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On a Class of Pseudodifferential Operators with Double Characteristics*

Louis Boutet de Monvel (Paris) and François Trèves (New Brunswick)

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Introduction

The present article is concerned with linear partial differential (or pseudodifferential) operators having double characteristics, those which are essentially of the form $P = X^* X + R$, where X satisfies a subelliptic estimate (with loss of 1/2 derivative) and R is a lower order term (actually we shall allow more general leading term than $X^* X$, but this will only introduce inessential modifications in the argument outlined below). Since the best hypoelliptic estimate for $X^* X$ involves the loss of *one* derivative, the "perturbation" R cannot be absorbed in the leading term. It has been observed in several particular cases ([4, 6, 7]) that, in order that P be hypoelliptic, the symbol of R must avoid a discrete set of values on the characteristic variety of X. We shall derive, here, necessary and sufficient conditions on R in order that P be hypoelliptic with loss of one derivative. They will turn out to be precisely of that kind. Our conditions will also imply a local existence theorem for P, with loss of one derivative.

The paper is an outgrowth of [10], in which the same phenomenon is described in the abstract set-up (i.e., P is a second-order ordinary differential operator whose coefficients are unbounded operators on an abstract Hilbert space).

Roughly speaking the idea of the proof is the following: we shall write $P = XX^* + R'$, $R' = R + [X^*, X]$, and observe that if P satisfies a hypo-

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elliptic estimate with a loss of one derivative, then R' must satisfy the same estimate, when restricted to $\operatorname{Ker} X^*$. Now, we dispose of a very precise description of $\operatorname{Ker} X^*$ (mod C^∞) due to Sjöstrand and Duistermaat ([2, 9]). It enables us to show that R' must be *elliptic* on the characteristic variety of X. The next step is to notice that the range of X, which is the "orthogonal complement" of $\operatorname{Ker} X^*$, is essentially stable by P (this is precisely formulated in Section 5) and, on it P behaves essentially as an operator P_1 of the same form but with R replaced by $R+[X^*, X]$. Repeating this procedure leads to a sequence ("concatenation") of operators P_j, where P_j is approximately equal to $P+j[X^*, X]$. Since $[X^*, X]$ is positive-elliptic on the characteristic set of P, P_j will be hypoelliptic for sufficiently large j, which makes possible an inductive proof.

However, in order to carry on such an induction, estimates in Sobolev spaces are not sufficient; more precise ones are needed, and these require the use of a scale of spaces of distributions (denoted by $\mathscr{H}^{s,k}$) closely linked to the characteristic manifold Σ (see Sections 2, 3, 4, and [1, 5]).

The present work is closely related to that of Grushin and Vishik (see [4–7]). As a matter of fact, in the particular case where Σ is the cotangent bundle of a submanifold of the base Ω, their results are considerably more general than ours. They introduce the spaces $\mathscr{H}^{s,k}$ relative to this situation: and also what we call *Hermite operators* and describe in Section 6. Their approach is very different from ours: they construct a parametrix and thus, in the "flat" special case, it is stronger than ours. The drawbacks of their approach is that the parametrices they describe dont seem to lend themselves to a "good" symbolic calculus. In particular, it is not clear how to "microlocalize" their computations. In our approach this is straightforward. The microlocalization reduces us to the flat case, where we prove or disprove the relevant estimates and there is no difficulty in patching the results together, in the cotangent bundle $T^*\Omega$. The difficulties inherent to handling this type of problem (where the characteristics are double) with parametrices, and in trying to devise a symbolic calculus for the latter, become apparent when one realizes that the commutators which recur lead to *non*negligible perturbation (these perturbations essentially determine whether the operator will be hypoelliptic or not).

The problem of the hypoellipticity (or solvability) of the operator P under study is transferred to that for another operator (which we denote by L_P), this time on the boundary – though in our case there is no boundary: only microlocally can we introduce a submanifold of the base which plays the role of a boundary. The *total* symbol of this pseudodifferential operator "lives" on the characteristic manifold Σ – it should be said however that the operator is uniquely defined only up to similitude (i.e. the class of operators $A^{-1}L_P A$ is defined, where A is elliptic arbitrary);

its similitude class, and the associated class of total symbols, are canonically attached to P (actually, there is a sequence L_{P_j} of such operators each attached to an element of the concatenation $\{P_j\}$; but, at each point of the characteristic manifold, only one of them is not elliptic, possibly and this is the one that controls the properties of P – at that point). In the last section (# 8) of the paper the reader will find a few remarks about these operators L_P and their construction.

0. Notations and Conventions

Throughout the article, Ω will stand for an N-dimensional C^∞ manifold, countable at infinity. The variable point in Ω will be denoted by x, its coordinates, in some local chart, by $x^1, \dots, x^N$. By $T^*\Omega$ we denote the cotangent bundle over Ω, by $\dot{T}^*\Omega$ the complement in it of the zero section. The variable along the fibres in $T^*\Omega$ will be called ξ, its coordinates in some local chart, $\xi_1, \dots, \xi_N$. The canonical symplectic form on $T^*\Omega$ will be called ω; in a local chart,

$$\omega = \sum_{j=1}^{N} d\xi_j \wedge dx^j.$$

If u, v are two functions, defined and C^1 in an open subset O of $T^*\Omega$, their Poisson bracket will be

$$\{u, v\} = \sum_{j=1}^{N} \left(\frac{\partial u}{\partial \xi_j} \frac{\partial v}{\partial x^j} - \frac{\partial u}{\partial x^j} \frac{\partial v}{\partial \xi_j} \right),$$

and as usual the Hamiltonian field H_u of u is defined by the formula $H_u v = \{u, v\}$.

Unless otherwise specified, all pseudodifferential operators considered in the present article have total symbols which are (asymptotic) sums of positive-homogeneous terms. "Positive-homogeneous" is always meant with respect to the fibre variable, ξ. Furthermore, the homogeneity degrees of any two terms in the above asymptotic series will always differ by an integer (the degrees themselves need not be integers, of course). The *principal* symbol of a pseudodifferential operator A will often be denoted by $\sigma(A)$. More generally, we use the standard notation of distribution theory and of the calculus of pseudodifferential operators.

In particular, we recall that a subset Γ of $\dot{T}^*\Omega$ is said to be *conic* if it is stable under the dilations $(x, \xi) \mapsto (x, t\xi)$, $t > 0$. This means that Γ is equal to the preimage of its (canonical) projection $\pi\Gamma$ into the cosphere bundle $S^*\Omega$ over Ω. If Γ' is another conic subset of $\dot{T}^*\Omega$, we shall write $\Gamma' \subset\subset \Gamma$ to express the fact that the closure of $\pi\Gamma'$ is a compact subset of $\pi\Gamma$.

Let $\Gamma \subset \dot{T}^*\Omega$ be conic. We denote by $S^0(\Gamma)$ the space of symbols of degree $\leqq 0$ (in $T^*\Omega$) whose support is contained in a conic subset $\Gamma' \subset\subset \Gamma$.

We depart somewhat from the standard conventions in the terminology used to describe *microlocal* (i.e., local in $T^*\Omega$) properties.

We shall say that a space of distributions in Ω, E, is *microlocal* if it stable under all the mappings $f \mapsto q(x, D) f$ as $q(x, \xi)$ ranges over $S^0(T^*\Omega)$, and if its topology can be defined by means of seminorms of the kind $\mathscr{P}(q(x,D)f)$, where q ranges over $S^0(T^*\Omega)$ and $\mathscr{P}$ over some set of continuous seminorms on E. One of the basic theorems in the theory of pseudodifferential operators states that $\mathscr{D}'(\Omega)$, $C^\infty(\Omega)$ and the local Sobolev spaces $H^s_{loc}(\Omega)$, $s \in \mathbb{R}$, are all microlocal.

Let then E be a microlocal space of distributions in Ω, and let Γ be a conic open subset of $\dot{T}^*\Omega$. We shall say that a distribution f in Ω *belongs to E in Γ* if $q(x, D) f \in E$ whatever $q \in S^0(\Gamma)$. In order to alleviate a little bit such expressions we shall omit any mention of Ω in the notation for E (for instance, we write C^∞ instead of $C^\infty(\Omega)$, H^s_{loc} instead of $H^s_{loc}(\Omega)$).

Incidentally, "$f \in C^\infty$ in Γ" means that the wave-front set $WF(f)$ does not intersect Γ.

We shall use a similar terminology in dealing with a pseudodifferential operator A in Ω.

1. Statement of the Main Theorem

One of the two basic ingredients in our analysis is a *conic* C^∞ submanifold Σ of $\dot{T}^*\Omega$, of codimension *two*, to which the restriction of the canonical form ω is *nondegenerate*.

Every point (x_0, ξ^0) of Σ has a conic open neighborhood Γ_0 in $\dot{T}^*\Omega$ in which Σ can be defined by two real equations:

(1.1) $$u = v = 0$$

where u and v are two real $\mathscr{C}^\infty$ functions, homogeneous of degree 0, whose Poisson bracket satisfies

(1.2) $$\{u, v\} > 0 \quad \text{in } \Gamma_0.$$

The second basic ingredient will be a pseudodifferential operator P of order (or degree) m in Ω, with principal symbol $\sigma(P) = p(x, \xi)$. It will be submitted to the following conditions:

(1.3) *the set of zeros of $p(x, \xi)$ in $\dot{T}^*\Omega$ is exactly Σ;*

(1.4) *p vanishes exactly of order two on Σ;*

(1.5) *the winding number of p about Σ is identically zero.*

(1.4) means that for any compact set $K \subset \dot{T}^* \Omega$, there exist two constants c, $C > 0$ such that, $d(x, \xi)$ denoting the distance to Σ

$$c \leqq p(x, \xi)/(d(x, \xi))^2 \leqq C \quad \text{for } (x, \xi) \in K - \Sigma .$$

Moreover, (1.5) means that, if (x_0, ξ^0) is an arbitrary point of $\dot{T}^* \Omega$ and if Σ' is an arbitrary two-dimensional C^∞ surface through (x_0, ξ^0) transversal to Σ, the restriction of p to Σ' defines a mapping into $\mathbb{R}^2$ whose winding number at (x_0, ξ^0) is zero ((1.4) implies that the hessian of $p|_{\Sigma'}$ at that point is nondegenerate).

Let us reason microlocally, in the conic neighborhood Γ_0 of $(x_0, \xi^0) \in \Sigma$ and use the functions u, v introduced above. The properties (1.3)–(1.4)–(1.5) mean that we have

$$p = a u^2 + 2 b u v + c v^2, \tag{1.6}$$

with a, b, c smooth in Γ_0, positive-homogeneous of degree $m = \deg p$ (we recall that u and v are homogeneous of degree zero). Moreover, the polynomial $a\lambda^2 + 2b\lambda + c$ must have exactly two roots α, β such that

$$\operatorname{Im} \alpha < 0, \quad \operatorname{Im} \beta > 0 \quad \text{in } \Gamma_0 . \tag{1.7}$$

Since these roots are distinct, they are smooth functions, homogeneous of degree zero, in Γ_0.

Let now X be any pseudodifferential operator in Ω whose principal symbol is of the form $q(u - \alpha v)$ in Γ_0, with q elliptic; let Y another operator such that $\sigma(Y) = q^{-1} a(u - \beta v)$ in Γ_0. Then

$$Z = P - XY \tag{1.8}$$

is of degree $\leqq m-1$ in Γ_0. We observe that, *on* $\Sigma \cap \Gamma_0$,

$$\sigma([X, Y]) = -i(\alpha - \beta)\, a \{u, v\}, \tag{1.9}$$

$$\sigma([X, X^*]) = -i(\alpha - \bar{\alpha})\, |q|^2 \{u, v\}, \tag{1.10}$$

$$\sigma([Y, Y^*]) = -i(\beta - \bar{\beta})\, |q^{-1} a|^2 \{u, v\}. \tag{1.11}$$

In view of (1.2) and (1.7), we derive from this, if Γ_0 is sufficiently "thin"

(1.12) $[X, Y]$ *is elliptic in* Γ_0;

(1.13) $[X^*, X]$ *is positive-elliptic in* Γ_0;

(1.14) $[Y^*, Y]$ *is negative-elliptic in* Γ_0.

We shall denote by $\sigma_{m-1}(Z)$ the principal symbol of Z regarded as a pseudodifferential operator of order $m-1$: this means that if Z happens to be of order $< m-1$, $\sigma_{m-1}(Z) = 0$. In passing, note that $[X, Y]$ is also of order $m-1$.

We set:

(1.15) $l_P = restriction\ to\ \Sigma \cap \Gamma_0\ of\ \sigma_{m-1}(Z)/\sigma([X, Y])$.

The omission in the notation l_P of any mention of Γ_0 is justified by the fact that l_P is defined in the whole of Σ — a consequence of the fact that its definition is independent of the decomposition $P = XY + Z$, as we now show:

Let $P = X' Y' + Z'$ denote another decomposition in Γ_0, such that $[X'^*, X']$ is positive-elliptic in Γ_0 and the order of Z' is $\leqq m-1$. Then necessarily the principal symbol of X' (resp. Y') is proportional to that of X (resp. Y), and we have:

$$X' = (X - R)Q, \qquad Y' = Q^{-1}(Y - S), \qquad Z' = Z + XS + RY - RS,$$

with Q elliptic in Γ_0, $\deg R \leqq \deg X - 1$, $\deg S \leqq \deg Y - 1$. Since $\deg XY = \deg P = m$, we see that $XS + RY$ is of order $m-1$. But its principal symbol vanishes on Σ; RS is of order $m-2$. From this it follows at once that $\sigma([X', Y']) = \sigma([X, Y])$, $\sigma(Z') = \sigma(Z)$ on $\Sigma \cap \Gamma_0$, whence our assertion.

Proposition 1.1. *Let P be a pseudodifferential operator in Ω, satisfying the conditions* (1.3), (1.4), (1.5). *Let l_P be the function on Σ defined above. Then:*

(1.16) *if P^* denotes the adjoint of P, $l_{P^*} = \bar{l}_P$:*

(1.17) *if A, B denote two elliptic pseudodifferential operators in Ω,*

$$l_{APB} = l_P.$$

The proof is very simple and we leave it to the reader.

We also note that l_P is "invariant" under canonical transformations. Let us state this more precisely. Let $\mathscr{T}$ be a canonical transformation of a conic open set $\Gamma \subset \dot{T}^*\Omega$ onto another such set, Γ', and U, an elliptic Fourier integral operator associated with $\mathscr{T}$. Then, whatever the pseudodifferential operator Ψ in Ω,

$$\sigma(U^{-1}\Psi U)|_{\Gamma} = (\sigma(\Psi)|_{\Gamma'}) \circ \mathscr{T}. \tag{1.18}$$

Then we have:

$$l_{U^{-1}PU}|_{\check{\Sigma}} = (l_P|_{\Sigma \cap \Gamma}) \circ \mathscr{T}, \tag{1.19}$$

where we have written $\check{\Sigma} = \overset{-1}{\mathscr{T}}(\Sigma \cap \Gamma')$. It suffices to recall that Poisson brackets are "preserved" under canonical transformations.

In summary, l_P is an invariant attached to our operator P: invariant under multiplication by elliptic operators and under canonical transformations. Note that it is essentially a symbol of degree zero but which is only defined on a proper submanifold of $\dot{T}^*\Omega$, namely Σ. Introduction of this invariant will now enable us to state the theorem proved in this article.

We are interested in the validity of the following *hypoellipticity* property:

(1.20) *Given any real number s, any open subset $\mathscr{U}$ of Ω and any distribution f in $\mathscr{U}$,*

$$Pf \in H^s_{loc}(\mathscr{U}) \Rightarrow f \in H^{s+m-1}_{loc}(\mathscr{U}).$$

Theorem 1.1. *Let P be a pseudodifferential operator in Ω, having the properties* (1.3), (1.4), (1.5).

The following conditions are equivalent:

(1.21) *P satisfies* (1.20);

(1.22) *P^* satisfies* (1.20);

(1.23) *whatever the integer $k \geqq 0$, $l_P - k$ does not vanish at any point of Σ.*

It is well known that Property (1.20) has various implications. Of course, it implies that P is hypoelliptic and that the adjoint P^* of P is locally solvable in Ω, also that the solutions of the homogeneous equation $Pf=0$ (in Ω) which have their support in a fixed compact subset K of Ω form a finite dimensional vector space.

2. The Spaces $\mathscr{H}^{s,k}(\Omega, \Sigma)$

Same notation as in Section 1 (in particular, Ω and Σ).

Definition 2.1. Let d be any real number, k any integer $\geqq 0$. We shall denote by $\mathscr{N}^{d,k} = \mathscr{N}^{d,k}(\Sigma)$ the linear space of pseudodifferential operators in Ω whose (total) symbol is of the form

$$a \sim \sum_{j=0}^{+\infty} a_j, \tag{2.1}$$

where, for each j, a_j is positive-homogeneous of degree $d-j$ (along the fibres) and vanishes at least of order $(k-2j)_+$ on Σ.

We have used the notation $n_+ = \sup(n, 0)$, $n \in \mathbb{Z}$. The condition on the homogeneous terms a_j (which, except for a_0, have no invariant meaning) is easily seen to be invariant, under canonical transformations.

Proposition 2.1. *Let $A \in \mathscr{N}^{d,k}$, $B \in \mathscr{N}^{d',k'}$, Then:*

$$A^* \in \mathscr{N}^{d,k}, \quad AB \in \mathscr{N}^{d+d',k+k'}; \tag{2.2}$$

$$[A, B] \in \mathscr{N}^{d+d'-1,k''} \quad \text{where } k'' = (k-1)_+ + (k'-1)_+ . \tag{2.3}$$

Note that, when $kk'=0$, we have $k''=(k+k'-1)_+$ in (2.3). In particular:

Corollary 2.1. *Let $A\in\mathcal{N}^{d,k}$ and Q be an elliptic pseudodifferential operator in Ω. Then:*

$$A-QAQ^{-1}\in\mathcal{N}^{d-1,k-1}. \tag{2.4}$$

Note also that we have, trivially,

$$\mathcal{N}^{d,k}\subset\mathcal{N}^{d',k'} \quad \text{if } d\leqq d',\ k\geqq k' \quad (d-d'\in\mathbb{Z}). \tag{2.5}$$

$\mathcal{N}^{0,k}$ is an ideal of the algebra of pseudodifferential operators of degree 0, $\mathcal{N}^{0,0}$. In fact it follows from (2.6) hereunder that $\mathcal{N}^{0,k}$ is the k-th power of $\mathcal{N}^{0,1}$.

Microlocally, a more "concrete" description of $\mathcal{N}^{d,k}$ is possible. We return to the Eqs. (1.1) of Σ in a conic neighborhood of one of its points, (x_0,ξ^0). We assume that (1.2) holds. Let U (resp. V) be a pseudodifferential operator in Ω whose principal symbol is equal to u (resp. v) in Γ_0. Then, if a pseudodifferential operator A belongs to $\mathcal{N}^{d,k}$, we must have

$$A=\sum_{\alpha+\beta\leqq k} A_{\alpha,\beta}U^\alpha V^\beta \quad \text{in } \Gamma_0, \tag{2.6}$$

where, for each (α,β), $A_{\alpha,\beta}$ is a pseudodifferential operator of degree $\leqq d-(\alpha+\beta)/2$. This property characterizes the elements of $\mathcal{N}^{d,k}$ in Γ_0.

Remark 2.1. It is worth noting that several choices of U, V are of some interest. We may select local *canonical* coordinates (for the symplectic structure of $\dot{T}^*\Omega$) in Γ_0 (if the latter is sufficiently "thin"), $(x^1,\dots,x^N,\xi_1,\dots,\xi_N)$, with the x's homogeneous of degree zero, the ξ's homogeneous of degree one, verifying the canonical relations

$$\{x^i,x^j\}=\{\xi_i,\xi_j\}=0,\qquad \{\xi_i,x^j\}=\delta_{ij}\quad \text{(Kronecker's index)},$$

and such that the N-th coordinate function x^N may be taken as u, while $v=-\xi_N/|\xi'|$, where $\xi'=(\xi_1,\dots,\xi_{N-1})$. Note that $\{u,v\}=|\xi'|^{-1}$ and that (1.2) is therefore satisfied. A more elaborate choice of local canonical coordinates will be made in § 7 (cf. also [2, 3]).

If we relinquish the demand that u and v be of degree zero, we may even achieve $u=x^N$, $v=-\xi_N$. Thus the operators belonging to $\mathcal{N}^{d,k}$ will be of the form (in Γ_0)

$$A=\sum_{\alpha+\beta\leqq k} A_{\alpha,\beta}(x^N)^\alpha(\partial/\partial x^N)^\beta, \tag{2.7}$$

with the orders of the operators $A_{\alpha,\beta}$ restricted in a manner analoguous to that indicated above.

The sets of operators $\mathcal{N}^{d,k}$ enable us to define certain special spaces of distributions in Ω, naturally associated with the submanifold Σ, which provide a precise description, and a "natural" proof of the *a priori estimates* leading to Th. 1.1:

Definition 2.2. Let s be any real number, k any integer $\geqq 0$. We shall denote by $\mathscr{H}_{loc}^{s,k} = \mathscr{H}_{loc}^{s,k}(\Omega, \Sigma)$ the space of distributions f in Ω having the following property:

(2.8) whatever $A \in \mathscr{N}^{d,l}$, $l \leqq k$, $Af \in H_{loc}^{s-d-(k-l)/2}(\Omega)$.

Observe that $\mathscr{H}_{loc}^{s,0}(\Omega, \Sigma) = H_{loc}^{s}(\Omega)$.

The proofs of the statements which now follow are routine, and left to the reader.

The space $\mathscr{H}_{loc}^{s,k}$ carries a natural topology: the coarsest locally convex one which renders all the mappings $f \mapsto Af$ into $H_{loc}^{s-d-(k-l)/2}(\Omega)$, with A as in (2.8), continuous. If then K is an arbitrary compact subset of Ω we denote by $\mathscr{H}_c^{s,k}(K)$ the (closed) linear subspace of $\mathscr{H}_{loc}^{s,k}$ consisting of the distributions f vanishing outside K, and by $\mathscr{H}_c^{s,k} = H_c^{s,k}(\Omega, \Sigma)$ the inductive limit of the Fréchet spaces $\mathscr{H}_c^{s,k}(K)$, $K \subset\subset \Omega$. The spaces $\mathscr{H}_{loc}^{s,k}$ and $\mathscr{H}_c^{s,k}$ are *normal* spaces of distributions in Ω, i.e., they contain $C_c^\infty(\Omega)$ as a dense subspace. This enables us to identify their respective duals with spaces of distributions in Ω. In all rigour we should use the language of currents, but we shall not, for the sake of simplicity.

Definition 2.3. Let s, k be as in Def. 2.2. We denote by $\mathscr{H}_{loc}^{s,-k} = \mathscr{H}_{loc}^{s,-k}(\Omega, \Sigma)$ the dual of $\mathscr{H}_c^{-s,k}$.

We can of course form $\mathscr{H}_c^{s,-k}(K)$, $\mathscr{H}_c^{s,-k} = \mathscr{H}_c^{s,-k}(\Omega, \Sigma)$. All these spaces closely mimick the classical Sobolev spaces. In particular, they are all reflexive; $\mathscr{H}_{loc}^{s,k}$ and $\mathscr{H}_c^{s,k}$ are normal, for all values of $s \in \mathbb{R}$, $k \in \mathbb{Z}$.

Proposition 2.2. *Every pseudodifferential operator* $A \in \mathscr{N}^{d,l}$ $(l \in \mathbb{Z}_+)$ *defines a continuous linear map* $\mathscr{H}_{loc}^{s,k} \to \mathscr{H}_{loc}^{s-d,k-l}$ $(s \in \mathbb{R}, k \in \mathbb{Z})$.

Proposition 2.3. *Let* $s, s' \in \mathbb{R}$, $k, k' \in \mathbb{Z}$ *be such that*

$$s' \leqq s, \quad s' - k'/2 \leqq s - k/2. \tag{2.9}$$

Then we have a continuous injection $\mathscr{H}_{loc}^{s,k} \hookrightarrow \mathscr{H}_{loc}^{s',k'}$.

Corollary 2.2. *We have* $\mathscr{H}_{loc}^{s,k} \hookrightarrow$ *or* $\hookleftarrow H_{loc}^{s-k/2}(\Omega)$ *according to whether* k *is* >0 *or* $\leqq 0$.

Let K be a compact subset of Ω; the spaces $H_c^s(K)$, $\mathscr{H}_c^{s,k}(K)$ are "normable" (in fact, they can be equipped with a Hilbert space structure). Let us select in each one of them a norm, $\| \; \|_s$ and $\| \; \|_{s,k}$, submitted to the sole requirement that it define the topology of the space.

Proposition 2.3′. *Suppose that*

$$s' < s, \quad s' - k'/2 < s - k/2. \tag{2.9′}$$

Then the injection $\mathscr{H}_c^{s,k}(K) \hookrightarrow \mathscr{H}_c^{s',k'}(K)$ *is compact and, given any real number* s_1 (arbitrarily close to $-\infty$) *and any* $\varepsilon > 0$, *there is* $C_\varepsilon > 0$ *such that, for all* $f \in \mathscr{H}_c^{s,k}(K)$,

$$\|f\|_{s',k'} \leqq \varepsilon \|f\|_{s,k} + C_\varepsilon \|f\|_{s_1}. \tag{2.10}$$

The easiest way to prove the above statements is by using a microlocal description of the spaces $\mathscr{H}_{loc}^{s,k}$ and the analoguous properties of the spaces H^s. Let Γ_0 be, as before, a conic open neighborhood of $(x_0, \xi^0) \in \Sigma$ and let u, v be the functions introduced earlier and appearing in (1.1) and (1.2). Let us now assume that they are real-valued and let U, V be two pseudodifferential operators in Ω whose principal symbols are respectively equal to u and v in Γ_0. We take U and V to be self-adjoint (in the whole of Ω if we wish).

Let then k be any integer $\geqq 0$. Possibly after some shrinking of Γ_0 about its axis, the following can be said.

If a distribution f belongs to $\mathscr{H}_{loc}^{s,k}$, then:

$$U^\alpha V^\beta f \in H_{loc}^{s-(k-\alpha-\beta)/2} \quad in\ \Gamma_0, \qquad \forall \alpha, \beta \in \mathbb{Z}_+, \alpha + \beta \leqq k. \tag{2.11}$$

If $f \in \mathscr{H}_{loc}^{s,-k}$, then:

(2.12) *there are distributions* $g_{\alpha,\beta} \in H_{loc}^{s-(|k|-\alpha-\beta)/2}(\Omega)$, $\alpha, \beta \in \mathbb{Z}_+$, $\alpha + \beta \leqq |k|$, *such that*

$$f - \sum_{\alpha+\beta \leqq |k|} U^\alpha V^\beta g_{\alpha,\beta} \in C^\infty \quad in\ \Gamma_0,$$

Furthermore, these properties are characteristic: if (2.11) holds, there is an element $f_1 \in \mathscr{H}_{loc}^{s,k}$ equal to f in a conic subneighborhood Γ_0' of (x_0, ξ^0); if every point of Σ has a conic neighborhood Γ_0 such that (2.11) holds, and if $f \in H_{loc}^s$ in every conic open set which does not intersect Σ, then f belongs to $\mathscr{H}_{loc}^{s,k}$. Same remark about (2.12).

In connection with these properties we mention the following result:

Proposition 2.4. *Let* $k \in \mathbb{Z}$ *be arbitrary.*

i) *A distribution* f *in* Ω *belongs to* $\mathscr{H}_{loc}^{s,k}$ *in* Γ_0 *if and only if both* Uf *and* Vf *belong to* $\mathscr{H}_{loc}^{s,k-1}$ *in* Γ_0;

ii) f *belongs to* $\mathscr{H}_{loc}^{s,-k}$ *in* Γ_0 *if and only if there are two distributions* g, h *belonging to* $\mathscr{H}_{loc}^{s,-k+1}$ *such that*

$$f - (Ug + Vh) \in \mathscr{C}^\infty \quad in\ \Gamma_0.$$

The complete proof of Prop. 2.4 will only be given in the next section. Here we shall limit ourselves to the case $k \geqq 1$, which will be needed in the proof of the general case, and in that of the statements of Section 3.

Proof. Thus $k \geqq 1$. We shall limit ourselves to the proof of i); ii) follows by duality. The "only if" part of i) is evident. Suppose that Uf and Vf

belong to $\mathscr{H}^{s,k-1}_{loc}$ in Γ_0. By (2.11) this means that

(2.13) $U^\alpha V^\beta f \in H^{s-(k-\alpha-\beta)/2}_{loc}$ *in* Γ_0, $\forall \alpha, \beta \in \mathbb{Z}_+$, $0 < \alpha + \beta \leqq k$,

The difference between (2.11) and (2.13) is that in the latter $\alpha+\beta$ must be >0. In particular we see that

$$Uf, Vf \in H^{s-(k-1)/2}_{loc} \text{ in } \Gamma_0. \tag{2.14}$$

We use now the fact that $W = i[U, V]$ is a pseudodifferential operator of order -1, whose principal symbol is equal to $\{u, v\} > 0$ in Γ_0. We may construct a pseudodifferential operator T of order $-1/2$ in Ω whose square is equal to W in Γ_0, and which is >0 in Γ_0. We see at once that, for all $q \in S^0(\Gamma_0)$, for a suitable constant $C > 0$ and all $\varphi \in \mathscr{C}^\infty_c$,

$$\left| \|Tq(x, D)\varphi\|^2_{s-\frac{k-1}{2}} - 2\,\mathrm{Re}\big(U q(x, D)\varphi, i\, Vq(x, D)\varphi\big)_{s-\frac{k-1}{2}} \right| \leqq C \|\varphi\|^2_{s'}$$

where s' can be chosen arbitrarily close to $-\infty$ (the constant C depends on the choice of q and s' but not on φ). It follows at once from this and from (2.14) that

$$Tf \in H^{s-(k-1)/2}_{loc} \quad \text{in } \Gamma_0.$$

By the ellipticity, of order $-1/2$, of T in Γ_0 we reach the conclusion that $f \in H^{s-k/2}_{loc}$ in Γ_0 which, together with (2.13), implies (2.11). Q.E.D.

Actually we may introduce the operator $L_0 = U + i\,V$ and form

$$L_0^* L_0 = U^2 + V^2 + i[U, V].$$

We make now Γ_0 range over a locally finite open covering of Σ by conic open sets Γ_i $(i = 1, 2, \ldots)$ of the kind we have been considering and introduce a partition of unity $\{q_i\}$ subordinate to this covering, and consisting of symbols $q_i(x, \xi) \geqq 0$ homogeneous of degree zero. Noting that the Γ_i might not form a covering of the whole $T^*\Omega$, we set $q_\infty = 1 - \sum_i q_i$. We form then

$$\mathscr{P} = \sum_i q_i(x, D)\, L_i^* L_i + q_\infty(x, D). \tag{2.15}$$

We see that $\mathscr{P}$ is a pseudodifferential operator of order zero in Ω, whose principal symbol is everywhere $\geqq 0$ and vanishes exactly of order two on Σ (and vanishes only on Σ). In particular, it has the properties (1.3), (1.4), (1.5); it belongs to $\mathscr{N}^{0,2}$. Thus (Prop. 2.2) it defines a continuous linear map $\mathscr{H}^{s,k}_{loc} \to \mathscr{H}^{s,k-2}_{loc}$, whatever $s \in \mathbb{R}, k \in \mathbb{Z}$. It can be shown (and will follow from the results of the next section) that $\mathscr{P}$ is actually an *isomorphism* of $\mathscr{H}^{s,k}_{loc}(\Omega, \Sigma)/\mathscr{C}^\infty(\Omega)$ onto $\mathscr{H}^{s,k-2}_{loc}(\Omega, \Sigma)/\mathscr{C}^\infty(\Omega)$. Among other things, this provides a simple description of the spaces $\mathscr{H}^{s,k}_{loc}$ when $k = 2j, j \in \mathbb{Z}$. Indeed, since $\mathscr{H}^{s,0}_{loc} = H^s_{loc}$, we see that $f \in \mathscr{H}^{s,2j}_{loc}$ is equivalent with the following property:

(2.16) *if* $j \geqq 0$, $\mathscr{P}^j f \in H^s_{loc}(\Omega)$; *if* $j<0$, *there is* $g \in H^s_{loc}(\Omega)$ *such that* $f-\mathscr{P}^{-j} g \in \mathscr{C}^\infty(\Omega)$.

When $k=2j+1$, the role of H^s_{loc} in (2.16) can be played by $\mathscr{H}^{s,1}_{loc}$.

Of course $\mathscr{P}$ is not unique in having these properties. Nevertheless the similarity with the elliptic situation is clear: in many respects $\mathscr{P}$ plays the role of $1-\Delta$, where Δ is the Laplace-Beltrami operator in some Riemannian metric in Ω.

If we do use a Riemmanian metric on Ω, we can define norms $\|j\|_{s,k}$ for distributions $f \in \mathscr{H}^{s,k}_c$. Indeed, we can then define norms $\|f\|_s$ for any s real and $f \in H^s_c$. We may then use a locally finite covering of Σ by conic open sets Γ_i and for each i, operators U_i, V_i of the kind described above. Let $\{q_i\}$ be a partition of unity in a neighborhood of Σ subordinate to the preceding covering and consisting of symbols which are homogeneous of degree zero. As before, let us set $q_\infty = 1-\sum_i q_i$. Clearly it suffices to define the norm $\| \ \|_{s,k}$ on each space $\mathscr{H}^{s,k}_c(K)$, K: an arbitrary compact subset of Ω, and later on, glue such norms together by means of a partition of unity in Ω. Let then $\chi \in \mathscr{C}^\infty_c(\Omega)$ be equal to 1 in a neighborhood of K. If $k \geqq 0$ and $f \in \mathscr{H}^{s,k}_c(K)$ we may set

$$\|f\|_{s,k} = \left\{ \sum_{i=1}^{+\infty} \sum_{\alpha+\beta \leqq k} \|\chi q_i(x,D)\, U_i^\alpha V_i^\beta f\|^2_{s-\frac{1}{2}(k-\alpha-\beta)} + \|\chi q_\infty(x,D) f\|^2_s \right\}^{\frac{1}{2}}. \tag{2.17}$$

Notice that, in the summation with respect to i, only finitely many terms are not zero—for $\chi(x)\, q_i(x,\xi) \equiv 0$ except for finitely many i's. Also observe that $\|f\|_{s,k}=0$ implies $q_i(x,D) f=0$ in a neighborhood of K for all $i \leqq +\infty$; but $q_\infty(x,D)+\sum_i q_i(x,D) = \text{Identity}$, hence $f=0$.

If $k<0$, the norm of $\mathscr{H}^{s,k}_c(K)$ can be defined by duality.

3. A Class of Stable Microlocal Estimates

Property (1.20) will be obtained via a microlocal estimate involving the $\mathscr{H}^{s,k}$-norms (we define norms $\| \ \|_s$ and $\| \ \|_{s,k}$ by means of a Riemannian metric chosen once for ever on Ω). In the present section we wish to investigate the interrelation between the estimate in question and microlocal existence and regularity results, and also the dependence of such results on the indices s and k.

Throughout the section Γ will denote a conic open set in $\dot{T}^*\Omega$. For the sake of simplicity we assume that the projection of Γ in the base Ω

is relatively compact and that there exist two symbols u, v in Γ, homogeneous of degree zero, real, such that $u=v=0$ define Σ and $\{u,v\}>0$ in Γ (for obvious reasons we are solely interested in sets which intersect Σ). Let us then list the three type of properties we are interested in. They will apply to an operator $P\in\mathcal{N}^{m,l}$, $l>0$ (in Section 1 we had $l=2$ but for our present purposes this limitation is unnecessary).

We begin by the property which is closest to (1.20) – which, as a matter of fact, is the microlocalization of a more precise version of (1.20):

$$(3.1)_{s,k} \qquad \forall f\in\mathcal{D}'(\Omega), \quad Pf\in\mathcal{H}^{s,k}_{loc} \text{ in } \Gamma \Rightarrow f\in\mathcal{H}^{s+m,k+l}_{loc} \text{ in } \Gamma.$$

Here s is any real number, k any integer >0 or $\leqq 0$. Note in passing that, if $(3.1)_{s,k}$ holds for all open conic sets $\Gamma'\subset\subset\Gamma$ (in the place of Γ), it holds for Γ itself. The converse (that if it holds for Γ it then holds also for all conic open sets $\Gamma'\subset\Gamma$) is not immediately apparent but will result from the forthcoming argument.

The next property is the microlocal estimate we have alluded to. In its formulation we use a real function $\psi\in\mathcal{C}^\infty_c(\Omega)$, $\psi=1$ in a neighborhood of the projection of Γ (ψ will be kept fixed from now on):

$(3.2)_{s,k}$ *to every symbol* $q\in S^0(\Gamma)$, *to every* $s'\in\mathbb{R}$ *and to every compact subset* K *of* Ω *there is a constant* $C>0$ *such that, for all* $\varphi\in\mathcal{C}^\infty_c(K)$,

$$(3.2') \qquad \|q(x,D)\varphi\|_{s+m,k+l}\leqq C(\|\psi P q(x,D)\varphi\|_{s,k}+\|\varphi\|_{s'}).$$

In applying (3.2′) one usually chooses s' close to $-\infty$. It is clear that, if $(3.2)_{s,k}$ holds, it also holds when we replace Γ by anyone of its conic open subsets. It is the glueing together of estimates (3.2′) which is not obviously possible. In fact, it follows from the next assertion:

Proposition 3.1. *If* $(3.2)_{s,k}$ *is true, it remains true after we have replaced* P *by* $P-R$, *where* R *is an arbitrary element of* $\mathcal{N}^{m-1,l-1}$.

Follows at once from Propositions 2.2 and 2.3′.

We come now to the third (and last) property, which is relative to the (microlocal) existence of solutions to the inhomogeneous adjoint equation (solutions modulo arbitrarily regular functions). Let us denote by P^* the adjoint of P:

$(3.3)_{s,k}$ *Let* $q\in S^0(\Gamma)$, $s'\in\mathbb{R}$ *be given arbitrarily. To every* $g\in\mathcal{H}^{-s-m,-k-l}_c$ *there is* $f\in\mathcal{H}^{-s,-k}_c$ *such that*

$$(3.3') \qquad q(x,D)(P^*f-g)\in H^{s'}_{loc} \quad \text{in } \Gamma.$$

The situation in what concerns the further localization of $(3.3)_{s,k}$ is the same as for $(3.2)_{s,k}$: if it holds, it also does when Γ is replaced by anyone of its open conic subsets; but it is not clear that the validity for all $\Gamma'\subset\subset\Gamma$

implies it for Γ. The latter will result from the main statement of this section, which is the following;

Theorem 3.1. *If* $(3.\mathrm{j})_{s,k}$ *is true for some* $j \in \{1, 2, 3\}$ *and for some* $(s,k) \in \mathbb{R} \times \mathbb{Z}$, *it is true for all such j's and* (s,k).

Proof. 1°) *for a fixed pair* (s,k) $(3.1)_{s,k} \Rightarrow (3.2)_{s,k}$.

Let us denote momentarily by E the space of distributions f in Ω which belong to $H_c^{s'}(K)$ (i.e., belong to $H_{loc}^{s'}$ and have their support in K) and which furthermore belong to $\mathscr{H}_{loc}^{s+m,k+l}$ in Γ. It is clear that E can be equipped with a natural Fréchet space topology: the coarsest locally convex topology which renders continuous the injection in $H_c^{s'}(K)$ and also all the mappings

$$f \to q(x,D) f \text{ from } E \text{ into } \mathscr{H}_{loc}^{s+m,k+l},$$

as q ranges over $S^0(\Gamma)$. We equip now E with a second topology (which we denote by $\mathscr{T}_P$): it is the coarsest one that renders continuous the injection in $H_c^{s'}(K)$ and also all the mappings

$$f \to q(x,D) Pf \text{ from } E \text{ into } \mathscr{H}_{loc}^{s,k}, \qquad q \in S^0(\Gamma f).$$

The topology $\mathscr{T}_P$ is metrizable. The main point is that $(3.1)_{s,k}$ implies that it is complete. Since it is obviously coarser than the natural topology on E, it is identical to it (by the open mapping theorem). We derive at once from this that, to every $q_1 \in S^0(\Gamma)$, there is $q_2 \in S^0(\Gamma)$ and a constant $C' > 0$ such that, for all $f \in E$,

$$\|q_1(x,D) f\|_{s+m,k+l} \leqq C'(\|q_2(x,D) Pf\|_{s,k} + \|f\|_{s'}). \tag{3.4}$$

Let us take $f = q(x,D)\varphi$, $\varphi \in \mathscr{C}_c^\infty$, $q \in S^0(\Gamma)$, and choose $q_1 \equiv 1$ in a neighborhood of $\operatorname{supp} q$: (3.2′) follows at once.

2°) *for a fixed pair* (s,k), $(3.2)_{s,k} \Leftrightarrow (3.3)_{s,k}$.

First we show that $(3.2)_{s,k} \Rightarrow (3.3)_{s,k}$.

Let q and g be given as in (3.3′) and let r denote the symbol of $q(x,D)^*$: $r \in S^0(\Gamma)$. We apply (3.2′) with $-s'$ substituted for s':

$$\begin{aligned} |\langle q(x,D) g, \bar\varphi \rangle| &= |\langle q(x,D) g, \psi \bar\varphi \rangle| \\ &\leqq \text{const}\, \|q(x,D)^*(\psi\varphi)\|_{s+m,k+l} \\ &\leqq \text{const}(\|r(x,D)(\psi\varphi)\|_{s+m,k+l} + \|\psi\varphi\|_{-s'}) \\ &\leqq \text{const}(\|\psi P r(x,D)\varphi\|_{s,k} + \|\psi\varphi\|_{-s'}), \end{aligned} \tag{3.5}$$

which shows that the antilinear functional $\varphi \mapsto \langle q(x,D) g, \bar\varphi \rangle$ is continuous (say on $\mathscr{C}^\infty$) for the seminorm in the last member of (3.5) and therefore (by the Hahn-Banach theorem) is equal to an antilinear

functional
$$\varphi \mapsto \langle r(x,D)^* P^*(\psi f_1), \overline{\varphi}\rangle + \langle \psi h, \overline{\varphi}\rangle,$$
where $f_1 \in \mathscr{H}_{loc}^{-s,-k}$, $h \in H_{loc}^{s'}$. Since $q(x,D) - r(x,D)^*$ is regularizing we obtain (3.3′) by setting $f = \psi f_1$.

Next we show that $(3.3)_{s,k} \Rightarrow (3.2)_{s,k}$. Let K be an arbitrary compact subset of Ω, K' another compact subset of Ω whose interior contains the closure of the base projection of Γ. Let us denote by F the space $\mathscr{C}_c^\infty(K)$ equipped with the single seminorm
$$\varphi \mapsto (\|\psi P q(x,D)\varphi\|_{s,k} + \|\varphi\|_{-s'})$$
where $q \in S^0(\Gamma)$ is given. Consider then the following sesquilinear functional
$$(g,\varphi) \mapsto \langle g, \overline{q(x,D)\varphi}\rangle$$
on $\mathscr{H}_c^{-s-m,-k-l}(K') \times F$. It follows at once from $(3.3)_{s,k}$ that it is separately continuous; but on the product of a Fréchet space with a metrizable space, any separately continuous sesquilinear functional is continuous, hence, for a suitable constant $C>0$,
$$\langle g, \overline{q(x,D)\varphi}\rangle \leqq C\,\|g\|_{-s-m,-k-l}(\|\psi P q(x,D)\varphi\|_{s,k} + \|\varphi\|_{-s'}).$$
Substitution of s' for $-s'$ yields at once (3.2′).

3°) *If* $(3.2)_{s_0,k_0}$ *holds for some* (s_0,k_0), $(3.2)_{s,k_0}$ *holds for all s.*

Let t be an arbitrary real number and Q an elliptic operator of order t in Ω. Setting $R = P - Q^{-1}PQ$, we may, thanks to Cor. 2.1, apply Prop. 3.1. It suffices then to observe that $(3.2)_{s,k_0}$ for $Q^{-1}PQ = P - R$ is equivalent with $(3.2)_{s+t,k_0}$ for P.

4°) *For a fixed* $k \in \mathbb{Z}$, *if* $(3.2)_{s,k}$ *holds for all s, so does* $(3.1)_{s,k}$.

Let $a, q \in S^0(\Gamma)$ with $q=1$ in an open subcone Γ' of Γ containing supp a. Let a_n $(n=1,2,\ldots)$ be a sequence of symbols[1] of degree $-\infty$ with support in Γ', which converge (weakly) to a in the space of symbols of degree zero while remaining bounded there (for instance, $a_n = a\chi_n$ where the χ_n are suitable cut-off functions in $T^*\Omega$). For simplicity we shall write $A = a(x,D)$, $A_n = a_n(x,D)$, $Q = q(x,D)$.

Let now $f \in \mathscr{D}'(\Omega)$ be such that $Pf \in \mathscr{H}_{loc}^{s,k}$ in Γ. If the real number σ is sufficiently close to $-\infty$ we have $f \in \mathscr{H}_{loc}^{\sigma+m,k+l}$ in Γ. Also:

(3.6) $$PQA_n f = A_n PQf - [A_n, P]\,Qf + P[Q, A_n]\,f.$$

We know that $QPf \in \mathscr{H}_{loc}^{s,k}$ and that $[P,Q]$ is of order $-\infty$ in Γ', therefore
$$A_n PQf = A_n QPf + A_n [P,Q]\,f$$

[1] Exceptionally, here, we are forced to handle symbols which are not asymptotic sums of homogeneous terms. Nevertheless the forthcoming reasoning should be clear.

converges to $APQf$ in $\mathcal{H}_{loc}^{s,k}$. The third term in the right-hand side of (3.6) converges to $P[Q, A]f$ in $\mathscr{C}^\infty(\Omega)$. As for the second term, we observe (cf. (2.3)) that $[A_n, P]$ "converges to $[A, P]$" (see footnote) in $\mathcal{N}^{m-1,l-1}$ and therefore (cf. Propositions 2.2, 2.3) $[A_n, P]Qf$ converges to $[A, P]Qf$ in $\mathcal{H}_{loc}^{\sigma+1,k+1} \subset \mathcal{H}_{loc}^{\sigma+\frac{1}{2},k}$. Altogether we see that PQA_nf converges to $PQAf$ in $\mathcal{H}_{loc}^{\sigma',k}$ where $\sigma' = \inf(s, \sigma+\frac{1}{2})$. We apply (3.2') to $\varphi = A_nf$ and with σ' substituted for s (and with s' sufficiently close to $-\infty$). We conclude that QAf, hence also Af, belongs to $\mathcal{H}_{loc}^{\sigma'+m,k+l}$. Since A is arbitrary, this means that $f \in \mathcal{H}_{loc}^{\sigma'+m,k+l}$ in Γ. By iterating this reasoning we eventually reach the stage where $\sigma' = s$.

5°) *If* $(3.1)_{s,k}$ *holds for some* k *and for all* s, $(3.1)_{s,k+1}$ *holds for all* s.

First, we restrict ourselves to the case where $k \geqq -l$.

Let $f \in \mathscr{D}'(\Omega)$ be such that $Pf \in \mathcal{H}_{loc}^{s,k+1}$ in Γ. Since, by Prop. 2.3, $\mathcal{H}_{loc}^{s,k+1} \subset \mathcal{H}_{loc}^{s-\frac{1}{2},k}$, we derive from $(3.1)_{s-\frac{1}{2},k}$ that f belongs to $\mathcal{H}_{loc}^{s+m-\frac{1}{2},k+l}$ in Γ. Let U, V be the pseudodifferential operators introduced in Section 2. We have $PUf = UPf + [P, U]f$. Since $Pf \in \mathcal{H}_{loc}^{s,k+1}$ in Γ we have $UPf \in \mathcal{H}_{loc}^{s,k}$ in Γ. On the other hand, $[P, U] \in \mathcal{N}^{m-1,l-1}$ in Γ (cf. Prop. 2.1), hence $[P, U]j \in \mathcal{H}_{loc}^{s+\frac{1}{2},k+1}$ in Γ. Thus $PUf \in \mathcal{H}_{loc}^{s,k}$ in Γ and by $(3.1)_{s,k}$ we see that $Uf \in \mathcal{H}_{loc}^{s+m,k+l}$ in Γ. Similarly, $Vf \in \mathcal{H}_{loc}^{s+m,k+l}$ in Γ. We apply Prop. 2.4 with $k+l+1$ in the place of k; our requirement that $k+l$ be $\geqq 0$ insures that we are in the case in which Prop. 2.4 has been proved. We conclude that $j \in \mathcal{H}_{loc}^{s+m,k+l+1}$ in Γ.

In order to remove the restriction k be $\geqq -l$, we must settle a certain number of particular cases of the general result we are seeking; we state the first one of them:

(3.7) *Suppose* $P = P^*$ *and let* k_0 *be an integer* $\geqq -l$. *If* $(3.1)_{s,k_0}$ *holds for all* s, *the same is true of* $(3.1)_{s,k}$ *whatever* $k \geqq k_0$ *or* $k \leqq -l-k_0$.

Proof of (3.7). Let us assume that $(3.1)_{s,k_0}$ holds for all s. We have already shown that $(3.1)_{s,k}$ holds then for all s and all $k \geqq k_0$. Let now k be $\leqq -l-k_0$, and let $g \in \mathcal{H}_c^{-s-m,-k-l} \subset \mathcal{H}_c^{-s-m,-k_0-l}$. By $(3.3)_{s,k_0}$ which we know to be equivalent to $(3.1)_{s,k_0}$, there is $f \in \mathcal{H}_c^{-s,-k_0}$ such that (3.3') holds. We choose $q=1$ on an open conic set $\Gamma' \subset\subset \Gamma$. We see that $Pf \in \mathcal{H}_c^{-s-m,-k-l}$ in Γ'. But $-k-l \geqq k_0$, hence $(3.1)_{s',k'}$ is true if

$$s' = -s-m, \qquad k' = -k-l$$

and if Γ' is substituted for Γ (in this connection we are exploiting the equivalence, already established in 1°) and 4°) between $(3.1)_{s',k'}$ and $(3.2)_{s',k'}$). Consequently we have $f \in \mathcal{H}_c^{-s,-k}$ in Γ', which shows that $(3.3)_{s,k}$, hence also $(3.1)_{s,k}$, is true.

We shall apply (3.7) to the operator $\mathscr{P}$ defined at the end of Section 2. It is not difficult to check that, given any $q \in S^0(\Gamma)$, any $s' \in \mathbb{R}$ and any compact set $K \subset\subset \Omega$, there is a constant $C_0 > 0$ such that, for all $\varphi \in \mathscr{C}_c^\infty(\Omega)$,

$$\|q(x,D)\varphi\|_{0,1}^2 \leqq C_0\{(\psi \mathscr{P} q(x,D)\varphi, q(x,D)\varphi)_0 + \|\varphi\|_{s'}^2\} \tag{3.8}$$

(we take $s' \sim -\infty$). We derive from (3.8):

$$\|q(x,D)\varphi\|_{0,1} \leqq C_0(\|\psi \mathscr{P} q(x,D)\varphi\|_{0,-1} + \|\varphi\|_{s'}), \tag{3.9}$$

which means that $(3.2)_{0,-1}$ hence (by 3°) $(3.2)_{s,-1}$ for all s, and therefore also $(3.1)_{s,-1}$ for all s. In the case of $\mathscr{P}$ we have $l=2$. Thus we see that $(3.1)_{s,k_0}$ holds for all s with $k_0=-1 \geqq -l$. From (3.7) we derive that $(3.1)_{s,k}$ also holds for all s and for $k \leqq -l-k_0=-1$, in other words for all k. We have thus obtained:

Lemma 3.1. *When $P=\mathscr{P}$, $(3.\mathrm{j})_{s,k}$ is true for all $j=1,2,3$, all $s\in\mathbb{R}$ and for all $k\in\mathbb{Z}$.*

This enables us to complete the proof of Prop. 2.4: let $k\in\mathbb{Z}$ be arbitrary and suppose that Uf and Vf belong to $\mathscr{H}_{loc}^{s,k-1}$ in Γ_0. Let Γ' be any conic open set, $\Gamma' \subset\subset \Gamma_0$. We may form $\mathscr{P}$ so that it is equal to $(U-iV)(U+iV)$ in Γ'. We have therefore $\mathscr{P} f \in \mathscr{H}_{loc}^{s,k-2}$ in Γ', hence (by Lemma 3.1) $f \in \mathscr{H}_{loc}^{s,k}$ in Γ'. As $\Gamma' \subset\subset \Gamma_0$ is arbitrary, we have $f \in \mathscr{H}_{loc}^{s,k}$ in Γ_0. Q.E.D.

But now that we know that Prop. 2.4 is true with no restriction on k we may repeat the argument at the beginning of 5°) without the assumption that $k+l$ is $\geqq 0$. This completes the proof of 5°).

We come now to the last stage of the proof of Th. 3.1:

6°) *If $(3.1)_{s,k}$ holds for all s and some k, it holds for all (s,k).*

By combining 5°) and (3.7) we see that the result is true when P is self-adjoint.

For a general $P \in \mathscr{N}^{m,l}$ we may as well assume that $(3.2)_{0,k}$ holds for some $k \geqq 0$. Setting

$$M = P^*(U-iV)^k(U+iV)^k P,$$

we may rewrite (3.2′) in the form

$$\|q(x,D)\varphi\|_{s+m,k+l}^2 \leqq C'\{|(\psi M q(x,D)\varphi, q(x,D)\varphi)| + \|\varphi\|_{s'}^2\}, \tag{3.10}$$

which, by Cauchy-Schwartz, implies at once that $(3.2)_{-s-m,-k-l}$ holds with M instead of P; the same is therefore true of $(3.1)_{-s-m,-k-l}$. Since M is self-adjoint, $(3.1)_{s',k'}$ for M is true whatever s', k'. Let then $f\in\mathscr{D}'(\Omega)$ such that $Pf \in \mathscr{H}_{loc}^{s_1,k_1}$ in Γ ($s_1\in\mathbb{R}$, $k_1\in\mathbb{Z}$ arbitrary). We derive from this:

$$Mf \in \mathscr{H}_{loc}^{s_1-m,k_1-2k-l} \quad \textit{in } \Gamma.$$

Since $M \in \mathscr{N}^{2m,2(k+l)}$ in Γ, we conclude that $f \in \mathscr{H}_{loc}^{s_1+m,k_1+l}$ in Γ.

The proof of Th. 3.1 is complete.

Remark 3.1. Suppose that both P and P^* satisfy anyone of the conditions $(3.j)_{s,k}$ $(j=1,2,3)$. Then, whatever $s\in\mathbb{R}$, $k\in\mathbb{Z}$, the following stronger version of $(3.3)_{s,k}$ is valid:

$(3.11)_{s,k}$ *Given any conic open set* $\Gamma'\subset\subset\Gamma$ *and any* $g\in\mathscr{H}_{loc}^{-s-m,-k-l}$ *there is* $f\in\mathscr{H}_{loc}^{-s,-k}$ *such that*

$$P^*f-g\in C^\infty \quad in\ \Gamma'.$$

This follows from exploitation of a priori estimates such as (3.2′) (or (3.4)) and standard Fréchet space techniques.

4. A Microlocal Subelliptic Estimate

We return now to the pseudodifferential operator P of Section 1. We shall assume throughout that its order is *zero*. We shall use the norms $\| \ \|_s$, $\| \ \|_{s,k}$ defined by means of a Riemannian metric on Ω.

Theorem 4.1. *Suppose that* P *has the properties* (1.3), (1.4), (1.5). *Let* (x_0,ξ^0) *be a point of* Σ *such that*

$$\mathrm{Re}\, l_P(x_0,\xi^0)+\tfrac{1}{2}\leqq 0. \tag{4.1}$$

There is an open conic neighborhood Γ_0 *of* (x_0,ξ^0) *such that the following is true:*

Given any $q\in S^0(\Gamma_0)$, *any real number* s' (arbitrarily close to $-\infty$) *and any compact subset* K *of* Ω, *there is* $C>0$ *such that, for all functions* $\varphi\in\mathscr{C}_c^\infty(K)$,

$$\|q(x,D)\varphi\|_{0,1}^2\leqq C\{|(Pq(x,D)\varphi,q(x,D)\varphi)_0|+\|\varphi\|_{s'}^2\}. \tag{4.2}$$

Proof. We use two real valued functions u, v as in (1.1) and (1.2). As already pointed out, the principal symbol p of P can be factorized in Γ_0 as $p=a(u-\alpha v)(u-\beta v)$ where a, α, β are elliptic (and homogeneous of degree zero). We assume that (1.7) holds. If we substitute $u+(\mathrm{Re}\,\alpha)v$ for u and v for $-(\mathrm{Im}\,\alpha)v$, these properties subsist, while we have now:

$$p=a(u+iv)(u-i\lambda v), \tag{4.3}$$

with λ homogeneous of degree zero, satisfying

$$\mathrm{Re}\,\lambda>0 \quad in\ \Gamma_0. \tag{4.4}$$

Let us set $w=u+iv$; (4.3) reads:

$$p=a(1+\lambda)\,w(\overline{w}+\zeta w), \qquad \zeta=(1-\lambda)/(1+\lambda). \tag{4.5}$$

Note that $\zeta=0$ (at some point) is equivalent with $\lambda=1$, *i.e.*, $p=a|w|^2$. Suppose that $\zeta(x_0,\xi^0)\neq 0$. In this case we may choose the neighborhood Γ_0 so that ζ stays in a simply connected subset of $\mathbb{C}\setminus\{0\}$ as (x,ξ) ranges over Γ_0; we may define $\theta=\frac{1}{2}\operatorname{Arg}\zeta$, and set

$$w'=u'+iv'=e^{-i\theta}w.$$

It is clear that u', v' have the same properties (listed above) as u, v. Let us set also

$$a'=a(1+\lambda)e^{2i\theta}, \qquad \zeta'=\zeta e^{-2i\theta}, \qquad \lambda'=(1-\zeta')/(1+\zeta').$$

In this notation,

$$p=a'w'(\overline{w}'+\zeta'w')=a'(1+\lambda')(u'+iv')(u'-i\lambda'v'). \tag{4.6}$$

Because of (4.4) we have $\zeta'=|\zeta|<1$ and therefore λ' is real positive in Γ_0. Summarizing, we may always assume that (4.3) holds, either with $\lambda=1$ at (x_0,ξ^0) or else with λ real >0 in the whole of Γ_0.

Let us also point out that both the hypotheses and the conclusion in Th. 4.1 are invariant under multiplication of P by elliptic operators (of order zero). This follows from (1.17) and from the definition of the spaces $\mathscr{H}^{s,k}$. We may therefore assume $a\equiv 1$.

We introduce two pseudodifferential operators of order zero in Ω, R, S, with principal symbols respectively equal to $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ in Γ_0. We may write:

$$P=(U+iV)(U+SV-iRV)+Z \qquad (\text{in } \Gamma_0). \tag{4.7}$$

Here U and V are the self-adjoint operators in Ω with principal symbols equal respectively to u and v in Γ_0, already used in Sections 2, 3. The operator Z is of order -1. The decomposition (4.7) is of the kind $P=XY+Z$ considered in Section 1. We have:

$$\operatorname{Re}P=\frac{P+P^*}{2}=U^2+VRV+\operatorname{Re}Z+\frac{1}{2}(USV+VSU)$$
$$+\frac{1}{2i}\{U(I+R)V-V(I+R)U\}.$$

First we exploit the fact that the principal symbol of R is >0 in Γ_0, by (4.4), and that the one of S vanishes at (x_0,ξ^0). Consequently, if we take Γ_0 sufficiently "narrow" about its axis, the norm of S, regarded as a linear operator in L^2 acting on functions of the form $q(x,D)\varphi$, $q\in S^0(\Gamma_0)$, $\varphi\in C_c^\infty(\Omega)$, will be as small as we wish, whereas that of the inverse of R will be bounded. Thus, for a suitable choice of Γ_0 and of the constant $C_0>0$, we will have, for all those functions $\psi=q(x,D)\varphi$,

$$\|U\psi\|_0^2+\|V\psi\|_0^2\leq C_0(\{U^2+VRV+\tfrac{1}{2}(USV+VSU)\}\psi,\psi)_0.$$

But then there is $\eta_0 > 0$ such that, for a suitable choice of $C_0' > 0$ and all $\eta \leqq \eta_0$,

$$\|U\psi\|_0^2 + \|V\psi\|_0^2$$

(4.8)
$$\leqq C_0' \left(\left\{U^2 + VRV + \frac{1}{2}(USV + VSU) + \frac{\eta}{2i}[U, V]\right\}\psi, \psi\right)_0.$$

This said, we observe that the principal symbol of

$$T = \operatorname{Re} Z + \frac{1}{2i}(1-\eta)[U, V] + \frac{1}{2i}(URV - VRU),$$

when restricted to Σ, is equal to

(4.9) $$\operatorname{Re}\sigma(Z) - \tfrac{1}{2}(1 - \eta + \operatorname{Re}\lambda)\{u, v\}.$$

Let us set $X = U + iV$, $Y = U + SV - iRV$. We see that

(4.10) $$\sigma([X, Y]) = -(1+\lambda)\{u, v\}.$$

If we combine (4.9) and (4.10) and use once more the fact that $\operatorname{Im}\lambda = 0$ at (x_0, ξ^0), we obtain, in view of (1.15):

(4.11) $$\sigma(T)(x_0, \xi^0) = -(1 + \operatorname{Re}\lambda)\{u, v\}(\operatorname{Re} l_P + \tfrac{1}{2} - \eta'),$$

where $\eta' = \frac{1}{2}(1 + \operatorname{Re}\lambda)^{-1}\eta$. The hypothesis (4.1) implies that $\sigma(T) > 0$ at (x_0, ξ^0), hence $\sigma(T) > 0$ in a full neighborhood of (x_0, ξ^0), neighborhood which we take to be Γ_0. A microlocal inversion of elliptic operators yields at once, for an arbitrary real numbers s', a suitable constant $C' > 0$ and all $\varphi \in \mathscr{C}_c^\infty(\Omega)$,

(4.12) $$\|q(x, D)\varphi\|_{-\frac{1}{2}}^2 \leqq C'\{(Tq(x, D)\varphi, q(x, D)\varphi)_0 + \|\varphi\|_{s'}^2\}.$$

The conjunction of (4.8) and (4.12) yields at once (4.2).

5. Concatenations

Same notation as in Section 1. We set d = degree of X so that degree of $Y = m - d$; degree of $Z = m - 1$. We begin by an easy result, stated and proved only in order to show that, under Condition (1.23) (in Th. 1.1), one does not really need the full strength of the (stronger) Prop. 5.2.

Proposition 5.1. *Assume that Z is elliptic* (of degree $m-1$). *Then there exist two pseudodifferential operators A, Q of degree $d-1$ and $m-1$ respectively, with $\sigma(Q) = \sigma([X, Y])$ on Σ, such that*

(5.1) $$X(P - Q) = P(X - A) \quad (\textit{mod operators of degree } -\infty).$$

Proof. Let M be an arbitrary operator of degree $m-2$ and set

$$A = Z^{-1}([Z, X] - XM)$$

where Z^{-1} is a parametrix of Z. We have

$$Z(X-A) = X(Z-M),$$

whence

$$P(X-A) = X\{Y(X-A) + Z - M\} = X(P-Q),$$

where $Q = [X, Y] + YA + M$. Our requirements are evidently satisfied.

We remove now the condition that Z be elliptic.

Proposition 5.2. *Whatever be the pseudodifferential operator Z of degree $m-1$ in Ω, there are two pseudodifferential operators A, Q of degree $d-1$ and $m-1$ respectively, such that $\sigma(Q) = \sigma([X, Y])$ and that*

(5.2) $\quad P(X-A) = (X-A)(P-Q) \quad$ (*mod. operators of degree* $-\infty$).

Proof. If we write $P = (X-A)Y + (Z + AY)$, we have:

$$[P, X-A] = -(X-A)[X-A, Y] + [Z + AY, X - A],$$

so that, setting $M = [X-A, Y] - Q$, we see that (5.2) is equivalent with

(5.3) $$(X-A)M = [Z + AY, X - A].$$

We see that M must be of degree $m-2$, which implies $\sigma(Q) = \sigma([X, Y])$. It will suffice to construct A and M, which yields Q. Let us write

$$A \sim \sum_{k=0}^{+\infty} A_k, \qquad M \sim \sum_{k=0}^{+\infty} M_k,$$

where the degree of A_k is $d-1-k$, and that of M_k, $m-2-k$, and where $\sim$ denotes an asymptotic sum as in [8]. For any $n \geqq 0$ let us also write

$$A_{(n)} = \sum_{k<n} A_k, \qquad M_{(n)} = \sum_{k<n} M_k,$$

$$R_{(n)} = [Z + A_{(n)} Y, X - A_{(n)}] - (X - A_{(n)}) M_{(n)}.$$

Suppose that $R_{(n)}$ is of order $\leqq m+d-2-n$ (this is certainly true if $n=0$, in which case $A_{(n)} = M_{(n)} = 0$ and $R_{(0)} = [Z, X]$). Let us then write

$$R_{(n+1)} = R_{(n)} + [A_n Y, X - A_{(n+1)}]$$
$$- [Z + A_{(n)} Y, A_n] - (X - A_{(n)}) M_n + A_n M_{(n+1)}.$$

The condition that $R_{(n+1)}$ be of degree $\leqq m+d-n-3$ translates into

(5.4) $$\sigma_{m+d-n-2}(R_{(n)} + [A_n Y, X] - XM_n) = 0,$$

where we have used the notation σ_ν to denote the principal symbol for operators of order ν. In going from the preceding expression of $R_{(n+1)}$ to (5.4) we have neglected all terms of degree $\leqq m+d-n-3$. Let us momentarily use the following notation:

$$\mathscr{X}=\sigma(X), \quad \mathscr{Y}=\sigma(Y), \quad r_{(n)}=\sigma(R_{(n)}), \quad a_n=\sigma(A_n), \quad \mu_n=\sigma(M_n).$$

Eq. (5.4) means:

(5.5) $$r_{(n)}-i\{a_n\mathscr{Y},\mathscr{X}\}-\mathscr{X}\,\mu_n=0.$$

We can find a_n, homogeneous of degree $d-1-n$, such that

(5.6) $r_{(n)}-i\{a_n\mathscr{Y},\mathscr{X}\}$ *vanishes of infinite order on* Σ.

Indeed, if a vanishes to the l-th order on Σ, it can be written

$$a=\sum_{j+k=l}\alpha_{j,k}\,\mathscr{X}^j\,\mathscr{Y}^k$$

with $a_{j,k}\in\mathscr{C}^\infty(\dot{T}^*\Omega)$ having the appropriate homogeneity degree. We have then

$$\{a\mathscr{Y},\mathscr{X}\}=\sum_{j+k=l}(j+1)\,\alpha_{j,k}\,\mathscr{X}^j\,\mathscr{Y}^k\{\mathscr{Y},\mathscr{X}\} \mod (\mathscr{X},\mathscr{Y})^{l+1},$$

and since $\{\mathscr{Y},\mathscr{X}\}$ does not vanish anywhere on Σ, this proves (again by successive approximations) that we can choose a_n so as to satisfy (5.6). Then

$$\mu_n=\mathscr{X}^{-1}(r_{(n)}-i\{a_n\mathscr{Y},\mathscr{X}\})$$

is a $\mathscr{C}^\infty$ function, homogeneous of degree $m-2-n$. If we take A_n and M_n with respective symbols a_n and μ_n, we see that (5.4) is verified, and the induction on n works. Q.E.D.

Let us set $P=P_0$, $P-Q=P_1$, $X_0=X-A$, where A and Q are chosen as indicated in Prop. 5.2; (5.2) reads $P_0X_0=X_0P_1$. We may repeat the argument with P_1 in the place of P, etc. We construct in this way a sequence of operators P_j, X_j which satisfy

(5.7) $$P_jX_j\sim X_jP_{j+1}.$$

This corresponds, in our setting, to the notion of concatenation introduced in [10] (Section II.4).

6. Hermite Operators

In this section we describe the operators introduced by Vishik and Grushin and Sjöstrand, used, in a similar context, in [4, 5, 6, 3, 9]. They provide a good microlocal description of the range of an operator such as

X of Section 1, and of the null-space of an operator such as Y (both image and kernel are to be understood modulo C^∞):

In this section we deal with the case $\Omega = \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}$, setting $n = N-1$. The variable in Ω will be denoted by (x, t), $x = (x^1, \ldots, x^n)$; the dual variable will be denoted by (ξ, τ), $\xi = (\xi_1, \ldots, \xi_n)$. We are going to assume that Σ is the submanifold $t = \tau = 0$ in $\dot{T}^*\Omega = \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$. We call Ω' the submanifold $t=0$ (i.e., $\mathbb{R}^n \subset \mathbb{R}^{n+1}$) of the base, and we identify Σ to $\dot{T}\Omega'$. The reader will notice that this is a particular case of the situation considered in the previous sections — although, *microlocally*, it is a model of that more general situation (this is not so *locally* since, in general, Σ will not be a cotangent bundle of a hypersurface in the base).

We shall call *Hermite operator* of degree m any linear operator $A: \mathscr{C}_c^\infty(\Omega') \to \mathscr{C}^\infty(\Omega)$ defined by an integral formula

$$Af(x,t) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,t,\xi) \hat{f}(\xi)\, d\xi, \tag{6.1}$$

where $\hat{f}$ stands for the Fourier transform of f and where $a \in \mathscr{C}^\infty(\Omega \times \mathbb{R}_n)$ satisfies the following condition:

(6.2) *Given any pair of integers $j, k \geqq 0$, any pair of n-tuples α, β and any compact subset K of Ω, there is a constant $C>0$ such that*

$$\left| t^j \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha \left(\frac{\partial}{\partial x}\right)^\beta a(x,t,\xi) \right| \leqq C(1+|\xi|)^{m-|\alpha|-(j-k)/2},$$
$$\forall (x,t) \in K,\ \xi \in \mathbb{R}^n.$$

We shall denote by $\mathfrak{s}^m$ the space of "symbols" a satisfying (6.2) and by $\mathfrak{S}^m$ the space of Hermite operators of order m,

Example 6.1. Let $a(x,t,\xi)$ be an element of $\mathscr{C}^\infty(\Omega \times \mathbb{R}^n)$ which coincides with $\exp(-\frac{1}{2}|\xi|\, t^2)$ for $|\xi|$ large (i.e., for $|\xi|$ larger than some number $C>0$). Then $a \in \mathfrak{S}^0$.

Example 6.2. This generalizes the example 6.1. Let $b \in \mathscr{C}^\infty(\dot{T}^*\Omega)$ have the following kind of homogeneity or, rather, of semihomogeneity:

$$b(x, \rho^{-\frac{1}{2}} t, \rho\xi) = \rho^m b(x,t,\xi), \qquad \rho > 0, \tag{6.3}$$

and if, moreover, as a function of t, $b(x,t,\xi)$ is in the space $\mathscr{S}$ of Schwartz ($\mathscr{C}^\infty$ functions rapidly decreasing at infinity), uniformly with respect to ξ when the latter remains in the unit sphere, then $b \in \mathfrak{s}^m$ at infinity (as usual, this means that any $\mathscr{C}^\infty$ function a which coincides with b for $|\xi| > R > 0$ belongs to $\mathfrak{s}^m$). The requirements on b are equivalent with the following property of its Fourier transform $\tilde{b}(x,\tau,\xi)$ with respect to t:

(6.4) $a(x,\tau,\xi) = \tilde{b}(x, |\xi|^{-\frac{1}{2}}\tau, \xi)$ *is positive-homogeneous of degree* $m - \frac{1}{2}$ *with respect to* (ξ,τ), *is* $\mathscr{C}^\infty$ *for* $(\xi,\tau) \neq 0$ *and vanishes of infinite order at* $\xi = 0$.

The following statements are proved in a similar way as the analogous statements for pseudo-differential operators (see also [2, 9]).

(6.5) *Consider a sequence of operators* $A_j \in \mathfrak{S}^{m_j}$, $m_j \searrow -\infty$. *Then there exists* $A \in \mathfrak{S}^m$, $m = \sup_j m_j$, *such that* $A \sim \sum_j A_j$ $\left(\text{i.e., } A - \sum_0^J A_j \in \mathfrak{S}^{m'_J} \text{ with } m'_J \to -\infty\right)$.

(In the present article the intervening Hermite operators will always be asymptotic sums $A \sim \sum_j A_j$ of the kind in (6.5) where the symbols a_j of the successive terms A_j will be *semihomogeneous* (in the sense of Example 6.2) of respective degrees $m - j/2$ (thus their degrees will differ by *half-integers*).)

If $A \in \mathfrak{S}^m$ is defined by $a(x, t, \xi)$ as in (6.1) we shall define the *principal symbol* $\sigma(A)$ of A as the class of a mod $\mathfrak{s}^{m-1/2}$. In our case we will be able to regard it as a semihomogeneous function (in the sense of (6.3)) of degree m. It determines A mod. $\mathfrak{S}^{m-1/2}$. We have:

(6.6) *Let* $A \in \mathfrak{S}^m$ *and let* Q *be a pseudodifferential operator of order* m' *in* Ω'. *Then we have:*

$$A Q \in \mathfrak{S}^{m+m'}, \qquad \sigma(AQ) = \sigma(A)\,\sigma(Q).$$

(6.7) *Let* $A \in \mathfrak{S}^m$, $P \in \mathcal{N}^{d,k}(\Sigma)$ (Def. 2.1). *Then:*

$$PA \in \mathfrak{S}^{m+d-k/2}.$$

We can complement (6.7) with the following information. Let $\sigma_\Sigma(P)$ denote the unique *differential operator* on $\mathbb{R}$:

$$\sigma_\Sigma(P) = \sum_{j+j' \leqq k} c_{j,j'}(x, \xi)\, t^j \left(\frac{\partial}{\partial t}\right)^{j'}, \tag{6.8}$$

where, for each (j, j'), $c_{j,j'}$ is positive-homogeneous in ξ of degree $d - (k - j + j')/2$, defined by the requirement that

$$P - \sigma_\Sigma(P) \in \mathcal{N}^{d,k+1}.$$

Then we have:

$$\sigma(PA) = \sigma_\Sigma(P)\,[\sigma(A)]. \tag{6.9}$$

In particular, if $k = 0$ (in which case the differential operator $\sigma_\Sigma(P)$ is simply multiplication by the homogenous symbol $\sigma(P)$), we have:

$$\sigma(PA) = \left(\sigma(P)|_\Sigma\right) \sigma(A). \tag{6.10}$$

Also:

(6.11) *Whatever* $s \in \mathbb{R}, k \in \mathbb{Z}$, $A \in \mathfrak{S}^m$ *can be extended as a continuous linear map*

$$H_c^{s-1/4}(\Omega') \to \mathscr{H}_{loc}^{s-m+k/2,k}(\Omega, \Sigma).$$

This is proved for $k=0$ in [8]. It follows from the inclusion

$$\mathscr{H}_{loc}^{s+k/2,k}(\Omega, \Sigma) \hookrightarrow H_{loc}^s(\Omega)$$

when $k \leqq 0$ (Cor. 2.2) and from (6.7) for $k>0$. It follows from (6.7) that this also holds microlocally, *i.e.*, if f is a compactly supported distribution in Ω' which belongs to $H^{s-1/4}$ near $(x, \xi) \in \dot{T}^* \Omega'$, then $Au \in \mathscr{H}_{loc}^{s-m+k/2,k}$ near $(x, 0, \xi, 0)$ (whatever the integer k). In particular, $WF(Af) \subset WF(f)$ (WF stands for *wave-front*; $WF(f)$ is a subset of $\dot{T}^* \Omega'$ but we have identified $\dot{T}^* \Omega'$ with the submanifold Σ of $\dot{T}^* \Omega$).

We continue to list the relevant properties of the Hermite operators.

(6.12) *Let* A_1, A_2 *be two Hermite operators of respective orders* $m_1 + 1/4$, $m_2 + 1/4$. *Then* $A_1^* A_2$ *is a pseudodifferential operator of order* $m_1 + m_2$ *on* Ω', *with principal symbol given by*

$$\sigma(A_1^* A_2) = \int_{-\infty}^{+\infty} \overline{\sigma(A_1)}\, \sigma(A_2)\, dt.$$

We recall that, in our situation, $\sigma(A)$ will be a semihomogeneous function of t, ξ, rapidly decreasing when $|t| \to +\infty$, so that the integral makes sense.

Let H_j ($j=0, 1, \ldots$) denote the Hermite operator of degree 1/4 defined by the "complete" symbol

(6.13) $$(|\xi|/\pi)^{1/4} \mathscr{H}_j(t|\xi|^{1/2}),$$

where $\mathscr{H}_j(u) = (2^j j!)^{-1/2} \left(\frac{\partial}{\partial u} - u\right)^j \exp(-u^2/2)$ is the j-th Hermite function. Let us also set

$$X_0 = \frac{\partial}{\partial t} - t|D_x|,$$

where $|D_x|$ is the pseudo-differential operator with total symbol $|\xi|$ (the behaviour of $|D_x|$ near $\xi = 0$ is irrelevant here). We have the following formulas, which follow immediately from the properties of the Hermite functions

(6.14) $$H_j^* H_{j'} = \delta_{j,j'} I,$$

where $\delta_{j,j'}$ is the Kronecker index and I the identity operator:

(6.15) $$X_0 H_j = (2(j+1)\,|D_x|)^{\frac{1}{2}}\, H_{j+1};$$

(6.16) $$X_0^* H_j = (2j|D_x|)^{\frac{1}{2}}\, H_{j-1}.$$

Let us denote by $\hat{f}(\xi, t)$ the Fourier transform of $f(x, t)$ with respect to x and define an operator E by

(6.17) $$Ef(x,t) = -(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \int_0^t e^{-\frac{1}{2}|\xi|(t^2-s^2)} \hat{f}(\xi, s)\, ds\, d\xi.$$

Let us set $E' = E^*(I - H_0 H_0^*)$. Then we have

(6.18) $$E' X_0 = X_0^* E'^* = I,$$

(6.19) $$X_0 E' = E'^* X_0^* = I - H_0 H_0^*.$$

Thus, $I - H_0 H_0^*$ is an "orthogonal" projector on the range of X_0 and $H_0 H_0^*$ is the „complementary projector" onto Ker X_0^*.

More generally, let $X \in \mathcal{N}^{1,1}(\Sigma)$ be a pseudodifferential operator with symbol $\sigma(X) = b(\tau + i\,t\,a)$, where a, b are elliptic, with degrees 1 and 0 respectively, and $\operatorname{Re} a > 0$ on Σ. Then there exists a Hermite operator K_X of degree 1/4 such that

(6.20) $\sigma(K_X) = (\alpha/\pi)^{1/4} \exp(-\alpha t^2/2), \quad \alpha = a|_\Sigma;$

(6.21) $K_X^* K_X \sim I$, the identity mapping on distributions in Ω';

(6.22) $X^* K_X \sim 0.$

The operator K_X^* essentially describes the range of X mod C^∞ near Σ, in the following sense: if g is any distribution in Ω and (x, ξ) any point of Σ, there exists $f \in \mathcal{D}'(\Omega)$ such that $Xf - g \in \mathcal{C}^\infty$ in a conic neighborhood of (x, ξ) if and only if (x, ξ) does not belong to the wave-front set of $K_X^* g$ (i.e., $K_X^* g \in \mathcal{C}^\infty$ near (x, ξ)). In fact, it is shown in [2, 9] that the more general X has a left-parametrix E_X such that

(6.23) $$X E_X \sim I - K_X K_X^*,$$

thus generalizing what is true for X_0 above.

Let us also mention the following fact, which reduces the case of more general X to that of X_0 [2]: the space $\mathfrak{S}^m$ is invariant under changes of coordinates in the base Ω which preserve that submanifold Ω', given by $t = 0$, and also the direction of the vector field $\frac{\partial}{\partial t}$ on Ω'. This is easy to check. What is more difficult is to show that, if Φ is a canonical transfor-

[2] This result has been established by the first author but is not yet published; it will not be used hereafter.

mation in $\dot{T}^*\Omega$ which preserves Σ, A (resp. B) a Fourier integral operator on Ω (resp. Ω') associated with Φ (resp. $\overset{-1}{\Phi}|_\Sigma$) of degree zero, then

$$A\,\mathfrak{S}^m\,B \subset \mathfrak{S}^m.$$

7. Proof of Theorem 1.1

Same notation as in Section 1, in particular for Ω, Σ, P, l_P. We are going to prove a stronger version of Th. 1.1, namely:

Theorem 7.1. *Same hypotheses as in Th.* 1.1. *The following assertions are then equivalent:*

(7.1) *Whatever* $s\in\mathbb{R}$ *and the point* $(x_0, \xi^0)\in\dot{T}^*\Omega$, *there is a conic open neighborhood* Γ_0 *of* (x_0, ξ^0) *such that*

$(7.1')_s$ $\forall f\in\mathscr{D}'(\Omega)$, *if* $Pf\in H^s_{loc}(\Omega)$ *and* $WF(f)\subset\Gamma_0$ *then* $f\in H^{s+m-1}_{loc}(\Omega)$.

(7.2) *Whatever the integer* $k\geqq 0$, *the function* l_P-k *does not vanish at any point of* Σ.

(7.3) *Whatever* $s\in\mathbb{R}$ *and* $k\in\mathbb{Z}$, *the operator* P *defines a microlocal isomorphism of* $\mathscr{H}^{s+m,k+2}_{loc}$ *onto* $\mathscr{H}^{s,k}_{loc}$.

(7.3)* *Same as* (7.3) *but with* P^* *substituted for* P.

Before proving Th. 7.1 a few remarks are in order. Observe that (7.1) is much weaker than (1.20). For instance, it is satisfied by strictly hyperbolic operators (which of course do *not* satisfy the hypotheses (1.3), (1.4), (1.5)). The advantage of (7.1) over (1.20) is that it is microlocal. By (1.16) we know that (7.2) is the same for P and P^*. The meaning of (7.3) is the following:

(7.4) *Whatever* $s\in\mathbb{R}, k\in\mathbb{Z}$ *and* $(x_0, \xi^0)\in\dot{T}^*\Omega$, *there is a conic open neighborhood* Γ_0 *of* (x_0, ξ^0) *such that, to every* $g\in\mathscr{H}^{s,k}_{loc}$ *there is* $f\in\mathscr{H}^{s+m,k+2}_{loc}$, *unique* mod $\mathscr{C}^\infty$, *such that* $Pf-g\in\mathscr{C}^\infty$ *in* Γ_0.

Of course (7.4) is equivalent with the following:

(7.5) *To every* $g\in\mathscr{D}'(\Omega)$ *there is* $f\in\mathscr{D}'(\Omega)$, *unique* mod $\mathscr{C}^\infty(\Omega)$, *such that* $Pf-g\in\mathscr{C}^\infty(\Omega)$. *Furthermore, given any conic open subset* Γ *of* $\dot{T}^*\Omega$, *if* $g\in\mathscr{H}^{s,k}_{loc}$ *in* Γ, *then* $f\in\mathscr{H}^{s+m,k+2}_{loc}$ *in* Γ (*s, k arbitrary*).

According to (7.5), if $Pf\in H^s_{loc}$ we will have $f\in\mathscr{H}^{s+m,2}_{loc}\subset H^{s+m-1}_{loc}$. This shows that (7.3)⇒(1.20), and also that (7.3)⇒(7.1). It suffices therefore to show that (7.1)⇒(7.2) and that (7.2)⇒(7.3). Before embarking on the proof, let us observe that all the statements are microlocal. We shall therefore be reasoning in a conic open subset Γ of $\dot{T}^*\Omega$ which intersects Σ (in the complement of Σ P is elliptic, and there the various statements are well known). Throughout the argument we are really dealing with *microfunctions*, that is to say, with distributions "in Γ" modulo C^∞ functions "in Γ". If Γ is sufficiently "thin" we may find local canonical coordinates $(x^1, \ldots, x^n, t, \xi_1, \ldots, \xi_n, \tau)$ which transform the whole situation in that of Section 6, where $\Omega = \mathbb{R}^{n+1}$, $\Sigma = \dot{T}^*\Omega'$, with $\Omega' = \mathbb{R}^n = \{(x, t) \in \mathbb{R}^{n+1};\ t=0\}$ (thus Σ is defined by $t=\tau=0$).

All the statements under scrutiny are unchanged if we multiply P by an elliptic operator, on the right and on the left, or if we replace P by CPC^{-1}, with C an elliptic Fourier integral operator associated with a canonical transformation Φ and C^{-1}, one of its parametrices (Σ must then be replaced by $\Phi(\Sigma)$).

1°) (7.1) ⇒ (7.2)

We shall apply right away the remarks we have just made. As is shown in [2] the canonical coordinates $(x^1, \ldots, x^n, t, \xi_1, \ldots, \xi_n, \tau)$ can be chosen in such a way that, if we write $P = XY + Z$, the principal symbol of Y is equal to $q(\tau - t|\xi|)$, with q elliptic. But since we may write

$$P = XA^{-1}A(Y-B) + (Z+XB),$$

with A elliptic and $\deg B \leqq \deg Y - 1$ and, writing X instead of XA^{-1}, Y instead of $A(Y-B)$ and Z instead of $Z+XB$, we may assume, not only that $\Omega = \mathbb{R}^{n+1}$, $\Sigma = \dot{T}^*\mathbb{R}^n$, but also, (cf. loc. cit.), that

$$Y = \frac{\partial}{\partial t} + t|D_x|. \tag{7.6}$$

On the other hand, we may always write $X = AY + BY^*$, where A, B are suitable pseudodifferential operators of order $m-2$ (totally unrelated to those which were so denoted above, and which may now be forgotten). We have:

$$\sigma([X, Y])|_\Sigma = \sigma(B)|_\Sigma \cdot \sigma([Y^*, Y])|_\Sigma = -2\,|\xi|\,\sigma(B)|_\Sigma. \tag{7.7}$$

We see that B is elliptic near Σ. We may multiply P by an elliptic operator of order $1-m$ whose symbol, near Σ, is equal to

$$(-2|\xi|\,\sigma(B)|_\Sigma)^{-1}.$$

We are thus reduced to the situation where P is of order *one* and

$$X = AY - (2|D_x|)^{-1}Y^*, \qquad \deg A \leqq 1.$$

In this situation we have $\sigma([X, Y])|_\Sigma = 1$, $l_P = \sigma(Z)|_\Sigma$.

Let then H_j ($j=0, 1, \ldots$) be the Hermite operators associated to the Hermite functions, as indicated in Section 6 (the symbol of H_j is equal to

(6.13)). We have, by virtue of (6.15) and (6.16),

$$PH_j = AY^2 H_j - (2|D_x|)^{-1} Y^* YH_j + ZH_j$$
$$(7.8) \qquad = (2j(j-1))^{\frac{1}{2}} AH_{j-2} + (Z-j) H_j.$$

We introduce the *matrix* Hermite operator

$$(7.9) \qquad G_J = \bigoplus_{j=0}^{J} H_j \colon \overbrace{\mathscr{C}_c^\infty(\mathbb{R}^n) \times \cdots \times \mathscr{C}_c^\infty(\mathbb{R}^n)}^{J+1} \to \mathscr{C}^\infty(\mathbb{R}^{n+1}).$$

Let us also set

$$(7.10) \qquad a_j = (2j(j-1))^{\frac{1}{2}} \sigma(A)|_\Sigma.$$

With such a notation, we may write

$$(7.11) \qquad PG_J = G_J \Lambda_J + R_J,$$

where R_J is a Hermite operator of order $\leqq -\frac{1}{4}$ whereas Λ_J is a triangular $(J+1)\times(J+1)$ matrix whose entries are pseudodifferential operators in Ω'. The symbol $\sigma(\Lambda_J)$ is the matrix (σ_{ij}) $(i, j = 0, \ldots, J)$ whose diagonal entries are given by

$$\sigma_{jj} = l_P - j \quad (j = 0, \ldots, J),$$

and whose only other possibly nonzero entries are

$$\sigma_{j-2,j} = a_j, \quad j = 2, \ldots, J.$$

If (7.2) does not hold, i.e., if l_P is equal to some integer $k \geqq 0$ at some point $(x_0, \xi^0) \in \Sigma$, Λ_J is not elliptic at (x_0, ξ^0) as soon as $J \geqq k$. Then, given any conic neighborhood Γ_0' of (x_0, ξ^0) we can find a distribution f in Ω' with the following properties:

$$(7.12) \quad f \in H_{loc}^{-1/2}(\Omega'),\ f \notin H_{loc}^0(\Omega'),\ WF(f) \subset \Gamma_0' \quad \textit{but} \quad \Lambda_J f \in H_{loc}^0(\Omega').$$

If we apply (6.11) we see that $R_J f \in H_{loc}^0(\Omega)$. On the other hand we have

$$G_J \Lambda_J f \in H_{loc}^0(\Omega),$$

also by (6.11) where we take $s = \frac{1}{4}$, $m = \frac{1}{4}$ (the Hermite operators H_j are of degree $\frac{1}{4}$). Thus

$$PG_J f \in H_{loc}^0(\Omega).$$

Since the Hermite operators "decrease" the wave-front set, we have $WF(G_J f) \subset \Gamma_0$ where Γ_0 is an arbitrary conic neighborhood of $\Gamma_0' \subset \Sigma$ in $\dot{T}^*\Omega$. Finally we observe that we have $G_J f \notin H_{loc}^0(\Omega)$, otherwise (by (6.12)) we should have $G_J^* G_J f = f$ (by (6.14)) also in $H_{loc}^0(\Omega)$. This contradicts $(7.1')_0$ if we recall that now P is of order *one*.

2°) (7.2)⇒(7.3)

In this part of the proof also the idea is to decompose the functions (or rather the distributions) space under study into the orthogonal sum of the range of X^J (J-th power of X) and the kernel of X^{*J}, and to use a description of these subspaces by means of Hermite operators. Here again we shall exploit the fact that, in this decomposition, the matrix representing P will be approximately triangular, with "good" diagonal coefficients. Actually we shall prove the assertion by induction on $J=0,1,\ldots$: we shall prove that, if we make the assumption that the assertion is true when $\operatorname{Re} l_P \leqq J-1$, then it is also true when $\operatorname{Re} l_P \leqq J$. We know that it is true when $\operatorname{Re} l_P \leqq -1$ (that is to say, for $J=0$). Indeed, in that case, $\operatorname{Re} l_P \leqq -\frac{1}{2}$, and consequently the subelliptic estimate (4.2) holds (if Γ_0 is sufficiently "small"). By virtue of Th. 3.1 and of Remark 3.1, we see that (7.3) holds in this case.

By virtue of Prop. 5.2 we see that we can modify X so as to have $PX \sim X(P-Q)$, which means essentially that the range of X is stable under P. Let $K=K_X$ be the Hermite operator associated with X at the end of Section 6. Modulo C^∞, KK^* is a projector on $\operatorname{Ker} X^*$ whereas $I-KK^*$ is a projector on the range of X. Then, in the corresponding "orthogonal" decomposition (the quotation marks recur, due to the fact that everything here is modulo C^∞ or modulo operators of order $-\infty$), P can be represented by a triangular matrix

$$P \sim \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

(here, $\sim$ means $\equiv$ mod an operator with C^∞ kernel), where

$$\begin{aligned} &A=(I-KK^*)P(I-KK^*) \sim P(I-KK^*), \\ (7.13) \qquad &B=(I-KK^*)P(KK^*), \\ &C=(KK^*)P(KK^*) \sim KK^*P. \end{aligned}$$

Thus P will satisfy (7.3) or (7.3)* if and only if both diagonal entries A and C satisfy the analogues on their respective domains.

We shall reason throughout in a conic open set $\Gamma_1 \subset \dot{T}^*\Omega$ whose projection in the cosphere bundle is sufficiently small. All the statements which now follow must be understood *in* Γ_1.[3]

Since (Prop. 5.2) $\sigma(Q)=\sigma([X,Y])$, we have $l_{P-Q}=l_P-1$, and therefore (7.3) holds for $P-Q$ by the induction hypothesis.

Suppose that $f=Xg$ mod C^∞ and that $Pf \in \mathscr{H}^{s,k}_{loc}$. Then $X(P-Q)g \in \mathscr{H}^{s,k}_{loc}$ and consequently, $(P-Q)g \in \mathscr{H}^{s+1,k+1}_{loc}$. By the induction hypothesis,

[3] This represents a further microlocalization: according to the remarks at the beginning of the proof we have already used a microlocalization, to reduce the situation to $\Omega=\mathbb{R}^{n+1}$, $\Sigma=\dot{T}^*\mathbb{R}^n$.

we derive from this that $g \in \mathscr{H}_{loc}^{s+m+1,k+3}$, which implies $f \in \mathscr{H}_{loc}^{s+m,k+2}$ (we recall that the order of P is m; also, in accordance with the statements in Section 6, we are tacitely assuming that $X \in \mathscr{N}^{1,1}$).

Suppose now that $h = Xg \bmod \mathscr{C}^\infty$ and that $h \in \mathscr{H}_{loc}^{s,k}$. Then $g \in \mathscr{H}_{loc}^{s+1,k+1}$ and there exists $f_1 \in \mathscr{H}_{loc}^{s+m+1,k+3}$ such that $(P-Q)f_1 = g \bmod C^\infty$. Then $f = Xf_1 \in \mathscr{H}_{loc}^{s+m,k+2}$ and satisfies

$$Pf = PXf_1 = X(P-Q)f_1 = h \bmod C^\infty.$$

This proves that the first diagonal entry A satisfies (7.3) on its domain (and terminates the role of the conic set Γ_1).

In what concerns the second diagonal entry, C, let us introduce

$$L_P = K^* PK; \tag{7.14}$$

L_P is a pseudodifferential operator of degree $m-1$ on Ω'. Writing, as usual, $P = XY + Z$ and using the fact that $K^* X \sim 0$, we obtain

$$L_P \sim K^* ZK, \tag{7.15}$$

and, consequently, by (6.6), (6.9) and (6.20),

$$\sigma(L_P) = \sigma(Z) = \sigma([X, Y])|_\Sigma \, l_P, \tag{7.16}$$

which shows that L_P is elliptic. Let E be a parametrix of L_P. We have

$$KK^* \sim K\, EK^*\, PK\, K^* \sim KK^*\, PKE\, K^*.$$

This, together with (6.11), implies easily that C satisfies (7.3) on its domain.

The proof of Th. 7.1, and therefore also that of Th. 1.1, is complete,

8. Remarks about the Operator L_P

If Condition (1.23) $(=(7.2))$ is not verified, it does not necessarily mean that P is not hypoelliptic or not locally solvable. In the "abstract" set-up, necessary and sufficient conditions for the latter to take place can be found in [9]. A similar approach in the present situation shows the following. The inductive proof, presented in Section 7, uses the concatenation of operators P_j, which satisfy $P_j X_j \sim X_j P_{j+1}$, and leads to the consideration of the associated pseudodifferential operators on the "boundary" Ω',

$$L_{P_j} = K_{X_j}^* P_j K_{X_j}.$$

We know that P_j satisfies (1.23) if j is large enough and therefore the microlocal behavior of P is completely controlled by the operators L_{P_j}. Observe that $\sigma(L_{P_j}) = \sigma([X, Y])|_\Sigma \cdot l_{P_j} = \sigma(L_{P_0}) - j\,\sigma([X, Y])|_\Sigma$. Thus, at any given point of Σ all the L_{P_j} are elliptic, except possibly one of them: at that point, the microlocal behaviour of P is completely determined by the corresponding operator L_{P_j}. For all practical purposes, we may assume that the index j in question is zero, in other words that $P_j = P$ and that it is the first L_{P_j}, L_P, which is not elliptic. There is nothing more to say about P unless one makes further assumptions, for it is quite clear that L_P can be *any* pseudodifferential operator on Ω' (of order $\leqq m-1$). The following can be asserted (the proof of this assertion is analoguous to those in Section 5, and we leave it to the reader):

(8.1) $L_P \sim 0$ *if and only if* $P \sim XY$ (i.e., $Z \sim 0$) *for a suitable choice of* X *and* Y.

By Prop. 5.2 we have seen that, if we choose X in a suitable manner, we may find a pseudodifferential operator Q of degree $\leqq m-1$ such that $PX = X(P-Q)$ which implies, of course, that P preserves the range (or image) of X. Similarly, we can choose Y such that, for a suitable pseudodifferential operator Q',

$$YP \sim (P+Q')\, Y, \tag{8.2}$$

which means that P preserves the null space (or kernel) of Y (in all this, images and kernels must be understood modulo C^∞ functions: we are really talking about microfunctions in a small conic open subset of $\dot{T}^*\Omega$).

