ is a nontrivial free-product with amalgamation, as one can easily show, since there is only one link with group $|x, y: xyx^{-1}y^{-1}|$ [16], and since K^3 is a knot manifold and Λ^3 is a 2-link manifold.

Together, the isomorphism $\pi_1(C_1) \approx \pi_1(C_2)$ and the asphericity of both C_1 and C_2 imply the existence of a homotopy equivalence $f: C_1 \to C_2$ inducing the given isomorphism $\pi_1(C_1) \xrightarrow{\approx} \pi_1(C_2)$; cf. [17, p. 93]. The manifold C_1 is orientable and compact, T^2 is bicollared in C_2, $\ker(\pi_j(T^2) \to \pi_j(C_2)) = 1$ $(j = 1, 2)$ since T^2 is incompressible in C_2 and $\pi_2(T^2) = 0$, and $\pi_2(K^3 - T^2) = \pi_2(\Lambda^3 - T^2) = 0$ since K^3 is a tame-knot manifold and Λ^3 is homotopic to the complement in S^3 of a tame unsplittable 2-link. By [26, Lemma 1.1, p. 506], there is a mapping $g: C_1 \to C_2$ with the following properties:

(1) $g \simeq f$;

(2) g is transverse with respect to T^2; that is, there exist product neighborhoods $U(g^{-1}(T^2))$ and $U(T^2)$ of $g^{-1}(T^2)$ and T^2, respectively, such that g maps each fiber of $U(g^{-1}(T^2))$ homeomorphically onto a fiber of $U(T^2)$;

(3) $g^{-1}(T^2)$ is a compact orientable (perhaps disconnected) surface properly imbedded in C_1;

(4) if F is any component of $g^{-1}(T^2)$, then

$$\ker(\pi_j(F) \to \pi_j(C_1)) = 1 \qquad (j=1,2).$$

We divide the remainder of the proof into six parts.

1. *$g^{-1}(T^2)$ is not empty.*

If $g^{-1}(T^2)=\emptyset$, we have either $g(C_1)\subset K^3$ or $g(C_1)\subset \Lambda^3$, since T^2 separates C_2. Let x be a point in C_1, and let y be a point in T^2. When a suitable path is chosen from y to $g(x)$, we have either $g_*(\pi_1(C_1, x))\subseteq \pi_1(K^3, y)$ or $g_*(\pi_1(C_1, x))\subseteq \pi_1(\Lambda^3, y)$, depending on whether $g(C_1)\subset K^3$ or $g(C_1)\subset \Lambda^3$. Thus, $g_*(\pi_1(C_1, x))$ is a *proper* subgroup of $\pi_1(C_2, y)$, since $\pi_1(C_2, y)$ is a nontrivial free-product with amalgamation. But g is a homotopy equivalence (by (1)), and, therefore, $g_*(\pi_1(C_1, x))=\pi_1(C_2, y)$, yielding a contradiction.

2. *We can assume that each component F of $g^{-1}(T^2)$ is a torus not parallel to ∂C_1. The knot K_1 is nontrivial.*

By (4) and (1), $\pi_1(F)$ is isomorphic to a subgroup of $\pi_1(T^2)$, and is, therefore, a free abelian group of rank 2 or less. By (3), F is either a 2-sphere, a disk, an annulus, or a torus. Property (4) implies that $\pi_2(F)=0$; thus, F is not a 2-sphere.

If F is a disk, it is properly imbedded in C_1; see (3). Since ∂C_1 is incompressible in C_1, ∂F bounds a disk D in ∂C_1. The 2-sphere $F\cup D$ bounds a 3-ball B in C_1; let N be a 3-cell regular neighborhood of B in C_1 such that $(N-B)\cap g^{-1}(T^2)=\emptyset$, and set $C_1^* = C_1 - \operatorname{Int} N$. Let N^* be a small regular neighborhood in C_1^* of $\partial N \cap C_1^*$ such that $N^*\cap g^{-1}(T^2)=\emptyset$. There is a homeomorphism h: $C_1 \to C_1^*$ such that $h|(C_1^* - \operatorname{Int} N^*)=1$ and $h\simeq 1_{C_1}$. Thus, $(g|C_1^*)h$ satisfies (1) through (4), and $[(g|C_1^*)h]^{-1}(T^2)$ has fewer components than $g^{-1}(T^2)$.