We can then construct a Hermite operator K_Y analoguous to K_X but with X replaced by Y^*, and define

$$L'_P = K_Y^* P K_Y.$$

We know that $g \in \operatorname{Im} X \Leftrightarrow K_X^* g = 0$ (both properties, and all the forthcoming are mod C^∞) and that P preserves $\operatorname{Im} X$; the projector on $\operatorname{Im} X$ is $I - K_X K_X^*$, whence

$$K_X^* P (I - K_X K_X^*) \sim 0. \tag{8.3}$$

Similarly, since P preserves $\operatorname{Ker} Y (\operatorname{mod} \mathscr{C}^\infty)$, we have:

$$(I - K_Y K_Y^*)\, P K_Y \sim 0. \tag{8.4}$$

By combining (8.3) and (8.4) we get

$$K_X^* P K_Y \sim L_P K_X^* K_Y \sim K_X^* K_Y L'_P. \tag{8.5}$$

It follows from (6.12), (6.20) and from the standard formula

$$(\lambda/\pi)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\lambda t^2}\, dt = 1 \qquad \textit{if } \operatorname{Re} \lambda > 0,$$

that $A = K_X^* K_Y$ is *elliptic* of degree zero in Ω'. Thus (8.5), which reads

$$L_P A \sim A L'_P, \tag{8.6}$$

shows that the *similitude classes* of L_P and $L_{P'}$ are the same (the similitude class of an operator S is the set of all operators T which can be written in the form QSQ^{-1}; as we deal here with pseudodifferential operators, Q must be elliptic). We notice that, in defining L_P and L'_P, we chose X and Y independently of each other – submitted only to the proviso that X be the hypoelliptic factor and Y the "solvable" one (i.e., the one with large kernel). Note also that the choices of the Hermite operators K_X, K_Y were arbitrary – provided that the appropriate relations held. We may summarize this as follows:

Proposition 8.1. *The similitude class of L_P only depends on that of P. If X or K_X is changed, L_P is replaced by $CL_P C^{-1}$ with C, an elliptic pseudodifferential operator on Ω'.*

We recall that we are reasoning microlocally and that we may therefore identify Σ to the cotangent bundle of a smooth submanifold Ω' of Ω with codim $\Omega' = 1$.

In fact, if we use the final remark in Section 6 we may even show that if we replace P by BPB^{-1} where B is an elliptic Fourier integral operator associated with a canonical transformation preserving Σ, L_P gets replaced by $CL_P C^{-1}$ where C is an elliptic Fourier integral operator on Ω' associated with the canonical transformation induced on $\Sigma = \dot{T}^* \Omega'$ by the one in $\dot{T}^* \Omega$.

Indeed, let us choose

$$X' = BXB^{-1}, \qquad K_{X'} = B^{*-1} K_X C^*,$$

where C is an elliptic Fourier integral operator (on Ω') such that

$$(C^* C)^{-1} \sim K_X^* B^{-1} B^{*-1} K_X.$$

We obtain; by virtue of this choice and of (8.3),

$$L_{BPB^{-1}} = K_{X'}^* (BPB^{-1}) K_{X'} = CK_X^* PK_X^{*-1} C^{-1} \sim CL_P C^{-1}.$$

For the sake of completeness, we wish to mention here the composition laws (modulo regularizing operators) for Hermite operators. Their proofs are similar to those for pseudodifferential operators, and we do not give them here. In the formulas below, σ denotes the *total* symbol; as usual, this can be interpreted as an element of an appropriate symbol space, or else as an asymptotic sum of semihomogeneous functions. Let K, K' be two Hermite operators, A a pseudodifferential operator on $\mathbb{R}^{n+1}$, B one on $\mathbb{R}^n$ (we are reasoning within the "flat" framework of

Section 6; here $N=n+1$). We have (with the notation of Section 6):

$$(8.7)\qquad \underline{\sigma}(K^* K')=\sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(\partial/\partial x)^\alpha \int_{-\infty}^{+\infty} \{(\partial/\partial\xi)^\alpha \overline{\underline{\sigma}(K)}\}\, \underline{\sigma}(K')\,dt,$$

$$(8.8)\qquad \underline{\sigma}(KB)=\sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}[(\partial/\partial\xi)^\alpha \underline{\sigma}(K)]\,(\partial/\partial x)^\alpha \underline{\sigma}(B);$$

$$(8.9)\quad \underline{\sigma}(AK)=\sum_{\alpha,p,q} \frac{i^{-|\alpha|-q}}{\alpha!}[(\partial/\partial\xi)^\alpha a_{p,q}(x,\xi)]\,\{t^p(\partial/\partial t)^q(\partial/\partial x)^\alpha \underline{\sigma}(K)\}$$

where $\underline{\sigma}(A)\sim \sum_{p,q} a_{p,q}(x,\xi)^{+p}\tau^q$ is the (formal) Taylor expansion of $\underline{\sigma}(A)$ about Σ (defined by $t=\tau=0$).

Together with the constructions in Section 6, with those of [2] and with the calculus of Fourier Integral operators ([3, 8]), the above rules provide, in principle, a tool for computing explicitely (by successive approximations) the pseudodifferential operator L_P (for an analogous construction, cf. [7]).

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Involutions on Homotopy Spheres

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In this paper we study differentiable orientation preserving involutions on S^{n+k} with non-trivial k dimensional fixed set. Define a cobordism relation between involutions T_0 and T_1 by requiring that there exist an involution T on $S^{n+k} \times I$ whose restriction to the ends yields T_0 and T_1. The set of cobordism classes becomes a group $\mathscr{A}(n+k, k)$ by taking connected sum about a fixed point. We assume throughout that $5 \leqq k \leqq n-3$.

By Smith theory the fixed set of T_0, $\mathrm{Fix}(T_0)$, is a differentiable k-manifold with the Z_2-homology of a sphere, and $\mathrm{Fix}(T)$ is a cobordism between $\mathrm{Fix}(T_0)$ and $\mathrm{Fix}(T_1)$ such that $H_*(\mathrm{Fix}(T), \mathrm{Fix}(T_i); Z_2) = 0$ for $i = 0, 1$. We call such a manifold a Z_2-*cobordism* and make the set of Z_2-homology k-spheres into a group $\theta_k^{(2)}$ under this relation via connected sum.

Let θ_k denote the group of homotopy k-spheres studied by Kervaire and Milnor [10]. To calculate $\theta_*^{(2)}$ we study the natural map $\theta_* \to \theta_*^{(2)}$ and prove

Theorem 1. *There is for each k an exact sequence*

$$0 \to A_k \to \theta_k \to \theta_k^{(2)} \to B_k \to 0$$

in which

$$A_k \approx \left(\frac{\Omega_k^{fr}}{\mathrm{Im}\, J}\right)_{\mathrm{odd}}$$

and

$$B_k \approx \begin{cases} \bigoplus_{\pi(m)-1} G \oplus (\oplus Z_2) \oplus (\oplus Z_4), & k = 4m-1 \\ 0, & \text{otherwise} \end{cases}$$

where $G = Z_{(2)}/Z$, $\pi(m)$ is the number of partitions of m and $Z_{(2)}$ denotes the integers localized at 2.

This result may be combined with results of Jones [9] and the Brieskorn-Hirzebruch examples to prove:

Theorem 2. *Every homotopy sphere Σ^k of odd order in θ_k may be written uniquely as a connected sum of an odd order element of bP_{k+1} and a homotopy sphere that bounds a $\mathbb{Z}_2$-disk. Thus each such Σ^k is the fixed set of an involution on S^{n+k}.*

Smith theory also says that if an involution on S^{n+k} extends to the disk D^{n+k+1}, then the Z_2-sphere fixed set in S^{n+k} must bound a Z_2-disk. We use this necessary condition to establish:

Theorem 3. *The non-trivial elements of* bP_{k+1} *do not bound* $\mathbb{Z}_2$*-disks. Thus any involution on a standard sphere leaving such an element fixed does not extend to the disk.*

In particular the Brieskorn-Hirzebruch involutions give non-zero elements of $\mathscr{A}(n+k, k)$. It has indeed been shown by Browder and Petrie [5] and Bredon [13] that some Brieskorn-Hirzebruch examples yield elements of infinite order.

We analyze the fixed sets more thoroughly by exploiting the natural homomorphism $\mathscr{A}(n+k, k) \to \theta_k^{(2)}$ that assigns to an involution its fixed set.

Theorem 4. *If* V^k *is a* $\mathbb{Z}_2$*-sphere, there is a homotopy sphere* Σ^k *belonging to the* 2*-primary component of* θ_k *and an involution on a homotopy sphere* Σ^{n+k} *belonging to the* 2*-primary component of* θ_{n+k} *with fixed set* $V^k \# \Sigma^k$, *where* $n \equiv 2 \bmod 4$. *If* V^k *has odd order in* $\theta_k^{(2)}$ *we may take* $\Sigma^k = S^k$ *and* $\Sigma^{n+k} = S^{n+k}$.

The paper is divided up as follows. In Sections 1 and 2 we develop the properties of Z_2-spheres analogous to those for homotopy spheres studied in [10]. In Section 3 we study the homomorphism $\theta_* \to \theta_*^{(2)}$ and prove Theorems 1, 2 and 3. In the last section we analyze the homomorphism $\mathscr{A}(n+k, k) \to \theta_k^{(2)}$ and prove Theorem 4.

In regard to Theorem 1, the referee has pointed out that in his thesis [14], P. Lynch studied framed G-homology cobordism classes of framed G-homology spheres and their relations to framed homotopy spheres.

1. Mod 2 Cobordism

Let V_1^k and V_2^k be differentiable oriented Z_2-homology spheres of dimension k. V_1 is *Z_2-cobordant* to V_2 if there exists a compact oriented manifold W^{k+1} such that

(i) ∂W^{k+1} is oriented diffeomorphic to $V_1 \cup -V_2$, and

(ii) $H_*(W, V_i; Z_2) = 0$ for $i = 1, 2$.

The set of equivalence classes under this relation is denoted by $\theta_k^{(2)}$. As in the case of homotopy spheres [10], the operation of connected sum gives $\theta_k^{(2)}$ the structure of an abelian group.

Recall from [1] that if M^k is an oriented differentiable manifold, an *odd framing* of M is a null homotopy of the composition

$$M^k \xrightarrow{\nu} BSO \xrightarrow{\ell} BSO_{(2)}$$

where ν is the classifying map for the stable normal bundle and ℓ is localization at the prime 2 [12]. If M_1^k and M_2^k are odd framed manifolds, then M_1 is odd framed cobordant to M_2 if there exists an odd framed W^{k+1} such that

(i) ∂W is oriented diffeomorphic to $M_1 \cup -M_2$, and

(ii) The odd framing on W extends the odd framings on M_1 and M_2.

If V^k is a Z_2-sphere, then the only obstruction to the existence of an odd framing of V is a class in $H^k(V; \pi_{k-1}(SO_{(2)}))$. Where $SO_{(2)}$ is the localization of SO whose homotopy is given by the following table

k mod 8	1	2	3	4	5	6	7	8
$\pi_{k-1}(SO_{(2)})$	Z_2	Z_2	0	$Z_{(2)}$	0	0	0	$Z_{(2)}$

where $Z_{(2)}$ denotes the integers localized at the prime ideal generated by 2. Using results of [1] this obstruction must vanish, so we have

(1.1) **Lemma.** *If V^k is an oriented differentiable Z_2-sphere then V^k admits an odd framing.* □

The cobordism group of oriented odd framed manifolds $\Omega_*^{\mathrm{fr}(2)}$ defined and computed in [1] is associated with the Thom spectrum of the bundle over $SO_{(2)}/SO$ given by the map $\varepsilon: SO_{(2)}/SO \to BSO$, where ε is the inclusion of the fibre in the fibration $SO_{(2)}/SO \xrightarrow{\varepsilon} BSO \xrightarrow{\ell} BSO_{(2)}$. There is an associated "J homomorphism"

$$J': \pi_k(SO_{(2)}) \to \Omega_k^{\mathrm{fr}(2)}$$

which assigns to any homotopy class the induced odd framing of the sphere.

Given a class $[V]$ in $\theta_k^{(2)}$ we choose an odd framing for V, producing a representative of a cobordism class in $\Omega_k^{\mathrm{fr}(2)}$. This class is well-defined modulo the image of J', so for each k there is a homomorphism

$$\Phi_k: \theta_k^{(2)} \to \Omega_k^{\mathrm{fr}(2)}/\mathrm{Im}\, J'.$$

If we denote by $bP_{k+1}^{(2)}$ the subgroup of $\theta_k^{(2)}$ consisting of all classes represented by k-dimensional Z_2-spheres which bound compact manifolds admitting an odd framing, then $bP_{k+1}^{(2)}$ is equal to the kernel of Φ_k. Since every homotopy sphere is a Z_2-sphere, there is a natural homomorphism

$$i: \theta_k \to \theta_k^{(2)}$$

which takes bP_{k+1} into $bP^{(2)}_{k+1}$. There is also the homomorphism

$$j\colon \Omega_k^{\mathrm{fr}}/\mathrm{Im}\, J \to \Omega_k^{\mathrm{fr}(2)}/\mathrm{Im}\, J'$$

studied in [1].

(1.2) **Lemma.** *For each k there is a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & bP_{k+1} & \longrightarrow & \theta_k & \xrightarrow{\Psi_k} & \Omega_k^{\mathrm{fr}}/\mathrm{Im}\, J \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ 0 & \longrightarrow & bP^{(2)}_{k+1} & \longrightarrow & \theta^{(2)}_k & \xrightarrow{\Phi_k} & \Omega_k^{\mathrm{fr}(2)}/\mathrm{Im}\, J' \end{array}$$

where the rows are exact sequences. □

2. Mod 2 Surgery

In order to analyze the diagram in (1.2) and obtain information about the groups $bP^{(2)}_{k+1}$ and $\theta^{(2)}_k$ we develop some surgery techniques for Z_2-spheres and odd framed manifolds which parallel those of Kervaire and Milnor [10] for homotopy spheres and framed manifolds. We will give only a few proofs since many of the arguments are similar to those in [10].

Since all of the Stiefel-Whitney classes of an odd framed manifold are zero, the following is not difficult to show:

(2.1) **Lemma.** *Every odd framed manifold V^k is odd framed cobordant to a simply connected manifold. Furthermore, if V is a Z_2-sphere then we can make the cobordism a Z_2-cobordism.* □

Let M^{k+1} be an odd framed manifold with $\partial M = V^k$ (possibly empty) such that both M and V are simply connected. Let $\mathscr{C}$ denote the Serre class of finite abelian groups of odd order. By [11] the following are equivalent:

(i) $\pi_q(M) \in \mathscr{C}$ for $1 \leqq q < r$

(ii) $H_q(M) \in \mathscr{C}$ for $1 \leqq q < r$

and either of these implies that $\pi_r(M) \approx H_r(M) \bmod \mathscr{C}$.

Suppose we have used surgery to make M $\mathscr{C}$-connected through dimension $r-1$. There is a subgroup Γ_r of $H_r(M)$ whose quotient is a group of finite odd order, and Γ_r is generated by spherical classes. Let $\phi\colon S^r \to M$ represent a generator of Γ_r. Since ν_M has odd order in $\widetilde{KO}(M)$, so does the stable tangent bundle τ_M. If $r < \left[\dfrac{k+1}{2}\right]$ then ϕ is homotopic to an imbedding. Now $\phi^*(\tau_M) = 0$ in $\widetilde{KO}(S^r)$ since KO^* has no elements of odd order; hence, $0 = \phi^*(\tau_M) = \tau_{S^r} \oplus \nu_\phi$ and ν_ϕ must be stably trivial. It is therefore possible to alter M by surgery to produce M' with $H_q(M')\in\mathscr{C}$ for $1\leqq q\leqq r$.

We must check that the trivializations chosen in each surgery will result in a new manifold with odd order normal bundle. So suppose

$$\phi\colon S^r\times D^{k+1-r}\to M$$

is a differentiable imbedding. Let $A=\phi(S^r\times D^{k+1-r})$. The odd framing on M induces an odd framing on A:

$$\begin{array}{ccccc} A & \subseteq & M & \to & BSO \\ \downarrow & & \downarrow & & \downarrow \ell \\ CA & \subseteq & CM & \to & BSO_{(2)} \end{array}$$

where CX stands for the cone on X. The obstruction to lifting the bottom composition to BSO is an element of

$$H^{r+1}(CA, A; \pi_r(SO_{(2)}/SO))\approx\pi_r(SO_{(2)}/SO)\approx\begin{cases} Z_{(2)}/Z & r\equiv 3 \bmod 4 \\ 0 & \text{otherwise.}\end{cases}$$

Let $\gamma(\phi)$ denote this obstruction. If $r\equiv 3 \bmod 4$ then $\gamma(\phi)=s/t$, where t is an odd integer and $(s,t)=1$. Pick a map $\psi\colon S^r\to S^r\times D^{k+1-r}$ which is of degree t from $H_r(S^r)$ to $H_r(S^r\times D^{k+1-r})$. By results of Haefliger [7] ψ is homotopic to an imbedding.

(2.2) **Lemma.** *If ν_ψ denotes the normal bundle to this imbedding, then ν_ψ is trivial.*

Proof. For $r<k+1-r$ the result follows since $\tau(S^r\times D^{k+1-r})$ is trivial. In the case $r=\dfrac{k+1}{2}\equiv 0 \bmod 2$, ν_ψ is stably trivial so it is determined by its Euler class. This is clearly zero. When $r=\dfrac{k+1}{2}\equiv 1 \bmod 2$, [10, Lemma 8.3] shows ν_ψ is trivial. □

Denote the imbedding given by the composition

$$S^r\times D^{k+1-r}\xrightarrow{\psi}S^r\times D^{k+1-r}\xrightarrow{\phi}M$$

by $t\cdot\phi$ and let $A_t=t\phi(S^r\times D^{k+1-r})\subseteq A$. By the commutativity of the diagram

$$\begin{array}{ccccccc} A_t & \subseteq & A & \subseteq & M & \to & BSO \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \ell \\ cA_t & \subseteq & cA & \subseteq & CM & \to & BSO_{(2)} \end{array}$$

the obstruction to lifting $CA_t\to BSO_{(2)}$, $\gamma(t\phi)$, is t times the obstruction to lifting $CA\to BSO_{(2)}$, $\gamma(\phi)$, and therefore is zero. Thus A_t may be given an actual framing compatible with its odd framing induced by M. By restricting $t\phi$ to a smaller disk we get a proper imbedding

$$\psi: S^r \times D^{k+1-r} \to A_t \subseteq M.$$

In [10] it is shown that the framing over A_t may be extended over the handle attached using this imbedding.

Using the above notation we have proven:

(2.3) **Lemma.** *Given an imbedding $\phi: S^r \times D^{k+1-r} \to M$ there exists an odd integer t such that*

$$W = M \bigcup_{t\phi} D^{r+1} \times D^{k+1-r}$$

is an odd framed manifold extending the odd framing on M. □

Following the program in [10], for odd dimensional manifolds we can establish:

(2.4) **Theorem.** *Let M be a compact oriented odd framed manifold of dimension $2\ell+1$, $\ell \geqq 2$, such that $\partial M = \emptyset$ or ∂M is a Z_2-sphere. By a sequence of odd framed surgeries M may be reduced to a manifold M_1 which is ℓ-connected mod $\mathscr{C}$.* □

There are obstructions to completing this surgery in the even dimensional cases. First let $k+1=4\ell$ and suppose that $M^{4\ell}$ has $\partial M^{4\ell} = V$ a Z_2-homology sphere and $\nu(M^{4\ell})$ has odd order in $\widetilde{KO}(M^{4\ell})$. By using the preceding techniques we can show that M is Z_2-cobordant mod V to a manifold $W^{4\ell}$ such that

(i) $\nu(W)$ has odd order, and

(ii) $H_q(W, V; Z_2) = 0$ for $0 \leqq q \leqq 2l-1$.

Let

$$Q: H_{2\ell}(W; Z_{(2)}) \otimes H_{2\ell}(W; Z_{(2)}) \to Z_{(2)}$$

be the intersection product on $Z_{(2)}$-homology. Since $\partial W = V$ is a $Z_{(2)}$-homology sphere, it follows that Q is a non-singular, even, symmetric bilinear form.

(2.5) **Lemma.** *W is Z_2-cobordant mod V to a Z_2-disk if and only if there is a subspace $F \subseteq H_{2\ell}(W; Z_{(2)})$ such that*

(i) $\operatorname{rank}_{Z_{(2)}} F = \frac{1}{2} \operatorname{rank}_{Z_{(2)}} H_{2\ell}(W; Z_{(2)})$ *and*

(ii) $Q(x, y) = 0$ *if $x, y \in F$.*

Proof. Let $x_1, \ldots, x_r$ be a $Z_{(2)}$-basis for $F \subseteq H_{2\ell}(W; Z_{(2)})$. By the Hurewicz-Serre theorem mod $\mathscr{C}$, the cokernel of

$$\pi_{2\ell}(W) \otimes Z_{(2)} \to H_{2\ell}(W; Z_{(2)})$$

is a finite group of odd order. It is possible to choose the basis elements $\{x_i\}$ so that they lie in the image of the natural map $\pi_{2\ell}(W) \to H_{2\ell}(W; Z_{(2)})$. Suppose x_i is represented by a map $\lambda_i: S^{2\ell} \to W$. Since W is simply connected and $x_i \cdot x_i = 0$, we can imbed $S^{2\ell}$ in W with a trivial normal bundle.

Now extend $\{x_1, \ldots, x_r\}$ to a basis by adding $\{y_1, \ldots, y_r\}$ where $y_i \in \text{Image}\,\{H_{2\ell}(W; Z) \to H_{2\ell}(W; Z_{(2)})\}$ and $x_i \cdot y_j = \delta_{ij} \cdot \alpha_i$, $\alpha_i \equiv 1 \bmod 2$. The diagram [10, p. 527] shows that surgery to kill x_1 also kills y_1, but adds a finite odd torsion group. That is

$$H_{2\ell}(W_1; Z) \approx H_{2\ell}(W; Z)/\langle x_1, y_1 \rangle \oplus Z_{\beta_1}$$

where $\beta_1 \equiv 1 \bmod 2$. After r surgeries we have $H_{2\ell}(W_r; Z)$ isomorphic to a finite odd torsion group or $H_{2\ell}(W_r; Z_2) = 0$. This implies W_r is a Z_2-sphere. □

Denote by G the Grothendieck group of even, symmetric bilinear forms over $Z_{(2)}$ with odd determinant, modulo the subgroup generated by those forms admitting a self-annihilating subspace of half the dimension. It follows from symmetry considerations that any such form must have even rank, hence even index. Given an element g of G we may represent g by a symmetric integral matrix with even entries on the diagonal and odd determinant. Using this matrix as a plumbing schematic, we produce a parallelizable 4ℓ-manifold M with $\partial M = V$ an element of $bP_{4\ell}^{(2)}$. From (2.5) this is a well-defined homomorphism which clearly maps G onto $bP_{4\ell}^{(2)}$.

In the next section the natural map $bP_{4\ell} \to bP_{4\ell}^{(2)}$ will be shown to be a monomorphism. With this in mind, we have the following result on the structure of $bP_{4\ell}^{(2)}$.

(2.6) **Lemma.** *$bP_{4\ell}^{(2)} \approx K \oplus (\oplus Z_2) \oplus (\oplus Z_4)$ where $bP_{4\ell} \subseteq K$ and $K/bP_{4\ell} \approx Z_4$.*

Proof. Let $V = \partial W \in bP_{4\ell}^{(2)}$. Since the Grothendieck group G maps onto $bP_{4\ell}^{(2)}$ we may use invariants of bilinear forms to classify V [8]. Specifically, V is classified modulo $bP_{4\ell}$ by the following invariants of the bilinear form $Q: H_{2\ell}(W; Z_{(2)}) \otimes H_{2\ell}(W; Z_{(2)}) \to Z_{(2)}$:

(i) rank of Q mod 8
(ii) index of Q mod 8
(iii) det Q mod squares in $Z_{(2)}$
(iv) the Hasse-Minkowski symbols of Q.

These are all Z_2 and Z_4 invariants. The element $V_0 \in bP_{4\ell}^{(2)}$ realizing the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has index 2 and $4V_0$ is Z_2-cobordant to the Milnor manifold generator for $bP_{4\ell}$. The difficulty in obtaining a precise calculation of $bP_{4\ell}^{(2)}$ lies in determining the relations among these invariants and is complicated by the fact that (iv) is not a homorphism. □

If the dimension of M is $4k+2$, there is a Kervaire invariant which appears as the obstruction to completing the surgery.

(2.7) **Lemma.** *The Kervaire invariant of an odd framed manifold is a cobordism invariant in* $\Omega_{4k+2}^{\mathrm{fr}(2)}$.

Proof. Let $B\langle v_{k+2}\rangle$ be the space obtained by killing the *Wu* class v_{k+2} in BSO. Browder has shown [4] that the Kervaire invariant may be defined in the cobordism theory corresponding to $B\langle v_{k+2}\rangle$. Now the fibre of the localization $B\langle v_{k+2}\rangle \xrightarrow{\ell} B\langle v_{k+2}\rangle_{(2)}$ is homotopy equivalent to $SO_{(2)}/SO$ and we have that odd framed cobordism maps naturally into this theory. □

Denoting this invariant by $c(M)\in Z_2$ we have the following analogs of theorems in [10].

(2.8) **Theorem.** *Let M be an odd framed manifold of dimension* $(4\ell+2)$ *such that* $\partial M=\emptyset$ *or* ∂M *is a* Z_2*-sphere. Then if* $c(M)=0$, *it is possible to alter M to be* $(2\ell+1)$*-connected by a sequence of odd framed surgeries.* □

(2.9) **Corollary.** *If* V_1 *and* V_2 *are* Z_2*-spheres of dimension* $(4k+1)$ *which bound odd framed manifolds* W_1 *and* W_2 *such that* $c(W_1)=c(W_2)$, *then* V_1 *is* Z_2*-cobordant to* V_2. □

3. Computation of Mod 2 Cobordism

In this section we apply the surgery results of § 2 to analyze the groups $\theta_k^{(2)}$ and $bP_{k+1}^{(2)}$ defined previously.

(3.1.) **Lemma.** $bP_{2k+1}^{(2)}=0$ *and* $\Phi_{2k+1}\colon \theta_{2k+1}^{(2)}\to\Omega_{2k+1}^{\mathrm{fr}(2)}/\mathrm{Im}\,J'$ *is an epimorphism for each* k.

Proof. This follows immediately from (2.2) since there is no obstruction to completing the surgery. □

(3.2) **Lemma.** $i\colon bP_{4k}\to bP_{4k}^{(2)}$ *is a monomorphism for each* k.

Proof. Consider the collection of all odd framed $4k$ manifolds having boundary a standard sphere. The signatures of these manifolds form a subgroup of the integers. Let q_k be a generator of that subgroup. Similarly, let σ_k be the least positive index of a stably parallelizable $4k$-manifold with standard sphere boundary so that $bP_{4k}\approx 8\,Z/\sigma_k Z$ [10].

It is apparent that q_k divides σ_k. The lemma will follow if we can show that $q_k=\sigma_k$.

Let W^{4k} be an odd framed manifold with standard sphere boundary. Fill in the sphere with a $4k$ disk to produce a closed manifold $\overline{W}^{4k}$ having $I(\overline{W}^{4k})=I(W^{4k})$. Since $\overline{W}^{4k}$ is an almost odd framed manifold, it must admit a spin structure. Furthermore all of its decomposable Pontryagin numbers vanish because the lower dimensional Pontryagin classes have odd order.

As observed by Brumfiel [6], any such manifold has the property that $a_k(2k-1)!\, j_k$ divides the indecomposable number $p_k(\overline{W}^{4k})$, where j_k is the denominator of $\frac{B_k}{4k}$, B_k is the k-th Bernoulli number, and a_k equals 1 or 2 according as k is even or odd.

By the Hirzebruch signature theorem

$$I(W^{4k})=I(\overline{W}^{4k})=\frac{2^{2k}(2^{2k-1}-1)\,B_k\,p_k(\overline{W}^{4k})}{(2k)!}.$$

So there exists an integer m with

$$\begin{aligned} I(\overline{W}^{4k}) &= \frac{2^{2k}(2^{2k-1}-1)\,B_k\cdot m\cdot a_k(2k-1)!\,j_k}{(2k)!} \\ &= m\cdot\left[2^{2k+1}(2^{2k-1}-1)\cdot a_k\cdot \operatorname{num}\left(\frac{B_k}{4k}\right)\right]. \end{aligned}$$

In particular the integer q_k can be written in this form for some integer m. Kervaire and Milnor [10] have shown that

$$\sigma_k=2^{2k+1}(2^{2k-1}-1)\cdot a_k\cdot \operatorname{num}\left(\frac{B_k}{4k}\right),$$

hence σ_k divides q_k. It follows that $\sigma_k=q_k$ and i is a monomorphism. □

From § 2 we see that $bP^{(2)}_{4k+2}$ will either be trivial or cyclic of order 2. Similarly the cokernel of $\Phi_{4k+2}\colon \theta^{(2)}_{4k+2}\to\Omega^{\mathrm{fr}(2)}_{4k+2}/\mathrm{Im}\,J'$ will either be 0 or Z_2.

(3.3) **Lemma.** *The following are equivalent:*

(a) *there is an odd framed* $(4k+2)$*-manifold* W *with non-zero Kervaire invariant having* ∂W *a standard sphere;*

(b) *the cokernel of* $\Phi_{4k+2}\colon \theta^{(2)}_{4k+2}\to\Omega^{\mathrm{fr}(2)}_{4k+2}/\mathrm{Im}\,J'$ *is* Z_2;

(c) *the cokernel of* $\Psi_{4k+2}\colon \theta_{4k+2}\to\Omega^{\mathrm{fr}}_{4k+2}/\mathrm{Im}\,J$ *is* Z_2;

(d) *there is a framed* $(4k+2)$*-manifold* W' *with nonzero Kervaire invariant having* $\partial W'$ *a standard sphere.*

Proof. We show a ⇒ b ⇒ c ⇒ d ⇒ a. Given W, attach a $(4k+2)$-disk to the boundary to give a closed manifold $\overline{W}$ with non-zero Kervaire invariant. As in (1.1) the odd framing on W extends to $\overline{W}$, so $\overline{W}$ represents a non-trivial element in the cokernel of Φ_{4k+2}.

Since the Kervaire invariant is additive, $\overline{W}$ must have even order in $\Omega_{4k+2}^{\mathrm{fr}(2)}/\mathrm{Im}\,J'$. It was shown in [1] that the homomorphism j: $\Omega_{4k+2}^{\mathrm{fr}}/\mathrm{Im}\,J \to \Omega_{4k+2}^{\mathrm{fr}(2)}/\mathrm{Im}\,J'$ is a 2-primary isomorphism. Thus there exists an element of $\Omega_{4k+2}^{\mathrm{fr}}/\mathrm{Im}\,J$ represented by a manifold having non-zero Kervaire invariant. Call this manifold $\overline{W}'$. Then the cokernel of Ψ_{4k+2} is Z_2.

Removing a disk from $\overline{W}'$ gives the desired manifold W', so c$\Rightarrow$d, and it is evident that d$\Rightarrow$a. □

Recall that for each k there exists a frames $(4k+2)$-manifold with non-zero Kervaire invariat having homotopy sphere boundary and that this homotopy sphere is a standard sphere if and only if $bP_{4k+2}=0$. Similar remarks hold for $bP_{4k+2}^{(2)}$, so we see by (3.3) that $bP_{4k+2} \approx bP_{4k+2}^{(2)}$ for each k.

To see that the isomorphism is induced by i, let Σ be a homotopy sphere bounding a framed manifold W with non-zero Kervaire invariant. Let B be any Z_2-cobordism between Σ and a Z_2-sphere V. The framing over W may be extended to an odd framing over $W \cup_{\Sigma} B$. Since the inclusion maps induce isomorphisms

$$H_*(W, \Sigma; Z_2) \xrightarrow{\approx} H_*(W \cup B, B; Z_2) \xleftarrow{\approx} H_*(W \cup B, V; Z_2)$$

the Kervaire invariant of $W \cup B$ must be non-zero. If V is diffeomorphic to the standard sphere, then $bP_{4k+2}^{(2)}=0$, hence $bP_{4k+2}=0$ and $[\Sigma]=0$. This shows i is a monomorphism, so we have established:

(3.4) **Corollary.** *i: $bP_{4k+2} \to bP_{4k+2}^{(2)}$ is an isomorphism for each k.* □

We now summarize the previous results.

(3.5) **Theorem.** *For each n there is a commutative diagram*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & bP_{n+1} & \longrightarrow & \theta_n & \xrightarrow{\Psi_n} & \Omega_n^{\mathrm{fr}}/\mathrm{Im}\,J & \longrightarrow & \mathrm{coker}\,\Psi_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \approx} & & \\
0 & \longrightarrow & bP_{n+1}^{(2)} & \longrightarrow & \theta_n^{(2)} & \xrightarrow{\Phi_n} & \Omega_n^{\mathrm{fr}(2)}/\mathrm{Im}\,J' & \longrightarrow & \mathrm{coker}\,\Phi_n & \longrightarrow & 0
\end{array}$$

with exact rows, where coker Ψ_n is either 0 or Z_2 and i: $bP_{n+1} \to bP_{n+1}^{(2)}$ is monic for $n \equiv 3 \bmod 4$ and an isomorphism for $n \not\equiv 3 \bmod 4$. □

(3.6) **Corollary.** *The Brieskorn-Hirzebruch examples of involutions on standard spheres leaving non-trivial elements of bP_{4n} fixed do not extend to the disk.*

Proof. The composition $bP_{4n} \to \theta_{4n} \to \theta_{4n}^{(2)}$ is monic. Hence no non-trivial element of bP_{4k} bounds *a* Z_2-disk. Therefore none of these actions may be extended to the disk. □

We now analyze the homomorphism $i\colon \theta_n \to \theta_n^{(2)}$ by applying the snake lemma to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & A_n & \longrightarrow & A'_n & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & bP_{n+1} & \longrightarrow & \theta_n & \xrightarrow{\Psi_n} & \text{image } \Psi_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & bP^{(2)}_{n+1} & \longrightarrow & \theta^{(2)}_n & \xrightarrow{\Phi_n} & \text{image } \Phi_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_n & \longrightarrow & B_n & \longrightarrow & B'_n & &
\end{array}$$

The top and bottom rows are the kernels and cokernels of the respective vertical homomorphisms. It was shown in [1] that j is a 2-primary isomorphism with kernel isomorphic to $\Omega^{fr}_{n(\mathrm{od})}/\mathrm{Im}\, J_{(\mathrm{od})}$. Since H_n is 2-primary by (2.6) it follows that $A_n \approx A'_n$ for each n.

From the computation of the cokernel B'_n in [1] we see that for $n \not\equiv 3 \mod 4$, $H_n = 0$ and $B'_n = 0$ so that $B_n = 0$. Since B'_{4k-1} is odd-primary, B_{4k-1} splits as a direct sum of H_{4k-1} and B'_{4k-1}. This establishes the following description of the groups $\theta_n^{(2)}$.

(3.7) **Theorem.** *For each n there is an exact sequence*

$$0 \to A_n \to \theta_n \xrightarrow{i} \theta_n^{(2)} \to B_n \to 0$$

in which $A_n \approx \Omega^{fr}_{n(\mathrm{od})}/\mathrm{Im}\, J_{(\mathrm{od})}$ *and*

$$B_n \approx \begin{cases} (\oplus Z_2) \oplus (\oplus Z_4) \oplus \left(\bigoplus_{\pi(k)-1} G \right) & \text{for } n = 4k-1 \\ 0 & \text{otherwise} \end{cases}$$

where $G = Z_{(2)}/Z$ *and* $\pi(k)$ *is the number of partitions of* k. □

4. Applications to Involutions

Lowell Jones proved the following in [9]:

(4.1) **Theorem.** *If* $2k+2 < r$ *and* K *is a* Z_2*-disk of dimension* k*, then there exists an involution* $T\colon D^r \to D^r$ *with fixed point set diffeomorphic to* K. □

His proof can easily be modified to give:

(4.2) **Theorem.** *If* T_0 *is an involution on* S^{n+k} *leaving* V_0^k *fixed,* $n > k+2$ *and* W^{k+1} *is a* Z_2*-cobordism from* V_0 *to* V_1 *then there exists an involution* T

on $S^{n+k} \times I$ *such that*

(i) $T_{S^{n+k} \times \{0\}} = T_0$, *and*

(ii) $\mathrm{Fix}(T)$ *is diffeomorphic to* W. □

There is a corresponding theorem for involutions on D^{n+k} as long as the cobordism of the fixed sets W^{k+1} has $\partial W = V_0 \cup W' \cup V_1$ where W' is an s-cobordism from ∂V_0 to ∂V_1.

Let $D_k^{(2)}$ be the set of cobordism classes of Z_2-disks with homotopy sphere boundary, where the relation is given by Z_2-cobordism of disks restricting to an h-cobordism of their boundaries. $D_k^{(2)}$ becomes an abelian group using connected sum along the boundary.

(4.3) **Lemma.** *The following sequence is exact*

$$\cdots \longrightarrow \theta_k \xrightarrow{\ i\ } \theta_k^{(2)} \xrightarrow{\ j\ } D_k^{(2)} \xrightarrow{\ \partial\ } \theta_{k-1} \xrightarrow{\ i\ } \cdots$$

where $j[V^k] = [V^k - D^k]$ *and* $\partial[\Delta^k] = [\partial \Delta^k]$.

Proof. This follows easily from the definitions. □

(4.4) **Lemma.** *Let* T_0 *and* T_1 *be involutions on* S^{n+k-1} *such that there exists an involution* T *on* $S^{n+k-1} \times I$ *having*

(i) $T|_{S^{n+k-1} \times \{i\}} = T_i$ *for* $i = 0, 1$, *and*

(ii) $\mathrm{Fix}(T)$ *is diffeomorphic to* $\mathrm{Fix}(T_0) \times I$.

Then T_0 *is equivariantly diffeomorphic to* T_1.

Proof. This is an application of the s-cobordism theorem with boundary. Let N be a tubular neighborhood of $\mathrm{Fix}(T_0)$ in S^{n+k-1}, identified with the normal disk bundle of $\mathrm{Fix}(T_0)$, so that $T_0|_N$ is the antipodal map in each fibre. Then $S^{n+k-1} = M \cup N$ where $M = \overline{S^{n+k-1} - N}$ and $T_0|_M$ is free. By (ii) there is an equivariant imbedding $g: N \times I \to S^{n+k-1} \times I$ such that $g(N \times I)$ is a tubular neighborhood of $\mathrm{Fix}(T)$, where the involution on $N \times I$ is given by $T_0 \times$ identity.

g defines a product structure on $\partial(\overline{S^{n+k-1} \times I - g(N \times I)}) \cong \partial N \times I \cong \partial M \times I$. Taking the quotient space under the action of T, there is a product structure on $\partial(\overline{S^{n+k-1} \times I - g(N \times I)}/T) \cong \partial M/T_0 \times I$. Since $Wh(Z_2) = 0$, the s cobordism theorem with boundary says that $\overline{S^{n+k-1} \times I - g(N \times I)}/T$ is diffeomorphic to $M/T_0 \times I$, extending the given product structure on $\partial M/T_0 \times I$.

Therefore there exists an equivariant diffeomorphism $h: M \times I \to \overline{S^{n+k-1} \times I - g(N \times I)}$ where the action on $M \times I$ is $T_0 \times$ identity. Since h and g agree on the common boundary, they may be pieced together to give an equivariant diffeomorphism $f: S^{n+k-1} \times I \to S^{n+k-1} \times I$, where the action on the left is $T_0 \times$ identity and, on the right, is T. Then f: $S^{n+k-1} \times \{1\} \to S^{n+k-1} \times \{1\}$ gives an equivariant diffeomorphism between T_0 and T_1. □

Define an equivalence relation on involutions on D^{n+k} with k dimensional fixed set as follows: $T_0 \sim T_1$ if there exists an involution T on $D^{n+k} \times I$ such that

(i) $T|_{D^{n+k} \times \{i\}} = T_i$ for $i=0,1$

(ii) $\mathrm{Fix}(T|_{S^{n+k-1} \times I})$ is diffeomorphic to $\mathrm{Fix}(T_0|_{S^{n+k-1}}) \times I$.

There is an immediate consequence of (4.4).

(4.5) **Corollary.** *If T_0 and T_1 are equivalent involutions on D^{n+k}, then $T_0|_{S^{n+k-1}}$ is equivariantly diffeomorphic to $T_1|_{S^{n+k-1}}$.* □

Denote by $\mathscr{A}(D^{n+k}, k)$ those equivalence classes of involutions having fixed set of dimension k such that the boundary of the fixed set is a homotopy sphere. $\mathscr{A}(D^{n+k}, k)$ becomes an abelian group via connected sum along the boundary. There is a well-defined homomorphism γ_k: $\mathscr{A}(D^{n+k}, k) \to D_k^{(2)}$ assigning to an involution T its fixed point set.

(4.6) **Lemma.** *γ_k is an isomorphism if $n > k+2$.*

Proof. Lowell Jones' theorem (4.1) shows that γ_k is onto. Suppose the fixed set of T_0 is Z_2-cobordant to (D^k, S^{k-1}) with an h-cobordism along the boundary. Then the disk version of (4.2) implies that T_0 is equivalent to an involution T_1 where $\mathrm{Fix}(T_1)$ is a standard disk. The proof will be complete if we can show:

(4.7) **Lemma.** *If T is an involution on D^{n+k} with D^k as fixed set, then T is diffeomorphic to the standard action.*

Proof. Let $x_0 \in D^k \subseteq D^{n+k}$ and suppose ϕ: $D^{n+k} \to D^{n+k}$ is an equivariant imbedding of D^{n+k} with the standard action into D^{n+k} with involution T such that $\phi(0) = x_0$. Remove this disk so that $\overline{D^{n+k} - \phi(D^{n+k})}$ is diffeomorphic to $S^{n+k-1} \times I$ where the action on $S^{n+k-1} \times \{0\}$ is diffeomorphic to the standard action.

As in the proof of Lemma (4.4) there is an equivariant diffeomorphism ϕ': $S^{n+k-1} \times I \to S^{n+k-1} \times I$ where the action on the left is the standard action on $S^{n+k-1} \times I$ leaving $S^{k-1} \times I$ fixed and the action on the right is the restriction of T. Using ϕ and ϕ' together gives a diffeomorphism of D^{n+k} with the standard action to D^{n+k} with action T. □

Two other groups of involutions play an important role. Let $\mathscr{A}(\Sigma^{n+k}, \Sigma^k)$ be the group of diffeomorphism classes of involutions on homotopy spheres with homotopy sphere fixed set and $\mathscr{A}(\Sigma^{n+k}, k)$ be the group of equivalence classes of involutions on homotopy spheres with k-dimensional fixed set, where the equivalent relation is given by: actions T_0 and T_1 on Σ^{n+k} are equivalent if there exists an action T on $\Sigma^{n+k} \times I$ such that $T|_{\Sigma^{n+k} \times \{i\}} = T_i$, $i=0,1$.

(4.8) **Lemma.** *There is an exact sequence*

$$\cdots \to \mathscr{A}(\Sigma^{n+k}, \Sigma^k) \xrightarrow{\alpha} \mathscr{A}(\Sigma^{n+k}, k) \xrightarrow{\beta} \mathscr{A}(D^{n+k}, k) \xrightarrow{\delta} \mathscr{A}(\Sigma^{n+k-1}, \Sigma^{k-1}) \to \cdots$$

where α *is the natural map,* $\beta[\Sigma^{n+k}, T]=[\overline{\Sigma^{n+k}-\phi(D^{n+k})}, T]$ *and* $\delta[D^{n+k}, T]=[S^{n+k-1}, T]$. *Here* $\phi\colon D^{n+k}\to\Sigma^{n+k}$ *is an equivariant imbedding of* D^{n+k} *with standard action (leaving* D^k *fixed) as a neighborhood of a fixed point.*

Proof. This follows easily from the definitions. □

Assume now that $n>k+2$. We can summarize our progress in the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\to \mathscr{A}(\Sigma^{n+k},\Sigma^k) & \xrightarrow{\alpha} & \mathscr{A}(\Sigma^{n+k},k) & \xrightarrow{\beta} & \mathscr{A}(D^{n+k},k) & \xrightarrow{\delta} & \mathscr{A}(\Sigma^{n+k-1},\Sigma^{k-1}) & \xrightarrow{\alpha} \\
\downarrow \phi_k & & \downarrow \psi_k & & \approx\downarrow \gamma_k & & \downarrow \phi_{k-1} & \\
\longrightarrow \theta_k & \xrightarrow{i} & \theta_k^{(2)} & \xrightarrow{j} & D_k^{(2)} & \xrightarrow{\partial} & \theta_{k-1} & \xrightarrow{i}
\end{array}$$

where ϕ_k, ψ_k and γ_k assign to an action its fixed point set.

It is clear from the diagram that any element of θ_k that lies in the kernel of i is the fixed set of an action on S^{n+k} that extends to the disk D^{n+k+1}. From the computations in [1] and the Brieskorn-Hirzebruch involutions described in [3] we have the following:

(4.9) **Theorem.** *If* Σ^k *is a homotopy sphere of odd order in* θ_k, *then there is an involution on* S^{n+k} *with fixed set diffeomorphic to* Σ^k. *Hence* coker ϕ_k *is a 2-primary group.*

Proof. For $k\not\equiv 3 \bmod 4$, any element of odd order is in the kernel of i. For $k=4\ell-1$, an element of odd order is either in the kernel of i or else comes from $bP_{4\ell}$. By [3] all of these occur as fixed sets in S^{n+k}. □

(4.10) **Lemma.** *There is an exact sequence*

$$0\to \operatorname{coker}\phi_k \to \operatorname{coker}\psi_k \xrightarrow{\rho_k} \ker\phi_{k-1} \to \ker\psi_{k-1}\to 0$$

where $\rho_k[M]=\delta\circ\gamma_k^{-1}\circ j[M]$.

Proof. This follows by diagram chasing, using the fact that γ_k is an isomorphism. □

The kernel of ϕ_{k-1} is easily identified with the group $\mathscr{A}(\Sigma^{n+k-1}, S^{k-1})$ of involutions on homotopy spheres with standard fixed sets. These groups have been studied by Browder and Petrie [5] and are known to be finitely generated.

(4.11) **Theorem.** *If* $k\equiv 3$ *mod 4, every element of* $bP_{k+1}^{(2)}$ *is the fixed set of an involution on a homotopy sphere* Σ^{n+k} *where* $n\equiv 2$ mod 4.

Proof. As in § 2, for any $V^k\in bP_{k+1}^{(2)}$ we get a bilinear form over $Z_{(2)}$ which may be represented by an integral matrix. In the usual way, we may plumb disk bundles over spheres to produce a parallelizable manifold W^{k+1} with ∂W equivalent to V^k.

If instead, we imbed these spheres as fixed sets of standard involutions on higher dimensional spheres and extend the involutions to the disk bundles, we may plumb equivariantly to produce a manifold M with involution fixing W, so that ∂M is a Z_2-sphere. Using equivariant surgery it is now possible to modify ∂M to make it a homotopy sphere, retaining the fixed set V. For details see [2]. The equivariant plumbing approach was suggested to us by P. E. Conner. □

(4.12) **Corollary.** *If* $n \equiv 2 \bmod 4$ *then* coker ψ_k/image coker ϕ_k *is divisible.*

Proof. If $k \not\equiv 3 \bmod 4$ the homomorphism i is onto, thus this quotient group is zero. If $k \equiv 3 \bmod 4$, then by (3.7) $\theta_k^{(2)}$/image θ_k is a direct sum of an odd primary part which is divisible and a 2-primary part arising from $bP_{k+1}^{(2)}$. By (4.11) this 2-primary part is in the image of ψ_k. Therefore coker ψ_k/image coker ϕ_k is odd primary and divisible. □

(4.13) **Corollary.** *If* $n \equiv 2 \bmod 4$ *the four term exact sequence of* (4.10) *breaks up into two isomorphisms, i.e.*

$$\operatorname{coker} \phi_k \approx \operatorname{coker} \psi_k \quad \textit{and} \quad \ker \phi_{k-1} \approx \ker \psi_{k-1}.$$

Proof. This follows from (4.12) since the only divisible subgroup of a finitely generated group is zero. □

We are now able to prove our main result.

(4.14) **Theorem.** *If* V^k *is a* Z_2*-sphere, there exist homotopy spheres* Σ^k *and* Σ^{n+k} *belonging to the 2-primary component of* θ_* *and an involution on* Σ^{n+k} *with fixed point set* $V^k \# \Sigma^k$, *where* $n \equiv 2 \bmod 4$.

Proof. First suppose $k \not\equiv 3 \bmod 4$ or $k \equiv 3 \bmod 4$ and V^k is 2-primary. V^k represents an element $[V^k]$ of coker ψ_k. By (4.9) and (4.13) there is a 2-primary homotopy sphere Σ^k such that $[-\Sigma^k]$ is mapped into $[V^k]$. Thus $[V^k \# \Sigma^k]$ is the zero coset in coker ψ_k and there is an involution on a homotopy sphere Σ^{n+k} with fixed set $V^k \# \Sigma^k$. Since V^k is 2-primary, we are free to multiply by a suitable odd number to insure that Σ^{n+k} is 2-primary.

If $k \equiv 3 \bmod 4$, recall from [3] that every element of bP_{k+1} is a fixed set in a standard sphere. Suppose V^k has odd order. It follows from (3.7) that modulo the image of bP_{k+1}, V^k is divisible. Let m_1 be the order of θ_k and m_2 be the order of θ_{n+k}. There exists an element $U^k \in \theta_k^{(2)}$ such that $m_1 m_2 \cdot U^k = V^k$ modulo bP_{k+1}. Now realize $U^k \# \Sigma^k$ as a fixed set in some Σ^{n+k} as above. Then multiplying by $m_1 m_2$ realizes V^k as a fixed set in S^{n+k}. □

(4.15) **Corollary.** *If* $V^k \in \theta_k^{(2)}$ *has odd order, then there is an involution on* S^{n+k} *with fixed set* V^k, *where* $n \equiv 2 \bmod 4$. □

We close with an application of the isomorphism $\ker \phi_k \approx \ker \psi_k$.

(4.16) **Theorem.** *If* Σ^{n+k} *is a homotopy sphere and* T *is an involution on* $\Sigma^{n+k} \times I$ *leaving each component of the boundary invariant such that*

$\mathrm{Fix}(T_i) = \mathrm{Fix}(T|_{\Sigma^{n+k} \times \{i\}})$ *for* $i = 0,1$ *are diffeomorphic homotopy spheres, then* T_0 *is equivariantly diffeomorphic to* T_1.

Proof. Let Σ_i^k be the fixed set of T_i and denote by $[T_i]$ the class represented by T_i in $\mathscr{A}(\Sigma^{n+k}, k)$. It is apparent that α takes $(\Sigma^{n+k}, \Sigma_i^k, T_i)$ into $[T_i]$ where the triple is an element of $\mathscr{A}(\Sigma^{n+k}, \Sigma^k)$.

Now by the hypothesis, $[T_1] - [T_0] = 0$, so the difference $(\Sigma^{n+k}, \Sigma_1^k, T_1) - (\Sigma^{n+k}, \Sigma_0^k, T_0)$ is the zero element in the kernel of ϕ_k. Since the equivalence relation here is equivariant diffeomorphism, it follows that T_0 is equivariantly diffeomorphic to T_1. □

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Differentiable S^1 Actions on Homotopy Spheres

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§ 0. Introduction

The purpose of this paper is to prove the following

Main Theorem. *There are infinitely many differentiably inequivalent free differentiable S^1 actions on standard $(2n-1)$-spheres for $n \geqq 4$.*

Throughout this paper we are working in the differentiable category. The first result in the area is due to W.C. Hsiang and W.Y. Hsiang. They constructed an infinite family of free S^1 actions on the standard 11-sphere. A joint paper of Montgomery and Yang followed soon. They gave complete information on free S^1 actions on homotopy 7-spheres. In particular, they proved that there are infinitely many free S^1 actions on the standard 7-sphere. Lately, Brumfiel gave a complete calculation of free S^1 actions on homotopy spheres and listed the homotopy spheres which support free S^1 actions in dimensions up to 11.

On the other hand, W.C. Hsiang applied surgery to construct infinitely many inequivalent free S^1 actions on homotopy $(2n-1)$-spheres for $n \geqq 4$. We will call these "Hsiang's actions". It is reasonable to conjecture that among the Hsiang's actions there could be infinitely many actions whose total spaces are the standard spheres. This is the main motivation of this research.

It is well known that any homotopy sphere is obtained by gluing $S^{2p-1} \times D^{2q}$ and $D^{2p} \times S^{2q-1}$ along the boundaries by a diffeomorphism f of $S^{2p-1} \times S^{2q-1}$. If the diffeomorphism f is also equivariant with respect to the linear S^1 action, then we can glue $S^{2p-1} \times D^{2q}$ and $D^{2p} \times S^{2q-1}$ equivariantly to get a free S^1 action on a homotopy sphere. As these actions are interesting to us they deserve a special name: "decomposable actions". For a more precise definition see Definition 2.1.

The advantage of the decomposable actions is this: Given a family of decomposable actions, we get a family of equivariant diffeomorphisms. Then we can construct further actions from the compositions of these equivariant diffeomorphisms.

This paper is organized as follows: In § 1 we will show that in this way we can construct actions on standard spheres. In § 2 we will state a necessary and sufficient condition for a free S^1 action to be decomposable. It is proved in §3 using the techniques of Livesay and Thomas. It turns out that all the Hsiang's actions in dimensions greater than 11 are decomposable. This is collected in §4. Let $\mathscr{D}(S^{2p-1}\times D^{2q}, A)$ be the group of diffeomorphisms of $S^{2p-1}\times D^{2q}$ equivariant with respect to the linear action. In §5 we will prove that $\mathscr{D}(S^{2p-1}\times D^{2q}, A)$ is finite for $p\leqq 2q$. In §6 we will prove the main theorem.

After this paper is complete, the author was informed that Burghelea, H.T. Ku and M.C. Ku had also proved the same theorem. During the Amherst Conference, Yang showed me another proof of the theorem when the spheres are of dimension $4n-1$. Our approach is completely different from the others and has further applications (see [22, 23]).

We omit all the discussions of S^3 actions. But the corresponding results are also true of which the proofs are completely analogous to S^1 actions.

Much of the material of this paper appeared first in the author's doctoral dissertation and was announced in the Proceedings of Conference on Transformation Groups (Amherst, 1971).

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§ 1. A Basic Lemma

We need to use some results in [13]. Let $D(S^p\times S^q)$ be the group of pseudo-isotopy classes of orientation preserving diffeomorphisms of $S^p\times S^q$ such that the restrictions to $x\times S^q$ are homotopic to the inclusion for x a fixed point in S^p. It is always assumed that $p\geqq q$.

Define subgroups D_1, D_2, D_3 and D_4 of $D(S^p\times S^q)$ to consist of those elements represented by $f\colon S^p\times S^q\to S^p\times S^q$ satisfying

(D_1): f extends to a diffeomorphism of $S^p\times D^{q+1}$,

(D_2): f extends to a diffeomorphism of $D^{p+1}\times S^q$,

(D_3): $f\,|\,D^p_+\times S^q=$ inclusion,

(D_4): for some $(p+q)$-disk $D\subset S^p\times S^q$, $f(D)\subset D$ and $f\,|(S^p\times S^q-D)=$ inclusion.

Lemma 1.1. (see [13]). (a) *D_4 is isomorphic to Γ^{p+q-1}, where Γ^{p+q-1} is the group of pseudo-isotopy classes of $(p+q)$-disk which is the identity on the boundary* (see [11]).

(b) *D_1 and D_3 are abelian.*

(c) *D_3 is normal in $D(S^p\times S^q)$.*

(d) *$D(S^p\times S^q)=D_2\cdot D_3$ and $D_2\cap D_3=\{identity\}$.*

(e) *$D_4\subset D_3$, $D_1\subset D_3$, $D_3=D_1\cdot D_4$ and $D_1\cap D_4=\{identity\}$.*

Lemma 1.2. *$u\cdot v^m\cdot u^{-1}\in D_1$ for any $v\in D_3$ and $u\in D(S^p\times S^q)$ where m is the order of Γ^{p+q-1}.*

Proof. Since D_3 is normal $u\cdot v\cdot u^{-1}\in D_3$, there is $w\in D_1$ and $r\in D_4$ such that $u\cdot v\cdot u^{-1}=w\cdot r$. Since D_3 is abelian

$$u\cdot v^m\cdot u^{-1}=(u\cdot v\cdot u^{-1})^m=w^m\cdot r^m=w^m.$$

Hence $u\cdot v^m\cdot u^{-1}\in D_1$.

Lemma 1.3. *Let f_j, $j=1,2$, be diffeomorphisms of $S^p\times S^q$ so that $\Sigma(f_j)=S^p\times D^{q+1}\cup_{f_j}D^{p+1}\times S^q$ are homotopy spheres. Then $\Sigma(f_1)$ and $\Sigma(f_2)$ are diffeomorphic if and only if there are diffeomorphisms g of $S^p\times D^{q+1}$ and h of $D^{p+1}\times S^q$ so that $f_2=h\cdot f_1\cdot g$.*

Proof. Easy.

Lemma 1.4. *Every element in D_1 for $p\leqq 2_q$ (respectively, D_2 for $q\leqq 2p$) contains a representative which is a bundle map.*

Proof. Let g be a diffeomorphism of $S^p\times D^{q+1}$. $g|S^p\times 0$ is homotopic to the inclusion. By the theorem of Haefliger [8] they are isotopic. By the isotopy extension theorem there is a diffeomorphism G of $S^p\times D^{q+1}$ which is isotopic to the identity such that $G|S^p\times S^q=\text{identity}$ and $G\cdot g|S^p\times 0=\text{inclusion}$.

Let $D_a^{q+1}=\{x\in D^{q+1}|\,\|x\|\leqq a\}$ where a is a small positive number. Then $S^p\times D_a^{q+1}$ is a tubular neighborhood of $S^p\times 0$ in $S^p\times D^{q+1}$ and so is $G\cdot g(S^p\times D_a^{q+1})$. Then by the uniqueness theorem of tubular neighborhoods there is a diffeomorphism H of $S^p\times D^{q+1}$ which is isotopic to the identity such that $H|S^p\times S^q=\text{identity}$ and

$$\begin{array}{ccc} S^p\times D^{q+1} & \xrightarrow{H\cdot G\cdot g} & S^p\times D_a^{q+1} \\ \downarrow & & \downarrow \\ S^p\times 0 & \xrightarrow{\text{identity}} & S^p\times 0 \end{array}$$

is a bundle map. Note that $S^p\times D^{q+1}$-interior $(S^p\times D_a^{q+1})$ is diffeomorphic to $S^p\times S^q\times[0,1]$. Up to diffeomorphisms

$$H\cdot G\cdot g|S^p\times S^q\times 0=H\cdot G\cdot g|S^p\times S^q=g$$

and

$$H\cdot G\cdot g|S^p\times S^q\times 1=H\cdot G\cdot g|S^p\times S_a^q$$

is a bundle map. Hence g is pseudo-isotopic to a bundle map.

Lemma 1.5. *Let f_j, $j=1,2$, be diffeomorphisms of $S^p\times S^q$ which are pseudo-isotopic. Then $\Sigma(f_1)$ is diffeomorphic to $\Sigma(f_2)$.*

Proof. Easy.

Lemma 1.6. *For* u: $S^q \to SO(p+1)$, *define* b_u: $D^{p+1} \times S^q \to D^{p+1} \times S^q$ *by* $b_u(x, y) = (u(y)\,x, y)$. *Then*

(a) *Every bundle map of* $D^{p+1} \times S^q$ *over the identity can be constructed in this way.*

(b) *The pseudo-isotopy class* $[b_u]$ *of* b_u *depends only on the homotopy class* $[u]$ *of* u.

Let $b_{[u]} = [b_u]$. *Then*

(c) $b_{[u][v]} = b_{[u]} \cdot b_{[v]}$.

Similar assertions hold for d_v: $S^p \times D^{q+1} \to S^p \times D^{q+1}$ *where* v: $S^p \to SO(q+1)$ *and* $d_v(x, y) = (x, v(x)\,y)$.

Proof. This is essentially proved in [4].

Lemma 1.7 (Basic Lemma). *For* $q \leqq p \leqq 2q$, *let* $\{f_j\}_{j \in N}$ *be an infinite set of orientation preserving diffeomorphisms of* $S^p \times S^q$ *such that* $f_j | x \times S^q$ *is homotopic to the inclusion. Then there exists an infinite subset* $\{f_{j_s}\}_{s \in N}$ *of* $\{f_j\}_{j \in N}$ *such that* $\Sigma(f_0 \cdot f_{j_0}^{-1} \cdot f_{j_s})$ *is diffeomorphic to* $\Sigma(f_0)$.

Proof. We may assume $\Sigma(f_j)$, $j \in N$, are the same homotopy sphere as the group of homotopy spheres is finite [11]. Then choose diffeomorphisms g_j and h_j as in Lemma 1.3 such that $f_j = h_j \cdot f_1 \cdot g_j$ and which will be fixed for the rest of the argument.

By Lemma 1.6, there are $u_j \in \pi_q(SO(p+1))$ such that $b_{u_j} = [h_j]$. Since $\pi_q(SO(p+1))$ is finitely generated, let $e_1, \ldots, e_s$ be its generators. Then $u_j = e_1^{t_{1,j}} \ldots e_s^{t_{s,j}}$ where $t_{s,j}$ are integers. We choose a map from the infinite set $\{f_j\}_{j \in N}$ to sZ_m, the direct sum of s copies of Z_m, where m is the order of Γ^{p+q-1}, by

$$f_j \to (t_{1,j} \bmod m, \ldots, t_{s,j} \bmod m).$$

Since sZ_m is finite, there are infinite number of $\{f_j\}_{j \in N}$ mapping into the same element. Let $\{f_{j_k}\}_{k \in N}$ be such a set, i.e.

$$t_{i,j_k} \equiv t_{i,j_0} \pmod m$$

for $i = 1, 2, \ldots, s$, and $k \in N$.

Let $c_{i,k} = (t_{i,j_k} - t_{i,j_0})/m$ and $b = b_{e_1}^{c_{1,k}} \ldots b_{e_s}^{c_{s,k}}$. Then

$$[f_0 \cdot f_{j_0}^{-1} \cdot f_{j_k}] = [f_0] \cdot [f_{j_0}]^{-1} \cdot b^m \cdot [f_{j_0}] \cdot [f_0]^{-1} \cdot [f_0] \cdot [g_{j_0}]^{-1} \cdot [g_{j_k}].$$

By Lemma 1.2, $[f_0] \cdot [f_{j_0}]^{-1} \cdot b^m \cdot [f_{j_0}][f_0]^{-1} \in D_1$. Hence by Lemma 1.3, $\Sigma(f_0 \cdot f_{j_0}^{-1} \cdot f_{j_k})$ is diffeomorphic to $\Sigma(f_0)$.

§ 2. A Criterion on the Decomposability

Let (M, G, F), or (M, F) if there is no confusion, be an action of a compact Lie group G on a smooth manifold M, i.e., F: $G \times M \to M$

such that if $m \in M$ and $g, h \in G$,

(a) $F(g, F(h, m)) = F(g \cdot h, m)$,

(b) $F(e, m) = m$ where e is the identity of G,

(c) F is a C^∞ map.

The action F is called free if for any $m \in M$

$$F(g, m) = m \quad \text{implies } g = e.$$

Let $S^1 = \{g \in C \mid |g| = 1\}$, $S^{2m-1} = \{u = (u_1, \ldots, u_m) \in C^m \mid \|u\| = 1\}$, $D^{2m} = \{u = (u_1, \ldots, u_m) \in C^m \mid \|u\| \leqq 1\}$. Let $g u = (g u_1, \ldots, g u_m)$ for $g \in S^1$ and $u \in C^m$. We define S^1 actions $(S^{2p-1} \times S^{2q-1}, A)$, $(S^{2p-1} \times D^{2q}, A)$ and $D^{2p} \times S^{2q-1}, A$ by the equation

$$A(g, (u, v)) = (g u, g v).$$

Let f be an equivariant diffeomorphism of $(S^{2p-1} \times S^{2q-1}, A)$. We can define an action $A(f)$ on $\Sigma(f)$ where

$$\Sigma(f) = S^{2p-1} \times D^{2q} \cup_f D^{2p} \times S^{2q-1}$$

so that $A(f) | S^{2p-1} \times D^{2q} = A$ and $A(f) | D^{2p} \times S^{2q-1} = A$ which is clearly a free S^1 action.

Definition 2.1. A free S^1 action (Σ^{2n-1}, F) on a homotopy sphere Σ^{2n-1} is decomposable if there is an equivariant diffeomorphism f of $(S^{2p-1} \times S^{2q-1}, A)$ such that (Σ^{2n-1}, F) is equivalent to $(\Sigma(f), A(f))$ for $p + q = n$.

In the next section we will prove the following theorem.

Theorem 2.2. *Let (Σ^{2n-1}, F) be a free S^1 action. If (Σ^{2n-1}, F) is equivalent to $(\Sigma(f), A(f))$ for an equivariant diffeomorphism f of $(S^{2p-1} \times S^{2q-1}, A)$ then*

$$p(\Sigma^{2n-1}/F) \equiv (1 + z^2)^n \pmod{z^p}.$$

Conversely, if

$$p(\Sigma^{2n-1}/F) \equiv (1 + z^2)^n \pmod{z^{[n+1/2]}},$$

then (Σ^{2n-1}, F) is decomposable, where z is a generator of $H^2(\Sigma^{2n-1}/F, Z)$ and p is the total Pontryagin class of the orbit space, $n \geqq 4$.

Let h_R denote the canonical real line bundle over RP^{2n+1} and h_C denote the canonical complex line bundle over CP^n. Let $c: KO(X) \to K(X)$ be the complexification and $r: K(X) \to KO(X)$ be the realification. Let $\omega = r(h_C - 1) \in KO(CP^n)$ be the generator.

Theorem 2.3 (Sanderson [17]). *$KO(CP^n)$ is the truncated polynomial ring over the integers with one generator ω and the following relations:*

(a) *If $n=2t$, then $\omega^{t+1}=0$,*

(b) *if $n=4t+1$, then $2\omega^{2t+1}=0$ and $\omega^{2t+2}=0$,*

(c) *if $n=4t+3$, then $\omega^{2t+2}=0$.*

Lemma 2.4 (Adams [1]). *Let π: $RP^{2n+1}\to CP^n$ be the canonical projection. Then $\pi^* h_C = c\, h_R$.*

Lemma 2.5. *Let $\omega=r(h_C-1)\in KO(CP^{4t+1})$. Then ω^{2t+1} is not fibre homotopy trivial.*

Proof. Let π: $RP^{8t+3}\to CP^{4t+1}$ be the canonical projection. If ω^{2t+1} were fibre homotopy trivial, then $\pi^*\omega^{2t+1}$ would be fibre homotopy trivial. But $\pi^*\omega=2(h_R-1)$, so

$$\pi^*\omega^{2t+1}=(\pi^*\omega)^{2t+1}=2^{4t+1}(h_R-1).$$

Since $KO(RP^{8t+3})$ is a cyclic group of order 2^{4t+2}, $\pi^*\omega^{2t+1}\neq 0$. But J: $KO(RP^n)\to J(RP^n)$ is an isomorphism [2, 3] so $J(\pi^*\omega^{2t+1})\neq 0$, i.e., $\pi^*\omega^{2t+1}$ is not fibre homotopy trivial.

Let the Pontryagin character $KO(X)\xrightarrow{c} K(X)\xrightarrow{Ch} H^*(X,Q)$ be defined to be the composition of the Chern character Ch and the complexification c and denote it by Ph. Since both c and Ch are ring homomorphisms so is the composition. The following proposition is well known [9].

Proposition 2.6. *Ph: $KO(X)\otimes Q\to \Sigma H^{4i}(X,Q)$ is an isomorphism.*

Let P^{2n} be a homotopy complex projective space of dimension $2n$.

Lemma 2.7. *Let f: $CP^n\to P^{2n}$ be a homotopy equivalence and let i: $CP^k\to CP^n$ be the usual inclusion such that $j=f\cdot i$ is an imbedding of CP^k into P^{2n}. If*

$$p(P^{2n})\equiv \bar{f}^*(CP^n) \quad (\operatorname{mod} z^{k+1}), \tag{1}$$

then $\nu(i(CP^k), CP^n)\overset{s}{\sim} f^*\nu(j(CP^k), P^{2n})$ where $\nu(N,M)$ is the normal bundle of N in M, $\bar{f}$ is the homotopy inverse of f and z is a generator of $H^2(P^{2n},Z)$.

Proof. i^*: $H^q(CP^n)\to H^q(CP^k)$ is an isomorphism for $q\leqq 2k$ and zero for $q>2k$. Hence (1) implies $i^*p(CP^n)=j^*p(P^{2n})$, so

$$Ph(i^*T(CP^n))=Ph(j^*T(P^{2n})).$$

For $k\neq 4t+1$, $KO(CP^k)$ has no torsion, so $i^*T(CP^n)\overset{s}{\sim} j^*T(P^{2n})$. For $k=4t+1$, by Lemma 2.3, either

$$i^*T(CP^n)\overset{s}{\sim} j^*T(P^{2n}) \quad\text{or}\quad i^*T(CP^n)+j^*T(P^{2n})^{-1}\overset{s}{\sim}\omega^{2t+1}.$$

But the left hand side is fibre homotopy trivial [3], and the right hand side is not fibre homotopy trivial. Thus we proved the lemma.

§ 3. The Proof of Theorem 2.2.

Let $M = S^{2p-1} \times D^{2q}/A$ and $N = D^{2p} \times S^{2q-1}/A$. Suppose (Σ^{2n-1}, F) is equivalent to $(\Sigma(f), A(f))$ for an equivariant diffeomorphism f of $(S^{2p-1} \times S^{2q-1}, A)$. Let $\bar{f}$ be the map of ∂M to ∂N induced from f. Then $\Sigma^{2n-1}/F = M \cup_{\bar{f}} N$. Hence we have an embedding j of CP^{p-1} into Σ^{2n-1}/F and the normal bundle is equal to $q\, h_C$.

$$j^* \, T(\Sigma^{2n-1}/F) = q\, h_C + T(CP^{p-1}) \overset{s}{\sim} n\, h_C.$$

$$j^* \, p(\Sigma^{2n-1}/F) = (1+z^2)^n.$$

Since $j^*\colon H^i(\Sigma^{2n-1}/F) \to H^i(CP^{p-1})$ is an isomorphism for $i \leqq 2p$ and zero for $i > 2p$,

$$p(\Sigma^{2n-1}/F) \equiv (1+z^2)^n \quad (\operatorname{mod} z^p).$$

Now we will adopt the techniques of Livesay and Thomas [14] to prove the converse.

Let $P^{2n-2} = \Sigma^{2n-1}/F$ which is a homotopy complex projective space. The bundle $S^1 \to \Sigma^{2n-1} \to P^{2n-2}$ is homotopically equivalent to the classical Hopf bundle [9]. Let $p = [n+1/2]$ and $q = n - p$. Let $\bar{e}$ be a homotopy equivalence of CP^{n-1} to P^{2n-2}. $\bar{e} \mid CP^{p-1}$ is a homotopic to an embedding. By the homotopy extension theorem, $\bar{e}$ is homotopic to a homotopy equivalence $\bar{g}$ so that $\bar{g} \mid CP^{p-1}$ is an embedding. Since $p(P^{2n-2}) \equiv (1+z^2)^n (\operatorname{mod} z^p)$, $\nu(CP^{p-1}, CP^{n-1}) \overset{s}{\sim} \nu(CP^{p-1}, P^{2n-2})$. In fact $\nu(CP^{p-1}, CP^{n-1})$ is bundle isomorphic to $\nu(CP^{p-1}, P^{2n-2})$ although for the case $n = $ odd it is not so obvious but the proof is well known. Hence $\bar{g} \mid M$ is an embedding or equivariantly, we have an equivariant embedding

$$(S^{2p-1} \times D^{2q}, A) \xhookrightarrow{g} (\Sigma^{2n-1}, F).$$

By the h-cobordism theorem it is easy to show that for $n \geqq 3$,

$$\Sigma^{2n-1} - g(\text{interior } S^{2p-1} \times D^{2q}) \simeq D^{2p} \times S^{2q-1}.$$

Consider

$$(S^{2p-1} \times D^{2q}, A) \xhookrightarrow{g} (\Sigma^{2n-1}, F) \xhookleftarrow{h} (D^{2p} \times S^{2q-1}, U)$$

where we define an action on the right hand side solid torus by $U = h^{-1} \cdot F \cdot h$. Let $S_L^{2q-1} \overset{\Delta}{\subset} S^{2q-1} \times S^{2q-1} \subset S^{2p-1} \times S^{2q-1} \subset S^{2p-1} \times D^{2q}$ which is an A-invariant submanifold and is mapped by $h^{-1} \cdot g$ onto a U-invariant sphere S_R^{2q-1} on the right. Notice that on the boundary of $D^{2p} \times S^{2q-1}$, U is equivalent to A. Equivariantly collar $S^{2p-1} \times S^{2q-1}$ in $(D^{2p} \times S^{2q-1}, U)$ and push S_R^{2q-1} a little away inside the boundary. U is equivalent to A on an equivariant tubular neighborhood K of this interior copy of S_R^{2q-1}. Interpreted in the orbit space $(D^{2p}\times S^{2q-1}$-interior $K)/U$ is an h-cobordism between $S^{2p-1}\times S^{2q-1}/A$ and $\partial K/A$, hence diffeomorphic to $(S^{2p-1}\times S^{2q-1})/A\times[0,1]$. Therefore U can be taken equivalent to A by a diffeomorphism $k: D^{2p}\times S^{2q-1}\to D^{2p}\times S^{2q-1}$ and finlly we have decomposed (Σ^{2n-1},F) as $(\Sigma(g^{-1}\cdot h\cdot k), A(g^{-1}\cdot h\cdot k))$.

Remark. Let (Σ^{2n-1},F) be a decomposable free S^1 action. Then we may always assume that (Σ^{2n-1},F) is equivalent to $(\Sigma(f), A(f))$ for an equivariant diffeomorphism f of $(S^{2p-1}\times S^{2q-1},A)$ with $p=[n+1/2]$.

§4. Corollaries

It is easy to check that all Hsiang's actions in dimensions greater than 11 are decomposable. Hence we have

Corollary 4.1. *There are infinitely many topologically inequivalent decomposable free S^1 actions on homotopy $(2n-1)$-spheres for $n\geqq 7$.*

Corollary 4.2. *There are only finitely many topologically inequivalent decomposable free S^1 actions on homotopy 9-spheres.*

Proof. Since the Hirzebruch L-genus of a manifold is a homotopy type invariant of the oriented manifold, it is easy to show that if there is a free S^1 action on a homotopy 9-sphere which is decomposable then its orbit space must be tangential homotopy equivalent to CP^4. But Sullivan [19] had proved that there are only finitely many such tangential homotopy complex projective spaces.

On the other hand, we can generalize the technique of Hsiang in [9] to prove

Corollary 4.3. *There are infinitely many topologically inequivalent non-decomposable free S^1 actions on homotopy $(2n-1)$-spheres for $n\geqq 5$. (See also* [21].)

§5. The Group of Pseudo-Isotopy Classes of Equivariant Diffeomorphisms of $(S^{2p-1}\times D^{2q}, A)$

Lemma 5.1. *Every equivariant diffeomorphism of $(S^{2p-1}\times D^{2q},A)$, for $p\leqq 2q$, is equivariantly pseudo-isotopic to an equivariant bundle map.*

Proof. Let g be an equivariant diffeomorphism of $(S^{2p-1}\times D^{2q},A)$ and $\bar{g}$ be the induced map on the orbit space $N=S^{2p-1}\times D^{2q}/A$. Since $S^{2p-1}\times D^{2q}\to N$ is $(2p-1)$-universal and both $\bar{g}|(S^{2p-1}\times 0)/A$ and the inclusion are classifying maps for $S^1\to S^{2p-1}\to CP^{p-1}$, they are homotopic. By the theorem of Haefliger they are isotopic, or equivariantly, i: $S^{2p-1}\to S^{2p-1}\times D^{2q}$ is equivariantly isotopic to $g|S^{2p-1}$: $S^{2p-1}\to S^{2p-1}\times D^{2q}$. By the equivariant isotopy extension theorem [16] there is an equivariant diffeomorphism G of $(S^{2p-1}\times D^{2q}, A)$ which is equivariantly isotopic to the identity and $G|S^{2p-1}\times S^{2q-1}=$ identity and $G\cdot g|S^{2p-1}=$ inclusion.

Let $D_a^{2q}=\{u\in C^q|\ \|u\|\leqq a\}$ where a is a small positive number. $S^{2p-1}\times D_a^{2q}$ is an equivariant tubular neighborhood of S^{2p-1} in $S^{2p-1}\times D^{2q}$, $G\cdot g(S^{2p-1}\times D_a^{2q})$ is also an equivariant tubular neighborhood of S^{2p-1} in $S^{2p-1}\times D^{2q}$. By the theorem of uniqueness of equivariant tubular neighborhoods [16] there is an equivariant diffeomorphism H of $(S^{2p-1}\times D^{2q}, A)$ which is equivariantly isotopic to the identity and $H|S^{2p-1}\times S^{2q-1}=$ identity. Furthermore, $H\cdot G\cdot g$: $S^{2p-1}\times D^{2q}\to S^{2q-1}\times D^{2q}$ is an equivariant bundle map. Note that $S^{2p-1}\times D^{2q}$-interior $(S^{2p-1}\times D_a^{2q})$ is equivariantly diffeomorphic to $S^{2p-1}\times S^{2q-1}\times[0,1]$ with the product action. Hence g is equivariantly pseudo-isotopic to an equivariant bundle map.

Let S^1 be embedded in $SO(2q)$ along the diagonal. Define an S^1 action F on $SO(2q)$ by $F(g,x)=g\cdot x\cdot g^{-1}$ where $g\in S^1$ and $x\in SO(2q)$. Let u: $(S^{2p-1},A)\to(SO(2q),F)$ be an equivariant map, i.e., $u(gx)=g\cdot u(x)\cdot g^{-1}$. Define b_u: $S^{2p-1}\times D^{2q}\to S^{2p-1}\times D^{2q}$ by $b_u(x,y)=(x,u(x)\,y)$. It is clear b_u is an equivariant bundle map. In fact every equivariant bundle map of $D^{2q}\to S^{2p-1}\times D^{2q}\to S^{2p-1}$ over the identity can be constructed in this way.

Lemma 5.2. *The equivariant pseudo-isotopy class of b_u is determined by the equivariant homotopy class of u.*

Proof. Let h_t: $(S^{2p-1},A)\to(SO(2q),F)$ be an equivariant homotopy between u and v. Then H_t: $S^{2p-1}\times D^{2q}\to S^{2p-1}\times D^{2q}$ defined by $H_t(x,y)=(x,h_t(x)\,y)$ is an equivariant pseudo-isotopy (in fact, an equivariant isotopy) between b_u and b_v.

Let $[S^{2p-1},SO(2q)]^{S^1}$ be the set of equivariant homotopy classes of equivariant maps of (S^{2p-1},A) to $(SO(2q),F)$.

Lemma 5.3. *$[S^{2p-1},SO(2q)]^{S^1}$ is finite.*

Proof. Let u: $S^{2p-1}\to SO(2q)$ be an equivariant map. For k large, let $S^{2p-1}\subset S^{2k-1}$ be the usual inclusion. Then

$$(u,i)\colon (S^{2p-1},A)\to(SO(2q)\times S^{2k-1},F\times A)$$

is an equivariant map. Note that $(SO(2q)\times S^{2k-1},F\times A)$ is free. Hence we have the following commutative diagram:

$$\begin{array}{ccccc}
S^{2p-1} & \xrightarrow{(u,i)} & SO(2q)\times S^{2k-1} & \xrightarrow{p} & S^{2k-1}\\
\downarrow & & \downarrow & & \downarrow\\
CP^{p-1} & \xrightarrow{\overline{(u,i)}} & SO(2q)\times_{S^1} S^{2k-1} & \xrightarrow{\bar p} & CP^{k-1}.
\end{array}$$

Conversely, let $\bar{v}$: $CP^{p-1} \to SO(2q) \times_{S^1} S^{2k-1}$ be a map such that $\bar{v}^* \cdot \bar{p}^*$: $H^2(CP^{k-1}) \to H^2(CP^{p-1})$ maps generator to generator. Then there is an equivariant map from S^{2p-1} to $SO(2q) \times S^{2k-1}$ which covers $\bar{v}$.

Let $[CP^{p-1}, SO(2q) \times_{S^1} S^{2k-1}]^\sim$ be the set of homotopy classes of $\bar{v}$ such that $v^* \cdot p^*$ maps the generator of $H^2(CP^{k-1})$ onto the generator of $H^2(CP^{p-1})$. We will show by induction on p that $[CP^{p-1}, SO(2q) \times_{S^1} S^{2k-1}]$ is a finite set.

For $p=2$, $CP^1 \simeq S^2$. Consider the fibration

$$SO(2q) \xrightarrow{i} SO(2q) \times_{S^1} S^{2k-1} \xrightarrow{\pi} CP^{k-1}$$

and the exact sequence of this fibration

$$\cdots \to \pi_{2m}(SO(2q)) \xrightarrow{i_*} \pi_{2m}(SO(2q) \times_{S^1} S^{2k-1}) \xrightarrow{\pi_*} \pi_{2m}(CP^{k-1}) \to \cdots$$

$$\cdots \to \pi_2(SO(2q)) \xrightarrow{i_*} \pi_2(SO(2q) \times_{S^1} S^{2k-1}) \xrightarrow{\pi_*} \pi_2(CP^{k-1}) \to \cdots.$$

So $[CP^1, SO(2q) \times_{S^1} S^{2k-1}]^\sim = \ker \pi_* = i_* \pi_2(SO(2q)) = 0$. Moreover, as it is well known that $\pi_{2m}(SO(2q))$ is finite, $\pi_{2m}(SO(2q) \times_{S^1} S^{2k-1})$ is finite. It follows that the homotopy classes of homotopy extensions of each element in $[CP^m, SO(2q) \times_{S^1} S^{2k-1}]$ to CP^{m+1} is finite [18]. Hence $[CP^{m+1}, SO(2q) \times_{S^1} S^{2k-1}]$ is finite, or equivariantly, $[S^{2p-1}, SO(2q) \times S^{2k-1}]^{S^1}$ is finite.

Let p: $SO(2q) \times S^{2k-1} \to SO(2q)$ be the projection. Then the map p_*: $[S^{2p-1}, SO(2q) \times S^{2k-1}]^{S^1} \to [S^{2p-1}, SO(2q)] S^1$ in onto. Therefore $[S^{2p-1}, SO(2q)]^{S^1}$ must be finite.

Let $\mathscr{D}(S^{2p-1} \times D^{2q}, A)$ be the group of equivariant pseudo-isotopy classes of equivariant diffeomorphisms of $(S^{2p-1} \times D^{2q}, A)$. Now the following theorem is clear.

Theorem 5.4. *For $p \leqq 2q$, $\mathscr{D}(S^{2p-1} \times D^{2q}, A)$ is finite.*

§ 6. The Main Theorem

Proposition 6.1. *Suppose $q \leqq p \leqq 2q$. Let f_j, $j=1, 2$, be two equivariant diffeomorphisms of $(S^{2p-1} \times S^{2q-1}, A)$. Then $(\Sigma(f_1), A(f_1))$ and $(\Sigma(f_2), A(f_2))$ are equivalent if and only if there exist equivariant diffeomorphisms g of $(S^{2p-1} \times D^{2q}, A)$ and h of $(D^{2p} \times S^{2q-1}, A)$ such that $f_2 = h \cdot f_1 \cdot g$.*

Proof. Let

$$(S^{2p-1} \times D^{2q}, A) \xhookrightarrow{r_j} (\Sigma(f_j), A(f_j)) \xhookleftarrow{k_j} (D^{2p} \times S^{2q-1}, A)$$

be equivariant decompositions such that $f_j = k_j^{-1} \cdot r_j$. Suppose that $(\Sigma(f_1), A(f_1)) \simeq (\Sigma(f_2), A(f_2))$. Let $\Sigma^{2p+2q-1} = \Sigma(f_j)$, $j=1, 2$. Since r_j are the classifying maps of $S^1 \to S^{2p-1} \to CP^{p-1}$, $\bar{r}_1 | CP^{p-1}$ and $\bar{r}_2 | CP^{p-1}$ are homotopic. Therefore $\bar{r}_1 | CP^{p-1}$ and $\bar{r}_2 | CP^{p-1}$ are isotopic, or equivariantly, $r_1|S^{2p-1}$ and $r_2|S^{2p-1}$ are equivariantly isotopic. By the equivariant isotopy extension theorem, there is an equivariant diffeomorphism H of $\Sigma^{2p+2q-1}$ which is equivariantly isotopic to the identity and $H \cdot r_1|S^{2p-1}=r_2|S^{2p-1}$. Both $r_2(S^{2p-1}\times D^{2q})$ and $H\cdot r_1(S^{2p-1}\times D^{2q})$ are equivariant tubular neighborhoods of $r_2(S^{2p-1})$ in $\Sigma^{2p+2q-1}$. By the theorem of uniqueness of equivariant tubular neighborhoods, there is an equivariant diffeomorphism G of $\Sigma^{2p+2q-1}$ which is equivariantly isotopic to the identity such that $G\cdot H\cdot r_1(S^{2p-1}\times D^{2q})=r_2(S^{2p-1}\times D^{2q})$ and $r_2^{-1}\cdot G\cdot H\cdot r_1$ is an equivariant bundle map. Let $g=r_2^{-1}\cdot G\cdot H\cdot r_1$ and $h=k_1^{-1}\cdot H^{-1}\cdot G^{-1}\cdot k_2$. Then g is an equivariant diffeomorphism of $(S^{2p-1}\times D^{2q}, A)$ and h is an equivariant diffeomorphism of $(D^{2p}\times S^{2q-1}, A)$ such that $h\cdot f_2\cdot g=f_1$.

The converse is trivial.

Theorem 6.2. *Let Σ^{2n-1} be a homotopy sphere which supports a free decomposable S^1 action. Then it supports infinitely many differentiably inequivalent free decomposable S^1 actions for $n\geqq 7$.*

Proof. Let f_0 be an equivariant diffeomorphism of $(S^{2p-1}\times S^{2q-1}, A)$ such that $(\Sigma(f_0), A(f_0))$ is equivalent to the given action. By Theorem 2.2, we may assume that $p=[n+1/2]$. By Corollary 4.1, there are infinitely many equivariant diffeomorphisms $\{f_j|j=1,2,\ldots\}$ of $(S^{2p-1}\times S^{2q-1}, A)$ which define inequivalent actions. By Theorem 1.7, there is an infinite subset $\{f_{j_s}|s=1,2,\ldots\}$ of $\{f_j|j=1,2,\ldots\}$ such that $\Sigma(f_0\cdot f_j^{-1}\cdot f_{j_s})$ is diffeomorphic to $\Sigma(f_0)$, i.e., Σ^{2n-1}. Note that the $f_0\cdot f_{j_0}^{-1}\cdot f_{j_s}$ are equivariant diffeomorphisms of $(S^{2p-1}\times S^{2q-1}, A)$. Hence we have infinitely many actions

$$\mathscr{A}=\{\Sigma(f_0\cdot f_{j_0}^{-1}\cdot f_{j_s}), A(f_0\cdot f_{j_0}^{-1}\cdot f_{j_s})|s=1,2,\ldots\}.$$

It suffices to show that each equivalence class of actions in $\mathscr{A}$ is finite. Suppose not, then there are infinitely many actions in $\mathscr{A}$ which are equivalent to one another, i.e., there are infinitely many actions

$$(\Sigma(f_0\cdot f_{j_0}^{-1}\cdot f_{i_s}), A(f_0\cdot f_{j_s}^{-1}\cdot f_{i_s})), \qquad s=0,1,\ldots$$

in $\mathscr{A}$ such that

$$(\Sigma(f_0\cdot f_{j_0}^{-1}\cdot f_{i_s}), A(f_0\cdot f_{j_0}^{-1}\cdot f_{i_s}))\sim(\Sigma(f_0\cdot f_{j_0}^{-1}\cdot f_{i_0}), A(f_0\cdot f_{j_0}^{-1}\cdot f_{i_0})).$$

By Theorem 6.1, there exists equivariant diffeomorphisms g_s of $(S^{2p-1}\times D^{2q}, A)$ and h_s of $(D^{2p}\times S^{2q-1}, A)$ such that

$$f_0\cdot f_{j_0}^{-1}\cdot f_{i_0}=h_s\cdot f_0\cdot f_{j_0}^{-1}\cdot f_{i_s}\cdot g_s.$$

By Theorem 5.4, for some $a\neq b$, h_a is equivariantly pseudo-isotopic to h_b and g_a is equivariantly pseudo-isotopic to g_b. It follows easily that f_{i_a} is equivariantly pseudo-isotopic to f_{i_b}. But $(\Sigma(f_{i_a}), A(f_{i_a}))$ is not equivalent to $(\Sigma(f_{i_b}), A(f_{i_b}))$. This is a contradiction. Hence we have proved that there are infinitely many differentiably inequivalent decomposable free S^1 actions on Σ^{2n-1}.

In particular, we take $\Sigma^{2n-1} = S^{2n-1}$ together with the known results, we have:

Main Theorem. *There are infinitely many differentiably inequivalent free S^1 actions on standard $(2n-1)$-spheres for $n \geqq 4$.*

Remark 6.3. In fact the actions can be chosen to be topologically inequivalent (see [23]).

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Formule de Poisson pour les variétés riemanniennes

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0. Introduction

La formule de Poisson classique

$$\sum_{k\in\mathbb{Z}} e^{ikt} = 2\pi \sum_{k\in\mathbb{Z}} \delta(t-2\pi k), \qquad (*)$$

peut s'interpréter comme une relation entre d'une part, le spectre $\lambda_k = k^2$ $(k=0,1,\ldots)$ du laplacien $-d^2/dx^2$ sur le tore plat $\mathbb{R}/\mathbb{Z}$ et d'autre part, les longueurs $2k\pi$ $(k=0,1,\ldots)$ des géodésiques périodiques de cette variété, c'est à dire les cercles.

On se propose de généraliser cette formule de Poisson à des variétés riemanniennes M compactes connexes. Notons $-\Delta$ le laplacien de M et soit

$$0=\lambda_0<\lambda_1\leqq\lambda_2\leqq\cdots,$$

la suite des valeurs propres de cet opérateur; on leur associe la distribution S défine par

$$S(t)=\tfrac{1}{2}\sum_{k\geqq 0}\exp(\pm i\sqrt{\lambda_k}\,t).$$

D'autre part, on désigne par $\mathscr{L}$ l'ensemble des longueurs (et de leurs opposées) des géodésiques périodiques de M; on a alors un premier résultat sur les singularités de la distribution S:

le support singulier de S est inclus dans $\mathscr{L}\cup\{0\}$.

De plus, si $\mathscr{L}$ est un ensemble discret de $\mathbb{R}$ et si les géodésiques de longueur donnée s'organisent en de «bonnes variétés», on obtient la généralisation de $(*)$ sous la forme

$$\boxed{\sum_{k\geqq 0}\exp(\pm i\sqrt{\lambda_k}\,t)=\sum_{l\in\mathscr{L}\cup\{0\}} T_l} \quad \text{(au sens de } \mathscr{D}'(\mathbb{R})) \qquad (*)'$$

où T_l désigne une distribution à support dans un petit voisinage de l et qui admet en l une singularité que l'on décrit au moyen de développements asymptotiques.

Ce travail est inspiré par des résultats récents de Colin de Verdière [5, 6] qui concernent une généralisation de la formule classique

$$\sum_{k\in\mathbb{Z}} \exp(-k^2/z) = \sqrt{\pi z} \sum_{k\in\mathbb{Z}} \exp(-\pi^2 k^2 z) \qquad (\operatorname{Re} z > 0) \qquad (**)$$

au cadre des variétés riemanniennes (sous les mèmes hypothèses que pour $(*)'$)

$$\sum_{k\geqq 0} \exp(-\lambda_k/z) = \sum_{l\in\mathscr{L}^+\cup\{0\}} f_l(z)\exp(-z\,l^2/4) \quad (\mathscr{F}) \qquad (**)'$$

où le signe $(\mathscr{F})$ indique que cette dernière égalité a lieu en fait en un certain sens qui est défini au moyen d'une transformation de Fourier non linéaire. Cet auteur démontre $(**)'$ par une méthode assez technique qui est basée sur la forme explicite de la solution élémentaire de l'opérateur de la chaleur $\partial/\partial t - \Delta_x$.

Dans ce travail, on utilise au contraire une paramétrix de l'operateur des ondes $\square = \partial^2/\partial t^2 - \Delta_x$, ce qui permet de faire intervenir les géodésiques périodiques de façon très naturelle à partir des bicaractéristiques de $\square$ et à l'avantage de donner une démonstration pratiquement sans calcul de $(*)'$. De plus, cette méthode se généralise au cas d'un opérateur elliptique auto-adjoint sur une variété compacte.

Ce travail est résumé dans une note aux Comptes rendus [4].

I. Spectre du laplacien et opérateur des ondes

On sait (cf. par exemple, Berger-Gauduchon-Mazet [1]) que le laplacien $-\Delta$ d'une variété riemannienne compacte est un opérateur auto-adjoint positif dans $L^2(M, dv)$ où dv désigne la densité canonique de M. Cet opérateur elliptique a une résolvante compacte et par conséquent son spectre est constitué d'une suite de valeurs propres

$$0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \cdots$$

que l'on répète selon leur multiplicité. Ce spectre est caractérisé par la distribution $S \in \mathscr{D}'(\mathbb{R})$ définie par

$$S(t) = \tfrac{1}{2} \sum_{k\geqq 0} \exp(\pm i\sqrt{\lambda_k}\, t) = \tfrac{1}{2}\mathscr{F}_\tau\Big(\sum_{k\geqq 0} \delta(\tau \pm \sqrt{\lambda_k})\Big) \qquad (1.1)$$

qui est une distribution tempérée (car le nombre de valeurs propres inférieures à λ est à croissance polynômiale en λ), paire, à valeurs réelles (on renvoie à Hörmander [8] pour les notations concernant les distributions, la transformation de Fourier ...).

Rappelons que dans une carte de M, le laplacien $-\Delta_x$ a pour symbole principal

$$g(x, \xi) = \sum_{i,j=1}^{n} g^{ij}(x)\, \xi_i \xi_j$$

où

$$\sum g_{ij}(x)\,dx^i\,dx^j$$

désigne la métrique riemannienne et n la dimension de la variété M. On définit l'opérateur des ondes sur l'espace $\mathbb{R}\times M$ par

$$\square = \partial^2/\partial t^2 - \Delta_x$$

et on lui associe le noyau du problème de Cauchy; c'est à dire la distribution $E_y(t,x)\in\mathscr{D}'(\mathbb{R}\times M)$ solution du problème de Cauchy suivant

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2}-\Delta_x\right)E_y(t,x)=0\\ &E_y|_{t=0}=\delta(x-y)\\ &\frac{\partial}{\partial t}E_y|_{t=0}=0 \end{aligned} \tag{1.2}$$

où y est arbitraire dans M; notons que les restrictions ont un sens car $t=0$ est non caractéristique (cf. Hörmander [11]).

On sait que l'on peut représenter le noyau distribution associé $E(t,x,y)\in\mathscr{D}'(\mathbb{R}\times M\times M)$ sous la forme

$$E(t,x,y)=\sum_{k\geq 0} w_k(x)\,w_k(y)\cos\left(\sqrt{\lambda_k}\,t\right)$$

où $(w_k)_{k\geq 0}$ constitue une base orthonormée de fonctions propres de Δ. D'où la relation entre S et E

$$S(t)=\int_M E(t,x,x)\,dv(x) \tag{1.3}$$

(cette écriture formelle, sera précisée plus loin, grâce aux résultats sur le spectre singulier de la distribution E).