Suppose that F is an annulus. The components of ∂F bound annuli in ∂C_1, since F is incompressible in C_1. We can assume that F is not boundary parallel, since otherwise g can be replaced by a map g' satisfying (1) through (4) and having the property that $g'^{-1}(T^2)$ has fewer components than $g^{-1}(T^2)$; a process analogous to that in the preceding paragraph will produce such a g'. Since $\pi_1(C_1)\approx\pi_1(C_2)$ and since C_2 is a knot manifold, C_1 is also a knot manifold. Lemma 1.1 says that $D(K_1; \rho, \eta)$ is either a composite knot or a cable knot. But a doubled

knot is either prime or trivial, hence, not composite. Thus, $D(K_1; \rho, \eta)$ is a cable knot. Since $D(K_1; \rho, \eta)$ is nontrivial, its genus is one [19, p. 237]. The only cable knot of genus one is the trefoil [19, Theorem 1, p. 247]; thus, $D(K_1; \rho, \eta)$ is a trefoil knot. Consequently, $\pi_1(C_2)(\approx \pi_1(C_1))$ has a nontrivial center, and $D(K_2; \rho, \eta)$ is, therefore, a torus knot [4]. This means that $D(K_2; \rho, \eta)$ has no companions [19, Theorem 2, p. 250]. Since K_2, by assumption, is nontrivial, we have a contradiction. We can, thus, assume that F is not an annulus. By previous arguments, we can, therefore, assume that each component of $g^{-1}(T^2)$ is a torus.

If a component F of $g^{-1}(T^2)$ is parallel to ∂C_1, then clearly F can be "removed" in the obvious way; cf. the last two paragraphs. Since $g^{-1}(T^2) \neq \emptyset$, we can assume that no component of $g^{-1}(T^2)$ is parallel to ∂C_1. Now a component of $g^{-1}(T^2)$ is also incompressible in C_1; hence, $D(K_1; \rho, \eta)$ has a companion. Therefore, K_1 is not trivial [19, Theorem 4, p. 242].

3. *For each component F of $g^{-1}(T^2)$, we can assume that $g|F$ is a homeomorphism.*

Let x be a point on F. The homomorphism $(g|F)_*$: $\pi_1(F, x) \to \pi_1(T^2, g(x))$ is a monomorphism, since g has properties (4) and (1) and since T^2 is incompressible in C_2. Hence, $g|F$ is homotopic to a covering map k: $F \to T^2$ ([22], [27, Lemma 1.4.3, p. 61]). Since g is transverse with respect to T^2, this homotopy extends to a homotopy $\{h_t\}$ $(0 \leqq t \leqq 1)$ of g that is constant off a small product neighborhood of F. Note that h_{1*} is an isomorphism. Now $\pi_1(F, x) \subseteq h_{1*}^{-1}(\pi_1(T^2, h_1(x)))$; therefore, $h_{1*}(\pi_1(F, x)) = \pi_1(T^2, h_1(x))$ ([6, Theorem 1, p. 575], or [21, Theorem 1.3, p. 161]). Thus, $k = h_1|F$ is a homeomorphism, and our claim **3** is verified.

4. *We can assume that $g^{-1}(T^2)$ is connected.*

A complete proof of this has been given in each of [7, §6, p. 155] and [21, pp. 164–165]. Both proofs are similar and involve essentially three steps; we shall merely list the steps. Suppose $g^{-1}(T^2)$ is not connected. (1) Construct a path α meeting $g^{-1}(T^2)$ only in its endpoints—each endpoint on a different component—so that $g\alpha$ is a null-homotopic loop in C_2. (2) Split C_1 along $g^{-1}(T^2)$. The path α belongs to some component X resulting from this splitting. With α, construct a homotopy from g to a map g' taking X onto T^2. (3) With g', construct a map g'': $C_1 \to C_2$ with the same properties as g, but with one less component in $g''^{-1}(T^2)$ than in $g^{-1}(T^2)$.