De (1.3), on déduit qu'il suffit de connaître une paramétrix du proproblème de Cauchy (1.2) pour connaître les singularités de S, c'est à dire qu'il suffit de raisonner modulo C^∞.

II. Support singulier de S

On sait que l'on a une représentation globale d'une paramétrix du problème de Cauchy (1.2) en utilisant les opérateurs intégraux de Fourier de Hörmander [9] (voir aussi, Duistermaat [7]). De façon plus précise, on utilise le

Théorème. *Il existe un opérateur intégral de Fourier* $\tilde{F}: M \to \mathbb{R} \times M$ *qui est tel que*

$$\begin{aligned} &\tilde{F} \in I^{-\frac{1}{4}}(\mathbb{R} \times M, M; C) \\ &\Box \cdot \tilde{F} \equiv 0 \\ &\tilde{F}|_{t=0} \equiv \textit{Identité de } M \\ &\frac{\partial}{\partial t} \tilde{F}|_{t=0} \equiv 0 \end{aligned} \tag{2.1}$$

où C *est une relation canonique dans* $(T^* \mathbb{R} \times M \smallsetminus 0) \times (T^* M \smallsetminus 0)$ *définie par*

$$C = \left\{ (t, \tau; y, \eta; x, \xi) \,\middle|\, \begin{array}{l} \textit{le point } (t, \tau; y, \eta) \textit{ est sur la bicaractéristique} \\ \textit{de } \Box \textit{ qui passe par l'un des points } (0, \tau_0; x, \xi) \\ \textit{où } \tau_0 = \pm \sqrt{g(x, \xi)}. \end{array} \right\} \tag{2.2}$$

On utilise sans les rappeler les notations de [7] pour tout ce qui concerne les opérateurs intégraux de Fourier.

Rappelons qu'une bicaractéristique de $\Box$ passant par $(0, \tau_0; x, \xi)$ est une application $s \in \mathbb{R} \to (t(s), \tau(s); y(s), \eta(s)) \in T^* \mathbb{R} \times M \smallsetminus 0$ qui vérifie l'équation différentielle

$$\begin{aligned} &\frac{dt}{ds} = 2\tau \qquad \frac{d\tau}{ds} = 0 \\ &-\frac{dy}{ds} = \frac{\partial g}{\partial \xi}(y, \eta) \qquad \frac{d\eta}{ds} = +\frac{\partial g}{\partial x}(y, \eta) \\ &(t, \tau; y, \eta)|_{s=0} = (0, \tau_0; x, \xi) \end{aligned} \tag{2.3}$$

et la condition $\tau^2 - g(y, \eta) = 0$ (on trouve immédiatement que $\tau = Cte = \tau_0$ et $t = 2\tau_0 s$).

Remarquons que la densité canonique dv sur M, permet d'identifier les densités d'ordre $\frac{1}{2}$ et les fonctions au moyen de l'isomorphisme

$$f \in C^\infty(M) \mapsto f \sqrt{dv} \in C^\infty(M, \Omega_{\frac{1}{2}}).$$

On désigne par $F(t, x, y) \in \mathscr{D}'(\mathbb{R} \times M \times M)$ le noyau distribution de l'opérateur $\tilde{F}$, alors on sait que le spectre singulier de F (noté ici S.S.F et que Hörmander [11] appelle « wave front set » $WF(F)$) vérifie l'inclusion

$$\text{S.S.}F \subset \{(t, \tau; y, \eta; x, \xi) \mid (t, \tau; y, \eta; x, -\xi) \in C\} = C'.$$

On remarque ici que $\tau = \pm \sqrt{g(x, \xi)} \neq 0$ puisque $(x, \xi) \in T^* M \smallsetminus 0$, alors, il découle des théorèmes sur le spectre singulier (cf. [11]) que l'on peut définir la restriction de la distribution F à la variété $\mathbb{R} \times D \subset \mathbb{R} \times M \times M$

où D désigne la diagonale de $M \times M$. Il vient

$$\text{S.S.}(F|_{\mathbb{R}\times D}) \subset \{(t, \tau; x, \eta - \xi) \mid (t, \tau; x, \eta; x, \xi) \in C\}$$

et par intégration sur le compact M on en déduit

$$\text{S.S.}\Big(\int_M F(t, x, x)\, dv(x)\Big) \subset \left\{(t, \tau) \,\middle|\, \begin{array}{l}\text{il existe } (x, \xi) \text{ tel que l'on ait}\\ (t, \tau; x, \xi; x, \xi) \in C\end{array}\right\}.$$

Mais d'autre part, comme $\tilde{F}$ est une paramétrix, on a aussi

$$S - \Big(\int_M F(t, x, x)\, dv(x)\Big) \in C^\infty(\mathbb{R}) \tag{2.4}$$

et par conséquent on en déduit que le support singulier de la distribution S vérifie l'inclusion

$$\text{supp. sing. } S \subset \left\{t \,\middle|\, \begin{array}{l}\text{il existe } (x, \xi) \in T^* M \setminus 0 \text{ tel que}\\ (t, \tau; x, \xi; x, \xi) \in C \text{ où } \tau = \pm\sqrt{g(x, \xi)}\end{array}\right\}. \tag{2.5}$$

Si on se reporte à la définition (2.2) de C, on constate que les valeurs de t qui apparaissent dans (2.5) sont 0 et $\pm$ les temps de parcours pour aller de (x, ξ) à (x, ξ) sur une bicaractéristique, c'est à dire les périodes des bicaractéristiques périodiques au sens de la

Définition 2.1. Une bicaractéristique $s \to (t(s), \tau(s); x(s), \xi(s))$ de $\square$ est dite «périodique», si sa projection $s \to (x(s), \xi(s))$ est une application périodique. Les temps de parcours d'une ou plusieurs boucles s'appellent les périodes de la bicaractéristique. Notons $\mathscr{L}^+$, l'ensemble de ces périodes >0 et posons $\mathscr{L} = \mathscr{L}^+ \cup (-\mathscr{L}^+)$.

En utilisant la structure riemannienne on peut donner une interprétation plus géométrique de ces périodes. Pour cela, calculons la longueur l de la courbe $s \to (x(s))$ pour un temps de parcours correspondant à une période $t > 0$. Avec le paramétrage par s, on obtient l'expression

$$l = \int_0^{t/2\tau_0} \left\| \frac{dx}{ds} \right\| ds,$$

par définition, on a $\left\| \frac{dx}{ds} \right\|^2 = G_x\left(\frac{dx}{ds}, \frac{dx}{ds}\right)$ où G_x désigne la forme bilinéaire associée à la métrique sur $T_x M$. Notons G_x^{-1} l'isomorphisme canonique associé $T_x^* M \to T_x M$, de (2.3) on déduit

$$\frac{dx}{ds} = 2 G_x^{-1} \cdot \xi(s)$$

d'où

$$\left\| \frac{dx}{ds} \right\|^2 = 4 g(x, \xi) = 4 \tau_0^2,$$

et finalement

$$l=t.$$

D'autre part, il est bien connu que les projections sur M des bicaractéristiques de $\square$ sont précisément les géodésiques de M; on peut alors énoncer la

Proposition 2.1. *L'ensemble $\mathscr{L}\subset\mathbb{R}\setminus\{0\}$ des périodes des bicaractéristiques périodiques, peut également s'interpréter comme l'ensemble des longueurs (et de leurs opposées) des géodésiques périodiques de M.*

L'inclusion (2.5) permet de formuler le

Théorème I. *La distribution $S=\sum_{k\geqq 0}\cos(\sqrt{\lambda_k}\,t)$ est C^∞ en dehors de l'ensemble $\{0\}\cup\mathscr{L}$.*

III. Étude des singularités de S

Soit $l\in\mathscr{L}$ que l'on suppose être isolé; on se propose d'étudier la singularité de la distribution S au point l. Pour cela, on se donne une fonction $\theta\in C_0^\infty(\mathbb{R})$ à support dans un petit voisinage de l qui ne rencontre pas l'ensemble $\mathscr{L}\setminus\{l\}$ et vérifiant $\theta(l)=1$. La singularité de S en l est caractérisée par le comportement asymptotique en τ de l'expression

$$(\widehat{\theta S})(\tau)=\langle S(t),\theta(t)\,e^{-i\tau t}\rangle. \tag{3.1}$$

Comme on raisonne module C^∞, il revient au même, grâce à (2.4), d'étudier le comportement asymptotique de l'expression

$$I(\tau)=\int_{\mathbb{R}\times M}\theta(t)\,F(t,x,x)\,e^{-i\tau t}\,dt\,dv(x). \tag{3.2}$$

Tout d'abord, remarquons que l'on peut se restreindre à étudier le comportement pour $\tau\to+\infty$ car $\widehat{\theta S}(-\tau)=\overline{\widehat{\theta S}(\tau)}$ puisque S est à valeurs reelles, de plus S étant paire il suffit de considérer le cas où $l\in\mathscr{L}^+$.

Pour étudier l'intégrale (3.2) on est amené à expliciter la distribution $\theta F\in I^{-\frac{1}{4}}(\mathbb{R}\times M\times M;\,C')$ en se reportant à la définition donnée par Hörmander dans [11]. Cette distribution s'écrit comme une somme finie

$$\theta F=\sum_{\alpha\in A}F_\alpha \tag{3.3}$$

(A est un ensemble fini car θF est à support compact) où les F_α sont des distributions définies au moyen d'intégrales oscillantes. A cet effet, on commence par définir un recouvrement de la relation canonique C par des domaines C_α de cartes T_α associées à des fonctions de phases de la forme

$$\phi_\alpha(t,x,\eta,y)=\varphi_\alpha(t,x,\eta)-y\cdot\eta$$

où $y \cdot \eta = \sum y_j(y)\eta_j$ et les $(y_j(y)) = \chi_\alpha(y)$ sont les coordonnées du point y dans une carte M_α de M. De façon plus précise, l'ouvert C_α est difféomorphe à un ouvert conique Z_α de $]l-\varepsilon, l+\varepsilon[\times M \times \Gamma_\alpha$ (avec Γ_α un cône ouvert de $\mathbb{R}^n \setminus 0$) au moyen de l'application T_α définie par

$$(t, x, \eta) \in Z_\alpha \xrightarrow{T_\alpha} (t, \varphi'_{\alpha t}; x, \varphi'_{\alpha x}; y, -\phi'_{\alpha y}) \in C_\alpha \subset C \tag{3.4}$$

avec y détermine par $\chi_\alpha(y) = \varphi'_{\alpha\eta}(t, x, \eta)$ et où $\varphi'_{\alpha x}$ désigne la dérivée en x de φ_α. Ceci posé, la distribution F_α est définie par l'intégrale oscillante

$$F_\alpha(t, x, y) = \int_{\Gamma_\alpha} (\exp i\, \phi_\alpha(t, x, \eta, y))\, a_\alpha(t, x, \eta)\, đ\eta \tag{3.5}$$

où $đ\eta = (2\pi)^{-n} d\eta$ et a_α désigne un symbole de degré 0 dont le support est une partie conique à base compacte incluse dans Z_α. On a une telle représentation de F_α car la relation canonique C peut être aussi considérée comme un graphe local dépendant du paramètre t (cf. Hörmander [11], p. 170, et Chazarain [3]).

A la somme (3.3) correspond pour $I(\tau)$ l'expression

$$I(\tau) = \sum_{\alpha \in A} I_\alpha(\tau)$$

avec

$$I_\alpha(\tau) = \int_{\mathbb{R} \times M} F_\alpha(t, x, x)\, e^{-i\tau t}\, dt\, dv(x),$$

soit encore, compte tenu de (3.5),

$$\begin{aligned} I_\alpha(\tau) &= \int_{Z_\alpha} (\exp[i\,\varphi_\alpha(t, x, \eta) - i\,x \cdot \eta - i\,\tau\, t])\, a_\alpha(t, x, \eta)\, dt\, dv(x)\, đ\eta \\ &= \tau^n \int_{Z_\alpha} (\exp i\,\tau\, \Phi_\alpha(t, x, \eta))\, a_\alpha(t, x, \tau\,\eta)\, dt\, dv(x)\, đ\eta \end{aligned} \tag{3.6}$$

où l'on a posé $\Phi_\alpha(t, x, \eta) = \varphi_\alpha(t, x, \eta) - x \cdot \eta - t = \phi_\alpha(t, x, \eta, x) - t$.

Pour étudier le comportement asymptotique de $I_\alpha(\tau)$, on commence par chercher les points critiques de la phase Φ_α. Pour alléger, omettons provisoirement l'indice α. Les points critiques de Φ dans Z sont donnés par les équations

$$\begin{aligned} &\varphi'_t(t, x, \eta) = 1, \\ &\phi'_x(t, x, \eta, x) = -\phi'_y(t, x, \eta, x), \\ &\phi'_\eta(t, x, \eta, x) = 0. \end{aligned} \tag{3.7}$$

Par l'application T (définie en (3.4)) ces points critiques sont en bijection avec des points de C de la forme

$$(t, 1; x, \xi; x, \xi) \quad \text{où } \xi = \varphi'_x(t, x, \eta)$$

et compte tenu de la définition de C, cela signifie qu'il existe une bicaractéristique de $\Box$ qui relie les points $(0, 1; x, \xi)$ et $(t, 1; x, \xi)$, ce qui prouve que t est une période de bicaractéristique périodique; mais t est dans le support de θ; par conséquent on en déduit que

$$t = l. \tag{3.8}$$

D'après (2.3) et (3.4) la phase φ vérifie l'équation

$$\varphi_t'^2 = g(x, \varphi_x'); \tag{3.9}$$

en reportant dans (3.7); il vient

$$g(x, \xi) = 1. \tag{3.10}$$

Désignons par W_l^+, la partie de C constituée par tous les points

$$(l, 1; x, \xi; x, \xi)$$

qui sont obtenus comme image, par les diverses cartes T_α, des points critiques des diverses phases Φ_α. Cet ensemble W_l^+, qui va jouer un rôle essentiel, admet une interprétation géométrique simple. Pour cela, désignons par $S^* M$ le fibré en sphères conormales de M, c'est à dire

$$S^* M = \{(x, \eta) \in T^* M \mid g(x, \eta) = 1\}$$

et soit D^* la diagonale de l'ensemble $S^* M \times S^* M$. Alors les équations (3.10) et (3.7) signifient exactement que

$$W_l^+ = (\{l, 1\} \times D^*) \cap C. \tag{3.11}$$

De plus, cet ensemble W_l^+ est en bijection avec l'ensemble des bicaractéristiques périodiques de $\Box$ qui admettent l pour période et qui sont parcourues dans le sens des t croissants (celles qui sont parcourues dans le sens des t décroissants correspondent à -1 dans $\{l, -1\}$ et n'interviennent que pour le comportement avec $\tau \to -\infty$ et sont en bijection avec l'ensemble W_l^- défini de façon évidente. On pose aussi $W_l = W_l^+ \cup W_l^-$).

On retrouve ainsi le fait que si l n'est pas une période, les phases Φ_α n'ont pas de points critiques et par conséquent $I(\tau)$ est à décroissance rapide, ce qui implique que S est C^∞ an voisinage de l et redémontre le théorème I. En revanche, si $l \in \mathscr{L}$ l'ensemble W_l est non vide par définition et il s'agit de déterminer le développement asymptotique de $I(\tau)$. Pour cela, on utilise une extension du théorème de la phase stationnaire que Colin de Verdière [6] a déjà utilisé dans un but analogue.

IV. Une extension du théorème de la phase stationnaire

On développe dans ce paragraphe, les notions nécessaires à l'étude des développements asymptotiques des intégrales du type (3.6).

On commence par définir, suivant Meyer [12], la notion de variété critique non dégénérée pour une phase.

Définition (Meyer). Soit Z une variété et Φ une application C^∞ de Z dans $\mathbb{R}$. On suppose que l'ensemble des points critiques de Φ est une sous variété connexe W de Z. Alors, on dit que W est une variété critique *non dégénérée* pour la phase Φ, si pour tout point $z \in W$ le hessien $\Phi''(z)$ induit sur l'espace normal $N_z = T_z Z / T_z W$ une forme quadratique non dégénérée $\Phi''(z)|_N$.

Pour $z \in W$, on désigne par σ la signature du hessien $= \operatorname{sgn} \Phi''(z) =$ (nombre de carrés $+$) $-$ (nombre de carrés $-$) et qui est aussi égale à la signature de la forme induite $\Phi''(z)|_N$.

Dans cette situation on dispose d'un analogue du lemme de Morse, c'est le

Lemme (Meyer). *Soit $\Phi(u, v)$ une application C^∞ de $\mathbb{R}^p \times \mathbb{R}^q$ dans $\mathbb{R}$, définie au voisinage de* $(0,0)$ *et telle que l'ensemble des points critiques de Φ est donné par l'équation $\{v = 0\}$. On suppose de plus que $\Phi''_{vv}(0,0)$ est non dégénérée, autrement dit que la variété critique $\{v=0\}$ est non dégénérée pour Φ au voisinage de* 0. *Alors il existe un difféomorphisme H*

$$\mathbb{R}^{p+q} \ni (u, v) \xrightarrow{H} (x, y) \in \mathbb{R}^{p+q}$$

de la forme $x = u$ et $y = h(u, v)$, tel que la nouvelle phase $\tilde{\Phi} = \Phi \circ H^{-1}$ vérifie *au voisinage de zéro*

$$\tilde{\Phi}(x, y) - \tilde{\Phi}(x, 0) = \tfrac{1}{2}{}^t y \cdot Q(x) \cdot y$$

où $Q(x) = \Phi''_{vv}(x, 0)$ et on a aussi $\det(H'(x, 0)) = 1$.

Démonstration. La formule de Taylor appliquée à Φ donne

$$\Phi(u, v) - \Phi(u, 0) = \tfrac{1}{2}{}^t v \cdot A(u, v) \cdot v$$

avec $A(u, 0) = \Phi''_{vv}(u, 0)$. On cherche une application

$$(u, v) \to B(u, v) \in \mathscr{L}(\mathbb{R}^q; \mathbb{R}^q)$$

défine au voisinage de 0 et telle qu'en posant $x = u$ et $y = B(u, v) \cdot v = h(u, v)$ on ait

$$\Phi(u, v) - \Phi(u, 0) = \tfrac{1}{2}{}^t y \cdot Q(x) \cdot y.$$

Il vient pour B l'équation suivante

$${}^t B(u, v) \circ A(u, 0) \circ B(u, v) - A(u, v) = 0,$$

on remarque que pour $v = 0$ on a la solution $B(u, 0) = I$, ensuite le théorème des fonctions implites donne une solution B définie au voisinage de 0 car $A(u, 0)$ est inversible. Enfin, on vérifie immédiatement que $\det(H'(x, 0)) = 1$.

Enonçons maintenant une généralisation du théorème de la phase stationnaire.

Théorème (Colin de Verdière [6]). *Soit Z une variété riemannienne de dimension d, soit $a \in C_0^\infty(Z)$ et soit une phase à valeurs réelles $\Phi \in C^\infty(Z)$. On suppose que les points critiques de Φ situés dans le support de a constituent une sous variété compacte connexe W de Z dont on note v la dimension. On suppose de plus que W est une variété critique non dégénérée pour Φ et soit σ sa signature.*

Alors on a le comportement asymptotique suivant

$$J(\tau) = \int_Z e^{i\tau\Phi(z)} a(z)\, dv(z) = \left(\frac{2\pi}{\tau}\right)^{\frac{d-v}{2}} e^{i\frac{\pi}{4}\sigma} e^{i\tau\Phi(w)} p(\tau) \tag{4.1}$$

où $p(\tau)$ admet pour $\tau \to \infty$ un développement asymptotique de la forme

$$p(\tau) \approx \sum_{k \geqq 0} a_k \tau^{-k} \tag{4.2}$$

avec

$$a_0 = \int_W a(z)\, |\det \Phi''(z)|_N|^{-\frac{1}{2}}\, dv_W(z) \tag{4.3}$$

et $d_W v$ désigne la mesure induite sur la sous variété W.

Esquissons la démonstration. Comme W est une variété compacte, on peut la recouvrir par un nombre fini d'ouverts de carte Z_α dans lesquels on peut utiliser le lemme précédent; une partition de l'unité subordonnée à ce recouvrement permet d'écrire

$$J(\tau) = \sum_{\text{finie}} J_\alpha(\tau)$$

avec

$$J_\alpha(\tau) = \int_{Z_\alpha} e^{i\tau\Phi} a_\alpha dv.$$

Dans la carte Z_α, l'intégrale J_α s'exprime par

$$J_\alpha(\tau) = \left(\int_{\mathbb{R}^d} e^{\frac{1}{2} i\tau({}^t y \cdot Q(x) \cdot y)} a_\alpha(x, y)\, c_\alpha(x, y)\, dx\, dy\right) e^{i\tau\Phi(w)}. \tag{4.4}$$

Au moyen d'un changement de coordonnées linéaire en y ($y \to z$) et conservant x, on ramène la forme quadratique $Q(x)$ à la forme canonique

$${}^t y \cdot Q(x) \cdot y = (z_1^2 + \cdots + z_j^2) - (z_{j+1}^2 + \cdots + z_h^2) = q(z)$$

où $j - h = \sigma = \operatorname{sgn} \Phi'' = \operatorname{sgn} q$. Notons que le jacobien de ce changement de coordonnées est égal à

$$|\det \Phi''(z)|_N|^{-\frac{1}{2}}$$

quand on se place sur W. L'intégrale (4.4) s'écrit alors

$$\int_{\mathbb{R}^d} e^{\frac{1}{2} i\tau q(z)} a_\alpha(x, z)\, c_\alpha(x, z) \left|\frac{dy}{dz}\right| dx\, dz$$

et une intégration partielle en x donne

$$\int_{\mathbb{R}^{d-\nu}} e^{\frac{1}{2}i\tau q(z)}\,\tilde{a}_\alpha(z)\,\tilde{c}_\alpha(z)\,dz. \tag{4.5}$$

On termine en appliquant le théoreme de la phase stationnaire usuel (cf. par exemple, Hörmander [11]) à l'intégrale (4.5); d'où le théorème en notant que

$$a_0=\sum_\alpha \tilde{a}_\alpha(0)=\int_W a(z)\,|\det \Phi''(z)|_N|^{-\frac{1}{2}}\,dv_W(z).$$

V. Développement asymptotique de $I(\tau)$

On se fixe $l\in\mathscr{L}$, supposé isolé et on va appliquer les résultats précédents aux intégrales (3.6) pour obtenir le développement de $I(\tau)$.

Pour cela, on est conduit à introduire l'hypothèse suivante sur l'ensemble W_l des bicaractéristiques admettant l pour période.

(H_l) L'ensemble W_l^+ est une réunion finie de sous variétés compactes connexes

$$W_l^+=\bigcup_{j\in J_l} W_{l,j}$$

on pose $\nu_j=\dim W_{l,j}$.

Ces variétés critiques $W_{l,j}$ sont non dégénérées[1] pour les phases $\Phi_\alpha\circ T_\alpha^{-1}$; nous notons $\sigma_{j,\alpha}$ la signature correspondante.

Commençons par montrer que l'on peut séparer dans $I(\tau)$ les contributions des diverses composantes $W_{l,j}$ de W_l^+. Pour cela, on utilise le lemme général suivant.

Lemme 5.1. *Soit une «distribution de Fourier» $F\in I^m(X;\Lambda)$ et un recouvrement fini (V_j) de la variété lagrangienne Λ par des ouverts coniques. Soit (r_j) une partition de l'unité subordonnée à ce recouvrement au moyen de fonctions homogènes de degré 0. Alors la distribution F se décompose en*

$$F\equiv\sum_j F_j$$

où $F_j\in I^m(X;\Lambda)$, avec S.S.$(F_j)\subset V_j$. De plus, si a désigne le symbole principal de F, celui de F_j est égal à $r_j\,a$.

Démonstration. Comme le support de r_j est dans V_j, on peut toujours trouver une distribution $F_j^{(0)}$ dans $I^m(X;\Lambda)$, dont le symbole principal est $r_j a$ et telle que S.S.$(F_j^{(0)})\subset V_j$. On définit alors

$$G^{(0)}=(F-\sum F_j^{(0)})\in I^{m-1}.$$

Le même procédé appliqué à $G^{(0)}$, permet de trouver $F_j^{(1)}\in I^{m-1}$ avec S.S.$(F_j^{(1)})\subset V_j$ et telle que

$$G^{(1)}=(G^{(0)}-\sum F_j^{(1)})\in I^{m-2}.$$

[1] *Note ajoutée dans l'épreuve.* Cela signifie exactement que W_l est une «non-degenerate fixed manifold» au sens de Weinstein (Ann. of Math. **93**, 396, 1973); pour les détails voir notre exposé no 16 au Séminaire Goulaouic-Schwartz, Ecole Polytechnique (1974).

On définit ainsi, des suites $F_j^{(k)}$ avec des propriétés évidentes; d'où le lemme, en posant

$$F_j \sim \sum_{k \geqq 0} F_j^{(k)}.$$

Quitte à raffiner le recouvrement (C_α) de C, on peut supposer que les ouverts C_α sont connexes et ne rencontrent jamais deux variétés critiques à la fois. Soit (r_α) une partition de l'unité subordonnée; alors le lemme précédent permet d'écrire

$$F \equiv \sum_{\text{finie}} F_\alpha,$$

où F_α a pour symbole principal $r_\alpha\, a$. Désignons par $I_\alpha(\tau)$, l'intégrale (3.2) relativement à F_α. On a immédiatement

$$I(\tau) \equiv \sum_\alpha I_\alpha(\tau),$$

où le signe $\equiv$ signifie ici l'égalité modulo un terme à décroissance rapide en τ. Pour j fixé, on désigne par $I_{j,\alpha}(\tau)$ les termes $I_\alpha(\tau)$, quand C_α rencontre $W_{l,j}$. On définit alors la contribution de $W_{l,j}$ par

$$I_j(\tau) = \sum_\alpha I_{j,\alpha}(\tau); \tag{5.1}$$

et on a

$$I(\tau) \equiv \sum_{j \in J_l} I_j(\tau). \tag{5.2}$$

On se propose maintenant, d'étudier le comportement d'un terme $I_j(\tau)$. Tout d'abord, on note que $I_{j,\alpha}(\tau)$ s'exprime par une intégrale du type (3.6), dans laquelle a_α^0 désigne l'image du symbole $r_\alpha\, a$ par la carte T_α. L'application du théorème du paragraphe précédent à cette intégrale (la généralisation aux intégrales oscillantes ne présente aucune difficulté) donne l'expression

$$I_{j,\alpha}(\tau) = e^{-i\tau l} \left(\frac{2\pi}{\tau}\right)^{(1-\nu_j)/2} e^{i\frac{\pi}{4}\sigma_{j,\alpha}}\, p_{j,\alpha}(\tau), \tag{5.3}$$

où $p_{j,\alpha}(\tau)$ admet un développement asymptotique

$$p_{j,\alpha}(\tau) \sim \sum_{k \geqq 0} p_{j,\alpha}^k\, \tau^{-k}; \tag{5.4}$$

avec pour le terme principal $p_{j,\alpha}^0$, l'expression donnée par (4.3), soit

$$p_{j,\alpha}^0 = \int_{T_\alpha^{-1}(W_{l,j} \cap C_\alpha)} a_\alpha^0(l, x, \eta)\, |\det(\Phi_\alpha''|_N)|^{-\frac{1}{2}}\, dv(x)\, đ\eta. \tag{5.5}$$

Notons que cette intégrale a bien un sens, car on intègre en fait sur le compact $T_\alpha^{-1}(W_{l,j} \cap (\text{supp. } r_\alpha\, a))$.

Pour regrouper les différents termes $e^{i\frac{\pi}{4}\sigma_{j,\alpha}}\, p^0_{j,\alpha}$ qui définissent la partie principale de $I_j(\tau)$, on s'appuie sur la

Proposition 5.1. *Les termes non nuls de la forme*

$$e^{i\frac{\pi}{4}\sigma_{j,\alpha}}\, p^0_{j,\alpha}$$

ont tous le même argument j fixé.

On commence par démontrer le

Lemme 5.2. *Il existe un entier n_α tel que l'on ait sur $T_\alpha^{-1}(W_{l,j} \cap C_\alpha)$*

$$a^0_\alpha(l, x, \eta) = e^{i n_\alpha \frac{\pi}{2}}\, |a^0_\alpha(l, x, \eta)|.$$

Démonstration du lemme. Rappelons (cf. Duistermaat [7]) que le symbole principal, a, de F est une section de $L \otimes \Omega_{\frac{1}{2}}$ au dessus de C. Comme Δ est auto-adjoint, la première égalité de (2.1) implique que a vérifie l'équation

$$\mathscr{L}_{H_{\tilde{\square}}} \cdot a = 0, \tag{5.6}$$

où $\mathscr{L}_{H_{\tilde{\square}}}$ désigne la dérivée de Lie associée à l'hamiltonien $H_\square$ remonté sur C. De plus, les conditions initiales (2.1), entrainent que la restriction $a|_{(\{0,1\} \times D^*)}$ est identique à 1. Compte tenu de (5.6), on en déduit que la restriction $a|_{(\{l,1\} \times D^*) \cap C = W_l^+}$ est une section constante la valeur de cette restriction, lue dans la trivialisation de L définie par la carte T_α, est donc une puissance entière de $\sqrt{-1}$ d'après la définition du fibré de Maslov L; soit $e^{i n_\alpha \pi/2}$ cette valeur. Par conséquent, l'amplitude $a^0_\alpha(l, x, \eta)$ a toujours le même argument sur le domaine d'intégration de (5.5), puisqu'elle est l'image de $r_\alpha \cdot a$ dans la carte T_α.

Comparons les arguments dans deux cartes qui se coupent, c'est le

Lemme 5.3. *Soient deux indices α, β tels que*

$$(\text{supp. } r_\alpha) \cap (\text{supp. } r_\beta) \cap (W_{l,j}) \neq \emptyset.$$

Alors on a

$$e^{\left(i\frac{\pi}{4}\sigma_{j,\alpha} + i n_\alpha \frac{\pi}{2}\right)} = e^{\left(i\frac{\pi}{4}\sigma_{j,\beta} + i n_\beta \frac{\pi}{2}\right)}. \tag{5.7}$$

Démonstration du lemme. Il existe donc $\lambda \in (\text{supp. } r_\alpha \cap \text{supp. } r_\beta \cap W_{l,j})$. Soit V un voisinage conique de λ inclus dans $C_\alpha \cap C_\beta$ et dans lequel r_α et r_β ne s'annulent pas. On se donne une fonction $r \geqq 0$, à support dans V, homogène de degré 0, C^∞, et égale à 1 au point λ. On peut écrire

$$\frac{r}{r_\alpha}(r_\alpha\, a) = \frac{r}{r_\beta}(r_\beta\, a) = r\, a.$$

Désignons par G, une distribution de $I^{-\frac{1}{4}}(\mathbb{R} \times M \times M; C')$, admettant $r \cdot a$ pour symbole principal. En calculant l'expression (3.2) relative à G, dans les deux cartes T_α et T_β, il vient l'égalité

$$\begin{aligned} & e^{i\frac{\pi}{4}\sigma_{j,\alpha}} \cdot \int_{T_\alpha^{-1}(W_{l,j}\cap C_\alpha)} \left(\widetilde{\frac{r}{r_\alpha}}\right) \cdot a_\alpha^0(l,x,\eta)\,|\det(\Phi_\alpha''|_N)|^{-\frac{1}{2}}\, dv(x)\, d\!\!\!{}^-\eta \\ & = e^{i\frac{\pi}{4}\sigma_{j,\beta}} \cdot \int_{T_\beta^{-1}(W_{l,j}\cap C_\beta)} \left(\widetilde{\frac{r}{r_\beta}}\right) \cdot a_\beta^0(l,y,\xi)\,|\det(\Phi_\beta''|_N)|^{-\frac{1}{2}}\, dv(y)\, d\!\!\!{}^-\xi \end{aligned} \tag{5.8}$$

où $\left(\widetilde{\frac{r}{r_\alpha}}\right)$ désigne l'image de la fonction $\left(\frac{r}{r_\alpha}\right)$ dans la carte T_α. Alors, l'égalité (5.7) découle immédiatement de l'égalité (5.8) et du lemme 5.2.

Démontrons enfin la proposition 5.1. Tout d'abord, notons que l'égalité (5.7) est encore valable pour tous les indices α, β tels que $(\text{supp.}\, r_\alpha)\cap W_{l,j}\neq\emptyset$ et $(\text{supp.}\, r_\beta)\cap W_{l,j}\neq\emptyset$. En effet, $W_{l,j}$ étant connexe, on peut relier de tels indices par des indices intermédiaires qui vérifient la condition du lemme 5.3. Dans ces conditions, l'égalité (5.7) montre que, compte tenu de (5.5), tous les termes

$$e^{i\frac{\pi}{4}\sigma_{j,\alpha}}\, p_{j,\alpha}^0$$

non nuls, ont un argument indépendant de l'indice α.

On introduit alors un entier σ_j tel que la valeur commune des deux membres de (5.7) s'écrive

$$e^{i\frac{\pi}{4}\sigma_j}.$$

On dira que σ_j est la *signature* de la *variété critique* $W_{l,j}$ et on définit l'entier $\frac{1}{2}(v_j-\sigma_j)$ comme étant *l'indice* de $W_{l,j}$.

Revenons à l'étude du comportement de $I_j(\tau)$. En combinant (5.1), (5.3), et la définition de σ_j, on trouve

$$I_j(\tau)=e^{-i\tau l}\left(\frac{2\pi}{\tau}\right)^{(1-v_j)/2} e^{i\frac{\pi}{4}\sigma_j}\, p_j(\tau), \tag{5.9}$$

où $p_j(\tau)$ admet un développement asymptotique de la forme

$$p_j(\tau)\sim \sum_{k\geqq 0} p_j^k\,\tau^{-k}. \tag{5.10}$$

Remarque 5.1. Il est important de noter que $p_j^0>0$. En effet, on a immédiatement

$$\begin{aligned} p_j^0 &= e^{-i\frac{\pi}{4}\sigma_j}\sum_\alpha e^{i\frac{\pi}{4}\sigma_{j,\alpha}}\, p_{j,\alpha}^0 \\ &= \sum_\alpha \int_{T_\alpha^{-1}(W_{l,j}\cap C_\alpha)} |a_\alpha^0(l,x,\eta)|\,|\det \Phi_\alpha''|_N|^{-\frac{1}{2}}\, dv(x)\, d\!\!\!{}^-\eta. \end{aligned}$$

D'où, $p_j^0>0$, puisque la restriction de a à $W_{l,j}$ est une constante non nulle.

Regroupons ce qui précède en un théorème.

Théorème II. *Soit l un point isolé de l'ensemble $\mathscr{L}$ des périodes de bicaractéristiques périodiques. On suppose que l'hypothése (H_l) est satisfaite, alors le comportement asymptotique de $\widehat{\theta S}(\tau)$ est donné par*

$$\widehat{\theta S}(\tau) \equiv I(\tau) \equiv \sum_{j \in J_l} I_j(\tau)$$

où I_j correspond à une composante connexe de dimension ν_j et de signature σ_j de la variété critique, et on a l'expression

$$I_j(\tau) = e^{-i\tau l} \left(\frac{2\pi}{\tau}\right)^{(1-\nu_j)/2} e^{i\frac{\pi}{4}\sigma_j} p_j(\tau)$$

avec p_j vérifiant (5.10).

VI. Formule de Poisson

Pour interpréter ces développements asymptotiques, il est commode d'introduire une classe de distributions sur $\mathbb{R}$ (en fait, une classe de germes de singularités à l'origine) qui correspondent grosso-modo aux distributions admettant un développement asymptotique en distributions homogènes.

Définition 6.1. Soit $r \in \mathbb{R}$, on désigne par H_r l'espace des germes à l'origine de distributions reelles sur $\mathbb{R}$ qui sont C^∞ en dehors de $\{0\}$ et telles que la transformée de Fourier $\hat{T}(\tau)$ d'un représentant d'un tel germe, admet un développement asymptotique de la forme

$$\hat{T}(\tau) \approx \tau^{(r-1)/2} \sum_{k \geqq 0} b_k \tau^{-k} \qquad (\tau \to +\infty). \tag{6.1}$$

Par exemble, les germes définis par $\delta(t=0)$, $|t|^{-(r+1)/2}$ appartiennent respectivement à H_1 et H_r pour $r > 0$.

Avec cette définition, les formules (5.9) et (5.10) montrent que la restriction de S à un petit voisinage V de l, peut s'écrire

$$S(t)|_V = \sum_{j \in J_l} e^{i\frac{\pi}{4}\sigma_j} T_j(t+l) \tag{6.2}$$

où T_j définît un germe dans H_{ν_j}.

Remarque 6.1. Si W_l^+ se réduit à une seule variété connexe, la somme (6.2) se réduit à un seul terme qui admet donc effectivement l comme point singulier à cause de la remarque 5.1; dans ce cas on a donc $l \in$ Supp. Sing. S.

Avant de passer à la formule de Poisson, il reste à étudier ce qui se passe si $l=0$. Notons tout d'abord que 0 est un point isolé dans $\mathscr{L} \cup \{0\}$.

L 'étude de $I(\tau)$, défini en (3.2), se fait de façon complètement analogue quand $l=0$. On sait que pour $|t|$ petit, les phases $\varphi_\alpha(t,x,\eta)$ vérifient, en plus de l'équation (3.9), la condition initiale

$$\varphi_\alpha(t,x,\eta)|_{t=0}=x\cdot\eta$$

(cf. par exemple, [3]), de sorte que les équations (3.10) se réduisent ici à

$$g(x,\eta)=1$$
$$t=0.$$

La variété critique correspondante W_0 est donc donnée par

$$W_0=W_0^+\cup W_0^-=(\{0,1\}\times D^*)\cup(\{0,-1\}\times D^*).$$

On vérifie immédiatement que W_0 est une variété critique non dégénérée de signature nulle, elle est compacte et constituée de deux composantes connexes (du moins si $n\geqq 2$) W_0^+ et W_0^- de dimension $2n-1$. Par conséquent, le développement de $\widehat{\theta S}(\tau)$ est donné dans ce cas par

$$\widehat{\theta S}(\tau)=(2\pi)^{1-n}\,\tau^{n-1}\,p(\tau)\qquad(\tau>0)\tag{6.3}$$

avec

$$p(\tau)\approx\sum_{k\geqq 0}p^k\,\tau^{-k}$$

où

$$p^0=\int_{g(x,\eta)=1}dx\,d\eta.$$

De (6.3), on déduit le

Théorème III. *Le germe de distribution à l'origine défini par S est dans l'espace* H_{2n-1}.

On est maintenant en mesure d'énoncer la formule de Poisson pour la variété riemannienne M.

Théorème IV. *On suppose que l'ensemble $\mathscr{L}$ des périodes est une partie discrète de $\mathbb{R}$ et que pour tout $l\in\mathscr{L}$ l'hypothése (H_l) est satisfaite. Alors on a la formule de Poisson suivante*

$$\sum_{k\geqq 0}\exp(\pm i\sqrt{\lambda_k}\,t)=\sum_{l\in\{0\}\cup\mathscr{L}}T_l\qquad(\text{au sens de }\mathscr{D}'(\mathbb{R}))\tag{$*$}'$$

où la somme de droite est localement finie. Pour chaque l, la distribution T_l est à support compact et s'écrit

$$T_l=\sum_{j\in J_l}e^{i\frac{\pi}{4}\sigma_j}\,T_{l,j}$$

et la translaté de $-l$ de $T_{l,j}$ définit un germe $T_{l,j}(t+l)$ dans l'espace H_{ν_j}; où ν_j et σ_j sont la dimension et la signature de la variété critique $W_{l,j}$.

Démonstration. D'après le théorème I et l'hypothèse que $\mathscr{L}$ est discret on peut, au moyen d'une partition de l'unité, décomposer S sous la forme

$$S=\sum_{l\in\{0\}\cup\mathscr{L}} T_l$$

où les T_l sont des distributions à support compact et formant une famille localement finie et qui sont telles que:

$$\text{Supp. Sing. } T_l\subset\{l\} \quad \text{et} \quad S-T_l \text{ est un germe } C^\infty \text{ en } l.$$

On applique ensuite les théorèmes II et III pour avoir la structure du germe de singularité en l défini par T_l.

Terminons par quelques remarques.

Remarque 6.2. Dans le cas particulier où pour tout l la variété critique W_l^+ est réduite à une seule composante connexe, la remarque 6.1 permet d'en déduire

$$\text{Supp. Sing. } S=\{0\}\cup\mathscr{L}. \tag{6.4}$$

Alors, dans ce cas (générique, d'après [6]) le spectre du laplacien détermine les longueurs des géodésiques périodiques, on renvoie, pour d'autres applications de ce genre de relation entre le spectre de Δ et les géodésiques, au travail de Colin de Verdiere [6].

Remarque 6.3. Si on suppose que $\inf_{l\neq l'\in\mathscr{L}}|l-l'|>0$, on peut faire en sorte que la série du 2ème membre de $(*)'$ converge dans $\mathscr{S}'(\mathbb{R})$. Ce qui permet d'appliquer les deux membres de $(*)'$ à la fonction test

$$\rho_z(t)=\sqrt{z/2\pi}\,\exp(-z t^2/4)$$

quand $\operatorname{Re} z>0$. Il vient

$$\langle e^{\pm i\sqrt{\lambda_k}\,t}, \rho_z(t)\rangle=e^{-\lambda_k/z}$$

et d'autre part, le développement asymptotique de $\hat{T}_l(\tau)$ permet d'écrire

$$\langle T_l, \rho_z(t)\rangle=f_l(z)\, e^{-z l^2/4}.$$

Ce qui démontre l'égalité $(**)'$; avec une convergence en tout point z tel que $\operatorname{Re} z>0$.

Remarque 6.4. Soit P, un opérateur elliptique de degré $2m$ sur une variété compacte connexe M, qui définit un opérateur auto-adjoint positif sur l'espace L^2 des densités d'ordre $\frac{1}{2}$. Alors, les théorèmes précédents se généralisent aisément à cette situation, en remplaçant $\sqrt{\lambda_k}$ par $(\lambda_k)^{1/2m}$ dans S. Pour cela, on utilise l'opérateur des ondes défini par

$$\partial^2/\partial t^2-(P)^{1/m}$$

en s'inspirant du procédé de réduction introduit par Hörmander dans [9].

Remarque 6.5. Il semble possible de faire une étude analogue dans le cas plus complexe d'une varieté à *bord* car on dispose d'une description de la paramétrix du problème mixte hyperbolique pour l'opérateur des ondes (cf. Chazarain [2]), ceci sera détaillé ailleurs.

Note. Hans Duistermaat vient de m'indiquer qu'il a obtenu indépendamment et simultanément des résultats trés voisins (The Spectrum and Periodic Geodesics, lecture on the A.M.S. Summer Institute on Differential Geometry, Stanford, August 1973).

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Cobordism of Manifolds with Odd Order Normal Bundle

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An oriented differentiable manifold M^k admits an *odd framing* if the composition $M^k \xrightarrow{\nu} \mathrm{BSO} \xrightarrow{\ell} \mathrm{BSO}_{(2)}$ is null homotopic, where ν classifies the stable normal bundle and ℓ is localization at 2. An odd framing of M is then a null homotopy of this composition. Under a suitable relation these give rise to the odd framed cobordism groups $\Omega_k^{\mathrm{fr}(2)}$. In analogy with framed cobordism, there is a Whitehead homomorphism

$$J': \pi_k(SO_{(2)}) \to \Omega_k^{\mathrm{fr}(2)}$$

where $SO_{(2)}$ denotes the localization of SO at 2.

In this paper we will compute the cobordism groups in terms of the 2-primary part of stable homotopy and determine the image of J'. The motivation for this study lies in the fact [2] that all Z_2-homology spheres admit odd framings. In [2] the structure of $\Omega_k^{\mathrm{fr}(2)}$ modulo the image of J' is employed to analyze the groups of Z_2-homology spheres and give applications to involutions.

Section 1 gives some lemmas concerning localizations. The second and third sections introduce the cobordism theory and the homomorphism J' and establish the main result:

Theorem. *$\Omega_k^{\mathrm{fr}(2)} \approx \Omega_{k_{(2)}}^{\mathrm{fr}} \oplus B_k$, where $B_k = 0$ for $k \not\equiv 3$ mod 4,*

$$B_{4n-1} = \bigoplus_{\pi(n)} Z_{(2)}/Z,$$

$\pi(n)$ is the number of partitions of n, and $Z_{(2)}$ is the integers localized at 2.

In the final section we show that the cobordism group is additively generated by odd-framed lens spaces and spheres together with framed manifolds.

1. Localizations

Let P denote any collection of primes. If A is an abelian group, we may form the P-localization $A_{(P)} = A \otimes Z_{(P)}$, where $Z_{(P)}$ is the set of rational numbers having denominator prime to each member of P. There is a natural localization homomorphism $\ell': A \to A_{(P)}$. The group A is said to be *P-local* if ℓ' is an isomorphism. Following Sullivan [7] there is an analogous geometric construction by which a simply connected space X may be localized at P to form $X_{(P)}$. Similarly there is a natural map $\ell: X \to X_{(P)}$ which localizes homotopy and homology groups.

1.1. **Lemma** [7]. *If X and Y are simply connected spaces and $f: X \to Y$ is a map that localizes the reduced homology of X, then Y is homotopy equivalent to $X_{(P)}$.* □

1.2. **Lemma.** *Suppose X is simply connected and let F be the fibre of ℓ made into a fibration*

$$F \xrightarrow{i} X \xrightarrow{\ell} X_{(P)}.$$

If F is also simply connected, there is an isomorphism $H_*(X, F) \approx \tilde{H}_*(X_{(P)})$ induced by the map $\ell: (X, F) \to (X_{(P)}, *)$.

Proof. Using the pair $(X_{(P)}, *)$ in the base, consider the relative Serre spectral sequence for ℓ

$$\{E^*_{p,q}\} \Rightarrow H_{p+q}(X, F),$$

where

$$E^2_{p,q} \approx H_p(X_{(P)}, *; H_q(F)).$$

From the exact homotopy sequence of the fibration, we have a short exact sequence

$$0 \to B \to \pi_q(F) \to A \to 0$$

where A is the torsion prime to P in $\pi_q(X)$ and B is the cokernel of $\ell_*: \pi_{q+1}(X) \to \pi_{q+1}(X_{(P)})$, which is isomorphic to $\pi_{q+1}(X) \otimes Z_{(P)}/Z$. Thus $\pi_*(F)$ is all torsion prime to P.

The Serre theorem implies that $\tilde{H}_*(F)$ is also torsion prime to P, hence $E^2_{p,q} = 0$ for $q > 0$. Therefore the spectral sequence lives entirely along the p-axis and the edge homomorphism gives the desired isomorphism. □

1.3. **Corollary.** *If F and X are as above, then $X_{(P)}$ is homotopy equivalent to the cofibre of the inclusion $i: F \to X$.* □

In the remainder of this paper we will be concerned with the case $P = \{2\}$.

2. Odd Framed Manifolds

If M^n is an oriented differentiable manifold, an *odd framing* of M is a null homotopy of the composition

$$M \xrightarrow{\nu} BSO(k) \xrightarrow{\ell} BSO_{(2)}(k)$$

where ν classifies the stable normal bundle of M and ℓ is localization at 2. Denoting by $B_k = SO_{(2)}(k)/SO(k)$ the fibre of ℓ, we have that odd framings of M correspond to homotopy classes of liftings of ν to B_k. Let γ_k be the universal bundle over $BSO(k)$ and denote by ξ_k the bundle over B_k induced by the map from B_k to $BSO(k)$. Following Lashof [4] we use the Thom spaces $T(\xi_k)$ to define a spectrum **T** whose homotopy is isomorphic via the Pontrjagin-Thom construction to the bordism group of odd framed manifolds. In this section we will analyze the homology and homotopy of this spectrum.

Note that by restricting to the fibre over the base point, there is a standard copy of S^k in $T(\xi_k)$. In this way the sphere spectrum **S** is contained in **T**. We begin by studying the quotient spectrum $\mathbf{T}/\mathbf{S} = \{T(\xi_k)/S^k\}$.

For each k there is a cofibration

$$T(\xi_k)/S^k \xrightarrow{\alpha} MSO(k)/S^k \xrightarrow{\beta} MSO(k)/T(\xi_k).$$

The corresponding exact homology sequence is related via the Thom isomorphism to the exact sequence of the triple $(BSO(k), B_k, *)$:

$$\begin{array}{ccccccccc}
\longrightarrow & \tilde{H}_*(T(\xi_k)/S^k) & \xrightarrow{\alpha_*} & \tilde{H}_*(MSO(k)/S^k) & \xrightarrow{\beta_*} & H_*(MSO(k)/T(\xi_k)) & \longrightarrow \\
& \uparrow \approx & & \uparrow \approx & & \uparrow \approx & \\
\longrightarrow & H_*(B_k, *) & \xrightarrow{i_*} & H_*(BSO(k), *) & \xrightarrow{j_*} & H_*(BSO(k), B_k) & \longrightarrow
\end{array}$$

By (1.2) j_* must be localization at 2, hence β_* is localization at 2.

Now consider the diagram

$$\begin{array}{ccccc}
F_k & \longrightarrow & MSO(k)/S^k & \xrightarrow{\ell} & (MSO(k)/S^k)_{(2)} \\
\uparrow g & & \| & & \uparrow h \\
T(\xi_k)/S^k & \xrightarrow{\alpha} & MSO(k)/S^k & \xrightarrow{\beta} & MSO(k)/T(\xi_k)
\end{array}$$

where ℓ is localization at 2 and F_k is the fibre of ℓ. Since β_* localizes the reduced homology, it follows from (1.1) that there is a homotopy equivalence h with $\ell \simeq h \circ \beta$. Thus the composition $\ell \circ \alpha$ is null homotopic and there exists a map g making the first square commute.

By using (1.3) we see that both rows produce long exact homology sequences and we apply the 5-lemma to conclude that g induces an isomorphism of reduced homology groups.

Therefore

$$g: T(\xi_k)/S^k \to F_k$$

is a homotopy equivalence.

Recall that the spectrum $\{MSO(k)/S^k\}$ defines a bordism group $\Omega_*^{SO,\mathrm{fr}}$ represented geometrically by compact oriented manifolds with framed boundaries [6].

2.1. **Lemma** [6]. *For $n>1$ there is a short exact sequence*

$$0 \to \Omega_n^{SO} \to \Omega_n^{SO,\mathrm{fr}} \to \Omega_{n-1}^{\mathrm{fr}} \to 0.$$

Proof. This follows immediately from the long exact sequence relating these groups and the fact that the characteristic numbers of a framed manifold are zero. □

The exact homotopy sequence of the fibration ℓ shows that $\pi_n(\mathbf{T}/\mathbf{S})$ splits as a direct sum of the cokernel of ℓ_*: $\Omega_{n+1}^{SO,\mathrm{fr}} \to (\Omega_{n+1}^{SO,\mathrm{fr}})_{(2)}$ and the kernel of $\ell_*: \Omega_n^{SO,\mathrm{fr}} \to (\Omega_n^{SO,\mathrm{fr}})_{(2)}$. For $n=4k-1$ the cokernel is isomorphic to $\bigoplus_{\pi(k)} Z_{(2)}/Z$ where $\pi(k)$ is the number of partitions of k. For $n \not\equiv 3 \bmod 4$ the cokernel is zero. The kernel is just the odd torsion in $\Omega_n^{SO,\mathrm{fr}}$, which for $n \not\equiv 0 \bmod 4$ is isomorphic to the odd torsion in $\Omega_{n-1}^{\mathrm{fr}} = \pi_{n-1}^S$.

For $n \equiv 0 \bmod 4$ we must determine the subgroup of the odd torsion in $\Omega_{n-1}^{\mathrm{fr}}$ which is the image of torsion in $\Omega_n^{SO,\mathrm{fr}}$. To do this we recall two theorems of Stong.

2.2. **Theorem** [6]. *A necessary and sufficient condition that an oriented manifold with framed boundary have the same Pontrjagin numbers as a closed oriented manifold is that the L genus be integral.* □

Now let M^{n-1} be a framed manifold. Pick a compact oriented manifold W^n with $\partial W^n = M^{n-1}$, and take the L genus of W reduced in $\mathbb{Q}/\mathbb{Z}$. By (2.2) this defines a homomorphism

$$L'': \Omega_{n-1}^{\mathrm{fr}} \to \mathbb{Q}/\mathbb{Z}.$$

2.3. **Theorem** [6, p. 215]. *The homomorphism L'' coincides with the odd primary part of the Adams invariant e_c* [1]. □

2.4. **Corollary.** *For $n \equiv 0 \bmod 4$ there is an exact sequence*

$$\operatorname{Tors} \Omega_n^{SO,\mathrm{fr}} \to \Omega_{n-1}^{\mathrm{fr}} \xrightarrow{L''} \mathbb{Q}/\mathbb{Z}.$$

Proof. Since the L genus of any torsion element is zero the composition must be zero. Suppose M^{n-1} is in the kernel of L''. Let W^n be an oriented manifold with $\partial W^n = M^{n-1}$. The L genus of W is integral so by (2.2) there is a closed oriented manifold $\overline{W}^n$ having the same Pontrjagin numbers as W^n. The element $(W^n - \overline{W}^n) \in \Omega_n^{SO,\mathrm{fr}}$ has all Pontrjagin numbers zero, hence must be a torsion element, and its image in $\Omega_{n-1}^{\mathrm{fr}}$ is M^{n-1}. □

For each integer n let $K_{n-1} \subseteq \Omega_{n-1}^{\mathrm{fr}}$ be the odd part of the kernel of e_c. If $J: \pi_{n-1}(SO) \to \Omega_{n-1}^{\mathrm{fr}}$ is the classical J-homomorphism, then $e_c: \Omega_{n-1(\mathrm{od})}^{\mathrm{fr}} \to \mathbb{Q}/\mathbb{Z}$ splits out the odd part of the image of J [1]. Thus $\Omega_{n-1(\mathrm{od})}^{\mathrm{fr}} = \operatorname{im} J_{(\mathrm{od})} \oplus K_{n-1}$. Note that when $n \equiv 1$, 2, or 3 mod 4, the odd part of the image of J is zero in $\Omega_{n-1(\mathrm{od})}^{\mathrm{fr}}$. Therefore for these cases $K_{n-1} = \Omega_{n-1(\mathrm{od})}^{\mathrm{fr}}$. We now can summarize the results of the previous discussion.

2.5. **Theorem.** *The homotopy groups of the spectrum* $\{T(\xi_k)/S^k\}$ *are given by*

$$\pi_n(\mathbf{T}/\mathbf{S}) \approx \begin{cases} (\bigoplus_{\pi(k)} G) \oplus K_{n-1} & \text{if } n=4k-1 \\ K_{n-1} = \Omega^{\mathrm{fr}}_{n-1(\mathrm{od})} & \text{if } n \not\equiv 3 \bmod 4, \end{cases}$$

where $G = Z_{(2)}/Z$ *and* $\pi(k)$ *is the number of partitions of the integer* k. □

To pass from this information to the homotopy of $\mathbf{T}$ consider the exact homotopy sequence

$$\cdots \to \pi_{n+1}(\mathbf{T}/\mathbf{S}) \to \pi_n(\mathbf{S}) \to \pi_n(\mathbf{T}) \to \pi_n(\mathbf{T}/\mathbf{S}) \to \cdots.$$

Since the subgroup K_n of $\pi_{n+1}(\mathbf{T}/\mathbf{S})$ arises from the kernel of localization it must inject into $\pi_n(S) = \Omega^{\mathrm{fr}}_n$. Therefore for $n \not\equiv 3 \bmod 4$ there is a split exact sequence

$$0 \to K_n \to \pi_n(\mathbf{S}) \to \pi_n(\mathbf{T}) \to 0$$

and since $K_n = \pi_n(\mathbf{S})_{(\mathrm{od})}$ we have

$$\pi_n(\mathbf{T}) \approx \pi_n(\mathbf{S})_{(2)}.$$

If $n = 4k-1$ it is evident that each copy of G in $\pi_{4k-1}(\mathbf{T}/\mathbf{S})$ must go to zero in $\pi_{4k-2}(\mathbf{S})$. This completes the proof of the following:

2.6. **Theorem.** *The bordism groups of odd framed manifolds* $\Omega^{\mathrm{fr}(2)}_n$, *defined as the homotopy groups of the spectrum* $\mathbf{T}$, *are isomorphic to the 2-primary part of stable homotopy if* $n \not\equiv 3 \bmod 4$. *For* $n = 4k-1$ *there is an exact sequence*

$$0 \to K_{4k-1} \to \pi^S_{4k-1} \to \Omega^{\mathrm{fr}(2)}_{4k-1} \to \bigoplus_{\pi(k)} G \to 0. \quad \square$$

The homology groups of the spectrum $\mathbf{T}$ may be computed directly. Let k be large and consider the group $H_{n+k}(T(\xi_k))$. By the Thom isomorphism this is isomorphic to $H_n(SO_{(2)}(k)/SO(k))$. By (1.5) this group appears in the exact cofibration sequence

$$\cdots \to H_n(SO_{(2)}(k)/SO(k)) \to H_n(BSO(k)) \xrightarrow{\ell_*} H_n(BSO_{(2)}(k)) \to \cdots$$

where ℓ_* is localization at 2. Since all of the torsion in $H_*(BSO(k))$ is of order 2 we conclude the following:

2.7. **Theorem.** *The reduced homology of the spectrum* $\mathbf{T}$ *is given by*

$$\tilde{H}_n(\mathbf{T}) \approx \begin{cases} \bigoplus_{\pi(k)} G & \text{if } n=4k-1 \\ 0 & \text{otherwise} \end{cases}$$

where $G = Z_{(2)}/Z$ *and* $\pi(k)$ *is the number of partitions of* k. □

3. The Homomorphism J'

In this section we study a homomorphism $J'\colon \pi_n(SO_{(2)}) \to \Omega_n^{\mathrm{fr}(2)}$ analogous to the Whitehead homomorphism $J\colon \pi_n(SO) \to \Omega_n^{\mathrm{fr}}$. A much more general treatment of J-homomorphisms in cobordism theory has been given by Harris [3].

Given spaces A, B and C and maps $f\colon A \to B$, $g\colon A \to C$, the *push out* of (f, g) is defined to be the space $D = M_f \cup_A M_g$, the union of the mapping cylinders of f and g identified along the copy of A contained in each. There is a natural way to define the unlabeled maps so that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

If $f'\colon A' \to B'$, $g'\colon A' \to C'$ is another pair of maps and $\alpha\colon A \to A'$, $\beta\colon B \to B'$, $\gamma\colon C \to C'$ have the property that $\beta \circ f = f' \circ \alpha$ and $\gamma \circ g = g' \circ \alpha$ then there is a natural map induced from the push out of (f, g) to the push out of (f', g').

Let $\varphi\colon S^n \to SO(k)_{(2)}$, k large, represent an element of $\pi_n(SO(k)_{(2)})$. Denote by β the composition

$$S^n \to SO(k)_{(2)} \to SO(k)_{(2)}/SO(k).$$

Since β factors through $SO(k)_{(2)}$ we have that $\beta^*(\xi_k)$ is trivial and its trivialization is uniquely determined up to fibre homotopy equivalence. Denoting by $S(\xi_k)$ the sphere bundle of ξ_k, we may identify the sphere bundle of $\beta^*(\xi_k)$ with $S^n \times S^{k-1}$ and find a map α so that the following diagram is homotopy commutative:

$$\begin{array}{ccc} S^n \times S^{k-1} & \xrightarrow{\alpha} & S(\xi_k) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1'} \\ S^n & \xrightarrow{\beta} & SO(k)_{(2)}/SO(k) \end{array}$$

There is also an obviously commutative diagram

$$\begin{array}{ccc} S^n \times S^{k-1} & \xrightarrow{\alpha} & S(\xi_k) \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2'} \\ S^{k-1} & \xrightarrow{\beta} & * \end{array}$$

where $*$ is a point and π_2 is projection on the second factor. Now the push out of (π_1, π_2) is $S^n \circ S^{k-1} = S^{n+k}$ and the push out of (π_1', π_2') is the Thom space $T(\xi_k)$. The map of push outs $S^{n+k} \to T(\xi_k)$ represents the element $J'(\varphi) \in \Omega_n^{\mathrm{fr}(2)}$.

It is not difficult to see that this is related to the classical J-homomorphism by the commutative diagram

$$\begin{array}{ccc} \pi_n(SO) & \xrightarrow{J} & \Omega_n^{\mathrm{fr}} \\ \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle h} \\ \pi_n(SO_{(2)}) & \xrightarrow{J'} & \Omega_n^{\mathrm{fr}(2)} \end{array}$$

From (2.6) it follows that for $n \equiv 0, 1$ or $2 \bmod 4$ the behavior of J' is identical to that of J. Thus we concentrate our attention on the case

$$J'\colon \pi_{4k-1}(SO_{(2)}) \approx Z_{(2)} \to \Omega_{4k-1}^{\mathrm{fr}(2)}.$$

As before let $G = Z_{(2)}/Z$ and define $\hat{Z}_{\mathrm{od}} = \mathrm{Hom}(G, G) \approx \prod_p \hat{Z}_p$, where the product is taken over all odd primes p. Since $\hat{Z}_{\mathrm{od}}$ is a ring containing $\frac{1}{2}$ it is a theorem of Milnor that $H^*(BSO; \hat{Z}_{\mathrm{od}}) \approx \hat{Z}_{\mathrm{od}}[p_1, p_2, \ldots]$ where the p_i are the Pontrjagin classes. Thus $H^{4n}(BSO; \hat{Z}_{\mathrm{od}})$ is a free $\hat{Z}_{\mathrm{od}}$-module with basis the monomials of degree n in the Pontrjagin classes.

3.1. **Lemma.** *The following is a sequence of isomorphisms of* $\hat{Z}_{\mathrm{od}}$ *modules for each* $k > 0$

$$H^{4k}(BSO; \hat{Z}_{\mathrm{od}}) \overset{i^*}{\approx} H^{4k}(SO_{(2)}/SO; \hat{Z}_{\mathrm{od}}) \overset{\beta^{-1}}{\approx} H^{4k-1}(SO_{(2)}/SO; G)$$
$$\approx \mathrm{Hom}\big(H_{4k-1}(SO_{(2)}/SO), G\big).$$

Proof. The exact cohomology sequence of (1.5) together with the fact that $\tilde{H}^*(BSO_{(2)}; \hat{Z}_{\mathrm{od}}) = 0$ imply the first isomorphism.

By tensoring the short exact sequence

$$0 \to Z \to Z_{(2)} \to G \to 0$$

with $\hat{Z}_{\mathrm{od}}$ we produce an exact sequence of $\hat{Z}_{\mathrm{od}}$ modules

$$0 \to \hat{Z}_{\mathrm{od}} \to \hat{Z}_{\mathrm{od}} \otimes Z_{(2)} \to G \to 0.$$

Now $\hat{Z}_{\mathrm{od}} \otimes Z_{(2)}$ is a rational vector space, thus $\tilde{H}^*(SO_{(2)}/SO; \hat{Z}_{\mathrm{od}} \otimes Z_{(2)}) = 0$.

This gives the second isomorphism as the inverse of the corresponding Bockstein operator.

The third isomorphism is a result of the universal coefficient theorem since G is injective. □

3.2. **Lemma.** *If* $A_k \subseteq H^{4k-1}(SO_{(2)}/SO; G)$ *is the subgroup generated by the Bockstein inverses of the monomials in the Pontrjagin classes, then the Kronecker index gives an isomorphism*

$$H_{4k-1}(SO_{(2)}/SO) \to \mathrm{Hom}(A_k, G).$$

Proof. From previous results we know that the Kronecker index defines a monomorphism

$$\theta\colon H_{4k-1}(SO_{(2)}/SO) \to \mathrm{Hom}(A_k, G).$$

These two groups are both isomorphic to a direct sum of the same number of copies of G, hence each group contains the same number of elements of order q for each odd integer q. This implies that θ is an isomorphism. □

The isomorphism of (3.2) gives us characteristic numbers for odd framed $(4k-1)$-manifolds. A null homotopy of the composition

$$M^{4k-1} \xrightarrow{\nu} BSO \xrightarrow{\ell} BSO_{(2)}$$

determines a lift $f\colon M \to SO_{(2)}/SO$ of ν. Evaluating $f_*[M]$ on the generators of A_k defines a collection of "numbers" in G, one for each monomial of degree k in the Pontrjagin classes. The decomposable numbers may be computed in the cohomology of M. Let $(i_1, \ldots, i_\ell)$ be a non-trivial partition of k, and $\alpha = \beta^{-1}(i^*(p_{i_1})) \in H^{4i_1-1}(SO_{(2)}/SO; G)$ (see 3.1). Then define an element of G by the Kronecker index

$$\langle f^*(\alpha) \cdot \bar{p}_{i_2} \cdots \bar{p}_{i_\ell}, [M] \rangle$$

where the $\bar{p}_j$ are Pontrjagin classes of M. Note that if M has the Z_2-homology of a sphere, the decomposable numbers are independent of the lifting f.

3.3. **Lemma.** *The image of the Hurewicz homomorphism*

$$H\colon \pi_{4k-1}(SO_{(2)}/SO) \to H_{4k-1}(SO_{(2)}/SO).$$

is the dual of $\beta^{-1}(i^*(p_k))$ *under the isomorphism of* (3.2).

Proof. It is clear that the image of H is contained in the dual of $\beta^{-1}(i^*(p_k))$ since all Pontrjagin classes of the sphere vanish. So it will be sufficient to show that H has finite kernel.

Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & \pi_{4k}(BSO) & \to & \pi_{4k}(BSO_{(2)}) & \to & \pi_{4k-1}(SO_{(2)}/SO) & \to 0 \\
& \downarrow{\scriptstyle H'} & & \downarrow{\scriptstyle H''} & & \downarrow{\scriptstyle H} & \\
0 \to & H_{4k}(BSO) & \to & H_{4k}(BSO_{(2)}) & \to & H_{4k-1}(SO_{(2)}/SO) & \to 0.
\end{array}$$

H'' is a monomorphism and the cokernel of H' maps to the cokernel of H'' with finite kernel. Thus by the snake lemma, the kernel of H is finite. □

3.4. **Lemma.** *The following diagram is commutative*

$$\begin{array}{ccccc} \pi_{4n-1}(SO_{(2)}) & \xrightarrow{J'} & \Omega^{\mathrm{fr}(2)}_{4n-1} & \xrightarrow{H} & H_{4n-1}(\mathbf{T}) \\ \downarrow & & & & \downarrow{\scriptstyle \Phi} \\ \pi_{4n-1}(SO_{(2)}/SO) & & \xrightarrow{\quad H \quad} & & H_{4n-1}(SO_{(2)}/SO) \end{array}$$

where Φ is the Thom isomorphism.

Proof. This is a consequence of the fact that the image of the fundamental class of $S^n \circ S^{k-1} = S^{n+k}$ in $T(\xi_k)$ corresponds via the Thom isomorphism with the image of the fundamental class of S^n in $SO(k)_{(2)}/SO(k)$. Similarly we may view an element of the image of J' as a sphere with a given odd framing. Then both compositions yield the image of the fundamental class of this sphere. □

3.5. **Theorem.** *The composition*

$$\pi_{4n-1}(SO_{(2)}) \xrightarrow{J'} \Omega^{\mathrm{fr}(2)}_{4n-1} \xrightarrow{H} H_{4n-1}(\mathbf{T})$$

takes $Z_{(2)}$ onto that copy of $G = Z_{(2)}/Z$ corresponding to $\beta^{-1}(i^*(p_n))$.

Proof. This follows from (3.3) and (3.4) and the fact that $\pi_{4n-1}(SO_{(2)}) \to \pi_{4n-1}(SO_{(2)}/SO)$ is onto. □

Note that the image of J' in $\Omega^{\mathrm{fr}(2)}_{4n-1}$ is divisible by all odd integers, so it must be a direct sum of the corresponding copy of G and a finite 2-torsion group. Thus from the commutative diagram

$$\begin{array}{ccccccccc} & & & & \pi_{4n-1}(SO) & \xrightarrow{\ell} & \pi_{4n-1}(SO_{(2)}) & & \\ & & & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J'} & & \\ 0 \to & K_{4n-1} & \to & & \pi_{4n-1}(\mathbf{S}) & \longrightarrow & \pi_{4n-1}(\mathbf{T}) & \to \bigoplus_{\pi(n)} G & \to 0 \end{array}$$

the odd part of the image of J must all be mapped into this copy of G. This completely determines the structure of $\pi_{4n-1}(\mathbf{T})$ in terms of stable homotopy.

3.6. **Theorem.** $\Omega^{\mathrm{fr}(2)}_{4n-1} \approx \left(\bigoplus_{\pi(n)} G\right) \oplus \pi^S_{4n-1_{(2)}}$. □

Together with (2.6) this gives the complete determination of the cobordism groups of odd framed manifolds. It will be important in our applications [2] to know the quotient of $\Omega^{\mathrm{fr}(2)}_k$ modulo the image of J'.

3.7. **Theorem.** *If* $k \not\equiv 3 \bmod 4$, *then* $\Omega_k^{\mathrm{fr}(2)}/\mathrm{im}\, J' \approx \Omega_{k(2)}^{\mathrm{fr}}/\mathrm{im}\, J_{(2)}$. *For each* $n>0$ $\Omega_{4n-1}^{\mathrm{fr}(2)}/\mathrm{im}\, J' \approx \left(\bigoplus_{\pi(n)-1} G\right) \oplus (\Omega_{4n-1(2)}^{\mathrm{fr}}/\mathrm{im}\, J_{(2)})$. □

4. Lens Spaces

In this section we will show that the homology of the spectrum **T** may be generated by odd framings of lens spaces and spheres. It is a consequence of (3.3) that all indecomposable characteristics numbers arise from odd framings of spheres. We will show that all decomposable numbers arise from appropriate connected sums of lens spaces.

Recall from § 3 that for each $k>0$ there are isomorphisms

$$H^{4k-1}(SO_{(2)}/SO;\, G) \xrightarrow[\beta]{\approx} H^{4k}(SO_{(2)}/SO;\, \hat{Z}_{\mathrm{od}}) \xleftarrow[i^*]{\approx} H^{4k}(BSO;\, \hat{Z}_{\mathrm{od}})$$

and we denote by $A_k \subseteq H^{4k-1}(SO_{(2)}/SO;\, G)$ the subgroup generated by $\{\beta^{-1}(i^*(\gamma)) \mid \gamma$ is a monomial in the Pontrjagin classes$\}$. If $f\colon M^{4k-1} \to SO_{(2)}/SO$ arises from an odd framing of M, then by evaluating $f_*[M]$ on the generators of A_k we arrive at the characteristic numbers (in G) of M, one for each partition of k.

Now let p be an odd prime. If $q_1, \ldots, q_{2k}$ are integers relatively prime to p, there is a free action of $Z_p r$ on S^{4k-1} generated by α where

$$\alpha(z_1, \ldots, z_{2k}) = (\lambda_1 z_1, \ldots, \lambda_{2k} z_{2k})$$

and $\lambda_j = \exp(q_j \cdot 2\pi i/p^r)$. The quotient space under this action is the lens space $L^{4k-1}(p^r; q_1, \ldots, q_{2k})$. It is a smooth orientable Z_2-sphere.

If x is the generator of $H^2(L^{4k-1}(p^r; q_1, \ldots, q_{2k})) \approx Z_p r$ then the total Pontrjagin class of L is given by [5]

$$p(L) = (1+q_1^2 x^2)(1+q_2^2 x^2) \ldots (1+q_{2k}^2 x^2).$$

Thus the i-th Pontrjagin class $p_i(L) = v_i(q_1^2, \ldots, q_{2k}^2)\, x^{2i}$ where v_i is the i-th elementary symmetric function of $2k$ variables and the coefficient is read modulo p^r.

It is not difficult to show that the decomposable characteristic numbers of L may be computed directly in the following manner. The Bockstein operator for the sequence $0 \to Z \to Z_{(2)} \to G \to 0$ defines for each i an isomorphism

$$\beta\colon\, H^{4i-1}(L;\, G) \xrightarrow{\approx} H^{4i}(L;\, Z).$$

If $(i_1, \ldots, i_\ell)$ is a non-trivial partition of k, then the corresponding number for L in G may be computed as the Kronecker index

$$\langle \beta^{-1}(p_{i_1}) \cdot p_{i_2} \cdots p_{i_\ell}, [L] \rangle.$$

Thus the number may be determined by taking the appropriate monomial in the elementary symmetric functions v_i, evaluating at $(q_1^2, \ldots, q_{2k}^2)$ and reducing mod p^r to give an element of $Z_{p^r} \subseteq Z_{p^\infty} \subseteq G$.

For each non-trivial partition $\pi = (i_1, \ldots, i_\ell)$ of k, let $\xi_\pi = v_{i_1} \ldots v_{i_\ell}$ be the corresponding monomial in the elementary symmetric functions. Let $X = \{4^i \mid i \geqq 0\}$ and $Y = \underbrace{X \times \cdots \times X}_{2k \text{ fold}}$. Then each ξ_π may be viewed as a real valued function with domain Y since each v_i is a function of $2k$ variables.

4.1. **Lemma.** *For any non-trivial partition π_0 of k there exist elements $y_1, \ldots, y_m \in Y$ and rational numbers $r_1, \ldots, r_m$ such that*

$$\sum_i r_i \, \xi_\pi(y_i) = \begin{cases} 0 & \text{for } \pi \neq \pi_0 \\ 1 & \text{for } \pi = \pi_0. \end{cases}$$

Proof. As π ranges over the non-trivial partitions of k, the polynomials ξ_π give a linearly independent set of homogeneous polynomials of degree $2k$ in $2k$ variables. One can show by a geometric argument that any linearly independent set of polynomials of degree n in ℓ variables must be independent on any $(n+1)^\ell$ lattice.

So pick a finite set $Y' \subseteq Y$ on which the $\{\xi_\pi\}$ are linearly independent. Let V be the rational vector space with basis the elements of Y'. By extending linearly, the set $\{\xi_\pi\}$ is a linearly independent set in V^*. There exist elements $\alpha_\pi \in V^{**}$ such that

$$\alpha_\pi(\xi_{\pi'}) = \begin{cases} 0 & \text{if } \pi \neq \pi' \\ 1 & \text{if } \pi = \pi'. \end{cases}$$

Under the isomorphism $V^{**} \to V$ the image of α_{π_0} is $\sum_{i=1}^{m} r_i y_i$, and these satisfy the conclusion of the lemma. □

Multiplying by a large integer we have for each π_0 an integral linear combination $\sum n_i y_i$ with

$$\sum n_i \cdot \xi_\pi(y_i) = \begin{cases} 0 & \text{if } \pi \neq \pi_0 \\ \neq 0 & \text{if } \pi = \pi_0. \end{cases}$$

For sufficiently large r the integer $\sum_{i=1}^{m} n_i \, \xi_{\pi_0}(y_i)$ is non-zero modulo p^r. We view $\sum n_i y_i$ as a connected sum of Z_{p^r} lens spaces. Since the characteristic numbers are additive, we have the following:

4.2. **Theorem.** *For any non-trivial partition π_0 of k and any odd prime p there is a connected sum of Z_{p^r} lens spaces whose π_0 characteristic number is non-zero in $Z_{p^\infty} \subseteq G$ for which all other characteristic numbers are zero.*

It is evident that all multiples of this non-zero element of Z_{p^∞} may be realized. In order to show that the entire subgroup may be realized, we must give a procedure that divides by p.

Any lens space $L = L^{4k-1}(p^r; q_1, \ldots, q_{2k})$ in the connected sum of (4.2) is naturally the p-fold cover of the lens space $L' = L^{4k-1}(p^{r+1}; q_1, \ldots, q_{2k})$. It may be checked directly that p times the characteristic numbers of L' gives the corresponding characteristic numbers of L. Doing this for each lens space in (4.2) and taking the connected sum of the resulting lens spaces gives the desired division process.

We are now able to conclude the main results:

4.3. **Theorem.** *Let $\mathscr{L} \subseteq \Omega_*^{\mathrm{fr}(2)}$ be the subgroup generated by lens spaces together with the image of J'. Then the Hurewicz homomorphism*

$$\Omega_*^{\mathrm{fr}(2)} \to H_*(\mathbf{T})$$

maps the subgroup $\mathscr{L}$ onto $H_*(\mathbf{T})$. That is, any element of $H_*(\mathbf{T})$ may be represented by a connected sum of lens spaces and spheres with odd framings. □

4.4. **Corollary.** *$\mathscr{L}$ + image $\pi_*^S = \Omega_*^{\mathrm{fr}(2)}$. In other words, the cobordism group is generated by odd framed lens spaces and spheres together with all framed manifolds.*

Proof. From (2.6) $\Omega_*^{\mathrm{fr}(2)}/\mathrm{im}\, \pi_*^S$ is divisible and it maps onto $H_*(\mathbf{T})$ with finite kernel. The image of $\mathscr{L}$ in $\Omega_*^{\mathrm{fr}(2)}/\mathrm{im}\, \pi_*^S$, denoted $\tilde{\mathscr{L}}$, also maps onto $H_*(\mathbf{T})$ by (4.3). Thus $\tilde{\mathscr{L}}$ is a subgroup of finite index in a divisible group, hence $\tilde{\mathscr{L}} = \Omega_*^{\mathrm{fr}(2)}/\mathrm{im}\, \pi_*^S$. This implies $\Omega_*^{\mathrm{fr}(2)} = \mathrm{im}\, \pi_*^S + \mathscr{L}$. □

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Subvarieties of Moduli Spaces

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In this paper we try to decide along algebraic lines whether moduli spaces of abelian varieties or of algebraic curves contain complete subvarieties. In Theorem (1.1) we consider abelian varieties and curves of genus three in characteristic $p \neq 0$. In Section 2 we use monodromy in order to show certain families of abelian varieties up to a purely inseparable isogeny are isotrivial; probably the results (2.1) and (2.2) are special cases of more general facts about l-adic monodromy. In Section 4 we answer a question raised by Manin concerning a possible generalization of the fact that two supersingular elliptic curves over an algebraically closed field (of characteristic p) are isogenous. We abbreviate abelian variety(ies) by AV; we use X^t for the dual, and $\hat{X}$ for the formal group of an AV X.

1. Moduli Spaces in Characteristic p which Contain Complete Curves

All moduli spaces in this section considered will be moduli spaces in the sense of [21] with a field k as base ring.

Suppose $k = \mathbb{C}$, the field of complex numbers; the coarse moduli scheme A_g of principally polarized abelian varieties of dimension g contains a projective subscheme of dimension $g-1$; this easily follows from the existence and the properties of the Satake compactification of A_g as for example Shafarevich remarked (cf. [32], p. 111).

As Mumford pointed out to me, along the same lines it follows that if $k = \mathbb{C}$, and $g \geqq 3$, then the coarse moduli scheme M_g or irreducible, non-singular, complete algebraic curves of genus g contains a complete algebraic curve; this can be seen as follows: let M'_g be the coarse moduli scheme of good curves of genus g (stable curves of genus g whose Jacobian variety is an abelian variety), and

$$j\colon\ M'_g \to A_g$$

the Jacobi mapping; let $A_g \subset \bar{A}_g \subset \mathbb{P}^N$ be the Satake compactification, and denote by C the closure of $j(M_g)$ in $\bar{A}_g$; each component of $C \smallsetminus j(M_g)$ has codimension at least two in C: this is seen to be true for components of $j(M'_g) \smallsetminus j(M_g)$ by counting moduli, and for components of $C \smallsetminus j(M'_g)$ it follows from $\bar{A}_g \smallsetminus A_g = \bigcup_{h<g} A_h$; thus we can intersect $j(M_g) \subset \mathbb{P}^N$ with a

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convenient linear space of dimension $N-3g+4$ which does not meet $C\setminus j(M_g)$.

By a result of Deligne and Mumford we know a natural compactification D of M_g exists by the use of stable curves, cf. [2]; note that $D\setminus M'_g$ and $M'_g\setminus M_g$ both have at least one component of codimension one (consider irreducible curves of genus g having one node, or consider a component U coming from curves consisting of an elliptic curve and a smooth curve of genus $g-1>1$ connected by one normal crossing); thus it is not clear the method above can be applied to this or another compactification of M'_g (and note that j "contracts" U to a lower dimensional $j(U)\subset A_g$).

Theorem (1.1). *Let k be field,* $\operatorname{char}(k)=p\neq 0$.

a) *The coarse moduli scheme $A_{g,d}$ of abelian varieties of dimension g plus a polarization of degree d^2 has a projective subscheme of dimension at least $\frac{1}{2}g(g-1)$.*

b) *Suppose k is algebraically closed. The coarse moduli scheme M_3 of complete irreducible non-singular algebraic curves of genus 3 contains a complete algebraic curve.*

Definition (1.2). Let X be an AV over a field k, $\operatorname{char}(k)=p$; we say the p-rank of X equals f, notation: $\operatorname{pr}(X)=f$, if

$$|{}_pX(K)|=p^f,$$

where K is an algebraically closed field containing k, and

$${}_pX=\operatorname{Ker}(p\colon X\to X);$$

we say X is *ordinary* if $f=\dim X$, and we say X is *very special* if $f=0$.

Let Y be a scheme over the prime field $\mathbb{F}_p$; we denote by $F=F_Y$: $Y\to Y$ the morphism obtained by raising all sections of $\mathcal{O}_Y$ to the p-th power; the induced homomorphism

$$F^*=h\colon H^1(Y,\mathcal{O}_Y)\to H^1(Y,\mathcal{O}_Y)$$

is usually called the *Hasse-Witt transformation.*

Lemma (1.3). *Let X be an* AV *over an algebraically closed field of characteristic p of p-rank f; then the number of elements $v\in H^1(X,\mathcal{O}_X)$ such that*

$$hv=v \quad \textit{equals } p^f.$$

Proof. By duality of abelian varieties, the p-rank of X equals f if and only if the semi-simple rank of h equals f, which, by [10], p. 488, Satz 10, yields the result (also cf. [22], p. 143, Corollary).

Lemma (1.4). *Let S be an irreducible algebraic k-scheme, and $\mathsf{X} \to S$ an abelian scheme over S; let f be the p-rank of the generic fibre; for any field K, and for any $s \in S(K)$,*

$$\operatorname{pr}(\mathsf{X}_s) \leqq f.$$

Corollary (1.5). *Let $\mathsf{X} \to T$ be an abelian scheme over a locally noetherian k-scheme T, and n an integer; the set of points s of T with $\operatorname{pr}(\mathsf{X}_s) \leqq n$ is a closed set in T.*

Lemma (1.6). *Assumptions as in* (1.4); *let W be the closed subset of S over which the fibre has p-rank at most $f-1$ (closed because of 1.5); then either W is empty or each component of W has codimension one in S.*

Proof of (1.4), (1.5), (1.6). The sheaf $\mathscr{H} = R^1(\mathsf{X} \to S)(\mathcal{O}_X)$ is locally free over S, thus each point of S has a neighborhood $\operatorname{Spec}(R) \hookrightarrow S$ over which $\mathscr{H} \mid \operatorname{Spec}(R)$ is a free coherent sheaf; let $G = (\mathbb{G}_{a,R})^g$ be the affine space of dimension $g = \dim(\mathsf{X}/S)$ and note $\mathscr{H}$ can be identified with the sheaf of germs of sections in G; thus the Hasse-Witt transformation can be viewed as a group scheme homomorphism; its kernel $N := \operatorname{Ker}(h\colon G \to G)$ can be given by the equations $(h v - v)_i = 0$, and if $v = \sum x_i v_i$, then

$$(h v - v)_i = \sum_j x_j^p h_{ij} - x_i;$$

N is quasi-finite over $\operatorname{Spec}(R)$, and N is smooth over $\operatorname{Spec}(R)$ because the derivations $\partial/\partial x_j$ of the defining equations have value 1, if $i = j$, or 0, if $i \neq j$, which implies smoothness. At a geometric point s of $\operatorname{Spec}(R)$, the rank of the finite group scheme equals p^n if and only if the p-rank of X_s equals n, thus (1.4) is proved.

This implies (1.5).

For the proof of (1.6) we restrict to $\operatorname{Spec}(R) \subset S$ as above; suppose k perfect and S reduced, let K be the field of fractions of R, and choose $L \supset K$, a separable finite extension, so that $N \otimes_R L$ is a constant L-group scheme; let S' be the normalization of $\operatorname{Spec}(R)$ in L, and $N' = N \times_S S'$, $G' = G \times_S S'$; choose some coordinate system for G' over S', let $v_i^{(m)} \in L$ be all the coordinates of the non-zero points of $N \otimes_R L$, and let W' be the union of all divisors defined by (the poles of the) $v_i^{(m)}$ on S'; because S' is normal each component of W' has codimension one in S', and for $s' \in S'$ the p-rank of the fibre of $\mathsf{X}' = \mathsf{X} \times_S S'$ is smaller than f if and only if $s' \in W'$; thus W' has the required properties, and because each fibre of $S' \to \operatorname{Spec}(R)$ is non-empty and finite, Lemma (1.6) is proved.

Remark (1.7). We should like to make (1.6) more precise in the following sense. Consider the set of 2×2-matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(consider the entries as unknown over some field k of characteristic p, i.e. A can be considered as the generic point of four-dimensional affine space), and consider those matrices, for which

$$AA^{(p)} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix} = 0;$$

is the set of such points given by two equations? (and generalize: A is a $g \times g$-matrix, is the set of points $AA^{(p)} \dots A^{(p^{g-1})} = 0$ given by g-equations?). For example

$$c \neq 0, \quad a^{p+1} + b\,c^p = 0 = c\,a^p + d\,c^p$$

define a closed set in $(c \neq 0)$ for which $AA^{(p)} = 0$ (if $d = 0$ it easily follows, if $d \neq 0$, then $a\,d = b\,c$ follows etc.), but if $a = 0 = c$, and $d \neq 0$, then $AA^{(p)} \neq 0$, although the matrix satisfies the two equations; in particular we were not able to check [19], p. 79, lines 11 and 12 (probably the lower line should read $b_{2p-1}\, b_{p-1}^p + b_{2p-2}\, b_{2p-1}^p = 0$, a misprint both in the Russian version and the translation): consider $p = 5$, and the curve given by

$$Y^2 = X^5 + X^3 + 1;$$

according to [19], p. 79, its Hasse-Witt matrix equals

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$$

thus $AA^{(p)} \neq 0$, although the coefficients of the curve satisfy the equations which are supposed to define the formal type $2G_{1,1}$; however, basically the claims in the middle of p. 79, [19], are correct: at every point of the moduli space M_2 the closed set corresponding to curves with very special Jacobian variety *locally* is given by 2 equations.

Proof of Theorem (1.1 a). Let $W_{g,d}$ be the subset of points in $A_{g,d}$ corresponding to very special abelian varieties; as there exists a proper morphism from a fine moduli scheme to $A_{g,d}$, by (1.5) we conclude $W_{g,d}$ is closed in $A_{g,d}$. Clearly $W_{g,d}$ is non-empty ($W_{g,1}$ is non-empty, contains e.g. a point corresponding to C^g, where C is a supersingular elliptic curve: take an α_p-covering of an AV corresponding to a point in $W_{g,d}$, this gives a point in $W_{g,dp}$, etc.). As each component of $A_{g,d}$ has dimension at least $\frac{1}{2}g(g+1)$ (cf. [24], Theorem 2.3.3), by (1.6) it follows that each component of $W_{g,d}$ has at least dimension $\frac{1}{2}g(g+1) - g = \frac{1}{2}g(g-1)$; as $A_{g,d}$ is quasi-projective and $W_{g,d}$ is closed in $A_{g,d}$, it follows $W_{g,d}$ is quasi-projective. Let R be a discrete valuation ring, which is a k-algebra, with field of fractions K; we want to show

$$\mathrm{Mor}_k(\mathrm{Spec}(R), W_{g,d}) \xrightarrow{\sim} \mathrm{Mor}_k(\mathrm{Spec}(K), W_{g,d}); \qquad (*)$$

this map is injective; suppose given $w \in W_{g,d}(K)$; then there exists a finite extension $K' \supset K$ and a (very special) abelian variety X over $\mathrm{Spec}(K')$ plus a polarization which (over $\bar{K}$) defines the point w; by the stable reduction theorem (cf. [9], I.6; or cf. [25] for other references), there exists a finite extension $L \supset K'$, a discrete valuation ring $U \subset L$ and a smooth stable group scheme Y over $\mathrm{Spec}(U)$ with generic fibre $\mathsf{Y} \otimes L = X \otimes L$; consider the p-Lie algebra $\mathrm{Lie}(\mathsf{Y})$ of the group scheme $\mathsf{Y} \to \mathrm{Spec}(U)$ (cf. [3], II.7.2); the p-operation on $\mathrm{Lie}(\mathsf{Y})$ is nilpotent because $\mathrm{Lie}(\mathsf{Y}) \hookrightarrow \mathrm{Lie}(\mathsf{Y}) \otimes_U L = \mathrm{Lie}(X \otimes L)$, and because X is very special; thus the p-operation on the Lie-algebra $\mathrm{Lie}(Y_0) = \mathrm{Lie}(\mathsf{Y}) \otimes l$ (l is the residue class field of U) is nilpotent; as Y is stable over $\mathrm{Spec}(U)$ this implies Y is an abelian scheme over $\mathrm{Spec}(U)$. Thus the polarization on $X \otimes L$ can be extended to Y, and a commutative diagram results

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(R) \\ {\scriptstyle w}\Big\downarrow \quad \searrow & & \Big\downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(U) \\ \swarrow & & \Big\downarrow \\ W_{g,d} & \hookrightarrow & A_{g,d}; \end{array}$$

thus w can be extended to $w'\colon \mathrm{Spec}(R) \to A_{g,d}$ and as $W_{g,d}$ is closed in $A_{g,d}$ this morphism factors through $W_{g,d}$; thus we have proved (*) to be an isomorphism; by the valuative criterion for properness (cf. EGA, II.7.3.8), this implies $W_{g,d}$ is proper over $\mathrm{Spec}(k)$, and (1.1a) is proved.

Proof of Theorem (1.1b). Denote by A_g the coarse moduli scheme of principally polarized abelian varieties $A_g = A_{g,1}$, and by

$$I_g = j(M_g), \qquad R_g = j(M_g') \setminus j(M_g)$$

the subsets corresponding to Jacobians of irreducible curves, respectively corresponding to Jacobians of reducible curves. We know $I_g \cup R_g$ is closed in A_g; moreover $I_1 = A_1$, and $I_g \cup R_g = A_g$ for $g = 2, 3$ for dimension reasons (cf. [27]). Note that R_2 is irreducible and has dimension 2, because it is contained in the image of the natural morphism $A_1 \times A_1 \to A_2$, and that each component of R_3 has dimension at most 4, because R_3 is contained in the images of $A_1 \times A_1 \times A_1$ and $A_1 \times A_2$ in A_3, and $\dim(A_1) = 1$, $\dim(A_2) = 3$. Moreover W_1 is zero-dimensional (the points of W_1 correspond to supersingular elliptic curves, $W_1 \subsetneq A_1$), and each component of W_2 has dimension at most one: suppose $V \subset W_2$, V irreducible of dimension 2; V cannot be contained in R_2, because the generic point of R_2 corresponds to an ordinary AV; moreover I_2 is affine (this follows from the fact that a curve of genus 2 is hyperelliptic, or, cf. [13]); thus $V \setminus (V \cap I_r)$ would be a closed set of dimension one, but $W_2 \cap R_2$

has dimension zero, contradiction, hence each component of W_2 has dimension at most one. From this we deduce that each component of $W_3 \cap R_3$ has dimension at most one: it is contained in the image of $W_1 \times W_2 \to A_3$. But each component of W_3 is proper and has at least dimension $\frac{1}{2} \cdot 3(3-1) - 3 = 3$ (use (1.1a)); as A_3 is quasi-projective, and k algebraically closed we can intersect with a linear space of dimension $N-2$, $L \hookrightarrow \mathbb{P}_k^N$, $A_3 \hookrightarrow \mathbb{P}_k^N$ such that $L \cap W_3$ is a (complete!) algebraic curve, and $L \cap W_3 \cap R_3 = \emptyset$; then $\overset{-1}{j}(L \cap W_3) \subset M_3$ is a complete algebraic curve, and (1.1b) is proved.

Remarks (1.8). If we could prove each component of W_3 has dimension three, and if we could prove $W_4 \cap I_4$ is non-empty (which looks very plausible, true e.g. if $\mathrm{char}(k)=3$, cf. [19], p. 78, Example 2), then it would follow that M_4 contains a complete algebraic curve.

It seems plausible that M_3 contains a complete rational algebraic curve.

Note that M_6 in case $k=\mathbb{C}$ contains a complete curve of genus 129 as was proved by Kodaira (cf. [15], $n=2$, $m=2$, $m(2n-1)=6$).

In case $\mathrm{char}(k) \neq 2$ the result (1.1a) (and probably also 1.1b) follows from an algebraic construction by Mumford of the Satake compactification (cf. Inventiones math. **3** (1967), p. 236, Main Theorem).

2. Families of AV over Curves

We denote by K^s the separable closure of a field K, and by $T_l X$ the l-Tate-group of an AV X.

Theorem (2.1). *Let k be a perfect field (no restriction on its characteristic), C a complete, smooth irreducible algebraic curve over k and X an* AV *over $K=k(C)$. Let l be a prime number, $l \neq \mathrm{char}(k)$, and suppose that*

$$\rho: \mathrm{Gal}(K^s/K) \to \mathrm{Aut}(T_l X)$$

has the property that

$$\rho[\mathrm{Gal}(K^s/K k^s)] \text{ is a commutative subgroup}$$

of Aut $(T_l X)$. *Then there exists a finite separable extension $L \supset K$, an* AV *Y over the algebraic closure of k in L, and a purely inseparable isogeny*

$$t: Y \otimes L \to X \otimes_K L;$$

the extension $L \supset K$ can be chosen in such a way that it is unramified at all places of C where X has good reduction. If moreover X is an abelian scheme over C such that $\mathsf{X} \otimes K = X$ and C has a k-rational point, i.e. $C(k) \neq \emptyset$, there exists an unramified k-covering $D \to C$, an AV *Y over k, and a purely inseparable isogeny t: $Y \otimes_k D \to \mathsf{X} \times_C D$.*

Corollary (2.2). *Suppose* X *is an abelian scheme over* C, *a curve with all the properties stated in* (2.1); *suppose moreover* genus$(C)\leqq 1$; *then for a suitable k-covering* $D\to C$ *the last conclusion of* (2.1) *holds; if* genus$(C)=0$, *then* $D \xrightarrow{\sim} C$.

Corollary (2.3). *Let k be a field of characteristic* zero; *any abelian scheme* $\mathsf{X}\to C$, *with* genus$(C)\leqq 1$ *becomes constant over a suitable unramified covering* $D\to C$, *and hence any* fine *moduli scheme of abelian schemes in characteristic zero does not contain a complete curve whose normalization has a component of genus zero or one.*

Remark (2.4). Because of the existence of Nmm (= Néron minimal model) the existence of an abelian scheme X over a smooth k-curve C is the same as the existence of an AV $X=\mathsf{X}\otimes K$ over the function field $K=k(C)$ having good reduction at all discrete k-valuations of K. The condition that $Y\otimes D$ and $X\times_C D$ are isogenous is the same as $Y\otimes L$ and $\mathsf{X}\otimes_K L$ being isogenous, where $L=k(D)$. The t as indicated in the last part of the theorem is an isogeny if it is an epimorphism with finite kernel, and it is called purely inseparable if Ker(t) is an infinitesimal D-group scheme (which is the same as $t\otimes L$ being a purely inseparable isogeny of L-group varieties). Coverings will be considered between smooth, irreducible curves; note that if $D\to C$ is an *unramified* covering with genus$(C)\leqq 1$, then

$$\text{genus}(C)=\text{genus}(D),$$

because in the unramified case

$$2\cdot\text{genus}(D)-2=\left(2\cdot\text{genus}(D)-2\right)\cdot n,$$

where n is the degree of the covering (the Hurwitz formula, e.g. cf. [5], p. 215).

Proof of (2.2). The (étale part of the pro-finite completion of the) fundamental group of an algebraic curve C with genus$(C)\leqq 1$ is known to be commutative (in characteristic p, use a result by Grothendieck, cf. [6], Exposé X, Theorem 2.6), thus if Gal$(K^s/K k^s)$ acts in an unramified way on $T_l X$ (which is equivalent to X having good reduction everywhere on C: the Néron-Ogg-Shafarevich criterion, cf. [31], Theorem 1 on p. 493) and if moreover genus$(C)\leqq 1$, the main condition of (2.1) is satisfied because $\pi_1(C)=\text{Gal}(K^s/K k^s)_{\text{unram.}}$, thus (2.2) follows from (2.1).

Proof of (2.3). The last conclusion of the theorem implies that all geometric fibres of $\mathsf{X}\to C$ are isogenous to $Y\otimes\bar{k}$; this isogeny, being purely inseparable, is an isomorphism if char$(k)=0$ because group schemes in characteristic zero are reduced (hence Ker$(t)\otimes L$, being local, is trivial); moreover a polarization of a constant family is constant (cf. [21], Corollary 6.2), thus (2.3) follows from (2.2).

Remark (2.5). As Ueno pointed out to me, (2.3) can be proved with topological-analytic methods: if C maps to a fine moduli scheme, the universal covering of C (completeness is essential) maps to the universal covering of that fine moduli scheme, which is a Siegel space, and analysis shows this lifted map to be constant (cf. Kas, [14], p. 790).

Remark (2.6). On the other hand the conclusion of the corollary does not hold in case $\operatorname{char}(k)=p\neq 0$: let Z be an AV over k such that

$$(\alpha_p)^3 \hookrightarrow Z^t,$$

where t denotes the dual abelian variety; e.g. Z is the product of three supersingular elliptic curves; for a point $a\in \mathbb{P}_k^2(R)$, where R is a k-algebra, we define

$$\alpha_p \otimes R \xrightarrow{\sim} N_a \subset Z^t \otimes R$$

as follows: let $b_0, b_1, b_2 \in \operatorname{Hom}_k(\alpha_p, (\alpha_p)^3)$ be linearly independent, $a=(a_0:a_1:a_2)$, then

$$N_a := \operatorname{Im}((a_0 b_0, a_1 b_1, a_2 b_2)\colon \alpha_p \to Z^t \otimes R);$$

we define

$$X_a := (Z^t/N_a)^t,$$

thus we have an exact sequence of group schemes over $\mathbb{P}_k^2$:

$$0 \to \mathsf{N}^D \to \mathsf{X} \xrightarrow{\pi} Z \times \mathbb{P}^2 \to 0$$

(cf. [23], Theorem 19.1; here N^D denotes the dual of N; note $\alpha_p^D \cong \alpha_p$); choose a polarization on Z (and hence on $Z\times \mathbb{P}^2$), and lift it to a polarization λ on X via π (and we can also choose a level n-structure on X); let M be the moduli space of polarized abelian varieties (respectively of polarized abelian varieties with level n-structure); then (X, λ) (plus the level n-structure) defines a k-morphism

$$f\colon \mathbb{P}_k^2 \to M,$$

and we claim the image of f has dimension 2: assume k is algebraically closed, suppose $E\subset \mathbb{P}_k^2$ is an irreducible k-algebraic curve such that $f(E)$ is one point on M; this implies there exists a finite k-morphism $g\colon E' \to E$, and an AV Y over k, and an E'-isomorphism

$$Y \otimes_k E' \xrightarrow{\sim} g^*(\mathsf{X}\,|\,E);$$

this yields a homomorphism

$$h\colon Z^t \otimes_k E' \to g^*(\mathsf{X}^t\,|\,E) \cong Y^t \otimes_k E'$$

and for $e\in E'(k)$, the kernel of $h_e\colon Z^t \to Y^t$

$$\operatorname{Ker}(h_e) = N_{g(e)};$$

take a point $e_0 \in E'(k)$, let $h_0\colon Z^t \to Y^t$ be the fibre of h at e_0; the homomorphisms h and $h_0 \otimes_k E'$ coincide over e_0, thus (cf. [21], Corollary 6.2) $h = h_0 \otimes_k E'$, which implies

$$\operatorname{Ker}(h_e) = \operatorname{Ker}(h_{e_0}) \qquad \text{for all } e \in E'(h);$$

as $\{b_0, b_1, b_2\}$ was a basis, different geometric points of $\mathbb{P}^2_k$ yield different subgroup schemes $N_a \subset Z^t$, a contradiction; thus $f(E)$ is not a point in M, and $f(\mathbb{P}^2_k)$ has dimension two; conclusion: *M contains curves E_0, E_1 such that the normalization of E_i has genus i* (it is not difficult to choose $E'_i \subset \mathbb{P}^2$, genus$(E'_i) = i$ such that $f \mid E'_i$ is birational). This example seems to contradict [32], Theorem 5, in case $\operatorname{char}(k) = p \neq 0$ and $\dim(A) = d > 2$ (in order to construct E_0 we can do the same procedure with $(\alpha_p)^2 \subset Z^t$, so in that case we can take $d \geqq 2$); in the proof of [32], Theorem 5 in case of characteristic $p \neq 0$ the trace A^* need not be a subvariety of A (cf. [16], VIII.3, Corollary 2 on p. 216).

Proof of (2.1). Suppose k', C', X' as in the theorem. Because C' is of finite type over k', and X' is of finite type over K (and X' of finite type over C'), there exist k, C, X such that $k \subset k'$, $C' = C \otimes k'$, $X' = \mathsf{X} \otimes k'(C')$, and such that under the identification $T_l X \cong T_l X'$ (choose a k-embedding $k^s \subset k'^s$) the groups $\operatorname{Gal}(K^s/K k^s)$ and $\operatorname{Gal}(K'^s/K' k'^s)$ have the same image, and such that k is the perfect closure of a field finitely generated over its prime field. Thus it suffices to prove (2.1) in case k has this property.

In order to prove the first conclusion of the theorem, we replace k by a finite extension, again denoted by k, so that all places of C where X has bad reduction become rational over the new field, and such that at least one more point of C is rational over k (note that these properties are satisfied already in the second part of the theorem). We choose $N = l^2$; the group scheme

$${}_N X = \operatorname{Ker}(\times N\colon X \to X)$$

is constant over an extension $L \supset K$, $L = k(D)$; because $(l, \operatorname{char}(p)) = 1$ this extension (or: the covering $D \to C$, $k(D) = L$) can be chosen such that it is separable and unramified at all places where X has good reduction; we denote the new fields again by k, K, with $K = k(C)$, and we arrive at a situation where ${}_N X$ is a K-constant group scheme, i.e. $\operatorname{Gal}(K^s/K)$ acts trivially on ${}_N X(K^s)$.

First Step. For every $a \in \operatorname{Gal}(K^s/K k^s)$ all eigenvalues of $\rho\, a \in \operatorname{Aut}(T_l X)$ are equal to one. This we prove as follows. Let $M \subset C(k)$ be the set of points where X has bad reduction. Let

$$K k^s \subset L \subset K^s,$$

where L is the union of all abelian extension of $K k^s$, of degree a power of l, unramified at all places (discrete valuations) of $K k^s/k^s$ outside M; clearly $\mathrm{Gal}(K^s/L)$ is a characteristic subgroup of $\mathrm{Gal}(K^s/K k^s)$ (i.e. invariant under every automorphism). Let

$$J = \mathrm{Jac}_M(C)$$

be the generalized Jacobian variety of C constructed with M as conductor (all multiplicities of points in M equal to one), cf. [30], Chap. 5; this is a k-group variety, there is an exact sequence of k-group varieties

$$0 \to (\mathbb{G}_m)^{|M|-1} = H \to J \to \mathrm{Jac}(C) \to 0$$

where $\mathrm{Jac}(C)$ is the (ordinary) Jacobian variety of C (and H has this form because all points in M have multiplicity one and $M \subset C(k)$). Because J is a k-group scheme, $T_l J$ is a $\mathrm{Gal}(k^s/k)$-module in a natural way. Because $\mathrm{Gal}(K^s/L)$ is characteristic in $\mathrm{Gal}(K^s/K k^s)$, and abelian,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Gal}(K^s/K k^s) & \longrightarrow & \mathrm{Gal}(K^s/K) & \longrightarrow & \mathrm{Gal}(k^s/k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \rho} & & & & \\
 & & \mathrm{Gal}(L/K k^s) \cong T_l J & \overset{\varrho}{\cdots\cdots\rightarrow} & \mathrm{Aut}(T_l X) & & & &
\end{array}$$

the action by $\mathrm{Gal}(k^s/k)$ via inner conjugation inside $\mathrm{Gal}(K^s/K)$ on $\mathrm{Gal}(K^s/K k^s)$ induces an action on $\mathrm{Gal}(L/K k^s)$; there exists a natural isomorphism of $\mathrm{Gal}(k^s/k)$-modules

$$\mathrm{Gal}(L/K k^s) \cong T_l J;$$

this can be seen as follows: a finite abelian extension of $K k^s$ is induced by an isogeny of a generalized Jacobian of C (cf. [30], VI.11, Prop. 9); if such an extension is contained in L, we can choose the support of the conductor to be contained in M (cf. [30], VI.12, Lemma 1); in fact in that case, the degree of the extension is a power of l, hence prime to $\mathrm{char}(k)$, thus the conductor can be chosen contained in M (cf. [30], p. 128, Example 1: if $M \subset N$, $\mathrm{support}(N) = \mathrm{support}(M)$, then

$$\mathrm{Ker}(\mathrm{Jac}_N(C) \to \mathrm{Jac}_M(C))$$

is a unipotent group scheme, hence has no l-torsion points); the isogenies $\times l^i$: $J \to J$ are cofinal in the set of all isogenies over J of degree a power of l (if $G \to J$ is an isogeny with kernel annihilated by $\times l^i$, then its factors multiplication by l^i on J), thus the natural action of l-power torsion points on J on the corresponding coverings of C induce an isomorphism (use [30], VI.11, Proposition 10) between $\mathrm{Gal}(L/K k^s)$ and $T_l J$; note that the canonical morphism $C \setminus M \to J$ is defined over k, because moreover $C(k) \setminus M \neq \emptyset$, and it follows that the action of $\mathrm{Gal}(k^s/k)$

commutes with the isomorphism. Thus we obtain a commutative diagram as indicated above.

Because k is the perfect closure of a field finitely generated over its prime field, there exists an element $\sigma\in \mathrm{Gal}(k^s/k)$ such that its action on $T_l J$

$$r\colon \mathrm{Gal}(k^s/k)\to \mathrm{Aut}(T_l J)$$

has the property:

$$r(\sigma)\in \mathrm{Aut}(T_l J\otimes \mathbb{Q}_l)$$

has no eigenvalues equal to a root of unity and the characteristic polynomial (of any power) of $r(\sigma)$ has integral coefficients (cf. [25], Lemma 3.2). Let $\sigma'\in \mathrm{Gal}(K^s/K)$ be such that

$$\mathrm{Gal}(K^s/K)\to \mathrm{Gal}(K k^s/K)\cong \mathrm{Gal}(k^s/k), \qquad \sigma'\mapsto\sigma,$$

and write $\rho(\sigma')=:S\in \mathrm{Aut}(T_l X)$; let $a\in T_l J$, $\rho\, a=:A\in \mathrm{Aut}(T_l X)$; choose $Q\supset \mathbb{Q}_l$ containing all eigenvalues of A and of S.

Consider $a_i:=\big(r(\sigma^{-i})\big)(a)$, and $A_i:=\rho\, a_i$; because of the hypothesis made, *the matrices A_i all commute;* because of the way σ acts on $\mathrm{Gal}(L/K k^s)\cong T_l J$,

$$A_i=S^{-i}AS^i, \qquad i\in\mathbb{Z}$$

hence any *two of these matrices have the same set of eigenvalues*. Consider all infinite sequences

$$u=(\dots, u_{-1}, u_0, u_1, u_2, \dots)$$

of eigenvalues of A; let $E:=T_l X\otimes_{\mathbb{Z}_l} Q$, and let

$$E_u:=\bigcap_{i\in\mathbb{Z}} \mathrm{Ker}(A_i-u_i)\subset E;$$

fix $i\in\mathbb{Z}$, and consider $\mathrm{Ker}(A_i-u)=E_{iu}$, where u is an eigenvalue; clearly

$$\bigoplus_u E_{iu}=\sum_u E_{iu}$$

(on E_{iu} the transformation A_i has eigenvalue u and on $\sum_{v\neq u} E_{iv}$ the map A_i has no eigenvalue equal to u); thus it follows (because the A_i commute, and $\dim(E)<\infty$) that the set U of sequence u with $E_u\neq 0$ is finite and their sum is a direct sum

$$\bigoplus_{u\in U} E_u \xrightarrow{\sim} F:=\Big(\sum_{u\in U} E_u\Big)\subset E.$$

Note that

$$S(E_u)=E_{S(u)},$$

where $(S(u))_i:=u_{i+1}$, i.e. it is the shift to the left:

$$x\in E_u \Rightarrow SS^{-1}S^{-i}AS^iSx=Su_{i+1}x;$$

thus $S(E_u) \subset E_{S(u)}$, thus $SF \subset F$; because S is invertible $SF = F$, thus $S(E_u) = E_{S(u)}$; thus S induces a permutation

$$S\colon\ U \to U\,.$$

As U is finite, there exists an integer n so that $S^n(E_u) = E_u$ for all $u \in U$. Let $T = S^n$, let $\tau = \sigma^n$, let P be the characteristic polynomial of $r(\tau)$; note P has coefficients in $\mathbb{Z}$, and $P(1) \neq 0$ (because $r(\sigma)$ has no eigenvalues equal to a root of units, hence $r(\tau)$ does not have 1 as eigenvalue). Let λ be an eigenvalue of $A = \rho(a)$; choose $u \in U$ with $u_0 = \lambda$ (this is possible because $\mathrm{Ker}(A - \lambda) \neq 0$, etc.); as $T(E_u) = E_u$, for all $x \in E_u$,

$$(T^{-i} A T^i)\, x = \lambda\, x;$$

note that

$$\big(P(\tau)\big)(a) = 0,$$

thus $\rho\big(P(\tau)(a)\big) = 1$,

$$\rho\big(P(\tau)(a)\big)(x) = \lambda^{P(1)}\, x = 1\,,$$

and $P(1)$ being non-zero, this implies that λ is a root of unity. Because $\mathrm{Gal}(K^s/K)$ acts trivially on ${}_N X(K^s) = T_l X / N(T_l X)$,

$$A \equiv 1 \ (\mathrm{mod}\ N)$$

(considered as $\mathbb{Z}_l$-linear maps), and λ being an eigenvalue of A, we conclude

$$\lambda \equiv 1 \ (\mathrm{mod}\ l)$$

(in the ring of integers of $\mathbb{Q}_l(\lambda)$), thus λ is an l-power root of unity. Suppose $\lambda \neq 1$; in the ring of integers of $\mathbb{Q}_l(\lambda)$ the element $\pi = 1 - \lambda$ divides l, and

$$\det(A - \lambda I) = 0,$$

thus

$$\pi^{2g} + \pi^{2g-1} \cdot N \cdot a_1 + \cdots + \pi^{2g-i} \cdot N^i \cdot a_i + \cdots = 0, \qquad a_i \in \mathbb{Z}_l,$$

which implies (because $\pi^2 \mid N = l^2$):

$$\pi^{2g} \equiv 0 \ (\mathrm{mod}\ \pi^{2g+1});$$

however, from $\lambda \neq 1$ it follows that π is a uniformizing element of $\mathbb{Q}_l(\lambda)$ (e.g. cf. [35], 7.4.1), contradiction, and hence $\lambda = 1$, which concludes the proof of the first step.

Remark. In case $l > 2g+1$, one can choose $N = l$: if $\mathrm{Gal}(K^s/K)$ acts trivially on ${}_l X(K^s)$, again λ is a l-power root of unity, its degree over $\mathbb{Q}_l$

is at most $2g$ because it is an eigenvalue of $A \in \mathrm{Aut}((\mathbb{Z}_l)^{2g})$ and $[\mathbb{Q}_l(\zeta):\mathbb{Q}_l] \geqq l-1$ for any l-power root of unity not equal to one, thus it follows that $\lambda = 1$.

Last Step of the Proof. Consider

$$Y := \mathrm{Tr}_{K/k} X,$$

i.e. this is an AV over k, plus a purely inseparable homomorphism

$$t\colon\ Y \otimes_k K \to X$$

(with finite kernel) which has a certain universal property (cf. [16], VIII.3, Theorem 8 on p. 213, and Corollary 2 on p. 216); we want to show t is an isogeny (i.e. t is epimorphic); let Z be an AV over K defined by the exact sequence

$$Y \otimes K \xrightarrow{\ t\ } X \longrightarrow Z \longrightarrow 0;$$

because t is purely inseparable, and $l \neq \mathrm{char}(k)$, we obtain an exact sequence

$$0 \to T_l Y \to T_l X \to T_l Z \to 0,$$

the action of $\mathrm{Gal}(K^s/K k^s)$ on $T_l Y$ is trivial and by the first step we conclude all images of

$$\rho\colon\ \mathrm{Gal}(K^s/K k^s) \to \mathrm{Aut}(T_l Z)$$

have eigenvalues all equal to one, and this image is commutative; this implies that if $Z \neq 0$, there exists

$$0 \neq v \in \bigcap \mathrm{Ker}(\rho(b) - 1),$$

the intersection taken over all $b \in \mathrm{Gal}(K^s/K k^s)$ (or all $b \in T_l J$); we define

$$Z' := \mathrm{Tr}_{K/k} Z, \quad \text{thus} \quad T_l Z' \hookrightarrow T_l Z;$$

by the Lang-Néron version of the Mordell-Weil theorem we conclude (cf. [17], p. 97, Theorem 1) that $v \in T_l Z'$, thus $Z' \neq 0$; let

$$Z'' := Z/\mathrm{Im}(Z' \otimes K \to Z);$$

by [16], II.1, Theorem 6, we conclude there exists an isogeny

$$(Y \otimes K) \oplus (Z' \otimes K) \oplus Z'' \to X,$$

thus the image of the K/k-trace of X contains the image of $(Y \oplus Z') \otimes K$ (universal property of the trace), the K/k-trace of X is $t\colon Y \otimes K \to X$, thus $Z' = 0$; thus $Z = 0$, thus t is epi, i.e. t is a purely inseparable isogeny, which concludes the proof of (2.1).

Remark (2.7). Probably the methods above can also be used to prove the statements (2.1) and (2.2) with C replaced by (the function field of) an AV over k.

Remark (2.8). In case genus $(C) = 0$, the result (2.2) was proved by Grothendieck (cf. [8], pp. 74/75, Proposition 4.4). Theorem (2.1) was inspired by the result of Grothendieck which says that an AV with sufficiently many CM is isogenous to an AV defined over an algebraic extension of the prime field (cf. [22], Theorem on p. 220, and cf. [26]). Possibly these theorems are special cases of more general results.

Remark (2.9). It is not difficult to construct examples of non-trivial families $\mathsf{X} \to C$ over a complete elliptic curve over any given base field: take an elliptic curve D, a point $d \in D(k)$ of order $(d) > 1$, and an AV Y over k with $a \in \mathrm{Aut}_k(Y)$, with order $(a) = $ order (d); let $C = D/\langle d \rangle$ and let X be the quotient of $D \times Y$ by the equivalence relation $(u, y) \sim (u + d, a(y))$.

3. Subvarieties of Moduli Spaces Defined by Kodaira Surfaces[1]

In the previous section we showed that a *fine* moduli scheme of algebraic curves in characteristic zero does not contain a complete elliptic or rational curve (use Corollary (2.3) plus Torelli's theorem which says that the Jacobi-morphism from moduli schemes of nonsingular irreducible curves to the moduli schemes of principally AV is injective); in this section we follow a construction suggested by Parshin of certain surfaces, so that we can prove:

Theorem (3.1). *In any characteristic there exists a number g such that the* coarse *moduli scheme M_g of (irreducible and nonsingular) algebraic curves contains a complete* rational *curve $E \subset M_g$.*

By a Kodaira surface, or an irregular algebraic surface, we mean a complete (nonsingular) algebraic surface M plus a smooth morphism $h: \mathsf{M} \to D$ onto a complete nonsingular irreducible algebraic curve D; such surfaces were constructed by Kodaira (cf. [15], also cf. Kas, [14]). The family h of curves parametrized by D defines a morphism $M: D \to M_g$, where g is the genus of each of the fibres of $\mathsf{M} \to D$; we construct M and D such that $D \to M_g$ factors through a rational curve.

(3.2) *Parshin's Construction* (cf. [28], pp. 1168/1169, and 1163–1167). Suppose given a field k, a smooth, irreducible, complete algebraic curve B over k, two positive integers m, n both prime to char(k); then we arrive at an etale covering $D \to B$ and a Kodaira surface $\mathsf{M} \to D$; moreover if genus $(B) \geqq 2$, and $m > 1$, and $n > 1$, then no (etale) covering of D makes the fibration $\mathsf{M} \to D$ trivial (equivalently: at least two fibres of $\mathsf{M} \to D$ are non-isomorphic). The construction is performed as follows:

$$\begin{array}{ccc} \mathsf{D}, \mathsf{P} \subset C & \hookrightarrow & J \times B \\ \downarrow \pi & \text{cartesian} & \downarrow (\times n) \times id \\ B \times B & \overset{i}{\hookrightarrow} & J \times B \end{array}$$

[1] I thank Knud Lønsted for drawing my attention to the paper by Parshin and for stimulating discussion on this topic.

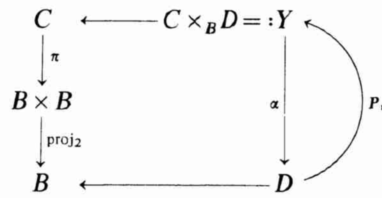
consider $J := \mathrm{Jac}(B)$, let $B \to J$ be the canonical embedding of B into its Jacobian variety (e.g. cf. [30], V. 6 and V. 9), and define i so that $i(b, b) = (0, b)$ (i.e. the "variable" point b is used to define the canonical morphism i_b: $B \times \{b\} \to J \times \{b\}$); let C be the pull-back of $i(B \times B)$ by multiplication by n on the B-group scheme $J \times B$, and let

$$\mathsf{D} \cup \mathsf{P} = \overset{-1}{\pi}(b, b)$$

be the fibre over (b, b) so that P: $B \times B \to C$ is the section defined by

$$\mathsf{P} = \{0\} \times B \subset C \subset J \times B;$$

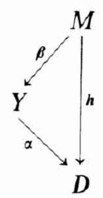
then C is irreducible (cf. [30], VI.11, Proposition 10), and π is an etale covering because $\mathrm{char}(k)$ does not divide n; note that the degree of the relative Cartier divisor $\mathsf{D} \subset C \to B \times B$ is $n^{2 \cdot \mathrm{genus}\ (B)} - 1$, so bigger than one if $n > 1$ and $\mathrm{genus}(B) > 0$; because D is an etale covering of $B \times B$, we can choose an etale covering $D \to B$



and sections P_i: $D \to Y$ so that

$$\mathsf{D} \times_B D = \cup P_i$$

(in case all points of ${}_nJ$ are rational over k, we can choose $D = B$); we write P for the section $\mathsf{P} \times_B D$: $D \to Y$, and using D, P, P_i, and the integer m we construct



as follows: first, let $\zeta \in D$ be the generic point, with fibre Y_ζ over this point; on this fibre there is a point $P(\zeta)$ and a divisor $\delta = \sum P_i(\zeta)$, thus there results a canonical morphism

$$Y_\zeta^0 = Y_\zeta \setminus \mathrm{Supp}(\delta) \to \mathrm{Jac}_\delta(Y_\zeta) = J_\delta$$

into its generalized Jacobian variety with respect to δ, sending $P(\zeta)$ onto the zero point of $0 \in J_\delta$ (and this morphism is defined over $k(\zeta)$, because

$P(\zeta)$ and all $P_i(\zeta)$ are rational over $k(\zeta)$); then we construct

$$\begin{array}{ccccc} M_\zeta & \supset & M_\zeta^0 & \hookrightarrow & J_\delta \\ \downarrow & & \downarrow & \text{cartesian} & \downarrow \times m \\ Y_\zeta & \supset & Y_\zeta^0 & \hookrightarrow & J_\delta \end{array}$$

the etale covering $M_\zeta^0 \to Y_\zeta^0$ defined by multiplication by m on J_δ (again we know the covering is irreducible, and etale because char(k) does not divide m); the curve M_ζ is the unique smooth complete irreducible curve containing M_ζ^0; the minimal fibring $h\colon \mathsf{M} \to D$ having M_ζ as generic fibre exists, it is smooth (use the fact that $Y \to D$ is smooth, and $\mathsf{D} \cup \mathsf{P}$ etale over $B \times B$, and apply [28], p. 1164, Lemma 10), and for every point d of D, the fibre M_d can be constructed as follows:

$$\begin{array}{ccccc} M_d & \supset & M_d^0 & \longrightarrow & J_{\delta(d)} \\ \downarrow & & \downarrow & & \downarrow \times m \\ Y_d & \supset & Y_d^0 & \xrightarrow{\ i\ } & J_{\delta(d)}, \ i(P(d))=0, \end{array}$$

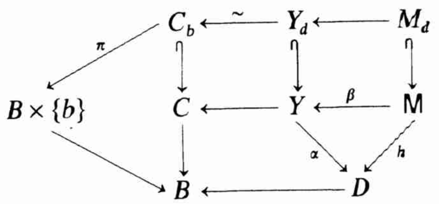
with $Y_d^0 = Y_d \setminus \operatorname{Supp} \delta(d)$ (this follows because M_d is the unique smooth complete curve containing $M_d^0 \subset \mathsf{M}$); note that if $\deg(\delta)>1$, and $m>1$ then the morphism

$$\beta_d\colon\ M_d \to Y_d$$

ramifies exactly at all points of $\delta(d)$; this can be seen as follows: all points in $\delta(d)$ have multiplicity one, thus $J_{\delta(d)}$ is an extension over $\operatorname{Jac}(Y_d)$ with kernel $L = (\mathbb{G}_m)^{\deg(\delta)-1}$; thus

$$L \cap \operatorname{Ker}(\times m\colon\ J_{\delta(d)} \to J_{\delta(d)}) \neq 0;$$

because the unramified coverings of Y_d correspond bijectively with isogenies over $\operatorname{Jac}(Y_d)$ (cf. [30], VI.12, Corollary on p. 128) it follows β_d ramifies at each of the points $P_i(d) \in Y_d$;



let d be a point of D mapping onto a point b of B; if genus$(B)>0$ and $n>1$ (so $\deg(\delta)>1$), and $m>1$ then the composed covering $M_d \to B \times \{b\}$ ramifies at the point (b, b).

Lemma (3.3). *Let k be a field, B and Z complete smooth irreducible algebraic curves over k, and T a k-prescheme. Let*

$$\varphi:\ Z\times T\to B\times T$$

be a family of morphisms of Z to B parametrized by T (i.e. φ commutes with the projections on T); if genus$(B)\geqq 2$, then φ is locally constant on T.

Proof. One way of proving the lemma is the following: suppose $T=\operatorname{Spec} k[\varepsilon]$, then all $\varphi: Z\times T\to B\times T$ deforming a fixed $(\varphi_0: Z\to B)=\varphi\otimes_{k[\varepsilon]}k$ are in 1-1-correspondence with $\Gamma(B, \mathscr{D}er(\mathcal{O}_B, \varphi_{0*}\mathcal{O}_Z))$ (here $\mathscr{D}er$ stands for the sheaf of germs of k-$\mathcal{O}_B$-derivations), and one can show this set of sections to be trivial in case $\mathscr{D}er(\mathcal{O}_B, \mathcal{O}_B)$ is a negative line bundle (its degree is $2-2g$). Another way of proving is the following. Because the functor of morphisms from Z to B is representable (Grothendieck, cf. Sém. Bourbaki **13**, p. 221–20 (1960/61)), and smooth (B is a curve), it suffices to prove the lemma in case k is algebraically closed and T is connected and reduced (and this is the case which we need in applying the lemma). Let $t\in T(k)$, and consider

$$\begin{array}{ccc} Z & \xhookrightarrow{\ j\ } & \operatorname{Jac}(Z) \\ \downarrow{\scriptstyle \varphi_t} & & \downarrow{\scriptstyle \psi_t=\lambda+a_t} \\ B & \xhookrightarrow{\ i\ } & \operatorname{Jac}(B); \end{array}$$

because of the Albanese property of the Jacobian variety (of B) the morphism ψ_t results, making commutative the diagram; let $a_t=\psi_t(0)$; because of the rigidity lemma ([21], Corollary 6.2) the morphism $\lambda=\psi_t-a_t$ (is a homomorphism which) does not depend on t; fix $s\in T(k)$ and consider

$$b\in B, \qquad (i\,b)\mapsto (i\,b+a_s-a_t)=:\gamma_t(b)$$

because

$$\lambda jZ+a_t=i\,B=\lambda jZ+a_s$$

we conclude we obtain a family of automorphisms

$$\gamma: B\times T\to B\times T, \qquad (b,t)\mapsto(\gamma_t b, t)$$

(we identified B and $B\simeq iB$); because genus$(B)\geqq 2$ we know B has only finitely many automorphisms thus T being connected (and reduced), the family γ is constant, thus $a_s=a_t$ for all t, thus φ is constant. Q.E.D.

Now suppose, notation as in (3.2): genus$(B)\geqq 2$, $m>1$ and $n>1$; let $T\to D$ be a covering, then $\mathsf{M}\times_D T$ is not trivial; in fact, suppose $\mathsf{M}\times_D T\simeq Z\times T$; then apply (3.3) to

$$(\mathsf{M}\to Y\to C\to B\times B)\times_D T,$$

thus concluding that for each $d \in D$ the covering

$$M_d = (\mathsf{M} \times_D T)_t = Z \to Y_d \to C_b \to B \times \{b\}$$

with $t \mapsto d \mapsto b$ does not depend on t; however, we have seen that this covering ramifies at b in the given situation, thus while t runs through T, the point b is not constant, and $M_d \to B \times \{b\}$ cannot be a constant morphism; hence the fibring $h\colon \mathsf{M} \to D$ cannot be trivialized by a covering of D, and the construction and the claim in the first sentence of (3.2) are established (the arguments essentially can be found on p. 1169 of [28]).

Proof of (3.1). Choose a curve B with genus$(B) \geqq 2$, and a (finite) group $G \subset \mathrm{Aut}(B)$ with

$$q\colon\ B \to B/G \simeq \mathbb{P}^1$$

(such an example is easy to construct, e.g. take any curve B' with genus$(B') \geqq 2$; its function field $k(B')$ is a separable extension of $k(\mathbb{P}^1)$, and take for B the normalization of B' in some finite Galois extension of $k(\mathbb{P}^1)$ containing $k(B')$). Using the construction explained above, with the help of integers m and n (prime to char(k) and both at least equal to two), we arrive at a Kodaira surface

$$\begin{array}{ccccc} & & \mathsf{M} & & \\ & & \downarrow{\scriptstyle h} & & \\ B & \xleftarrow{\ f\ } & D & & \\ {\scriptstyle q}\downarrow & & & \searrow{\scriptstyle M} & \\ \mathbb{P}^1 & \dashrightarrow\!\!\!\!\rightarrow & E & \hookrightarrow & M_g; \end{array}$$

thus we obtain a morphism $M\colon D \to M_g$, with $g = \text{genus}(\mathsf{M}/D)$, and we claim this factors through $\mathbb{P}^1$; because h is not locally trivial in the etale topology, the image $M(D) = E \subset M_g$ is not a point, but a curve, and because of Lüroth's theorem we conclude, $\mathbb{P}^1 \twoheadrightarrow E \subset M_g$, this to be a rational curve; the proof of the factorization follows because: if $d, e \in D(k)$, and $qfd = qfe$ then $M_d \simeq M_e$; indeed in that case there exists $\sigma \in G$ with $fd = \sigma fe$; the morphism

$$\sigma\colon\ B \times \{b\} \to B \times \{\sigma b\}$$

extends to a commutative diagram

$$\begin{array}{ccc} B \times \{b\} & \longrightarrow & \mathrm{Jac}(B) \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma_*} \\ B \times \{\sigma b\} & \longrightarrow & \mathrm{Jac}(B) \end{array}$$

thus arriving at a morphism

$$\sigma_*\colon\ C_b \xrightarrow{\sim} C_{\sigma b},\ \sigma_* P_b = P_{\sigma b},\ \sigma_* D_b = D_{\sigma b};$$

this results in an isomorphism

$$\sigma_*\colon\ M_d^0 \xrightarrow{\sim} M_e^0,$$

and because M_d is the unique smooth curve containing M_d^0 (and the analogous statement for M_e^0 and M_e), we conclude $M_d \simeq M_e$, and Theorem (3.1) is proved.

4. Supersingular AV

In this section all fields considered will be of characteristic $p \neq 0$.

Consider an elliptic curve E over an algebraically closed field k; the following properties are equivalent:

a) E is very special (i.e. E has no points of order p);

b) $\mathrm{End}_k(E)$ has rank 4 as $\mathbb{Z}$-module (note k is algebraically closed);

c) the formal group of E is isomorphic to $G_{1,1}$ (notation of [19], p. 35);

d) α_p is a subgroup scheme of E (here α_p denotes the kernel of the Frobenius homomorphism $F\colon \mathbb{G}_a \to \mathbb{G}_a$).

An elliptic curve E over a field l is said to be *supersingular* if it satisfies these properties over an algebraic closure k of l; note that a supersingular curve is isomorphic over k with an elliptic curve defined over the field $\mathbb{F}_{p^2}$, the quadratic extension of the prime field; note that any two supersingular curves are isogenous over an algebraically closed field (in fact they are already isogenous over an extension of degree 12 of a common field of definition, cf. [34], pp. 537/538).

Definition (4.1). An AV X over a field l is called *supersingular* if the formal group $\hat{X}$ of X is isogenous over an algebraic closure k of l to $(G_{1,1})^g$, i.e.

$$\hat{X} \otimes k \sim \hat{E}^g \otimes k,$$

where E is a supersingular curve.

Theorem (4.2). *Let X be a supersingular* AV *over a field l; then X is isogenous to E^g over k,*

$$X \otimes k \sim E^g \otimes k,$$

where k is an algebraic closure of l, and E is a supersingular curve.

Remark (4.3). This gives a positive answer to a question posed by Manin in the case $g = 2$ (cf. [19], p. 79). In general the isogeny type of the formal group of an AV does not imply the AV can be decomposed up to isogeny in the same manner; e.g. there exists an abelian surface X whose formal group is isogenous with $G_{1,0} \oplus G_{1,1}$ such that X is a simple

AV; this can be seen as follows: consider the closed subscheme $V \subset A_2$ corresponding to abelian surfaces with p-rank equal to one (notation as in Section 1); as $\dim(A_2)=3$, from (1.6) we conclude $\dim(V)=2$; an abelian surface with p-rank equal to one has no positive dimensional families of finite subgroup schemes, thus if such an abelian surface is isogenous to a product of two elliptic curves, it can be defined over a field of transcendence degree one over the prime field; thus the generic point of V corresponds to a simple AV (this fact was already proved by Honda, cf. [11], p. 93). More generally: for any isogeny type of abelian varieties not equal to a power of $G_{1,1}$ there exist a simple AV having that isogeny type (cf. [18]).

Notation (4.4). Let X be an AV over a field l; by $a(X)$ we denote the dimension

$$a(X) := \dim_k \mathrm{HOM}(\alpha_p, X \otimes k),$$

where k is an algebraically closed field containing l (note that $\mathrm{End}_k(\alpha_p) \cong k$, a ring isomorphism, thus $\mathrm{HOM}(\alpha_p, G)$ is a right-k-module for any k-group scheme G).

Lemma (4.5). *Let X be an* AV *over a field l such that $a(X)=\dim X$; then X can be defined over a finite field.*

Proof. Take a polarization on X over l, let its degree be d, choose a large integer n prime to p so that $A_{g,d,n}$ is a fine moduli scheme, $g=\dim X$, take a level n-structure on X (make a finite extension of l if necessary), and consider the corresponding point $x \in A_{g,d,n}(l)$. Let V be the closure of x in $A_{g,d,n}$, and let $y \in V(k)$, where k is an algebraic closure of the prime field; we want to show $\dim V=0$, thus it suffices to show the tangent space to V at y to be zero. Note that $a(X)=\dim(X)$ is equivalent by saying that the p-operation in the Lie-algebra $\mathrm{Lie}(X)$ is zero. Let X be the abelian scheme over $V \subset A_{g,d,n}$ induced from the universal family over this fine moduli scheme; $\mathrm{Lie}(\mathsf{X})$ is a locally free sheaf of p-Lie algebras over V, and clearly the p-operation is identically zero; let

$$t:\ \mathrm{Spec}(k[\varepsilon]) \to V \subset A_{g,d,n}$$

be a tangent vector, $\varepsilon^2=0$, to V at y; then the fibre $X_t := \mathsf{X} \times_V t$ has the property $\mathrm{Lie}(X_t)$ has p-operation equal to zero; from this we are going to conclude t is a zero vector. Consider $Y=X_y$ the AV over k which is the fibre of X at $y \in V$; let $\mathscr{L}=\mathscr{L}(Y; k[\varepsilon] \to k)$ be the set of infinitesimal deformations of Y (notations of [24], p. 274 and p. 277), this set can be canonically identified with the tangent space of the local moduli space M of Y, i.e. $\mathscr{L}=M(k[\varepsilon])$, and the isomorphism class X_t can be considered as an element $t \in \mathscr{L}$. Let $G_0=\mathrm{Lie}(Y)$ be the Lie algebra of Y, and write $R=k[\varepsilon]$, $\varepsilon^2=0$; consider the set $\mathscr{K}=\mathscr{K}(G_0; R \to k)$ of infinitesimal deformation of G_0:

$\mathscr{K} := \{\cong \text{classes of } (G, \varphi_0) \mid G \text{ is a } p\text{-Lie algebra over } R, \text{ and}$
$\varphi_0\colon G \otimes_R k \xrightarrow{\sim} G_0\}$;

this is a k-vector space in a natural way (see below, or use [29], Lemma 2.10, exactness of the infinitesimal deformation functor of G_0 is immediate); an infinitesimal deformation of Y yields the same for G_0, thus a natural map

$$\mathscr{L} \to \mathscr{K}$$

results, which is k-linear (because the map which associates with Y' over R its Lie-algebra $\mathrm{Lie}(Y') \in \mathscr{K}$ is functorial). We claim:

a) the k-linear map $\mathscr{L} \to \mathscr{K}$ is surjective, and

b) because $a(Y) = \dim Y$, this map is an isomorphism.

To prove this, consider the group scheme $N_0 = \mathrm{Ker}(F\colon Y \to Y^{(p)})$, i.e. N_0 is the group scheme of height one uniquely determined by its p-Lie algebra G_0. Consider the scheme $N := N_0 \otimes_k R$. By [4], III.3.5 we have a natural identification

$$\mathscr{K} \cong H^2_{\mathrm{symm}}(N_0, G_0)$$

(symm denotes the symmetric cocycles; they correspond to the commutative group scheme structures on N; a group scheme structure on N lifting the one on N_0 can be identified with a p-Lie algebra structure on $G_0 \otimes_k R$ extending the p-Lie structure on G_0: cf. [3], II.7.3.5), moreover

$$\mathscr{L} \cong H^2_{\mathrm{symm}}(Y, G_0)$$

(cf. [4], III.3.7), and the map $\mathscr{L} \to \mathscr{K}$ is induced by $i\colon N_0 \hookrightarrow Y$. After choice of a k-basis $G_0 \cong k^g$, one can make the following identifications, resulting in a commutative diagram:

$$\begin{array}{ccccc} \mathscr{L} & \cong & H^2_{\mathrm{symm}}(Y, G_0) & \cong & (\mathrm{Ext}(Y, \mathbb{G}_a))^g \\ \downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ \mathscr{K} & \cong & H^2_{\mathrm{symm}}(N_0, G_0) & \cong & (\mathrm{Ext}(N_0, \mathbb{G}_a))^g \end{array}$$

(where Ext stands for the group of isomorphism classes of k-group scheme extensions); the exact sequence

$$0 \longrightarrow N_0 \longrightarrow Y \xrightarrow{F} Y^{(p)} \to 0$$

results into an exact sequence

$$\ldots \to \mathrm{Ext}(Y, \mathbb{G}_a) \to \mathrm{Ext}(N_0, \mathbb{G}_a) \to E^2(Y^{(p)}, \mathbb{G}_a) = 0$$

(the last equality because of [23], Lemma 12.8), thus Claim (a) is proved. Because $a(Y)=\dim Y=g$, we have $N_0\cong(\alpha_p)^g$, thus $\dim_k \mathcal{K}=g^2$ (because $\mathrm{Ext}(\alpha_p, \mathbb{G}_a)\cong k$ as k-modules, the action of k is via $\mathrm{End}(\mathbb{G}_a, \mathbb{G}_a)$, cf. [23], Proposition 10.5); moreover $\dim_k \mathcal{L}=g^2$ (because $\dim_k \mathrm{Ext}(Y, \mathbb{G}_a)=\dim Y$, cf. [30], VII.17), thus Claim (b) is proved. The fact that $\mathrm{Lie}(X_t)$ has p-operation zero, i.e. $\mathrm{Lie}(X_t)=0\in\mathcal{K}$, implies via (b) that $t=0$. Thus $\dim V=0$, the point $x\in A_{g,d,n}$ is rational over the algebraic closure k of the prime field, and $X=\mathsf{X}_x$; thus the lemma is proved.

Remark (4.6) (Mumford). From the lemma one can deduce that abelian varieties of dimension two can be lifted to characteristic zero.

Proof of (4.2). Let X and l be as in the theorem; we assume l is algebraically closed; we can replace X within isogeny by an AV over l (again denoted by X) such that $\hat{X}\cong(G_{1,1})^g$; then $a(X)=g$, because $\alpha_p\hookrightarrow G_{1,1}$; thus by (4.5) we know X *can be defined over a finite field* K (and this AV again is denoted by X); let $f\colon X\to X$ be the geometric Frobenius of X over K, i.e. if K has $q=p^a$ elements, raising to the q-th power of the elements of $\mathcal{O}_X$ is a K-endomorphism:

$$f=(X\xrightarrow{F} X^{(p)}\to\cdots\to X^{(p^a)}\cong X).$$

Let P be the characteristic polynomial of f. Because P is monic and has integral coefficients (cf. [16], p. 187, Corollary 2; [22], p. 180, Theorem 4), the zeros of P are algebraic integers. Let v be an extension of the p-adic valuation on $\mathbb{Q}$ to $\mathbb{C}$, thus $v(p)=1$. Let $\lambda_1,\ldots,\lambda_{2g}$ be the zeros of P in $\mathbb{C}$. By the Riemann-Weil hypothesis we know

$$|\lambda_i|=\sqrt{q},\qquad 1\leqq i\leqq 2g$$

(cf. [16], p. 139, Theorem 2; [22], p. 206, Theorem 4). Because the isogeny type of the formal group of X is $(G_{1,1})^g$ we conclude by a theorem of Manin (cf. [19], 4.1) that

$$v(\lambda_i=v(\sqrt{q}),\qquad 1\leqq i\leqq 2g.$$

Thus the elements $\lambda_i\cdot q^{-\frac{1}{2}}$ are integral, they form complete sets of conjugates and they have absolute value one; this implies they are roots of unity (e.g. cf. [1], p. 105, Theorem 2). Thus replacing K by a finite extension (the new field again denoted by K, same for X, f and P) we achieve $\lambda_i\cdot q^{-\frac{1}{2}}=1$, i.e.

$$P=(T-\sqrt{q})^{2g};$$

by a result of Tate, this implies X is isogeneous to the g-th power of a supersingular curve (cf. [33], Theorem 2.d), which ends the proof of the theorem.

Alternative proof of (4.2): Let X be defined over a finite field having $q=p^a$ elements, and f its geometric Frobenius as above; if $\hat{X}\cong(G_{1,1})^g$, we know $\mathrm{Ker}(f)$ equals the kernel of multiplication by $p^{a/2}$ (if necessary, extend K so that a is even), thus there exists a K-automorphism α of X with

$$f=p^{a/2}\cdot\alpha.$$

Clearly α is compatible with any polarization on X, and because a polarized AV has only finitely many automorphisms (cf. [20]), the order of α is finite, i.e. replacing K by a finite extension we see the geometric Frobenius of X has eigenvalues equal to $q^{\frac{1}{2}}$, and we conclude by the theorem due to Tate as above.

Corollary (4.7). *The coarse moduli scheme $A_2=A_{2,1,1}$ of abelian varieties of dimension two with a principal polarization (over a field of characteristic $p\neq 0$) contains a complete rational curve.*

Proof. By Theorem (1.1 a) A_2 contains a complete subscheme E of dimension $\frac{1}{2}\cdot 2(2-1)=1$ (which, by the way, contradicts a suggestion by Grothendieck, cf. [8], p. 77, lines 18–20); suppose the base field k is algebraically closed, suppose E is irreducible and reduced (if necessary take one of the components of E); we now show E is a rational curve, i.e. $\mathrm{genus}(E)=0$. Let X_0 be an AV with a principal polarization λ_0, both defined over an algebraic closure of $k(E)$ such that (X_0,λ_0) corresponds to the generic point of $E\subset W_{2,1}$, cf. Sect. 1), and $\dim X_0=2$, thus the isogeny type of $\hat{X}_0$ is $2\cdot G_{1,1}$; by (4.2) we conclude there exists an isogeny $\beta\colon Z\to X_0$ with

$$\hat{Z}\cong 2\cdot G_{1,1},$$

thus $a(Z)=2$, and Z defined over k (cf. 3.5); we may assume that $Z\to X_0$ is purely inseparable (replace Z by $Z/(\mathrm{Ker}\,\beta)_{\mathrm{red}}$), let its degree be p^n, write $Z=X_n$, construct isogenies

$$Z=X_n\to X_{n-1}\to\cdots\to X_{i+1}\to X_i\to\cdots\to X_0$$

each of degree p (and thus each having α_p as kernel); lifting back λ_0 we obtain polarizations λ_i on X_i, $0\leqq i\leqq n$; let $E=E_0$, and let $E_i\subset A_{2,p^{2i}}$ be the closure of the point given by (X_i,λ_i); let $0<m\leqq n$ be such that $\dim(E_i)\neq 0$ for $0\leqq i<m$ and $a(X_m)=2$ (and thus $\dim(E_m)=0$ by 4.5). We claim the isogeny correspondence between E_{i+1} and E_i, $0\leqq i<m-1$ given by $X_{i+1}\to X_i$ is birational (because $a(X_i)=1=a(X_{i+1})$ for $0\leqq i\leqq m-1$ there is only isogeny $\alpha_p\to X_{i+1}\to X_i$ possible, thus the isogeny correspondence is generically $1-1$, and because of $\alpha_p^D\to X_i^t\to X_{i+1}^t$ it is birational); next we show E_{m-1} is a rational curve: let C be the isogeny

correspondence containing $X_{m-1} \to X_m$,

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ A_{2,p^{(m-1)}} \supset E_{m-1} & & E_m \subset A_{2,p^m} \\ a(X_{m-1})=1,\ \dim(E_{m-1})=1 & & \dim(E_m)=0,\ a(X_m)=2, \end{array}$$

note that E_m is a point, a point of C corresponds to an embedding $\alpha_p \hookrightarrow X_m$, thus it follows that C is birationally equivalent with $\mathbb{P}^1$, thus E_{m-1} is a rational curve; hence $E=E_0$ is a rational curve by what is said before, and the corollary is proved.

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Algebraische Eigenschaften der lokalen Ringe in den Spitzen der Hilbertschen Modulgruppen

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Einleitung

In dieser Arbeit wird die Untersuchung der Spitzen Hilbertscher Modulgruppen in mehr als zwei Variablen weitergeführt. Wir bauen auf der Arbeit [3] auf.

Im Vordergrund werden algebraische Eigenschaften der lokalen analytischen Ringe in den Spitzen stehen.

Wir wollen uns in der Einleitung damit begnügen, die Komplettierung dieser Ringe zu beschreiben.

Gegeben seien:

1) Ein total reeller algebraischer Zahlkörper L.

2) Ein Gitter $\mathfrak{t} \subset L$ vom Rang n.

3) Eine Untergruppe Λ von endlichem Index in der Gruppe aller total positiven Einheiten, welche auf $\mathfrak{t}$ operiert

$$\Lambda \cdot \mathfrak{t} \subset \mathfrak{t}.$$

4) Eine Gruppe G von Automorphismen von L, welche $\mathfrak{t}$ und Λ in sich überführt.

Diesen Daten ordnen wir einen Ring R zu.

Zunächst sei $\mathfrak{t}_+$ die additive Halbgruppe aller *total positiven Elemente* aus $\mathfrak{t}$ vereinigt mit 0.

Wir bilden dann den formalen Gruppenring $\mathbb{C}[[\mathfrak{t}_+]]$, welcher aus *allen* Abbildungen

$$f\colon\ \mathfrak{t}_+ \to \mathbb{C}$$

besteht. Das Produkt zweier Abbildungen ist durch

$$f \cdot f'(a) = \sum_{a'+a''=a} f(a') f'(a'')$$

erklärt. Diese Summe ist endlich! Auf dem Ring $\mathbb{C}[[\mathfrak{t}^+]]$ operieren die Gruppen Λ und G. Der Invariantenring sei

$$R = \mathbb{C}[[\mathfrak{t}_+]]^{\Lambda, G}.$$

Dieser Ring R ist ein *normaler lokaler vollständiger noetherscher Ring der Dimension* $n = [L : \mathbb{Q}]$.

Seine Tiefe (homologische Kodimension) kann berechnet werden. Es kommen nur die Werte 1, 2, 3, 4 vor. Im allgemeinen handelt es sich also um keine Cohen-Macauley-Ringe. Die Divisorenklassengruppe des Ringes R wird bestimmt. Sie ist endlich, wenn $L/\mathbb{Q}$ galoisch ist und wenn G die volle Galoisgruppe ist. Genau dann, wenn überdies G mit seiner Kommutatorgruppe übereinstimmt, kann man $\mathfrak{t}$ und Λ so konstruieren, daß R ein ZPE-Ring ist. Mit Hilfe der Invariantentheorie werden galoissche total reelle Körper L mit Galoisgruppe A_5 (alternierende Gruppe) konstruiert. Man erhält dann Beispiele von 60dimensionalen ZPE-Ringen der Tiefe drei.

Damit ist eine seit längerer Zeit offene Frage beantwortet, ob es ZPE-Ringe der Charakteristik 0 gibt, welche nicht Cohen-Macauley sind[1].

Wenn die Gruppe G nur aus der Identität besteht, ist der Ring starr im Sinne der Deformationstheorie analytischer Singularitäten ($n \geqq 3$ vorausgesetzt). Andere Beispiele starrer normaler Singularitäten, welche nicht Cohen-Macauley sind, scheinen nicht bekannt zu sein. Obwohl die obigen Resultate rein algebraisch formuliert sind, erfordert ihr Beweis analytische Hilfsmittel.

Die vorliegende Arbeit steht im Zusammenhang mit Untersuchungen von U. Christian.

Er hat sich mit der Frage nach den Automorphiefaktoren des Stabilisators einer Spitze (auch für allgemeine Gruppen) beschäftigt.

Es besteht ein enger Zusammenhang zwischen diesen Faktoren und der lokalen Divisorenklassengruppe (s. § 4).

§ 1. Gitter und Multiplikatoren

Ausgangspunkt unserer Untersuchung ist ein Gitter $\mathfrak{t}$ von maximalem Rang in $\mathbb{R}^n$.

$$\mathfrak{t} \subset \mathbb{R}^n; \quad \text{Rang } \mathfrak{t} = n.$$

Unter einem *Multiplikator* (von $\mathfrak{t}$) versteht man ein n-Tupel $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ von positiven reellen Zahlen mit der Eigenschaft

$$\varepsilon \mathfrak{t} = \mathfrak{t}.$$

(Das Produkt zweier Vektoren ist komponentenweise zu bilden

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = (a_1 \cdot b_1, \dots, a_n \cdot b_n).)$$

Für jeden Multiplikator ε gilt

$$\varepsilon_1 \cdot \dots \cdot \varepsilon_n = 1,$$

da $\mathfrak{t}$ maximalen Rang hat.

[1] Ein Charakteristik-p-Beispiel findet man im Ergebnisbericht von Fossum über Divisorenklassengruppen (Springer, 1973).

Es sei $\Lambda \subset \mathbb{R}^{\times n}$ eine diskrete Gruppe von Multiplikatoren. Mit Hilfe der Logarithmusabbildung transformiert man Λ in eine diskrete Gruppe

$$\log \Lambda \subset \mathbb{R}^n.$$

Diese ist in der Hyperebene

$$x_1 + \cdots + x_n = 0$$

enthalten. Daher ist Λ eine freie abelsche Gruppe vom Rang $\leqq n-1$.

Wir setzen im folgenden voraus, daß Λ Maximalrang hat.

$$\operatorname{Rang} \Lambda = n-1.$$

Konstruktion von $(\mathfrak{t}, \Lambda)$

Wir fassen im folgenden $\mathbb{C}^n$ als einen Ring auf. Zwei n-Tupel von Zahlen werden komponentenweise addiert und multipliziert. Man hat die Spur- und Normabbildungen:

$$S\colon\ \mathbb{C}^n \to \mathbb{C}, \qquad S z = z_1 + \cdots + z_n$$

$$N\colon\ \mathbb{C}^n \to \mathbb{C}, \qquad N z = z_1 \cdot \cdots \cdot z_n.$$

Wir betrachten nun einen total reellen algebraischen Zahlkörper vom Grad n. Es gibt n verschiedene Einbettungen von L in den Körper der reellen Zahlen

$$L \to \mathbb{R}, \qquad a \to a_\nu, \qquad 1 \leqq \nu \leqq n,$$

die wir zu einer Einbettung

$$L \to \mathbb{R}^n, \qquad a \to (a_1, \ldots, a_n)$$

zusammenfassen. Wir identifizieren der Einfachheit halber a mit $(a_1, \ldots, a_n)$. Damit ist L ein Unterring von $\mathbb{R}^n$. Dann ist der Ring $\mathfrak{t}$ der ganzen Zahlen in L ein solches Gitter vom Rang n in $\mathbb{R}^n$ und die Gruppe Λ der total positiven Einheiten eine Multiplikatorengruppe. Diese hat nach dem Dirichletschen Einheitensatz den Rang $n-1$. Ist allgemeiner $\mathfrak{t} \subset L$ irgendein Gitter vom Rang n, so existiert einer Untergruppe Λ von endlichem Index in der Gruppe aller total positiven Einheiten, welche auf $\mathfrak{t}$ operiert.

1.1. Bemerkung. *Es sei* $\mathfrak{t} \subset \mathbb{R}^n$ *ein Gitter und* Λ *eine diskrete Gruppe von Multiplikatoren*

$$\operatorname{Rang} \mathfrak{t} = n; \qquad \operatorname{Rang} \Lambda = n-1.$$

Der von den Multiplikatoren ε *erzeugte* $\mathbb{Q}$*-Vektorraum*

$$L = \mathbb{Q} \cdot \Lambda$$

ist ein total reeller Körper vom Grad n.

Es gilt

a) *Die Einheitengruppe von L enthält Λ als Untergruppe von endlichem Index.*

b) *Es existiert ein Vektor $a \in \mathbb{R}^{\times n}$ mit der Eigenschaft*

$$a \cdot \mathfrak{t} \subset L.$$

Beweis. Jedem Multiplikator $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ kann man das Polynom

$$P_\varepsilon(x) = (x - \varepsilon_1) \ldots (x - \varepsilon_n)$$

zuordnen. Dies ist das charakteristische Polynom der Abbildung

$$\mathbb{R}^n \to \mathbb{R}^n, \qquad a \to \varepsilon a.$$

Da das Gitter $\mathfrak{t}$ bei dieser Abbildung in sich überführt wird, ist das Polynom P_ε ganzzahlig. Seine Wurzeln $\varepsilon_1, \ldots, \varepsilon_n$ sind daher algebraische Zahlen. Ihr Produkt ergibt Eins, es handelt sich also um algebraische Einheiten. Das Polynom P_ε kann unter Umständen reduzibel sein. Dann gilt aber

$$\varepsilon_{j_1} \ldots \varepsilon_{j_\nu} = 1$$

für ein ν-Tupel

$$1 \leqq j_1 < \cdots < j_\nu \leqq n, \qquad 1 \leqq \nu \leqq n.$$

Das Gitter

$$\log \Lambda \subset H = \{a \in \mathbb{R}^n, S\,a = 0\}$$

kann nicht in der Vereinigung von endlich vielen echten Unterräumen aus H enthalten sein. Daher existiert ein Multiplikator ε, so daß das Polynom P_ε irreduzibel ist. Die Komponenten $\varepsilon_1, \ldots, \varepsilon_n$ bilden dann ein vollständiges System von konjugierten algebraischen Zahlen. Der Modul $\mathbb{Z}[\varepsilon]$ hat den Rang n.

Wir wählen nun irgendeinen Vektor

$$a = (a_1, \ldots, a_n) \in \mathbb{R}^n, \qquad a_\nu \neq 0 \quad \text{für } \nu = 1, \ldots, n$$

mit der Eigenschaft

$$(1, \ldots, 1) \in a \cdot \mathfrak{t}$$

aus. Dann gilt

$$\mathbb{Z}[\varepsilon] \subset \mathbb{Z} \cdot \Lambda \subset a \cdot \mathfrak{t}.$$

Da auch $a \cdot t$ ein Modul vom Rang n ist, existiert eine natürliche Zahl r mit der Eigenschaft

$$r\,a\,\mathfrak{t} \subset \mathbb{Z}[\varepsilon].$$

Hieraus folgt

$$\mathbb{Q}[\varepsilon] = \mathbb{Q} \cdot \Lambda = \mathbb{Q} \cdot a \cdot \mathfrak{t},$$

und Bemerkung 1.1 ist bewiesen.

Wir können hieraus einfache Folgerungen ziehen.

Wenn eine Komponente a_ν eines Gittervektors $a \in \mathfrak{t}$ verschwindet, so ist $a=(0, \ldots, 0)$.

Wenn eine Komponente ε_ν eines Multiplikators $\varepsilon \in \Lambda$ Eins ist, so gilt: $\varepsilon=(1, \ldots, 1)$.

Ein Vektor $a \in \mathbb{R}^n$ heißt *total positiv* – in Zeichen $a>0$ – wenn alle Komponenten positiv sind.

Entsprechend wird die Schreibweise

$$a \geqq 0 \Leftrightarrow a_1 \geqq 0, \ldots, a_n \geqq 0$$

verwendet.

Offenbar gilt für einen Vektor $a \in \mathfrak{t}$:

$$a \geqq 0 \Leftrightarrow a>0 \quad \text{oder} \quad a=0.$$

§ 2. Fourierreihen

Es sei

$$H^n = H \times \cdots \times H, \qquad H=\{z \in \mathbb{C}, \operatorname{Im} z>0\},$$

das kartesische Produkt von n oberen Halbebenen. Auf H^n sei eine Gruppe Γ von Transformationen der Form

$$z \to \varepsilon z + a$$

gegeben.

Die *Translationen* in Γ definieren eine Untergruppe von $\mathbb{R}^n$

$$\mathfrak{t}=\{a \in \mathbb{R}^n, z \to z+a \text{ in } \Gamma\}.$$

Wir setzen voraus, daß $\mathfrak{t}$ ein Gitter vom Rang n ist.

Wenn die Transformation $z \to \varepsilon z + a$ in Γ liegt, so ist offensichtlich ε ein Multiplikator von $\mathfrak{t}$.

Die Gesamtheit dieser Multiplikatoren bildet eine Gruppe

$$\Lambda=\{\varepsilon \in \mathbb{R}^{\times n}, z \to \varepsilon z + a_\varepsilon \text{ in } \Gamma \text{ für ein } a_\varepsilon \in \mathbb{R}^n\}.$$

(Der Vektor a_ε ist natürlich nur modulo t bestimmt.)

Wir setzen weiterhin voraus, daß Λ eine diskrete Gruppe vom Rang $n-1$ ist.

Die Gruppe Γ operiert auf H^n *diskontinuierlich und fixpunktfrei.*

Die Spitze ∞

Im folgenden ist zu beachten, daß die Hyperfläche $N y=1$ unter Γ stabil ist und daß ihr Bild im Quotientenraum H^n/Γ kompakt ist.

Dies gibt uns die Möglichkeit, den Raum H^n/Γ durch Hinzufügen eines Punktes ∞

$$X = H^n/\Gamma \cup \{\infty\}$$

so zu erweitern, daß folgende Eigenschaften erfüllt sind:

1) *X ist lokal kompakt.*
2) *H^n/Γ ist ein offener Unterraum.*
3) *Eine Umgebungsbasis von ∞ bilden die Mengen*

$$U_C/\Gamma \cup \{\infty\} \quad \textit{mit} \quad U_C = \{z \in H^n, N\,y > C\}.$$

2.1. Theorem. *Der Raum X trägt eine Struktur als normaler komplexer Raum, H^n/Γ ist eine offene analytische Untermannigfaltigkeit.*

Einen Beweis dieses Satzes findet man in [6].

Eine in einer Umgebung U der Spitze $\infty \in X$ definierte Funktion ist genau dann holomorph, wenn sie stetig ist und wenn sie in $U \cap H^n/\Gamma$ analytisch ist.

Den lokalen analytischen Ring in der Spitze ∞ bezeichnen wir mit

$$R = R(\Gamma) = \mathcal{O}_{X,\infty}.$$

Beschreibung von R mit Hilfe von Fourierreihen:

Jedes Element von R kann durch eine Γ-invariante holomorphe Funktion

$$f\colon\ U_C \to \mathbb{C}, \qquad C \text{ hinreichend groß}$$

repräsentiert werden. Eine solche Funktion ist insbesondere periodisch

$$f(z+a) = f(z) \qquad \text{für } a \in \mathfrak{t}$$

und kann daher in eine Fourierreihe entwickelt werden.

$$f(z) = \sum_{g \in \mathfrak{t}^0} a_g\, e(g\,z), \qquad e(\ldots) = e^{2\pi i S(\ldots)}.$$

Dabei ist

$$\mathfrak{t}^0 = \{g \in \mathbb{R}^n,\ S(g\,x) \in \mathbb{Z} \ \text{ für } x \in \mathfrak{t}\}$$

das zu $\mathfrak{t}$ duale Gitter.

Die Invarianz

$$f(\varepsilon\,z + a) = f(z)$$

bedeutet

$$a_{g\varepsilon} = a_g \cdot e(g\,a).$$

Die Funktion f ist genau dann in die Spitze ∞ stetig (und damit auch analytisch) fortsetzbar, wenn

$$a_g \neq 0 \Rightarrow g \geqq 0$$

gilt. Dies ist im Falle $n \geqq 2$ automatisch der Fall nach dem *Götzky-Koecher-Prinzip* oder nach dem *Riemannschen Hebbarkeitssatz.*

Es ist gelegentlich zweckmäßig, die Konvergenzbedingung zu vernachlässigen und formale Fourierreihen

$$\sum_{g \in \mathfrak{t}^0,\, g \geqq 0} a_g\, e(g\, z) \quad \text{mit} \quad a_{g\varepsilon} = a_g\, e(g\, a)$$

zu betrachten. Diese bilden einen Ring $\hat{R}$. Die Summationsbedingung $g \geqq 0$ gestattet es nämlich, solche Reihen formal zu multiplizieren

$$\sum_{g'} a_{g'}\, e(g'\, z) \sum_{g''} b_{g''}\, e(g''\, z) = \sum_{g} c_g\, e(g\, z)$$

mit

$$c_g = \sum_{g'+g''=g} a_{g'}\, b_{g''}.$$

Die letzte Summe ist endlich.

2.2. Satz. *Der Ring $\hat{R}$ ist die Komplettierung von $R = \mathcal{O}_{X,\infty}$.*

Insbesondere ist $\hat{R}$ ein lokaler, vollständiger, noetherscher, normaler Ring der Dimension n.

Beweis. Die Menge der Spuren von Elementen aus $\mathfrak{t}$ bildet eine zyklische Gruppe

$$S(\mathfrak{t}) = \mathbb{Z} \cdot r_0, \qquad r_0 > 0.$$

Mit Hilfe der Spur definiert man gewisse Idealketten

$$\hat{\mathfrak{m}}_r = \{f \in R,\ a_g = 0 \text{ für } S\, g \leqq r\, r_0\}$$
$$\mathfrak{m}_r = R \cap \hat{\mathfrak{m}}_r,$$

offenbar gilt

a) $\mathfrak{m}_1 = \mathfrak{m} =$ maximales Ideal in R,

b) $\mathfrak{m}_1 \supset \mathfrak{m}_2 \supset \cdots$,

c) $\mathfrak{m}_r \cdot \mathfrak{m}_s \subset \mathfrak{m}_{r+s}$.

Diese Filtrierung von R ist insbesondere gröber als die m-adische Filtrierung, welche durch die Potenzen des maximalen Ideals definiert wird.

$$\mathfrak{m}^r \subset \mathfrak{m}_r \qquad \text{für } r = 1, 2, \ldots.$$

2. . Hilfssatz. *Der Ring $\hat{R}$ ist die Komplettierung von R in bezug auf die Topologie, die durch die Filtrierung $\{\mathfrak{m}_r\}$ definiert wird.*

(Satz 2.2 bezieht sich natürlich auf die $\mathfrak{m}$-adische Filtrierung.)

Beweis. Die „Poincaréreihen"

$$P_g(z) = \sum_{\varepsilon \in A} e(g\, \varepsilon\, z + g\, a_\varepsilon), \qquad g \in \mathfrak{t}^0,\ g \geqq 0$$

konvergieren in ganz H^n und sind Γ-invariant. Offenbar wird der Vektorraum $\hat{R}/\hat{\mathfrak{m}}_r$ von endlich vielen dieser Reihen erzeugt. Die kanonische Abbildung

$$R/\mathfrak{m}_r \to \hat{R}/\hat{\mathfrak{m}}_r$$

ist also ein Isomorphismus von endlich dimensionalen Vektorräumen.

Die Schwierigkeit beim Beweis von Satz 2.2 besteht also im Vergleich der beiden Filtrierungen $\{\mathfrak{m}_r\}$ und $\{\mathfrak{m}^r\}$. Wir müssen zeigen, daß sie dieselbe Topologie definieren. Dazu bezeichnen wir mit

$$\bar{R} = \varprojlim R/\mathfrak{m}^r$$

die m-adische Komplettierung von R. Durch stetige Fortsetzung erhält man einen Homomorphismus

$$\varphi\colon\ \bar{R} \to \hat{R}.$$

Mit Hilfe einer „Mittag-Leffler-Schlußweise“ zeigen wir nun, daß der Homomorphismus $\varphi\colon \bar{R} \to \hat{R}$ surjektiv ist.

Da $R/\mathfrak{m}^r$ endlich dimensional ist, kann man eine Folge von natürlichen Zahlen $r_1 < r_2 < \cdots$ mit der Eigenschaft

$$\mathfrak{m}_{r_1} \subset \mathfrak{m}^2 + \mathfrak{m}_{r_2}$$
$$\mathfrak{m}_{r_2} \subset \mathfrak{m}^3 + \mathfrak{m}_{r_3}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

finden.

Jedes Element $f \in \hat{R}$ läßt sich als Reihe der Form

$$f = \sum f_j; \quad f_j \in \mathfrak{m}_{r_j}$$

schreiben.

Man kann nun induktiv Folgen

$$a_j \in \mathfrak{m}^j; \quad g_j \in \mathfrak{m}_{r_j}$$

mit der Eigenschaft

$$f_1 + \cdots + f_{j-1} = a_1 + \cdots + a_j + g_j$$

konstruieren. Die Reihe $a_1 + a_2 + \cdots$ konvergiert in $\bar{R}$. Ihr Bild in $\hat{R}$ ist gerade f.

Damit ist gezeigt, daß φ surjektiv ist. Insbesondere ist $\hat{R}$ ein noetherscher vollständiger Ring.

Um zu zeigen, daß φ auch injektiv ist, benutzen wir den bekannten Satz, daß die Komplettierung eines nullteilerfreien analytischen Ringes nullteilerfrei ist.

Hieraus und aus der Formel

$$\dim \hat{R} = \dim \bar{R} + \text{Höhe (Kern } \varphi)$$

folgert man

$$\text{Kern}\ \varphi = 0 \Leftrightarrow \dim \hat{R} \geqq n.$$

Das *Hilbert-Samuelpolynom* P des Ringes $\hat{R}$ ist durch die Eigenschaft

$$P(r) = \dim \hat{R}/\hat{\mathfrak{m}}^r \quad \text{für hinreichend große } r$$

charakterisiert. Sein Grad stimmt mit der Dimension von $\hat{R}$ überein.

Es gilt

$$\dim \hat{R}/\hat{\mathfrak{m}}^r \geqq \dim \hat{R}/\hat{\mathfrak{m}}_r.$$

Daher genügt es zu zeigen, daß

$$\frac{1}{r^{n-1}} \dim \hat{R}/\hat{\mathfrak{m}}_r$$

unbeschränkt ist.

Nun ist $\dim \hat{R}/\hat{\mathfrak{m}}_r$ offenbar genau die Maximalzahl von nicht assoziierten Elementen

$$g \in \mathfrak{t}^0, \quad g \geqq 0, \quad S g < r r_0.$$

(Zwei Gitterelemente a, b heißen assoziiert, wenn es einen Multiplikator $\varepsilon \in \Lambda$ mit der Eigenschaft $\varepsilon a = b$ gibt.)

Eine einfache Abzählung von Gitterpunkten beendet den Beweis von Satz 2.2.

§ 3. Die Kohomologie der Gruppe Γ

Ordnet man einer Transformation $z \to \varepsilon z + a$ den Multiplikator ε zu, so erhält man einen Homomorphismus von Γ auf Λ, dessen Kern aus den Translationen besteht

$$0 \to \mathfrak{t} \to \Gamma \to \Lambda \to 1.$$

Wir werden im folgenden das Gitterelement a mit der Translation $z \to z + a$ identifizieren.

3.1. Bemerkung. *Die Kommutatorgruppe von Γ ist eine Untergruppe von endlichem Index von* $\mathfrak{t}$.

Beweis. Die Kommutatorgruppe von Γ ist offenbar in $\mathfrak{t}$ enthalten. Der Kommutator zweier Transformationen

$$z \to \varepsilon z + a_\varepsilon \quad \text{und} \quad z \to z + a$$

ist die Translation

$$z \to z + (\varepsilon - 1)\, a.$$

Bereits für einen Multiplikator $\varepsilon \neq 1$ ist $(\varepsilon - 1)\,\mathfrak{t} \subset \mathfrak{t}$ ein Untergitter vom Rang n und damit von endlichem Index.

Wir benötigen im folgenden die Kohomologiegruppen $H^{\cdot}(\Gamma, \mathbb{C})$. Dabei operiert Γ trivial auf $\mathbb{C}$.

3.2. Satz. *Die natürliche Abbildung*

$$H^r(\Lambda, \mathbb{C}) \to H^r(\Gamma, \mathbb{C})$$

ist im Falle $0 \leqq r < n$ *ein Isomorphismus.*

Der Beweis ergibt sich aus der Analyse der Spektralsequenz

$$H^p(\Lambda, H^q(t, \mathbb{C})) \Rightarrow H^r(\Gamma, \mathbb{C}), \qquad r = p + q.$$

Die Gruppe Γ operiert auf dem Vektorraum $\mathcal{O}(U_C)$ der holomorphen Funktionen

$$f\colon U_C \to \mathbb{C}, \qquad U_C = \{z \in H^n, N(\operatorname{Im} z) > C\}.$$

Die Kohomologiegruppen

$$H^{\cdot}(\Gamma, \mathcal{O}(U_C)) = H^{\cdot}(U_C/\Gamma, \mathcal{O})$$

wurden in [3] berechnet.

Die kanonischen Abbildungen

$$\mathbb{C} \hookrightarrow \mathcal{O}(U_C) \quad \text{und} \quad \Gamma \to \Lambda$$

induzieren Homomorphismen

$$H^r(\Lambda, \mathbb{C}) \to H^r(\Gamma, \mathcal{O}(U_C)).$$

Aus [3] übernehmen wir

3.3. Satz. *Die natürliche Abbildung*

$$H^r(\Lambda, \mathbb{C}) \to H^r(\Gamma, \mathcal{O}(U_C))$$

ist injektiv. Im Falle $0 < r < n - 1$ *ist sie sogar ein Isomorphismus.*

Wir betrachten nun gewisse Erweiterungen der Gruppe Γ, indem wir auch Permutationen der Variablen zulassen

$$\sigma(z_1, \dots, z_n) = (z_{\sigma(1)}, \dots, z_{\sigma(n)}).$$

Es sei $\hat{\Gamma}$ eine Gruppe von Transformationen

$$z \to \varepsilon \cdot \sigma(z) + a.$$

Ordnet man jeder dieser Transformationen die Permutation σ zu, so erhält man einen Homomorphismus auf eine gewisse Untergruppe $G \subset S_n$ der symmetrischen Gruppe.

Der Kern dieses Homomorphismus sei genau die oben untersuchte Gruppe Γ. Wir haben also die exakte Sequenz

$$1 \to \Gamma \to \hat{\Gamma} \to G \to 1.$$

Die Gruppe G operiert in bekannter Weise auf Γ. Bei dieser Operation bleibt $\mathfrak{t}$ stabil und ist daher ein G-Modul. Ebenfalls ist $\Lambda = \Gamma/\mathfrak{t}$ ein G-Modul.

Aufgrund von Bemerkung 1.1 hat man eine Einbettung von G in die Automorphismengruppe des Körpers L

$$G \hookrightarrow G(L/\mathbb{Q}).$$

Daher ist die Ordnung von G nicht größer als n

$$|G| \leqq n.$$

Wenn die Gruppe $\hat{\Gamma}$ auf einem $\mathbb{C}$-Vektorraum M linear operiert, so erhält man die Kohomologie $H^{\cdot}(\hat{\Gamma}, M)$ aus $H^{\cdot}(\Gamma, M)$ durch Invariantenbildung:

$$H^r(\hat{\Gamma}, M) = H^r(\Gamma, M)^G.$$

Es ist zu berücksichtigen, daß die höheren Kohomologiegruppen einer endlichen Gruppe, welche auf einem $\mathbb{C}$-Vektorraum linear operiert, verschwinden.

Dieses Prinzip kann man auf die Moduln $M = \mathbb{C}$ und $\mathcal{O}(U_C)$ anwenden.

(Wenn die Substitution $z \to \varepsilon\sigma(z) + a$ in $\hat{\Gamma}$ liegt, so ist eine gewisse Potenz schon in Γ enthalten. Hieraus folgert man $N\varepsilon = 1$. Die Gruppe $\hat{\Gamma}$ operiert also auf U_C.)

Wir ziehen aus den Sätzen 3.2 und 3.3 einige Folgerungen:

Man hat im Falle $n \geqq 3$ aufgrund der Sätze 3.2 und 3.3 natürliche Isomorphismen.

$$\operatorname{Hom}(\Gamma, \mathbb{C}) = H^1(\Gamma, \mathbb{C}) = H^1(\Lambda, \mathbb{C}) = H^1(\Gamma, \mathcal{O}(U_C)).$$

Durch Invariantenbildung erhält man

3.4. Bemerkung. *Im Falle $n \geqq 3$ existiert ein natürlicher Isomorphismus*

$$\operatorname{Hom}(\hat{\Gamma}, \mathbb{C}) = H^1(\hat{\Gamma}, \mathcal{O}(U_C)).$$

In § 1 wurde der lokale Ring

$$R(\Gamma) = \mathcal{O}_{X_C, \infty}; \qquad X_C = U_C/\Gamma \cup \{\infty\}$$

eingeführt. Auf diesem operiert die endliche Gruppe G und wir können

$$R(\hat{\Gamma}) = R(\Gamma)^G$$

betrachten.

Dies ist der lokale Ring des Raumes

$$U_C/\hat{\Gamma} \cup \infty = X_C/G$$

im Punkt ∞.

Wir wollen nun die Tiefe τ des lokalen Ringes $R(\hat{\Gamma})$ ausrechnen.

Unter der Tiefe eines lokalen noetherschen Ringes R versteht man die Länge einer maximalen Nichtnullteilerfolge

$$a_0, \ldots, a_\tau \in \mathfrak{m}(R).$$

(Das Bild von a_j in $R/(a_1, \ldots, a_{j-1})$ ist Nichtnullteiler für $0 \leqq j \leqq \tau$.)

Ist X ein komplexer Raum, $x \in X$, so kann man die Tiefe τ des lokalen Ringes

$$R = \mathcal{O}_{X,x}$$

folgendermaßen mit Hilfe der Kohomologie mit Träger in x charakterisieren

$$\begin{aligned} H^r_{\{x\}}(X, \mathcal{O}_X) &= 0 \quad \text{für } r < \tau \\ &\neq 0 \quad \text{für } r = \tau \ [8]. \end{aligned}$$

Im Falle $R = R(\hat{\Gamma})$, $n \geqq 2$, erhält man hieraus die folgende Beschreibung der Tiefe

$$\begin{aligned} H^r(\hat{\Gamma}, \mathcal{O}(U_C)) &= 0 \quad \text{für } 0 < r \leqq \tau - 2 \\ &\neq 0 \quad \text{für } r = \tau - 1. \end{aligned}$$

3.5. Theorem. *Es sei* $n \geqq 3$.

1) *Die Gruppe G habe nicht die Ordnung n. Dann ist die Tiefe von $R(\hat{\Gamma})$ zwei.*

2) *Die Gruppe G habe die maximale Ordnung n. Dann ist die Tiefe von $R(\hat{\Gamma})$ drei, mit Ausnahme des Falles*

$$G = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2.$$

In diesem Falle ist die Tiefe 4.

3.6. Folgerung. *Im Falle $n > 4$ ist $R(\hat{\Gamma})$ niemals ein Cohen-Macauley-Ring.*

(Ein lokaler noetherscher Ring R heißt Cohen-Macauley-Ring, wenn die Tiefe gleich der Dimension ist.)

Beweis von Theorem 3.5. Die Tiefe ist genau dann größer als zwei, wenn

$$\mathrm{Hom}(\hat{\Gamma}, \mathbb{C}) = H^1(\hat{\Gamma}, \mathcal{O}(U_C))$$

verschwindet. Nun ist

$$\mathrm{Hom}(\hat{\Gamma}, \mathbb{C}) = \mathrm{Hom}(\Gamma, \mathbb{C})^G = \mathrm{Hom}(\mathfrak{t}, \mathbb{C})^G.$$

Nach Wahl einer Basis von $\mathfrak{t}$ erhält man einen Isomorphismus

$$\mathrm{Hom}(\mathfrak{t}, \mathbb{C})^G = \mathfrak{t}^G \otimes \mathbb{C}.$$

Wenn ein G-invarianter Multiplikator $\varepsilon \neq 1$ existiert, so kann G nicht eine volle Galoisgruppe sein.

Ist umgekehrt die Ordnung von G kleiner als n, so ist der Körper L^G total reell $\neq \mathbb{Q}$. Es existiert daher eine total positive Einheit $\varepsilon \neq 1$ in L^G. Eine Potenz von ε liegt in Λ^G.

Die Ordnung von G sei jetzt $n(\geqq 3)$. Dann ist die Tiefe von $R(\hat{\Gamma})$ größer als zwei. Im Falle $n=3$ ist also $R(\hat{\Gamma})$ ein Cohen-Macauleyring. Wir können daher $n \geqq 4$ annehmen.

Wir haben zu untersuchen, wann die Gruppe

$$H^2(\hat{\Gamma}, \mathcal{O}(U_C)) = H^2(\Lambda, \mathbb{C})^G$$

verschwindet.

Die Kohomologie einer freien abelschen Gruppe ist bekannt

$$H^2(\mathrm{t}, \mathbb{C}) = \Lambda^2 \operatorname{Hom}(\Lambda, \mathbb{C}).$$

Den G-Modul $\operatorname{Hom}(\Lambda, \mathbb{C})$ kann man mit Hilfe der regulären Darstellung $\mathbb{Q}[G]$ beschreiben. Bekanntlich zerfällt die Gruppenalgebra in eine direkte Summe von zwei G-Moduln

$$\mathbb{Q}[G] = J(G) + \mathbb{Q},$$

wobei G auf $\mathbb{Q}$ trivial operiert.

Man kann zeigen, daß die beiden G-Moduln

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{und} \quad J(G) \otimes_{\mathbb{Q}} \mathbb{R}$$

isomorph sind.

Dazu nur folgende Bemerkungen:

1) Die Gruppe G operiert auf $L \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^n$ durch Permutation der Variablen. Dieser G-Modul ist isomorph zum Gruppenring $\mathbb{R}[G]$.

2) Der Untermodul

$$M = \{x \in \mathbb{R}^n, S x = 0\}$$

ist isomorph zu $J(G) \otimes_{\mathbb{Q}} \mathbb{R}$.

3) Die Logarithmusabbildung

$$\log\colon \Lambda \to \mathbb{R}^n$$

ist mit der Operation von G verträglich.

Der Vektorraum $H^2(\mathrm{t}, \mathbb{C})^G$ ist also isomorph zu $(\Lambda^2 J(G) \otimes_{\mathbb{Q}} \mathbb{C})^G$. Ob dieser Raum verschwindet, hängt nur von der abstrakten Gruppe G ab.

Bemerkung. *Ist G eine endliche Gruppe der Ordnung $n \geqq 4$, so gilt*

$$(\Lambda^2 J(G) \otimes_{\mathbb{Q}} \mathbb{C})^G = 0 \Leftrightarrow G = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2.$$

Beweis. Seien a, b nur Elemente aus G. Die Summe

$$\sum_{g\in G} g\,a \wedge g\,b$$

ist invariant unter G.

Sie verschwindet offenbar genau dann, wenn

$$a\,b^{-1} = (a\,b^{-1})^{-1}$$

gilt. Wenn diese Relation für alle $(a, b) \in G \times G$ erfüllt ist, so muß G vom Typ $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$ sein.

In diesem Falle kann man die Operation von G auf $\Lambda^2 J(G) \otimes_{\mathbb{Q}} \mathbb{C}$ leicht überblicken.

Eine einfache Überlegung zeigt, daß die Gruppe $(\Lambda^r J(G) \otimes_{\mathbb{Q}} \mathbb{C})^G$ im Falle $r=2$ tatsächlich verschwindet. Sie verschwindet aber nicht im Falle $r=3$, wenn G mehr als zwei Faktoren vom Typ $\mathbb{Z}/2$ enthält.

Damit ist Theorem 3.5 bewiesen.

§4. Die lokale Divisorenklassengruppe

Sei X ein normaler komplexer Raum, X^0 sein regulärer Ort. Im folgenden verwenden wir die folgende modifizierte Picardgruppe.

Pic X *sei die Gruppe der Isomorphieklassen analytischer Geradenbündel auf* X^0, *welche sich auf ganz* X *als kohärente Garben fortsetzen lassen.*

Wenn die Kodimension des singulären Ortes größer oder gleich drei ist, so gilt

$$\operatorname{Pic} X = \operatorname{Pic} X^0.$$

Im folgenden sei

$$X_C = U_C/\hat{\Gamma},$$
$$X_C^0 = \text{regulärer Ort von } X_C,$$
$$U_C^0 = \text{Urbild von } X_C^0 \text{ in } U_C.$$

4.1. Bemerkung. *Im Falle* $n \geqq 3$ *operiert* $\hat{\Gamma}$ *frei auf* U_C^0.

Beweis. Wenn die Abbildung $U_C^0 \to X_C^0$ überhaupt verzweigt ist, so hat der Verzweigungsort die genaue Kodimension Eins. Daher genügt es zu zeigen, daß im Falle $n \geqq 3$ keine Spiegelung

$$z \to \varepsilon\,\sigma(z) + a$$

enthalten ist. Wenn diese Substitution eine Fixpunktmannigfaltigkeit der Kodimension eins haben soll, so muß σ eine Transposition sein. Im Falle $n \geqq 3$ kann aber niemals eine Transposition in der Automorphismengruppe des Körpers L liegen. Für $a \in L$ würden nämlich gewisse Komponenten von $a - \sigma(a)$ verschwinden. Hieraus folgt aber schon $a = \sigma(a)$.

Im folgenden sei stets

$$n \geqq 3.$$

4.2. Bemerkung. *Wenn die Gruppe* Pic X_C *für jedes* $C>0$ *nur aus dem trivialen Geradenbündel besteht, so ist der lokale Ring* $R=\mathcal{O}_{X_C,\infty}$ *ein ZPE-Ring.*

Beweis. Ein beliebiges Element $f \in R$ kann als holomorphe Funktion

$$f\colon X_C \to \mathbb{C}, \qquad C \text{ hinreichend groß},$$

aufgefaßt werden. Den Nullstellendivisor von f kann man in Primdivisoren zerlegen

$$(f)=n_1 P_1+\cdots+n_r P_r.$$

Da X_C^0 singularitätenfrei ist, kann man jedem Divisor D auf X_C ein Geradenbündel aus Pic X_C zuordnen. Wenn dieses trivial ist, existiert eine holomorphe Funktion mit genauem Nullstellendivisor D.

In unserem Falle kann man also holomorphe Funktionen

$$f_1,\ldots,f_r \quad \text{mit} \quad (f_\nu)=P_\nu \qquad \text{für } \nu=1,\ldots,r$$

finden.

Es gilt

$$f=h\, f_1^{n_1}\ldots f_r^{n_r}$$

mit einer holomorphen invertierbaren Funktion h. Dies ist die Primfaktorzerlegung von f.

Man kann Bemerkung 4.2 auch anders interpretieren.

In der kommutativen Algebra wird jedem normalen noetherschen Ring R die Gruppe der Divisoren zugeordnet.

Ein Divisor ist ein Element der von den Primidealen der Höhe 1 erzeugten freien abelschen Gruppe. Jedem von Null verschiedenen Element des Quotientenkörpers wird ein „Hauptdivisor" zugeordnet. Die Gruppe der *Divisorenklassen* ist genau dann trivial, wenn R *ZPE*-Ring ist.

Bemerkung. *Im Falle* $n \geqq 3$ *ist die Divisorenklassengruppe des lokalen Ringes* $R(\hat{\Gamma})$ *isomorph zu* $\varinjlim$ Pic X_C^0.

Manchmal ist es nützlich, die Geradenbündel durch Automorphiefaktoren zu beschreiben. Unter einem solchen Faktor versteht man eine Abbildung

$$J\colon U_C \times \Gamma \to \mathbb{C}$$

mit folgenden Eigenschaften

a) $J(z,\gamma)$ ist als Funktion von z holomorph,

b) $J(z,\gamma\gamma')=J(z,\gamma')\,J(\gamma' z,\gamma)$.

Ein Automorphiefaktor ist also nichts anderes als ein 1-Kozykel der Gruppe $\hat{\Gamma}$ im Modul

$$A^* = H^0(U_C, \mathcal{O}^*)$$

der holomorphen invertierbaren Funktionen auf U_C. Jedem Automorphiefaktor J wird eine kohärente Garbe zugeordnet. Ein Schnitt über einer offenen Menge $V \subset X_C$ ist eine holomorphe Funktion

$$f\colon U \to \mathbb{C}, \qquad U = \text{Urbild von } V \text{ in } U_C,$$

mit der Eigenschaft

$$f(z) = J(z, \gamma) f(\gamma z).$$

Die Einschränkung auf X_C^0 ist ein Geradenbündel $L_J \in \operatorname{Pic} X_C$. Dieses Bündel ist genau dann trivial, wenn J ein Korand ist.

$$J(z, \gamma) = \frac{h(\gamma z)}{h(z)}, \qquad h \in \mathcal{O}^*(U_C).$$

4.3. Bemerkung. *Im Falle $n \geqq 3$ ist die natürliche Abbildung*

$$H^1(\hat{\Gamma}, A^*) \to \operatorname{Pic} X_C^0, \qquad A^* = H^0(U_C, \mathcal{O}^*)$$

ein Isomorphismus.

Beweis. Sei L ein analytisches Geradenbündel aus $\operatorname{Pic} X_C$. Sein reziprokes Bild $p^* L$ in U_C^0 kann zu einem Geradenbündel auf ganz U_C fortgesetzt werden, da L kohärent auf X_C fortsetzbar ist. Auf U_C ist jedes Geradenbündel trivial, denn U_C ist Steinsch und zusammenziehbar.

Es existiert daher ein globaler nirgends verschwindender Schnitt s von $p^* L$.

Der Automorphiefaktor $J(\cdot, \gamma)$ wird nun durch

$$J(\cdot, \gamma) = \gamma(s) \otimes s^{-1}$$

definiert.

Spezielle Automorphiefaktoren sind die Gruppenhomomorphismen

$$v\colon \hat{\Gamma} \to \mathbb{C}^*, \qquad J(z, \gamma) = v(\gamma).$$

Damit ist insbesondere jedem solchen Gruppenhomomorphismus ein Geradenbündel auf X_C^0 zugeordnet.

4.4. Theorem. *Im Falle $n \geqq 3$ ist die natürliche Abbildung*

$$\operatorname{Hom}(\hat{\Gamma}, \mathbb{C}^*) \to \operatorname{Pic} X_C$$

ein Isomorphismus.

Beweis. Wir setzen

$$A = H^0(U_C, \mathcal{O})$$
$$A^* = H^0(U_C, \mathcal{O}^*).$$

Aus der kurzen exakten Sequenz

$$0 \to \mathbb{Z} \to A \xrightarrow{\exp} A^* \to 0$$

resultiert eine exakte Kohomologiesequenz

$$H^1(\hat{\Gamma}, A) \to H^1(\hat{\Gamma}, A^*) \to H^2(\hat{\Gamma}, \mathbb{Z}) \to H^2(\hat{\Gamma}, A).$$

Nach 3.4 hat man einen natürlichen Isomorphismus

$$H^1(\hat{\Gamma}, A) \to \operatorname{Hom}(\hat{\Gamma}, \mathbb{C}).$$

Mit Hilfe der Exponentionalabbildung erhält man einen Homomorphismus

$$\operatorname{Hom}(\hat{\Gamma}, \mathbb{C}) \to \operatorname{Hom}(\hat{\Gamma}, \mathbb{C}^*).$$

Es ist leicht nachzurechnen, daß das Diagramm

$$\begin{array}{ccc} H^1(\hat{\Gamma}, A) & \longrightarrow & \operatorname{Pic} X_C^0 = H^1(\hat{\Gamma}, A^*) \\ \| \wr & & \uparrow \\ \operatorname{Hom}(\hat{\Gamma}, \mathbb{C}) & \longrightarrow & \operatorname{Hom}(\hat{\Gamma}, \mathbb{C}^*) \end{array}$$

kommutativ ist.

Das Bild von $H^1(\hat{\Gamma}, A)$ *in* Pic X_C^0 *besteht also nur aus Geradenbündeln der Form* L_v, $v \in \operatorname{Hom}(\hat{\Gamma}, \mathbb{C}^*)$.

Als nächstes untersuchen wir das Bild von $H^1(\hat{\Gamma}, A^*)$ in $H^2(\hat{\Gamma}, \mathbb{Z})$, also den Kern von

$$H^2(\hat{\Gamma}, \mathbb{Z}) \to H^2(\hat{\Gamma}, A).$$

Dieser Kern ist endlich, denn aufgrund der Sätze 3.2 und 3.3 ist die Abbildung

$$H^2(\Gamma, \mathbb{C}) \to H^2 \Gamma, A)$$

injektiv. Hieraus folgt, daß auch

$$H^2(\hat{\Gamma}, \mathbb{C}) \to H^2(\hat{\Gamma}, A)$$

injektiv ist.

Das Bild von $H^1(\hat{\Gamma}, A)$ *in* Pic X_C *ist also eine Untergruppe von endlichem Index.*

Insbesondere ist eine geeignete Potenz L^r eines Geradenbündels $L \in \operatorname{Pic} X_C^0$ im Bild der Abbildung

$$\operatorname{Hom}(\hat{\Gamma}, \mathbb{C}^*) \to \operatorname{Pic} X_C$$

enthalten.

Dies kann man auch folgendermaßen ausdrücken:

Zu jedem Automorphiefaktor J existiert eine natürliche Zahl r, ein Homomorphismus $v: \hat{\Gamma} \to \mathbb{C}^*$ und eine invertierbare holomorphe Funktion $h: U_C \to \mathbb{C}$ mit der Eigenschaft

$$J(z, \gamma)^r = v(\gamma) \cdot \frac{h(\gamma z)}{h(z)}.$$

Es gibt eine holomorphe Funktion

$$h_0: U_C \to \mathbb{C}^* \quad \text{mit} \quad h_0^r = h.$$

Wir erhalten

$$J_0(z, \gamma)^r = v(\gamma) \quad \text{mit} \quad J_0(z, \gamma) = J(z, \gamma) \frac{h_0(z)}{h_0(\gamma z)}.$$

Die Funktion $J_0(z, \gamma)$ ist bei festem γ konstant. Der Automorphiefaktor J ist daher zu einem Homomorphismus

$$v_0: \hat{\Gamma} \to \mathbb{C}, \qquad v_0(\gamma) = J_0(z, \gamma)$$

äquivalent.

Damit ist bewiesen, daß die Abbildung

$$\operatorname{Hom}(\hat{\Gamma}, \mathbb{C}^*) \to \operatorname{Pic} X_C$$

surjektiv ist.

Wir müssen noch zeigen, daß sie auch injektiv ist.

Sei $v: \hat{\Gamma} \to \mathbb{C}^*$ ein Homomorphismus, so daß das assoziierte Geradenbündel trivial ist. Es existiert dann eine holomorphe invertierbare Funktion

$$h: U_C \to \mathbb{C}^* \quad \text{mit} \quad h(z) = v(\gamma) h(\gamma z).$$

Nach Bemerkung 3.1 ist die Funktion h periodisch bei einem Untergitter $\tilde{\mathfrak{t}} \subset \mathfrak{t}$ von endlichem Index. Man kann daher h in eine Fourierreihe entwickeln

$$h(z) = \sum_{g \in \tilde{\mathfrak{t}}^0} a_g e(gz).$$

Eine Variante des Götzky-Koecher-Prinzips besagt [3]

$$a_g \neq 0 \Rightarrow g \geqq 0.$$

Auf die Funktion $1/h$ kann man dieselbe Überlegung anwenden. Es folgt $a_0 \neq 0$.

Andererseits ist

$$a_0 = v(\gamma) \cdot a_0 \qquad \text{für } \gamma \in \hat{\Gamma}.$$

Daher ist

$$v(\gamma) = 1 \qquad \text{für alle } \gamma \in \hat{\Gamma}.$$

§5. Die Konstruktion einiger Gruppen $\hat{\Gamma}$, für die $R(\hat{\Gamma})$ ZPE-Ring ist

Gegeben sei eine endliche Gruppe G. Wir bezeichnen mit $[G, G]$ die Kommutatoruntergruppe von G, mit $\mathbb{Z}[G]$ die Gruppenalgebra von G über dem Ring $\mathbb{Z}$ der ganzen Zahlen und mit J folgendes $\mathbb{Z}[G]$-Ideal:

$$J = \sum_{\delta \in G} \mathbb{Z}(\delta - 1).$$

Aus der homologischen Algebra benötigen wir den bekannten

5.1. Hilfssatz. *Die abelschen Gruppen $G/[G, G]$ und J/J^2 sind isomorph.*

Beweis. Es genügt folgendes zu zeigen:

Für jede abelsche Gruppe A ist die abelsche Gruppe der Homomorphismen von $G/[G, G]$ in A isomorph zur abelschen Gruppe der Homomorphismen von J/J^2 in A. Äquivalent dazu ist:

Die Gruppe der Homomorphismen von G in A ist isomorph zur Gruppe der Homomorphismen von J in A, die

$$J^2 \cong \sum_{\delta, \tau \in G} \mathbb{Z}(\delta - 1)(\tau - 1)$$

annullieren.

Man hat eine natürliche Bijektion

$$\chi \leftrightarrow \tilde{\chi}$$

zwischen der Menge aller Abbildungen

$$\chi\colon G \to A, \qquad \chi(1) = 0,$$

und der Menge aller $\mathbb{Z}$-linearen Abbildungen

$$\tilde{\chi}\colon J \to A$$

mit

$$\tilde{\chi}(\delta - 1) = \chi(\delta).$$

Für δ, τ aus G gilt:

$$(\delta - 1) + (\tau - 1) - (\delta\tau - 1) = -(\delta - 1)(\tau - 1),$$

damit

$$\chi(\delta) + \chi(\tau) - \chi(\delta\tau) = -\tilde{\chi}((\delta - 1)(\tau - 1)).$$

Es ist also $\chi(\delta\tau) = \chi(\delta)\chi(\tau)$ genau dann, wenn $\tilde{\chi}((\delta - 1)(\tau - 1))$ verschwindet.