We now assume that $g^{-1}(T^2)$ has exactly one component F. The surface F separates C_1 into a knot manifold K_1^3 and a 2-link manifold Λ_1^3. We have $C_1 = K_1^3 \cup_F \Lambda_1^3$; F is not parallel to ∂C_1 (by **2**).

5. $K_1^3 \cong K^3$, *a core of* $S^3 - \operatorname{Int} K_1^3$ *belongs to the same isotopy type as* K_1, *and* $\Lambda_1^3 \cong \Lambda^3$.

Let x be a point in F. Then $\pi_1(C_1, x) = \pi_1(K_1^3, x) \underset{\pi_1(F,x)}{*} \pi_1(\Lambda_1^3, x)$. Because of **4**, exactly one of the groups $g_*(\pi_1(K_1^3, x))$ and $g_*(\pi_1(\Lambda_1^3, x))$ is contained in $\pi_1(K^3, g(x))$, and the other is contained in $\pi_1(\Lambda^3, g(x))$. We also have $g_*(\pi_1(F, x)) = \pi_1(T^2, g(x))$, because of **3**. Since K_1^3 and K^3 are knot manifolds, since Λ_1^3 and Λ^3 are 2-link manifolds, and since g_* is an isomorphism, we have $g_*(\pi_1(K_1^3, x)) = \pi_1(K^3, g(x))$ and $g_*(\pi_1(\Lambda_1^3, x)) = \pi_1(\Lambda^3, g(x))$ [3, Proposition 2.5, p. 485]. Thus, $g(\Lambda_1^3) \subseteq \Lambda^3$, $g(K_1^3) \subseteq K^3$, and each of $(g|\Lambda_1^3)_*$ and $(g|K_1^3)_*$ is an isomorphism. This implies the existence of a homotopy from g to a map $g'\colon C_1 \to C_2$ such that $g'|K_1^3\colon K_1^3 \to K^3$ is a homeomorphism and such that the homotopy is constant on Λ_1^3 [27, Theorem 6.1, p. 77]. Thus, $K_1^3 \cong K^3$, and we can assume that $g|K_1^3$ is a homeomorphism.

We have $C_1 = K_1^3 \cup_F \Lambda_1^3$. From the definition of $D(K_1; \rho, \eta)$, we also have a torus $\mathscr{J} \subset C_1$, a K_1-knot manifold X, and a space $Y \cong \Lambda^3$ such that $C_1 = X \cup_{\mathscr{J}} Y$. Now, by [19, Theorem 1, p. 216; Lemma 3, p. 238], we can assume, after an ambient isotopy of C_1, that $K_1^3 \subset X$; for details, see the first paragraph in the proof of Theorem 3, p. 241 of [19].

Let $\tau'\colon C_2 \to C_1$ denote a homotopy inverse of g. There is a map $\tau\colon C_2 \to C_1$ satisfying four properties: (1) $\tau \simeq \tau'$; (2) τ is transverse with respect to $\mathscr{J}$; (3) $\tau^{-1}(\mathscr{J})$ is a compact orientable surface properly imbedded in C_2; (4) if S is any component of $\tau^{-1}(\mathscr{J})$, then

$$\ker(\pi_j(S) \to \pi_j(C_2)) = 1 \qquad (j = 1, 2).$$

Since $\pi_1(X) \underset{\pi_1(\mathscr{J})}{*} \pi_1(Y)$ is a nontrivial free-product with amalgamation, we have $\tau^{-1}(\mathscr{J}) \neq \emptyset$; the proof is similar to that given in step **1** above. We can assume that $\tau^{-1}(\mathscr{J})$ has exactly one component S, that S is a torus not parallel to ∂C_2, and that $\tau|S\colon S \to \mathscr{J}$ is a homeomorphism; the proofs justifying these assumptions are analogous to those given in steps **2**, **3**, and **4**. Now there is a knot manifold X_2 and a 2-link manifold Y_2 such that $C_2 = X_2 \cup_S Y_2$, and we can assume that $\tau(X_2) = X$ and that $\tau|X_2$ is a homeomorphism; see the first paragraph of the present step **5**.

Recall that $C_1 = K_1^3 \cup_F \Lambda_1^3 = X \cup_{\mathscr{J}} Y$ and that $C_2 = K^3 \cup_{T^2} \Lambda^3 = X_2 \cup_S Y_2$. A core of $S^3 - \operatorname{Int} X_2$ is obviously a companion of $D(K_2; \rho, \eta)$, and the diagonal K_2 of $D(K_2; \rho, \eta)$ is knotted. We can, therefore, move the torus S by an ambient isotopy of C_2 so that $X_2 \subset K^3$ [19, Theorem 1, p. 216; Lemma 3, p. 238]; again, for details, see the first paragraph in the proof of Theorem 3, p. 241 of [19]. But $K_1^3 \subset X$, $X \cong X_2$, and $K_1^3 \cong K^3$. Thus, $K_1^3 \subset X \cong X_2 \subset K^3$, and so the space K^3 contains a knot manifold L_1 homeomorphic to K^3.

If $\partial K_1^3 = F$ is not parallel to $\partial X = \mathscr{J}$, then Lemma 1.2 says that ∂L_1 is incompressible in K^3 and not parallel to ∂K^3. We thus obtain an infinite sequence $K^3 \supset L_1 \supset L_2 \supset \cdots$ of homeomorphic knot manifolds such that, by Lemma 1.3, each torus ∂L_i $(i \geqq 1)$ is incompressible in each L_j $(0 \leqq j < i, L_0 = K^3)$ and is not parallel to ∂L_j $(0 \leqq j < i)$. This shows that the knot K_2 has an infinity of companions. But a knot has only a finite number of companions [20], and we have a contradiction. Hence, F is parallel to $\mathscr{J}$. Therefore, $\Lambda_1^3 \cong Y \cong \Lambda^3$, and a core of $S^3 - \operatorname{Int} K_1^3$ is of the same isotopy type as K_1.

6. *The knots K_1 and K_2 belong to the same isotopy type.*

We have fixed integers ρ and η; ρ, arbitrary; $\eta = \pm 2$. We shall prove **6** for ρ arbitrary and $\eta = +2$. The claim then holds for the pair $(\rho, -2)$: if $\pi_1(C^3(K_1; \rho, -2)) \approx \pi_1(C^3(K_2; \rho, -2))$, then $\pi_1(C^3(K_1^*; -\rho, +2)) \approx \pi_1(C^3(K_2^*; -\rho, +2))$, whence $\{K_1^*\} = \{K_2^*\}$ and, therefore, $\{K_1\} = \{K_2\}$.

Set $\Lambda^3 = \Lambda_2^3$, and choose basepoints of $G_1 = \pi_1(\Lambda_1^3)$ and $G_2 = \pi_1(\Lambda_2^3)$ on F and T^2, respectively. Now read off a presentation for G_i from Fig. 2.

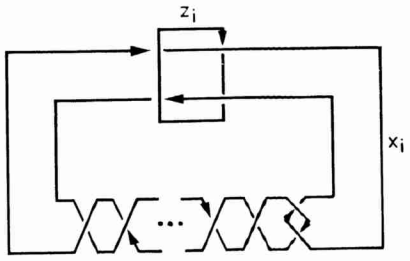

Fig. 2. A Whitehead link

Remark. As in Fig. 1, the orientation of S^3 is assumed fixed so that, with the designated orientation of the twist knot in Fig. 2, $\eta = +2$. The twisting shown in Fig. 2 is negative, but this is accounted for in the presentation (*) below. Fig. 2 depicts a so-called Whitehead link; see [28].