5.2. Folgerung. *M sei ein beliebiger G-Modul,* $P = JM = \sum_{\delta \in G} (\delta - 1) M$. *Wenn die Gruppe G mit ihrer Kommutatorgruppe* $[G, G]$ *übereinstimmt, so gilt*

$$P = JP.$$

Es sei L ein total reeller galoischer Zahlkörper vom Grad $n > 1$ mit der Galoisgruppe G. E sei eine Gruppe total positiver Einheiten, von endlichem Index in der Gruppe aller Einheiten von L. Nach dem Dirichletschen Einheitensatz ist dann E eine endlich erzeugte freie abelsche Gruppe vom Rang $n-1$.

5.3. Hilfssatz. *Die Gruppe total positiver Einheiten* $\Lambda = \prod_{\delta \in G} (\delta - 1) E$ *ist G-stabil und hat einen endlichen Index in der Gruppe aller Einheiten. Damit ist* Λ *ebenfalls eine freie abelsche Gruppe vom Rang* $n-1$.

Beweis. Die Norm $N\varepsilon = \prod_{\delta \in G} \delta\varepsilon$ jeder Einheit ε aus E ist Eins, damit $\varepsilon^{-n} = \prod_{\delta \in G} \delta\varepsilon/\varepsilon = \prod_{\delta \in G} (\delta - 1)\varepsilon$ in Λ enthalten. Also ist E^n in Λ enthalten!

5.4. Hilfssatz. *Gegeben sei ein Gitter M in L, d.h. ein endlicher* $\mathbb{Z}$*-Untermodul vom Rang n in L.* Λ *sei eine G-stabile total positive Untergruppe von endlichem Index in der Gruppe aller Einheiten von L. Wir setzen:*

$$N = \sum_{\delta \in G} (\delta - 1) M, \qquad P = \sum_{\varepsilon \in E} \varepsilon N.$$

Die Gruppe G stimme mit ihrer Kommutatoruntergruppe $[G, G]$ *überein. Dann ist der* $G-\Lambda$*-Modul P ein Gitter und es gilt:*

$$P = \sum_{\delta \in G} (\delta - 1) P + \sum_{\varepsilon \in \Lambda} (\varepsilon - 1) P.$$

Beweis. Wegen Hilfssatz 5.2 gilt:

$$N = \sum_{\delta \in G} (\delta - 1) N.$$

Für eine Einheit f aus Λ gilt:

$$\begin{aligned} fN \subseteq N + (f-1) N &\subseteq \sum_{\delta \in G} (\delta - 1) N + \sum_{\varepsilon \in \Lambda} (\varepsilon - 1) N \\ &\subseteq \sum_{\delta \in G} (\delta - 1) P + \sum_{\varepsilon \in \Lambda} (\varepsilon - 1) P. \end{aligned}$$

Es bleibt noch zu zeigen, daß P ein Gitter ist.

N ist ein endlicher $\mathbb{Z}$-Modul. Weil M nicht im Körper $\mathbb{Q} = K^G$ der rationalen Zahlen enthalten ist, muß N von Null verschieden sein.

$\mathfrak{o} = \sum_{\varepsilon \in \Lambda} \mathbb{Z}\,\varepsilon$ ist ein Ring ganzer Größen in L. Sein Quotientenkörper K enthält $n-1$ unabhängige Einheiten. Nach dem Dirichletschen Einheitensatz muß der Grad von L mindestens n sein. Also stimmen K und L überein und $\mathfrak{o}$ ist eine Ordnung von L. P ist ein (vielleicht gebrochenes) von Null verschiedenes $\mathfrak{o}$-Ideal!

5.5. Lemma. *Gegeben sei ein total reeller galoisscher Zahlkörper vom Grad $n \geqq 2$ über dem Körper $\mathbb{Q}$ der rationalen Zahlen und G seine Galoisgruppe. Die Gruppe G stimme mit ihrer Kommutatoruntergruppe $[G, G]$ überein, sei also z.B. eine einfache Gruppe.*

Dann gibt es eine G-stabile Gruppe Λ total positiver Einheiten von L vom Rang $n-1$ und ein $\Lambda - G$-stabiles Gitter $\mathfrak{A}$ in L mit folgenden Eigenschaften:

$$\Lambda = \prod_{\delta \in G} (\delta - 1)\,\Lambda$$

$$\mathfrak{A} = \sum_{\delta \in G} (\delta - 1)\,\mathfrak{A} + \sum_{\varepsilon \in \Lambda} (\varepsilon - 1)\,\mathfrak{A}.$$

Sei Γ das semidirekte Produkt von $\mathfrak{A}$ und Λ, $\hat{\Gamma}$ das semidirekte Produkt von Γ und G

$$\Gamma = \mathfrak{A} \times \Lambda, \qquad \hat{\Gamma} = \Gamma \times G.$$

Dann stimmt die Gruppe $\hat{\Gamma}$ mit ihrer Kommutatoruntergruppe $[\hat{\Gamma}, \hat{\Gamma}]$ überein.

Anhang. Die Existenz total reeller galoisscher Zahlkörper mit einfacher Galoisgruppe

Über die Existenz galoisscher Zahlkörper mit vorgegebener Galoisgruppe ist sehr wenig bekannt. Nach Hilbert [5] gibt es galoissche Zahlkörper zu allen alternierenden Gruppen. Leider sind diese Hilbertschen Beispiele nicht total reell. Es soll daher die Konstruktion einiger Beispiele total reeller Zahlkörper mit einfacher Galoisgruppe kurz skizziert werden.

Gegeben sei eine rein transzendente endlich erzeugte Körpererweiterung $K = \mathbb{Q}(t_1, \ldots, t_n)$ des Körpers $\mathbb{Q}$ der rationalen Zahlen erzeugt durch die algebraisch unabhängigen Elemente $t_1, \ldots, t_n$ und eine endliche galoische Körpererweiterung L von K mit der Galoisgruppe G. $K = L^G$.

Sei $A = \mathbb{Z}[t_1, \ldots, t_n] = \mathbb{Z}[t]$; der ganze Abschluß B von A in L ist dann endlicher A-Modul. Es gilt: $A = B^G$.

Gegeben sei ein Punkt $a = (a_1, \ldots, a_n)$ aus dem Zahlraum $\mathbb{R}^n$; $\mathbb{Q}(a)$ sei der von den Elementen $a_1, \ldots, a_n$ über $\mathbb{Q}$ erzeugte Körper. Man hat einen natürlichen Homomorphismus

$$\mathbb{Z}[t] \to \mathbb{Q}(a)$$
$$t_i \mapsto a_i$$

und $\mathbb{Q}(a)$ ist über diesen Homomorphismus ein A-Modul. Wir setzen:

$$L(a) = B \otimes_{\mathbb{Z}[t]} \mathbb{Q}(a).$$

Die Gruppe G operiert auf $L(a)$. Hilberts Irreduzibilitätssatz [5] sagt aus, daß die Menge H derjenigen $a \in \mathbb{Q}^n$, für die $K(a)$ ein Körper, damit $K(a)$ eine galoissche Erweiterung von $\mathbb{Q}$ mit Galoisgruppe G ist, dicht im Raum $\mathbb{R}^n$ liegt.

Die Menge U derjenigen Elemente $a \in \mathbb{R}^n$, für die $L(a)$ ein Produkt total reeller Körper ist (insbesondere ist dann der Kern des natürlichen Homomorphismus $\mathbb{Z}[t] \to \mathbb{Q}(a)$ unverzweigt in B), ist offen. Wir wollen annehmen, daß U nicht leer ist. Diese Bedingung ist immer erfüllt, wenn L über $\mathbb{Q}$ rein transzendent ist. Dann ist $H \cap U$ nicht leer. Es gibt also total reelle galoissche Zahlkörper $L(a)$ mit Galoisgruppe G.

Sei $\mathbb{Q}$ algebraisch abgeschlossen in L. Dann ist L immer verzweigt über $\mathbb{Q}[t]$: Es gibt keine unverzweigten Überlagerungen des affinen Raumes über einem algebraisch abgeschlossenen Körper der Charakteristik Null. Wir wollen unter diesen Voraussetzungen zeigen:

Es gibt unendlich viele total reelle galoissche Zahlkörper mit Galoisgruppe G, die untereinander nicht isomorph sind.

Der Beweis wird zunächst mit Hilfe von Hilberts Irreduzibilitätssatz auf den Fall $n=1$ zurückgeführt.

Wir wollen also annehmen, daß $n=1$ ist. Nach einer geeigneten Substitution $t \to \dfrac{1}{t-\alpha}$ kann man annehmen, daß alle genügend großen reellen Zahlen a, insbesondere alle natürlichen Zahlen $n \geqq n_0$ in U liegen.

$\mathbb{Q}[t]$ ist verzweigt in L. Die Diskriminante $d(t)$ ist also ein nicht konstantes Polynom in $\mathbb{Q}[t]$. Nach Multiplikation mit einer passenden natürlichen Zahl kann man annehmen, daß $d(t)$ ganzzahlig ist. Wir zerlegen $d(t)$ in $\mathbb{Q}[t]$ in irreduzible Faktoren:

$$d(t) = d_1(t)^{e_1} \dots d_r(t)^{e_r}.$$

Auch die nicht konstanten Polynome $d_i(t)$ können ganzzahlig angenommen werden.

Man zeigt:

Es gibt eine endliche Menge S von Primzahlen, so daß folgendes gilt:

Für eine ganze Zahl $m \in H$ und eine Primzahl $p \notin S$, die in $d_1(m)$ in erster Ordnung aufgeht, ist $B \otimes_{\mathbb{Z}[t]} \mathbb{Z}_{(p)}$ ganz abgeschlossen in $L(m) = B \otimes_{\mathbb{Z}[t]} \mathbb{Q}$, damit Lokalisierung der Hauptordnung von $L(m)$ nach dem Primideal (p), und $d(m)$ ist bis auf eine Einheit in $\mathbb{Z}_{(p)}$ Diskriminante von $B \otimes_{\mathbb{Z}[t]} \mathbb{Z}_{(p)}$ über $\mathbb{Z}_{(p)}$. Damit ist eine solche Primzahl p Teiler der Körperdiskriminante $d_{L(m)/\mathbb{Q}}$ von $L(m)$ über $\mathbb{Q}$.

Andererseits gibt es eine unendliche Menge $\mathfrak{M}$ von Primzahlen p, die nicht in S enthalten sind, so daß das Polynom $d_1(t)$ modulo p in die „richtige“ Anzahl von teilerfremden Linearfaktoren zerfällt [4].

Es gibt dann zu einer solchen Primzahl p aus $\mathfrak{M}$ eine natürliche Zahl m mit

$$p \mid d_1(m), \qquad p^2 \nmid d_1(m).$$

Betrachte die Folge natürlicher Zahlen $(m+q\,p^2)$, $q \in \mathbb{N}$. Nach Hilberts Irreduzibilitätssatz gibt es beliebig große natürliche Zahlen q, so daß $L(m+q\,p^2)$ ein galoisscher Zahlkörper mit Galoisgruppe G ist. Für genügend große q liegt $m+q\,p^2$ nach Voraussetzung in U, ist also $L(m+q\,p^2)$ total reell. Die Diskriminante von $L(m+q\,p^2)$ wird durch p geteilt.

Also gibt es unter den total reellen galoisschen Zahlkörpern $L(a)$ $(a \in U \cap H)$ mit Galoisgruppe G solche mit beliebig großer Diskriminante.

Es soll ein Beispiel für die angegebene Situation diskutiert werden:

5.7. Satz. *Es gibt unendlich viele nicht untereinander isomorphe galoissche total reelle Zahlkörper, deren Galoisgruppen isomorph sind zur Gruppe A_5 der alternierenden Permutationen von fünf Elementen.*

Dazu betrachten wir den Raum der binären Formen

$$f(x,y) = \sum_{\kappa=1}^{5} a_\kappa x^{5-\kappa} y^\kappa = \prod_{\kappa=1}^{5} (\alpha_\kappa x - \beta_\kappa y)$$

vom Grad fünf.

Auf diesem Raum operiert die lineare Gruppe Gl_2: $\gamma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Dann transformiert γ die Form $f(x,y)$ in die Form $f(a\,x+b\,y,\, c\,x+d\,y)$.

Damit operiert G auf dem Körper Δ der über $\mathbb{Q}$ rationalen Funktionen $h(a_0, \ldots, a_5)$ der Koeffizienten a_κ, die homogen sind vom Grad Null. Außerdem operiert Gl_2 auf natürliche Weise auf dem Oberkörper Ω der über $\mathbb{Q}$ rationalen Funktionen

$$f\left(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_5}{\alpha_5}\right)$$

in den Wurzeln $\beta_\kappa/\alpha_\kappa$ der Formen. Der Invariantenkörper $K_0 = \Delta^{Gl_2}$ der Formen fünften Grades ist nach dem Hauptsatz der Invariantentheorie rein transzendent über $\mathbb{Q}$ mit den algebraisch unabhängigen Erzeugenden

$$u = \frac{D}{I_4^2}, \qquad v = \frac{I_{12}}{I_4^3}$$

$$K_0 = Q(u, v).$$

Dabei ist die Diskriminantenform $D=D(a_0, \dots, a_5)$ eine invariante Form vom Grad 8, $I_4=I_4(a_0, \dots, a_5)$ eine invariante Form vom Grad 4, $I_{12}=I_{12}(a_0, \dots, a_5)$ eine invariante Form vom Grad 12 [7].

Es ist leicht zu sehen, daß $L=\Omega^{Gl_2}$ rein transzendent über $\mathbb{Q}$ ist und daß auf L die volle Permutationsgruppe S_5 (durch Vertauschung der Wurzeln β_i/α_i) operiert. Es gilt

$$L^{S_5}=K_0.$$

Die in L enthaltene quadratische Erweiterung

$$K=K_0(\sqrt{u})=\mathbb{Q}\left(\frac{\sqrt{D}}{I_4}, v\right)$$

ist ebenfalls rein transzendent und wird von der Untergruppe A_5 der alternierenden Permutationen aus S_5 invariant gelassen. Also ist L/K galoissch mit der Galoisgruppe A_5. Für die Erweiterung K liegt unsere Ausgangssituation vor!

Aus Bemerkung 4.2 und den Theoremen 4.4, 5.5 und 5.7 erhält man

5.8. Theorem. *Es gibt unendlich viele untereinander nicht isomorphe ZPE-Ringe der Tiefe* 3 *von z.B. der Dimension* 60.

§6. Die Starrheit der Spitzen

Gegeben sei eine analytische Algebra A. Eine *Deformation* dieser Algebra A ist ein flacher Homomorphismus $B \to C$ analytischer Algebren zusammen mit einem Isomorphismus

$$C/\mathfrak{m}_B C \cong A.$$

Dabei ist $\mathfrak{m}_B$ das maximale Ideal von B. Man erklärt auf natürliche Weise die Äquivalenz von Deformationen.

Die Deformation $B \to C$ heißt *trivial*, wenn sie zur trivialen Deformation

$$C \to C \tilde{\otimes} A$$

äquivalent ist. $C \tilde{\otimes} A$ bezeichnet das *analytische* Tensorprodukt der Algebren C und A über $\mathbb{C}$.

6.1. *Definition.* Die analytische Algebra A heißt starr, wenn jede Deformation trivial ist. Ein analytischer Raum X heißt starr im Punkt $a \in X$, wenn der Ring $\mathcal{O}_{X,a}$ der Keime holomorpher Funktionen in a starr ist.

Nach einer mündlichen Mitteilung von Herrn Schuster gilt folgendes Starrheitskriterium:

6.2. Lemma. *Gegeben sei ein Steinscher normaler Raum X der Dimension $n \geqq 2$ in einem Punkt $a \in X$. X sei außerhalb von a glatt. Ω_X sei die Garbe der holomorphen Differentialformen vom Grad Eins auf X, $\mathcal{D}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ die dazu duale Garbe. Es verschwinde die Kohomologiegruppe $H^1(X - \{a\}, \mathcal{D}_X)$. Dann ist X im Punkt a starr.*

Der *Beweis* soll kurz skizziert werden:

Wir können annehmen, daß X ein durch die kohärente Idealgarbe $\mathcal{J}$ definierter abgeschlossener Unterraum eines komplexen Zahlraumes $\mathbb{C}^m$ ist. j sei die natürliche Einbettung von $U = X - \{a\}$ in X, $\underline{\underline{\mathcal{R}}}^{\cdot} j_* \mathcal{O}_U$ der abgeleitete direkte Bildkomplex von $\mathcal{O}_U$ im Sinne von Verdier, $\mathcal{H}^0_{\{a\}} \mathcal{O}_X$ die Garbe der Schnitte von $\mathcal{O}_X$ mit Trägern in $\{a\}$ und $\underline{\underline{\mathcal{R}}}^{\cdot} \mathcal{H}^0 \mathcal{O}_X$ der abgeleitete Komplex im Sinne von Verdier.

Man hat auf natürliche Weise einen Komplex $\mathcal{K}^{\cdot}$:

$$\mathcal{K}^1 = \mathcal{J}/\mathcal{J}^2 | X \to \mathcal{K}^0 = \Omega_{\mathbb{C}^m}/\mathcal{J}\,\Omega_{\mathbb{C}^m} | X .$$

Wir betrachten die exakte Folge von Hyperkohomologiegruppen:

$$\cdots \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{K}^{\cdot}, \underline{\underline{\mathcal{R}}}^{\cdot} \mathcal{H}^0_{\{a\}} \mathcal{O}_X) \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{K}^{\cdot}, \mathcal{O}_X) \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{K}^{\cdot}, \underline{\underline{\mathcal{R}}}^{\cdot} j_* \mathcal{O}_U) \to \cdots .$$

$\mathcal{K}^{\cdot} | U$ ist quasiisomorph zu Ω_U, also gilt:

$$\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{K}^{\cdot}, \underline{\underline{\mathcal{R}}}^{\cdot} j_* \mathcal{O}_U) \cong \operatorname{Ext}^1_{\mathcal{O}_U}(\mathcal{K}^{\cdot} | U, \mathcal{O}_U) \cong \operatorname{Ext}^1_{\mathcal{O}_U}(\Omega_U, \mathcal{O}_U) \cong H^1(U, D_X) = 0 .$$

Weil die Tiefe von $\mathcal{O}_{X,a}$ mindestens Zwei ist, verschwindet $H^i(\underline{\underline{\mathcal{R}}}^{\cdot} \mathcal{H}^0_{\{a\}} \mathcal{O}_X) = H^i_{\{a\}}(X, \mathcal{O}_X)$ für $i \leqq 1$ [8], damit $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{K}^{\cdot}, \underline{\underline{\mathcal{R}}} \mathcal{H}^0_{\{a\}} \mathcal{O}_X)$. Wir erhalten:

$$\operatorname{Ext}^1_{\mathcal{O}_{X,a}}(\mathcal{K}^{\cdot}_a, \mathcal{O}_{X,a}) \cong \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{K}^{\cdot}, \mathcal{O}_X) = 0 .$$

$\operatorname{Ext}^1_{\mathcal{O}_{X,a}}(\mathcal{K}^{\cdot}_a, \mathcal{O}_{X,a})$ ist aber isomorph zur Gruppe der infinitesimalen Deformationen von $\mathcal{O}_{X,a}$.

Gegeben sei nun eine Transformationsgruppe Γ des Hilbertschen Halbraumes H^n im Sinne von § 2. Wir bezeichnen mit p die Projektion von H^n auf den Restklassenraum $X^0 = H^n/\Gamma$. Auf X^0 erklären wir die Geradenbündel $\mathcal{G}_1, \ldots, \mathcal{G}_n$ und das Vektorraumbündel $\mathcal{M}$ vom Rang n durch ihre Schnitte über offenen Teilmengen von X^0.

Sei U eine offene Teilmenge von X^0. Das Urbild $Y = p^{-1}(U)$ ist eine Γ-saturierte offene Teilmenge von H^n.

$\mathcal{G}_i(U)$ ist der Vektorraum der holomorphen Funktionen $f\colon Y \to \mathbb{C}$, die folgendes Transformationsverhalten bei Substitutionen $\gamma\colon z \to \varepsilon z + a$ der Gruppe Γ haben:

$$f(\gamma z) = \varepsilon_i^{-1} f(z).$$

$\mathcal{M}(U)$ ist der Vektorraum der Γ-invarianten holomorphen Differentialformen

$$f_1(z)\,dz_1+\cdots+f_n(z)\,dz_n$$

auf Y.

Die Γ-Invarianz bedeutet:

Für jede Transformation

$$\gamma\colon\ z\to\varepsilon z+a$$

aus Γ gilt

$$f_1(\gamma z)\,d(\varepsilon_1 z_1)+\cdots+f_n(\gamma z)\,d(\varepsilon_n z_n)=f_1(z)\,dz_1+\cdots+f_n(z)\,dz_n,$$

d.h.

$$f_1(\gamma z)=\varepsilon_1^{-1}f_1(z),\ldots,f_n(\gamma z)=\varepsilon_n^{-1}f_n(z).$$

Klar ist folgender Hilfssatz:

6.3. Hilfssatz. *Die Garbe $\mathcal{M}$ ist isomorph zur Garbe Ω_{X^0} der holomorphen Differentialformen vom Grade Eins auf X^0. $\mathcal{M}$ zerfällt in die direkte Summe der Geradenbündel $\mathcal{G}_i$:*

$$\mathcal{M}=\mathcal{G}_1\ \oplus\cdots\oplus\mathcal{G}_n,$$
$$\mathcal{D}_{X^0}=\mathcal{G}_1^{-1}\oplus\cdots\oplus\mathcal{G}_n^{-1}.$$

Sei $X=X^0\cup\{\infty\}$ die „Kompaktifizierung“ von X^0 durch die Spitze ∞. $R(\Gamma)$ ist die Algebra der Keime holomorpher Funktionen in der Spitze ∞.

6.4. Theorem. *Sei $n\geqq 3$. Dann ist die Algebra $R(\Gamma)$ starr.*

Der Raum X ist ein Steinscher Raum: Die Γ-invariante Funktion

$$H^n\to\mathbb{C}$$
$$z\to N(\operatorname{Im} z)^{-1}$$

definiert eine plurisubharmonische Funktion auf X^0, die sich stetig mit dem Wert Null auf ∞ fortsetzen läßt.

Wegen Kriterium 6.2 und Hilfssatz 6.3 genügt es deshalb zu zeigen, daß $H^1(X^0,\mathcal{G}_i^{-1})$ für alle i verschwindet.

Allgemeiner zeigen wir:

6.5. Theorem. *Gegeben sei ein n-Tupel ganzer Zahlen $(r_1,\ldots,r_n)\neq(0,\ldots,0)$. Wir betrachten folgendes Geradenbündel automorpher Formen. Sei U eine offene Teilmenge von X^0, $Y=p^{-1}(U)$ das Urbild in H^n. $\mathcal{G}(U)$ ist der Raum der holomorphen Funktionen $f\colon Y\to\mathbb{C}$, die folgendes Transformationsverhalten bei Substitutionen $\gamma\colon z\to\varepsilon z+a$ der Gruppe haben:*

$$f(\gamma z)=\varepsilon_1^{-r_1}\ldots\varepsilon_n^{-r_n}f(z).$$

Dann verschwinden die Kohomologiengruppen

$$H^{\nu}(X^{0}, \mathscr{G}); \qquad 1 \leqq \nu \leqq n-2.$$

Der Beweis verläuft ähnlich wie die in [3] durchgeführte Bestimmung der Kohomologiegruppen des trivialen Bündels auf X^{0}.

Sei M der Vektorraum aller holomorphen Funktionen auf dem Halbraum H^{n}. Auf M operiert Γ folgendermaßen:

$$\gamma^{-1}: \; z \to \varepsilon z + a$$

sei eine Substitution aus Γ, f eine holomorphe Funktion auf dem Halbraum H^{n}

$$(\gamma f)(z) = \varepsilon_1^{r_1} \dots \varepsilon_n^{r_n} f(\gamma z).$$

Es gilt:

$$H^{\nu}(X^{0}, \mathscr{G}) = H^{\nu}(\Gamma, M) = H^{\nu}(\Lambda, M^{t}).$$

M^{t} kann als Vektorraum gewisser Fourierreihen beschrieben werden:

$$f(z) = \sum_{g \in t^{0}} a_g e(g z).$$

Λ operiert auf M^{t}:

Das Element $\varepsilon^{-1} \in \Lambda$ werde durch die Substitution

$$z \to \varepsilon^{-1}(z+a)$$

aus Γ repräsentiert

$$(\varepsilon f)(z) = \sum_{g \in t^{0}} \varepsilon_1^{r_1} \dots \varepsilon_n^{r_n} e(g a) a_{g\varepsilon} e(g z).$$

M^{t} zerfällt in eine direkte Summe des Λ-Moduls N aller Fourierreihen ohne konstanten Term und den Λ-Modul $\mathbb{C}$ der konstanten Funktionen. Man beachte, daß im Gegensatz zum Falle des in [3] behandelten trivialen Bündels Λ *nicht trivial* auf $\mathbb{C}$ operiert!

$$\varepsilon c = \varepsilon_1^{r_1} \dots \varepsilon_n^{r_n} c.$$

Deshalb gilt

$$H^{\nu}(\Lambda, \mathbb{C}) = 0 \qquad \text{für alle } \nu.$$

Wörtlich wie in [3] folgert man das *Verschwinden* der Kohomologiegruppen $H^{\nu}(\Lambda, N)$ $1 \leqq \nu \leqq n-2$ aus der *Endlichkeit* [8] dieser Vektorräume.

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An Analogue of Stickelberger's Theorem for the Higher K-Groups

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Introduction

Let F be a finite extension of the rational field $\mathbf{Q}$, $\mathfrak{O}$ the ring of algebraic integers of F, and $K_{2k}\mathfrak{O}$ $(k=0, 1, \ldots)$ the even K-groups of $\mathfrak{O}$ in the sense of Bass-Milnor-Quillen (see [11, 13]). In particular, $K_0\mathfrak{O}$ is just the ideal class group of $\mathfrak{O}$ (see Chapter 1 of [11]). All the groups $K_{2k}\mathfrak{O}$ have been proven to be finite by the work of Garland [7], Borel [1], and Quillen [12]. However, little is known so far about their structure, and the present note is intended to be a contribution to this problem in the special case in which F is abelian over $\mathbf{Q}$. In that case, it has been known since last century that there is a canonical relation in $K_0\mathfrak{O}$, namely, the Stickelberger relation (see, for example, [9]). Are there similar canonical relations in the higher K-groups? We prove that this is indeed the case for $K_2\mathfrak{O}$, and we conjecture a relation for the $K_{2k}\mathfrak{O}$ with $k>1$. It may be worth noting that these relations were suggested to us both by Lichtenbaum's conjectures [10] about the order of $K_{2k}\mathfrak{O}$, and by the earlier work of one of us [5] on these conjectures.

We now state our result explicitly. Let f be an integer >1. For each integer a with $(a,f)=1$, we define the partial zeta function of a modulo f by

$$\zeta_f(a, s) = \sum_{\substack{n \equiv a \bmod f \\ n>0}} n^{-s} \qquad (R(s)>1),$$

where, as indicated, the summation is taken over all positive integers n with $n \equiv a \bmod f$. Plainly $\zeta_f(a,s)$ depends only on the residue class of $a \bmod f$. It is well known that $\zeta_f(a,s)$ can be analytically continued over the whole complex plane, except for a pole at $s=1$. Moreover, for each integer $k \geqq 0$, it is known (see § 1) that $\zeta_f(a,-k)$ is a rational number. Our conjectural canonical relations are defined in terms of these rational numbers. For each integer $m \geqq 1$, let μ_m be the group of m-th roots of unity (in some fixed algebraic closure of F). Assume, as above, that F is an abelian extension of $\mathbf{Q}$. By class field theory, F is contained in $\mathbf{Q}(\mu_m)$ for some integer m. Let f be the least positive integer m with this property (in fact, f is then also the non-archimedean part of the conductor of $F/\mathbf{Q}$,

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in the sense of class field theory). For each positive integer b with $(b,f)=1$, we write (b,F) for the restriction to F of the automorphism of $\mathbf{Q}(\mu_f)$ which maps each element of μ_f to its b-th power (of course, (b,F) is the Artin symbol attached to the ideal (b)). If k is an integer $\geqq 0$, we let $w_{k+1}(\mathbf{Q})$ denote the largest integer m such that the Galois group of $\mathbf{Q}(\mu_m)$ over $\mathbf{Q}$ has exponent dividing $k+1$. Let G be the Galois group of F over $\mathbf{Q}$. Now it follows from Theorem 1.2 that, for each positive integer b with $(b,f)=1$, the element

$$S_k(b)=w_{k+1}(\mathbf{Q})\left(b^{k+1}-(b,F)\right)\sum_{\substack{a \bmod f\\(a,f)=1}}\zeta_f(a,-k)(a,F)^{-1},$$

where the summation is taken over any complete set $\{a\}$ of representatives of the relatively prime residue classes mod f, belongs to the integral group ring $\mathbf{Z}[G]$. Since G acts on $K_{2k}\mathfrak{O}$ in the natural way, we can therefore consider the action of $S_k(b)$ on $K_{2k}\mathfrak{O}$.

Conjecture 1. *For each positive integer b with $(b,f)=1$, $S_k(b)$ annihilates $K_{2k}\mathfrak{O}$.*

It should be noted that, when F is totally real and k is even, this conjecture is entirely vacuous since $S_k(b)=0$ for all $b\geqq 1$ with $(b,f)=1$. On the other hand, excluding this case, it is easy to see (cf. Lemma 1.7) that $S_k(b)\neq 0$ for all $b>1$ with $(b,f)=1$.

We first remark that, when $k=0$, the above conjecture is just a slightly modified form of Stickelberger's theorem [9], and so it is certainly valid in this case. The main result of the present note is the following. We write F^+ for the maximal totally real subfield of F, and $\mathfrak{O}^+$ for the ring of algebraic integers of F^+.

Theorem 2. *Assume that $(b,f)=1$, and that, in addition, b is prime to the order of $K_2\mathfrak{O}$ or $K_2\mathfrak{O}^+$, according as F is totally real or totally imaginary. Then $S_1(b)$ annihilates the l-primary subgroup of $K_2\mathfrak{O}$ for all odd primes l.*

The additional restriction on b is rather mild, especially as one can always compute a finite set of primes which contains all the primes dividing the order of K_2 of the ring of integers of a totally real abelian number field (see Theorem 4.4 of [5]). Also, we strongly suspect that the proof of Theorem 2 can be extended to show that $S_1(b)$ annihilates the 2-primary subgroup of $K_1\mathfrak{O}$ as well. However, we do not discuss this in the present paper, because it would introduce considerable technical complications into our arguments. Finally, we note that, when k is odd and >1, and F is totally real, we show that $S_k(b)$ annihilates an l-primary group $(l\neq 2)$ which is conjectured to be isomorphic as a G-module to the l-primary subgroup of $K_{2k}\mathfrak{O}$ (see Theorem 2.1).

1. A Congruence for the Values of the Partial Zeta Functions

We suppose throughout this section that f is an arbitrary integer >1. Our first lemma is obvious from the definition of the partial zeta functions.

Lemma 1.1. *Assume that f divides g, and that f and g are divisible by the same primes. Then, for each integer b with $(b,f)=1$, we have*

$$\zeta_f(b,s)=\sum \zeta_g(a,s),$$

where the summation on the right is taken over a complete set $\{a\}$ of representatives of the residue classes mod g which are mapped to the residue class of b mod f by the canonical surjection from $\mathbf{Z}/g\mathbf{Z}$ to $\mathbf{Z}/f\mathbf{Z}$.

Now let k be any integer $\geqq 0$. It is well known (see [15], p. 17), that, for each b with $(b,f)=1$, $\zeta_f(b,-k)$ is given by the Fourier expansion

$$\zeta_f(b,-k)=(2\pi i)^{-k-1}k!\,f^k \sum_{n=-\infty}^{\infty}{}' e^{2\pi i n b/f}\, n^{-k-1}, \tag{1}$$

where the summation is taken over all non-zero integers n; when $k=0$, it is understood that the terms with $-n$ and n are always taken together in the summation. Moreover, when $0<b<f$ the sum of the Fourier series is known to be given by

$$\zeta_f(b,-k)=-f^k(k+1)^{-1}B_{k+1}(b/f), \tag{2}$$

where $B_{k+1}(x)$ denotes the $(k+1)$-th Bernoulli polynomial (see [15], p. 17). If b lies outside this range, we use the periodicity of the Fourier series to obtain the value of $\zeta_f(b,-k)$.

If b, c are positive integers with $(b,f)=(c,f)=1$, and k is an integer $\geqq 0$, we define

$$\delta_{k+1}(b,c;f)=c^{k+1}\zeta_f(b,-k)-\zeta_f(bc,-k).$$

For each rational prime p, let v_p be the corresponding valuation of $\mathbf{Q}$, normalized so that $v_p(p)=1$. In order to bring out the analogy of our present results with those in [6], we introduce the integers $f_0=f$ and

$$f_k=f\prod_{p\mid f} p^{v_p(k)} \qquad (k\geqq 1),$$

where the product is taken over all primes p which divide f. Let $w_k(\mathbf{Q}(\mu_f))$ be the largest positive integer m such that the Galois group of $\mathbf{Q}(\mu_f,\mu_m)$ over $\mathbf{Q}(\mu_f)$ has exponent dividing k. Then, in connexion with the results of [6], we make the following observation. Suppose that f is the conductor of $\mathbf{Q}(\mu_f)$ over $\mathbf{Q}$ (of course, this is the same as assuming that f is either odd or divisible by 4). Then it is an easy consequence of the irreducibility of the cyclotomic equation over $\mathbf{Q}$ that, for each prime p

which divides f, we have

$$v_p(f_k)=v_p(w_k(\mathbf{Q}(\mu_f))) \quad (k\geqq 1). \tag{3}$$

We now state our main congruence and integrality assertion about the values of the partial zeta functions. It seems to be natural to break the result into two separate theorems. In both theorems, it is assumed that b, c, f are arbitrary positive integers with $(b,f)=(c,f)=1$, and that k is any integer $\geqq 0$.

Theorem 1.2. (i) *$w_{k+1}(\mathbf{Q})\,\delta_{k+1}(b,c;f)$ is an integer, and*

(ii) $w_{k+1}(\mathbf{Q})\,\delta_{k+1}(b,c;f)\equiv w_{k+1}(\mathbf{Q})(bc)^k\,\delta_1(b,c;f) \bmod f_k$.

Theorem 1.3. *Assume that $(c, w_{k+1}(\mathbf{Q}(\mu_f)))=1$. Then*

(i) *$\delta_{k+1}(b,c;f)$ is an integer, and*

(ii) $\delta_{k+1}(b,c;f)\equiv(bc)^k\,\delta_1(b,c;f) \bmod f_k$.

Corollary 1.4. *$w_{k+1}(\mathbf{Q}(\mu_f))\,\zeta_f(b,-k)$ is an integer.*

We first note that Corollary 1.4 is a consequence of (i) of Theorem 1.3. For, let b be any positive integer with $(b,f)=1$. Then, for each positive integer c with $c\equiv 1 \bmod f$, we have plainly $\zeta_f(b,-k)=\zeta_f(bc,-k)$, whence, for such c,

$$\delta_{k+1}(b,c;f)=(c^{k+1}-1)\,\zeta_f(b,-k).$$

The above assertion is then plain from the following lemma (cf. Lemma 2 of [6]).

Lemma 1.5. *Let d be the greatest common divisor of the numbers $c^{k+1}-1$, where c runs over all the positive integers satisfying $c\equiv 1 \bmod f$ and $(c, w_{k+1}(\mathbf{Q}(\mu_f)))=1$. Then $d=w_{k+1}(\mathbf{Q}(\mu_f))$.*

Proof. Put $m=w_{k+1}(\mathbf{Q}(\mu_f))$. Note that f divides m. We first show that m divides d. Let c be any positive integer with $c\equiv 1 \bmod f$ and $(c,m)=1$. Since $c\equiv 1 \bmod f$, we see that $(c,\mathbf{Q}(\mu_m))$ belongs to $G(\mathbf{Q}(\mu_m)/\mathbf{Q}(\mu_f))$. Hence, by the definition of m, $(c^{k+1},\mathbf{Q}(\mu_m))=1$, and so $c^{k+1}\equiv 1 \bmod m$. This proves that m divides d. To prove the converse, let σ be any element of $G(\mathbf{Q}(\mu_d)/\mathbf{Q}(\mu_f))$. There exists a positive integer c with $(c,d)=1$ such that $\sigma=(c,\mathbf{Q}(\mu_d))$. Since σ fixes $\mathbf{Q}(\mu_f)$ by hypothesis, we have $c\equiv 1 \bmod f$. Also the condition $(c,d)=1$ implies that $(c,m)=1$ because m divides d. Thus, by the definition of d, $c^{k+1}\equiv 1 \bmod d$, and so $\sigma^{k+1}=1$. This shows that $G(\mathbf{Q}(\mu_d)/\mathbf{Q}(\mu_f))$ has exponent dividing $k+1$, whence d divides m. The proof of the lemma is complete.

We next establish the following elementary result, which will be needed in the proof of Theorems 1.2 and 1.3.

Lemma 1.6. *For each integer $k\geqq 1$, and each integer i with $1\leqq i\leqq k$, we have that $\binom{k}{i} f^i$ is divisible by f_k.*

Proof. Let p be any prime number, and put $e=v_p(k)$. We first observe that, for $1\leqq i\leqq p^e$, we have

$$v_p\left(\binom{k}{i}\right)=v_p(k)-v_p(i). \tag{4}$$

The proof is by induction on i, the assertion being plainly true for $i=1$. Assume it true for $i=1,\ldots,j$, where $j<p^e$. Since $j<p^e\leqq k$, it is plain that $k-j\neq 0$ and $v_p(k-j)=v_p(j)$. Thus the identity

$$(j+1)\binom{k}{j+1}=(k-j)\binom{k}{j},$$

together with the validity of (4) for $i=j$, implies that (4) is true with $i=j+1$, as required. We next claim that, for each i with $1\leqq i\leqq k$, we have

$$v_p\left(\binom{k}{i}p^i\right)\geqq v_p(k)+1. \tag{5}$$

Indeed, if $i>p^e$ this is true since $i>e$ and

$$v_p\left(\binom{k}{i}p^i\right)\geqq i\geqq e+1=v_p(k)+1.$$

On the other hand, if $i\leqq p^e$, (5) is also valid by virtue of (4) and the fact that $v_p(i)<i$. Finally, on applying (5) to the primes p which divide f, the conclusion of Lemma 1.6 follows.

We now commence the proof proper of Theorems 1.2 and 1.3. We recall that the Bernoulli numbers and polynomials are defined by

$$t/(e^t-1)=\sum_{n=0}^{\infty}B_n t^n/n!,\qquad B_n(x)=\sum_{j=0}^{n}\binom{n}{j}B_j x^{n-j}.$$

From these definitions, one deduces easily the well known identities

$$B_{k+1}(x+a)=\sum_{j=0}^{k+1}\binom{k+1}{j}B_{k+1-j}(a)\,x^j, \tag{6}$$

$$B_{k+1}(x)=(-1)^{k+1}B_{k+1}(1-x). \tag{7}$$

$$\left(B_{k+1}(n)-B_{k+1}\right)/(k+1)=\sum_{j=0}^{n-1}j^k\quad (n \text{ an integer }\geqq 1). \tag{8}$$

As usual, if u and v are two rational numbers, the notation $u\equiv v \bmod f_k\mathbf{Z}$ will mean that $u-v$ belongs to $f_k\mathbf{Z}$. We first claim that, if $k\geqq 1$,

$$\zeta_f(bc,-k)\equiv -f^k(k+1)^{-1}B_{k+1}(bc/f)-(bc)^k\left(B_1(r)-B_1\right) \bmod f_k\mathbf{Z}, \tag{9}$$

where r denotes the unique integer such that $g=bc+rf$ satisfies $1\leqq g<f$. For (6), together with the equation

$$(k+1)^{-1}\binom{k+1}{j}=(k+1-j)^{-1}\binom{k}{j}, \tag{10}$$

shows that

$$f^k(k+1)^{-1}\left(B_{k+1}\left(\frac{bc}{f}+r\right)-B_{k+1}(bc/f)\right)=W,$$

where

$$W=\sum_{j=0}^{k}\binom{k}{j}(k+1-j)^{-1}(B_{k+1-j}(r)-B_{k+1-j})\left(\frac{bc}{f}\right)^j f^k. \tag{11}$$

Now we observe that

$$(B_{k+1-j}(r)-B_{k+1-j})/(k+1-j) \qquad (0\leqq j\leqq k)$$

is an integer. If $r\geqq 0$, this is plain from (8). If $r<0$ and $j<k$, it follows from (7) and (8) since $B_{k+1-j}=0$ when $k+1-j$ is odd and >1. Finally, the assertion is also true when $j=k$ and $r<0$ because $B_1(x)=x-\frac{1}{2}$. In view of this observation, we conclude from Lemma 1.6 that all terms in the sum (11), except perhaps that given by $j=k$, are integers divisible by f_k, whence

$$W\equiv(bc)^k(B_1(r)-B_1) \mod f_k\mathbf{Z}.$$

The congruence (9) then follows from the equation above (11) on recalling that, by (2),

$$\zeta_f(bc,-k)=-f^k(k+1)^{-1}B_{k+1}\left(\frac{bc}{f}+r\right).$$

Again using (10), we have

$$f^k(k+1)^{-1}B_{k+1}(bc/f)=(k+1)^{-1}(bc)^{k+1}/f+\sum_{j=0}^{k}\binom{k}{j}\frac{B_{k+1-j}}{k+1-j}(bc)^j f^{k-j}.$$

Also, since $\delta_{k+1}(b,c;f)$ depends only on the residue class of $b \bmod f$ (note, however, that $\delta_{k+1}(b,c;f)$ does depend on the particular choice of c), we can assume that $0<b<f$, so that $\zeta_f(b,-k)$ is given by the right hand side of (2). Therefore we conclude from (9) that, assuming $k\geqq 1$,

$$\delta_{k+1}(b,c;f)\equiv V_{k+1}(b,c;f)+(bc)^k(B_1(r)-B_1) \mod f_k\mathbf{Z}, \tag{12}$$

where

$$V_{k+1}(b,c;f)=\sum_{j=0}^{k}\alpha(k,j,c)\,b^j f^{k-j},$$

and

$$\alpha=\alpha(k,j,c)=\binom{k}{j}\frac{B_{k+1-j}}{k+1-j}c^j(1-c^{k+1-j}).$$

Let p be any prime number. We now proceed to show that, for each $k \geqq 1$, $\alpha = \alpha(k, j, c)$ is integral at p unless the following four conditions hold: (i) p divides c, (ii) $j = 0$, (iii) $k+1$ is even, and (iv) $p-1$ divides $k+1$. If these four conditions are satisfied, we shall show that $v_p(\alpha) \geqq -v_p(k+1) - 1$. If $p-1$ does not divide $k+1-j$, α is integral at p by Kummer's theorem (see, for example, [2], p. 385). Hence we may assume that $p-1$ divides $k+1-j$. Put

$$m = v_p(k+1), \qquad n = v_p(k+1-j), \qquad t = \min(m, n).$$

Suppose first that p does not divide c. Then, since $(p-1)p^n$ divides $k+1-j$, we have $c^{k+1-j} \equiv 1 \mod p^{n+1}$. But, by the Clausen-von Staudt theorem, p divides the denominator of B_{k+1-j} to (precisely) the first power, and so α is integral at p in this case. Suppose next that p divides c. Now we can use (10) to rewrite α as

$$\alpha = \binom{k+1}{j} B_{k+1-j} c^j (1 - c^{k+1-j})(k+1)^{-1}.$$

Hence, again using the Clausen-von Staudt theorem, we deduce that $v_p(\alpha) \geqq j - t - 1$. Now, if $j \neq 0$, $t < j$ since p^t divides j, and thus α is integral at p. If $j = 0$, we have $\alpha = 0$ unless $k+1$ is even because the Bernoulli numbers with odd subscripts > 1 are all 0. This proves the above assertion.

We can now complete the proofs of Theorems 1.2 and 1.3. Let us note first that both theorems are true for $k = 0$ because

$$\delta_1(b, c; f) = c/2 + r - \tfrac{1}{2}, \tag{13}$$

and the condition $(c, w_1(\mathbf{Q}(\mu_f))) = 1$ in Theorem 1.3 implies that c is odd. Hence, we can assume that $k \geqq 1$. We consider Theorem 1.2 first. It follows easily from the irreducibility of the cyclotomic equation over $\mathbf{Q}$ that $w_{k+1}(\mathbf{Q}) = 2$ or

$$w_{k+1}(\mathbf{Q}) = 2 \prod_{p-1 \mid k+1} p^{1 + v_p(k+1)}$$

according as k is even or odd; here the product is taken over all primes p such that $p-1$ divides $k+1$. Thus it is plain from the description of the denominator of $\alpha(k, j, c)$ obtained above that $w_{k+1}(\mathbf{Q})\alpha(k, j, c)$ is integral for $0 \leqq j \leqq k$. Hence, by (12), $w_{k+1}(\mathbf{Q})\delta_{k+1}(b, c; f)$ is an integer. To derive the congruence, let p be any prime number which divides f. Then p does not divide c because $(c, f) = 1$, whence it follows easily from Lemma 1.6 and the argument used to show that $\alpha(k, j, c)$ is integral at p that

$$v_p\big(V_{k+1}(b, c; f) - B_1 b^k c^k (1 - c)\big) \geqq v_p(f_k). \tag{14}$$

Assertion (ii) of Theorem 1.2 is now clear from (12), (13), (14) and the fact that $w_{k+1}(\mathbf{Q})\delta_{k+1}(b, c; f)$ is an integer (recall that $B_1(x) = x - \tfrac{1}{2}$). Finally, assume that $(c, w_{k+1}(\mathbf{Q}(\mu_f)))=1$ (this is the same as saying that c is prime to both f and $w_{k+1}(\mathbf{Q})$). Therefore, each prime p which divides c does not have the property that $p-1$ divides $k+1$. Therefore $\alpha(k,j,c)$ is an integer in this case, and the rest of the proof of Theorem 1.3 follows in an entirely similar manner to that of Theorem 1.2.

Finally, we give an alternative expression for the element $S_k(b)$ defined in the Introduction, which will be needed for the proof of Theorem 2. We use the notation of the Introduction. Thus F will denote an abelian extension of $\mathbf{Q}$, f will be its conductor, and G will be the Galois group of F over $\mathbf{Q}$. We write $\hat{G}$ for the group of homomorphisms of G into the multiplicative group of $\mathbf{C}$. Let χ be any element of $\hat{G}$. As usual, we can view χ as a Dirichlet character mod f by defining $\chi(a)$ to be $\chi((a, F))$ for each integer a with $(a,f)=1$, and we can then form the complex L-function

$$L_f(\chi, s)=\prod_{\substack{p \\ (p,f)=1}} (1-\chi(p)p^{-s})^{-1},$$

where the product is taken over all primes p with $(p,f)=1$; we have included the subscript f in $L_f(\chi, s)$ to make it quite clear that we are not assuming that χ is necessarily primitive. Plainly

$$L_f(\chi, s)=\sum_{\substack{a \bmod f \\ (a,f)=1}} \chi(a)\zeta_f(a,s). \tag{15}$$

We write

$$\alpha_k(F)=\sum_{\substack{a \bmod f \\ (a,f)=1}} \zeta_f(a,-k)(a,F)^{-1}.$$

Also we let e_χ be the orthogonal idempotent of χ in the group ring $\mathbf{C}[G]$. If $\chi\in\hat{G}$, $\bar{\chi}$ will denote its complex conjugate character.

Lemma 1.7. *We have* $\alpha_k(F)=\sum_{\chi\in\hat{G}} L_f(\bar{\chi},-k)\,e_\chi$.

Proof. Since

$$e_\chi\alpha_k(F)=\sum_{\substack{a \bmod f \\ (a,f)=1}} \zeta_f(a,-k)\,\bar{\chi}(a),$$

the assertion is clear on putting $s=-k$ in (15).

2. Proof of Theorem 2

We first establish an auxiliary result, from which Theorem 2 will follow easily. Throughout this section, l will be an odd prime number and k an *odd* positive integer. Moreover, F will denote an abelian extension of $\mathbf{Q}$, which will be assumed to be totally real until further notice.

Let $F_0 = F(\mu_l)$, and let q_0 be the order of the group of l-power roots of unity in F_0. For each integer $n \geqq 0$, put $q_n = q_0 l^n$, and define

$$F_n = F(\mu_{q_n}), \qquad F_\infty = \bigcup_{n=0}^{\infty} F_n.$$

Thus F_n/F_0 is cyclic of degree l^n. Let A_n be the l-primary subgroup of the ideal class group of F_n. If $n \leqq m$, the natural inclusion of the divisor group of F_n in the divisor group of F_m induces a homomorphism $A_n \to A_m$ (which is not, in general, injective), and we put $A_\infty = \varinjlim A_n$. Since each F_n is abelian over $\mathbf{Q}$, there is a natural action of complex conjugation (which we denote by J) on both A_n and A_∞, which is independent of any particular embedding of F_∞ into $\mathbf{C}$. Put

$$A_n^- = (1-J)A_n, \qquad A_\infty^- = (1-J)A_\infty.$$

Now it is well known that, when $n \leqq m$, the natural map $A_n^- \to A_m^-$ is always injective. Hence, regarding these maps as inclusions, we have

$$A_\infty^- = \bigcup_{n=0}^{\infty} A_n^-. \tag{16}$$

We next need the notion of twisting by roots of unity. Let $T = \varprojlim \mu_{l^n}$ be the Tate module, viewed as a $G(F_\infty/\mathbf{Q})$-module in the natural way. If M is any $\mathbf{Z}_l$-module, which is also a $G(F_\infty/\mathbf{Q})$-module, we define $M(k)$ to be $M \otimes_{\mathbf{Z}_l} T^{\otimes k}$, where $T^{\otimes k}$ is the k-fold tensor product of T with itself over $\mathbf{Z}_l$. Here it is understood that $G(F_\infty/\mathbf{Q})$ acts on $M(k)$ via the diagonal action, that is $\sigma(m \otimes t_1 \otimes \cdots \otimes t_k) = (\sigma m) \otimes (\sigma t_1) \otimes \cdots \otimes (\sigma t_k)$. Of course, $T^{\otimes k}$ is a free $\mathbf{Z}_l$-module of rank 1. In particular, if we choose a $\mathbf{Z}_l$-basis $\{\gamma\}$ of T, then $\{\gamma^{\otimes k}\}$, where $\gamma^{\otimes k} = \gamma \otimes \cdots \otimes \gamma$ (k times), is a $\mathbf{Z}_l$-basis of $T^{\otimes k}$. Thus each element of $M(k)$ can be written uniquely in the form $\alpha \otimes \gamma^{\otimes k}$, where $\alpha \in M$. Finally, put $G_\infty = G(F_\infty/F)$.

Theorem 2.1. *Assume that F is a totally real abelian extension of $\mathbf{Q}$. For each odd positive integer k, $(A_\infty^-(k))^{G_\infty}$ is annihilated by $S_k(b)$ for all $b > 0$ with $(b, fl) = 1$.*

Proof. In the theorem, it is, of course, understood that the $G(F/\mathbf{Q})$-structure on $(A_\infty^-(k))^{G_\infty}$ is the natural one coming from the $G(F_\infty/\mathbf{Q})$-structure on $A_\infty^-(k)$. Also $A_\infty^-(k)$ is an l-torsion group because each A_n^- is l-torsion. Fix a basis $\{\gamma\}$ of T as a $\mathbf{Z}_l$-module. Let $\eta \otimes \gamma^{\otimes k}$ be any element of $(A_\infty^-(k))^{G_\infty}$. For each $n \geqq 0$, we write g_n for the conductor of F_n over $\mathbf{Q}$. For the rest of the proof, we shall assume that the integer n is so large that (i) $g_n(\eta \otimes \gamma^{\otimes k}) = 0$, and (ii) $\eta \in A_n^-$. Assertion (i) is valid for all sufficiently large n because q_n divides g_n. Let b be any positive integer with $(b, fl) = 1$. Note that this last condition is equivalent to the property that $(b, g_n)=1$ for all $n \geqq 0$. We define

$$\alpha_n(b)=w_{k+1}(\mathbf{Q})(b^{k+1}-(b,F_n)) \sum_{\substack{a=1 \\ (a,g_n)=1}}^{g_n} \zeta_{g_n}(a,-k)(a,F_n)^{-1}.$$

It is easy to see that $\alpha_n(b)$ can be written in the alternative form

$$\alpha_n(b)=w_{k+1}(\mathbf{Q}) \sum_{\substack{a=1 \\ (a,g_n)=1}}^{g_n} \delta_{k+1}(a,b;g_n)(a,F_n)^{-1},$$

where $\delta_{k+1}(a,b;g_n)$ is as defined in § 1. Hence, by Theorem 1.2, $\alpha_n(b)$ belongs to the integral group ring $\mathbf{Z}[G(F_n/\mathbf{Q})]$. We claim that

$$\alpha_n(b)(\eta \otimes \gamma^{\otimes k})=0. \tag{17}$$

For, since $g_n(\eta \otimes \gamma^{\otimes k})=0$, we conclude from the second assertion of Theorem 1.2 that $\alpha_n(b)(\eta \otimes \gamma^{\otimes k})=\theta_n(b)(\eta \otimes \gamma^{\otimes k})$, where

$$\theta_n(b)=w_{k+1}(\mathbf{Q}) \sum_{\substack{a=1 \\ (a,g_n)=1}}^{g_n} (ba)^k \delta_1(a,b;g_n)(a,F_n)^{-1}.$$

Let (a, F_∞) be the element of $G(F_\infty/\mathbf{Q})$ which is determined by the (a, F_m) for all $m \geqq 0$. Then (a, F_∞) acts on T by multiplication by a. Recalling that $\eta \in A_n^-$, and that the action of $G(F_\infty/\mathbf{Q})$ on $A_\infty^-(k)$ is the diagonal one, it follows that

$$(a,F_n)^{-1}(\eta \otimes \gamma^{\otimes k})=a^{-k}((a,F_n)^{-1}\eta)\otimes \gamma^{\otimes k}.$$

Thus

$$\theta_n(b)(\eta \otimes \gamma^{\otimes k})=w_{k+1}(\mathbf{Q})\, b^k((\rho_n(b)\,\eta)\otimes \gamma^{\otimes k}),$$

where

$$\rho_n(b)=\sum_{\substack{a=1 \\ (a,g_n)=1}}^{g_n} \delta_1(a,b;g_n)(a,F_n)^{-1}.$$

But, by the classical Stickelberger theorem for F_n over $\mathbf{Q}$ (see [9]), $\rho_n(b)\,\eta=0$. This proves (17).

On the other hand, since $\eta \otimes \gamma^{\otimes k}$ is in $(A_\infty^-(k))^{G_\infty}$, it is plain that

$$\alpha_n(b)(\eta \otimes \gamma^{\otimes k})=\alpha_n^*(b)(\eta \otimes \gamma^{\otimes k}),$$

where $\alpha_n^*(b)$ is the element of $\mathbf{Z}[G]$ obtained by replacing (c, F_n) by (c, F) (c any integer prime to $f\,l$) throughout in the definition of $\alpha_n(b)$. Noting that g_n and g_0 are divisible by precisely the same primes, it follows from Lemma 1.1 that

$$\alpha_n^*(b)=w_{k+1}(\mathbf{Q})(b^{k+1}-(b,F)) \sum_{\substack{a=1 \\ (a,g_0)=1}}^{g_0} \zeta_{g_0}(a,-k)(a,F)^{-1}.$$

If l divides f, the same argument allows us to deduce that $\alpha_n^*(b) = S_k(b)$. If l does not divide f, we conclude easily from Lemma 1.7 that

$$\alpha_n^*(b) = (1 - l^k(l, F)^{-1})\, S_k(b).$$

Since $k \geqq 1$, $1 - l^k(l, F)^{-1}$ is clearly a unit in the group ring $\mathbf{Z}_l[G]$. Thus, in either case, we conclude from (17) that $S_k^*(b)(\eta \otimes \gamma^{\otimes k}) = 0$. This completes the proof of Theorem 2.1.

We can now prove Theorem 2. Suppose first that F is totally real. Then, for each odd prime number l, the l-primary subgroup of $K_2\mathfrak{O}$ is known to be isomorphic as a G-module to $(A_\infty^-(1))^{G_\infty}$. This result can be derived from the work of Tate [16], Quillen [13], and others, as is shown in [3, 4]. Thus, by Theorem 2.1, $S_1(b)$ annihilates the l-primary subgroup of $K_2\mathfrak{O}$ for all odd primes l and all positive integers b which are prime to f and the order of $K_2\mathfrak{O}$. Assume next that F is totally imaginary. Let F^+ be the maximal totally real subfield of F, and $\mathfrak{O}^+$ the ring of algebraic integers of F^+. Then, of course, $G(F/F^+)$ is a cyclic group of order 2 which is generated by J. Let l be an odd prime number. If B is any abelian group, we write B_l for its l-primary subgroup. Since l is odd, it follows immediately from the existence of the trace map in K-theory (see [11]) that the inclusion of $\mathfrak{O}^+$ in $\mathfrak{O}$ induces an isomorphism

$$K_2\mathfrak{O}_l^+ \xrightarrow{\sim} (K_2\mathfrak{O}_l)^{G(F/F^+)}, \tag{18}$$

where both groups are regarded as modules over $G(F^+/\mathbf{Q})$. Moreover,

$$K_2\mathfrak{O}_l = (1-J)\,K_2\mathfrak{O}_l \oplus (1+J)\,K_2\mathfrak{O}_l,$$

and the direct summand on the right is the same as $(K_2\mathfrak{O}_l)^{G(F/F^+)}$. Now, if χ is a character of G with $\chi(J) = -1$, it follows from the functional equation for $L_f(\chi, s)$ that $L_f(\chi, -1) = 0$. It is therefore plain from Lemma 1.7 that $S_1(b)$ automatically annihilates $(1-J)\,K_2\mathfrak{O}_l$. On the other hand, $S_1(b)$ has the same effect on $(1+J)\,K_2\mathfrak{O}_l$ as the image $\bar{S}_1(b)$ of $S_1(b)$ under the canonical map from $G(F/\mathbf{Q})$ onto $G(F^+/\mathbf{Q})$. Let $S_1^+(b)$ denote the analogue of $S_1(b)$ for the field F^+, and let f^+ be the conductor of F^+ over $\mathbf{Q}$. Then it follows easily from Lemma 1.7 that

$$\bar{S}_1(b) = S_1^+(b) \prod (1 - q(q, F^+)^{-1}), \tag{19}$$

where q runs over all primes which divide f but not f^+. The assertion of Theorem 2 is now plain from (18), (19), and the corresponding result for F^+.

3. Concluding Remarks

We conclude by mentioning several related questions. Iwasawa [8] has shown that the classical Stickelberger element for $K_0\mathfrak{O}$ can be used to construct the p-adic L-functions of Kubota-Leopoldt. In a similar manner, one can prove that, for each fixed integer $k \geqq 1$, the conjectural Stickelberger element for $K_{2k}\mathfrak{O}$ discussed in this paper gives rise to p-adic functions which are very simply related to the Kubota-Leopoldt functions. Also one suspects that Conjecture 2 is just a special case of a vastly more general phenomenon. Indeed, as has already been pointed out by Brumer (unpublished, but see [14]) for the ideal class group, the work of Klingen and Siegel [15] enables one to at least formulate an analogue of Conjecture 2 in which $\mathfrak{O}$ is the ring of integers of an abelian extension of an arbitrary totally real base field. Needless to say, very little is known about this more general question. While it has no direct bearing on the problem of the annihilation of the K-groups, it is shown in [6] that these proposed Stickelberger elements for abelian extensions of a real quadratic field do, in fact, give rise to p-adic functions which are the precise analogues of the Kubota-Leopoldt functions.

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Time Dependent Stable Diffeomorphisms

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The study of structurally stable dynamical systems has been motivated by the following considerations. If one studies a differential equation arising from a physical situation where it is necessary to make measurements then, since the measurements are only approximations, one is really studying only an approximation to the true equation. In this case it is important to know the qualitative behavior of the approximation and the true dynamical system are the same. Structural stability of a dynamical system guarantees this if the approximation is sufficiently good.

It seems likely, however, that in most physical situations the measured quantities would not be completely independent of time but would instead be only approximately constant; their values staying near to some constant approximate value. In other words, the "true" dynamical system is not really autonomous but is instead to a certain extent time dependent. Under these assumptions we can ask when the approximate autonomous system is qualitatively equivalent to the true time dependent system. We are then asking when is an automous system structurally stable allowing perturbations to time dependent systems.

In this paper we give a solution to this problem for smooth (C^2) discrete time dynamical systems (diffeomorphisms) on compact manifolds.

Definition. A diffeomorphism $f: M \to M$ on a compact manifold is time dependent stable (*TD* stable) if there is a neighborhood N of f in $\mathrm{Diff}^1(M)$ with the property that if $g_1, g_2, \ldots, g_k \in N$ then there exists a homeomorphism $h: M \to M$ such that $h^{-1} \circ f^k \circ h = g_1 \circ \cdots \circ g_k$. N is independent of k.

Our main result is the following.

Theorem. *If $f: M \to M$ is a C^2 diffeomorphism of a compact manifold then f is TD stable if and only if f satisfies Axiom A and the strong transversality condition.*

The definitions of Axiom A and strong transversality can be found, for example, in [4]. We give the proof of this theorem in a sequence of lemmas. Throughout M will denote a C^∞ compact manifold and $|\,|$ the norm arising from a smooth Riemannian metric.

Lemma 1. *If $f: M \to M$ is a C^2 diffeomorphism which satisfies Axiom A and the strong transversality property then f is TD stable.*

Proof. Here we depend heavily on the results and techniques of Robbin [3] and assume considerable familiarity with results and definitions from this paper.

Let $\bar{M} = M \times \{1, 2, 3, \ldots, k\}$ and consider the diffeomorphism $\bar{f}: \bar{M} \to \bar{M}$ given by

$$\bar{f}(x, i) = \begin{cases} (f(x), i+1), & i \neq k \\ (f(x), 1), & i = k. \end{cases}$$

Since f satisfies Axiom A and the strong transversality condition, it is easy to see that $\bar{f}$ does also. Hence by the main result of [3] there exists an $\delta > 0$ such that if $\bar{g}$ is within δ of $\bar{f}$ in the C^1 metric then there is a homeomorphism $\bar{h}: \bar{M} \to \bar{M}$ close to the identity such that $\bar{f} \circ \bar{h} = \bar{h} \circ \bar{g}$. If N is the δ neighborhood of f in $\mathrm{Diff}^1(M)$ and if $g_1, \ldots, g_k \in N$ then we define

$$\bar{g}(x, i) = \begin{cases} (g_i(x), i+1), & i \neq k \\ (g_i(x), 1), & i = k. \end{cases}$$

In this case there exists an $\bar{h}$ such that $\bar{h}^{-1} \circ \bar{f}^k \circ \bar{h} = \bar{g}^k$. We let $h: M \to M$ be given by $h = \bar{h} | M \times 1$. Then since $\bar{g}^k | M \times 1 = g_k \circ g_{k-1} \circ \cdots \circ g_1$ and $\bar{f}^k | M \times 1 = f^k$ we have $h^{-1} \circ f^k \circ h = g_k \circ \cdots \circ g_1$. We need only show that δ can be chosen to be independent of k.

To do this it is necessary to check some of the estimates in Robbin's paper [3] and show that they are independent of k.

Robbin obtains the conjugacy h, between f and a perturbation g, by finding a fixed point γ of a certain map $J \circ R_\xi$ defined on d_f Lipschitz tangent vector fields of M and showing that $h = \exp(\gamma)$ is a homeomorphism which satisfies $g \circ h = h \circ f$. We first consider the map R_ξ which is defined by $R_\xi = Q - P_\xi$ where $\exp(\xi) = f^{-1} \circ g$ and Q and P_ξ are two other maps defined on the space of d_f-Lipschitz vector fields. We summarize the properties of Q and P_ξ:

Let $| \, |_0$ denote the C^0 sup norm on continuous tangent vector fields and let $\Lambda(\eta)$ denote the d_f-Lipschitz constant for a d_f-Lipschitz vector field η as in [3]. Then we have the following: Given $\varepsilon > 0$, there exist $\delta_0, \delta_1 > 0$ such that if $|\eta|_0, |\eta'|_0 < \delta_0$ and $|\xi|_1 < \delta_1$, then

$$|Q(\eta)|_0 \leqq \varepsilon |\eta|_0, \tag{3.1 A}$$

$$|Q(\eta) - Q(\eta')|_0 \leqq \varepsilon |\eta - \eta'|_0, \tag{3.1 B}$$

$$\Lambda(Q(\eta)) < \varepsilon + \varepsilon |\eta|_f, \tag{3.2}$$

$$|P_\xi(\eta)|_0 < |\xi|_0 + \varepsilon |\eta|_0, \tag{3.3 A}$$

$$|P_\xi(\eta) - P_\xi(\eta')|_0 \leqq \varepsilon |\eta - \eta'|_0, \tag{3.3 B}$$

$$\Lambda(P_\xi(\eta)) < \varepsilon + \varepsilon|\eta|_f \qquad \text{where } |\eta|_f = \Lambda(\eta) + |\eta|_0. \tag{3.4}$$

The numbers (3.1 A) etc. are those assigned to each equation in [3].

If we let $\bar{Q}$ and $\bar{P}_\xi$ be the corresponding functions for the diffeomorphism $\bar{f}$ then from their construction in [3] it follows that if η is a tangent vector field on $\bar{M}$ and we let $\eta(i) = \eta|_{M\times\{i\}}$ then

$$\bar{Q}(\eta)(i) = \begin{cases} Q(\eta(i-1)), & i \neq 1 \\ Q(\eta(k)), & i = 1 \end{cases}$$

and

$$\bar{P}_\xi(\eta)(i) = P_{\xi(i)}(\eta(i)).$$

Thus the Eqs. (3.1 A), (3.1 B), (3.2), (3.3 A), (3.3 B), and (3.4) all hold for $\bar{P}_\xi$ and $\bar{Q}$ with the same δ_0 and δ_1 for a given ε; i.e., δ_0, δ_1 depend only on ε not on k ($\bar{M} = M \times \{1, 2, \ldots, k\}$).

We must also get an estimate on J independent of k. J as constructed in [3] is a continuous linear map on the space of C^0 vector fields on M with norm $|\;|_0$. It also preserves the subspace of d_f-Lipschitz vector fields and is continuous when the norm $|\;|_f$ is used.

In order to state some estimates on J we must define some universal constants. If $\Omega_1, \Omega_2, \ldots, \Omega_l$ are the basic sets of f and $W_1, \ldots, W_l$ the unrevisited neighborhoods constructed in [3], then if $\bar{\Omega}_i = \Omega_i \times \{1, \ldots, k\}$ and $\bar{W}_i = W_i \times \{1, \ldots, k\}$, the basic sets of $\bar{f}$ are $\bar{\Omega}_1, \ldots, \bar{\Omega}_l$ and $\bar{W}_1, \ldots, \bar{W}_l$ are unrevisited neighborhoods of the $\bar{\Omega}_i$'s. In [3] an integer r is given such that $\{f^n(W_j) \mid -r \leqq n \leqq r, j = 1, \ldots, l\}$ cover M. Note that $\{\bar{f}^n(\bar{W}_j) \mid -r \leqq n \leqq r, j = 1, \ldots, l\}$ cover $\bar{M}$. Let $q = 2lr + r$. Let $b = \sup_{x\in M} \{|Df_x|, |Df_x^{-1}|, |D^2 f|\}$, then if $\bar{b}$ is the corresponding number for $\bar{f}$, $b = \bar{b}$. Finally let D be greater than the norm of any of the linear maps constructed in [3] which send η to η_{iu} or η to η_{is} ($i = 1, \ldots, l$). Then in [3] (6.14 and following) Robbin considers $J = J_u + J_s$ and shows

$$|J_u(\eta)|_0 \leqq C(1-\rho)^{-1} D|\eta|_0$$

and

$$\Lambda(J_u(\eta)) \leqq C(1-\rho)^{-1} D\Lambda(\eta) + bC(1-\rho)^{-2} D|\eta|_0$$

where ρ is a constant depending only on the hyperbolicity of f on basic sets and $C = b^q \rho^{-q}$. A similar result holds for J_s.

As already mentioned b and q would be the same for $\bar{f}$ but also ρ and D are the same so if $\bar{J}$ is the corresponding map for $\bar{f}$ we have

$$|\bar{J}(\eta)|_0 \leqq 2C(1-\rho)^{-1} D|\eta|_0$$

and

$$\Lambda(\bar{J}(\eta)) \leqq 2C(1-\rho)^{-1} D\Lambda(\eta) + 2bC(1-\rho)^{-2} D|\eta|_0,$$

with the constants b, C, ρ, D all independent of k.

Now given $\varepsilon > 0$, (3.1 A), (3.3 A) added give

$$|\bar{R}_\xi(\eta)|_0 \leqq |\xi|_0 + 2\varepsilon|\eta|_0$$

and from (3.2) and (3.4)

$$\Lambda(\bar{R}_\xi(\eta)) \leqq 2\varepsilon + 2\varepsilon \Lambda(\eta).$$

When these are added we obtain

$$|\bar{R}_\xi(\eta)|_f \leqq |\xi|_0 + 2\varepsilon + 2\varepsilon|\eta|_f$$

when $|\xi|_1$ and $|\eta|_0$ are sufficiently small.

We now repeat the argument given by Robbin in [3]. The above inequality implies

$$|\bar{J} \circ \bar{R}_\xi(\eta)|_f \leqq |\bar{J}_f| \, (|\xi|_0 + 2\varepsilon + 2\varepsilon|\eta|_f).$$

For a fixed ξ, *a* C^1 vector field on $\bar{M}$, define the d_f-Lipschitz vector field η_n inductively by $\eta_0 = 0$ and $\eta_{n+1} = \bar{J} \circ \bar{R}_\xi(\eta_n)$.

Given a sufficiently small $\gamma > 0$, by the last inequality there exist $\delta_0, \delta_1 > 0$, independent of k, such that

$$|\eta_{n+1}|_f \leqq \gamma \quad \text{if} \quad |\eta_n|_f \leqq \gamma, \quad |\eta_n|_0 < \delta_0 \quad \text{and} \quad |\xi|_1 < \delta_1.$$

Choose $\varepsilon > 0$ so that $2\varepsilon|\bar{J}|_0 < 1/2$. By (3.1 A) and (3.3 A) (shrinking δ_0 and δ_1 if necessary)

$$|\eta_{n+1}|_0 < |\bar{J}|_0 \, |\xi|_0 + 1/2 \, |\eta_n|_0$$

which by induction implies

$$\begin{aligned} |\eta_{n+1}|_0 &\leqq |\bar{J}|_0 \, |\xi|_0 \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\right) \\ &\leqq 2|\bar{J}|_0 \, |\xi|_0 . \end{aligned}$$

But if δ_1 (and hence $|\xi|_1$) is sufficiently small $2|\bar{J}|_0 \, |\xi|_0 < \delta_0$ so $|\eta_n|_0 < \delta_0$ for all n.

By (3.1 B) and (3.3 B)

$$|\eta_{n+1} - \eta_n| \leqq \tfrac{1}{2} |\eta_n - \eta_{n-1}|_0$$

so η_n converges uniformly to η. Since $\Lambda(\eta_n) \leqq \gamma$ for all n, η is d_f-Lipschitz and $\Lambda(\eta) \leqq \gamma$. Clearly $\gamma = \bar{J} \circ \bar{R}_\xi(\gamma)$. Notice that δ_1 is completely independent of k since it depends only on the ε of (3.1 A), (3.1 B), (3.2), (3.3 A), (3.3 B), and (3.4); and on $|\bar{J}|_f$ and $|\bar{J}|_0$.

Now if

$$N = \{g \in \mathrm{Diff}^1(M) \,|\, f^{-1} \circ g = \exp(\xi), \ |\xi|_1 < \delta_1\}$$

and if $g_1, \ldots, g_k \in N$ with $f^{-1} \circ g_i = \exp(\xi(i))$ then we define the vector field ξ on $\bar{M}$ by $\xi|_{M \times \{i\}} = \xi(i)$ and note $|\xi|_1 < \delta_1$. So there exists η satisfying

$\eta = J \circ \bar{R}_{\xi}(\eta)$. If $\bar{h} = \exp(\eta)$ then by Robbin's proof $\bar{h}$ is a topological conjugacy between $\bar{f}$ and $\bar{g}$. Since N is independent of k we have by our earlier remarks shown that f is *TD* stable. q.e.d.

Remarks. 1) The hypothesis of the main theorem that f be C^2 is used only in the proof of this lemma and for the proof of the converse it is sufficient that f be C^1.

2) By the techniques of [1] one can show that the h constructed depends in a C^1 fashion on $g_1, \ldots, g_k$. Also there exists a $K > 0$ depending only on f such that

$$\sup_{x \in M} d(h(x), x) \leqq K \max_{1 \leqq i \leqq k} \sup_{x \in M} d(f(x), g_i(x)).$$

Lemma 2. *If U is a neighborhood of the identity in $\mathrm{Diff}^1(M)$ there exists $\varepsilon > 0$ such that if $v \in TM$ and $w \in TM$ is such that $d(v, w) \leqq \varepsilon |v|$ then there is $g \in U$ such that $Dg(v) = w$.*

Here $d(\ ,\ )$ is a metric on *TM* induced by a Riemannian metric. The proof of this lemma is straightforward and hence will be omitted.

Lemma 3. *Suppose X is a compact metric space and $g: X \to X$ is a homeomorphism. Then given $\varepsilon > 0$, there exists an integer $n_0 > 0$ and a neighborhood N of g in the C^0 (uniform) topology such that: if $h \in N$ and $h^m(x) = y$ for some $m > 0$, then there exist $x_0, x_1, \ldots, x_n \in X$ with $x = x_0$, $y = x_n$, $d(h(x_i), x_{i+1}) < \varepsilon$, and $n \leqq n_0$.*

The importance of this lemma is that n_0 depends only on ε not on the points x and y. So, while x and y are on the same orbit it might take many applications of h to get from x to y and the lemma allows us to take a short cut in fewer than n_0 steps if we are satisfied with an ε approximation.

Proof. Let $N = \{h \mid \sup_{x \in X} d(h(x), g(x)) < \varepsilon/6\}$ and let $\delta > 0$ be a number such that $d(x_1, x_2) < \delta$ implies $d(g(x_1), g(x_2)) < \varepsilon/6$ (X is compact so g is uniformly continuous). Thus if $d(x_1, x_2) < \delta$, and $h \in N$

$$\begin{aligned} d(h(x_1), h(x_2)) &\leqq d(h(x_1), g(x_1)) + d(g(x_1), g(x_2)) + d(g(x_2), h(x_2)) \\ &< \varepsilon/6 + \varepsilon/6 + \varepsilon/6 = \varepsilon/2. \end{aligned}$$

Since X is compact there is a finite set S in X such that if $w \in X$ there exists $z \in S$ with $d(w, z) < \min(\varepsilon/2, \delta)$.

Let n_0 be the cardinality of S plus 2. Now given $h \in N$ and $y = h^m(x)$, we pick a sequence z_i, $1 \leqq i \leqq m-1$ such that $d(h^i(x), z_i) < \min(\delta, \varepsilon/2)$ and $z_i \in S$. Clearly $d(h(x), z_1) < \varepsilon$, $d(h(z_{m-1}), y) < \varepsilon$ and

$$\begin{aligned} d(h(z_i), z_{i+1}) &\leqq d(h(z_i), h^{i+1}(x)) + d(h^{i+1}(x), z_{i+1}) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all i, $1 \leqq i \leqq m-2$. If $m-1$ is less than the cardinality of S we are finished letting $x_i = z_i$. If not there exist k, j with $z_k = z_j$, $k<j$. Hence we can shorten the sequence of z's by setting $z'_i = z_i$, for $i \leqq k$ and $z'_i = z_{i+(j-k+1)}$ for $i>k$. If the new sequence has fewer than n_0 elements we are finished, otherwise it too contains repetitions and can be shortened further. Ultimately we achieve the desired sequence. q.e.d.

Recall that a periodic point p of a diffeomorphism f is said to be hyperbolic if there is a direct sum splitting $TM_p = E^s_p \oplus E^u_p$ and constants $C>0$, $0<\lambda<1$ such that $|Df^n(v)| \leqq C\lambda^n|v|$ for $v \in E^s_p$ and $|Df^{-n}(v)| \leqq C\lambda^n|v|$ for $v \in E^u_p$. It is shown in [2] that all periodic points of a structurally stable diffeomorphism must be hyperbolic. Since *TD* stable diffeomorphisms are structurally stable and open, each *TD* stable diffeomorphism has a neighborhood of diffeomorphisms all of whose periodic points are hyperbolic.

Definition. For any diffeomorphism $g\colon M \to M$ we will denote by $\hat{g}$ the diffeomorphism of $SM = \{x \in TM\ |v|=1\}$ given by $\hat{g}(v) = Dg(v)/|Dg(v)|$. A point $v \in SM$ will be called *ε-non-wandering* if for any integer N and $\delta>0$, there exists an integer $n>N$ and points $u, w \in SM$ such that $d(u,v)<\delta$, $d(w,v)<\varepsilon$ and $\hat{g}^n(u)=w$. The set of ε-non-wandering points will be denoted $\Sigma(\varepsilon)$.

Lemma 4. *If g is TD stable then there exists an integer $m_0>0$ and numbers $\varepsilon>0$, and $0<\rho<1$ with the following properties: If $v \in \Sigma(\varepsilon)$ then either $|Dg^n(v)|^{1/n}<\rho$ or $|Dg^n(v)|^{1/n}>\rho^{-1}$ for all $n \geqq m_0$.*

Proof. Choose a constant $K>0$ such that

$$K > \sup |Dg_x| \quad \text{and} \quad K > \sup |Dg_x^{-1}|.$$

Choose $\delta>0$ so that if g' is within δ of g in the C^1 metric then g' is also *TD* stable.

Applying Lemma 2 we can obtain an ε such that for each $v \in SM$ and $w \in SM$ with $d(w, \hat{g}(v))<\varepsilon$ there is a g' such that C^1 distance from g to g' is less than $\delta/4$ and $\hat{g}'(v)=w$.

We now apply Lemma 3 and obtain an integer $n_0>0$ such that, if $x, y \in SM$ with y on the forward $\hat{g}$ orbit of x, then there exists a sequence $\{x_i\}$, $0 \leqq i \leqq n \leqq n_0$, with $d(x_{i+1}, \hat{g}(x_i))<\varepsilon$ and $x_0=x$, $x_n=y$.

Now choose an integer $l>0$ such that

$$\frac{1}{n}\log K^{n_0} < \frac{1}{2}\log(1+\varepsilon)$$

and

$$\frac{1}{n}\log K^{-n_0} > \frac{1}{2}\log(1-\varepsilon)$$

for $n > l$.

We now assume the lemma is false and derive a contradiction. If the lemma is false, there is a $v' \in \Sigma(\varepsilon)$ such that

$$\limsup_m \frac{1}{m}\log|Dg^m(v')| \geqq 0$$

and

$$\liminf_m \frac{1}{m}\log|Dg^m(v')| \leqq 0.$$

From this and the easily verified fact the difference in two successive terms of the sequence $\frac{1}{m}\log|Dg^m(v')|$ will be small when m is large we can conclude that it is possible to choose a subsequence m_i such that

$$\lim_{i\to\infty} \frac{1}{m_i}\log|Dg^{m_i}(v')| = 0.$$

Thus we can choose $m > l$ such that

$$\frac{1}{2}\log(1-\varepsilon) < \frac{1}{m}\log|Dg^m(v')| < \frac{1}{2}\log(1+\varepsilon).$$

Since $v' \in \Sigma(\varepsilon)$ there are $v, y \in SM$ such that $d(v, v') < \varepsilon/2$, $d(y, v') < \varepsilon/2$ and $\hat{g}^k(v) = y$ for some $k > m$. Moreover we can choose v so close to v' that

$$\frac{1}{2}\log(1-\varepsilon) < \frac{1}{m}\log|Dg^m(v)| < \frac{1}{2}\log(1+\varepsilon).$$

Let $x = \hat{g}^m(v)$. Now by our application of Lemma 3 above there exists a sequence $\{x_j\} \subset SM$, $0 \leqq j \leqq n \leqq n_0$ such that $d(\hat{g}(x_i), x_{i+1}) < \varepsilon$ and $x_0 = x$, $y = x_n$.

Applying Lemma 2 as mentioned above there exist diffeomorphisms h_j such that the C^1 distance from h_j to g is less than $\delta/4$ and $\hat{h}_j(x_j) = x_{j+1}$. Altering h_n by another $\delta/4$ perturbation we can assume that $\hat{h}_n(x_{n-1}) = v$. Now let $\bar{M} = M \times \{1, 2, \ldots, m+n\}$ and define $F: \bar{M} \to \bar{M}$ by

$$F(x, i) = \begin{cases} (g(x), i+1) & 1 \leqq i < m \\ (h_{i-m}(x), i+1) & m \leqq i < m+n \\ (h_n(x), 1) & i = m+n. \end{cases}$$

Let $\bar{v}=(v,1)\in S\bar{M}=SM\times\{1,\ldots,m+n\}$, then if $q=m+n$, $\hat{F}^q(\bar{v})=\bar{v}$. Also if $\bar{g}\colon \bar{M}\to\bar{M}$ is given by $\bar{g}(x,i)=(g(x),i+1)$ for $1\leqq i<q$ and $\bar{g}(x,q)=(g(x),1)$ then the C^1 distance from F to $\bar{g}$ is less than $\delta/2$.

We wish now to alter F to G for which $DG^q(\bar{v})=\bar{v}$. We have

$$\begin{aligned}\frac{1}{q}\log|DF^q(\bar{v})| &= \frac{1}{q}\log|Dh_n\circ\cdots\circ Dh_1\circ Dg^m(v)|\\ &\leqq\frac{1}{q}\log|K^n Dg^m(v)|\leqq\frac{1}{q}\log K^n+\frac{1}{q}\log|Dg^m(v)|\\ &\leqq\frac{1}{2}\log(1+\varepsilon)+\frac{1}{2}\log(1+\varepsilon)=\log(1+\varepsilon).\end{aligned}$$

Similarly $\log(1-\varepsilon)\leqq\frac{1}{q}\log|DF^q(\bar{v})|$. Define γ by

$$\log(1+\gamma)=\frac{1}{q}\log|DF^q(\bar{v})|$$

so $-\varepsilon\leqq\gamma\leqq\varepsilon$. Let $\bar{v}_i=\hat{F}^i(\bar{v})$. By another application of Lemma 3 we can obtain $G\colon\bar{M}\to\bar{M}$ such that the C^1 distance from G to F is less than $\delta/4$ and $DG(\bar{v}_i)=(1+\gamma)^{-1}DF(\bar{v}_i)$. Now

$$|DG^q(\bar{v})|=(1+\gamma)^{-q}|DF^q(\bar{v})|=(1+\gamma)^{-q}(1+\gamma)^q=1\quad\text{so}\quad DG^q(\bar{v})=\bar{v}$$

and G has a non-hyperbolic periodic point. However, since the C^1 distance from G to $\bar{g}$ is less than $3\delta/4$ it follows from the choice of δ that G is structurally stable and hence should have only hyperbolic periodic points. We have arrived at a contradiction from the assumption that the lemma is false. q. e. d.

Lemma 5. *If f is TD stable there exist constants $C>0$ and $0<\lambda<1$ such that if $p\in\operatorname{Per}(f)$*

$$\begin{aligned}|Df^n(v)|&\leqq C\lambda^n|v|\quad\text{for } v\in E^s_p\\ |Df^{-n}(v)|&\leqq C\lambda^n|v|\quad\text{for } v\in E^u_p.\end{aligned}$$

The constants C and λ are independent of p.

Proof. Let m_0 and ρ be as in Lemma 4. We prove the inequality for E^s by deriving a contradiction from the assumption it is false. If the lemma is false, there exists a point p and a vector $v\in E^s_p$ such that

$$|Df^n(v)|\geqq\lambda^n|v|\qquad\text{for some } n\geqq m_0$$

where λ is a number less than 1 satisfying $\varepsilon/4\,\lambda^n>\rho^n$ for the ε of the conclusion of Lemma 4 (we assume $\varepsilon<\frac{1}{2}$).

We can assume $|v|=1$. If p has period k, an examination of the Jordan canonical form of Df^k: $E_p \to E_p$ shows there is $w \in E_p$ such that $|w|=1$ and $|\hat{f}mk(v)-\hat{f}mk(w)|$ tends to 0 as mk tends to ∞ (recall $\hat{f}(v)=Df(v)/|Df(v)|$). In fact w will either be in the kernel of $(A-\lambda I)\circ(A-\bar{\lambda} I)$ where λ is a complex eigenvalue of Df^k and A is the complexification of Df^k or else w will be a real eigenvector of Df^k. In either case w is a recurrent point of $\hat{f}^k$.

Now let $u'=w+\frac{\varepsilon}{4}v$ and $u=u'/|u'|$. Since $|\hat{f}mk(v)-\hat{f}mk(w)|$ goes to zero as m tends to infinity it follows there is a sequence of integers n_i such that $\lim_{i\to\infty} \hat{f}^{n_i}(u)=w$. We know that $|u-w|<\varepsilon/2$ so by Lemma 4 u must satisfy $|Df^n(u)|\leqq \rho^n|u|$ for $n>m$. On the other hand, since $0<\varepsilon<\frac{1}{2}$

$$|Df^n(u)|=\frac{1}{|u'|}|Df^n(u')|\geqq\frac{1}{1+\varepsilon}|Df^n(u')|\geqq\frac{2}{3}|Df^n(u')|.$$

So

$$\begin{aligned}|Df^n(u)|&\geqq\frac{2}{3}\left|\frac{\varepsilon}{2}Df^n(v)+Df^n(w)\right|\\&\geqq\frac{2}{3}\left(\frac{\varepsilon}{2}\lambda^n-\rho^n\right)\\&>\frac{\varepsilon}{2}\lambda^n-\rho^n\geqq 2\rho^n-\rho^n=\rho^n.\end{aligned}$$

Hence we have concluded $|Df^n(u)|>\rho^n$ and $|Df^n(u)|\leqq\rho^n$ which contrdicts the assumption that the lemma is false. A similar argument proves the inequality for vectors in E^u. q.e.d.

Lemma 6. *If f: $M\to M$ is TD stable then f satisfies Axiom A and the strong transversality property.*

Proof. We must show that the non-wandering set Ω of f has a hyperbolic structure. Since it is generically true that the non-wandering set of a diffeomorphism is the closure of its periodic points and since f is structurally stable, we know, by the general density theorem of C. Pugh, that $\Omega=$closure Per(f), so this part of Axiom A is satisfied.

Let $x\in\Omega$ and choose $p_i\in$ Per(f) such that $\lim_{i\to\infty} p_i=x$. Let $E_i^s=E_{p_i}^s$ and $E_i^u=E_{p_i}^u$; by choosing a subsequence we can assume dim E_i^s and dim E_i^u are constant, that $\lim_{i\to\infty} E_i^s=E_x^s\subset TM_x$ and that $\lim_{i\to\infty} E_i^u=E_x^u\subset TM_x$.

Applying Lemma 5 we see that, since each $v\in E_x^s$ is the limit of some sequence $v_i\in E_i^s$, we have

$$|Df^n(v)|\leqq C\lambda^n|v| \quad \text{for } v\in E_x^s.$$

Since $|Df^n(v)| \geqq C^{-1}\lambda^{-n}|v|$ for $v \in E^u_p$ it follows that $|Df^n(v)| \geqq C^{-1}\lambda^{-n}|v|$ for $v \in E^u_x$. Thus $E^s_x \cap E^u_x = 0$ and since $\dim E^s_x + \dim E^u_x = \dim E^s_i + \dim E^u_i = \dim M$ we know $E^s_x \oplus E^u_x = TU_x$.

In this fashion we can assign vector spaces $E^s_y \oplus E^u_y = TM_y$ to each point $y \in \Omega$ which satisfy the hyperbolicity requirements and it is only necessary to show they vary continuously.

Note that for each x,

$$E^s_x = \{v \in TM_x \,|\, |Df^n(v)| \leqq C\lambda^n |v| \text{ for all } n\}.$$

Hence if $\lim_{i\to\infty} x_i = x$ then the limit of any convergent subsequence of the sequence $E^s_{x_i}$ is contained in E^s_x. But we can assume by choosing a further subsequence that $E^u_{x_i}$ converges to a subspace of E^u_x and thus dimension considerations imply that the limit of the subsequence of E^s_x is actually E^s_x. Since this is true for any convergent subsequence we actually have $\lim_{i\to\infty} E^s_{x_i} = E^s_x$. So $E^s = \bigcup_{x\in\Omega} E^s_x$ and $E^u = \bigcup_{x\in\Omega} E^u_x$ are continuous vector bundles with $TM|_\Omega = E^s \oplus E^u$, and we have shown that f satisfies Axiom A.

By a theorem stated in [5] which is not difficult to prove, if f is structurally stable and satisfies Axiom A then f satisfies this strong transversality property. q.e.d.

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Algebraic Vector Bundles over $\mathbb{A}^3$ Are Trivial

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Introduction

Let k be a field and $A=k[X_1, \ldots, X_n]$. Serre [9] posed the question: is every finitely generated projective A-module free. For $n \leqq 1$, it is classical that the answer is in the affirmative. The case $n=2$ was settled in the affirmative by Seshadri [10]. Bass' cancellation theorem [3, Chapter IV, 3.5] (together with the fact that projectives are stably free over A which is consequence of Grothendieck's theorem) implies that projective A-modules of rank $>n$ are free. Of course, rank 1 projectives are free over A since A is factorial. By Bass [4] it also follows that if n is odd and P is a projective A-module of rank n, then $P \approx P' \oplus A$, for some P'. Also, projective modules of infinite rank are free over A by Bass [5]. We prove here the following.

Theorem. *Let k be an algebraically closed field. Any finitely projective module over $k[X, Y, Z]$ is free.*

Our method of proof is roughly as follows. Let $A=k[X, Y, Z]$ with k algebraically closed and let P be a projective A-module of rank r. The case $r=1$ is trivial. The case $r \geqq 4$ is covered by ([3, Chapter IV, 35]). If $r=3$, then $P \approx P' \oplus A$. We may thus suppose $r=2$.

In Section 1, it is proved that if P is "close" to a free module F in the following sense:

$$P \supset F \supset \alpha P$$

where $\alpha \in k[X]$ has no triple roots, then P is free (Lemma 1.4).

Using Kleiman's Bertini theorems for vector bundles (see [6, Corollary 3.6] and [7, Theorem 7.3]) one maps P onto a prime ideal $\mathfrak{p}$ of a non-singular curve C in $\mathbb{A}^3$, i.e., there is an exact sequence

$$0 \to A \to P \to \mathfrak{p} \to 0 \qquad \text{(E)}$$

and we shall study P by means of this sequence. In this situation, we have $\operatorname{Ext}^1_A(\mathfrak{p}, A) \approx A/\mathfrak{p}$. It is well known (cf. [2]) that the module of differentials on C is isomorphic to $\operatorname{Ext}^1_A(\mathfrak{p}, A)$ and hence free.

Sections 2 and 3 are devoted to the study of the prime ideal $\mathfrak{p}$. A central role is played in this investigation by a construction of Abhyankar's which gives a specific set of three generators for $\mathfrak{p}$, as follows.

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By choosing a suitable coordinate system in $\mathbb{A}^3$, we may assume C lies nicely over its projection Γ in the (X, Y)-plane. (More precisely: Γ has only nodes as singularities and $C \to \Gamma$ is birational and integral.) In such a coordinate system Abhyankar [1] constructs a set of generators for $\mathfrak{p}$ of degrees 0, 1, 2, respectively:

$$\gamma, \alpha Z-\beta, a Z^2+b Z+c. \tag{1}$$

Here $\mathfrak{p} \cap k[X, Y]=(\gamma)$ any $\alpha \in B=k[X, Y]$ may be chosen, such that the image $\bar{\alpha}$ of α in $A/\mathfrak{p}$ generates the conductor $\mathfrak{c}$ from $A/\mathfrak{p}$ to $B/\gamma B$, and β, a, b, $c \in B$ are chosen suitably. For the sake of completeness we include a proof of this result.

In Section 2, we also show (Lemma 2.7) that Abhyankar's result is available in the present situation, i.e., that conductor $\mathfrak{c}$ is principal by showing that $\mathfrak{c}$ is isomorphic to the module of differentials on C. (It should be noted that, in [1], Abhyankar gives a construction which works without the assumption that $\mathfrak{c}$ is principal; however, it is the more special construction which is needed in the present proof; Section 2 concludes with a technical result (Lemmas 2.4 and 2.6) which asserts that α, β, in 1.) Maybe chosen such that they intersect transversally; this is needed at a crucial point in the end of the proof.

In Section 3, we use Abhyankar's generators to construct a projective resolution $E(\bar{\alpha})$ for $\mathfrak{p}$ which turns out to depend only on the image $\bar{\alpha}$ of α in $A/\mathfrak{p}$. We then show that for a suitable choice of generator $\bar{\alpha}$ for $\mathfrak{c}$, the extension $E(\bar{\alpha})$ is isomorphic to the extension

$$0 \to A \to P \to \mathfrak{p} \to 0$$

with which we began. This gives us an extremely explicit description of P in terms of Abhyankar's generators for $\mathfrak{p}$ (as a sub-module of $\mathbb{A}^3$ generated by four quite explicit elements). We then conclude the proof in Section 4, by using this description of P to show it satisfies the hypothesis of Lemma 1.4, and so is free.

As Serre observed in [8], it is a consequence of the theorem thus obtained that any nonsingular curve in $\mathbb{A}^3$ with trivial canonical line bundle (e.g., any nonsingular rational or elliptic curve) is an idealtheoretic complete intersection.

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1. Some Lemmas on Projective Modules

The object of this section is to prove the following lemma which will be used at the end of the proof of the theorem.

Lemma (see Lemma 1.4). *Let k be an algebraically closed field and P a finitely generated projective module of rank 2 over $k[X, Y, Z]$. Suppose there exists a free sub-module F of P and an $\alpha \in k[X]$ without triple roots such that $\alpha P \subset F$. Then P is free.*

Throughout this paper k will denote an algebraically closed field and A the ring $k[X, Y, Z]$.

Lemma 1.1. *Let F be a free A-module of rank 2, P a sub-A-module such that $F \supset P \supset XF$; then P is A-free.*

Proof. Let $\{f_1, f_2\}$ be a free A-basis for F, and let

$$\bar{F} = \{a_1 f_1 + a_2 f_2 \mid a_1, a_2 \text{ in } k[Y, Z]\}.$$

$\bar{F}$ is free over $k[Y, Z]$ of rank 2, and $F = \bar{F} \oplus XF$ (direct sum as $k[Y, Z]$-modules). Thus, with $\bar{P} = P \cap \bar{F}$, $P = \bar{P} \oplus XF$ (direct sum as $k[Y, Z]$-modules. Clearly $\bar{P}$ is projective, hence free (by Seshadri's theorem) over $k[Y, Z]$ of rank $r \leqq 2$. We must consider three cases:

$r=0$. $P = XF$ is free.

$r=1$. Let $\bar{P}$ be free over $k[Y, Z]$ on $e = a_1 f_1 + a_2 f_2$ (a_1, a_2 in $k[Y, Z]$); then $P = \bar{P} \oplus XF$ shows that P has the generating set $\{e, Xf_1, Xf_2\}$ over A, with all relations on these generated by $(X, -a_1, -a_2)$. Since P is projective, this row $(X, -a_1, -a_2)$ must be unimodular, i.e., there exist b, b_1, b_2 in A with $bX - a_1 b_1 - a_2 b_2 = 1$. Setting $X=0$, we get $-a_1 b_1(0) - a_2 b_a(0) = 1$; then the row $(X, -a_1, -a_2)$ may be completed via the identity

$$\begin{vmatrix} X & -a_1 & -a_2 \\ 0 & -b_2(0) & -b_1(0) \\ 1 & 0 & 0 \end{vmatrix} = 1$$

which exhibits the fact that P is A-free (namely, on $-b_2(0) X f_1 + b_1(0) X f_2$ and e).