We have

$$(*) \quad G_i = |u_i, z_i, x_i : z_i u_i z_i^{-1} u_i^{-1},\ u_i = x_i u_i^{\rho} z_i^{-1} x_i^{-1} z_i u_i^{-\rho} x_i^{-1} u_i^{\rho} z_i^{-1} x_i z_i u_i^{-\rho}|.$$

The pair (u_i, z_i) is a meridian-longitude pair for the knot K_i. Now abolish the subscripts in G_2. The map $g_* | G_1 : G_1 \to G$ is an isomorphism, and we can assume that $g_*(z_1) = z^{\pm 1}$ and $g_*(u_1) = u^{\pm 1} z^q$ for some integer q, since $g | K_1^3 : K_1^3 \to K^3$ is a homeomorphism. Consider the relation

$$u^{\pm 1} z^q = g_*(x_1)(u^{\pm 1} z^q)^{\rho} z^{\mp 1} g_*(x_1^{-1}) z^{\pm 1} (u^{\pm 1} z^q)^{-\rho} \cdot g_*(x_1^{-1})(u^{\pm 1} z^q)^{\rho} z^{\mp 1} g_*(x_1) z^{\pm 1} (u^{\pm 1} z^q)^{-\rho}$$

in G. Under an abelianization α: $G \to G/G'$, this relation becomes $[\alpha(z)]^q = 1$. But $\alpha(z)$ is a free generator of G/G'; thus, $q = 0$. Consequently, $g_*|\pi_1(K_1^3)$ matches meridians with meridians or their inverses and matches longitudes with longitudes or their inverses; hence, K_1 and K_2 are equivalent.

The orientation of S^3 induces an orientation in each of $\mathcal{V}_1$ $(= A_1^3 \cup V_1)$ and $\mathcal{V}$; in turn, the oriented solid torus $\mathcal{V}_1$ induces an orientation in F; $\mathcal{V}$, an orientation in T^2. A pair of transverse, simple, closed curves on F oriented with respect to an oriented core of $\mathcal{V}_1$ represents the meridian-longitude pair (u_1, z_1); a similar pair of curves on T^2 oriented with respect to K_2 represents the pair (u, z). Each of these pairs of curves has intersection number $+1$ or each has intersection number -1, because there is an orientation-preserving homeomorphism e: $\mathcal{V}_1 \to \mathcal{V}$ satisfying $e_*(u_1) = u$ and $e_*(z_1) = z$. Therefore, to see that $\{K_1\} = \{K_2\}$, we suppose that g does not preserve the intersection numbers; that is, we suppose that one of the following two cases holds: (A) $g_*(z_1) = z$, $g_*(u_1) = u^{-1}$; (B) $g_*(z_1) = z^{-1}$, $g_*(u_1) = u$.

Set $\Gamma = G_1/\langle z_1 u_1^{-(\rho+1)} \rangle$ and $\Omega = G/\langle z u^{\rho+1} \rangle$. A straightforward argument shows that $g_*(\langle z_1 u_1^{-(\rho+1)} \rangle) = \langle z u^{\rho+1} \rangle$ in either case (A) or case (B). Hence, g_* induces an isomorphism $\Gamma \to \Omega$. The corresponding Alexander polynomials are $\Delta_\Gamma(t) = t^2 - t + 1$ and $\Delta_\Omega(t) = -(2\rho+1)t^2 + (4\rho+3)t - (2\rho+1)$, which obviously are distinct unless $\rho = -1$. Therefore, neither case (A) nor case (B) can hold when $\rho \neq -1$.

To show that neither (A) nor (B) holds when $\rho = -1$, set $\rho = -1$ in the presentation (*). The isomorphism g_*, in either case, induces an isomorphism of

$$\Sigma = G_1/\langle z_1 u_1^{-1} \rangle = |u_1, x_1 : u_1 = x_1 u_1^{-2} x_1^{-1} u_1^2 x_1^{-1} u_1^{-2} x_1 u_1^2|$$

onto $\Theta = G/\langle z u \rangle \approx Z$. But $\Delta_\Sigma(t) = 2t^2 - 3t + 2$ and $\Delta_\Theta(t) = 1$, showing that $\Sigma \not\approx \Theta$. Consequently, neither (A) nor (B) can occur, and, therefore, $\{K_1\} = \{K_2\}$, completing the proof of the theorem.