$r=2$. Since P/XP and $P/XF \cong \bar{P}$ are both free $k[Y, Z]$-modules of rank 2, $XP = XF$, i.e., $P = F$ and we are done.

Lemma 1.2. *Let C, D be 2×2 matrices over $k[Y, Z]$ with $D^2 = 0$. Let the 2×4 matrix*

$$M = (XI_2 + D \mid C)$$

over A be unimodular, i.e., *let the 2×2 subdeterminants of M generate the unit ideal in A. Then there exists a 2×4 matrix N over A such that*

$$\det \left(\frac{M}{N} \right) = 1. \qquad (1)$$

Proof. Let $D=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$; since the characteristic polynomial of D is $\det(XI_2+D)=X^2$, we have $0=\operatorname{trace} D=\det D$, i.e.,

$$p=-s, \quad p^2=-qr.$$

Since $k[Y,Z]$ is factorial, there exists λ, μ, v in $k[Y,Z]$ such that

$$p=-s\ \lambda\mu v, \quad q=\lambda\mu^2, \quad r=-\lambda v^2.$$

Let p_{ij} denote the subdeterminant formed from the i-th row and j-th column of

$$M=\begin{pmatrix} X+\lambda\mu v & \lambda\mu^2 & a & b \\ -\lambda v^2 & X-\lambda\mu v & c & d \end{pmatrix}, \quad \text{where} \quad C=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since M is unimodular, there exist q'_{ij} in A such that $\sum_{i<j} P_{ij} q'_{ij}=1$; setting $X=0$ in this equation, we see that also

$$M_0=\begin{pmatrix} \lambda\mu v & \lambda\mu^2 & a & b \\ -\lambda v^2 & -\lambda\mu v & c & d \end{pmatrix}$$

is unimodular; note that the entries in M_0 lie in $k[Y,Z]$. It follows that also

$$M_1=\begin{pmatrix} \lambda\mu & a & b \\ -\lambda v & c & d \end{pmatrix}$$

is unimodular; indeed, the 2×2 subdeterminants of M_0 are linear combinations of the 2×2 subdeterminants of M_1. Thus, there exists α, β, γ in $k[Y,Z]$ such that

$$1=\begin{vmatrix} \lambda\mu & a & b \\ -\lambda\mu & c & d \\ \alpha & \beta & \gamma \end{vmatrix}=\alpha\begin{vmatrix} a & b \\ c & d \end{vmatrix}-\beta\lambda(bv+d\mu)+\gamma\lambda(av+c\mu). \tag{2}$$

It is easy, using (2), to compute specific linear combinations of the 2×2 subdeterminants of M_0, and then of M, which equal 1; a pleasant surprise awaits us if we do it.

We begin by computing the 2×2 subdeterminants $p_{ij}^{(0)}$ of M_0; they are:

$$p_{12}^{(0)}=0,\ p_{13}^{(0)}=\lambda v(av+c\mu),\ p_{14}^{(0)}=\lambda v(bv+d\mu),\ p_{23}^{(0)}=\lambda\mu(av+c\mu),$$

$$p_{24}^{(0)}=\lambda\mu(bv+d\mu) \quad \text{and} \quad p_{34}^{(0)}=\begin{vmatrix} a & b \\ c & d \end{vmatrix};$$

we then obtain

$$p_{12}=X^2,\ p_{13}=p_{13}^{(0)}+cX,\ p_{14}=p_{14}^{(0)}+dX,$$
$$p_{23}=p_{24}^{(0)}-aX,\ p_{24}=p_{24}^{(0)}-bX,\ p_{34}=p_{34}^{(0)}.$$

If we square the right-hand side of (2) and express the resulting 9 terms as linear combinations of the above $p_{ij}^{(0)}$, we obtain $1=\sum_{i<j} p_{ij}^{(0)} q_{ij}$ with

$$q_{12}=0,\ q_{13}=\lambda\gamma(a\gamma-b\beta),\ q_{14}=-\lambda\beta(a\gamma-b\beta),\ q_{23}=\lambda\gamma(c\gamma-d\beta),$$

$$q_{24}=-\lambda\beta(c\gamma-d\beta) \quad \text{and} \quad q_{34}$$

$$=\alpha^2\begin{vmatrix} a & b \\ c & d\end{vmatrix}-2\alpha\beta\lambda(bv+d\mu)-2\alpha\gamma\lambda(av+c\mu).$$

Note that all q_{ij} lie in $k[Y,Z]$. Very simple computations suffice to verify the two following remarkable coincidences:

a) $1=\sum_{i<j} p_{ij} q_{ij}$;

b) The q_{ij} satisfy the Plucker conditions, i.e.,

$$q_{12}q_{34}-q_{13}q_{24}+q_{14}q_{23}=0.$$

We may now demonstrate the existence of a matrix N satisfying (1). It suffices to show there exists a 2×4 matrix N whose 2×2 subdeterminants n_{ij} are given by

$$n_{12}=q_{34},\ n_{13}=-q_{24},\ n_{14}=q_{23},\ n_{23}=q_{14},\ n_{24}=-q_{13},\ n_{34}=q_{12}$$

since then (1) follows from (a). Since (b) implies

$$n_{12}n_{34}-n_{13}n_{24}+n_{14}n_{23}=0,$$

the existance of N results by applying to the ring $k[Y,Z]$, Theorem 1.1 of [11].

Lemma 1.3. *Let F be a free A-module of rank 2, P a projective sub-A-module of F such that $F\supset P\supset X^2F$; then P is A-free.*

Proof. Let $\{f_1, f_2\}$ be a free A-basis for F, and let $\bar{F}$ be the set of all $a_1 f_1+a_2 f_2$ with a_1, a_2 elements of A whose degrees in X are $\leqq 1$. Thus $\bar{F}$ is a free $k[Y,Z]$-module of rank 4, and $F=\bar{F}\oplus X^2F$ (direct sum as $k[Y,Z]$-modules). Thus, with $\bar{P}=P\cap\bar{F}$,

$$P=\bar{P}\oplus X^2F \tag{3}$$

(direct sum as $k[Y,Z]$-modules). Clearly $\bar{P}$ is projective, hence free, over $k[Y,Z]$ of rank $\leqq 4$. We must consider five cases, depending on the $k[Y,Z]$-rank r of $\bar{P}\cong P/X^2F$.

$r=0$: $P=X^2F$ is free.

$r=1$: Let P/X^2F be $k[Y,Z]$-free on e, and let $Xe=ce$, $c\in k[Y,Z]$; then $0=X^2e=c^2e$, $c=0$, i.e., $Xe=0$. Thus $X^2F\supset XP, XF\supset P\supset X^2F$, and P is free over A by Lemma 1.1.

$r=2$. (This case is the critical one.) Let $\bar{P}$ be free over $k[Y,Z]$ on e_1, e_2 with $e_i=a_{i1}f_1+a_{i2}f_2$ ($i=1$ or 2, a_{ij} in A of degree in $X\leqq 1$). As

(3) shows, P is generated over A by the set $S=\{e_1, e_2, X^2 f_1, X^2 f_2\}$. We next compute a generating set for module of A-relations on S. We have, $\deg_X(X a_{ij}) \leqq 2$; thus we may write, in a unique fashion,

$$X a_{ij} = b_{ij} + X^2 c_{ij}$$

with $c_{ij} \in k[Y, Z]$, b_{ij} in A of degree in $X \leqq 1$. Thus

$$X e_i = p_i + c_{i1} X^2 f_1 + c_{i2} X^2 f_2 \qquad (i=1, 2)$$

with

$$p_i = b_{i1} f_1 + b_{i2} f_2$$

an element of $\bar{P}$; thus we may write $p_i = d_{i1} e_1 + d_{i2} e_2$, and so

$$X e_i = d_{i1} e_1 + d_{i2} e_2 + c_{i1} X^2 f_1 + c_{i2} X^2 f_2 \quad (c_{ij} \text{ and } d_{ij} \text{ in } k[Y, Z],\ i=1, 2). \tag{4}$$

Thus, if C, D denote the 2×2 matrices $(-c_{ij})$, $(-d_{ij})$ over $k[Y, Z]$, the rows of the 2×4 matrix $M=(XI_2+D|C)$ are relations on the generating set S for P.

Suppose $\alpha_1 e_1 + \alpha_2 e_2 + \beta_1 X^2 f_1 + \beta_2 X^2 f_2$, α_i and β_i in A for $i=1, 2$. We claim that $r=(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is an A-linear combination of the rows of M, i.e., that these two rows generated over A the module of A-relations on S. Indeed, by subtracting from r a linear combination of these two rows, we may reduce to the case where α_i, β_i do not involve X, whence $\alpha_1 e_1 + \alpha_2 e_2 \in \bar{P}$; since the sum (3) is direct, we then have

$$\beta_1 = \beta_2 = 0,\ \alpha_1 e_1 + \alpha_2 e_2 = 0,\ \alpha_1 = \alpha_2 = 0$$

and our claim is justified.

Since the 2×4 matrix M is a relation-matrix for the rank 2 projective module, P, M must be unimodular, i.e., the 2×2 subdeterminants of M generate the unit ideal in M. Eq. (4) shows that multiplication by X on $P/X^2 F$ is represented by the matrix $(d_{ij}) = -D$ with respect to the free basis $e_1 + X^2 F$, $e_2 + X^2 F$; thus $D^2 = 0$. The hypotheses of Lemma 2 are all satisfied by M; thus M is completable to a 4×4 matrix of determinant 1, whence P is free.

$r=3$. Consider the $k(Y, Z]$-endomorphism M_X of $P/X^2 F \cong \bar{P}$ consisting of multiplication by X. Clearly $M_X^2 = 0$; this, together with the fact $\bar{P}$ is $k[1, Z]$-free of rank 3, shows that the ker M_X must have rank 2 or 3 over $k[Y, Z]$ depending on which of the two possible Jordan canonical forms:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

M_X has over $k(Y, Z)$. If ker M_X has rank 3 then $M_X=0$, $XP \subseteq X^2 F$, i.e., $XF \supseteq P \supseteq X^2 F$, which is impossible since then $r \leqq$ rank $XF/X^2 F = 2$; thus ker M_X must have rank 2 over $k[Y, Z]$. Then also Coker

$$M_X = P/(XP + X^2 F)$$

has rank 2 over $k[Y, Z]$. Tensoring with A/XA the exact sequence

$$0 \to F \xrightarrow{X^2} P \to P/X^2 F \to 0. \tag{5}$$

We obtain the exact sequence

$$F/XF \xrightarrow{i} P/XP \xrightarrow{j} P/XP + X^2 F \to 0.$$

Since the domain and range of j are both rank 2 $k[Y, Z]$-modulus, the domain being $k[Y, Z]$-free, it follows that j is an isomorphism, and hence $i=0$; therefore $X^2 F \subseteq XP$, $XF \subseteq P \subseteq F$ and P is free by Lemma 1.1.

$r=4$. Tensoring the exact sequence (5) again, this time with $A/X^2 A$, we obtain the exact sequence

$$F/X^2 F \xrightarrow{\theta} P/X^2 P \xrightarrow{\varphi} P/X^2 F.$$

The domain and range of φ are rank 4 $k[Y, Z]$-modules, the domain being free; as before, this implies $\theta=0$, $X^2 F \subseteq X^2 P$, and so $P=F$ is free. This completes the proof of Lemma 1.3.

Lemma 1.4. *Let F be a free A-module of rank 2, $\alpha \in k[X]$ a polynomial with no triple roots, P a projective sub-A-module of F such that*

$$F \supset P \supset \alpha(X) F;$$

then P is A-free.

Proof. Let $\alpha(X) = \prod_{i=1}^{m} (X - a_i)^{n_i}$ with $a_1, \ldots, a_m$ distinct, and each $n_i = 1$ or 2. Let M_X denote multiplication by X on $\bar{P} = P/\alpha(X) F$; then $\alpha(M_X)=0$. Thus, if we denote by $\bar{P}_1$, $\bar{P}_2$ the submodules of $\bar{P}$ annihilated respectively by $(X-c_1)^{n_1}$, $\dfrac{\alpha(X)}{(X-c_1)^{n_1}} = \alpha_1(X)$ a standard argument shows that $\bar{P} = \bar{P}_1 \oplus \bar{P}_2$. (We may assume that $\bar{P}_1 \neq 0$ without loss; otherwise, replace α by α_1.)

Note that $\bar{P}$ is projective over $k[Y, Z]$; indeed, if F_1, F_2 are free A-modules such that $P \oplus F_1 = F_2$ then $\bar{P} \oplus F_1/\alpha F_1 \cong F_2/\alpha F_2$, and $F_1/\alpha F_1$, $F_2/\alpha F_2$ are clearly $k[Y, Z]$-free. (We may also see this by taking $k[Y, Z]$-module compositon $P \cong \bar{P} \oplus \alpha F$ as in Lemmas 1.1 and 1.3.) Thus $\bar{P}_2$ is projective over $k[Y, Z]$, hence of homological dimension 1 over A. Let

$j\colon P \to P/\alpha(X)F$ be the canonical map and let $j^{-1}(\bar{P}_2)=P_1$. The exact sequence $0 \to P_1 \to P \to \bar{P}_2 \to 0$ then shows that P_1 is A-projective. Since $P_1 \supseteq \alpha(X)F \supseteq (X-a_1)^2 P_1$, Lemma 1.3 implies that P_1 is A-free. Now $P \supseteq P_1 \supseteq \left(\prod_{i=2}^{m}(X-a_i)^{n_i}\right)P$, and an obvious induction on m completes the proof.

Remark 1. To say there exists a free F with $F \supset P \supset \alpha F$ is of course the same as to say there exists a free F' with $P \supset F' \supset \alpha P$; the latter implies $F' \supset \alpha P \supset \alpha F'$ and $P \cong \alpha P$.

Remark 2. A second proof is available for Lemma 3 which works for any $A=R[X]$, with the condition that stably projectives over R are free. Swan has recently shown the following: Let R be any commutative ring and P a finitely generated projective $R[X]$ module of rank 2. Suppose exists a projective R-module F_0 of rank 2 such that $X^2 P \subset F_0[X] \subset P$. Then $P \approx F_0[X]$.

2. Some Lemmas and Known Results on Space Curves

In what follows k always denotes an algebraically closed field. Let $\alpha \in B = k[X, Y]$ be a nonconstant polynomial. We refer to the affine scheme spec $B/\alpha B$ as the "curve α" or simply as "α" when there is no confusion.

Let $\mathbf{C}$ be a closed irreducible nonsingular curve in the affine 3-space $\mathbb{A}^3$ over k and $\mathfrak{p}$ its prime ideal in $A=k[X, Y, Z]$. Let $\bar{A}=A/\mathfrak{p}$ denote the coordinate ring of $\mathbf{C}$. Let Γ be the closure of the projection of $\mathbf{C}$ onto the (X, Y)-plane so that the prime ideal of Γ in $B=k[X, Y]$ is $\mathfrak{p} \cap B = B\gamma$. We denote by $\bar{B}$ the coordinate ring $B/B\gamma$ of Γ. We fix a coordinate system X, Y, Z, for A such that the projection $\mathbf{C} \to \Gamma$ is birational, integral and such that the only singularities of Γ are nodes (see for example [1, p. 23]). Let $\mathfrak{c}$ be the conductor from $\bar{A}$ to $\bar{B}$. For $\alpha \in A$, we denote by $\bar{\alpha}$ its image in $\bar{A}$. The following result of Abhynkar [1] is basic for the proof of our theorem and we include its proof for the sake of completeness.

Proposition 2.1 (Abhyankar). *With the hypothesis and notation as above, assume further that the conductor $\mathfrak{c}$ is $\bar{A}$-principal. Let $\alpha \in B$ be such $\mathfrak{c}=\bar{A}\bar{\alpha}$ and $\beta \in B$ such that $\bar{\alpha}\bar{Z}=\bar{\beta}$. Then*

1. *The curves α, β meet transversally at all the nodes of Γ and do not meet anywhere else on Γ. (When Γ in nonsingular (1) means that α and β do not meet on Γ.)*

2. *There exists $\gamma' \in B$ such that $B\gamma + B\gamma' = B$ and $\gamma\gamma' = a\beta^2 + b\alpha\beta + c\alpha^2$ with $a, b, c, \in B$. For any such choice of $\gamma', a, b, c,$ the prime ideal $\mathfrak{p}$ of $\mathbf{C}$ is generated by*

$$(\alpha Z - \beta,\ aZ^2 + bZ + c,\ \gamma).$$

Proof. (1) Since $\bar{\alpha}$ is the generator of the conductor, it is clear that α and γ meet precisely at the nodes of γ and *a fortiori* α, β meet on γ precisely at the nodes of γ. Since γ has only nodes as singularities $\mathfrak{c}$ is the intersection of maximal ideals in B of the nodes of γ. Further, if M is the maximal ideal in B of a node of γ, then $\gamma \in M^2 B_M$. Hence to prove that α and β meet transversally at the nodes of γ, it is sufficient to show that $\mathfrak{c} = \bar{B}\bar{\alpha} + \bar{B}\bar{\beta}$. This again is enough to check locally at the completion of local rings of the nodes. Hence we may assume $\bar{B} = k[[X, Y]]/(XY)$ and

$$\bar{A} = k[[X]] \times k[[Y]]$$

and $\bar{B}$ can be identified as a subring of $\bar{A}$:

$$\bar{B} = \{(f, g) \;\; \bar{A} \mid f(0) = g(0)\}.$$

We may also assume $\alpha = (X, Y)$ and $\bar{Z}$ is of the form (f, g) with $f(0) \neq g(0)$. Now it is easy to check that $\bar{A}\bar{\alpha} = \bar{B}\bar{\alpha} + \bar{B}\bar{\beta}$.

2. It is easy to see that we can find $\gamma'' \in B$ such that $B\gamma + B\gamma'' = B$ and γ'' passes through each point of intersection of α and β which is not on γ. For example, if $P_1, \ldots, P_r$ are points of intersection of α and β outside γ, take $\gamma'' = \prod_{i=1}^{r} (\gamma - \gamma(P_i))$. Since γ has a double point at every point of intersection of α and β on γ, it follows that $\gamma\gamma''^N \in (\alpha, \beta)^2 B$ for sufficiently large N. We set $\gamma' = \gamma''^N$ and $\gamma\gamma' = a\beta^2 + b\alpha\beta + c\alpha^2$, with $a, b, c, \in B$. This proves the existence of γ' as in (2).

Let now $\gamma' \in B$ be an element such that $B\gamma + B\gamma' = B$ and

$$\gamma\gamma' = a\beta^2 + b\alpha\beta + c\alpha^2,$$

with $a, b, c \in B$. Set $f = \alpha Z - \beta$ and $g = aZ^2 + bZ + c$ and $I = Af + Ag + A\gamma$. It is obvious that $f, \gamma \in \mathfrak{p}$ ($\mathfrak{p}$ = prime ideal of C in A). Since

$$\gamma\gamma' = a\beta^2 + b\alpha\beta + c\alpha^2,$$

we have $\alpha^2 g \equiv \gamma\gamma' (\bmod f)$. Since $\alpha \notin \mathfrak{p}$, we have $g \in \mathfrak{p}$. Hence $I \subset \mathfrak{p}$. Since $\alpha^2 g \equiv \gamma\gamma' (\bmod f)$, we have $\gamma\gamma' \in Af + Ag$. Hence for every maximal ideal M of A containing I, we have $IA_M = A_M f + A_M g$. Hence I is locally generated by two elements. Also height $I \geqq 2$ (for example, since γ and f have no common factor). Hence I is an unmixed ideal of height 2. Let M be a maximal ideal of B. If $\gamma \notin M$, then $I_M = \mathfrak{p}_M = A_M$. If $\gamma \in M$ and M is the maximal ideal of a simple point of γ, it is easy to see that

$$I_M = \mathfrak{p}_M = A_M \gamma + A_M f.$$

Hence, $I = \mathfrak{p} \cap \mathfrak{q}$, where $\mathfrak{q} = A$ or all the irreducible components of the variety $V(\mathfrak{q})$ of $\mathfrak{q}$ are of the form $Q \times k$, where Q is a node of γ. If $Q \times k$ is a component of $V(\mathfrak{q})$, then the polynomial $a(Q)Z^2+b(Q)Z+c(Q)$ is identically zero. Hence $a(Q)=b(Q)=c(Q)=0$. But then, the relation $\gamma\gamma'=a\beta^2+b\alpha\beta+c\alpha^2$ implies that γ has a triple point at Q. Contradiction. Hence $q=A$ and $I=\mathfrak{p}$.

Remark 2.2 (Abhyankar). If γ is nonsingular, we may take $\alpha=1$ and it is easy to see that $\mathfrak{p}=A\gamma+A(Z-\beta)$. Also if α, β do not intersect outside γ, then in the above argument we may take $\gamma'=1$ and then $\mathfrak{p}=Af+Ag$.

Lemma 2.3 (Zariski [12, Lemma 5]). *Let k be an infinite field and K/k a finitely generated field extension. Let $x_1, x_2, \in K$ be algebraically independent over k. Let Ω be the algebraic closure of $k(x_1, x_2)$ in K. Suppose that Ω is separable over $k(x_1, x_2)$. Then $k(x_1+cx_2)$ is algebraically closed in K for all but finitely many $c\in k$.*

Lemma 2.4. *Let k be an algebraically closed field and $B=k[X, Y]$.*

(1) Let $\alpha, \gamma\in B$ be nonconstant polynomials with $gcd(\alpha, \gamma)=1$. Then there exists an $a\in B$ such that $\alpha+a\gamma$ is irreducible.

(2) Let $\alpha\in B$ be irreducible and let $\beta, \gamma\in B$ be not divisible by α. Then there exists a $b\in B$ such that α and $\beta+b\gamma$ intersect transversally outside γ.

Proof. (1) We may assume $d(\alpha/\gamma)\neq 0$. For if $d(\alpha/\gamma)=0$, then

$$d(\alpha/\gamma+X)=dX\neq 0.$$

Thus if necessary by changing α to $\alpha+X\gamma$, we may assume $d(\alpha/\gamma)\neq 0$. Then α/γ is a part of a separating base for $k(X, Y)/k$. Choose $\theta\in k[X, Y]$ such that $k(X, Y)/k(\alpha/\gamma, \theta)$ is a finite separable extension. Then by Lemma 2.3 there exists a $c\in k$ such that $k(\alpha/\gamma+c\theta)$ is algebraically closed in $k(X, Y)$. Hence by changing α to $\alpha+c\theta\gamma$, we may assume $k(\alpha/\gamma)$ is algebraically closed in $k(X, Y)$. Hence the generic member of the pencil defined by α/γ is reduced and irreducible. Since $gcd(\alpha, \gamma)=1$, we see that $\alpha+c\gamma$ is irreducible for all but finitely many $c\in k$.

(2) Let $R=B/\alpha B$ and K the quotient field of R. Let $\bar{\beta}$, $\bar{\gamma}$ denote the images of β and γ in R. As in (1), by adding a multiple of γ to β if necessary, we may assume $d(\bar{\beta}/\bar{\gamma})\neq 0$ so that $K/k(\bar{\beta}/\bar{\gamma})$ is separably algebraic. Let C denote the nonsingular model of the function field K. Then for almost all $c\in k$, the divisor $(\bar{\beta}/\bar{\gamma}+c)_0$ of zeroes of the function $\bar{\beta}/\bar{\gamma}+c$ consists of distinct points of C with multiplicity 1. Hence for almost all $c\in k$, $\beta+c\gamma$ and α meet transversally outside γ.

Remark 2.5. (i) The proof shows that in (1) and (2), we may take a, b to be linear polynomials. (ii) The proof of (2) is inspired by an argument in Abhyankar [1].

Corollary 2.6. *With the hypothesis and notation as in Proposition 2.1 there exists, $\alpha, \beta \in B$ such that the ideal $\mathfrak{p}$ is generated by*

$$(\alpha Z-\beta, aZ^2+bZ+c, \gamma),$$

where (1) $\mathfrak{c}=\bar{A}\bar{\alpha}$, $\bar{\alpha}\bar{Z}=\bar{\beta}$, (2) there exists $\gamma' \in B$ such that $B\gamma+B\gamma'=B$ and $\gamma\gamma'=a\beta^2+b\alpha\beta+c\alpha^2$, with $a, b, c \in B$, (3) α is irreducible and α, β intersect transversally at every point of their intersection.

Proof. Since Proposition 2.1 is valid for any lifts α, β of $\bar{\alpha}$ and $\bar{\beta}$, by Lemma 2.4 modifying α, β by multiples of γ, we may assume that α is irreducible and α, β intersect transversally outside γ. But by Proposition 2.1, (1) α, β intersect transversally on γ (for arbitrary lifts α, β of $\bar{\alpha}$ and $\bar{\beta}$). Hence α, β intersect transversally at all points of their intersection. The rest of the corollary is just Proposition 2.1.

Lemma 2.7. *Let $\Gamma \subset \mathbb{A}^2$ be an irreducible plane curve with only nodes as singularities. Let $\gamma \in B=k[X, Y]$ be an irreducible polynomial of Γ and $\bar{A}$ the integral closure of $\bar{B}=B/\gamma B$. Then $\Omega_{\bar{A}/k} \approx \mathfrak{c}$ as $\bar{A}$-modules, where $\Omega_{\bar{A}/k}$ is the module of k-differentials of $\bar{A}$ and $\mathfrak{c}$ the conductor from $\bar{A}$ to $\bar{B}$.*

Proof. Since the only singular points of Γ are nodes, the curves $\partial\gamma/\partial X$ and $\partial\gamma/\partial Y$ meet transversally at the singular points of Γ. Let γ_x, γ_y denote the images of $\partial\gamma/\partial X$ and $\partial\gamma/\partial Y$ in $\bar{B}$. Then $\mathfrak{c}=\bar{B}\gamma_x+\bar{B}\gamma_y$. Now, $\Omega_{\bar{B}/k} \approx \bar{B}dx+\bar{B}dy$ with the relation $\gamma_x dx+\gamma_y dy=0$. We have a natural map $\phi: \Omega_{\bar{B}/k} \to \bar{B}\gamma_x+\bar{B}\gamma_y=\mathfrak{c}$ by sending $dx \to \gamma_y$, $dy \to -\gamma_x$. *Since* $\Omega_{\bar{B}/k}$ is a $\bar{B}$-module of rank 1, the composite map

$$\bar{A} \otimes_{\bar{B}} \Omega_{\bar{B}/k} \xrightarrow{1\otimes\phi} \bar{A} \otimes_{\bar{B}} \mathfrak{c} \xrightarrow{\text{multiplication}} \bar{A}\mathfrak{c}=\mathfrak{c}$$

is surjective and is an isomorphism of $\bar{A}$-modulo torsion. But the extension $\bar{A}/\bar{B}$ is unramified (Γ has only nodes as singularities) so that the natural map $\bar{A} \otimes_{\bar{B}} \Omega_{\bar{B}/k} \to \Omega_{\bar{A}/k}$ is surjective and hence an isomorphism modulo torsion. Hence

$$\Omega_{\bar{A}/k} \approx \frac{\bar{A} \otimes \Omega_{\bar{B}/k}}{\text{torsion}} \approx \mathfrak{c}$$

(as $\bar{A}$-modules), since $\mathfrak{c}$ is torsion free.

3. Projective Resolution of $\mathfrak{p}$

We construct in this section projective resolutions of the prime ideals of nonsingular curves in $\mathbb{A}^3$ with trivial canonical line-bundles; for this purpose, we need the following lemma.

Lemma 3.1. *Let R be any Noetherian ring, and let f_i, g_i $(1 \leqq i \leqq 3)$ be elements of R such that $\sum f_i g_i=0$, $f_1 R+g_1 R=1$; assume also that f_2, f_3*

form an R-sequence; then

(1) *The module* $\mathscr{R}_R(f_1, f_2, f_3)$ *of R-relations on* f_1, f_2, f_3 *is projective, and generated over R by* $p=(g_1, g_2, g_3)$ *and the "trivial relations"*

$$p_1=(0, f_3, -f_2), \quad p_2=(-f_3, 0, f_1), \quad p_3=(f_2, -f_1, 0).$$

(2) *If also* f_1, f_2 *form an R-sequence then there is an exact sequence*

$$0 \longrightarrow R \xrightarrow{i} \mathscr{R}_R(f_1, f_2, f_3) \xrightarrow{\pi} Rf_1+Rf_2+Rg_3 \longrightarrow 0$$

with $i(1)=(f_2, -f_1, 0)$, $\pi(r_1, r_2, r_3)=r_3$ *which furnishes a projective resolution of* $I=Rf_1+Rf_2+Rg_3$.

Proof. Suppose $\lambda f_1+\lambda f_2+\lambda_3 f_3=0$; then $\lambda_1 f_1 \in Rf_2+Rf_3$; since f_1 is a unit modulo g_1, this shows that $\lambda_1 \in g_1 R+f_2 R+f_3 R$, whence subtracting from $(\lambda_1, \lambda_2, \lambda_3)$ a suitable linear combination of p, p_2, p_3 we obtain an element in $R_R(f_1, f_2, f_3)$ of the form $(0, \mu_2, \mu_3)$ since f_2, f_3 form an R-sequence, $(0, \mu_2, \mu_3)$ is a multiple of p_1. Thus $R_R(f_1, f_2, f_3)$ has the indicated generators.

In proving $R_R(f_1, f_2, f_3)$ is projective, we may assume R is local. In this case, either f_1 or g_1 is a unit in R; $R_A(f_1, f_2, f_3)$ is free over R on p_2, p_3 if f_1 is a unit, and is free over R on p, p_1 if g_1 is a unit.

From what has been proved (2) is now immediate.

Now let C be nonsingular curve in $\mathbb{A}^3$ with prime ideal $\mathfrak{p}$, and assume the module of differentials on C is free. As in Section 2, we fix a coordinate system, X, Y, Z for $\mathbb{A}^3$ such that the projection of C onto the (X, Y)-plane is birational, integral, and maps C onto a curve which has only nodes as singularities. We again follow the notation of Section 2, i.e.,

$$B=k[X, Y],$$

$$\mathfrak{p} \cap B=\gamma B$$

$\bar{A}=A/\mathfrak{p}$, $\bar{B}=B/\gamma B$, and for δ in A, $\bar{\delta}$ denotes its image in $\bar{A}$

$$\mathfrak{c}=\text{conductor from } \bar{A} \text{ to } \bar{B}.$$

We next observe that the generating set of $\mathfrak{p}$ constructed in Proposition 2.1 leads to a projective resolution, as follows. Thus, choose any α in B such that $\bar{A}\bar{\alpha}=\mathfrak{c}$ (this is possible, since $\mathfrak{c}$ is $\bar{A}$-principal by Lemma 2.7), choose β in B such that $\overline{\alpha Z}=\bar{\beta}$; by Proposition 2.1, we may then choose γ', a, b, c in B such that

$$\gamma\gamma'=a\beta^2+b\alpha\beta+c\alpha^2, \quad B\gamma+B\gamma'=B. \tag{1}$$

$\mathfrak{p}$ will then be generated over A by the three elements

$$\gamma, f=\alpha Z-\beta, \quad g=aZ^2+bZ+c.$$

Note the following relation between γ, f, and g:

If we replace αZ by $f+\beta$ in

$$\alpha^2 g = a\alpha^2 Z^2 + b\alpha^2 Z + c\alpha^2$$

and observe (1) we obtain

$$\alpha^2 g = \gamma\gamma' + ff' \tag{2}$$

with

$$f' = af + 2a\beta + b\alpha. \tag{3}$$

We may now apply Lemma 3.1 with $I=\mathfrak{p}$.

$$f_1=\gamma, \quad f_2=\alpha Z=\beta, \quad f_3=\alpha^2$$

$$g_1=\gamma', \quad g_2=f', \quad g_3=-g$$

to obtain the following projective resolution of $\mathfrak{p}$. (To verify that f_1, f_2 and f_2, f_3 are A-sequences simply observe $\alpha Z-\beta$ is irreducible):

$$0 \longrightarrow A \xrightarrow{\ i\ } R_A(\gamma, \alpha Z-\beta, \alpha^2) \xrightarrow{\ \pi\ } \mathfrak{p} \longrightarrow 0 \tag{4}$$

with $i(1)=(\alpha Z-\beta, -\gamma, 0)$, π = projection onto 3rd coordinate. Let us denote by E the element in $\operatorname{Ext}^1_A(\mathfrak{p}, A)$ corresponding to (4). *We next observe that E depends only on $\bar{\alpha} \in \bar{A}$*, not on the choice of representation $\alpha \in A$, nor on the particular choices made of β, γ', a, b, c. It will be convenient to prove the following more general lemma, from which this observation follows immediately (with $a_1=\gamma$, $a_2=\alpha Z-\beta$, $a_3=\alpha^2$, and for alternative choices α', β' with $\bar{\alpha}'=\bar{\alpha}$, $a_2'=\alpha' Z-\beta'$ and $a_3'=\alpha'^2$).

Lemma 3.2. *Let a_1, a_2, a_3, a_2', a_3' be elements of a commutative ring R such that (a_1, a_2) form an R-sequence, and such that $a_2 \equiv a_2' \pmod{a_1}$ $a_3 \equiv a_3' \pmod{a_1}$. Then the following diagram has exact rows, and there exists an R-homomorphism ϕ for which the diagram is commutative:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\ i\ } & \mathscr{R}_R(a_1, a_2, a_3) & \xrightarrow{\ p\ } & \operatorname{Im}(p) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\
0 & \longrightarrow & R & \xrightarrow{\ i'\ } & \mathscr{R}_R(a_1, a_2', a_3') & \xrightarrow{\ p'\ } & \operatorname{Im}(p') & \longrightarrow & 0
\end{array}$$

where p and p' indicate restrictions of the projection $p_3\colon R^3 \to R$ onto the third coordinate, $i(1)=(a_2, -a_1, 0)$, $i'(1)=(a_2', -a_1, 0)$.

Proof. Suppose

$$a_2' = a_2 + \lambda_2 a_1, \qquad a_3' = a_3 + \lambda_3 a_1.$$

It is readily verified that the isomorphism

$$R^3 \to R^3, \qquad (r_1, r_2, r_3) \to (r_1 - \lambda_2 r_2 - \lambda_3 r_3, r_2, r_3)$$

maps $\mathscr{R}_R(a_1, a_2, a_3)$ onto $\mathscr{R}_R(a_1, a_2', a_3')$ and its restriction renders the diagram above commutative. The remainder is immediate.

We are now entitled to denote by $E(\bar{\alpha})$ the element in $\mathrm{Ext}_A^1(\mathfrak{p}, A)$ corresponding to the extension (4); this symbol makes sense for any $\bar{A}$-generator $\bar{\alpha}$ of $\mathfrak{c}$. We conclude this section with the following lemma, which shows how $E(\bar{\alpha})$ depends on the choice of $\bar{A}$-generator for $\mathfrak{c}$.

Lemma 3.3. *If* $\alpha \in B$, $\lambda \in A$, $\mathfrak{c} = \bar{A}\bar{\alpha}$, *and* $\bar{\lambda}$ *is a unit in* $\bar{A}$, *then* $E(\overline{\lambda\alpha}) = \lambda E(\bar{\alpha})$.

Proof. Observe that $\mathrm{Ext}_A^1(\mathfrak{p}, A)$ is isomorphic to $\Omega_{\bar{A}/k}$ ([2, 8]), and so is isomorphic to $A/\mathfrak{p}$. Hence, for any maximal ideal M of A corresponding to a point Q of **C**, the natural map $\mathrm{Ext}_A^1(\mathfrak{p}, A) \to \mathrm{Ext}_{A_M}^1(\mathfrak{p}_M, A_M)$ is injective. Thus, we are done if we show the images of $E(\overline{\lambda\alpha})$, $\lambda E(\bar{\alpha})$ coincide for any one such maximal ideal M.

For this purpose, we pick any point Q of C which does not lie above any node of γ, thus $N = M \cap B$ corresponds to regular point of γ, and so $\bar{A}_{\bar{M}} = \bar{B}_{\bar{N}}$. Accordingly, there exists λ' in B_N such that $\bar{\lambda} = \bar{\lambda}'$. By Lemma 3.2 the sequence

$$0 \longrightarrow A_M \xrightarrow{i'} \mathscr{R}_{A_M}(\gamma, \lambda'\alpha Z - \lambda'\beta, \lambda'^2\alpha^2) \xrightarrow{p'} \mathfrak{p} A_M \longrightarrow 0 \qquad (\mathrm{E}')$$

$$i(1) = (\lambda'\alpha Z - \lambda'\beta, -\gamma, 0), \qquad p' = \text{projection on third coordinate}$$

is exact and represents the element $E(\overline{\lambda\alpha})_M$ in $\mathrm{Ext}_{A_M}^1(\mathfrak{p} A_M, A_M)$.

It thus suffices to verify that (E') also represents the element $[\lambda E(\bar{\alpha})]_M$; this is immediate from the following commuting diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_M & \xrightarrow{i'} & \mathscr{R}_{A_M}(\gamma, \lambda'\alpha Z - \lambda'\beta, \lambda'^2\alpha^2) & \xrightarrow{p'} & \mathfrak{p} A_M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \psi} & & \| & & \\
0 & \longrightarrow & A_M & \xrightarrow{i_M} & \mathscr{R}_{A_M}(\gamma, \alpha Z - \beta, \alpha^2) & \xrightarrow{p_M} & \mathfrak{p} A_M & \longrightarrow & 0
\end{array}$$

with $\Psi(r_1, r_2, r_3) = (\lambda'^2 r_1, \lambda' r_2, r_3)$.

4. Projective $k[X, Y, Z]$-Modules Are Free

Theorem 4.1. *Let k be an algebraically closed field; then every projective $k[X, Y, Z]$-module is free.*

Proof. As observed in the introduction, it suffices to consider rank 2 projective modules. Thus, assume P is a rank 2 projective module over $k[X, Y, Z] = A$.

Consider the rank 2 vector bundle $\mathscr{B}$ associated to $P^* = \mathrm{Hom}_A(P, A)$ [i.e., P^* is the module of sections of $\mathscr{B}$]. It follows from Kleiman's Bertini theorems for vector bundles ([6, 3.6] and [7, Theorem 7.3]) that there exists a section of $\mathscr{B}$ which intersects the zero section s transversely, in an irreducible nonsingular curve. Let **C** be the projection of this curve in $\mathbb{A}^3$, and let $\mathfrak{p}$ be the prime ideal of C; then $s \in P^*$ maps P onto $\mathfrak{p}$. $\mathfrak{p}$ is therefore locally generated by 2 elements, and so of homological dimension 1; we then have an exact sequence

$$0 \longrightarrow A \longrightarrow P \xrightarrow{s} \mathfrak{p} \longrightarrow 0. \qquad \text{(E)}$$

Serre proves [8], that, in this situation, E generates $\mathrm{Ext}^1_A(\mathfrak{p}, A)$ which is isomorphic to $\Omega_{\bar{A}/k}$ (see [2]). The results of Sections 2 and 3 are now available, since $\Omega_{\bar{A}/k}$ is monogenic, and we next compare (E) with the generator $E(\bar{\alpha})$ of $\mathrm{Ext}^1_A(\mathfrak{p}, A)$ introduced in Section 3. Since $\mathrm{Ext}^1_A(\mathfrak{p}, A) \cong A/\mathfrak{p}$, there is an element λ in A such that $\bar{\lambda}$ is a unit in $\bar{A}$ and $E = \lambda E(\bar{\alpha})$ in $\mathrm{Ext}^1_A(\mathfrak{p}, A)$. By Lemma 3.3, it follows that $E = E(\overline{\lambda\alpha})$ thus, replacing α by $\lambda\alpha$; we may assume without loss of generality that E is isomorphic to the extension $E(\bar{\alpha})$ for a suitable generator $\bar{\alpha}$ of $\mathfrak{c}$ over $\bar{A}$. Lemma 2.4 then shows that we may choose a representative α and β in B for $\bar{\alpha}$ and $\bar{Z}\bar{\alpha}$ such that α and β intersect transversely; assuming this done, we then choose γ', a, b, c as in Proposition 2.1. Since E is isomorphic to

$$E(\bar{\alpha}){:}\ 0 \longrightarrow A \xrightarrow{i} \mathscr{R}_A(\gamma, \alpha Z - \beta, \alpha^2) \xrightarrow{p} \mathfrak{p} \longrightarrow 0$$

where $i(1) = (\alpha Z - \beta, -\gamma, 0)$, p = projection on third coordinate. We may assume that

$$P = \mathscr{R}_A(\gamma, \alpha Z - \beta, \alpha^2).$$

By Lemma 3.1, P is then generated by the elements

$$(-\gamma', -f', aZ^2 + bZ + c), \qquad (0, -\alpha^2, \alpha Z - \beta),$$
$$(-\alpha^2, 0, \gamma), \qquad (\alpha Z - \beta, -\gamma, 0),$$

where $f' = af + 2a\beta + b\alpha$.

If p_1 denotes projection onto the first coordinate, we thus have an epimorphism

$$P \xrightarrow{p_1} J = (\gamma', \alpha^2, \alpha Z - \beta).$$

As before, the kernel of this map is then isomorphic to A, and we have an exact sequence

$$0 \longrightarrow A \xrightarrow{i_1} P \xrightarrow{p_1} J \longrightarrow 0.$$

J contains the ideal $(\alpha^2, \alpha\beta, \beta^2)$; indeed, since it contains α^2, it contains $\alpha^2 Z - \alpha(\alpha Z - \beta) = \alpha\beta$ and so also contains $\alpha\beta Z - \beta(\alpha Z - \beta) = \beta^2$. By a k-linear change of coordinates X, Y we may ensure the points

$$Q_i = (a_i, b_i), \qquad (1 \leqq i \leqq N)$$

where α and β intersect have disdinct X-coordinates; then since α and β intersect transversely,

$$\theta = \prod_{i=1}^{N} (X - a_i)^2$$

lies in $(\alpha, \beta)^2$ and so in J. Choose e in P so $p_1(e) = \theta$; then $p_1^{-1}(A\theta)$ is A-free on e and $i_1(1)$, and $P \supset p_1^{-1}(A\theta) \supset \theta P$. By Lemma 1.4 P is A-free.

Corollary 4.2. *If Q is an ideal in $k[X, Y, Z]$ of homological dimension $\leqq 1$, it may be generated by 2 elements, if and only if* $\mathrm{Ext}_A^1(\mathfrak{p}, A)$ *is monogenic.*

Corollary 4.3. *A nonsingular curve (possibly reducible) in $\mathbb{A}^3$ is an ideal-theoretic complete intersection, if and only if its canonical line bundle is trivial. In particular, all nonsingular rational and elliptic curves are ideal-theoretic complete intersections.*

Proof. Serre proves in [8] that the preceding theorem implies these two corollaries.

Corollary 4.4. *A reducible curve in $\mathbb{A}^3$ is a set-theoretic complete intersection, if and only if its connected components are.*

Proof. This is a consequence of the following observation: if an ideal Q is an intersection of finitely many pairwise comaximal Q_i, then

$$\mathrm{Ext}_A^1(Q, A) = \bigoplus_i \mathrm{Ext}_A^1(Q_i, A)$$

is monogenic, if and only if each $\mathrm{Ext}_A^1(Q_i, A)$ is.

Remark 4.5. To prove Corollary 4.3, one does not have to know that all projectives over $k[X, Y, Z]$ are free. One has to only show that $R_A(\gamma, \alpha Z - \beta, \alpha^2)$ is free with α, β as in Corollary 2.6. This follows from Lemma 1.4 and discussion just preceeding Corollary 4.2.

Addendum. After this work was done, Serre recently informed us via Hyman Bass that A. Souslin (Leningrad University) has independently obtained Theorem 4.1. During the preparation of this work we also learned from Bass that Moshe Roitman (Hebrew University) has shown that projectives of rank n over $k[X_1, \ldots, X_n]$ are free (k-algebraically closed). Swan has recently generalized Roitman's argument and proved the following: Let k be an infinite field. Then projectives of rank $> \frac{n+1}{2}$ are free over $k[X_1, \ldots, X_n]$. We have also learned from Bass that this has been independently proved by A. Souslin.

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Groups with a (B, N)-Pair of Rank 2. II

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Introduction

A group G has a Tits system (B, N) of rank n if there exist subgroups B and N of G satisfying the following conditions:

(i) $G=\langle B, N\rangle$ and $H=B\cap N\trianglelefteq N$.

(ii) $W=N/H$ is generated by a set S of n involutions $s_1, \ldots, s_n$.

(iii) $wBs_i\subseteq BwB\cup Bws_iB$ for $s_i\in S$, $w\in W$.

(iv) $s_iBs_i\neq B$ for $s_i\in S$.

The study of finite groups with a Tits system of rank 2 satisfying the additional condition

(*) there exists a normal nilpotent subgroup U of B such that $B=HU$

was begun in Part I of this paper [1], where we refer the reader for the notation, statements of the main results, and references. In part II we complete the classification of these groups.

In the cases to be considered, W will be indecomposable, $U\cap H=1$, and it must be shown that G is isomorphic to a Chevalley group. This we do by constructing enough of the multiplication table of G to identify G either by a classification theorem or by the multiplication table itself. Since G has a Tits system (B, N), we proceed as follows. Representatives s_1, s_2 for the generators of $W=N/H$ are chosen so that $s_i\in R_i=\langle U_i, U_i^{s_i}\rangle$. The elements of W are then to be represented by fixed representatives in $\langle s_1, s_2\rangle$. Since G is the disjoint union of double cosets BwB, $w\in W$, the product of any two elements can be calculated once all products in B, in N, and of the form wbs_i, where $w\in W$, $b\in B$, are known.

The procedure for deciding which of BwB or Bws_iB contains wbs_i is well-known. We repeat it in order to point out two consequences. Let ℓ be the length function on W, and express $b=uvh$, where $u\in U_i$, $v\in U_{s_i}$, $h\in H$. If $\ell(ws_i)>\ell(w)$, then $wbs_i=(u)^{w^{-1}}(ws_i)(vh)^{s_i}$ is in Bws_iB. If $\ell(ws_i)<\ell(w)$, then $wbs_i=(ws_i)(s_i^{-1}us_i)(vh)^{s_i}$. But $s_i^{-1}us_i\in R_i$ and $R_i\subseteq HU_i\cup HU_is_iU_i$. If $s_i^{-1}us_i\in HU_is_iU_i$, then there exist elements c, $d\in B$ such that $s_i^{-1}us_i=cs_id$, whence $wbs_i=(ws_ics_i)(d(vh)^{s_i})$. The product ws_ics_i can then be

[1] Inventiones math. 21, 1–57 (1973).

calculated by the first case. If $s_i^{-1} u s_i \in HU_i$, then $u=1$ so that wbs_i is in Bws_iB.

The following consequences are thus implied from the (B, N) axioms in the present situation. Firstly, the multiplication table of G is determined once the following is given: a labeling of the elements of G; the multiplication tables of B, N, R_1, R_2 with respect to this labeling; the action of s_1, s_2 on the elements of the subgroups U_1^w, U_2^w for $w \in W$. Secondly, suppose $\tilde{H}$ is a subgroup of H normalized by $\langle s_1, s_2\rangle$, $\tilde{W} = \langle s_1, s_2\rangle \tilde{H}/\tilde{H}$ is isomorphic to W, and $R_i \cap H \leqq \tilde{H}$ for $i=1, 2$. Then the set $\tilde{G} = \bigcup_{w \in \tilde{W}} \tilde{H} U w U$ is a subgroup of G. Moreover, if $\tilde{B} = \tilde{H}U$ and $\tilde{N} = \langle \tilde{H}, s_1, s_2\rangle$, then $(\tilde{B}, \tilde{N})$ is a Tits system for $\tilde{G}$.

§ 7. The Structure of H

We show in this section that H is abelian. In §6 we showed that the subgroups R_1 and R_2 are isomorphic to $SL(2, q)$, $PSL(2, q)$, $SU(3, q)$, $PSU(3, q)$, or $Sz(q)$. Using an additional subscript i we can label elements $u_i(\alpha)$ or $u_i(\alpha, \beta)$, $h_i(\gamma)$, and s_i in R_i as in §4. In case R_i is isomorphic to $PSL(2, q)$ or $PSU(3, q)$, we make the usual identification of elements of R_i modulo $Z(R_i)$. This labeling can in fact be done so that $U_i = \langle u_i(\alpha)\rangle$ or $\langle u_i(\alpha, \beta)\rangle$ and $H_i = H \cap R_i = \langle h_i(\gamma)\rangle$, since U_i is a Sylow p-subgroup of R_i and H_i is an abelian p'-complement of U_i in $N(U_i) \cap R_i$.

(7A) *Let $|W| = 2m$ and $t = (s_1 s_2)^m$. Then H_1, H_2 are normal cyclic p'-subgroups of H, $H = H_1 H_2 \langle t\rangle$, and $t^2 \in H_1 H_2$.*

Proof. We have $H_i \trianglelefteq H$ since $H_i = H \cap R_i \trianglelefteq H$. Moreover, s_i centralizes H/H_i since $[s_i, H] \leqq R_i \cap H = H_i$. Thus $\langle s_1, s_2\rangle$ normalizes $H_1 H_2 \langle t\rangle$. But s_1^2 and s_2^2 are in $H_1 H_2$. Thus $\langle s_1, s_2\rangle H_1 H_2/H_1 H_2$ is a dihedral group containing $tH_1 H_2$ in its center, and so $t^2 \in H_1 H_2$. The Bruhat decomposition of G then implies that the set $\tilde{G}$ of elements in G of the form $uhwv$, where $h \in H_1 H_2 \langle t\rangle$, $u, v \in U$, and $w \in \langle s_1, s_2\rangle$, is a subgroup. Since $G = U^G$, $U \leqq \tilde{G}$, and $G = \tilde{G}H$, it follows that $G = \tilde{G}$, $H = H_1 H_2 \langle t\rangle$, and (7A) holds.

Let $P\hat{G}L(2, 9)$ be the subgroup $PSL(2, 9)\langle ab\rangle$ of $P\Gamma L(2, 9)$, where $a \in PGL(2, 9) - PSL(2, 9)$ and b is an involutory field automorphism of $\mathbf{F}_9$ extended to an automorphism of $PSL(2, 9)$. In $P\hat{G}L(2, 9)$ the Sylow 2-subgroups of the normalizer of a Sylow 3-subgroup are quaternion.

(7B) *Suppose H is generated by normal cyclic subgroups. Then one of the following holds:*

(a) *HK_i/K_i is cyclic.*

(b) *HK_i/K_i is quaternion and $P_i/K_i \simeq P\hat{G}L(2, 9)$.*

(c) *HK_i/K_i is a 2-group of order 16 and P_i/K_i is isomorphic to a subgroup of $P\Gamma U(3, 5)$ of index 3.*

Proof. Using (2A) we can apply (4G)(b) to the group HK_i/K_i. Since H is the product of normal cyclic subgroups so that $H=F(H)$ is nilpotent, it follows by (4G)(b) that HK_i/K_i is cyclic, or $P_i/K_i > L_i$ and $L_i \simeq PSL(2, p^2)$ with p a Mersenne prime, or $L_i \simeq PSU(3, p)$ with p a Mersenne prime or a prime such that $p+1=2^b \cdot 3$. Suppose HK_i/K_i is not cyclic. If $p \neq 5$, then the Sylow 2-subgroups of the normalizer of a Sylow p-subgroup in $P\Gamma L(2, p^2)$ or $P\Gamma U(3, p)$ are quasidihedral. However, the Sylow 2-subgroup of HK_i/K_i, being also a product of normal cyclic subgroups, is then necessarily quaternion, so that $p=3$, $P_i/K_i \simeq P\hat{G}L(2, 9)$, and (b) holds. If $p=5$, then P_i/K_i is isomorphic to a subgroup of $P\Gamma U(3, 5)$ containing $PSU(3, 5)$. Then the nilpotency of H implies that (c) holds.

(7C) *If* $|W|=8$, *then H is abelian.*

Proof. We consider cases in accordance with (6A)(a), (b), and (c). Suppose (6A)(a) holds. We may choose notation so that $[U_2, U_2^{s_1}]=1$. Since $[U_1, U_2^{s_1}]=[U_1^{s_2}, U_2^{s_1}]=1$ by (3G), we have $U_2^{s_1} \leqq Z(U)$. Represent G as a permutation group on the cosets of P_2. By (1I) this permutation group has rank 3 with suborbits P_2, $P_2 s_1 P_2$, $P_2 s_1 s_2 s_1 s_2 P_2$ of lengths 1, $q(q+1)$, q^3 respectively. Now $P_2 s_1 P_2 = P_2 s_1(B \cup B s_2 B) = P_2 s_1 U_1 \cup P_2 s_1 s_2 U_2 U_1^{s_2}$, and $P_2 s_1 x U_2^{s_1} = P_2 s_1 x$ for every $x \in P_2$. If $q>2$, we can apply a theorem of Higman ([16], Theorem 2) to conclude that G has a normal subgroup $G_0 \cong PSp(4, q)$. Since $G=U^G$, it follows that $G=G_0$. If $q=2$, then G has degree 15 and $|G|=720$ by (1H)(c). It follows by [26] that $G \cong S_6 \cong PSp(4, 2)$. Thus $G \cong PSp(4, q)$ and H is abelian.

Suppose (6A)(b) holds. Then

$$L_1 \cong PSL(2, q), \quad R_2 \cong PSL(2, q^2),$$
$$[U_1, U_1^{s_2}]=1, \quad [U_2, U_2^{s_1}]=U_1^{s_2}.$$

In particular, $[R_1, R_1^{s_2}]=1$ and $[s_1, s_1 s_2 s_1]=1$. Setting $s_1^2=j$, we then have $t=(s_1 s_2)^4=(s_1(s_2 s_1 s_2))^2=j j^{s_2}$. We claim $t=1$. This is so if $j=1$. Suppose then that $j \neq 1$ so that $\langle j \rangle = Z(R_1) \trianglelefteq H$ and $j \in Z(H)$. Then j induces a diagonal automorphism of R_2. Since $C(j) \geqq \langle R_1, H \rangle$ and $Z(G)=1$, j necessarily inverts U_2 and $U_2^{s_2}$. But j^{s_2} also inverts U_2 and $U_2^{s_2}$. In particular, $j j^{s_2} \in C(R_2)$ and $j j^{s_2} \in K_2$. Since $j=j^{s_1}$ inverts $U_2^{s_1}$ and $j^{s_2}=j^{s_1 s_2}$ inverts $U_2^{s_1}$, we also have $j j^{s_2} \in C(U_2^{s_1})$. Thus $U_1^{s_2}=[U_2, U_2^{s_1}] \leqq C(j j^{s_2})$. Transforming by s_2 then gives $j j^{s_2} \in C(U_1)$ and so $j j^{s_2} \in K_1$. Thus $t=j j^{s_2} \in H \cap K_1 \cap K_2$ and $t=1$ as claimed. By (7A) $H=H_1 H_2$ is then the product of normal cyclic subgroups. (7B) applies, so the embedding of H into $P_1/K_1 \times P_2/K_2$ embeds H into the direct product of two cyclic groups unless P_1/K_1 or P_2/K_2 is isomorphic to $P\hat{G}L(2, 9)$. If $P_2/K_2 \cong P\hat{G}L(2, 9)$, then $L_1 \cong PSL(2, 3)$ and $|H_1| \leqq 2$. In particular, $H_1 \leqq Z(H)$, and since H_2 is cyclic, it follows that H is abelian. If $P_1/K_1 \cong P\hat{G}L(2, 9)$, then $R_2 \cong PSL(2, 81)$. (4G) implies that P_2/K_2 is isomorphic to a subgroup of $PGL(2, 81)$ and so $H' \leqq H \cap K_2$. However, H/H_2 is abelian so that $H' \leqq H_2$. Since $R_2 \cong PSL(2, 81)$, $H' \leqq H_2 \cap K_2 = 1$ and $H' = 1$.

Suppose (6A)(c) holds. Then

$$R_1 \cong SL(2, q^2), \qquad R_2 \cong SU(3, q),$$
$$[U_1, U_1^{s_2}] = Z(U_2^{s_1}), \quad [U_2, U_2^{s_1}] = U_1^{s_2}.$$

For the remainder of the proof of (7C) we will depart from the notation in §4 and assume that $s_2 \in \langle Z(U_2), Z(U_2)^{s_2} \rangle$. Set $s_2^2 = j$ so that j has order 1 or 2 according as p is even or odd, and set $s_1^2 = i$, where $Z(R_1) = \langle i \rangle$. Since R_2 acts irreducibly on V_2, $H \cap K_2$ is faithful on V_2, and $[R_2, H \cap K_2] = 1$, it follows that $H \cap K_2$ is a normal cyclic subgroup of H. We claim $i i^{s_2} \in C(U_2)$. This is so if $i = 1$. Suppose then that $i \neq 1$. Now $\langle i \rangle = Z(R_1) \trianglelefteq H$ and so $C(i) \cap (U_2/\Phi(U_2))$ admits H. Since $i \in K_1$, this implies that i inverts $U_2/\Phi(U_2)$. The same argument implies that i inverts $(U_2/\Phi(U_2))^{s_2}$ so that i^{s_2} also inverts $U_2/\Phi(U_2)$. Thus $i i^{s_2} \in C(U_2)$ as claimed. By order considerations R_2 acts trivially on $[U_{s_2}, U_{s_2}] = Z(U_2^{s_1})$, so $\langle Z(U_2), Z(U_2^{s_2}) \rangle^{s_1} \leqq C(R_2)$. In particular, $s_2 \in C(R_2^{s_1})$. Let $t = (s_1 s_2)^4$. Then $u^t = u^{(s_1 s_2)^4} = ((u^{s_1})^{s_2})^{(s_1 s_2)^3} = ((u^{i s_2})^{s_1})^{s_2 s_1 s_2} = u^{i s_2 i s_2} = u^{i j i^{s_2}} = (u^j)^{i i^{s_2}} = u^j$ for $u \in U_2$. If $p = 2$, then $u^t = u$, $t \in H \cap K_2$, and $\langle t \rangle \trianglelefteq H$, so that $H = H_1 H_2 \langle t \rangle$ is the product of normal cyclic subgroups. If $p \neq 2$, then j is an involution. A consideration of the group $R_2 \simeq SU(3, q)$ shows that j inverts $U_2/\Phi(U_2)$. Consequently t inverts $U_2/\Phi(U_2)$. We have already observed that i inverts $U_2/\Phi(U_2)$ and that $i \in Z(H)$. It follows by (7A) that ti is a p'-element centralizing $U_2/\Phi(U_2)$. Thus $ti \in H \cap K_2$. Since $H \cap K_2$ is a normal cyclic subgroup of H, the subgroup $\langle ti \rangle \trianglelefteq H$. Thus $H = H_1 H_2 \langle t \rangle = H_1 H_2 \langle ti \rangle$ is also the product of normal cyclic subgroups.

As before H is abelian unless $P_1/K_1 \simeq P\hat{G}L(2, 9)$ or $P_2/K_2 \simeq PSU(3, 5)\langle h \rangle$ with $h^2 \in PSU(3, 5)$. Suppose $P_1/K_1 \simeq P\hat{G}L(2, 9)$. Then $P_2/K_2 \simeq PSU(3, 3) \simeq PGU(3, 3)$ so that $H = H_2(H \cap K_2)$ by (7B). Since $[H_2, H \cap K_2] \leqq [R_2, H \cap K_2] = 1$, it follows that H is abelian. Now suppose $P_2/K_2 \simeq PSU(3, 5)\langle h \rangle$. Since $H \cap K_2$ is isomorphic to a subgroup of HK_1/K_1 and $R_1 \simeq SL(2, 25)$, it follows that $Z(H_2) = O_3(H \cap K_2) = O_3(H)$ is normal in N. Then $O_3(H)$ acts fixed-point freely on U_1 and $U_1^{s_2}$, and $O_3(H)$ centralizes U_2 and $U_2^{s_1}$. But this is impossible since $[U_2, U_2^{s_1}] = U_1^{s_2}$. This completes the proof of (7C).

(7D) *If* $|W| = 12$, *then* H *is abelian.*

Proof. Suppose (6E)(a) holds. Then

$$L_1 \cong L_2 \cong PSL(2, q), \quad [U_{s_1}, U_{s_1}] \neq 1, \quad [U_{s_2}, U_{s_2}] \neq 1.$$

If $[U_1^{s_2 s_1}, U_1^{s_2}] = U_2^{s_1 s_2}$ and $[U_2^{s_1 s_2}, U_2^{s_1}] = U_1^{s_2 s_1}$, then $U_1^{s_2 s_1} \leqq [U_{s_1}, U_{s_1}]$. Since R_1 normalizes $[U_{s_1}, U_{s_1}]$ it follows that $[U_1^{s_2 s_1}, U_1] = [U_1^{s_2 s_1}, U_1^{s_2}]^{s_2} = U_2^{s_1} \leqq [U_{s_1}, U_{s_1}] \leqq U_1^{s_2} U_2^{s_1 s_2} U_1^{s_2 s_1}$, which is impossible. We may choose notation so that $[U_2^{s_1}, U_2^{s_1 s_2}] = 1$. Suppose $[U_1^{s_2 s_1}, U_1^{s_2}] = 1$ as well. If $[U_2^{s_1}, U_1^{s_2}] \neq 1$, then $Z(U_{s_1}) = U_2^{s_1 s_2}$ and so $[U_2^{s_1 s_2}, R_1] = 1$, $[U_2^{s_1 s_2}, U_1] = 1$, $[U_2^{s_1}, U_1^{s_2}] = 1$, which is impossible. Thus $[U_2^{s_1}, U_1^{s_2}] = 1$, $[U_2^{s_1 s_2}, U_1] = 1$. This implies $[U_2^{s_1 s_2}, R_1] = 1$, $[R_2^{s_1 s_2}, R_1] = 1$. Moreover, $Z(U_{s_1}) = U_1^{s_2 s_1} U_2^{s_1 s_2} U_1^{s_2}$ and $Z(U_{s_1})$ admits R_1. Thus R_1 is irreducible on $U_{s_1}/Z(U_{s_1})$ and $H \cap K_1$ is faithful on $U_{s_1}/Z(U_{s_1})$. In particular $H \cap K_1$ is a normal cyclic subgroup of H. A similar argument with the roles of U_{s_1} and U_{s_2} interchanged shows $H \cap K_2$ is also a normal cyclic subgroup of H. Set $s_1^2 = i$, $s_2^2 = j$, where $Z(R_1) = \langle i \rangle$, $Z(R_2) = \langle j \rangle$. We note that $\langle i, j \rangle \leqq Z(H)$. Since $[R_1, R_2^{s_1 s_2}] = [R_1^{s_2 s_1}, R_2] = 1$, we have

$$j^{(s_1 s_2)^4} = j^{s_1 j s_1 s_2 s_1 s_2} = j^{s_1 s_2},$$

$$i^{s_2 (s_1 s_2)^3} = i^{s_2 i s_2 s_1 s_2} = i^{s_2}.$$

Thus if $u \in U_2$, then

$$\begin{aligned} u^t &= u^{(s_1 s_2)^6} \\ &= u^{(s_1 s_2) s_1 s_2 (s_1 s_2)^4} \\ &= u^{s_1 j (s_1 s_2)^4} \\ &= u^{i j^{s_1} s_2 (s_1 s_2)^3} \\ &= u^{s_2 (s_1 s_2)^3 i^{s_2} j^{s_1 s_2}} \\ &= u^{s_2 s_1 j s_1 s_2 i^{s_2} j^{s_1 s_2}} \\ &= u^{s_2 i j^{s_1} s_2 i^{s_2} j^{s_1 s_2}} \\ &= u^{j (i^{s_2} j^{s_1 s_2})^2} \\ &= u, \end{aligned}$$

and so $t \in H \cap K_2$ and $\langle t \rangle \trianglelefteq H$.

Suppose $[U_1^{s_2 s_1}, U_1^{s_2}] = U_2^{s_1 s_2}$ so that $[U_1^{s_2 s_1}, U_1] = U_2^{s_1}$. In particular, $\langle U_2^{s_1 s_2}, U_2^{s_1} \rangle \leqq [U_{s_2}, U_{s_2}]$. Since U_1 normalizes $[U_{s_1}, U_{s_1}]$ and $[U_{s_1}, U_{s_1}] \leqq U_1^{s_2} U_2^{s_1 s_2} U_1^{s_2 s_1}$, it follows that $U_1^{s_2 s_1} \not\leqq [U_{s_1}, U_{s_1}]$. This implies that $[U_1^{s_2}, U_2^{s_1}] \leqq U_2^{s_1 s_2}$ since $U_2^{s_1 s_2} \leqq [U_{s_1}, U_{s_1}]$ and $[U_1^{s_2}, U_2^{s_1}] \leqq U_1^{s_2 s_1} U_2^{s_1 s_2}$. But $U_2^{s_1} \not\leqq [U_{s_1}, U_{s_1}]$ and so $[U_1, U_2^{s_1 s_2}] = 1$. This implies that $[R_1, R_2^{s_1 s_2}] = 1$, and in particular, $[U_1^{s_2 s_1}, R_2] = 1$. Now $U_2^{s_1} U_1^{s_2 s_1} U_2^{s_1 s_2}$ is a normal subgroup of U_{s_2} admitting R_2, and thus R_2 acts irreducibly on $U_{s_2}/U_2^{s_1} U_1^{s_2 s_1} U_2^{s_1 s_2}$. Since $H \cap K_2$ is faithful on this factor group, it follows that $H \cap K_2$ is a normal cyclic subgroup. We now proceed as before to show $\langle t \rangle \trianglelefteq H$.

In both cases H is then the product of normal cyclic subgroups, so by (7B) H is abelian unless $P_i/K_i \cong P\hat{G}L(2, 9)$ for $i=1$ or 2. It was shown above that R_2 acts faithfully and irreducibly on $V=U_{s_2}/U_2^{s_1} U_1^{s_2 s_1} U_2^{s_1 s_2}$. We claim that P_2/U_{s_2} is faithful on V. Let K be the kernel of P_2 on V. Since $P_2=HR_2 U_{s_2}$, we have $K=(K \cap HR_2) U_{s_2}$. But $[R_2, K \cap HR] \leqq R_2 \cap K=1$, so that $K \cap HR_2 \leqq K \cap K_2$. Since $K_2=(H \cap K_2) U_{s_2}$ and $H \cap K_2$ is faithful on V, it follows that $K \cap K_2=U_{s_2}$ and the claim is proved. Since H has only two irreducible constituents on this module, it follows by (4B) that P_2/U_{s_2} is isomorphic to a subgroup of $GL(2, 9)$. Now $H \leqq N(U_2)$ and in $GL(2, 9)$ the normalizer of a Sylow 3-subgroup has no quaternion section. Thus H is abelian.

Suppose (6E)(b) holds. Then R_2 is irreducible on V_2, $H \cap K_2$ is faithful on V_2, and so $H \cap K_2$ is a normal cyclic subgroup of H. Moreover, $[R_1, R_2^{s_1 s_2}]=1$. We proceed as above to show $t \in H \cap K_2$ and $\langle t\rangle \trianglelefteq H$. Then H is abelian unless $P_1/K_1 \cong P\hat{G}L(2, 9)$. But in this case $R_2 \cong SL(2, 3^6)$ by (6E)(b) and so H_2 contains an element h of order 8. Since $h^4 \in H \cap K_2$, it follows that $h^4 \notin K_1$, which contradicts $P_1/K_1 \cong P\hat{G}L(2, 9)$. This completes the proof of (7D).

(7E) *If* $|W|=16$, *then* H *is abelian.*

Proof. By (6H), $s_1^2=s_2^2=1$ and $[U_2, U_2^{s_1 s_2 s_1}]=1$ so that $[R_2, R_2^{s_1 s_2 s_1}]=1$. The group R_2 acts faithfully on $M=U_{s_2}/[U_{s_2}, U_{s_2}] U_2^{s_1 s_2 s_1}$. Indeed for $q>2$ this is clear since $Sz(q)$ is simple and $[s_2, M] \neq 1$. For $q=2$ $Sz(2)$ is the holomorph of $\mathbf{Z}_5$. Since s_2 and the involution in U_2 have different actions on M, it follows $Sz(2)$ acts faithfully on M. Thus (4F) implies R_2 acts irreducibly on M. Since $H \cap K_2$ is faithful on M, $H \cap K_2$ is then a normal cyclic subgroup of H. If $u \in U_2$, then

$$
\begin{aligned}
u^t &= u^{(s_1 s_2)^8} \\
&= u^{(s_1 s_2 s_1) s_2 (s_1 s_2 s_1) s_2 (s_1 s_2)^4} \\
&= u^{s_2 (s_1 s_2)^4} \\
&= u^{s_2 (s_1 s_2 s_1) s_2 (s_1 s_2 s_1) s_2} \\
&= u^{s_2 s_2} \\
&= u.
\end{aligned}
$$

Thus $t \in H \cap K_2$, $\langle t\rangle \trianglelefteq H$, and H is then abelian by (7B).

(7F) *H is abelian and the following hold:*

(a) *If* $L_i \cong PSL(2, q)$, *then* $P_i/K_i \cong PSL(2, q)$ *or* $PGL(2, q)$.

(b) *If* $L_i \cong PSU(3, q)$, *then* $P_i/K_i \cong PSU(3, q)$ *or* $PGU(3, q)$.

(c) *If* $L_i \cong Sz(q)$, *then* $P_i/K_i \cong Sz(q)$.

Proof. This is immediate.

§ 8. Identification of G in the Case $|W|=8$

In the next three sections we identify the group G. Although the methods used for the three cases $|W|=8$, $|W|=12$, $|W|=16$ do not differ significantly, the details required in each case are of such a nature that a separation of the cases is desirable. It is possible to identify G by showing that its multiplication table is unique, and we do so for some of the groups, as for example $PSU(5, q)$ and ${}^2F_4(q)$. But where it is possible to appeal to existing characterization theorems, we have done so in the interest of shortening the paper.

In §7 we have labeled the elements of R_i so that $U_i=\langle u_i(\alpha)\rangle$ or $\langle u_i(\alpha, \beta)\rangle$ and $H_i=H\cap R_i=\langle h_i(\gamma)\rangle$. If σ is a field automorphism of R_i, then replacement of $u_i(\alpha)$ or $u_i(\alpha, \beta)$ by $u_i(\alpha^\sigma)$ or $u_i(\alpha^\sigma, \beta^\sigma)$, $h_i(\gamma)$ by $h_i(\gamma^\sigma)$, and s_i by s_i is another labeling of the elements of R_i. We shall call this a change of labeling by the field automorphism σ.

(8A) Suppose $|W|=8$ *and* (a) *of* (6A) *holds. Then* $G\cong PSp(4, q)$.

Proof. This was shown in the proof of (7C).

(8B) *Suppose* $|W|=8$ *and* (b) *of* (6A) *holds. Then* $G\cong PSU(4, q)$.

Proof. By (6A) we have

$$\begin{aligned} &[U_1, U_1^{s_2}]=1, \quad [U_2, U_2^{s_1}]=U_1^{s_2} \\ &L_1\cong PSL(2, q), \quad R_2\cong PSL(2, q^2). \end{aligned} \tag{8.1}$$

Transforming the first commutator in (8.1) by s_1 gives $[U_1^{s_1}, U_1^{s_2}]=1$ so that $[R_1, U_1^{s_2}]=1$. Transforming this in turn by $s_2 s_1 s_2$ gives

$$[R_1, U_1^{s_1 s_2}]=1,$$

and thus

$$[R_1, R_1^{s_2}]=1. \tag{8.2}$$

Consider the HR_1-module $V_1=U_{s_1}/U_1^{s_2}$ of order q^4. As an H-module V_1 is the direct sum of irreducible H-submodules $U_2 U_1^{s_2}/U_1^{s_2}$ and $U_2^{s_1} U_1^{s_2}/U_1^{s_2}$, each of order q^2. Since H is abelian, the H_1-submodules in $U_2 U_1^{s_2}/U_1^{s_2}$ are all isomorphic, as are the H_1-submodules in $U_2^{s_1} U_1^{s_2}/U_1^{s_2}$. Since s_1 inverts H_1, the eigenvalues of $h_1(\gamma)$ on $U_2 U_1^{s_2}/U_1^{s_2}$ are the inverses of the eigenvalues of $h_1(\gamma)$ on $U_2^{s_1} U_1^{s_2}/U_1^{s_2}$.

Suppose V_1 is a reducible HR_1-module. Then there exists an HR_1-submodule V_0 of order q^2, which cannot be $U_2^{s_1} U_1^{s_2}/U_1^{s_2}$ since the latter does not admit s_1. By (3G)

$$[U_1, U_2^{s_1}]=1, \text{ and } [U_1, U_2]\leqq U_2^{s_1} U_1^{s_2}. \tag{8.3}$$

If $[U_1, V_0] \neq 1$, then the equality $[U_1, V_0] = [U_1, V_0]^H = V_0$ and (8.3) imply that $V_0 = U_2^{s_1} U_1^{s_2}/U_1^{s_2}$, which is impossible. If $[U_1, V_0] = 1$, then

$$[U_1, U_2] \geqq U_1^{s_2} \quad \text{and so} \quad [U_1, V_1] = 1.$$

But then $[R_1, V_1] = 1$, which is impossible. Thus V_1 is an irreducible HR_1-module.

Suppose j is an involution in $Z(R_1)$ and $j \in C(V_1)$. Then $[j, U_2] = 1$ so that $C(j) \geqq \langle R_1, R_2, H \rangle = G$, which is impossible. Thus R_1 is faithfully represented on each R_1-submodule of V_1 by Clifford's Theorem.

Suppose p is odd and $R_1 \cong PSL(2, q)$. Now $|H : H \cap K_1| = \frac{1}{2}(q-1)$ or $(q-1)$, and $|H : H \cap K_2| = \frac{1}{2}(q^2-1)$ or (q^2-1). If $|H \cap K_1|$ is odd, then an S_2-subgroup of H is cyclic and $q \equiv 1 \pmod 4$. But if $q \equiv 1 \pmod 4$, then H_1 and $H_1^{s_2}$ contain involutions. Since we are supposing $R_1 \cong PSL(2, q)$, we would then have by (8.2) that $\langle R_1, R_1^{s_2} \rangle = R_1 \times R_1^{s_2}$, so H would contain a subgroup of type (2, 2), which is impossible. Thus $|H \cap K_1|$ is even. Let j be an involution in $H \cap K_1$. Since $C(j) \geqq \langle R_1, H \rangle$ and $Z(G) = 1$, j must invert U_2 and $U_2^{s_2}$. Thus j^{s_2} inverts $U_2^{s_2}$ and U_2, and so $jj^{s_2} \in C(R_2)$. In particular, $jj^{s_2} \in H \cap K_2$. Now $j = j^{s_1}$ inverts $U_2^{s_1}$, and $j^{s_2} = j^{s_1 s_2}$ inverts $U_2^{s_1}$. Thus $jj^{s_2} \in C(U_2^{s_1})$, so by (8.1) we have $jj^{s_2} \in C(U_1^{s_2})$. Transforming this by s_2 gives $jj^{s_2} \in C(U_1)$ so that $jj^{s_2} \in C(R_1)$. Thus $jj^{s_2} \in H \cap K_1 \cap K_2 = 1$ and $j = j^{s_2}$. In particular, $C(j) \geqq R_1 \times R_1^{s_2}$. If $q \equiv 1 \pmod 4$, then H_1 and $H_1^{s_2}$ contain involutions. Since the S_2-subgroup of H has 2 generators, $j \in R_1 \times R_1^{s_2}$, which is impossible. Thus $R_1 \cong PSL(2, q)$ and $q \equiv 1 \pmod 4$ cannot hold simultaneously.

Suppose p is odd and V_1 is an irreducible R_1-module. If $R_1 \cong SL(2, q)$, then the representation of R_1 on V_1, which is faithful, is of type (c) in the notation of (4.7). Properties of this representation, which occurs only for $p \geqq 5$, can be found in [9] § 1, (1.1). In particular, $|C_{V_1}(U_1)| = q$, which contradicts (8.3). If $R_1 \cong PSL(2, q)$, so that $q \equiv -1 \pmod 4$, then R_1 on V_1 is of type (f) in the notation of (4.7), type (g) being excluded by $q \equiv -1 \pmod 4$. Thus n is odd and $n \geqq 3$. In the representation $\Gamma \otimes \Gamma^{p^i}$ of $SL(2, q)$, where $1 \leqq i \leqq n-1$, the element $h(\gamma)$ has eigenvalues $\gamma^{\pm 1 \pm p^i}$. In particular, there exists j, $0 \leqq j \leqq n-1$, such that $\gamma^{(1+p^i)p^j} = \gamma^{1-p^i}$ or $\gamma^{p^i - 1}$. Thus

$$(1+p^i)\, p^j \equiv \pm (p^i - 1) \pmod{p^n - 1}.$$

If $i + j < n$, this congruence is clearly impossible. If $i + j = n + k$, where $k \geqq 0$, then $k \leqq n-2$ and the congruence becomes $p^k + p^j \equiv \pm (p^i - 1) \pmod{p^n - 1}$, which is also impossible. Thus V_1 is a reducible R_1-module.

Suppose p is odd, and $V_1 = W_1 \oplus W_2$ is the decomposition of V_1 as a direct sum of conjugate irreducible R_1-submodules W_1 and W_2. If $R_1 \cong PSL(2, q)$, then R_1 on W_1 is necessarily of type (d) in the notation of

(4.7), and n is even. Thus $q \equiv 1 \pmod 4$, which is impossible. Thus we have

$$R_1 \cong SL(2, q) \tag{8.4}$$

and R_1 is faithful on W_1 and W_2.

Suppose p is odd, and let $Z(R_1) = \langle j \rangle$. If $R_1 \cap R_1^{s_2} = 1$, then $j j^{s_2} \neq 1$ and $C(j j^{s_2}) \geqq \langle R_1, H \rangle$. One of the three involutions j, j^{s_2}, $j j^{s_2}$ centralizes R_2. If j or $j^{s_2} \in C(R_2)$, then both lie in $C(R_2)$ since $C(R_2)^{s_2} = C(R_2)$. Thus we always have $j j^{s_2} \in C(R_2)$ so that $C(j j^{s_2}) \geqq \langle R_1, R_2, H \rangle = G$, which is impossible. Thus $R_1 \cap R_1^{s_2} = \langle j \rangle$ and $C(j) \geqq R_1 R_1^{s_2} H \langle s_2 \rangle$. Since $Z(G) = 1$, j inverts U_2. It is now possible to calculate $C(j) \cap BwB$ for each (B, B)-coset of G. We find that $|C(j)| = 2|H| q^2 (q+1)^2$. However $|R_1 R_1^{s_2} H \langle s_2 \rangle| = 2|H| q^2 (q+1)^2$, and so

$$C(j) = R_1 R_1^{s_2} H \langle s_2 \rangle. \tag{8.5}$$

By (3B) $|H|$ divides $(q^2-1)(q-1)$. But since $|H \cap R_1 R_1^{s_2}| = \frac{1}{2}(q-1)^2$ and $|H \cap R_2| = \frac{1}{2}(q^2-1)$, it follows that

$$|H| = \frac{1}{d}(q^2-1)(q-1), \qquad d = 1, 2, \text{ or } 4.$$

(1H) implies that $|G| = |H| q^6 (q^3+1)(q^2+1)(q+1)$ and thus j is in the center of an S_2-subgroup of G.

We continue with the assumption p is odd. The group H is a subdirect product of $H/H \cap K_1$ and $H/H \cap K_2$, $|H/H \cap K_1| = \frac{1}{2}(q-1)$ or $q-1$, and $|H/H \cap K_2| = \frac{1}{2}(q^2-1)$ or q^2-1. Let $T = H^{q-1}$. So T is a characteristic subgroup of order $\frac{1}{2}(q+1)$ or $(q+1)$ of H. Since s_2 inverts $H/H \cap K_2$ and $(H \cap K_2)^{q-1} = 1$, s_2 inverts T. Thus $\langle T, s_2 \rangle$ is a dihedral group of order $q+1$ or $2(q+1)$. From the definition of T we have $[T, R_1] = 1$, and so $[T, R_1 R_1^{s_2}] = 1$. The representation of R_2 on U_{s_2} is of type (d) in the notation of (4.7). In particular, the eigenvalues of $h_2(\gamma)$ are the algebraic conjugates of $\gamma^{\pm 1 \pm q}$. Since $U_1^H = U_1$ and $|U_1| = q$, $h_2(\gamma)$ acting on U_1 has eigenvalues which are algebraic conjugates of γ^{1+q} or γ^{-1-q}, if γ is an element of order q^2-1. Thus $h_2(\gamma)$ induces a diagonal automorphism of R_1 which is not inner. This implies

$$|H : H \cap K_1| = q-1 \quad \text{and} \quad H_0 < H, \tag{8.6}$$

where $H_0 = R_1 R_1^{s_2} T \cap H$.

Suppose $|H : H \cap K_2| = q^2-1$. The group H is then a subdirect product of $\mathbf{Z}_{q-1} \times \mathbf{Z}_{q^2-1}$, and $|T| = q+1$. Let $\tilde{H}$ be the characteristic subgroup of index 2 of H defined by $\tilde{H} = \langle h \in H : h^{\frac{1}{2}(q^2-1)} = 1 \rangle$. Then $\tilde{H}$ contains all involutions of H, so in particular $s_1^2 \in \tilde{H}$, $s_2^2 \in \tilde{H}$. Also H_0 has order divisible by $\frac{1}{4}(q^2-1)(q-1)$, so $|H : H_0| = 1, 2$, or 4. Since $s_2^2 = 1$, the exponent of $\langle H/H_0, s_2\rangle$ is 2 or 4. Choose $\varepsilon=\pm 1$, so that $q\equiv\varepsilon \pmod 4$, and let $2^a \| q-\varepsilon$. By (8.5) $C(j)$ and hence G contain no elements of order 2^{a+3}. Thus $(s_1 s_2)^4\in\tilde{H}$ and $\langle\tilde{H}, s_1, s_2\rangle/\tilde{H}\cong W$. Let $\tilde{B}=\tilde{H}U$, $\tilde{N}=\langle\tilde{H}, s_1, s_2\rangle$, and $\tilde{G}=\bigcup_s \tilde{B}s\tilde{B}$ where $s\in\tilde{N}/\tilde{H}$. Since $s_i U_i s_i^{-1}\subseteq\tilde{B}s_i U_i$ for $i=1, 2$, it follows from the (B, N)-properties of G that $\tilde{G}$ is a subgroup of G of index 2, which is impossible. Thus $|H:H\cap K_2|=\frac{1}{2}(q^2-1)$, $|T|=\frac{1}{2}(q+1)$, and $d=2$ or 4. Since $|H_2|=\frac{1}{2}(q^2-1)$, we also have $H=H_2\times(H\cap K_2)$.

We next show that $d=(4, q+1)$. If $q\equiv 1 \pmod 4$, then $|H_0|=\frac{1}{4}(q^2-1)(q-1)$. Then $d=(4, q+1)$ by (8.6). Suppose $q\equiv -1 \pmod 4$ and $d=2\neq(4, q+1)$. Then $|H\cap K_1|=\frac{1}{2}(q^2-1)$, $|H\cap K_2|=q-1$, and $H=H_2\times(H\cap K_2)$. Let i be an involution in T. If $i\neq j$, then $\langle i, j\rangle$ would be a subgroup of type (2, 2) in $H\cap K_1$. Since $|H\cap K_2|$ is even, an involution of $H\cap K_1$ would be in K_2, which is impossible. Thus $j\in T$. Let $\tilde{H}=H_2\times(H\cap K_2)^2$, so $\tilde{H}$ is a subgroup of H of index 2 containing the Hall subgroup of order $\frac{1}{4}(q-1)^2$ of H. Moreover, $j\in\tilde{H}$ since $j\in H^2$. Thus $H_1\leqq\tilde{H}$. Clearly $\tilde{H}^{s_2}=\tilde{H}$ from the definition of $\tilde{H}$. We claim $\tilde{H}^{s_1}=\tilde{H}$ as well. Since $|\tilde{H}:H^2|=2$ and $s_1\in N(H^2)$, it will be sufficient to show that $z^{s_1}\in\tilde{H}$ where $\langle z\rangle=H_2$. Now $H_1(H\cap K_1)$ admits s_1, and $|H:H_1(H\cap K_1)|=2$. Thus $z^{-1}s_1^{-1}zs_1=xy$ for some $x\in H_1$, $y\in H\cap K_1$. If $y\notin(H\cap K_1)^2$, then $|\langle y\rangle|_2=2^b$ where $2^b\|q+1$. But $|\langle z\rangle|_2=2^b$ as well, and so $j\in\langle z\rangle\cap\langle y\rangle$. Thus $|\langle zxy\rangle|_2<2^b$, which is impossible. Thus $z^{s_1}\in\tilde{H}$ and $s_1, s_2\in N(\tilde{H})$. Now $s_1^2\in\tilde{H}$, and $s_2^2=1$. We claim $(s_1 s_2)^4\in\tilde{H}$. The group H_0 has order $\frac{1}{8}(q^2-1)(q-1)$ and $H=\langle H_0, k, z\rangle$, where $\langle z\rangle=H_2$ and k is the involution in $H\cap K_2$. Thus s_2 centralizes H/H_0 and $\langle H/H_0, s_2\rangle$ has exponent 2. By (8.5) $C(j)$ and hence G contain no elements of order 2^{b+2}. Thus $(s_1 s_2)^4\in\tilde{H}$. As in the previous paragraph, we obtain a contradiction. Thus $d=(4, q+1)$ in all cases.

The subgroup $R_1 R_1^{s_2} T\langle s_2\rangle$ has a unique structure. Since it has index 2 in $C(j)$, the structure of $C(j)$ will be determined if we describe the extension in terms of an element $h\in H-H_0$. Suppose $q\equiv 1 \pmod 4$. By (3B)(c) $H\cap K_2$ is faithfully represented on R_1 as a cyclic group of order $q-1$. Let $H\cap K_2=\langle h\rangle$. Then h^2 induces an inner automorphism of R_1, so there exists $t\in H_1$ such that $h^2t^{-1}=v\in H\cap K_1$. Now $[h, s_2]=1$, $t\in C(R_1^{s_2})$, $t^{s_2}\in C(R_1)$. Thus $h^2=tv=t^{s_2}v^{s_2}$, and $h^2t^{-1}t^{-s_2}=vt^{-s_2}=v^{s_2}t^{-1}\in C(R_1 R_1^{s_2})$. Express $t=xy$, where $x\in H_2$, $y\in H\cap K_2$. Then $t^{s_2}=x^{-1}y$, $tt^{s_2}=y^2\in H\cap K_2$, so that $h^2t^{-1}t^{-s_2}\in H\cap K_1\cap K_2=1$ and $h^2=tt^{s_2}$. Now t has order $q-1$ since h does. Since h induces a diagonal automorphism of R_1, we have enough information to determine the structure of $C(j)$. Suppose $q\equiv -1 \pmod 4$. The group $H\cap R_2$ is cyclic of order $\frac{1}{2}(q^2-1)$. Let $\langle h\rangle$ be its subgroup of order $q+1$. Then $\langle h, s_2\rangle$ is dihedral of order $2(q+1)$. Since $h^2\in(H\cap R_2)^{q-1}$, it follows that $T=\langle h^2\rangle$. Also $\langle h\rangle$ contains an S_2-subgroup of H, and so h induces a diagonal automorphism of order 2 of R_1 and $R_1^{s_2}$. Again this determines the structure of $C(j)$. By a theorem of Phan [20], it follows that $G \cong PSU(4, q)$.

We now suppose $p=2$. Then (8.1) and (8.2) imply that $|H|$ is divisible by (q^2-1) and $(q-1)^2$. Since q is even, we conclude that $|H|=(q^2-1)(q-1)$. The group $H \cap K_1$ is represented faithfully on V_1. Since $[H \cap K_1, HR_1]=1$, $|H \cap K_1| = q^2-1$, and HR_1 is irreducible on V_1, it follows that V_1 may be regarded as a vector space over $\mathbf{F}_{q^2}$ of dimension 2 on which HR_1 acts linearly. In particular, R_1 acts irreducibly on this space, and necessarily as $\Gamma^\sigma \otimes \mathbf{F}_{q^2}$ where σ is some field automorphism of $\mathbf{F}_q$. We may assume by [9] §1 that a natural basis for Γ^σ has the form $\{u_2(1), u_2^{s_1}(1)\}$. Now viewing V_1 as a vector space over $\mathbf{F}_{q^2}$, we have a scalar multiplication given by $\alpha u_2(1) = u_2(\alpha^*)$ for $\alpha \in \mathbf{F}_{q^2}$. Also $\alpha u_2(\beta) = (u_2(\beta))^h$ for some $h \in H \cap K_1$ and all $\beta \in \mathbf{F}_{q^2}$. Since h induces a diagonal automorphism on R_2, we have $\alpha u_2(\beta) = u_2(\alpha^* \beta)$ for all $\beta \in \mathbf{F}_{q^2}$. We claim that the mapping $\tau_1: \alpha \to \alpha^*$ is a field automorphism of $\mathbf{F}_{q^2}$. To see this we first note that $(\alpha+\beta) u_2(1) = u_2(\alpha^*) u_2(\beta^*) = u_2(\alpha^* + \beta^*)$, so that τ_1 is additive. Also $\alpha\beta u_2(1) = \alpha u_2(\beta^*) = u_2(\alpha^* \beta^*)$, so that τ_1 is multiplicative. Similarly $\alpha u_2^{s_1}(\beta) = u_2^{s_1}(\alpha^{\tau_2}\beta)$ for $\beta \in \mathbf{F}_{q^2}$, where τ_2 is an automorphism of $\mathbf{F}_{q^2}$. Applying s_1 we see that $\tau_1 = \tau_2 = \tau$. The matrix form of Γ^σ then implies that

$$[u_2(\alpha), u_1(\beta)] = u_2^{s_1}(\alpha \beta^{\sigma\tau}) \pmod{U_1^{s_2}}. \tag{8.7}$$

The group R_2 acts irreducibly on U_{s_2} and $[H \cap K_2, R_2]=1$. Since $H \cap K_2$ has order $q-1$ and $H \cap K_2$ is faithful on U_{s_2} by (3B)(c), it follows that U_{s_2} may be regarded as a vector space over $\mathbf{F}_q$ of dimension 4 on which R_2 acts linearly and irreducibly. H has three constituents on U_{s_2}, so by (4C) R_2 necessarily acts on U_{s_2} as an $\mathbf{F}_q$-form of $(\Gamma \times \bar{\Gamma})^\rho$, where Γ in this instance is the natural representation of $SL(2, q^2)$, $\bar{\ }$ is the field automorphism $x \to x^q$, and ρ is some field automorphism of $\mathbf{F}_{q^2}$. We fix a choice of θ in $\mathbf{F}_{q^2} - \mathbf{F}_q$. By the discussion following (4C) in §4, we may assume R_2 on U_{s_2} has the form Ξ^ρ, where Ξ is as in §4 and ρ is a field automorphism of $\mathbf{F}_q$. Moreover, a natural basis for Ξ^ρ has the form

$$\{u_1(1), u_2^{s_1}(x), u_2^{s_1}(y), u_1^{s_2}(1)\},$$

for some $x, y \in \mathbf{F}_{q^2}$. Proceeding as above we see that there are field automorphisms τ_1, τ_2, τ_3 of $\mathbf{F}_q$ such that $\alpha u_1(\beta) = u_1(\alpha^{\tau_1}\beta)$, $\alpha u_1^{s_2}(\beta) = u_1^{s_2}(\alpha^{\tau_1}\beta)$, $\alpha u_2^{s_1}(\beta x) = u_2^{s_1}(\alpha^{\tau_2}\beta x)$, $\alpha u_2^{s_1}(\beta y) = u_2^{s_1}(\alpha^{\tau_3}\beta y)$ for all $\alpha, \beta \in \mathbf{F}_q$. Also $\alpha u_2^{s_1}(\beta x) = (u_2^{s_1}(\beta x))^h$ for some $h \in H$. Since h induces a diagonal automorphism on $R_2^{s_1}$, it follows that $\tau_2 = \tau_3$. We also note that $\{\alpha^{\tau_i}: \alpha \in \mathbf{F}_q\}$ for $i=1, 2$ is a subfield of $\mathbf{F}_{q^2}$ and hence is precisely the subfield $\mathbf{F}_q$. From the matrix form of Ξ^ρ we see that the cyclic subgroup T of order $q+1$ in H_2 centralizes U_1 and $U_1^{s_2}$. Moreover

$$[u_1(\beta), u_2(\alpha)] \equiv u_2^{s_1}(\beta^{\tau_1^{-1}\tau_2} \alpha^{\rho\tau_2} x + \beta^{\tau_1^{-1}\tau_2} \alpha^{\rho\tau_2} y) \pmod{U_1^{s_2}}, \tag{8.8}$$

where $\alpha=\alpha_1+\alpha_2\theta$ and $\alpha_1, \alpha_2\in\mathbf{F}_q$. (8.7) and (8.8) imply that $\alpha\beta^{\sigma\tau}=\beta^{\tau_1^{-1}\tau_2}(\alpha_1^{\rho\tau_2}x+\alpha_2^{\rho\tau_2}y)$ for all $\alpha\in\mathbf{F}_{q^2}$ and all $\beta\in\mathbf{F}_q$ (here we use the fact $p=2$). Setting $\alpha=\beta=1$, we have $x=1$. Setting $\alpha=\theta$, $\beta=1$, we get $y=\theta$. Therefore $\alpha\beta^{\sigma\tau}=\beta^{\tau_1^{-1}\tau_2}\alpha_1^{\rho\tau_2}+\beta^{\tau_1^{-1}\tau_2}\alpha_2^{\rho\tau_2}\theta$. First setting $\alpha=1$ and then $\beta=1$ in this equation, we obtain $\sigma\tau=\tau_1^{-1}\tau_2$ and $\rho\tau_2=1$ respectively. Extend the field automorphism $\tau_2\tau_1^{-1}$ of $\mathbf{F}_q$ to an automorphism η of $\mathbf{F}_{q^2}$. We then relabel the elements of R_2 by the automorphism η so that $\alpha u_2^{s_1}(\beta)=u_2^{s_1}(\alpha^{\tau_1}\beta)$ for each $\alpha\in\mathbf{F}_q$, $\beta\in\mathbf{F}_{q^2}$. With this relabeling of the elements of R_2, the matrix form of Ξ^ρ now implies that

$$[u_1(\beta), u_2(\alpha)]=u_2^{s_1}(\alpha\beta)\,u_1^{s_2}(\alpha\bar{\alpha}\beta),$$
$$[u_2^{s_1}(\beta), u_2(\alpha)]=u_1^{s_2}(\beta(\alpha+\bar{\alpha})),$$
$$[u_2^{s_1}(\beta\theta), u_2(\alpha)]=u_1^{s_2}(\beta(\bar{\theta}\alpha+\theta\bar{\alpha})),$$

for $\beta\in\mathbf{F}_q$. The commutator identity $[xy, z]=[x, z]$. $[x, z, y]\,[y, z]$ then gives

$$[u_2^{s_1}(\beta), u_2(\alpha)]=u_1^{s_2}(\bar{\alpha}\beta+\bar{\beta}\alpha) \quad \text{for } \alpha, \beta\in\mathbf{F}q^2. \tag{8.9}$$

We have now determined the structure of U.

The structure of $R_1 U_{s_1}$ is now determined since we have the action of R_1 on V_1 and hence by (8.9) the action of R_1 on U_{s_1}. In particular, $U_1^{s_2}=Z(U_{s_1})=Z(R_1 U_{s_1})$. Let j be any involution in $U_1^{s_2}$. The elements of $H_1^{s_2}$ act regularly on $U_1^{s_2}$ while the elements of $(H\cap K_1)^{s_2}$ centralize j. Thus $C_H(j)=(H\cap K_1)^{s_2}$, and $C_B(j)=(H\cap K_1)^{s_2}U$. We consider next the set $C(j)\cap BwB$. j commutes with an element $uhwv$, where $u\in U$, $h\in H$, $v\in U_w^-$ if and only if $jhwj=hw$. Equivalently, j commutes with $uhwv$ if and only if $h^{-1}jh=wjw^{-1}$. This last equation holds only if $U_1^{s_2w}=U_1^{s_2}$, that is, $w=1$ or s_1. We easily obtain

$$C_G(j)=(H\cap K_1)^{s_2}\,U\cup(H\cap K_1)^{s_2}\,U s_1\,U_1,$$

and thus $|C_G(j)|=q^6(q^2-1)(q+1)$. Since $C_G(j)\geqq TR_1 U_{s_1}$ and $T\cap R_1 U_{s_1}=1$, it follows that $C_G(j)=TR_1 U_{s_1}$. Also T acts faithfully as scalar multiplication on V_1, and $[T, U_1^{s_2}]=1$. Thus $C_G(j)$ has a unique structure and is independent of the choice of j in $U_1^{s_2}$. By a theorem of Suzuki [32] it follows that $G\cong PSU(4, q)$.

(8C) *Suppose* $|W|=8$ *and* (c) *of* (6A) *holds. Then* $G\cong PSU(5, q)$.

Proof. From (6A) we have

$$\begin{aligned}&[U_2, U_2^{s_1}]=U_1^{s_2}, \qquad [U_1, U_1^{s_2}]=Z(U_2^{s_1})\\ &R_1\cong SL(2, q^2), \qquad R_2\cong SU(3, q).\end{aligned} \tag{8.10}$$

Thus $|H:H\cap K_1|=\frac{1}{2}(q^2-1)$ or (q^2-1), $|H:H\cap K_2|=\frac{1}{3}(q^2-1)$ or (q^2-1), and $|H|$ must divide $(q^2-1)^2$. Now R_2 acts trivially on $Z(U_2^{s_1})$. Transforming $[R_2, Z(U_2^{s_1})]=1$ by $s_1 s_2 s_1$ gives $[R_2, Z(U_2^{s_2})^{s_1}]=1$. Thus if we set $\tilde{R}_2=\langle Z(U_2), Z(U_2^{s_2})\rangle$, then $[R_2, \tilde{R}_2^{s_1}]=1$ so by (4.3)

$$\langle R_2, \tilde{R}_2^{s_1}\rangle = R_2 \times \tilde{R}_2^{s_1} \cong SU(3, q) \times SL(2, q). \tag{8.11}$$

In particular, $|H \cap R_2 \tilde{R}_2^{s_1}|=(q^2-1)(q-1)$. If $p \neq 2$, there exist elements in $H_2^{s_1}$ inducing outer automorphisms of $\tilde{R}_2^{s_1}$. Thus

$$|H| \equiv 0 \begin{cases} \bmod (q^2-1)(q-1) \\ \bmod 2(q^2-1)(q-1) & \text{if } p \neq 2. \end{cases} \tag{8.12}$$

The group R_i is irreducible on $V_i=U_{s_i}/\Phi(U_{s_i})$, and $H \cap K_i$ is faithful on V_i. Since $[R_i, H \cap K_i]=1$, the commuting algebra E_i of R_i on V_i is a finite field, say of order p^{m_i}, containing a multiplicative subgroup of order $|H \cap K_i|$. Then n divides m_i since $q-1$ divides $|H \cap K_i|$. If $p \neq 2$, then $2n$ divides m_i since $2(q-1)$ divides $|H \cap K_i|$ by (8.12). If $p=2$ and $|H|>(q^2-1)(q-1)$, we can also conclude that $2n$ divides m_i since $|H|$ divides $(q^2-1)^2$. Thus we have

$$|E_i| \equiv 0 \pmod{q^2} \quad \text{if } \begin{cases} p \neq 2 \\ p=2, \quad |H|>(q^2-1)(q-1). \end{cases} \tag{8.13}$$

In such cases V_1 and V_2 can be regarded as vector spaces over $\mathbf{F}_{q^2}$ of dimensions 2 and 3, on which R_1 and R_2 act linearly and irreducibly. The proofs of (4B), (4D) imply that we may assume R_1 on V_1 has the form Γ^ρ and R_2 on V_2 has the form Θ^σ for suitable field automorphisms ρ and σ of $\mathbf{F}_{q^2}$.

We shall assume $p \neq 2$, or $p=2$ and $|H|>(q^2-1)(q-1)$ unless stated otherwise. From §4 and [9] §1, we may take

$$\begin{gathered} \{\bar{u}_2(1), \bar{u}_2^{s_1}(1)\} \\ \{u_1^{s_2}(1), \bar{u}_2^{s_1}(x), u_1(1)\} \end{gathered} \tag{8.14}$$

as natural bases for Γ^ρ and Θ^σ respectively. Here x is some element in $\mathbf{F}_{q^2}$ and $\bar{u}_2(\alpha)=u_2(\alpha, \beta) Z(U_2)$. As in (6B) there are field automorphisms τ, τ_1, τ_2, of $\mathbf{F}_{q^2}$ such that $\alpha \bar{u}_2(\beta)=\bar{u}_2(\alpha^\tau \beta)$, $\alpha \bar{u}_2^{s_1}(\beta)=\bar{u}_2^{s_1}(\alpha^\tau \beta)$,

$$\alpha u_1^{s_2}(\beta)=u_1^{s_2}(\alpha^{\tau_1} \beta), \; \alpha \bar{u}_2^{s_1}(\beta)=\bar{u}_2^{s_1}(\alpha^{\tau_2} \beta),$$

and $\alpha u_1(\beta)=u_1(\alpha^{\tau_1} \beta)$ for all $\alpha, \beta \in \mathbf{F}_{q^2}$. The matrix forms of Γ^ρ and Θ^σ then give

$$\begin{aligned} [\bar{u}_2(\alpha), u_1(\gamma)] &\equiv \bar{u}_2^{s_1}(\alpha \gamma^{\rho\tau}) && (\bmod\ [U_{s_1}, U_{s_1}]), \\ [u_1(\gamma), u_2(\alpha, \beta)] &\equiv \bar{u}_2^{s_1}(\gamma^{\tau_1^{-1}\tau_2} \bar{\alpha}^{\sigma\tau_2} x) && \big(\bmod\ U_1^{s_2} Z(U_2^{s_1})\big), \end{aligned}$$

where $^-$ in this case is the field automorphism $y \to y^q$. Thus it must be the case that $\alpha \gamma^{\rho\tau}=-\gamma^{\tau_1^{-1}\tau_2} \bar{\alpha}^{\sigma\tau_2} x$ for all $\alpha, \gamma \in \mathbf{F}_{q^2}$. This implies that

$x=-1$, $\sigma\tau_2=\bar{\ }$, and $\rho\tau=\tau_1^{-1}\tau_2$. We now relabel the elements of R_2, replacing $u_2(\alpha,\beta)$ by $u_2(\alpha^{\tau_2\tau_1^{-1}},\beta^{\tau_2\tau_1^{-1}})$. With this relabeling we have from the matrix forms of $\bar{\Theta}$ and Γ

$$\begin{aligned}
&\text{(i)} && [u_1^{s_2}(\gamma),u_2(\alpha,\beta)]=1,\\
&\text{(ii)} && [\bar{u}_2^{s_1}(-\gamma),u_2(\alpha,\beta)]\equiv u_1^{s_2}(\gamma\bar{\alpha}) && \left(\operatorname{mod} Z(U_2^{s_1})\right),\\
&\text{(iii)} && [u_1(\gamma),u_2(\alpha,\beta)]\equiv \bar{u}_2^{s_1}(-\alpha\gamma)\,u_1^{s_2}(\bar{\beta}\gamma) && \left(\operatorname{mod} Z(U_2^{s_1})\right).
\end{aligned} \tag{8.15}$$

Since $[U_2,U_2^{s_1}]\leqq U_1^{s_2}$ and $[U_2,Z(U_2^{s_1})]=1$, (8.15) (ii) implies that

$$[u_2^{s_1}(\gamma,\delta),u_2(\alpha,\beta)]=u_1^{s_2}(-\bar{\alpha}\gamma). \tag{8.16}$$

Let γ be an element of order q^2-1 in $\mathbf{F}_{q^2}$. Then H_1 is $\langle h_1(\gamma)\rangle$. If $h_1(\gamma)\in H_2(H\cap K_2)$, express $h_1(\gamma)=h_2(\alpha)\cdot b$ where $h_2(\alpha)\in H_2$ and $b\in H\cap K_2$. If $h_1(\gamma)\notin H_2(H\cap K_2)$, then $h_1(\gamma)^3\in H_2(H\cap K_2)$ and we may express $h_1(\gamma)^3=h_2(\alpha)\cdot b$. The eigenvalues of $h_2(\alpha)$ on the $\mathbf{F}_{q^2}$-spaces $\bar{U}_2$, $\bar{U}_2^{s_1}$ are α^{2-q}, α^{1-q} respectively; those of b are 1, β respectively, where $\beta\in\mathbf{F}_{q^2}$; and those of $h_1(\gamma)$ are γ and γ^{-1} respectively. Thus $\alpha^{1-q}\beta=\alpha^{q-2}$ and $\beta=\alpha^{2q-3}$. Since $(q^2-1,\ 2q-3)=(5,q+1)$, it follows $|\langle\alpha\rangle|=|\langle\beta\rangle|$ or $d\,|\,|\langle\beta\rangle|$, where $d=(5,q+1)$. Also, $\alpha^{2-q}=\gamma$ or γ^3. Since $(q^2-1,2-q)$ $=(3,q+1)$, it follows that $|\langle\alpha\rangle|=q^2-1$ or $\frac{1}{e}(q^2-1)$, where $e=(3,q+1)$. Thus $\frac{1}{de}(q^2-1)$ divides $|\langle\beta\rangle|$ and $|H\cap K_2|\equiv 0\ \left(\operatorname{mod}\frac{1}{de}(q^2-1)\right)$. If $\alpha^{2-q}=\gamma$, then $|H\cap K_2|\equiv 0\ \left(\operatorname{mod}\frac{1}{d}(q^2-1)\right)$; if $\alpha^{2-q}=\gamma^3$, then

$$|H:H_2(H\cap K_2)|=3.$$

In either case we have

$$|H|\equiv 0\ \left(\operatorname{mod}\frac{1}{de}(q^2-1)^2\right). \tag{8.17}$$

We repeat this argument interchanging the roles of R_1 and R_2. Let $H_2=\langle h_2(\gamma)\rangle$. If $h_2(\gamma)\in H_1(H\cap K_1)$, express $h_2(\gamma)=h_1(\alpha)\cdot b$ where $h_1(\alpha)\in H_1$, $b\in H\cap K_1$. If $h_2(\gamma)\notin H_1(H\cap K_1)$, then $h_2(\gamma)^2\in H_1(H\cap K_1)$, and we may express $h_2(\gamma)^2=h_1(\alpha)\cdot b$. The eigenvalues of $h_1(\alpha)$ on the $\mathbf{F}_{q^2}$-spaces $\bar{U}_2$, $\bar{U}_2^{s_1}$ are α, α^{-1} respectively; those of b are β, β respectively; and those of $h_2(\gamma)$ are γ^{2-q}, γ^{1-q} respectively. Thus $\alpha\beta=\gamma^{2-q}$ or γ^{4-2q}, and

$$\alpha^{-1}\beta=\gamma^{1-q}\quad\text{ or }\ \gamma^{2-2q}$$

respectively. This implies that $\beta^2=\gamma^{(3-2q)}$ or $\gamma^{2(3-2q)}$, so that $|\langle\beta\rangle|$ is divisible by $\frac{1}{d}(q^2-1)$ or $\frac{1}{2d}(q^2-1)$ respectively. Since the second case occurs only when $|H:H_1(H\cap K_1)|=2$, we have in either case that

$$|H|\equiv 0\begin{cases}\bmod \dfrac{1}{2d}(q^2-1)^2 & \text{if } p\neq 2\\ \bmod \dfrac{1}{d}(q^2-1)^2 & \text{if } p=2\end{cases} \tag{8.18}$$

(8.17) and (8.18) now imply that $|H|\equiv 0\left(\bmod \dfrac{1}{d}(q^2-1)^2\right)$.

Suppose $d=5$ and ζ is an element of order 5 in $\mathbf{F}_{q^2}$. Since

$$h_2(\zeta)\in H_1(H\cap K_1),$$

we may express $h_2(\zeta)=h_1(\alpha)b$ for some $h_1(\alpha)\in H_1$ and some $b\in H\cap K_1$. The calculations in the previous paragraph show that $b^2=1$ since $H\cap K_1$ is faithful on V_1. If $b\neq 1$, then $b=h_1(-1)$. If $b=1$, then $h_2(\zeta)=h_1(\alpha)$. In either case $h_2(\zeta)\in H_1\cap H_2$. Conversely, if $h_1(\alpha)=h_2(\beta)\in H_1\cap H_2$, then $\alpha=\beta^{2-q}$ and $\alpha^{-1}=\beta^{1-q}$ from our previous calculations. Thus $\beta^{2q-3}=1$ and $\beta\in\langle\zeta\rangle$. Hence if $d=5$, then $|H_1\cap H_2|=5$, the common elements being of the form $h_1(\zeta^{2-q})=h_2(\zeta)$.

Suppose $d=5$ and $|H|=(q^2-1)^2$. The group $\hat{H}=H_1H_2$ is then a subgroup of index 5 in H. For $i=1,2$, $[s_i,H]\leqq H\cap R_i=H_i$ so that s_i centralizes H/H_i. In particular $s_1, s_2\in N(\hat{H})$ and $s_i^2\in\hat{H}$. Moreover, $\langle s_1,s_2\rangle\,\hat{H}/\hat{H}$ has order 8, for otherwise $\langle s_1,s_2\rangle\,\hat{H}/\hat{H}\geqq H/\hat{H}$ and s_i inverts $H/\hat{H}$, whereas we have seen that $[s_i,H]\leqq\hat{H}$. Let $\hat{B}=\hat{H}U$, $\hat{N}=\langle\hat{H},s_1,s_2\rangle$, and $\hat{G}=\bigcup_s \hat{B}s\hat{B}$ as $s\in\hat{N}/\hat{H}$. Then as in the proof of (8B) we see that $\hat{G}$ is a normal subgroup of index 5 in G, which is impossible. Thus

$$|H|=\frac{1}{d}(q^2-1)^2 \quad\text{and}\quad H=H_1H_2. \tag{8.19}$$

The following table gives the action of $h_1(\alpha)$ and $h_2(\beta)$ on the root subgroups.

	$\bar{U}_2$	$Z(U_2)$	U_1	$\bar{U}_2^{s_1}$	$Z(U_2^{s_1})$	$U_1^{s_2}$
$h_1(\alpha)$	α	α^{1+q}	α^{-2}	α^{-1}	α^{-1-q}	—
$h_2(\beta)$	β^{2-q}	β^{1+q}	β^{-1}	β^{1-q}	1	β^{q}

(8.20)

The entries other than those giving the action of $h_1(\alpha)$ on $Z(U_2)$ and $Z(U_2^{s_1})$ come from (8.10) or the matrix forms of Γ and Θ. The two exceptions come from the relations $[u_2(\alpha,\beta),u_2(\gamma,\delta)]=u_2(0,\bar{\alpha}\gamma-\alpha\bar{\gamma})$,

$$s_1^{-1}h_1(\alpha)s_1=h_1(\alpha^{-1}),$$

and the fact that $Z(U_2)$ is a vector space over $\mathbf{F}_q$ of dimension 1 on which $h_1(\alpha)$ acts linearly. In particular, $\langle h_1(\alpha)\,h_2(\alpha^{-2})\colon \alpha\in\mathbf{F}_{q^2}\rangle \leqq H\cap K_1$ so that $H\cap K_1$ induces an operator group on $Z(U_2)$ of order $q-1$.

The group R_1 is irreducible on $[U_{s_1}, U_{s_1}]$, $[R_1, H\cap K_1]=1$, and $H\cap K_1$ induces a group of automorphisms of order $q-1$ on $[U_{s_1}, U_{s_1}]$. Thus $[U_{s_1}, U_{s_1}]$ may be regarded as a vector space over $\mathbf{F}_q$ of dimension 4 on which R_1 acts linearly and irreducibly. Since H_1 has 3 constituents on $[U_{s_1}, U_{s_1}]$, it follows by (4C) that R_1 on $[U_{s_1}, U_{s_1}]$ may be assumed to have the form Ξ^τ, where Ξ is an $\mathbf{F}_q$-form of $\Gamma\times\bar{\Gamma}$ constructed with respect to θ, and τ is a field automorphism of $\mathbf{F}_q$. We shall choose θ so that $\bar{\theta}=-\theta$ if $p\neq 2$ and $\bar{\theta}=\theta+1$ if $p=2$. Moreover, let $\theta^\varepsilon=\theta$ if $p\neq 2$, $\theta^\varepsilon=1$ if $p=2$. Then we may take

$$\{u_2(0,\theta^\varepsilon),\ u_1^{s_2}(x),\ u_1^{s_2}(y),\ u_2^{s_1}(0,\theta^\varepsilon)\} \tag{8.21}$$

as a natural basis for Ξ^τ, where $x, y\in\mathbf{F}_{q^2}$. As before there are field automorphisms τ_1, τ_2, τ_3 of $\mathbf{F}_q$ such that

$$\alpha u_2(0,\theta^\varepsilon\beta)=u_2(0,\theta^\varepsilon\beta\alpha^{\tau_1}),\qquad \alpha u_2^{s_1}(0,\theta^\varepsilon\beta)=u_2^{s_1}(0,\theta^\varepsilon\beta\alpha^{\tau_1}),$$

$\alpha u_1^{s_2}(\beta x)=u_1^{s_2}(\alpha^{\tau_2}\beta x)$, and $\alpha u_1^{s_2}(\beta y)=u_1^{s_2}(\alpha^{\tau_3}\beta y)$ for all α, $\beta\in\mathbf{F}_q$. There is an element $h\in H_1^{s_2}$ such that $(u_1^{s_2}(x))^h=u_1^{s_2}(\delta x)\,u_1^{s_2}(\eta y)$ for some δ, η such that $\eta\neq 0$. There is also an element $h_\alpha\in H\cap K_1$ such that $\alpha u_1^{s_2}(\gamma)=(u_1^{s_2}(\gamma))^{h_\alpha}$ for all $\gamma\in\mathbf{F}_{q^2}$. Thus

$$u_1^{s_2}(\alpha^{\tau_2}\delta x)\,u_1^{s_2}(\alpha^{\tau_3}\eta y)=(u_1^{s_2}(x))^{hh_\alpha}=(u_1^{s_2}(x)^{h_\alpha})^h=(u_1^{s_2}(\alpha^{\tau_2}x))^h=u_1^{s_2}(\alpha^{\tau_2}(\delta x+\eta y)).$$

It follows that $\tau_2=\tau_3$. The matrix form of Ξ^τ then gives

$$[u_2(0,\theta^\varepsilon\beta),\ u_1(\gamma)]=u_1^{s_2}(x\beta^{\tau\tau_1^{-1}\tau_2}\gamma_1^{\tau\tau_2}+y\beta^{\tau\tau_1^{-1}\tau_2}\gamma_2^{\tau\tau_2})\,u_2^{s_1}(0,\theta^\varepsilon\beta(\gamma\bar{\gamma})^{\tau\tau_1})$$

where $\gamma=\gamma_1+\gamma_2\theta$ for $\gamma_1, \gamma_2\in\mathbf{F}_q$ and $\beta\in\mathbf{F}_q$. Comparing this with (8.15) (iii) we see that $-\bar{\theta}^\varepsilon\beta\gamma=\beta^{\tau\tau_1^{-1}\tau_2}(x\gamma_1^{\tau\tau_2}+y\gamma_2^{\tau\tau_2})$ holds for all β and γ. Since $-\bar{\theta}^\varepsilon=\theta^\varepsilon$ we easily obtain $x=\theta^\varepsilon$, $y=\theta\theta^\varepsilon$, $\tau_1^{-1}\tau_2=1$, and $\tau\tau_2=1$. The missing entry in (8.20) is thus α^{q-1}. The matrix form of Ξ^τ now gives

$$\begin{aligned}
&\text{(i)} && [u_2^{s_1}(0,\theta^\varepsilon\beta),\ u_1(\gamma)]=1,\\
&\text{(ii)} && [u_1^{s_2}(\theta\theta^\varepsilon\beta),\ u_1(\gamma)]=u_2^{s_1}(0,\theta^\varepsilon\beta(\bar{\gamma}\theta+\bar{\theta}\gamma)),\\
&\text{(iii)} && [u_1^{s_2}(\theta^\varepsilon\beta),\ u_1(\gamma)]=u_2^{s_1}(0,\theta^\varepsilon\beta(\gamma+\bar{\gamma})),\\
&\text{(iv)} && [u_2(0,\theta^\varepsilon\beta),\ u_1(\gamma)]=u_1^{s_2}(\theta^\varepsilon\beta\gamma_1+\theta\theta^\varepsilon\beta\gamma_2)\,u_2^{s_1}(0,\theta^\varepsilon\beta\gamma\bar{\gamma})
\end{aligned} \tag{8.22}$$

where $\gamma=\gamma_1+\gamma_2\theta$ for $\gamma_1, \gamma_2\in\mathbf{F}_q$ and $\beta\in\mathbf{F}_q$. Let $\delta\in\mathbf{F}_{q^2}$ and express $\delta=\delta_1\theta\theta^\varepsilon+\delta_2\theta^\varepsilon$, where $\delta_1, \delta_2\in\mathbf{F}_q$. The commutator identity on $[xy, z]$

implies that

$$[u_1^{s_2}(\delta), u_1(\gamma)] = u_2^{s_1}(0, \theta^\varepsilon \delta_1(\bar{\gamma}\theta + \bar{\theta}\gamma) + \theta^\varepsilon \delta_2(\gamma + \bar{\gamma})) \\ = u_2^{s_1}(0, \bar{\gamma}\delta - \gamma\bar{\delta}). \tag{8.23}$$

We can now complete the multiplication table of U. By (8.15) there exists a function $f: \mathbf{F}_{q^2} \times \mathbf{F}_{q^2} \times \mathbf{F}_{q^2} \to \mathbf{F}_{q^2}$ such that

$$[u_1(\gamma), u_2(\alpha, \beta)] = u_2^{s_1}(-\alpha\gamma, f(\alpha, \beta, \gamma))\, u_1^{s_2}(\bar{\beta}\gamma). \tag{8.24}$$

The commutator identities on $[xy, z]$ and $[x, yz]$ give the following functional equations:

$$\begin{aligned} &\text{(i)} \quad f(\alpha, \beta, \gamma+\delta) = f(\alpha, \beta, \gamma) + f(\alpha, \beta, \delta) + \bar{\beta}(\bar{\gamma}\delta + \gamma\bar{\delta}), \\ &\text{(ii)} \quad f(\alpha+\alpha', \beta+\beta'+\bar{\alpha}\alpha', \gamma) = f(\alpha, \beta, \gamma) + f(\alpha', \beta', \gamma) + \bar{\alpha}'\alpha\gamma\bar{\gamma}. \end{aligned} \tag{8.25}$$

We note that in these relations $\alpha\bar{\alpha} = \beta + \bar{\beta}$, $\alpha'\bar{\alpha}' = \beta' + \bar{\beta}'$. Transforming (8.24) by $h_1(\lambda)\, h_2(\mu)$ and using (8.20), we obtain the relation

$$f(\lambda\mu^{2-q}\alpha, \lambda^{1+q}\mu^{1+q}\beta, \lambda^{-2}\mu^{-1}\gamma) = \lambda^{-1-q} f(\alpha, \beta, \gamma) \quad \text{for } \lambda, \mu \in \mathbf{F}_{q^2}. \tag{8.26}$$

(Note the action of an element $h \in H$ on U_2 is determined by its action on $\bar{U}_2$ since this calculation can be made in P_2/K_2.) In particular, setting $\lambda = 1$, $\mu = \gamma$ gives the special case

$$f(\alpha, \beta, \gamma) = f(\gamma^{2-q}\alpha, \gamma^{1+q}\beta, 1). \tag{8.27}$$

Also, setting $\lambda = \mu^{q-2}$ in (8.26) gives $f(\alpha, \beta, \mu^{3-2q}\gamma) = \mu^{1+q} f(\alpha, \beta, \gamma)$. Since $(3-2q, q^2-1) = d$ and $(\mu^{3-2q})^{1+q} = \mu^{1+q}$, we obtain

$$f(\alpha, \beta, \mu\gamma) = \mu^{1+q} f(\alpha, \beta, \gamma) \qquad \text{for } \mu \in L \tag{8.28}$$

where L is the cyclic subgroup of index d in the multiplicative subgroup of $\mathbf{F}_{q^2}$. The following is a special case of (8.25) (ii).

$$f(\alpha, \beta, \gamma) = \begin{cases} f(\alpha, \beta_1, \gamma) + f(0, \theta\beta_2, \gamma) & \text{if } p \neq 2 \\ f(0, \beta_1, \gamma) + f(\alpha, \theta\beta_2, \gamma) & \text{if } p = 2 \end{cases} \tag{8.29}$$

where $\beta = \beta_1 + \theta\beta_2$ for $\beta_1, \beta_2 \in \mathbf{F}_q$. But (8.22) (iv) implies

$$f(0, \beta, \gamma) = \bar{\beta}\gamma\bar{\gamma}. \tag{8.30}$$

Thus it suffices by (8.27), (8.29), (8.30) to determine $f(\alpha, \beta, 1)$ if $p \neq 2$, and $f(\alpha, \theta\beta, 1)$ if $p = 2$, for $\beta \in \mathbf{F}_q$. If $p \neq 2$, then (8.25) (i) implies that $f(\alpha, \beta, 2) = 2f(\alpha, \beta, 1) + 2\beta$. Since $2 \in \mathbf{F}_q^\# \subseteq L$, it follows by (8.28) that $f(\alpha, \beta, 1) = \beta$. If $p = 2$, we may assume $\theta \in L$ since $|H| > (q^2-1)(q-1)$. Again (8.25) (i) and (8.28) imply that $f(\alpha, \theta\beta, \theta + \bar{\theta}) = \bar{\theta}\bar{\beta}(\bar{\theta}^2 + \theta^2)$. Since $\theta + \bar{\theta} \in \mathbf{F}_q^\# \subseteq L$, it follows by (8.28) that $f(\alpha, \theta\beta, 1) = \bar{\theta}\bar{\beta}$. Thus the function f is completely determined, and indeed

$$f(\alpha, \beta, \gamma) = \bar{\beta}\gamma\bar{\gamma}. \tag{8.31}$$

In particular, the multiplication tables of U and hence of $B=HU$ are determined.

Suppose $p=2$ and $|H|=(q^2-1)(q-1)$. V_1 and V_2 are then vector spaces over $\mathbf{F}_q$ of dimensions 4 and 6 on which R_1 and R_2 act linearly and irreducibly. By (4C) and (4E) we may assume that R_1 on V_1 has the form $\Gamma^{[2]\rho}$, and R_2 on V_2 has the form $\Theta^{[2]\sigma}$ for suitable field automorphisms ρ and σ of $\mathbf{F}_q$. As natural bases for $\Gamma^{[2]\rho}$ and $\Theta^{[2]\sigma}$ we may take respectively

$$\begin{gathered}\{\bar{u}_2(1), \bar{u}_2(w), \bar{u}_2^{s_1}(1), \bar{u}_2^{s_1}(w)\}\\ \{u_1^{s_2}(1), u_1^{s_2}(x), \bar{u}_2^{s_1}(y), \bar{u}_2^{s_1}(z), u_1(1), u_1(x)\}\end{gathered} \tag{8.32}$$

where $w, x, y, z\in\mathbf{F}_{q^2}$. Thus there are field automorphisms τ, τ_1, τ_2 of $\mathbf{F}_q$ such that $\alpha\bar{u}_2(\beta)=\bar{u}_2(\alpha^\tau\beta)$, $\alpha\bar{u}_2(\beta w)=\bar{u}_2(\alpha^\tau\beta w)$, $\alpha\bar{u}_2^{s_1}(\beta)=\bar{u}_2^{s_1}(\alpha^\tau\beta)$, $\alpha\bar{u}_2^{s_1}(\beta w)=\bar{u}_2^{s_1}(\alpha^\tau\beta w)$, $\alpha u_1^{s_2}(\beta)=\alpha u_1^{s_2}(\alpha^{\tau_1}\beta)$, $\alpha u_1^{s_2}(\beta x)=u_1^{s_2}(\alpha^{\tau_1}\beta x)$,

$$\alpha\bar{u}_2^{s_1}(\beta y)=\bar{u}_2^{s_1}(\alpha^{\tau_2}\beta y),\ \alpha\bar{u}_2^{s_1}(\beta z)=\bar{u}_2^{s_1}(\alpha^{\tau_2}\beta z),\ \alpha u_1(\beta)=u_1(\alpha^{\tau_1}\beta),$$

$\alpha u_1(\beta x)=u_1(\alpha^{\tau_1}\beta x)$ for all $\alpha, \beta\in\mathbf{F}_q$. The matrix forms of $\Gamma^{[2]\rho}$ and $\Theta^{[2]\sigma}$ then give

$$\begin{aligned}
&\text{(i)}\quad [\bar{u}_2(\alpha_1), u_1(\gamma)]\equiv\bar{u}_2^{s_1}(\alpha_1\gamma_1^{\rho\tau}+\alpha_1 w\gamma_2^{\rho\tau}),\\
&\text{(ii)}\quad [\bar{u}_2(w\alpha_2), u_1(\gamma)]\equiv\bar{u}_2^{s_1}(\alpha_2\gamma_2^{\rho\tau}\mu^{\rho\tau}+\alpha_2 w(\gamma_1+\gamma_2\nu)^{\rho\tau}),\\
&\text{(iii)}\quad [u_1(\gamma_1), u_2(\alpha,\beta)]\equiv\bar{u}_2^{s_1}(y\gamma_1^{\tau_1^{-1}\tau_2}(\alpha_1+\alpha_2)^{\sigma\tau_2}+z\gamma_1^{\tau_1^{-1}\tau_2}\alpha_2^{\sigma\tau_2}),\\
&\text{(iv)}\quad [u_1(x\gamma_2), u_2(\alpha,\beta)]\\
&\qquad\equiv\bar{u}_2^{s_1}(y\gamma_2^{\tau_1^{-1}\tau_2}\alpha_2^{\sigma\tau_2}\mu^{\sigma\tau_2}+z\gamma_2^{\tau_1^{-1}\tau_2}(\alpha_1+\alpha_2+\alpha_2\nu)^{\sigma\tau_2})
\end{aligned} \tag{8.33}$$

where the congruences in (i) and (ii) are taken modulo $[U_{s_1}, U_{s_1}]$ and the congruences in (iii) and (iv) are taken modulo $U_1^{s_2}Z(U_2)^{s_1}$. Also

$$\gamma=\gamma_1+\gamma_2\theta, \qquad \alpha=\alpha_1+\alpha_2\theta$$

for $\gamma_1, \gamma_2, \alpha_1, \alpha_2\in\mathbf{F}_q$, and $\theta^2=\mu+\nu\theta$. Setting $\alpha=\alpha_1$, $\gamma=\gamma_1$, and comparing (8.33)(i) and (iii) we find that $y=1$, $\rho\tau=\tau_1^{-1}\tau_2$, $\sigma\tau_2=1$. Now we extend the automorphism $\tau_2\tau_1^{-1}$ of $\mathbf{F}_q$ to an automorphism η of $\mathbf{F}_{q^2}$ and relabel the elements of R_2 using the automorphism η. Thus we replace $u_2(\alpha,\beta)$ by $u_2(\alpha^\eta,\beta^\eta)$. This relabeling has the effect of equating τ_1 and τ_2. The equations in (8.33) remain valid and are simplified by the additional conditions that $\rho\tau=\tau_1^{-1}\tau_2=\sigma\tau_2=1$.

Let T be the unique subgroup of order $q+1$ in H. Since

$$|H|=(q^2-1)(q-1)$$

and $|H_i|=q^2-1$ for $i=1,2$, it follows that $T\leqq H_1\cap H_2$. Let $T=\langle h\rangle$ and express $h=h_1(\alpha)=h_2(\beta)$. Consider h acting on the $\mathbf{F}_q$-spaces $\bar{U}_2$ and U_1. The eigenvalues of $h_1(\alpha)$ are $\alpha, \bar{\alpha}$ on $\bar{U}_2$ and $\alpha^{-2}, \bar{\alpha}^{-2}$ on U_1; the eigenvalues of $h_2(\beta)$ are $\beta^{2-q}, \beta^{2q-1}$ on $\bar{U}_2$ and $\beta^{-1}, \bar{\beta}^{-1}$ on U_1. Thus $\beta^{-1}=\beta^{-2(2-q)}$

or $\beta^{-2(2q-1)}$, and so $\beta^{2q-3}=1$ or $\beta^{3-4q}=1$. β has order $q+1$. Since $(q+1, 3-4q)=1$ or 7, $(q+1, 2q-3)=1$ or 5, and q is a power of 2, we conclude that $q=4$.

Now $\theta^4=\theta+1$. It is easily checked that $\theta^2=\theta+\omega$, where ω is a primitive cube root of unity in $\mathbf{F}_4$. Also $h_1(\theta)$ acts linearly on the $\mathbf{F}_{16}$-spaces $\bar{U}_2$ and $\bar{U}_2^{s_1}$. From the action of R_1 on V_1, we see that the eigenvalue of $h_1(\theta)$ on $\bar{U}_2$ is θ or θ^4, the eigenvalue of $h_1(\theta)$ on $\bar{U}_2^{s_1}$ is θ^{-1} or θ^{-4}. Transforming (8.33)(i) by $h_1(\theta)$ shows that θ and θ^{-1} are the correct values. Since $h_1(\theta)$: $\bar{u}_2(1)\to\bar{u}_2(w)$, we conclude that $w=\theta$. Setting $\gamma=\gamma_1$ in (8.33)(i), (ii), (iii), we find that $z=\bar{\theta}$. Now $h_2(\theta^{-4})$ acts linearly on the $\mathbf{F}_{16}$-spaces U_1 and $\bar{U}_2^{s_1}$. From the action of R_2 on V_2, we obtain the eigenvalues θ^4 or θ on U_1, and θ^3 or θ^{-3} on $\bar{U}_2^{s_1}$. As above the correct values can be seen to be θ^4 and θ^{-3}. Since $h_2(\theta^{-4})$: $u_1(1)\to u_1(x)$, we find that $x=\bar{\theta}$. The bases in (8.32) are now completely determined, and (8.15) and (8.16) follow. Moreover, the arguments giving (8.20), (8.21), (8.22) and (8.23) carry over.

To complete the multiplication table of U, we define the function $f(\alpha,\beta,\gamma)$ as in (8.24). The relations (8.25), (8.26), (8.27), (8.28), (8.29), and (8.30) remain valid and it remains to determine $f(\alpha,\theta\beta,1)$ for $\beta\in\mathbf{F}_4$. In (8.26) choose λ so that $\lambda^5=\beta$, and set $\mu=\lambda^{-2}$. This gives

$$f(\alpha,\theta\beta,1)=\beta f(\beta\alpha,\theta,1),$$

and so it suffices to compute $f(\rho,\theta,1)$, where $\rho^5=\theta^4+\theta=1$. Now $(\omega\theta)$ is a primitive 5th root of unity in $\mathbf{F}_{16}$. Let $x_i=f((\omega\theta)^i,\theta,1)$ for $0\leqq i\leqq 4$. Setting α,α' in (8.25)(ii) equal to $1,\omega\theta$; $1,(\omega\theta)^2$; and $1,(\omega\theta)^3$ respectively, we obtain the equations

$$\begin{aligned} x_0+x_1+\quad\; +\omega x_3 \quad &=(\omega\theta)^4\\ x_0+\omega^2 x_1+x_2 \quad &=(\omega\theta)^3+1 \qquad (8.34)\\ x_0 \quad +x_3+\omega^2 x_4 &=(\omega\theta)^2+\omega. \end{aligned}$$

Setting $\lambda=(\omega\theta)^2$, $\mu=(\omega\theta)^3$ in (8.26) we have $x_0=f(\omega\theta,\theta,\theta\omega^2+\omega)$ which by (8.25)(i) and (8.27) becomes $x_0+\omega^2x_1+\omega^2x_4=\theta^4$. Next setting $\lambda=\mu=\omega\theta$ in (8.26) we obtain

$$x_0=f\big((\omega\theta)^4,\theta,(\omega\theta)^2\big)=f\big((\omega\theta)^4,\theta,\omega^2\theta+1\big),$$

which again by (8.25)(i) and (8.27) becomes $x_0+\omega^2x_2+x_4=\omega^2\theta^4$. We now have the system of equations

$$\begin{aligned} x_0+x_1 \quad +\omega x_3 \quad &=(\omega\theta)^4\\ x_0+\omega^2 x_1+x_2 \quad &=(\omega\theta)^3+1\\ x_0+ \quad x_3+\omega^2 x_4 &=(\omega\theta)^2+\omega \qquad (8.35)\\ x_0+\omega^2 x_1 \quad +\omega^2 x_4 &=\theta^4\\ x_0+ \quad \omega^2 x_2 \quad + \quad x_4 &=\omega^2\theta^4. \end{aligned}$$

The determinant of this system is non-zero, so there exists a unique set of solutions. Since $x_0 = x_1 = x_2 = x_3 = x_4 = \overline{\theta + 1}$ is a set of solutions for (8.35), we conclude that $f(\alpha, \theta\beta, 1) = \overline{\theta\beta}$. Thus (8.31) holds and the multiplication tables of U and B are determined.

We now complete the proof of (8C). No restrictions are assumed on p or $|H|$. We recall that $s_1^2 = h_1(-1)$ and $s_2^2 = 1$. Let $(h_2(\beta))^{s_1} = h_1(x)\,h_2(y)$ for some $x, y \in \mathbf{F}_{q^2}$. Using (8.20) to evaluate both sides of this expression on $\overline{U}_2$ and $\overline{U_2^{s_1}}$, we find that $\beta^{1-q} = x y^{2-q}$ and $\beta^{2-q} = x^{-1} y^{1-q}$. Thus $\beta^{3-2q} = y^{3-2q}$. If $d = 1$, then x and y are determined from β, and indeed

$$h_2(\beta)^{s_1} = h_1(\beta^{-1})\,h_2(\beta). \tag{8.36}$$

If $d = 5$ and ζ is a primitive 5th root of unity in $\mathbf{F}_{q^2}$, then $y = \beta\zeta^i$,

$$x = \beta^{-1}\,\zeta^{i(q-2)}$$

for some integer i. Since $h_1(\zeta^{q-2})\,h_2(\zeta) = 1$, we see that (8.35) still holds. Now let $h_1(\alpha)^{s_2} = h_1(x)\,h_2(y)$ for some $x, y \in \mathbf{F}_{q^2}$. Using (8.20) on U_1 and $U_1^{s_2}$, we find that $\alpha^{q-1} = x^{-2} y^{-1}$ and $\alpha^{-2} = x^{q-1} y^q$, so that $\alpha^{q-3} = x^{q-3} y^{q-1}$. The action of R_2 on V_2 shows that s_2 inverts $\overline{U_2^{s_1}}$, and s_2 of course centralizes $Z(U_2^{s_1})$. Using (8.20) on $\overline{U_2^{s_1}}$ and $Z(U_2^{s_1})$, we obtain the additional relations $\alpha^{-1} = x^{-1} y^{1-q}$ and $\alpha^{1+q} = x^{1+q}$. These all imply that $\alpha^{2q-3} = x^{2q-3}$. As above, we obtain

$$h_1(\alpha)^{s_2} = h_1(\alpha)\,h_2(\alpha^{-1-q}). \tag{8.37}$$

We now determine the action of s_2 on $U_2^{s_1}$. We know that s_2 inverts $\overline{U_2^{s_1}}$ and centralizes $Z(U_2^{s_1})$. Now $\tilde{R} = \langle Z(U_2), Z(U_2)^{s_2}\rangle$ centralizes $U_2^{s_1}$ by (8.11). If $p = 2$, then $s_2 \in \tilde{R}_2$ and so s_2 centralizes $U_2^{s_1}$. Thus we have $u_2^{s_1 s_2}(\alpha, \beta) = u_2^{s_1}(-\alpha, \beta)$. By (4.4) $h_2(\theta^{-1}) s_2 \in \tilde{R}_2$ so s_2 and $h_2(\theta)$ then have the same action on $U_2^{s_1}$. By (8.20) we again have $u_2^{s_1 s_2}(\alpha, \beta) = u_2^{s_1}(-\alpha, \beta)$. Also s_2 normalizes $U_2^{s_1}$, $U_2^{s_2 s_1}$, and $R_2^{s_1} \cong SU(3, q)$. Since the automorphisms of $SU(3, q)$ with these properties are readily computed, we find that $u_2(\alpha, \beta)^{s_2 s_1 s_2} = u_2(-\alpha, \beta)^{s_2 s_1}$. Since R_1 acts on $[U_{s_1}, U_{s_1}]$ as Ξ, it follows that $u_1(\gamma)^{s_2 s_1} = u_1(\gamma^q)^{s_2}$. Now s_1 normalizes $U_1^{s_2}$, $U_1^{s_1 s_2}$, and $R_1^{s_2} \cong SL(2, q^2)$. Arguing as above, we find that $u_1(\gamma)^{s_1 s_2 s_1} = u_1(\gamma^q)^{s_1 s_2}$. It now follows that $C((s_1 s_2)^4) \geqq \langle U_1, U_2\rangle$, and using (8.20) we conclude that $(s_1 s_2)^4 = 1$. We have now computed the action of s_1 and s_2 on the subgroups $U_1, U_2, U_1^{s_2}$, $U_2^{s_1}, U_1^{s_1}, U_2^{s_2}, U_1^{s_1 s_2}, U_2^{s_2 s_1}$. It now follows from the (B, N)-properties that G has a unique multiplication table. Since $PSU(5, q)$ satisfies (6A)(c), we have $G \cong PSU(5, q)$.

§ 9. Identification of G in the Case $|W| = 12$

(9A) *Suppose* $|W| = 12$ *and* (a) *of* (6E) *holds. Then* $G \cong G_2(q)$.

Proof. Let $T_1 = U_1^{s_2 s_1} U_2^{s_1 s_2} U_1^{s_2}$ and $T_2 = U_2^{s_1 s_2} U_1^{s_2 s_1} U_2^{s_1}$. By (6E) we have the following:

$$\begin{aligned} &T_1 \geqq [U_{s_1}, U_{s_1}] \neq 1, \quad T_2 \geqq [U_{s_2}, U_{s_2}] \neq 1 \\ &L_1 \cong PSL(2, q), \qquad L_2 \cong PSL(2, q). \end{aligned} \tag{9.1}$$

In particular, $|H|$ divides $(q-1)^2$, $H^{q-1}=1$, and

$$|G| = |H|\, q^6 (q^4 + q^2 + 1)(q+1)^2.$$

Suppose $[U_1^{s_2 s_1}, U_1^{s_2}] = U_2^{s_1 s_2}$ and $[U_2^{s_1 s_2}, U_2^{s_1}] = U_1^{s_2 s_1}$. Then $U_1^{s_2 s_1} \leqq [U_{s_1}, U_{s_1}]$, and since R_1 normalizes $[U_{s_1}, U_{s_1}]$, it follows that $[U_1^{s_2 s_1}, U_1] = [U_1^{s_2 s_1}, U_1^{s_2}]^{s_2} = U_2^{s_1} \leqq [U_{s_1}, U_{s_1}] \leqq T_1$ by (9.1), which is a contradiction. Choosing appropriate notation, we may assume

$$[U_2^{s_1 s_2}, U_2^{s_1}] = 1. \tag{9.2}$$

Suppose $[U_1^{s_2 s_1}, U_1^{s_2}] = 1$. If $[U_2^{s_1}, U_1^{s_2}] \neq 1$, then $[U_2, U_1^{s_2 s_1}] \neq 1$ and $[U_2^{s_1 s_2}, U_1] \neq 1$. Also $Z(U_{s_1}) = U_2^{s_1 s_2}$. Thus $U_2^{s_1 s_2}$ admits R_1, and indeed, $[U_2^{s_1 s_2}, R_1] = 1$ so in particular $[U_2^{s_1 s_2}, U_1] = 1$, which is impossible. Thus $[U_2^{s_1}, U_1^{s_2}] = [U_2^{s_1 s_2}, U_1] = [U_2, U_1^{s_2 s_1}] = 1$ and $T_1 \leqq Z(U_{s_1})$. Since $Z(U_{s_1}) \neq U_{s_1}$ by (9.1), it follows that $T_1 = Z(U_{s_1})$. Thus R_1 normalizes T_1. By (3G) $[U_1, U_1^{s_2}] \leqq T_1 \cap U_2^{s_1} U_1^{s_2 s_1} U_2^{s_1 s_2} = U_1^{s_2 s_1} U_2^{s_1 s_2}$. Transforming this inclusion by s_2 gives $[U_1, U_1^{s_2}] \leqq U_1^{s_2 s_1} U_2^{s_1}$. Thus $[U_1, U_1^{s_2}] \leqq U_1^{s_2 s_1}$. Since R_1 is irreducibly represented on $T_1 / U_2^{s_1 s_2}$, it must be the case that $[U_1, U_1^{s_2}] = U_1^{s_2 s_1}$. Interchanging the roles of the subscripts we have $T_2 = Z(U_{s_2})$ and $[U_2, U_2^{s_1}] = U_2^{s_1 s_2}$. In particular, $[u_2^{s_1}(1), u_2(1)] = u_2^{s_1 s_2}(\delta)$, where $\delta \neq 0$. Now R_1 acts irreducibly on the $\mathbf{F}_p$-space U_{s_1}/T_1, and $[U_2^{s_1}, U_1] = [U_2, U_1^{s_1}] = 1$. Since H has two constituents on U_{s_1}/T_1 which are conjugate under s_1, this representation is of type (*a*) in the notation of (4.7). Suppose $p=2$. Thus $u_2(1)^{u_1(1)} = u_2(1)\, u_2^{s_1}(1)\, u_1^{s_2 s_1}(x)\, u_2^{s_1 s_2}(y)\, u_1^{s_2}(z)$ for suitable $x, y, z \in \mathbf{F}_q$. Using the identity $s_1 = u_1(1)\, u_1^{s_1}(1)\, u_1(1)$, we calculate that

$$u_2^{s_1}(1) \equiv u_2^{s_1}(1)\, u_2^{s_1 s_2}(\delta) \pmod{U_1^{s_2 s_1} U_1^{s_2}}, \tag{9.3}$$

so that $\delta = 0$, which is impossible.

Continuing with the supposition $[U_1^{s_2 s_1}, U_1^{s_2}] = 1$ we may then suppose $p \neq 2$. Since R_i acts irreducibly on U_{s_i}/T_i and H has two constituents on U_{s_i}/T_i which are conjugate under s_i, R_i on U_{s_i}/T_i is of type (*a*) in the notation of (4.7). Thus $R_1 \cong R_2 \cong SL(2, q)$. Now $[U_1, U_2^{s_1 s_2}] = 1$ and (1F) imply that $[R_1, R_2^{s_1 s_2}] = 1$. Let $\langle j \rangle = Z(R_1)$. If $R_1 \cap R_2^{s_1 s_2} = 1$, or if $R_1 \cap R_2^{s_1 s_2} = \langle j \rangle$ and $H \leqq R_1 R_2^{s_1 s_2}$, then G has a Sylow 2-subgroup which is a direct product or a central product of two generalized quaternion groups. By [11] this implies that $j \in Z^*(G)$ and so

$$[R_2, j] \leqq R_2 \cap Z^*(G) \leqq R_2 \cap \langle j \rangle.$$

Thus $C(j) \geqq \langle R_1, R_2, H\rangle = G$, which is impossible. We may then suppose $|H : H \cap R_1 R_2^{s_1 s_2}| = 2$. As in the proof of (8B), we can show that $C(j) = HR_1 R_2^{s_1 s_2}$ by computing $|C(j) \cap BsB|$ for all $s \in N \pmod{H}$. Thus $C(j)$ satisfies the condition (*) of [9] so that $G \cong G_2(q)$. We may henceforth assume that

$$[U_1^{s_2 s_1}, U_1^{s_2}] = U_2^{s_1 s_2}, \qquad [U_1^{s_2 s_1}, U_1] = U_2^{s_1}, \tag{9.4}$$

the second relation coming from the first one by transformation by s_2.

The group U_1 normalizes $[U_{s_1}, U_{s_1}]$, so $U_1^{s_2 s_1} \nleqq [U_{s_1}, U_{s_1}]$ by (9.4). Now $[U_1^{s_2}, U_2^{s_1}] \leqq U_1^{s_2 s_1} U_2^{s_1 s_2}$ by (3G) and $U_2^{s_1 s_2} \leqq [U_{s_1}, U_{s_1}]$. Thus

$$[U_1^{s_2}, U_2^{s_1}] \leqq U_2^{s_1 s_2}.$$

Transforming this relation by s_2 gives $[U_1, U_2^{s_1 s_2}] \leqq U_2^{s_1}$. Since

$$U_2^{s_1} \cap [U_{s_1}, U_{s_1}] = 1,$$

it follows that $[U_1, U_2^{s_1 s_2}] = 1$. This leads to the relations

$$[U_1, U_2^{s_1 s_2}] = [U_1^{s_2}, U_2^{s_1}] = [U_1^{s_2 s_1}, U_2] = 1. \tag{9.5}$$

It now follows from (9.5) and (1F) that

$$[R_1, R_2^{s_1 s_2}] = [R_1^{s_2}, R_2^{s_1}] = [R_1^{s_2 s_1}, R_2] = 1. \tag{9.6}$$

Now $U_2^{s_1 s_2}$ and $U_2^{s_1}$ are in $[U_{s_2}, U_{s_2}]$ by (9.4). Since R_2 normalizes $[U_{s_2}, U_{s_2}]$, it follows by (9.5) that R_2 normalizes T_2. The representation of R_2 on U_{s_2}/T_2 is of type (*a*) in the notation of (4.7) since H has two constituents on U_{s_2}/T_2 which are conjugate under s_2. Thus

$$[T_2, R_2] \leqq T_2, \qquad [U_1, U_2] \nleqq T_2, \qquad R_2 \cong SL(2, q). \tag{9.7}$$

The representation of R_2 on $T_2/U_1^{s_2 s_1}$ is likewise of type (*a*). Since

$$[U_2^{s_1}, U_2] \leqq U_1^{s_2 s_1} U_2^{s_1 s_2},$$

it follows by transforming this relation by s_1 that

$$[U_2^{s_1}, U_2] = U_2^{s_1 s_2}. \tag{9.8}$$

Thus $[U_{s_1}, U_{s_1}] = Z(U_{s_1}) = U_2^{s_1 s_2}$ by (9.2), (9.5), (9.8).

The group R_2 acts irreducibly on $V_2 = U_{s_2}/T_2$. By (9.6) $H \cap K_2 \geqq H_1^{s_2 s_1}$ so that $|H \cap K_2|$ is divisible by $\frac{1}{d}(q-1)$ where $d = (2, q-1)$. Since $[R_2, H \cap K_2] = 1$ and $H \cap K_2$ is faithful on V_2, the commuting algebra E_2 of R_2 on V_2, which is a finite field, contains the multiplicative subgroup $H \cap K_2$. Thus $\mathbf{F}_q \subseteq E_2$ and V_2 can be regarded as a vector space of dimension 2 over $\mathbf{F}_q$ on which R_2 acts linearly and irreducibly. R_2 on V_2 then has the form Γ^{ρ} for some field automorphism ρ of $\mathbf{F}_q$, and

$$\{u_1(1), u_1^{s_2}(1)\} \tag{9.9}$$

may be taken as a natural basis. There is then an automorphism τ_0 of $\mathbf{F}_q$ such that $\alpha u_1(\beta)=u_1(\alpha^{\tau_0}\beta)$ and $\alpha u_1^{s_2}(\beta)=u_1^{s_2}(\alpha^{\tau_0}\beta)$ for all $\alpha, \beta\in\mathbf{F}_q$.

The group R_1 acts nontrivially on $V_1=U_{s_1}/U_2^{s_1 s_2}$. Suppose $q>2$ and V_0 is a proper minimal HR_1-submodule of V_1. Since H has exactly 4 constituents on V_1 of order q, it follows that $|V_0|=q, q^2$, or q^3. If $[V_0, R_1]=1$, then $|V_0|=q$. Moreover, $V_0\leqq T_1$. Otherwise there exists an element $v\in V_0$ of the form $v=x x^{s_1} t$, where $1\neq x\in U_2$ and $t\in T_1$, and then $[U_1, x]\leqq T_2$, contrary to (9.7). The form of Γ^{ρ} and (9.9) show that H_2 is represented faithfully and fixed-point freely on $U_1^{s_2}$ and on $U_2^{s_1}$. Since $[U_1^{s_2 s_1}, H_2]=1$ and $V_0^H=V_0$, it follows that $U_1^{s_2} U_2^{s_1 s_2}/U_2^{s_1 s_2}\leqq V_0$. But then $V_0^{s_1}=V_0$ implies that $V_0=T_1/U_2^{s_1 s_2}$, which is impossible. Thus $[V_0, R_1]=V_0$ and $|V_0|=q^2$ or q^3. If $V_0\cap T_1/U_2^{s_1 s_2}\neq 1$, the argument just used shows that $T_1/U_2^{s_1 s_2}\leqq V_0$. But (9.4) and $V_0^{s_1}=V_0$ would then imply that $V_0=V_1$, which is impossible. Thus $V_0\cap T_1/U_2^{s_1 s_2}=1$ and $V_0(T_1/U_2^{s_1 s_2})=V_1$. There is an element $v\in V_0$ of the form $x y z$ (mod $U_2^{s_1 s_2}$), where $x\in U_2^{s_1}$, $y\in U_1^{s_2 s_1}$, $z\in U_1^{s_2}$, and $x\neq 1$. If $z=1$, then (9.4) implies that $U_2^{s_1}$ and U_2 are in V_0 (mod $U_2^{s_1 s_2}$). If $V_0>\langle U_2^{s_1}, U_2\rangle/U_2^{s_1 s_2}$, then $V_0\cap T_1/U_2^{s_1 s_2}\neq 1$, which is impossible. If $V_0=\langle U_2^{s_1}, U_2\rangle/U_2^{s_1 s_2}$, then R_1 normalizes $\langle U_2^{s_1}, U_2\rangle$. Since $C(\langle U_2^{s_1}, U_2\rangle)\cap U_{s_1}=T_1$, R_1 then normalizes T_1, which is impossible by (9.4). Thus $z\neq 1$. Replacing v by $v^{-1} v^h$ for a suitable h in H_2, we may assume v has the form xz (mod $U_2^{s_1 s_2}$). If there exists an h in $H_2^{s_1}$ such that $x^h\neq x$, then $1\neq v^{-1} v^h=x^{-1} x^h$ is in V_0 (mod $U_2^{s_1 s_2}$), and as before, this is impossible. Thus HR_1 is irreducible on V_1 if $q>3$. The same conclusion holds in case $q=3$ if there exists an h in H inducing an outer diagonal automorphism of $R_2^{s_1 s_2}$ and centralizing R_1.

Suppose $p\neq 2$ and $R_1\cong PSL(2, q)$. Then $\langle R_1, R_2^{s_1 s_2}\rangle=R_1\times R_2^{s_1 s_2}$. If $H\leqq R_1 R_2^{s_1 s_2}$, then a Sylow 2-subgroup of G is the direct product of a dihedral group and a generalized quaternion group. Let $\langle j\rangle=Z(R_2)^{s_1 s_2}$. Then $\langle j\rangle\leqq Z^*(G)$ by [11], and so $[R_2, j]\leqq R_2\cap O(G)=1$. Thus

$$C(j)\geqq\langle R_1, R_2, H\rangle=G,$$

which is impossible. Thus $|H:H\cap R_1 R_2^{s_1 s_2}|=2$ and $|H|=(q-1)^2$. Since $(H\cap K_1)\cap(H\cap K_2)=1$ and $P_i/K_i\cong PSL(2, q)$ or $PGL(2, q)$, it follows that $|H\cap K_1|=|H\cap K_2|=q-1$. Moreover $H\cap K_i$ is cyclic and

$$H=(H\cap K_1)\times(H\cap K_2).$$

In particular there exists an $h\in H$ such that h induces an outer diagonal automorphism on $R_2^{s_1 s_2}$. If h induces an inner automorphism of R_1, we may replace h by $h h_1$ for suitable $h_1\in H_1$ and assume $[R_1, h]=1$. By order considerations $H=\langle H_1, H_2^{s_1 s_2}, h\rangle$. But then no element of H induces an outer diagonal automorphism on R_1. This is a contradiction. Thus h induces outer diagonal automorphisms on R_1 and on $R_2^{s_1 s_2}$.

We may choose h so that its order is a power of 2. Let S be a Sylow 2-subgroup of $HR_1 R_2^{s_1 s_2}$ containing h. Then S has the form $\langle h\rangle PQ$, where P, Q are Sylow 2-subgroups of R_1 and $R_2^{s_1 s_2}$ respectively. We may choose generators for P and Q so that

$$P=\langle a, b\colon a^{2^n}=b^2=1,\ b^{-1}ab=a^{-1}\rangle$$
$$Q=\langle c, d\colon c^{2^{n+1}}=1,\ d^2=c^{2^n},\ d^{-1}cd=c^{-1}\rangle,$$
$$h^{-1}ah=a^{-1},\quad h^{-1}bh=ba,\quad h^{-1}ch=c^{-1},\quad h^{-1}dh=dc$$

where $n\geqq 1$. Let $i=a^{2^{n-1}}$, $j=c^{2^n}$. Then $Z(S)=\langle i, j\rangle$ and $h^2\in Z(S)$. Since $[S,S]=\langle a, c\rangle$, it follows that $\langle j\rangle$ is characteristic in $[S,S]$ and hence in S. Thus distinct involutions in $Z(S)$ are not fused in G. Since R_1 has exactly one class of involutions, this implies j and ij cannot be fused to elements other than themselves in PQ and P respectively. If $j\in Z^*(G)$, we obtain a contradiction as before. Thus $j\notin Z^*(G)$, and there exists an involution t in $S-PQ$ fused in G to j. The element t must have the form $ha^\alpha c^\gamma$ or $ha^\alpha bc^\gamma$, the latter occurring only if $n=1$. Transforming t by $a^{2^{n-2}}$ and $a^{2^{n-2}}c^{2^{n-1}}$ when $n>1$, or by b and $bc^{2^{n-1}}$ when $n=1$, we see that t, ti, tij are conjugate in S. Choose $g\in G$ so that $t^g=j$, $C_s(t)^g\leqq S$. Since $Z(S)\leqq C_s(t)$, one of the involutions in $Z(S)$ is fused by g to an element of $PQ-\langle j\rangle$. Thus $i^g\in P$ or $(ij)^g\in Pj$, so that ti or tij is fused by g to an element of $PQ-\langle j\rangle$. Thus the same would hold for j, which is a contradiction. Hence $R_1\cong SL(2,q)$ when $p\neq 2$. This situation occurred earlier in the proof and it was shown that $G\cong G_2(q)$.

We may henceforth assume $p=2$. In particular, $R_1 R_2^{s_1 s_2}=R_1\times R_2^{s_1 s_2}$ and $|H|=(q-1)^2$. Moreover, $C_H(U_2^{s_1 s_2})=H_1$ and $C_B(U_2^{s_1 s_2})=H_1 U$, $H_2^{s_1 s_2}$ acts regularly on $U_2^{s_1 s_2}$. As in the proof of (6B) we find that

$$C(j)=R_1 U_{s_1} \tag{9.10}$$

for every involution j in $U_2^{s_1 s_2}$. Suppose $q=2$. Then $D'=\langle U_1^{s_2 s_1}, U_1^{s_2}\rangle$ and $D''=\langle U_2^{s_1}, U_2\rangle$ are dihedral groups of order 8, $[D', D'']=1$ and $D'\cap D''=U_2^{s_1 s_2}$. Thus U_{s_1} is the central product of D' and D''. U_{s_1} is also the central product of the quaternion subgroups

$$Q'=\langle u_1^{s_2 s_1}(1)\,u_1^{s_2}(1),\ u_1^{s_2 s_1}(1)\,u_2^{s_1}(1)\,u_2(1)\rangle,$$
$$Q''=\langle u_2^{s_1}(1)\,u_2(1),\ u_2^{s_1}(1)\,u_1^{s_2 s_1}(1)\,u_1^{s_2}(1)\rangle.$$

Every automorphism of U_{s_1} either fixes Q' and Q'', or interchanges Q' and Q''. Now s_1 fixes Q' and Q''. Thus R_1 normalizes Q' and Q''. If

$$[s_1 u_1(1), Q'']=1,$$

then $[u_1(1), u_2^{s_1}(1)\,u_2(1)] \in U_2^{s_1 s_2}$ and so $[U_1, U_2] \leqq T_2$, which is impossible by (9.7). If $[s_1 u_1(1), Q'] \neq 1$, then $[u_1(1), u_1^{s_2 s_1}(1)\,u_2^{s_1}(1)\,u_2(1)]$ or

$$[u_1(1), u_1^{s_2}(1)\,u_2^{s_1}(1)\,u_2(1)]$$

is in $U_2^{s_1 s_2}$. In either case, $[U_1, U_2] \leqq T_2$, which is impossible. Thus R_1 is faithful on Q'', non-trivial and non-faithful on Q'. This determines the structure of $C(j)$ completely. By a theorem of Thomas [33], $G \cong G_2(2)$.

We may henceforth assume $q \geqq 4$, so that HR_1 is irreducible on V_1. Since $HR_1 = R_1 \times (H \cap K_1)$, the elements of $H \cap K_1$ act as scalar multiplications on V_1, so V_1 may be regarded as a vector space of dimension 4 over $\mathbf{F}_q$ on which HR_1 and hence R_1 act linearly and irreducibly. We may then assume R_1 on V_1 has the form $(\Gamma \times \Gamma^\sigma)^\tau$, where σ and τ are field automorphisms of $\mathbf{F}_q$ and $\sigma \neq 1$. In particular, $u_1(\alpha)$, $h_1(\alpha)$, and s_1 are represented by

$$\begin{pmatrix} 1 & \alpha & \alpha^\sigma & \alpha^{1+\sigma} \\ & 1 & 0 & \alpha^\sigma \\ & & 1 & \alpha \\ & & & 1 \end{pmatrix}^\tau, \quad \begin{pmatrix} \alpha^{1+\sigma} & & & \\ & \alpha^{\sigma-1} & & \\ & & \alpha^{-\sigma+1} & \\ & & & \alpha^{-1-\sigma} \end{pmatrix}^\tau, \quad \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \tag{9.11}$$

respectively. The set of lines $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4$ generated over $\mathbf{F}_q$ by the vectors in the corresponding basis are distinguished by the action of the triple (H_1, U_1, s_1). Since $\Gamma \times \Gamma^\sigma$ and $\Gamma^\sigma \times \Gamma$ are equivalent representations of R_1, we may assume that

$$\{u_2(1), u_1^{s_2}(x), u_1^{s_2 s_1}(x), u_2^{s_1}(1)\} \pmod{U_2^{s_1 s_2}} \tag{9.12}$$

is a natural basis in V_1 for (9.11), where $x \in \mathbf{F}_q$. Thus there are field automorphisms τ_1, τ_2 of $\mathbf{F}_q$ such that $\alpha u_2(\beta) = u_2(\alpha^{\tau_1}\beta)$, $\alpha u_1^{s_2}(\beta x) = u_1^{s_2}(\alpha^{\tau_2}\beta x)$, $\alpha u_1^{s_2 s_1}(\beta x) = u_1^{s_2 s_1}(\alpha^{\tau_2}\beta x)$, $\alpha u_2^{s_1}(\beta) = u_2^{s_1}(\alpha^{\tau_1}\beta)$ for all $\alpha, \beta \in \mathbf{F}_q$. The matrix form of R_1 on V_1 and the matrix form Γ^ρ of R_2 on V_2 give

$$\left.\begin{aligned} [u_2(\beta), u_1(\alpha)] &\equiv u_1^{s_2}(x\,\beta^{\tau\tau_1^{-1}\tau_2}\,\alpha^{\tau\tau_2}) \\ [u_1(\alpha), u_2(\beta)] &\equiv u_1^{s_2}(\alpha\,\beta^{\rho\tau_0}) \end{aligned}\right\} \pmod{T_2}.$$

Thus $x\,\beta^{\tau\tau_1^{-1}\tau_2}\,\alpha^{\tau\tau_2} = \alpha\,\beta^{\rho\tau_0}$ for all $\alpha, \beta \in \mathbf{F}_q$, which implies that $x = 1$, $\tau\tau_2 = 1$, and $\tau_1^{-1}\tau_2 = \rho\tau_0$. Relabeling the elements of R_1 by $\tau_2\tau_1^{-1}$ we then have the relations

$$\begin{aligned} &\text{(i)} \quad [u_2(\beta), u_1(\alpha)] \equiv u_1^{s_2}(\alpha\beta)\,u_1^{s_2 s_1}(\alpha^\sigma\beta)\,u_2^{s_1}(\alpha^{1+\sigma}\beta) \pmod{U_2^{s_1 s_2}}, \\ &\text{(ii)} \quad [u_1^{s_2}(\beta), u_1(\alpha)] = u_2^{s_1}(\alpha^\sigma\beta) \pmod{U_2^{s_1 s_2}}, \\ &\text{(iii)} \quad [u_1^{s_2 s_1}(\beta), u_1(\alpha)] = u_2^{s_1}(\alpha\beta). \end{aligned} \tag{9.13}$$

Equality holds in (iii) by (9.4). Transforming (ii), (iii) by s_2 implies

$$\begin{aligned} &\text{(i)} \quad [u_1^{s_2}(\beta), u_1(\alpha)] = u_2^{s_1}(\alpha^\sigma \beta)\, u_2^{s_1 s_2}(\alpha \beta^\sigma), \\ &\text{(ii)} \quad [u_1^{s_2 s_1}(\beta), u_1^{s_2}(\alpha)] = u_2^{s_1 s_2}(\alpha \beta). \end{aligned} \tag{9.14}$$

The following table gives the action of $h_1(\alpha)$ and $h_2(\beta)$ on the root subgroups.

	U_1	$U_1^{s_2}$	$U_1^{s_2 s_1}$	U_2	$U_2^{s_1}$	$U_2^{s_1 s_2}$
$h_1(\alpha)$	α^{-2}	$\alpha^{\sigma-1}$	$\alpha^{1-\sigma}$	$\alpha^{1+\sigma}$	$\alpha^{-1-\sigma}$	1
$h_2(\beta)$	β	β^{-1}	1	β^{-2}	β	β^{-1}

(9.15)

The entries other than those giving $h_2(\beta)$ on $U_2^{s_1}$ and $U_2^{s_1 s_2}$ come from (9.1) or the matrix forms of $\Gamma \times \Gamma^\sigma$ and Γ. The two exceptions come from the relations $s_2^{-1} h_2(\beta) s_2 = h_2(\beta^{-1})$ and (9.13) (iii). In particular,

$$h_1(\alpha)\, h_2(\alpha^2) \in C(U_1)$$

and hence $h_1(\alpha)\, h_2(\alpha^2) \in H \cap K_1$. The values of $h_1(\alpha)\, h_2(\alpha^2)$ on $U_1^{s_2}$ and $U_1^{s_2 s_1}$ are $\alpha^{\sigma-3}$ and $\alpha^{1-\sigma}$ respectively. Thus $\alpha^{2\sigma} = \alpha^4$ for all $\alpha \in \mathbf{F}_q$ and σ is the automorphism $x \to x^2$.

We can now complete the multiplication table of U. By (9.8) and (9.13) (i) there exist functions f and g from $\mathbf{F}_q \times \mathbf{F}_q \to \mathbf{F}_q$ such that

$$\begin{aligned} &\text{(i)} \quad [u_2(\beta), u_2^{s_1}(\alpha)] = u_2^{s_1 s_2}(f(\alpha, \beta)), \\ &\text{(ii)} \quad [u_2(\beta), u_1(\alpha)] = u_1^{s_2}(\alpha \beta)\, u_1^{s_2 s_1}(\alpha^2 \beta)\, u_2^{s_1}(\alpha^3 \beta)\, u_2^{s_1 s_2}(g(\alpha, \beta)). \end{aligned} \tag{9.16}$$

The commutator identities on $[x y, z]$ and $[x, y z]$, and (9.13), (9.14) imply that

$$\begin{aligned} &\text{(i)} \quad f(\alpha+\alpha', \beta+\beta') = f(\alpha, \beta) + f(\alpha', \beta) + f(\alpha, \beta') + f(\alpha', \beta'), \\ &\text{(ii)} \quad g(\alpha+\alpha', \beta) = g(\alpha, \beta) + g(\alpha', \beta) + \alpha \alpha'^2 \beta^2 + \alpha' \alpha^2 \beta^2. \end{aligned} \tag{9.17}$$

Transforming (9.16) (i) by $h_1(\lambda)\, h_2(\mu)$ gives

$$f(\lambda^{-3} \mu \alpha, \lambda^3 \mu^{-2} \beta) = \mu^{-1} f(\alpha, \beta).$$

In particular, setting $\lambda = 1$, $\mu = \alpha^{-1}$, we obtain $f(\alpha, \beta) = \alpha^{-1} f(1, \alpha^2 \beta)$, and setting $\mu = \lambda^3$, we obtain $f(1, \alpha^2 \beta \lambda^{-3}) = \lambda^{-3} f(1, \alpha^2 \beta)$. Now the additive subgroup of $\mathbf{F}_q$ generated by $(\mathbf{F}_q)^3$ is a subfield of $\mathbf{F}_q$, and consequently is $\mathbf{F}_q$ unless $q = 4$. Thus it follows from (9.17)(i) that if $q > 4$, then

$$f(\alpha, \beta) = \alpha \beta (f(1, 1)). \tag{9.18}$$

If $q = 4$, (9.18) remains valid. For let ω be a primitive cube root of unity in $\mathbf{F}_4$. Expanding $f(\omega, \omega) = f(1+\omega^2, 1+\omega^2)$ and $f(1, 1) = f(1, \omega+\omega^2)$

by (9.17)(i) and using the identity $f(\alpha, \beta) = \alpha^{-1} f(1, \alpha^2 \beta)$, we find that $f(1, \omega^2) = \omega f(1, \omega) = \omega^2 f(1, 1)$ so (9.18) does hold. The group R_2 is irreducible on $U_2^{s_1} U_2^{s_1 s_2}$ and $[U_2^{s_1 s_2}, U_2] = [U_2^{s_1}, U_2^{s_2}] = 1$. Since H has two constituents on $U_2^{s_1} U_2^{s_1 s_2}$ which are conjugate under s_2, this representation is of type (a) in the notation of (4.7). Hence

$$[u_2^{s_1}(1), u_2(1)] = u_2^{s_1 s_2}(1)$$

and $f(1, 1) = 1$. Thus

$$[u_2(\beta), u_2^{s_1}(\alpha)] = u_2^{s_1 s_2}(\alpha \beta). \tag{9.19}$$

Transforming (9.16)(ii) by $h_1(\lambda) h_2(\mu)$ gives

$$g(\lambda^{-2} \mu \alpha, \lambda^3 \mu^{-2} \beta) = \mu^{-1} g(\alpha, \beta).$$

In particular, setting $\mu = \lambda^2$, we obtain $g(\alpha, \lambda^{-1} \beta) = \lambda^{-2} g(\alpha, \beta)$, and setting $\lambda = \nu^2$, $\mu = \nu^3$, we obtain $g(\nu^{-1} \alpha, \beta) = \nu^{-3} g(\alpha, \beta)$. Thus

$$g(\alpha, \beta) = \alpha^3 \beta^2 g(1, 1). \tag{9.20}$$

It follows from this and (9.17)(ii) that $(\alpha^2 \alpha' + \alpha'^2 \alpha) g(1, 1) = \alpha^2 \alpha' + \alpha'^2 \alpha$. Since $q \geqq 4$ we can choose $\alpha' = 1$ and $\alpha \in \mathbf{F}_q$ so that $\alpha^2 + \alpha \neq 0$. Thus $g(1, 1) = 1$ and

$$g(\alpha, \beta) = \alpha^3 \beta^2. \tag{9.21}$$

It is now clear that $R_1 U_{s_1}$ has a unique multiplication table, and that $U_2^{s_1 s_2} = Z(U)$. By (9.10) the theorem of Thomas [33] applies, so $G \cong G_2(q)$.

(9B) *Suppose* $|W| = 12$ *and* (b) *of* (6E) *holds. Then* $G \cong {}^3D_4(q)$.

Proof. By (6E) we have the following relations:

$$\begin{aligned}
&\text{(i)} && [U_1^{s_2 s_1}, U_1^{s_2}] = [U_1, U_1^{s_2 s_1}] = 1, \\
&\text{(ii)} && [U_2^{s_1 s_2}, U_2^{s_1}] = U_1^{s_2 s_1}, \qquad [U_2, U_2^{s_1 s_2}] = U_1^{s_2}, \\
&\text{(iii)} && [U_2, U_1^{s_2 s_1}] = [U_2^{s_1}, U_1^{s_2}] = [U_2^{s_1 s_2}, U_1] = 1, \\
&\text{(iv)} && [U_1, U_1^{s_2}] = U_1^{s_2 s_1}.
\end{aligned} \tag{9.22}$$

In particular, $[U_{s_2}, U_{s_2}] = U_1^{s_2 s_1}$. Moreover, by (6E) $V_2 = U_{s_2}/U_1^{s_2 s_1}$ is an irreducible R_2-module. We also have by (6E)

$$L_1 \cong PSL(2, q), \qquad R_2 \cong SL(2, q^3). \tag{9.23}$$

In particular, $|H|$ divides $(q^3 - 1)(q - 1)$ and

$$|G| = |H| \, q^{12} (q^3 + 1)(q + 1)(q^8 + q^4 + 1).$$

It follows from (9.22) that $Z(U_{s_1}) = U_1^{s_2} U_1^{s_2 s_1}$ and thus R_1 normalizes $U_1^{s_2} U_1^{s_2 s_1}$. This action of R_1 is irreducible, and since H has two constituents on $U_1^{s_2} U_1^{s_2 s_1}$ which are conjugate under s_1, this representation is of type (a) in the notation of (4.7). In particular,

$$R_1 \cong SL(2, q). \tag{9.24}$$

Suppose $p \neq 2$. By (9.22) (iii) and (1 F) we have

$$[R_2, R_1^{s_2 s_1}] = [R_2^{s_1}, R_1^{s_2}] = [R_2^{s_1 s_2}, R_1] = 1. \tag{9.25}$$

Let $\langle j \rangle = Z(R_2)$. If $R_2 \cap R_1^{s_2 s_1} = 1$, or if $R_2 \cap R_1^{s_2 s_1} = \langle j \rangle$ and $H \leqq R_2 R_1^{s_2 s_1}$, then G has a Sylow 2-subgroup which is the direct product or the central product of two generalized quaternion groups. As in the proof of (9A) this implies $j \in Z(G)$ which is impossible. Thus $R_2 \cap R_1^{s_2 s_1} = \langle j \rangle$ and $|H : H \cap R_2 R_1^{s_2 s_1}| = 2$. As in the proof of (8B) we can show that

$$C(j) = HR_2 R_1^{s_2 s_1}$$

by computing $C(j) \cap BsB$ for all $s \in N$ (mod H). Thus $C(j)$ satisfies condition (*) of [9] and $G \cong {}^3D_4(q)$.

We henceforth suppose $p = 2$. In particular, $R_2 R_1^{s_2 s_1} = R_2 \times R_1^{s_2 s_1}$ and $|H| = (q^3 - 1)(q - 1)$. We noted earlier that $Z(U_{s_1}) = U_1^{s_2} U_1^{s_2 s_1}$ and

$$[U_1^{s_2}, U_1] = U_1^{s_2 s_1}.$$

Thus $Z(U) = U_1^{s_2 s_1}$. Moreover, $H_1^{s_2 s_1}$ acts regularly on $U_1^{s_2 s_1}$, $C_H(U_1^{s_2 s_1}) = H_2$, and $C_B(U_1^{s_2 s_1}) = H_2 U$. As in the proof of (8B) we find that

$$C(j) = R_2 U_{s_2} \tag{9.26}$$

for every involution $j \in U_1^{s_2 s_1}$.

The group R_2 acts irreducibly on V_2. Since $[R_2, H \cap K_2] = 1$ and $H \cap K_2$ is faithful on V_2, V_2 may be regarded as a vector space of dimension 8 over $\mathbf{F}_q$ on which R_2 acts irreducibly. Thus R_2 on V_2 is necessarily of type (b) in the notation of (4.7). An $\mathbf{F}_q$-form Γ_0 of the representation of type (b) can be found in [9], § 1, and we shall refer to [9] for properties of this matrix form. We may assume R_2 acts on V_2 as Γ_0^σ, where σ is a field automorphism of $\mathbf{F}_q$. There exists an $h \in H_2$ such that if v is the second vector in a natural basis for Γ_0^σ, then v^h, v^{h^2} are the third and fourth vectors respectively in this basis. In particular,

$$\{u_1(1), u_2^{s_1}(x), u_2^{s_1}(xy), u_2^{s_1}(xy^2), u_2^{s_1 s_2}(x), u_2^{s_1 s_2}(xy), u_2^{s_1 s_2}(xy^2), u_1^{s_2}(1)\}$$

may be taken as a natural basis for Γ_0, where $x, y \in \mathbf{F}_{q^3}$. Also R_1 is irreducible on $V_1 = U_{s_1}/U_1^{s_2 s_1} U_2^{s_1 s_2} U_1^{s_2}$. Since $[R_1, H \cap K_1] = 1$ and $H \cap K_1$ is faithful, V_1 may be regarded as a vector space of dimension 2 over $\mathbf{F}_{q^3}$ on which R_1 acts irreducibly. Then R_1 on V_1 is necessarily of the form Γ^ρ, where ρ is a field automorphism of $\mathbf{F}_q$. We may take as a natural basis

$$\{u_2(1), u_2^{s_1}(1)\}.$$

As usual there are field automorphisms τ_1, τ_2, τ_3 of $\mathbf{F}_q$ such that τ_1 is associated with U_1 and $U_1^{s_2}$, τ_2 is associated with $U_2^{s_1}$ and $U_2^{s_1 s_2}$, and τ_3 is associated with U_2 and $U_2^{s_1}$.

The definition of Γ_0 depends on the choice of an element $\theta \in \mathbf{F}_{q^3}$ of order q^3-1. Given $\alpha \in \mathbf{F}_{q^3}$, we may express $\alpha = \alpha_0 + \alpha_1 \theta + \alpha_2 \theta^2$ for suitable $\alpha_0, \alpha_1, \alpha_2 \in \mathbf{F}_q$. The matrix forms of Γ^ρ and Γ_0^σ then give

$$\left.\begin{aligned} [u_2(\alpha), u_1(\beta)] &\equiv u_2^{s_1}(\alpha \beta^{\rho \tau_3}) \\ [u_1(\beta), u_2(\alpha)] &\equiv u_2^{s_1}(x \beta^{\tau_1^{-1} \tau_2}(\alpha_0^{\sigma \tau_2} + \alpha_1^{\sigma \tau_2} y + \alpha_2^{\sigma \tau_2} y^2)) \end{aligned}\right\} \pmod{U_1^{s_2 s_1} U_2^{s_1 s_2} U_1^{s_2}}.$$

Thus $\alpha \beta^{\rho \tau_3} = x \beta^{\tau_1^{-1} \tau_2}(\alpha_0^{\sigma \tau_2} + \alpha_1^{\sigma \tau_2} y + \alpha_2^{\sigma \tau_2} y^2)$ for all $\alpha \in \mathbf{F}_{q^3}$ and $\beta \in \mathbf{F}_q$. This implies that $x=1$, $\rho \tau_3 = \tau_1^{-1} \tau_2$, $\sigma \tau_2 = 1$, and $y = \theta$. We relabel the elements of R_1 by replacing $u_1(\beta)$ by $u_1(\beta^{\tau_1 \tau_2^{-1}})$. Let $\bar{\ }$ be the automorphism of $\mathbf{F}_{q^3}$ defined by $x \to x^q$. The matrix form of Γ_0^σ then gives the following relations:

$$\begin{aligned} &\text{(i)} & [u_2^{s_1 s_2}(\beta_0), u_2(\alpha)] &= u_1^{s_2}(\alpha \beta_0 + \bar{\alpha} \beta_0 + \bar{\bar{\alpha}} \beta_0), \\ &\text{(ii)} & [u_2^{s_1 s_2}(\beta_1 \theta), u_2(\alpha)] &= u_1^{s_2}(\theta \alpha \beta_1 + \overline{\theta \alpha} \beta_1 + \overline{\overline{\theta \alpha}} \beta_1), \qquad (9.27) \\ &\text{(iii)} & [u_2^{s_1 s_2}(\beta_2 \theta^2), u_2(\alpha)] &= u_1^{s_2}(\theta^2 \alpha \beta_2 + \bar{\theta}^2 \bar{\alpha} \beta_2 + \bar{\bar{\theta}}^2 \bar{\bar{\alpha}} \beta_2) \end{aligned}$$

for $\beta_0, \beta_1, \beta_2 \in \mathbf{F}_q$ and $\alpha \in \mathbf{F}_{q^3}$. Using the commutator identity on $[x y, z]$ we obtain

$$\begin{aligned} &\text{(i)} & [u_2^{s_1 s_2}(\beta), u_2(\alpha)] &= u_1^{s_2}(\alpha \beta + \overline{\alpha \beta} + \overline{\overline{\alpha \beta}}), \\ &\text{(ii)} & [u_2^{s_1 s_2}(\beta), u_2^{s_1}(\alpha)] &= u_1^{s_2 s_1}(\alpha \beta + \overline{\alpha \beta} + \overline{\overline{\alpha \beta}}) \end{aligned} \qquad (9.28)$$

the second relation coming from the first one by transformation by s_1. The same argument applied to the remaining relations given by the matrix form of Γ_0^σ gives

$$\begin{aligned} &\text{(i)} \ [u_2^{s_1}(\beta), u_2(\alpha)] = u_2^{s_1 s_2}(\bar{\alpha} \bar{\bar{\beta}} + \bar{\bar{\alpha}} \bar{\beta})\, u_1^{s_2}(\bar{\alpha} \bar{\bar{\alpha}} \beta + \bar{\bar{\alpha}} \alpha \bar{\beta} + \alpha \bar{\alpha} \bar{\bar{\beta}})\, u_1^{s_2 s_1}(f(\alpha, \beta)), \\ &\text{(ii)} \ [u_1(\beta), u_2(\alpha)] = u_2^{s_1}(\alpha \beta)\, u_2^{s_1 s_2}(\bar{\alpha} \bar{\bar{\alpha}} \beta)\, u_1^{s_2}(\alpha \bar{\alpha} \bar{\bar{\alpha}} \beta)\, u_1^{s_2 s_1}(g(\alpha, \beta)) \qquad (9.29) \end{aligned}$$

where f and g are functions from $\mathbf{F}_{q^3} \times \mathbf{F}_{q^3}$ and $\mathbf{F}_{q^3} \times \mathbf{F}_q$ into $\mathbf{F}_q$. Transforming (9.29) (i) by s_1 shows that

$$f(\alpha, \beta) = \alpha \bar{\beta} \bar{\bar{\beta}} + \bar{\alpha} \bar{\bar{\beta}} \beta + \bar{\bar{\alpha}} \beta \bar{\beta}. \qquad (9.30)$$

The following table gives the action of $h_1(\alpha), h_2(\beta)$ on selected root subgroups.

	U_1	U_2	$U_1^{s_2 s_1}$
$h_1(\alpha)$	α^{-2}	α	α^{-1}
$h_2(\beta)$	$\beta \bar{\beta} \bar{\bar{\beta}}$	β^{-2}	1

(9.31)

The entries other than that of $h_1(\alpha)$ on $U_1^{s_2 s_1}$ come from the matrix forms of Γ_0 and Γ, and from (9.25). The exception comes from (9.25) and (9.28). Transforming (9.29)(ii) by $h_1(\lambda)\,h_2(\mu)$ gives

$$g(\lambda\mu^{-2}\alpha, \lambda^{-2}\mu\bar{\mu}\bar{\bar{\mu}}\beta)=\lambda^{-1}g(\alpha,\beta).$$

Setting $\mu^2=\lambda$, we obtain $g(\alpha,\mu^{-1}\beta)=\mu^{-2}g(\alpha,\beta)$. Setting $\lambda=\mu\bar{\mu}\bar{\bar{\mu}}$, we obtain $g(\bar{\mu}\bar{\bar{\mu}}\mu^{-1}\alpha,(\mu\bar{\mu}\bar{\bar{\mu}})^{-1}\beta)=(\mu\bar{\mu}\bar{\bar{\mu}})^{-1}g(\alpha,\beta)$. Since

$$\mu\bar{\mu}\bar{\bar{\mu}}=(\bar{\mu}\bar{\bar{\mu}}\mu^{-1})^{1+q+q^2} \quad\text{and}\quad (q^3-1, q^2+q-1)=1,$$

it follows that

$$g(\alpha,\beta)=\alpha\bar{\alpha}\bar{\bar{\alpha}}\beta^2 g(1,1). \tag{9.32}$$

The commutator identity on $[x, yz]$, (9.28), (9.29)(i), and (9.30) then give

$$\big(N(\alpha+\alpha')+N(\alpha)+N(\alpha')\big)\,g(1,1)=T(\alpha'\bar{\alpha}\bar{\bar{\alpha}})+T(\alpha\bar{\alpha}'\bar{\bar{\alpha}}') \tag{9.33}$$

where N and T are the norm and trace functions from $\mathbf{F}_{q^3}$ into $\mathbf{F}_q$. If $q\geqq 4$, we choose $\alpha'=1$, and $\alpha\in\mathbf{F}_q$ so that $\alpha^2+\alpha\neq 0$. Then (9.33) becomes $(\alpha^2+\alpha)\,g(1,1)=\alpha^2+\alpha$ so that $g(1,1)=1$. If $q=2$, then $\mathbf{F}_8$ is the splitting field over $\mathbf{F}_2$ of each of the irreducible polynomials $f'(x)=x^3+x^2+1$ and $f''(x)=x^3+x+1$. Let α be a primitive 7th root of unity in $\mathbf{F}_8$ and set $\alpha'=\alpha^{-1}$. Then $\alpha+\alpha'\neq 0$ and (9.33) becomes $g(1,1)=T(\alpha^5)+T(\alpha^{-5})$. Since the inverses of the roots of $f'(x)$ are the roots of $f''(x)$, it follows that $T(\alpha^5)+T(\alpha^{-5})=1$, so that $g(1,1)=1$. Thus

$$g(\alpha,\beta)=\alpha\bar{\alpha}\bar{\bar{\alpha}}\beta^2. \tag{9.34}$$

The group R_1 is irreducible on $Z(U_{s_1})$ and $[R_1, H_2^{s_1 s_2}]=1$. Using (9.31), we can see that $H_2^{s_1 s_2}$ induces a faithful group of operators of order $q-1$ on $Z(U_{s_1})$. Thus $Z(U_{s_1})$ can be regarded as a vector space of dimension 2 over $\mathbf{F}_q$ on which R_1 acts irreducibly. We may assume R_1 on $Z(U_{s_1})$ has the form Γ^τ for some field automorphism τ of $\mathbf{F}_q$, and take as a natural basis

$$\{u_1^{s_2}(1), u_1^{s_2 s_1}(1)\}.$$

Then there is a field automorphism τ_0 of $\mathbf{F}_q$ such that $\alpha u_1^{s_2}(\beta)=u_1^{s_2}(\alpha^{\tau_0}\beta)$ and $\alpha u_1^{s_2 s_1}(\beta)=u_1^{s_2 s_1}(\alpha^{\tau_0}\beta)$ for all $\alpha,\beta\in\mathbf{F}_q$. We obtain as usual the relation

$$[u_1^{s_2}(\alpha), u_1(\beta)]=u_1^{s_2 s_1}(\alpha\beta^{\tau\tau_0}).$$

If we transform this by s_2 and compare the two relations, we see that $\tau\tau_0=1$ and thus

$$[u_1^{s_2}(\alpha), u_1(\beta)]=u_1^{s_2 s_1}(\alpha\beta). \tag{9.35}$$

It is now clear that $R_2 U_{s_2}$ has a unique multiplication table. It follows from (9.26) that the theorem of Thomas [34] applies, so $G\cong {}^3D_4(q)$.

The groups $G_2(q)$ and ${}^3D_4(q)$ have also been characterized by Tits in unpublished work. Indeed his methods do not require that G be finite, but it is assumed that G has a root data system in the sense of [37].

§ 10. Identification of G in the Case $|W|=16$

(10 A) *Suppose* $|W|=16$. *Then* $G\cong {}^2F_4(q)$.

Proof. By (6 H) we have that

$$R_1\cong SL(2,q) \qquad R_2\cong Sz(q). \tag{10.1}$$

In particular, $|H|=(q-1)^2$ and $q=2^n$, where n has the form $n=2a+1$, $a\geqq 0$. We recall that in the definition of $Sz(q)$, the automorphism θ of $\mathbf{F}_q$ given by $x\to x^{2^{a+1}}$ plays an essential role. We note that

$$(1+\theta^{-1})(2-\theta)=(\theta-\theta^{-1})\,\theta=(\theta-1)(\theta+1)=1$$

so in particular, all the factors are bijections. By (6 H) the following relations hold:

$$\begin{array}{ll}
\text{(i)} & [U_1, U_1^{s_2 s_1}]=1,\\
\text{(ii)} & [U_2, U_2^{s_1 s_2}]=U_1^{s}, \qquad [\Omega_1(U_2), U_2^{s_1 s_2}]=1,\\
\text{(iii)} & [U_2, U_1^{s_2 s_1 s_2}]=\Omega_1(U_2^{s_1 s_2}), \qquad [\Omega_1(U_2), U_1^{s_2 s_1 s_2}]=1,\\
\text{(iv)} & [U_1, U_1^{s_2 s_1 s_2}]=[U_2, U_2^{s_1 s_2 s_1}]=1,\\
\text{(v)} & [\Omega_1(U_2), U_1^{s_2 s_1}]\leqq \Omega_1(U_2^{s_1 s_2}).
\end{array} \tag{10.2}$$

The relations (10.2) also imply the additional relations obtained by transformation by elements of $N=\langle H, s_1, s_2\rangle$. Finally we have by (6 H)

$$\begin{array}{ll}
\text{(i)} & [U_{s_2}, U_{s_2}]=\Phi(U_{s_2})\\
& \quad =\Omega_1(U_2^{s_1 s_2})\, U_1^{s_2 s_1 s_2}\, \Omega_1(U_2^{s_1 s_2 s_1})\, U_1^{s_2 s_1}\, \Omega_1(U_2^{s_1}),\\
\text{(ii)} & \Phi(\Phi(U_{s_2}))=1,\\
\text{(iii)} & [\Phi(U_{s_2}), U_{s_2}]=\Omega_1(U_2^{s_1 s_2 s_1})=Z(U_{s_2})=Z(U).
\end{array} \tag{10.3}$$

It follows from (10.2)(iv) that R_2 normalizes $T_2=[U_{s_2}, U_{s_2}]\, U_2^{s_1 s_2 s_1}$, and so R_2 acts on $V_2=U_{s_2}/T_2$. Now R_2 is faithful on V_2 by (5 D). Thus R_2 is irreducible on V_2 by (4 F). As $[R_2, H\cap K_2]=1$, $|H\cap K_2|=q-1$, and $H\cap K_2$ is faithful on V_2, it follows that V_2 may be regarded as a vector space of dimension 4 over $\mathbf{F}_q$ on which R_2 acts linearly. We may assume R_2 on V_2 has the form Λ^σ, where Λ is the natural representation of $Sz(q)$ discussed in § 4, and σ is a field automorphism of $\mathbf{F}_q$. We note that if $\{v_1, v_2, v_3, v_4\}$ is a natural basis for Λ^σ, then the lines $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4$ spanned over $\mathbf{F}_q$ by v_1, v_2, v_3, v_4 are distinguished by the action of the triple (H_2, U_2, s_2) when $q>2$. If $q=2$, $\mathscr{L}_1$ is the unique line in the underlying space $\mathscr{V}$ of Λ^σ centralized by U_2. The line $\mathscr{L}_2$ is determined (mod $\mathscr{L}_1$) since $\langle \mathscr{L}_1, \mathscr{L}_2\rangle = \langle \mathscr{V}(1-u)^2$ for $u \in U_2\rangle$. Now $v_2^{s_2}=v_3$, $v_1^{s_2}=v_4$. Thus v_1+v_2 and v_2 can be distinguished since

$$\langle (v_1+v_2)^{s_2}(1-u); u\in U_2\rangle \nsubseteq \langle \mathscr{L}_1, \mathscr{L}_2\rangle$$

whereas $\langle v_2^{s_2}(1-u)$: $u\in U_2\rangle \subseteq \langle \mathscr{L}_1, \mathscr{L}_2\rangle$. With these remarks it follows by (3G) and (10.2) that a natural basis in V_2 for Λ^σ has the form

$$\{u_1^{s_2}(1), \bar{u}_2^{s_1 s_2}(x), \bar{u}_2^{s_1}(x), u_1(1)\} \pmod{T_2} \tag{10.4}$$

where $\bar{u}_2^{s_1 s_2}(x) = u_2^{s_1 s_2}(x, 0)\,\Omega_1(U_2^{s_1 s_2})$, and $x\in \mathbf{F}_q$. There are field automorphisms τ_1, τ_2 of $\mathbf{F}_q$ such that $\alpha u_1^{s_2}(\beta) = u_1^{s_2}(\alpha^{\tau_1}\beta)$,

$$\alpha \bar{u}_2^{s_1 s_2}(\beta x) = \bar{u}_2^{s_1 s_2}(\alpha^{\tau_2}\beta x), \qquad \alpha \bar{u}_2^{s_1}(\beta x) = \bar{u}_2^{s_1}(\alpha^{\tau_2}\beta x),$$

$\alpha u_1(\beta) = u_1(\alpha^{\tau_1}\beta)$ for all $\alpha, \beta\in \mathbf{F}_q$. The matrix form of Λ^σ implies

$$[U_2^{s_1}, U_2] \equiv U_2^{s_1 s_2}\, U_1^{s_2} \pmod{T_2}. \tag{10.5}$$

The relations (10.2)(ii) and (iii) imply that

$$[U_{s_1}, U_{s_1}] \geqq U_1^{s_2 s_1}\,\Omega_1(U_2^{s_1 s_2 s_1})\, U_1^{s_2 s_1 s_2}\,\Omega_1(U_2^{s_1 s_2})\, U_1^{s_2}.$$

Moreover (3G) and (10.5) imply that

$$[U_2^{s_1}, U_2] \equiv U_2^{s_1 s_2}\, U_1^{s_2} \left(\mathrm{mod}\ U_1^{s_2 s_1}\, U_2^{s_1 s_2 s_1}\, U_1^{s_2 s_1 s_2}\,\Omega_1(U_2^{s_1 s_2})\right).$$

In particular, there exists a function $f\colon \mathbf{F}_q \to \mathbf{F}_q$ such that

$$\bar{u}_2^{s_1 s_2}(\alpha)\,\bar{u}_2^{s_1 s_2 s_1}(f(\alpha)) \in [U_{s_1}, U_{s_1}]$$

for all $\alpha\in\mathbf{F}_q$. Since H_2 centralizes $U_2^{s_1 s_2 s_1}$ by (10.2)(iv) and H_2 acts regularly and faithfully on $\bar{U}_2^{s_1 s_2}$ from the matrix form of Λ^σ, it follows that $U_2^{s_1 s_2} \leqq [U_{s_1}, U_{s_1}]$ whenever $q>2$. If we set

$$T_1 = U_1^{s_2 s_1}\, U_2^{s_1 s_2 s_1}\, U_1^{s_2 s_1 s_2}\, U_2^{s_1 s_2}\, U_1^{s_2},$$

then $T_1 \leqq [U_{s_1}, U_{s_1}]$ and $\Phi(U_{s_1}) = \Omega_1(U_2^{s_1})\, T_1\, \Omega_1(U_2)$ whenever $q>2$. In particular, $\Omega_1(U_2^{s_1})\, T_1\, \Omega_1(U_2)$ is characteristic in U_{s_1} wheever $q>2$. The last conclusion holds when $q=2$. Indeed, if $U_2^{s_1 s_2} \leqq [U_{s_1}, U_{s_1}]$, then the above argument applies. On the other hand, if $U_2^{s_1 s_2} \nleqq [U_{s_1}, U_{s_1}]$, then (10.5) implies

$$[U_{s_1}, U_{s_1}] = \langle U_1^{s_2 s_1}\,\Omega_1(U_2^{s_1 s_2 s_1})\, U_1^{s_2 s_1 s_2}\,\Omega_1(U_2^{s_1 s_2})\, U_1^{s_2},\ u_2^{s_1 s_2 s_1}(1,0)\, u_2^{s_1 s_2}(1,0)\rangle.$$

Then $U_{s_1}/[U_{s_1}, U_{s_1}]$ has order 2^5, and $\Omega_1(U_2^{s_1})\, T_1\, \Omega_1(U_2)$ is the inverse image in U_{s_1} of $\Omega_1(U_{s_1}/[U_{s_1}, U_{s_1}])$.