References

1. Bing, R. H.: Necessary and sufficient conditions that a 3-manifold be S^3. Ann. of Math. **68**, 17–37 (1958)
2. Bing, R. H., Martin, J. M.: Cubes with knotted holes. Trans. Amer. Math. Soc. **155**, 217–231 (1971)
3. Brown, E. M.: Unknotting in $M^2 \times I$. Trans. Amer. Math. Soc. **123**, 480–505 (1966)
4. Burde, G., Zieschang, H.: Eine Kennzeichnung der Torusknoten. Math. Ann. **167**, 169–176 (1966)
5. Connor, A. C.: Splittable knots. Ph. D. thesis. University of Georgia, 1969
6. Feustel, C. D.: Some applications of Waldhausen's results on irreducible surfaces. Trans. Amer. Math. Soc. **149**, 575–582 (1970)

7. Feustel, C.D.: A splitting theorem for closed orientable 3-manifolds. Topology **11**, 151–158 (1972)
8. Fisher, G.M.: On the group of all homeomorphisms of a manifold. Trans. Amer. Math. Soc. **97**, 193–212 (1960)
9. Fox, R.H.: On the complementary domains of a certain pair of inequivalents knots. Nederl. Akad. Watensch. Proc. Ser. A **55**, 37–40 (1952)
10. Gluck, H.: The reducibility of embedding problems. Topology of 3-manifolds and related topics (Proc. the Univ. of Georgia Institute, 1961), pp. 182–183. Englewood Cliffs, N.J.: Prentice-Hall 1962
11. González-Acuña, F.: Dehn's construction on knots. Bol. Soc. Math. Mexicana **15**, 58–79 (1970)
12. Hempel, J.: A simply connected 3-manifold is S^3 if it is the sum of a solid torus and the complement of a torus knot. Proc. Amer. Math. Soc. **15**, 154–158 (1964)
13. Kinoshita, S.: On Fox's property of a surface in a 3-manifold. Duke Math. J. **33**, 791–794 (1966)
14. Lyon, H.C.: Simple knots with unique spanning surfaces. (to appear)
15. Nakagawa, Y.: A new class of knots with Property P. (to appear)
16. Neuwirth, L.P.: A note on torus knots and links determined by their groups. Duke Math. J. **28**, 545–551 (1961)
17. Neuwirth, L.P.: Knot Groups. Annals of Mathematics Studies 56. Princeton, N.J.: Princeton University Press 1965
18. Noga, D.: Über den Außenraum von Produktknoten und die Bedeutung der Fixgruppen. Math. Z. **101**, 131–141 (1967)
19. Schubert, H.: Knoten und Vollringe. Acta Math. **90**, 131–286 (1953)
20. Schubert, H.: Über eine numerische Knoteninvariante. Math. Z. **61**, 245–288 (1954)
21. Scott, G.P.: On sufficiently large 3-manifolds. Quart. J. Math. Oxford Ser. **23**, 159–172 (1972)
22. Seifert, H.: Bemerkung zur stetigen Abbildung von Flächen. Abh. Math. Sem. Univ. Hamburg **12**, 29–37 (1938)
23. Simon, J.: Some classes of knots with Property P. Topology of manifolds (Proc. The Univ. of Georgia Institute, 1969), pp. 195–199. Chicago, Ill.: Markham Publishing Company 1970
24. Simon, J.: On knots with non-trivial interpolating manifolds. Trans. Amer. Math. Soc. **160**, 467–473 (1971)
25. Simon, J.: An algebraic classification of knots in S^3. Ann. of Math. **97**, 1–13 (1973)
26. Waldhausen, F.: Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten. Topology **6**, 505–517 (1967)
27. Waldhausen, F.: On irreducible 3-manifolds which are sufficiently large. Ann. of Math. **87**, 56–88 (1968)
28. Whitehead, J.H.C.: On doubled knots. J. London Math. Soc. **12**, 63–71 (1937)
29. Whitehead, J.H.C.: Combinatorial homotopy, I. Bull. Amer. Math. Soc. **55**, 213–245 (1949)

Wilbur Whitten
Fine Hall, Box 37
Princeton University
Princeton, N.J. 08540, USA

Current address:
Department of Mathematics
University of Southwestern Louisiana
Lafayetta, La. 70501, USA

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