The quotient group $V_1 = U_{s_1}/\Omega_1(U_2^{s_1})\, T_1\, \Omega_1(U_2)$ has order q^2 and R_1 is faithful and irreducible on V_1 by (5D). Thus R_1 is irreducible on V_1. Now $[R_1, H\cap K_1]=1$ and $|H\cap K_1| = q-1$. Since $H\cap K_1$ is faithful on

V_1, it follows that V_1 may be regarded as a vector space of dimension 2 over $\mathbf{F}_q$ on which R_1 acts linearly. We may assume R_1 has the form Γ^ρ on V_1, where ρ is a field automorphism of $\mathbf{F}_q$, and take as a natural basis for Γ^ρ on V_1

$$\{\bar{u}_2(1), \bar{u}_2^{s_1}(1)\} \pmod{\Phi(U_{s_1})}. \tag{10.6}$$

There is a field automorphism τ_3 of $\mathbf{F}_q$ such that modulo $\Phi(U_{s_1})$ we have the scalar action $\alpha\bar{u}_2(\beta)=\bar{u}_2(\alpha^{\tau_3}\beta)$, $\alpha\bar{u}_2^{s_1}(\beta)=\bar{u}_2^{s_1}(\alpha^{\tau_3}\beta)$ for all $\alpha, \beta\in\mathbf{F}_q$. The matrix forms of Λ^σ and Γ^σ now give the following congruences

$$\left.\begin{aligned} [u_1(\gamma), u_2(\alpha,\beta)] &\equiv \bar{u}_2^{s_1}(x\gamma^{\tau_1^{-1}\tau_2}\alpha^{\sigma\tau_2}) \\ [\bar{u}_2(\alpha), u_1(\gamma)] &\equiv \bar{u}_2^{s_1}(\alpha\gamma^{\rho\tau_3}) \end{aligned}\right\} \bmod \Phi(U_{s_1}).$$

Thus $\alpha\gamma^{\rho\tau_3}=x\gamma^{\tau_1^{-1}\tau_2}\alpha^{\sigma\tau_2}$ for all $\alpha,\gamma\in\mathbf{F}_q$, and so $x=1$, $\sigma\tau_2=1$, and $\rho\tau_3=\tau_1^{-1}\tau_2$. We now relabel the elements of R_2 by the field automorphism $\tau_2\tau_1^{-1}$, thereby replacing $u_2(\alpha,\beta)$ by $u_2(\alpha^{\tau_2\tau_1^{-1}}, \beta^{\tau_2\tau_1^{-1}})$. The matrix form of Λ^σ then gives the following:

$$\left.\begin{aligned} &\text{(i)}\ [\bar{u}_2^{s_1s_2}(\gamma), u_2(\alpha,\beta)] = u_1^{s_2}(\alpha\gamma), \\ &\text{(ii)}\ [\bar{u}_2^{s_1}(\gamma), u_2(\alpha,\beta)] \equiv \bar{u}_2^{s_1s_2}(\alpha^\theta\gamma)\, u_1^{s_2}((\alpha^{1+\theta}+\beta)\gamma) \\ &\text{(iii)}\ [u_1(\gamma), u_2(\alpha,\beta)] \\ &\qquad \equiv \bar{u}_2^{s_1}(\alpha\gamma)\,\bar{u}_2^{s_1s_2}(\beta\gamma)\, u_1^{s_2}((\alpha^{2+\theta}+\alpha\beta+\beta^\theta)\gamma) \end{aligned}\right\} \pmod{T_2}. \tag{10.7}$$

We may clearly rewrite (10.7) (i) as

$$[u_2^{s_1s_2}(\gamma,\delta), u_2(\alpha,\beta)] = u_1^{s_2}(\alpha\gamma). \tag{10.8}$$

We may also rewrite (10.7) (ii) in the form

$$[u_2^{s_1}(\gamma,\delta), u_2(\alpha,\beta)] \equiv \bar{u}_2^{s_1s_2}(\alpha^\theta\gamma)\, u_1^{s_2}((\alpha^{1+\theta}+\beta)\gamma)$$
$$\pmod{U_1^{s_2s_1}\, U_2^{s_1s_2s_1}\, U_1^{s_2s_1s_2}\, \Omega_1(U_2^{s_1s_2})}.$$

Transforming this congruence by s_1, we derive

$$\begin{aligned} &[u_2^{s_1}(\gamma,\delta), u_2(\alpha,\beta)] \\ &\quad \equiv u_1^{s_2s_1}(\alpha(\gamma^{1+\theta}+\delta))\,\bar{u}_2^{s_1s_2s_1}(\alpha\gamma^\theta)\,\bar{u}_2^{s_1s_2}(\alpha^\theta\gamma)\, u_1^{s_2}((\alpha^{\theta+1}+\beta)\gamma) \\ &\qquad \pmod{\Omega_1(U_2^{s_1s_2s_1})\, U_1^{s_2s_1s_2}\, \Omega_1(U_2^{s_1s_2})}. \end{aligned} \tag{10.9}$$

We have the following table giving the action of $h_1(\alpha)$ and $h_2(\beta)$ on the following root subgroups.

	U_1	$U_1^{s_2}$	$\bar{U}_2$	$\bar{U}_2^{s_1}$	$\bar{U}_2^{s_1s_2}$	$\bar{U}_2^{s_1s_2s_1}$
$h_1(\alpha)$	α^{-2}	α^θ	α	α^{-1}	$\alpha^{\theta-1}$	$\alpha^{1-\theta}$
$h_2(\beta)$	$\beta^{-1-\theta^{-1}}$	$\beta^{1+\theta^{-1}}$	β	$\beta^{-\theta^{-1}}$	$\beta^{\theta^{-1}}$	1

(10.10)

The entries other than those of $h_1(\alpha)$ on $\bar{U}_2^{s_1 s_2}, \bar{U}_2^{s_1 s_2 s_1}, U_1^{s_2}$ come from (10.1) and the matrix forms of Λ and Γ. The exceptions come from (10.8) and (10.9). In particular $h_1(\alpha)\,h_2(\alpha^{-1})\in C(\bar{U}_2)$ and so $h_1(\alpha)\,h_2(\alpha^{-1})\in C(U_2)$. Thus $h_1(\alpha)\,h_2(\alpha^{-1})\in C(R_2)$ and $H\cap K_2=\langle h_1(\alpha)\,h_2(\alpha^{-1})\colon \alpha\in\mathbf{F}_q\rangle$. In the proof of (6H) it was proved that $C_H(U_r)\cap C_H(U_s)=1$ for independent roots r, s. It follows that $H\cap K_2$ is faithful on $\Phi(U_{s_2})/Z(U_{s_2})$.

Consider R_2 acting on $\Phi(U_{s_2})/Z(U_{s_2})$. Since s_2 and $u_2(0,1)$ have distinct actions on $\Phi(U_{s_2})/Z(U_{s_2})$ by (10.2)(iii), R_2 is faithful and hence by (4F) irreducible on $\Phi(U_{s_2})/Z(U_{s_2})$. Now $[R_2, H\cap K_2]=1$ and $H\cap K_2$ is faithful on $\Phi(U_{s_2})/Z(U_{s_2})$. Thus $\Phi(U_{s_2})/Z(U_{s_2})$ can be regarded as a vector space of dimension 4 over $\mathbf{F}_q$ on which R_2 acts linearly. We may assume R_2 on $\Phi(U_{s_2})/Z(U_{s_2})$ has the form Λ^τ where τ is a field automorphism of $\mathbf{F}_q$. As in the earlier discussion of R_2 on V_2, we may make

$$\{u_2^{s_1 s_2}(0,1),\, u_1^{s_2 s_1 s_2}(y),\, u_1^{s_2 s_1}(y),\, u_2^{s_1}(0,1)\}\ \left(\operatorname{mod} Z(U_{s_2})\right)$$

as a natural basis for Λ^τ, where $y\in\mathbf{F}_q$. There are then field automorphisms τ_4, τ_5 of $\mathbf{F}_q$ such that τ_4 is associated with $\Omega_1(U_2^{s_1 s_2})$ and $\Omega_1(U_2^{s_1})$, and τ_5 is associated with $U_1^{s_2 s_1 s_2}$ and $U_1^{s_2 s_1}$. The matrix form of Λ^τ gives

$$\begin{aligned}[u_2^{s_1}(0,\delta), u_2(\alpha,\beta)] &\equiv u_1^{s_2 s_1}(y\,\delta^{\tau_4^{-1}\tau_5}\,\alpha^{\tau\tau_5})\, u_1^{s_2 s_1 s_2}(y\,\delta^{\tau_4^{-1}\tau_5}\,\beta^{\tau\tau_5})\\ &\quad\cdot u_2^{s_1 s_2}(0, \delta(\alpha^{2+\theta}+\alpha\beta+\beta^\theta)^{\tau\tau_4})\ \left(\operatorname{mod} Z(U_{s_2})\right).\end{aligned}$$

Comparing this with (10.9) we see that $y\,\delta^{\tau_4^{-1}\tau_5}\,\alpha^{\tau\tau_5}=\alpha\delta$ for all α, δ in $\mathbf{F}_q$. Thus $y=1$, $\tau_4^{-1}\tau_5=\tau\tau_5=1$. The matrix form of Λ^τ then gives

$$\begin{aligned}
&\text{(i)}\ \ [u_1^{s_2 s_1 s_2}(\gamma), u_2(\alpha,\beta)]=u_2^{s_1 s_2}(0,\alpha\gamma),\\
&\text{(ii)}\ \ [u_1^{s_2 s_1}(\gamma), u_2(\alpha,\beta)]\\
&\qquad \equiv u_1^{s_2 s_1 s_2}(\alpha^\theta\gamma)\, u_2^{s_1 s_2}(0, (\alpha^{1+\theta}+\beta)\gamma)\ \left(\operatorname{mod} Z(U_{s_2})\right), \qquad (10.11)\\
&\text{(iii)}\ \ [u_2^{s_1}(0,\delta), u_2(\alpha,\beta)]\\
&\qquad \equiv u_1^{s_2 s_1}(\alpha\delta)\, u_1^{s_2 s_1 s_2}(\beta\delta)\, u_2^{s_1 s_2}(0, (\alpha^{2+\theta}+\alpha\beta+\beta^\theta)\,\delta)\ \left(\operatorname{mod} Z(U_{s_2})\right).
\end{aligned}$$

Now $[u_2(\alpha,\beta), u_2(\gamma,\delta)]=u_2(0,\alpha\gamma^\theta+\alpha^\theta\gamma)$, and $[U_2, U_2]=\Omega_1(U_2)$ if $q>2$. If an element $h\in H$ induces a multiplication by ζ on $\bar{U}_2$, it then induces a multiplication by $\zeta^{1+\theta}$ on $\Omega_1(U_2)$, this holding for $q\geqq 2$. Thus (10.10) and (10.11)(iii) give the following table of the action of $h_1(\alpha)$ and $h_2(\beta)$.

	$\Omega_1(U_2)$	$\Omega_1(U_2^{s_1})$	$\Omega_1(U_2^{s_1 s_2})$	$\Omega_1(U_2^{s_1 s_2 s_1})$	$U_1^{s_2 s_1}$	$U_1^{s_2 s_1 s_2}$
$h_1(\alpha)$	$\alpha^{1+\theta}$	$\alpha^{-1-\theta}$	α	α^{-1}	$\alpha^{-\theta}$	1
$h_2(\beta)$	$\beta^{1+\theta}$	$\beta^{-1-\theta^{-1}}$	$\beta^{1+\theta^{-1}}$	1	$\beta^{-\theta^{-1}}$	$\beta^{\theta^{-1}}$

(10.12)

In particular, (10.12) implies that $H_1\cap H_2=1$ so $H=H_1H_2$.

We may rewrite (10.11) (ii) and (iii) in the form

$$\begin{aligned}
&\text{(i)}\ [u_1^{s_2 s_1}(\gamma), u_2(\alpha, \beta)] \\
&\quad = u_1^{s_2 s_1 s_2}(\alpha^\theta \gamma)\, u_2^{s_1 s_2}(0, (\alpha^{1+\theta}+\beta)\gamma)\, u_2^{s_1 s_2 s_1}(0, g(\alpha, \beta, \gamma)), \\
&\text{(ii)}\ [u_2^{s_1}(0, \delta), u_2(\alpha, \beta)] \\
&\quad = u_1^{s_2 s_1}(\alpha\delta)\, u_1^{s_2 s_1 s_2}(\beta\delta)\, u_2^{s_1 s_2}(0, (\alpha^{2+\theta}+\alpha\beta+\beta^\theta)\delta)\, u_2^{s_1 s_2 s_1}(0, h(\alpha, \beta, \delta))
\end{aligned} \tag{10.13}$$

where g and h are functions from $\mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q \to \mathbf{F}_q$. The commutator identity on $[x, yz]$ gives

$$\begin{aligned}
&\text{(i)}\ g(\alpha+\alpha', \alpha\alpha'^\theta+\beta+\beta', \gamma) = g(\alpha, \beta, \gamma) + g(\alpha', \beta', \gamma), \\
&\text{(ii)}\ h(\alpha+\alpha', \alpha\alpha'^\theta+\beta+\beta', \delta) = h(\alpha, \beta, \delta) + h(\alpha', \beta', \delta) + g(\alpha', \beta', \alpha\delta).
\end{aligned} \tag{10.14}$$

In particular, $g(\alpha, \beta, \gamma) = g(\alpha, 0, \gamma) + g(0, \beta, \gamma)$ and

$$h(\alpha, \beta, \delta) = h(\alpha, 0, \delta) + h(0, \beta, \delta).$$

Setting $\alpha = \alpha'$, $\beta = \beta'$ in (10.14), we find that $g(0, \alpha^{1+\theta}, \gamma) = 0$,

$$h(0, \alpha^{1+\theta}, \delta) = g(\alpha, \beta, \alpha\delta).$$

Since $1+\theta$ is a bijection, it follows that $g(0, \beta, \gamma) = 0$ for all $\beta, \gamma \in \mathbf{F}_q$. Transforming (10.13) by $h_1(\lambda)\, h_2(\mu)$ gives

$$g(\lambda\mu\alpha, 0, \lambda^{-\theta}\mu^{-\theta^{-1}}\gamma) = \lambda^{-1} g(\alpha, 0, \gamma),$$

and $h(\lambda\mu\alpha, 0, \lambda^{-1-\theta}\mu^{-1-\theta^{-1}}\delta) = \lambda^{-1} h(\alpha, 0, \delta)$. Setting $\mu = \lambda^{-1}$, we obtain $g(\alpha, 0, \lambda^{\theta^{-1}-\theta}\gamma) = (\lambda^{\theta^{-1}-\theta})^\theta g(\alpha, 0, \gamma)$ and $h(\alpha, 0, \lambda^{\theta^{-1}-\theta}\delta) = (\lambda^{\theta^{-1}-\theta})^\theta h(\alpha, 0, \delta)$. Setting $\mu = \lambda^{-2}$ and $\mu = \lambda^{-\theta}$, we obtain respectively

$$g(\lambda^{-1}\alpha, 0, \gamma) = \lambda^{-1} g(\alpha, 0, \gamma) \quad \text{and} \quad h(\lambda^{1-\theta}\alpha, 0, \delta) = (\lambda^{1-\theta})^{\theta+1} h(\alpha, 0, \delta).$$

Thus $g(\alpha, 0, \gamma) = \alpha\gamma^\theta g(1, 0, 1)$, $h(0, \beta, \delta) = \beta\delta^\theta g(1, 0, 1)$, and

$$h(\alpha, 0, \delta) = \alpha^{\theta+1}\delta^\theta h(1, 0, 1).$$

If we transform (10.13) (ii) by s_1 and set $\alpha = 0$, we see that $g(1, 0, 1) = 1$. Thus

$$g(\alpha, \beta, \gamma) = \alpha\gamma^\theta. \tag{10.15}$$

Choosing $\beta = \beta' = 0$, $\delta = 1$ in (10.14) (ii) we obtain

$$(\alpha\alpha'^\theta + \alpha^\theta\alpha')\, h(1, 0, 1) = \alpha\alpha'^\theta + \alpha^\theta\alpha'.$$

If $q > 2$, then it must be the case that $h(1, 0, 1) = 1$. Thus if $q > 2$, then

$$h(\alpha, \beta, \delta) = \alpha^{\theta+1}\delta^\theta + \beta\delta^\theta. \tag{10.16}$$

We can show (10.16) holds for $q=2$ as follows. R_2 acts on the elementary 2-group spanned by $\{u_2^{s_1 s_2 s_1}(0,1),\ u_2^{s_1}(0,1),\ u_2^{s_1 s_2}(0,1),\ u_1^{s_2 s_1}(1),\ u_1^{s_2 s_1 s_2}(1)\}$. Relative to this basis s_2 and $u_2(1,0)$ are represented by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \tag{10.17}$$

where $x=1$ if and only if (10.16) holds. Since $s_2 u_2(1,0)$ has order 4, the product of the above matrices has order 4. This is possible if and only if $x=1$.

Consider the characteristic subgroup $X_1=\Omega_1(U_2^{s_1})\,T_1\,\Omega_1(U_2)$ of U_{s_1}. Now (10.1), (10.2), and (10.11) imply that

$$\Phi(X_1)=\Omega_1(U_2^{s_1 s_2 s_1})\,U_1^{s_2 s_1 s_2}\,\Omega_1(U_2^{s_1 s_2})$$

and $[U_{s_1}, X_1]=U_1^{s_2}\,U_1^{s_2 s_1}\,\Phi(X_1)$. Thus R_1 acts on $U_1^{s_2}\,U_1^{s_2 s_1}\,\Phi(X_1)/\Phi(X_1)$. Since s_1 and $u_1(1)$ have distinct actions, this representation of R_1 is faithful and hence irreducible. Now (10.10) implies that $h_1(\alpha)\,h_2(\alpha^{2\theta-4})$ centralizes U_1 and hence R_1. Thus $H\cap K_1=\langle h_1(\alpha)\,h_2(\alpha^{2\theta-4})\colon \alpha\in\mathbf{F}_q\rangle$. Also (10.12) implies $H\cap K_1$ is faithful on $U_1^{s_2 s_1}$. Thus we may assume R_1 acts as Γ^ω on $U_1^{s_2}\,U_1^{s_2 s_1}\,\Phi(X_1)/\Phi(X_1)$, where ω is a field automorphism of $\mathbf{F}_q$, and take as natural base

$$\{u_1^{s_2}(1),\ u_1^{s_2 s_1}(1)\}\ \left(\operatorname{mod}\Phi(X_1)\right).$$

There is a field automorphism τ_6 of $\mathbf{F}_q$ such that $\alpha u_1^{s_2}(\beta)=u_1^{s_2}(\alpha^{\tau_6}\beta)$ and $\alpha u_1^{s_2 s_1}(\beta)=u_1^{s_2 s_1}(\alpha^{\tau_6}\beta)$ for all α,β in $\mathbf{F}_q$. We then have the congruence

$$[u_1^{s_2}(\beta), u_1(\alpha)]\equiv u_1^{s_2 s_1}(\alpha^{\omega\tau_6}\beta)\ \left(\operatorname{mod}\Phi(X_1)\right).$$

Transforming by s_2 implies

$$[u_1^{s_2}(\beta), u_1(\alpha)]\equiv u_1^{s_2 s_1}(\alpha^{\omega\tau_6}\beta)\,u_1^{s_2 s_1 s_2}(\alpha\beta^{\omega\tau_6})\ \left(\operatorname{mod}\Omega_1(U_2^{s_1 s_2 s_1})\right). \tag{10.18}$$

If we transform (10.18) by $h_1(\gamma)$, we obtain $(\gamma^\theta\beta)(\gamma^{-2}\alpha)^{\omega\tau_6}=\gamma^{-\theta}(\beta\alpha^{\omega\tau_6})$ for all $\alpha,\beta,\gamma\in\mathbf{F}_q$. Thus $\gamma^{2\omega\tau_6}=\gamma^{2\theta}$ and $\omega\tau_6=\theta$.

We may rewrite (10.18) in the form

$$[u_1^{s_2}(\beta), u_1(\alpha)]=u_1^{s_2 s_1}(\alpha^\theta\beta)\,u_1^{s_2 s_1 s_2}(\alpha\beta^\theta)\,u_2^{s_1 s_2 s_1}(0, f(\alpha,\beta)) \tag{10.19}$$

where f is a function from $\mathbf{F}_q\times\mathbf{F}_q\to\mathbf{F}_q$. The commutator identity on $[x, yz]$ shows that

$$f(\alpha+\alpha',\beta)=f(\alpha,\beta)+f(\alpha',\beta). \tag{10.20}$$

Transforming (10.19) by $h_1(\lambda)\,h_2(\mu)$ gives

$$f(\lambda^{-2}\mu^{-1-\theta^{-1}}\alpha,\ \lambda^{\theta}\mu^{1+\theta^{-1}}\beta)=\lambda^{-1}f(\alpha,\beta).$$

Setting $\lambda=\mu^{-\theta^{-1}-\theta^{-2}}$, we obtain $f(\mu^{\theta^{-1}}\alpha,\beta)=(\mu^{\theta^{-1}})^{(1+\theta^{-1})}f(\alpha,\beta)$, and setting $\lambda^2=\mu^{-1-\theta^{-1}}$, we obtain $f(\alpha,\lambda^{\theta-2}\beta)=(\lambda^{\theta-2})^{(1+\theta^{-1})}f(\alpha,\beta)$. Thus $f(\alpha,\beta)=(\alpha\beta)^{1+\theta^{-1}}f(1,1)$. Choosing $\beta=1$ in (10.20) we obtain

$$(\alpha\,\alpha'^{\theta^{-1}}+\alpha^{\theta^{-1}}\alpha')\,f(1,1)=0.$$

If $q>2$, it must be the case that $f(1,1)=0$, whence

$$f(\alpha,\beta)=0. \tag{10.21}$$

We can show (10.21) holds for $q=2$ as follows. R_1 acts on the elementary 2-group spanned by $\{u_1^{s_2s_1s_2}(1),\ u_2^{s_1s_2s_1}(0,1),\ u_2^{s_1s_2}(0,1),\ u_1^{s_2s_1}(1),\ u_1^{s_2}(1)\}$. Relative to this basis s_1 and $u_1(1)$ are represented by

$$\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix},\quad \begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&1&1&0&0\\0&0&0&1&0\\1&x&0&1&1\end{pmatrix}, \tag{10.22}$$

where $x=0$ if and only if (10.21) holds. Since $s_1u_1(1)$ has order 3, the product of the above matrices has order 3. This is possible if and only if $x=0$. We note that the multiplication table of U_{s_2} is now complete.

In order to determine $[u_2^{s_1}(\gamma,\delta),u_2(\alpha,\beta)]$, it suffices to determine $[u_2^{s_1}(\gamma,0),u_2(\alpha,0)]$ since the previous calculations and the commutator identities on $[xy,z]$ and $[x,yz]$ will then give the expansion for

$$[u_2^{s_1}(\gamma,\delta),u_2(\alpha,\beta)].$$

We may rewrite (10.9) in the form

$$\begin{aligned}&[u_2^{s_1}(\gamma,0),u_2(\alpha,0)]\\&\quad=u_1^{s_2s_1}(\alpha\gamma^{1+\theta})\,u_2^{s_1s_2s_1}(\alpha\gamma^{\theta},\ell(\alpha,\gamma))\,u_1^{s_2s_1s_2}(m(\alpha,\gamma))\,u_2^{s_1s_2}(\alpha^{\theta}\gamma,n(\alpha,\gamma))\\&\qquad\cdot u_1^{s_2}(\alpha^{\theta+1}\gamma)\end{aligned} \tag{10.23}$$

where ℓ,m,n are functions from $\mathbf{F}_q\times\mathbf{F}_q\to\mathbf{F}_q$. Transforming (10.23) by $h_1(\lambda)\,h_2(\mu)$ gives

$$\begin{aligned}\ell(\lambda\mu\alpha,\lambda^{-1}\mu^{-\theta^{-1}}\gamma)&=\lambda^{-1}\ell(\alpha,\gamma)\\ m(\lambda\mu\alpha,\lambda^{-1}\mu^{-\theta^{-1}}\gamma)&=\mu^{\theta^{-1}}m(\alpha,\gamma)\\ n(\lambda\mu\alpha,\lambda^{-1}\mu^{-\theta^{-1}}\gamma)&=\lambda\mu^{1+\theta^{-1}}n(\alpha,\gamma).\end{aligned} \tag{10.24}$$

Setting $\lambda=\mu^{-1}$, we obtain $m(\alpha, \mu^{1-\theta^{-1}}\gamma)=(\mu^{1-\theta^{-1}})^{1+\theta} m(\alpha, \gamma)$ and

$$\ell(\alpha, \mu^{1-\theta^{-1}}\gamma)=(\mu^{1-\theta^{-1}})^{\theta+2}\,\ell(\alpha, \gamma).$$

Setting $\lambda=\mu^{-\theta^{-1}}$ we obtain $m(\mu^{1-\theta^{-1}}\alpha, \gamma)=(\mu^{1-\theta^{-1}})^{1+\theta} m(\alpha, \gamma)$ and $\ell(\mu^{1-\theta^{-1}}\alpha, \gamma)=(\mu^{1-\theta^{-1}})^{\theta+1}\,\ell(\alpha, \gamma)$. Thus $m(\alpha, \gamma)=\alpha^{\theta+1}\gamma^{\theta+1} m(1,1)$ and $\ell(\alpha, \gamma)=\alpha^{\theta+1}\gamma^{\theta+2}\,\ell(1,1)$. Transforming (10.23) by s_1 implies that

$$\ell(\alpha, \gamma)=n(\gamma, \alpha)+\alpha^{\theta+1}\gamma^{\theta+2}.$$

Thus $n(\alpha, \gamma)=\alpha^{\theta+2}\gamma^{\theta+1} n(1,1)$ and $\ell(1,1)+n(1,1)=1$. The commutator identity on $[x, yz]$ used for $yz=u_2(\alpha, 0)\,u_2(\alpha', 0)$ and for

$$yz=u_2(\alpha+\alpha', 0)\,u_2(0, \alpha\alpha'^{\theta})$$

implies that

$$m(\alpha+\alpha', \gamma)=m(\alpha, \gamma)+m(\alpha', \gamma),$$

$$\ell(\alpha+\alpha', \gamma)=\ell(\alpha, \gamma)+\ell(\alpha', \gamma).$$

If we choose $\gamma=1$, we find that $(\alpha\alpha'^{\theta}+\alpha^{\theta}\alpha')\,m(1,1)=0$ and

$$(\alpha\alpha'^{\theta}+\alpha'\alpha^{\theta})\,\ell(1,1)=0.$$

If $q>2$, then $m(1,1)=\ell(1,1)=0$ and $n(1,1)=1$. If $q=2$, we can use the identity $[x^2, y]=[x, y]^x\,[x, y]$ with $x=u_2^{s_1}(1,0)$, $y=u_2(1,0)$ to show that $m(1,1)=0$. Thus for $q\geqq 2$, we have

$$m(\alpha, \gamma)=0. \tag{10.25}$$

The values of $\ell(1,1)$ and $n(1,1)$ in case $q=2$ will not be determined at this point. We have

$$\begin{array}{lll} \ell(\alpha, \gamma)=0, & n(\alpha, \gamma)=\alpha^{\theta+2}\gamma^{\theta+1} & \text{if } q>2 \\ \ell(1,1)=\varepsilon, & n(1,1)=1+\varepsilon & \text{if } q=2 \end{array} \tag{10.26}$$

where $\varepsilon=0$ or 1. The multiplication table of U_{s_1} is now complete aside from the ε in case $q=2$.

We now consider the commutator $[u_1(\gamma), u_2(\alpha, \beta)]$. We may rewrite (10.7)(iii) in the form

$$[u_1(\gamma), u_2(\alpha, \beta)]\equiv \bar{u}_2^{s_1}(\alpha\gamma)\,\bar{u}_2^{s_1 s_2}(\beta\gamma)\,u_1^{s_2}((\alpha^{2+\theta}+\alpha\beta+\beta^{\theta})\gamma)\,\bar{u}_2^{s_1 s_2 s_1}(e(\alpha, \beta, \gamma)) \pmod{[U_{s_2}, U_{s_2}]}. \tag{10.27}$$

The commutator identity on $[x, yz]$ implies that

$$e(\alpha+\alpha', \alpha\alpha'^{\theta}+\beta+\beta', \gamma)=e(\alpha, \beta, \gamma)+e(\alpha', \beta', \gamma)+\alpha'\alpha^{\theta}\gamma^{\theta}. \tag{10.28}$$

In particular, $e(\alpha, \beta, \gamma)=e(\alpha, 0, \gamma)+e(0, \beta, \gamma)$. Transforming (10.27) by $h_1(\lambda)h_2(\mu)$ gives $e(\lambda\mu\alpha, \lambda^{1+\theta}\mu^{1+\theta}\beta, \lambda^{-2}\mu^{-1-\theta^{-1}}\gamma)=\lambda^{1-\theta} e(\alpha, \beta, \gamma)$. Setting $\mu=\lambda^{-1}$, we obtain $e(\alpha, \beta, \lambda^{-1+\theta^{-1}}\gamma)=(\lambda^{-1+\theta^{-1}})^{\theta} e(\alpha, \beta, \gamma)$. Setting

$\mu=\lambda^{2\theta-4}$ we obtain $e(\lambda^{2\theta-3}\alpha, 0, \gamma)=(\lambda^{2\theta-3})^{\theta+1} e(\alpha, 0, \gamma)$ and

$$e(0, \lambda^{1-\theta}\beta, \gamma)=\lambda^{1-\theta} e(0, \beta, \gamma).$$

Thus $e(\alpha, 0, \gamma)=\alpha^{\theta+1}\gamma^{\theta} e(1, 0, 1)$ and $e(0, \beta, \gamma)=\beta\gamma^{\theta} e(0, 1, 1)$. If we choose $\alpha=\alpha'=\beta=\beta'=\gamma=1$ in (10.28), we find that $e(0, 1, 1)=1$. If we choose $\beta=\beta'=0$, $\gamma=1$ in (10.28), we find that $(\alpha\alpha'^{\theta}+\alpha^{\theta}\alpha') e(1, 0, 1)=\alpha\alpha'^{\theta}+\alpha^{\theta}\alpha'$. If $q>2$, this implies that $e(1, 0, 1)=1$. Thus

$$e(\alpha, \beta, \gamma)=\alpha^{\theta+1}\gamma^{\theta}+\beta\gamma^{\theta}. \tag{10.29}$$

If $q=2$, we can show (10.29) remains valid as follows: R_2 acts on the elementary 2-group spanned by $\{\bar{u}_2^{s_1 s_2 s_1}(1), u_1(1), u_1^{s_2}(1), \bar{u}_2^{s_1}(1), \bar{u}_2^{s_1 s_2}(1)\}$ (mod $[U_{s_2}, U_{s_2}]$). Relative to this basis, s_2 and $u_2(1, 0)$ are represented by the matrices of (10.17), where $x=1$ if and only if (10.29) holds. But we showed that $x=1$ in (10.17) and thus (10.29) holds for $q=2$.

We may now rewrite (10.27) in the form

$$\begin{aligned}
[u_1(\gamma), u_2(\alpha, \beta)] = {} & u_2^{s_1}(\alpha\gamma, r(\alpha, \beta, \gamma))\, u_1^{s_2 s_1}(s(\alpha, \beta, \gamma)) \\
& \cdot u_2^{s_1 s_2 s_1}(\alpha^{\theta+1}\gamma^{\theta}+\beta\gamma^{\theta}, t(\alpha, \beta, \gamma))\, u_1^{s_2 s_1 s_2}(v(\alpha, \beta, \gamma)) \qquad (10.30) \\
& \cdot u_2^{s_1 s_2}(\beta\gamma, w(\alpha, \beta, \gamma))\, u_1^{s_2}((\alpha^{2+\theta}+\alpha\beta+\beta^{\theta})\gamma)
\end{aligned}$$

where r, s, t, v, w are functions from $\mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q \to \mathbf{F}_q$. The following table gives the action of $h_1(\lambda) h_2(\lambda^{-1})$ and $h_1(\lambda) h_2(\lambda^{2\theta-4})$ on the relevant subgroups.

	$\bar{U}_2$	$\Omega_1(U_2)$	U_1	r	s	t	v	w
$h_1(\lambda) h_2(\lambda^{-1})$	1	1	$\lambda^{\theta^{-1}-1}$	$\lambda^{-\theta^{-1}}$	$\lambda^{-\theta^{-1}}$	λ^{-1}	$\lambda^{-\theta^{-1}}$	$\lambda^{-\theta^{-1}}$
$h_1(\lambda) h_2(\lambda^{2\theta-4})$	$\lambda^{2\theta-3}$	$\lambda^{1-\theta}$	1	$\lambda^{1-\theta}$	$\lambda^{\theta-2}$	λ^{-1}	$\lambda^{2-2\theta}$	λ^{-1}

Since $(\theta^{-1}-1)(\theta+1)=-\theta^{-1}$, $(\theta^{-1}-1)(\theta+2)=-1$, and

$$(2\theta-3)(\theta+1)=1-\theta, \quad (2\theta-3)(\theta+2)=\theta-2,$$

$(2\theta-3)(2\theta+3)=-1$, and $(1-\theta)\theta=\theta-2$, $(1-\theta)(\theta+1)=-1$, we find that

$$\begin{aligned}
& r(\alpha, 0, \gamma)=(\alpha\gamma)^{\theta+1} r(1, 0, 1), && r(0, \beta, \gamma)=\beta\gamma^{\theta+1} r(0, 1, 1) \\
& s(\alpha, 0, \gamma)=\alpha^{\theta+2}\gamma^{\theta+1} s(1, 0, 1), && s(0, \beta, \gamma)=\beta^{\theta}\gamma^{\theta+1} s(0, 1, 1) \\
& t(\alpha, 0, \gamma)=\alpha^{2\theta+3}\gamma^{\theta+2} t(1, 0, 1), && t(0, \beta, \gamma)=\beta^{\theta+1}\gamma^{\theta+2} t(0, 1, 1) \qquad (10.31) \\
& v(\alpha, 0, \gamma)=\alpha^{2\theta+2}\gamma^{\theta+1} v(1, 0, 1), && v(0, \beta, \gamma)=\beta^{2}\gamma^{\theta+1} v(0, 1, 1) \\
& w(\alpha, 0, \gamma)=\alpha^{2\theta+3}\gamma^{\theta+1} w(1, 0, 1), && w(0, \beta, \gamma)=\beta^{\theta+1}\gamma^{\theta+1} w(0, 1, 1).
\end{aligned}$$

We shall determine these functions in the order r, s, v, w, t.

The commutator identity on $[x, yz]$ with

$$yz = u_2(\alpha, \beta)\, u_2(\alpha', \beta') = u_2(\alpha+\alpha', \alpha\alpha'^{\theta} + \beta + \beta')$$

gives

$$r(\alpha+\alpha', \alpha\alpha'^{\theta} + \beta + \beta', \gamma) = r(\alpha, \beta, \gamma) + r(\alpha', \beta', \gamma) + \alpha'\alpha^{\theta}\gamma^{1+\theta}. \quad (10.32)$$

In particular, $r(\alpha, \beta, \gamma) = r(\alpha, 0, \gamma) + r(0, \beta, \gamma)$. Choosing $1 = \alpha = \alpha' = \beta = \beta' = \gamma$ in (10.32) we find that $r(0, 1, 1) = 1$. Choosing $\beta = \beta' = 0$, $\gamma = 1$ in (10.32) we find that $(\alpha\alpha'^{\theta} + \alpha^{\theta}\alpha')\, r(1, 0, 1) = \alpha\alpha'^{\theta} + \alpha^{\theta}\alpha'$. If $q > 2$, this implies that $r(1, 0, 1) = 1$, and thus

$$r(\alpha, \beta, \gamma) = (\alpha^{\theta+1} + \beta)\, \gamma^{\theta+1}. \quad (10.33)$$

We can show that (10.33) remains valid for $q = 2$ as follows. $s_1 u_1(1)$ is an element of order 3, and $(s_1 u_1(1))^2 \colon u_2(1, 0) \to u_2^{s_1}(1, x+1)\, u_2(1, 0) \pmod{T_1}$, where $x = 1$ if and only if (10.33) holds. On the other hand,

$$u_1(1)\, s_1 \colon u_2(1, 0) \to u_2^{s_1}(1, 0)\, u_2(1, x+1) \pmod{T_1}.$$

Thus $x = 1$ and (10.33) holds for $q = 2$.

The commutator identity on $[x, yz]$ shows that

$$\begin{aligned} &s(\alpha+\alpha', \alpha\alpha'^{\theta} + \beta + \beta', \gamma) \\ &\quad = s(\alpha, \beta, \gamma) + s(\alpha', \beta', \gamma) + (\alpha\alpha'^{\theta+1} + \alpha\beta' + \alpha'\beta)\, \gamma^{\theta+1}. \end{aligned} \quad (10.34)$$

In particular, $s(\alpha, \beta, \gamma) = s(\alpha, 0, \gamma) + s(0, \beta, \gamma) + \alpha\beta\gamma^{\theta+1}$. Choosing

$$\alpha = \alpha' = \beta = \beta' = \gamma = 1$$

in (10.34), we find that $s(0, 1, 1) = 1$. Choosing $\beta = \beta' = 0$, $\gamma = 1$ in (10.34), we find that $(\alpha^{\theta}\alpha'^2 + \alpha^2\alpha'^{\theta})\, s(1, 0, 1) = \alpha^{\theta}\alpha'^2 + \alpha^2\alpha'^{\theta}$. If $q > 2$, this implies that $s(1, 0, 1) = 1$ and thus

$$s(\alpha, \beta, \gamma) = (\alpha^{\theta+2} + \beta^{\theta} + \alpha\beta)\, \gamma^{\theta+1}. \quad (10.35)$$

We can show (10.35) holds for $q = 2$ exactly as before. Indeed

$$\begin{aligned} (s_1 u_1(1))^2 \colon u_2(1, 0) &\to u_2^{s_1}(1, 0)\, u_1^{s_2 s_1}(x)\, u_2^{s_1 s_2 s_1}(0, t(1, 1, 1)) \\ &\quad \cdot u_1^{s_2 s_1 s_2}(v(1, 1, 1))\, u_2^{s_1 s_2}(1, w(1, 1, 1))\, u_1^{s_2}(1)\, u_2(1, 0), \end{aligned}$$

where $x = 1$ if and only if (10.35) holds. On the other hand,

$$\begin{aligned} u_1(1)\, s_1 \colon u_2(1,0) &\to u_2(1, 0)\, u_1^{s_2}(x)\, u_2^{s_1 s_2}(0, t(1, 1, 1))\, u_1^{s_2 s_1 s_2}(v(1, 1, 1)) \\ &\quad \cdot u_2^{s_1 s_2 s_1}(1, w(1, 1, 1))\, u_1^{s_2 s_1}(1)\, u_2^{s_1}(1, 0). \end{aligned}$$

After rearranging the elements in the last expansion in proper order, we find that $x = 1$. Thus (10.35) holds for $q = 2$.

We shall not compute the functions v, w, t explicitly when $q>2$. It will suffice to show they are uniquely determined. The commutator identity on $[x, yz]$ implies relations of the form

$$\begin{aligned} v(\alpha+\alpha', \alpha\alpha'^{\theta}+\beta+\beta', \gamma) &= v(\alpha, \beta, \gamma)+v(\alpha', \beta', \gamma)+v_0(\alpha, \alpha', \beta, \beta', \gamma) \\ w(\alpha+\alpha', \alpha\alpha'^{\theta}+\beta+\beta', \gamma) &= w(\alpha, \beta, \gamma)+w(\alpha', \beta', \gamma)+w_0(\alpha, \alpha', \beta, \beta', \gamma) \\ t(\alpha+\alpha', \alpha\alpha'^{\theta}+\beta+\beta', \gamma) &= t(\alpha, \beta, \gamma)+t(\alpha', \beta', \gamma)+t_0(\alpha, \alpha', \beta, \beta', \gamma). \end{aligned} \tag{10.36}$$

Here v_0 is an explicit function computable from the commutator identities; w_0 can be computed once v is known; and t_0 can be computed once v and w are known. In particular,

$$v(\alpha, \beta, \gamma)=v(\alpha, 0, \gamma)+v(0, \beta, \gamma)+v_0(\alpha, 0, 0, \beta, \gamma).$$

Choosing $\alpha=\alpha'=\beta=\beta'=\gamma=1$ in (10.36), we see that $v(0, 1, 1)$ is determined. Choosing $\beta=\beta'=0$, $\gamma=1$ in (10.36) we find an equation for $v(1, 0, 1)$ with a coefficient of the form $(\alpha^{2\theta}\alpha'^{2}+\alpha^{2}\alpha'^{2\theta})$. If $q>2$, we can solve for $v(1, 0, 1)$ and thus determine $v(\alpha, \beta, \gamma)$. The arguments for w and t are similar. The relevant coefficients in these cases are of the form $\alpha^3\alpha'^{2\theta}+\alpha^{2\theta}\alpha'^{3}$. If $q>2$, we can determine $w(\alpha, \beta, \gamma)$ and $t(\alpha, \beta, \gamma)$. Thus U has a uniquely determined multiplication table for $q>2$.

Suppose $q=2$. For notational convenience we set

$$\begin{aligned} &x_1=u_1(1), && x_3=u_1^{s_2 s_1}(1), && x_5=u_1^{s_2 s_1 s_2}(1), && x_7=u_1^{s_2}(1), \\ &x_2=u_2^{s_1}(1, 0), && x_4=u_2^{s_1 s_2 s_1}(1, 0), && x_6=u_2^{s_1 s_2}(1, 0), && x_8=u_2(1, 0). \end{aligned} \tag{10.37}$$

It remains to determine $[x_1, x_8]$ and ε, since $[x_1, x_8^2]$ can be determined by the identity $[x, y^2]=[x, y][x, y]^y$. Thus far we have the identity

$$[x_1, x_8]=x_2^{-1}x_3x_4^{1+2\rho}x_5^{\sigma}x_6^{2\tau}x_7, \tag{10.38}$$

where ρ, σ, τ are 0 or 1. We proceed as follows: s_1x_1 is an element of order 3. It can be checked that

$$\begin{aligned} (s_1x_1)^2&: \; x_8^{-1}\to x_2^{-1}x_3x_4^{1+2\rho}x_5^{\sigma}x_6^{2\tau}x_7x_8^{-1}, \\ (x_1s_1)&: \; x_8^{-1}\to x_2^{-1}x_3x_4^{1+2\tau+2\varepsilon+2\sigma}x_5^{\sigma}x_6^{2\rho+2\sigma+2\varepsilon}x_7x_8^{-1}. \end{aligned}$$

Thus

$$\rho+\sigma+\tau+\varepsilon\equiv 0 \pmod 2). \tag{10.39}$$

x_1 has order 2. It can be checked that the following holds:

$$x_1^2: \; x_8^{-1}\to x_4^{2\sigma+2\tau}x_8^{-1}.$$

Thus by (10.39) we have

$$\sigma=\tau, \qquad \rho=\varepsilon. \tag{10.40}$$

Finally $s_2 x_8$ has order 4. It can be checked that if $\sigma=\tau=0$, then $(s_2 x_8)^4$: $x_2 x_6 \to x_2^{-1} x_3^\rho x_4^{2\rho} x_5^\rho x_6$, which is impossible. If $\sigma=\tau=1$, then

$$(s_2 x_8)^4\colon\; x_2 x_6 \to x_2 x_3^\rho x_4^{2\rho} x_6.$$

Thus

$$\sigma=\tau=1, \quad \rho=\varepsilon=0, \tag{10.41}$$

and the multiplication table of U is completely determined.

We can now complete the proof of (10A). Let $h_2^{s_1}(\beta)=h_1(x)h_2(y)$, where $x, y\in \mathbf{F}_q$. Using (10.10) and (10.12) to evaluate both sides of this expression on $\bar{U}_2$ and $\Omega_1(U_2^{s_1 s_2 s_1})$, we find that $\beta^{-\theta^{-1}}=xy$, $\beta^{1+\theta^{-1}}=x^{-1}$. Thus

$$h_2^{s_1}(\beta)=h_1(\beta^{-1-\theta^{-1}})\,h_2(\beta). \tag{10.42}$$

Now let $h_1^{s_2}(\alpha)=h_1(x)h_2(y)$, where $x, y\in\mathbf{F}_q$. Evaluating both sides of the expression on $U_1^{s_2 s_1 s_2}$ and $\bar{U}_2^{s_1}$, we find that $\alpha^{-\theta}=y^{\theta^{-1}}$ and $\alpha^{\theta-1}=x^{-1}y^{-\theta^{-1}}$. Thus

$$h_1^{s_2}(\alpha)=h_1(\alpha)\,h_2(\alpha^{-2}). \tag{10.43}$$

We label the root subgroups by

$$U_1,\ U_2,\ U_1^{s_2},\ U_2^{s_1},\ U_1^{s_2 s_1},\ U_2^{s_1 s_2},\ U_1^{s_2 s_1 s_2},\ U_2^{s_1 s_2 s_1},$$

$$\bar{U}_1^{s_1},\ \bar{U}_2^{s_2},\ \bar{U}_1^{s_1 s_2},\ \bar{U}_2^{s_2 s_1},\ \bar{U}_1^{s_1 s_2 s_1},\ \bar{U}_2^{s_2 s_1 s_2},\ \bar{U}_1^{s_1 s_2 s_1 s_2},\ \bar{U}_2^{s_2 s_1 s_2 s_1}.$$

Since $[s_1, R_1^{s_2 s_1 s_2}]=[s_2, R_2^{s_1 s_2 s_1}]=1$, it follows that the action of s_1 and s_2 on these root subgroups is completely determined. Moreover, $C((s_1 s_2)^8) \geqq \langle U_1, U_2\rangle$. Using (10.10) we see that $(s_1 s_2)^8=1$. It now follows from the (B, N)-properties that G has a unique multiplication table. Since ${}^2F_4(q)$ satisfies (10A), we have that $G\cong {}^2F_4(q)$.

§ 11. Some Consequences

We prove two results in this section. The first of these is the following extension of Theorem A to a group with a Tits system of rank $n\geqq 2$.

Theorem C. *Let G be a finite group with a Tits system (B, N) of rank $n\geqq 2$ satisfying $(*)$. Let $G_0=U^G$, $B_0=B\cap G_0$, and $Z=\bigcap_{g\in G_0} B_0^g$. If the Weyl group $W=N/H$ of G is indecomposable, then G_0/Z is isomorphic to a Chevalley group of normal or twisted type. In particular, if G is simple, then G is isomorphic to a Chevalley group of normal or twisted type.*

We first establish several lemmas. We may assume that $n\geqq 3$ by Theorem A, and that $G=G_0$, $Z=1$, $U=F(B)$, and (B, N) is saturated by (1A), (1B), (1C), and (1D). It should be noted at this point that the Weyl group of a Tits system is unchanged by the successive reductions (1A), (1B), (1C), (1D). Moreover, if $K=\bigcap_{g\in G} B^g$, then it easily follows that

$K \cap G_0 = Z$ so that $G_0 K/K \simeq G_0/Z$. Set $P_{ij} = \langle B, s_i, s_j \rangle$, $K_{ij} = \bigcap_{g \in P_{ij}} B^g$, and $L_{ij} = U^{P_{ij}} K_{ij}/K_{ij}$ for $i \neq j$, $i, j \in \{1, \ldots, n\}$. Then $(B, \langle H, s_i, s_j \rangle)$ is a Tits system of P_{ij} of rank 2. This system satisfies $(*)$ and has Weyl group isomorphic to $W_{ij} = \langle s_i, s_j \rangle H/H$. The structure of L_{ij} is then known by Theorem A.

(11 A) *$W = N/H$ is isomorphic to a finite reflection group corresponding to a root system Δ of type A_n, B_n, D_n, E_6, E_7, E_8, or F_4.*

Proof. This is immediate from [3].

(11 B) *Let $U_w = U \cap B_w$, $U_w^- = U \cap B_w^-$, and $U_i = U \cap B_i$. The following hold:*

(a) *There is a prime p such that $U = O_p(B)$.*

(b) *If W_{ij} is non-abelian and $w \in W_{ij}$, then*

$$UK_{ij} \cap U^w K_{ij} = (U \cap U^w) K_{ij}.$$

(c) $U_{s_i} \trianglelefteq U$, $U = U_{s_i} U_i$, $U_{s_i} \cap U_i = U \cap H = 1$ *for* $i = 1, \ldots, n$.

Proof. (11 A) implies that any two distinct integers from $\{1, \ldots, n\}$ are connected by a sequence of integers from $\{1, \ldots, n\}$ such that W_{ij} is non-abelian whenever i, j are consecutive terms of the sequence. Applying Theorem A to L_{ij}, we see that UK_{ij}/K_{ij} is a p-group for some prime p depending on i, j. Since $U_i \cap K_{ij} \leqq B_i \cap B_{s_i} = U \cap H$ and $U_j \cap K_{ij} \leqq B_j \cap B_{s_j} \leqq U \cap H$, it follows that $|U_i : U \cap H|$ and $|U_j : U \cap H|$ are powers of p. Hence there exists a prime p such that $|U_i : U \cap H|$ is a power of p for $1 \leqq i \leqq n$. By (1 E) there is a sequence $i_1, i_2, \ldots, i_t$ of integers from $\{1, \ldots, n\}$ such that

$$B = B_{i_1} B_{i_2}^{s_{i_1}} B_{i_3}^{s_{i_2} s_{i_1}} \ldots = H U_{i_1} U_{i_2}^{s_{i_1}} U_{i_3}^{s_{i_2} s_{i_1}} \ldots.$$

Thus $|B : H| = \prod_{j=1}^{t} |B_{i_j} : H|$ so that $|U : U \cap H| = \prod_{j=1}^{t} |U_{i_j} : U \cap H|$ and $|U : U \cap H|$ is a power of p. In particular, $O_{p'}(U) \leqq H$. Let $g \in G$ and express $g^{-1} = bnb'$, where $b, b' \in B$ and $n \in N$. Then $O_{p'}(U)^{g^{-1}} = O_{p'}(U)^{bnb'} = O_{p'}(U)^{nb'} \leqq H^{nb'} \leqq H^{b'} \leqq B$ and so $O_{p'}(U) \leqq B^g$. Thus $O_{p'}(U) \leqq \bigcap_{g \in G} B^g = 1$. Since $U = F(B)$, (a) then holds.

Let $w \in W_{ij}$. We have $UK_{ij} \cap U^w K_{ij} \leqq B \cap B^w = B_w = U_w H = (U \cap U^w) H$ by (3 A). Since $U \cap U^w \leqq UK_{ij} \cap U^w K_{ij}$, it follows that $UK_{ij} \cap U^w K_{ij} = (U \cap U^w)(H \cap UK_{ij} \cap U^w K_{ij})$. The structure of L_{ij} implies that $H \cap UK_{ij} \leqq K_{ij}$ and (b) follows.

(3 A)(c) implies $U = U_{s_i} U_i$ and $U_{s_i} \cap U_i = U \cap H$ for $i = 1, \ldots, n$. Also $U \cap H \leqq K_{ij}$ since $UK_{ij} \cap HK_{ij} = K_{ij}$ by the structure of L_{ij}. But $U \cap K_{ij} = O_p(K_{ij})$ and $U \cap K_{ij} \leqq U \cap U^{s_i} = U_{s_i}$. Thus $U \cap H = U_i \cap K_{ij}$ so that

$U \cap H \trianglelefteq U_i$ for $i=1, \ldots, n$. But $U \cap H = O_p(H) \trianglelefteq N$ as well. Thus

$$N(U \cap H) \geqq \langle U, N \rangle = G \quad \text{and} \quad U \cap H = 1.$$

In particular, $U_{s_i} = U \cap U_{s_i} K_{ij}$. Since $U_{s_i} K_{ij} = (U \cap U^{s_i}) K_{ij} \trianglelefteq U K_{ij}$ by (b) and the structure of L_{ij}, we then have $U_{s_i} = U \cap U_{s_i} K_{ij} \trianglelefteq U$ and (c) holds. This completes the proof of (11 B).

Let $i, j \in \{1, \ldots, n\}$, $i \neq j$, let Δ_{ij} be the set of all roots in Δ which are linear combinations of r_i and r_j, and let $U_{ij} = \langle U_r : r \in \Delta_{ij} \text{ and } r > 0 \rangle$. If W_{ij} is non-abelian, let $w_{ij} H$ be an element which as a word in $s_i H$ and $s_j H$ has greatest length.

(11 C) *If W_{ij} is non-abelian, then the following hold:*

(a) $U_{w_{ij}} = U \cap K_{ij} \trianglelefteq U$

(b) *U_{ij} is a subgroup of U, $U_{ij} U_{w_{ij}} = U$, and $U_{ij} \cap U_{w_{ij}} = 1$.*

(c) $U \cap U^{w_{ij} s_i} = U_i U_{w_{ij}}$.

Proof. The structure of L_{ij} implies that $U_{w_{ij}} \leqq K_{ij}$, and since $U \cap K_{ij} = O_p(K_{ij}) \leqq U \cap U^w$ for $w \in W_{ij}$, it follows that $U_{w_{ij}} = U \cap K_{ij}$ and (a) then holds. (b) holds since $U = U_{w_{ij}} U^-_{w_{ij}}$, $U_{w_{ij}} \cap U^-_{w_{ij}} = 1$, and $U^-_{w_{ij}} = U_{ij}$ by (3 C)(a). Finally the structure of L_{ij} implies that $U K_{ij} = U_{s_i} (U \cap U^{w_{ij} s_i}) K_{ij}$ and $U_{s_i} K_{ij} \cap (U \cap U^{w_{ij} s_i}) K_{ij}$. Since $U = U_{s_i} U_i$ by (11 B)(c) and $U_i \leqq U \cap U^{w_{ij} s_i}$ by (3 C)(a), we then have $U \cap U^{w_{ij} s_i} = U_i (U \cap U^{w_{ij} s_i} \cap U_{s_i} K_{ij}) \leqq U_i (U \cap K_{ij})$. But $U \cap K_{ij} = O_p(K_{ij})$ is contained in $U \cap U^{w_{ij} s_i}$. Thus (c) holds.

(11 D) *If W_{ij} is abelian, then $[U_i, U_j] = 1$.*

Proof. Since $U_i \leqq U_{s_j} \trianglelefteq U$ and $U_j \leqq U_{s_i} \trianglelefteq U$, it follows that $[U_i, U_j] \leqq U_{s_i} \cap U_{s_j} \cap U_{ij}$. But W_{ij} is abelian and so $U_{ij} = U_i U_j$ by (3 c)(a). Thus $[U_i, U_j] \leqq U_{s_i} \cap U_{s_j} \cap U_i U_j = 1$.

We note that (11 C) and (11 D) imply the following: Let r and s be roots in Δ_{ij}. If W_{ij} is non-abelian, then the commutator relation $[U_r, U_s]$ is determined by $[U_r, U_s] \pmod{K_{ij}}$ in L_{ij}. If W_{ij} is abelian, then $[U_r, U_s] = 1$.

(11 E) *There exists a power q of p and an ordering of the fundamental reflections $s_1, \ldots, s_n$ such that one of the following holds:*

(a) *U_i is elementary abelian of order q for $1 \leqq i \leqq n$.*

(b) *U_1 is elementary abelian of order q^2 and U_i is elementary abelian of order q for $i > 1$.*

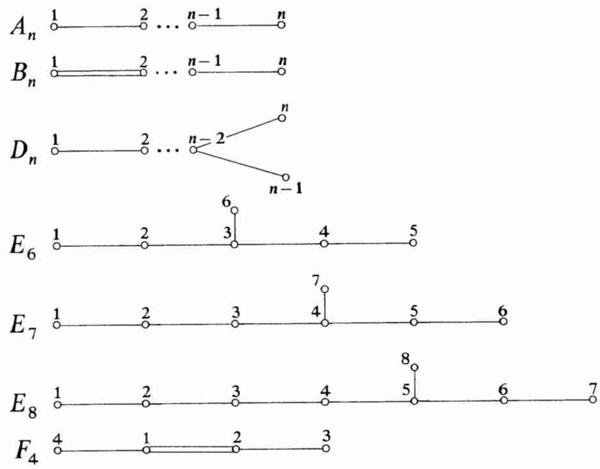
(c) *U_1 is elementary abelian of order $\sqrt{q}$ and U_i is elementary abelian of order q for $i > 1$.*

(d) *U_1 is non-abelian of order $\sqrt{q}^3$ and U_i is elementary of order q for $i > 1$.*

(e) *U_1, U_4 are elementary abelian of order q, and U_2, U_3 are elementary abelian of order q^2.*

In cases (b), (c), (d) *the Weyl group W is of type B_n and in case* (e) *W is of type F_4.*

Proof. By (11 A) W is of type A_n, B_n, D_n, E_6, E_7, E_8, or F_4. We order the reflections as in the diagrams below



Note that given $i>1$ there exists a unique $j<i$ such that W_{ij} is non-abelian. Moreover, such a W_{ij} is dihedral of order 6 except in types B_n and F_4 where W_{12} is dihedral of order 8.

Suppose W is of type A_n, D_n, E_6, E_7, or E_8. If W_{ij} is non-abelian, then Theorem A implies $L_{ij} \cong PSL(3, q)$ for some power q of p, so in particular, U_i and U_j are elementary abelian of order q. Since any two distinct integers in $\{1, \dots, n\}$ are connected by a sequence of integers $i_1, i_2, \dots, i_k$ from $\{1, \dots, n\}$ such that $W_{i_j i_{j+1}}$ is non-abelian for $1 \leqq j \leqq k-1$, we see that (a) holds.

Suppose W is of type B_n. The above argument shows there is a power q of p such that U_i is elementary abelian of order q for $i>1$. By Theorem A $L_{12} \cong PSp(4, q_0)$, $PSU(4, q_0)$, or $PSU(5, q_0)$ for some power q_0 of p. Since $|U_2| = q$, we see that (a) holds if $L_{12} \cong PSp(4, q_0)$, (b) or (c) hold if $L_{12} \cong PSU(4, q_0)$, and (d) holds if $L_{12} \cong PSU(5, q_0)$.

Finally, suppose W is of type F_4. As above there exist powers q' and q'' of p such that U_1, U_4 are elementary abelian of order q', and U_2, U_3 are elementary abelian of order q''. By Theorem A $L_{12} \cong PSp(4, q_0)$ or $PSU(4, q_0)$ for some power q_0 of p, and so (a) or (e) holds. This completes the proof of (11 E).

The following are consequences of (2 D) and the proof of (11 E). If W_{ij} is non-abelian, then $L_{ij} \cong PSL(3, q)$, $PSp(4, q)$, or $PSU(5, q)$ for some power q of p, the first case occurring whenever $\{i, j\} \neq \{1, 2\}$. If W_{ij} is abelian, then $L_{ij} \cong R_i K_{ij}/K_{ij} \times R_j K_{ij}/K_{ij}$, where $R_i K_{ij}/K_{ij} \cong PSL(2, q_i)$ or $PSU(3, q_i)$, $R_j K_{ij}/K_{ij} \cong PSL(2, q_j)$ or $PSU(3, q_j)$, and q_i, q_j are powers of p. Here $R_i = \langle U_i, U_i^{s_i} \rangle$.

Each of the cases of (11F) corresponds to a Chevalley group $\bar{G}$. Indeed, if (a) holds, then W is of type A_n, B_n, D_n, E_6, E_7, E_8, or F_4. We take respectively the Chevalley group of normal type $A_n(q)$, $B_n(q)$ or $C_n(q)$, $D_n(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, or $F_4(q)$. The choice of $B_n(q)$ or $C_n(q)$ for $\bar{G}$ when W is of type B_n is decided as follows: for $p=2$ $B_n(q) \cong C_n(q)$; for $p \neq 2$ we choose $B_n(q)$ or $C_n(q)$ according as $[U_2, U_2^{s_1}] = 1$ or $[U_1, U_1^{s_2}] = 1$. Since $L_{12} \cong PSp(4, q)$ and $n \geqq 3$, one and only one of these commutators is trivial so that the choice is unambiguous. If (b), (c), (d), or (e) holds, we take respectively the Chevalley group of twisted type ${}^2D_{n+1}(q^2)$, ${}^2A_{2n-1}(q^2)$, ${}^2A_{2n}(q^2)$, or ${}^2E_6(q^2)$.

The Chevalley group $\bar{G}$ has a Tits system of rank n satisfying (*). Applying the previous notation and results to $\bar{G}$, we may define subgroups $\bar{U}_i$, $\bar{K}_{ij}$ and the section $\bar{L}_{ij}$. We assume the root systems used in the indexing of G and $\bar{G}$ are the same. By the choice of $\bar{G}$ there exists for each unordered pair $i, j \in \{1, \dots, n\}$, $i \neq j$, an isomorphism θ^{ij} of $\bar{L}_{ij}$ onto L_{ij} mapping $\bar{U}_r \bar{K}_{ij}/\bar{K}_{ij}$ onto $U_r K_{ij}/K_{ij}$ for all $r \in \Delta_{ij}$. Since $\bar{U}_r \cong \bar{U}_r \bar{K}_{ij}/\bar{K}_{ij}$ and $U_r \cong U_r K_{ij}/K_{ij}$, θ^{ij} then induces an isomorphism θ_r^{ij} of $\bar{U}_r$ onto U_r.

Now $\Delta_{ij} \cap \Delta_{k\ell} = \emptyset$ if $\{i, j\} \cap \{k, \ell\} = \emptyset$ and $\Delta_{ij} \cap \Delta_{k\ell} = \{\pm r_i\}$ if $\{i, j\} \cap \{k, \ell\} = \{i\}$. We shall show by induction that the isomorphism θ^{ij} can be chosen so that

$$\theta_{r_i}^{ij} = \theta_{r_i}^{ik} \quad \text{and} \quad \theta_{-r_i}^{ij} = \theta_{-r_i}^{ik} \tag{11.1}$$

whenever $i, j, k \in \{1, \dots, n\}$, $i \neq j$, $i \neq k$. Since (11.1) certainly holds for $i, j, k \leqq 2$, we may suppose by induction that the θ^{ij} for $i, j \leqq d-1$ have been chosen so that (11.1) holds for all $i, j, k \leqq d-1$. We shall change the θ^{id} for $1 \leqq i \leqq d-1$ so that (11.1) holds when $i = d$ or $k = d$.

There is a unique positive integer $c < d$ such that W_{cd} is non-abelian. In fact W_{cd} is dihedral of order 6, $\bar{L}_{cd} \simeq L_{cd} \simeq PSL(3, q)$, and $\bar{R}_c \simeq R_c \simeq SL(2, q)$ since $d \geqq 3$. For the same reason there exists a positive integer $b \neq c$, $1 \leqq b \leqq d-1$, such that W_{bc} is non-abelian. By induction (11.1) holds for $i = c$, $k = b$, and $1 \leqq j \leqq d-1$. Hence (11.1) will hold for $i = c$, $k = d$, and $1 \leqq j \leqq d$ if it holds for the special case $i = c$, $j = b$, $k = d$. We shall choose θ^{cd} so that this is so. Isomorphisms φ and $\bar{\varphi}$ in the diagram below may be defined by

$$\bar{\varphi}: \bar{x}\bar{K}_{cd} \to \bar{x}\bar{K}_{bc} \quad \text{and} \quad \varphi: xK_{cd} \to xK_{bc},$$

since $\bar{R}_c \cap \bar{K}_{cd} = \bar{R}_c \cap \bar{K}_{bc} = 1$ and $R_c \cap K_{cd} = R_c \cap K_{bc} = 1$ by the structure of $\bar{L}_{cd} \simeq L_{cd}$ and $\bar{L}_{bc} \simeq L_{bc}$. In particular, we have the following (not necessarily commutative) diagram:

$$\begin{array}{ccc} \bar{R}_c \bar{K}_{bc}/\bar{K}_{bc} & \xrightarrow{\theta^{bc}} & R_c K_{bc}/K_{bc} \\ {\scriptstyle \bar{\varphi}} \Big\uparrow & & \Big\uparrow {\scriptstyle \varphi} \\ \bar{R}_c \bar{K}_{cd}/\bar{K}_{cd} & \xrightarrow{\theta^{cd}} & R_c K_{cd}/K_{cd}. \end{array}$$

The composition $\theta^{cd} \varphi (\theta^{bc})^{-1} \bar{\varphi}^{-1}$ is then an automorphism of $\bar{R}_c \bar{K}_{cd}/\bar{K}_{cd}$ fixing $\bar{U}_c \bar{K}_{cd}/\bar{K}_{cd}$ and $\bar{U}_c^{\bar{s}_c} \bar{K}_{cd}/\bar{K}_{cd}$. The structure of $PSL(3, q)$ implies that such an automorphism extends to an automorphism λ of $\bar{L}_{cd}$, where λ is the product of a diagonal automorphism and a field automorphism. In particular, λ fixes $\bar{U}_r \bar{K}_{cd}/\bar{K}_{cd}$ for all $r \in \Delta_{cd}$. Replacing θ^{cd} by $\lambda^{-1} \theta^{cd}$, we see that (11.1) holds in this special case.

If $i \neq c$ and $1 \leqq i \leqq d-1$, then W_{id} is abelian

$$\bar{L}_{id} \simeq \bar{R}_i \bar{K}_{id}/\bar{K}_{id} \times \bar{R}_d \bar{K}_{id}/\bar{K}_{id},$$

and $\theta^{id} = (\theta^{id})' \times (\theta^{id})''$, where the factors are respectively isomorphisms of $\bar{R}_i \bar{K}_{id}/\bar{K}_{id}$ onto $R_i K_{id}/K_{id}$ and of $\bar{R}_d \bar{K}_{id}/\bar{K}_{id}$ onto $R_d K_{id}/K_{id}$. We may change $(\theta^{id})'$ so that (11.1) holds whenever $i \neq c$, $k = d$. Finally we may change $(\theta^{jd})''$ for $j \neq c$ and $1 \leqq j \leqq d-1$ so that (11.1) holds whenever $i = d$, $k = c$. Then (11.1) holds for all possible cases of $i, j, k \leqq d$.

We can now complete the proof of Theorem C. We first claim that G and $\bar{G}$ satisfy the hypothesis of [7], Theorem 1.4. Indeed, this is immediate once the following has been shown: let s, t be independent roots in Δ, and let $r \in \Delta$ be expressible in the form $r = i' s + j' t$ with $i', j' \geqq 0$. Then

$$U_r \leqq \langle U_{is+jt} \colon i, j \geqq 0 \text{ and } is + jt \neq r \rangle.$$

It suffices to check this in the groups $PSL(3, q)$, $PSp(4, q)$, $PSU(4, q)$, and $PSU(5, q)$. Since the case $PSL(3, q)$ is obvious, it will be enough to show

$$U_1^{s_2} \leqq \langle U_2, U_2^{s_1}, U_1 \rangle \quad \text{and} \quad U_2^{s_1} \leqq \langle U_1, U_1^{s_2}, U_2^{s_1} \rangle \tag{11.2}$$

in the remaining cases. The roots in $PSp(4, q)$ may be labeled so that

$$[u_1^{s_2}(y), u_1(x)] = u_2^{s_1}(2xy),$$
$$[u_2(y), u_1(x)] = u_1^{s_2}(xy)\, u_2^{s_1}(x^2 y).$$

If $p \neq 2$, then (11.2) holds. If $p = 2$, then R_1 and R_2 act non-trivially on $U_{s_1}/U_1^{s_2}$ and $U_{s_2}/U_2^{s_1}$ respectively, so that $[U_1, U_2] \equiv U_2^{s_1} \pmod{U_1^{s_2}}$ and $[U_1, U_2] \equiv U_1^{s_2} \pmod{U_2^{s_1}}$, whence (11.2) holds. In $PSU(5, q)$ (11.2) holds by (6A)(c), (8.15), and (8.33). With suitable labeling in $PSU(4, q)$, we may assume by (6A)(b) that $[U_2, U_2^{s_1}] = U_1^{s_2}$. Moreover, R_1 acts non-trivially on $U_{s_1}/U_1^{s_2}$ so that $[U_1, U_2] \equiv U_2^{s_1} \pmod{U_1^{s_2}}$. Thus (11.2) holds in $PSU(4, q)$.

Let G^* and $\bar{G}^*$ be the central extensions of G and $\bar{G}$ respectively given by [7], Theorem 1.4, and let

$$\varphi: G^* \to G, \quad \bar{\varphi}: \bar{G}^* \to \bar{G}$$

be the corresponding epimorphisms. Since $Z(G)=1$, $Z(\bar{G})=1$, it follows that $\ker\varphi = Z(G^*)$, $\ker\bar{\varphi} = Z(\bar{G}^*)$. Thus $G \cong \bar{G}$ is implied by $G^* \cong \bar{G}^*$. But G^* and $\bar{G}^*$ are generated respectively by subgroups U_r^* and $\bar{U}_r^*$, where $r \in \bigcup_{i,j} \Delta_{ij}$. By the remark following (11 D) the defining relations are the commutator relations on these root subgroups obtained from the L_{ij}. Since the θ^{ij} have been chosen so that these generators and relations are isomorphic, it follows that $G^* \cong \bar{G}^*$. This completes the proof of Theorem C.

The final result is a variant of Theorem A, where the hypothesis that U is nilpotent is replaced by a condition related to the splitting of U, U_{s_1}, and U_{s_2}.

Theorem D. *Let G be a finite group with a saturated Tits system (B, N) of rank 2 such that B has a normal subgroup U satisfying $B = HU$, $H \cap U = 1$, and $U_{s_i} = U \cap U^{s_i} \trianglelefteq U$ for $i = 1, 2$. Let $G_0 = U^G$, $B_0 = B \cap G_0$, and $Z = \bigcap_{g \in G_0} B_0^g$. If the Weyl group $W = N/H$ of G is indecomposable, then G_0/Z is a Chevalley group of normal or twisted type.*

Proof. It will suffice by Theorem A to show that U is nilpotent. We may assume that $G = G_0$, $Z = 1$, and $|W| = 8$, 12, or 16 as well. Let $w \in W$. By (1 E) one of the factorizations

$$B_w^- = B_1 B_2^{s_1} B_1^{s_2 s_1} \ldots, \quad B_w^- = B_2 B_1^{s_2} B_2^{s_1 s_2} \ldots$$

holds. We claim the corresponding factorization

$$U_w^- = U_1 U_2^{s_1} U_1^{s_2 s_1} \ldots, \quad U_w^- = U_2 U_1^{s_2} U_2^{s_1 s_2} \ldots$$

also holds. Indeed, $B_{w'} = HU_{w'}$ for $w' \in W$, do in particular,

$$HU_w^- = HU_1 U_2^{s_1} U_1^{s_2 s_1} \ldots \quad \text{or} \quad HU_w^- = HU_2 U_1^{s_2} U_2^{s_1 s_2} \ldots$$

holds. The claim will follow from $H \cap U = 1$ once we show

$$U_w^- \supseteq U_1 U_2^{s_1} U_1^{s_2 s_1} \ldots \quad \text{or} \quad U_w^- \supseteq U_2 U_1^{s_2} U_2^{s_1 s_2} \ldots.$$

Since $U_w^- = U \cap B_w^-$, this is equivalent to showing $U \supseteq U_1 U_2^{s_1} U_1^{s_2 s_1} \ldots$ or $U \supseteq U_2 U_1^{s_2} U_2^{s_1 s_2} \ldots$. Now $U_2 = U \cap B_2 \leqq U \cap B_{s_1} = U_{s_1} = U \cap U^{s_1}$ and so $U_2^{s_1} \leqq U_{s_1} \leqq U$. Similarly U_1, $U_1^{s_2} \leqq U$. Next, $U_1^{s_2} \leqq U \cap B_1^{s_2} \leqq U \cap B_{s_1} = U_{s_1}$ and so $U_1^{s_2}$, $U_1^{s_2 s_1} \leqq U$. Similarly $U_2^{s_1}$, $U_2^{s_1 s_2} \leqq U$. Continuing in this way, we have the desired inclusions and the claim. Consequently (3 G) applies to G since its proof remains valid. In particular, U_1 and U_2 are subnormal in U. Moreover, we have the factorization $U = U_1 U_2^{s_1} U_1^{s_2 s_1} \dots U_2^{s_1 s_2 \dots s_1 s_2}$. Thus U is nilpotent if we show U_1 and U_2 are p-groups for some prime p.

Now $G_i = HU_i \cup HU_i s_i U_i$ is a subgroup of G by (1G). Moreover, G_i is 2-transitive on the cosets of HU_i, and U_i is transitive on the cosets of HU_i different from HU_i. Since $U_i \cap U_i^{s_i} \leqq U_i \cap B \cap B^{s_i} \leqq U \cap H = 1$, U_i acts regularly on these cosets. By [14] Theorem 1.1 U_i is either a p-group for some prime p or a Frobenius complement. In either case there is a prime p_i such that $O_{p_i}(U_i) \neq 1$ for $i = 1, 2$. Since U_i is subnormal in U, we have $O_{p_i}(U_i) \leqq O_{p_i}(U)$.

Let $p = p_1$ and let C_1 be the kernel of G_1 acting by transformation on $U_{s_1}/O_p(U_{s_1})$. Since $[O_p(U_1), U_{s_1}] \leqq O_p(U) \cap U_{s_1} \leqq O_p(U_{s_1})$, it follows that $O_p(U_1) \leqq C_1$. Let $T_1 = \bigcap_{g \in G_1} B_1^g$. Then G_1/T_1 is faithfully represented as a 2-transitive group on the cosets of B_1 in G_1, and since $U_1 \cap U_1^{s_1} = 1$, we have $T_1 < O_p(U_1) T_1$. Thus $T_1 < C_1 T_1 \trianglelefteq G_1$, $C_1 T_1/T_1$ is a transitive subgroup of G_1/T_1, and $C_1 T_1 B_1 = C_1 B_1 = G_1$. In particular, $C_1 s_1 = C_1 b$ for some $b \in B_1$. Suppose $q \neq p$ is a prime divisor of $|U_2|$. Then

$$1 < U_2^{s_1} O_p(U_{s_1})/O_p(U_{s_1}) \leqq U_{s_1}/O_p(U_{s_1}).$$

Now (3G) implies that b and hence s_1 normalize $U_2^{s_1} O_p(U_{s_1})/O_p(U_{s_1})$. If $X_1 = \prod_r U_r$, where $U_r \leqq U_{s_1}$, $U_r \neq U_2$, $U_r \neq U_2^{s_1}$, then $U_2^{s_1} X_1 \trianglelefteq U_{s_1}$ by (3G). But then $U_2 \leqq U_2^{s_1} X_1 O_p(U_{s_1})$, and in particular, $1 \neq O^p(U_2) \leqq U_2^{s_1} X_1$, which is impossible. Thus p is the unique prime divisor of $|U_2|$. Interchanging the roles of U_1 and U_2, we see that U_1 is also a p-group. This completes the proof of Theorem D.

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Concordance and Bordism of Line Fields*

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Introduction

Let ξ be a line bundle over a smooth, compact n-dimensional manifold M, and assume that an injective vector bundle homomorphism $j\colon \xi|\partial M \to TM|\partial M$ represents ξ as a line field, i.e., a subline bundle of the tangent bundle TM, along the boundary ∂M of M. One purpose of this paper is to establish some sort of "ultimate Poincare-Hopf theorem", that is, to relate the behavior of a (singular) extension $v\colon \xi \to TM$ of j around its isolated zeroes to the global topology of M, ξ and j.

In § 1 we introduce invariants $\theta \in \mathbb{Z}_2$ (for the case when n is odd and ξ is nontrivial) and $\tilde{\theta} \in \mathbb{Z}$ (otherwise), which both depend only on M, ξ and the regular homotopy class of j. For the isolated zeroes of any extension v of j over all of M we can define an index, and the sum of these indices equals θ, resp. $\tilde{\theta}$; also when M is connected, then j admits an extension v without zeroes if and only if θ, resp. $\tilde{\theta}$, vanishes. Furthermore, if n is even (or ξ is trivial), we have an invariant β which measures how far j is from being everywhere tangential to ∂M; and we give an identity (in Proposition 1.3) relating β to the difference between $\tilde{\theta}$ and the Euler number $\chi(M)$. Our methods are rather geometric and do not directly involve standard obstruction theory; however, we indicate the connection with the latter in Remark 1.9.

When we restrict our attention to closed manifolds, the identity mentioned above leads to certain relations among the Stiefel-Whitney numbers of ξ and M (see Remarks 2.3 and 2.9). Using Wu's formulas, we explore such relations somewhat more systematically, and in several cases we can considerably simplify the invariants relevant to existence and concordance questions for vector bundle imbeddings $j\colon \xi \to TM$. As a result, the answers often do not depend fully on M, ξ and j, but only on the bordism class of (M, ξ), or on M or its dimension n. E.g., if M is connected, then ξ occurs as a line field over M if and only if $\tilde{\theta} = \chi(M) = 0$ (when n is even), resp. the Stiefel-Whitney number $\theta = w(\xi)^{-1} w(M)[M]$ vanishes (when it is odd), see Theorem 2.1. As a consequence, if $n \equiv 1$ (4) and $w_1(M)^2 = w_{n-1}(M) = 0$, then every line bundle over M embeds into TM (Proposition 2.16). Also, the orientation bundle $\xi_M = \Lambda^n TM$ of M embeds always when n is odd (Theorem 3.1). Furthermore, for odd n, any

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two embeddings of any fixed line bundle ξ into TM are concordant (Proposition 2.6). For even n and connected M, an embedding j of ξ into TM is concordant to its negative $-j$, if and only if $\phi = w(\xi)^{-2}\, w(M)\,[M]$ vanishes (Proposition 2.8); in particular, j and $-j$ are concordant if M is orientable (Proposition 2.13). Also, for arbitrary M of dimension $n \not\equiv 0\,(4)$, every embedding of the orientation bundle ξ_M into TM is concordant to its negative (Proposition 3.2).

As a side result, we obtain in this context a simple and homogeneous proof of the theorem of Wu and Massey to the effect that $w_{n-1}(M)=0$ if M is orientable and of even dimension (Remark 2.15). More generally, we characterize those polynomials of dimension $n-1$ which vanish for the Stiefel-Whitney classes of all orientable n-manifolds, in terms of the relations among the Stiefel-Whitney numbers of arbitrary $(n-1)$-manifolds (Remark 3.4).

Now we can apply our general treatment of singular "twisted vectorfields" with prescribed boundary behavior to concordance and bordism questions of line fields on closed manifolds M. In § 4 we prove that every such M carries only a finite number $a(M)$ of concordance classes of line fields, and we characterize this number in terms of solutions of certain polynomial equations defined for elements in $H^1(M, \mathbb{Z}_2)$. Using the computations of § 2, we also determine $a(M)$ explicitly when M is orientable and of dimension $n \not\equiv 3\,(4)$.

In § 5 we define and discuss bordism groups of manifolds with q-plane fields. In § 6 and § 7 we study these groups, for $q=1$, by means of the natural forgetful homomorphism f into the bordism groups $\Omega_n(B(S)\,O(1))$ and $\mathfrak{N}_n(B(S)\,O(1))$ of manifolds with arbitrary line bundles (not necessarily sitting in the tangent bundle). f fits into a long exact sequence with third term O or $\mathbb{Z}_2$, and we can determine the bordism groups of line fields completely; (this has already been applied to certain bordism questions concerning foliated manifolds, cf. Remark 6.4.) An interesting phenomenon here is the occurence of 4-torsion (in § 7); to detect it, we have to draw on some crucial computations in § 3.

Finally, in § 8, we sketch the construction of a "singularity homomorphism" which generalizes the homomorphism σ, derived in § 6 from our invariants θ and $\tilde{\theta}$, to the case of (singular) q-plane fields for higher q. The underlying approach has already proved to be quite powerful [9].

In § 8 we also describe B. Reinhart's refined bordism groups within our framework.

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§ 1. Imbedding Line Bundles into the Tangent Bundle

Throughout this paper let M be consistently a compact, smooth, not necessarily orientable manifold of dimension $n>0$. Also, in § 1, M may have a boundary ∂M. Later, from § 2 on, we will assume M to be closed unless otherwise stated.

To any isolated zero z (off ∂M) of a smooth vector field v on M we can assign a well-defined integer, the index. It is the degree of the map

$$\frac{d\phi \circ v \circ \phi^{-1}}{\|d\phi \circ v \circ \phi^{-1}\|}: S^{n-1} \to S^{n-1},$$

where ϕ is any chart from a small neighborhood of z onto $\mathbb{R}^n$ taking z to 0. We have the classical

Poincaré-Hopf Theorem. *Let v be a smooth vector field on M with isolated zeroes, and pointing outward at all boundary points.*

Then the sum of indices at the zeroes of v is equal to the Euler number $\chi(M)$ of M.

For an elegant geometric proof see Milnor [14].

Phrased differently, the theorem deals with zeroes of vector bundle maps from the trivial line bundle into the tangent bundle TM of M.

So we may ask the following more general question.

Question 1.1. Let ξ be an arbitrary line bundle over M, and let j: $\xi|\partial M \to TM|\partial M$ be a smooth bundle imbedding. Clearly there exists a smooth bundle map $v: \xi \to TM$ over all of M, extending j and having only isolated zeroes. How does the behavior of v near its zeroes relate to the global topology of M, ξ and j?

First we assume that n is even. Near a zero z of v we can trivialize ξ and therefore consider v as a vector field. Its index at z does not depend on the trivialization, since the antipodal map on S^{n-1} has mapping degree $+1$. In particular the sum of indices at the zeroes of v is defined without ambiguity.

On the other hand think of j as being split into a tangential component j': $\xi|\partial M \to T(\partial M)$ and a normal component j'': $\xi|\partial M \to \partial M \times \mathbb{R}$. Here the trivial line bundle is imbedded into $TM|\partial M$ by means of a field of outward pointing vectors. Note that j' does not vanish anywhere on the zero set of j'', and therefore j' can be deformed into a bundle map j'_1: $\xi|\partial M \to T(\partial M)$ with only isolated zeroes, and such that j'_1 agrees with j' on a neighborhood of the zero set of j''. Near a zero z of j'_1 trivialize $\xi|\partial M$ by j''. So j'_1 can be interpreted as a vector field on ∂M around z, and its index $i(z)$ at z is well defined.

Now attach a collar $C=\partial M \times [0, 1]$ to M by the identity map $\partial M \times \{0\} = \partial M$. Extend the bundles ξ, $T(\partial M)$ etc. to C in the obvious way. Then j can also be extended to a map $v_C\colon \xi_C \to TC \cong T(\partial M)\times \mathbb{R}$ without zeroes on $\partial M \times [0,1)$, and such that on $\partial M \times [1/2, 1]$ v_C is given by

$$v_C(x,t) = \big(j_1'(x), (1-t)\cdot j''(x)\big).$$

Using v_C and v we get a smooth bundle map from ξ (extended over $C \cup M$) to $T(C\cup M)$. If we lift it to the double cover $S^0(\xi)$ which makes (the extended) ξ trivial, we obtain a vector field $\tilde{v}$ with two types of zeroes. Those which come from v keep their index unchanged (but there are twice as many as before). And then we have also the zeroes $(z_l, 1)$, $l = 1, \ldots, N$, on the boundary, coming from the zeroes $z_1, \ldots, z_N$ of j_1'. Each $(z_l, 1)$ lifts to two points: there is the point $\tilde{z}_l$ such that around $\tilde{z}_l$ $\tilde{v}$ corresponds to the vector field around $(z_l, 1)$ mapping (x,t) to $\big(j_1'\circ j''^{-1}(x), 1-t\big)$. And there is the point $\tilde{\tilde{z}}_l$ where $\tilde{v}$ corresponds to the negative of this latter vector field. If in a suitable neighborhood of the boundary of $S^0(\xi)$ we add an outward pointing vector field to $\tilde{v}$, the zero at $\tilde{z}_l$ disappears, while $\tilde{\tilde{z}}_l$ gives rise to a zero with index equal to minus the index $i(z_l)$ of $j_1'\circ j''^{-1}$ at z_l.

Now we can apply the Poincaré-Hopf Theorem. We obtain that twice the sum of the indices of v at its zeroes, minus the sum of the indices $i(z_l)$ of $j_1'\circ j''^{-1}$, equals the Euler number of $S^0(\xi)$ (which is twice the Euler number of $C\cup M \cong M$). Since we can vary v while leaving j_1' etc. unchanged, and vice versa, each of these sums of indices is independent of the choices involved in their formulation. We summarize:

Definition 1.2. Let ξ be a line bundle over the even-dimensional manifold M, together with a bundle imbedding $j\colon \xi|\partial M \to TM|\partial M$.

(i) Define $\tilde{\theta}(M,\xi,j)\in\mathbb{Z}$ to be the sum of indices of a bundle map $v\colon \xi \to TM$ at its zeroes, where v extends j and has only isolated zeroes.

Let $\theta(M,\xi,j)\in\mathbb{Z}_2$ be $\tilde{\theta}(M,\xi,j)$ reduced mod 2.

(ii) Define $\beta(M,\xi,j)\in\mathbb{Z}$ to be the sum $\sum_{l=1}^{N} i(z_l)$ of indices of the modified tangential component j_1' of j (as constructed above; note that this sum also depends on the normal component of j).

Proposition 1.3. *The numbers $\tilde{\theta}(M,\xi,j)$, $\theta(M,\xi,j)$ and $\beta(M,\xi,j)$ are well-defined invariants depending only on M, ξ and the regular homotopy class of j. Moreover, we have the following relation with the Euler number $\chi(M)$ of M:*

$$2\cdot\tilde{\theta}(M,\xi,j) - \beta(M,\xi,j) = 2\cdot\chi(M).$$

This follows from the discussion above. Furthermore, we have

Proposition 1.4. *Assume M is connected. Then j extends to a nowhere vanishing bundle map from ξ to TM over all of M if and only if*

$$\tilde{\theta}(M, \xi, j) = 0.$$

Proof. Assume $\tilde{\theta}(M, \xi, j) = 0$. Choose an extension v of j with isolated zeroes. We may use isotopies of M (which leave ∂M fixed) to push all the zeroes of v inside a small disc D over which ξ and TM are trivial; e.g. move along the trajectories of a suitable vector field in the tubular neighborhood of an imbedded curve. Over D, v can be viewed as a map into $\mathbb{R}^n$, and the mapping degree of $\frac{v}{\|v\|}: \partial D \to S^{n-1}$ is just the sum of indices of v at its zeroes, and hence, by assumption, it vanishes. Now by a theorem of Hopf, v can be altered inside of D so as to have no more zero.

Corollary 1.5. *If the image of j lies entirely in the tangent bundle of ∂M, or if it is everywhere normal to $T(\partial M)$, then $\beta(M, \xi, j)$ is zero. Hence in both cases j extends to a bundle imbedding of ξ into TM over all of M if and only if each connected component of M has vanishing Euler number.*

Proof. If the image of j is normal to $T(\partial M)$, then $\xi | \partial M$ must be trivial, and j viewed as a vector field can be rotated into a nowhere vanishing vector field on the odd dimensional manifold ∂M. The rest of the corollary follows from the definition of β and from the propositions above.

Now we turn to the case when $n = \dim M$ is odd. Here the situation is quite different. If locally we trivialize ξ and interpret v as a vector field, the sign of its index at a zero depends on the trivialization. Moreover, the double cover argument, which works so well in the even dimensional case, now gives us nothing better than the well-known identity $\chi(\partial M) = 2 \cdot \chi(M)$.

So our previous approach centering around $\tilde{\theta}(M, \xi, j)$ and $\beta(M, \xi, j)$ will in general not be of too much use here. However, an appropriate generalization of the invariant $\theta(M, \xi, j)$ to the odd-dimensional case will help us.

Definition 1.6. Let ξ be a line bundle over the odd-dimensional manifold M, together with a bundle imbedding $j: \xi | \partial M \to TM | \partial M$.

Define $\theta(M, \xi, j) \in \mathbb{Z}_2$ to be the number of zeroes of v taken mod 2, where $v: \xi \to TM$ is an extension of j which, when considered as a section in the homomorphism bundle $\mathrm{Hom}(\xi, TM)$, is transverse regular to the zero section.

The existence of such a v follows from standard density theorems for transversal sections. Also note that this definition of $\theta(M, \xi, j)$ would be equivalent to the one in Definition 1.2 in case M were even dimensional. This is a consequence of Lemma 4 in § 6 of [14].

Proposition 1.7. *$\theta(M, \xi, j)$ is a well-defined invariant, depending only on M, ξ and the regular homotopy class of j.*

Proof. Let $\bar{v}$ be another section of $\mathrm{Hom}(\xi, TM)$ transverse regular to zero and extending some bundle imbedding $\bar{j}\colon \xi|\partial M \to TM|\partial M$, which is regular homotopic to j. Then, on $M \times \mathbb{R}$, we can find a section w of the lifted bundle $\pi_1^*(\mathrm{Hom}(\xi, TM))$ which coincides with v' on $M \times (-\infty, \varepsilon]$ and with $\bar{v}$ on $M \times [1-\varepsilon, \infty)$ for some small $\varepsilon > 0$. Furthermore, w may be required to be transverse regular to the zero section, and to have no zeroes on $\partial M \times \mathbb{R}$. Then the set of zeroes of w in $M \times [0, 1]$ is a 1-dimensional compact submanifold of M, and its boundary is just the union of the zero set of v in $M \times \{0\}$ with the zero set of $\bar{v}$ in $M \times \{1\}$. Hence this union consists of an even number of points.

Proposition 1.8. *Assume that the odd-dimensional manifold M is connected, and that ξ is non-trivial. Then j can be extended to a bundle imbedding of ξ into TM over all of M if and only if $\theta(M, \xi, j) = 0$.*

Proof. For $n = 1$ this is clear, so suppose $n \geqq 3$. Let v be as in Definition 1.6. We may assume that all the zeroes of v lie in a small open disc $D \subset M - \partial M$. If we fix a trivialization of ξ over D, these zeroes have well-defined indices. Since ξ is non-trivial, we can find a circle S^1 imbedded into the interior of M such that $\xi|S^1$ is non-trivial, and such that S^1 contains just one zero of v. (If v has no zero at all there is nothing to prove.) Let N be a small compact tubular neighborhood of S^1 in $M - \partial M$; moreover, let $z' \neq z$ lie in the connected component of z in $S^1 \cap D$, and let $N_{z'} \subset N$ be the normal disk at z'. Then over $N - N_{z'}$ we can trivialize ξ and consider v as a vector field. Also we can construct another vector field which is parallel to S^1 inside $N - N_{z'}$ and vanishes everywhere else. The isotopy resulting from such a vector field can be used to move the zero z of v around S^1 by nearly a whole circuit and back into D. However, the index of this new zero (with respect to the fixed trivialization of ξ on D) is the negative of the index of z. Hence this modification changes the index sum of v by ± 2, and we can continue until the index sum is either 0 or $+1$. In the first case we can modify v further within D to cancel all zeroes.

To complete our analysis of the odd-dimensional situation let us take a quick look at the case when ξ is trivial. If we choose a trivialization, any extension v of j with isolated zeroes can be considered as a vector field and has a well-defined index $i(v, x)$ at each zero x. Also, as in the even dimensional case, we can split j into a tangential component j' and a normal component j'', and modify j' outside of the zero set of j'' so as to obtain a vector field j'_1 on ∂M with only isolated zeroes. If $y_1, \ldots, y_r$ are those zeroes of j'_1 where the normal vector field j'' points inward, we have:

$$\sum_{x \text{ zero of } v} i(v, x) + \sum_{k=1}^{r} i(j'_1, y_k) = \chi(M).$$

Hence the integer

$$\tilde{\theta}(M,\xi,j)=\sum_{M'}\Big|\sum_{x \text{ zero of } v|M'} i(v,x)\Big|$$

(the first summation goes over the set of connected components M' of M) is a well-defined invariant, depending only on M, ξ, and the regular homotopy class of j. Clearly $\tilde{\theta}(M,\xi,j)$, when reduced mod 2, is just $\theta(M,\xi,j)$. Furthermore we have: j can be extended to a bundle imbedding of ξ into TM over all of M if and only if $\tilde{\theta}(M,\xi,j)=0$.

Remark 1.9. Much of our treatment of Question 1.1 can also be obtained from classical obstruction theory (e.g. as presented in [17]). Indeed, let us assume that M is triangulated and connected. Then the obstruction to extending j to a nowhere vanishing section of Hom (ξ, TM) lies in $H^n(M,\partial M;\tilde{\mathbb{Z}})$, where the coefficient bundle $\tilde{\mathbb{Z}}=\widetilde{\pi_{n-1}}(S^{n-1})$ has fiber $\mathbb{Z}$ and can be viewed as being associated with the orientation bundle $\Lambda^n(\mathrm{Hom}(\xi, TM))$ of Hom (ξ, TM). Now pick an n-simplex σ of M, and let $\tilde{\mathbb{Z}}_x$ be the fiber of $\tilde{\mathbb{Z}}$ at some reference point $x\in\sigma$. Then the homomorphism

$$h: \tilde{\mathbb{Z}}_x \to H^n(M,\partial M;\mathbb{Z}),$$

taking $\lambda\in\tilde{\mathbb{Z}}_x$ to the cohomology class of $\lambda\cdot\sigma$, is onto. Its kernel is zero iff $\tilde{\mathbb{Z}}$ is associated with the orientation bundle of TM; otherwise its kernel is $2\cdot\tilde{\mathbb{Z}}_x$ (see e.g. [15], p. 461).

Now, if n is even, there is a natural isomorphism between $\widetilde{\pi_{n-1}}(S^{n-1})$ and the corresponding coefficient bundle for TM, and therefore we can canonically identify $H^n(M,\partial M;\tilde{\mathbb{Z}})$ with $\mathbb{Z}$. If n is odd and ξ is non-trivial, then the first Stiefel-Whitney class $w_1(\Lambda^n(\mathrm{Hom}(\xi, TM)))$ does not equal $w_1(\Lambda^n(TM))$, and hence $H^n(M,\partial M;\mathbb{Z})=\mathbb{Z}_2$. Finally, for odd n and trivial ξ we have $H^n(M,\partial M;\tilde{\mathbb{Z}})\cong\mathbb{Z}$, but the isomorphism depends on the trivialization of ξ. We choose it so as to make our obstruction non-negative. So in each case the obstruction to extending j can be expressed by either an integer number or by an integer mod 2, and these numbers are precisely $\tilde{\theta}(M,\xi,j)$ resp. $\theta(M,\xi,j)$.

§ 2. The Invariants $\theta(M,\xi)$ and $\phi(M,\xi)$ for Line Bundles on Closed Manifolds

From now on we will assume that M is closed, i.e. $\partial M=\phi$ (unless we explicitly state otherwise). So the "boundary condition" j plays no more role, and we will drop it from our notations. Thus, for instance, $\theta(M,\xi)$ denotes the number, taken mod 2, of zeroes of any section in Hom (ξ, TM) which is transverse regular to the zero section.

Theorem 2.1 [1]. *Let ξ be a line bundle over the closed connected n-dimensional manifold M.*

[1] I have learned that this result was obtained simultaneously by D. Toledo.

Then ξ is isomorphic to a sub-bundle of TM if and only if Euler number $\chi(M)$ of M vanishes (when n is even), resp. if and only if $\theta(M,\xi)=0$ (when n is odd).

This follows from Corollary 1.5 and from Proposition 1.8 except in the case when n is odd and ξ is trivial. But then $\chi(M)=0$, and a vector field on M generates the sub-bundle of TM in question.

Next we compute $\theta(M,\xi)$ in terms of the Stiefel-Whitney classes of M and ξ and of the mod 2 fundamental class $[M]$ of M.

Proposition 2.2. *For a line bundle ξ over a closed manifold M we have*

$$\theta(M,\xi)=\big(w(\xi)^{-1}\,w(M)\big)\,[M]=\sum_{i=0}^{n}\big(w_1(\xi)^{n-i}\,w_i(M)\big)\,[M].$$

Proof. This follows from Proposition 5.2, 5.1 and Theorem 5.2 in [5]. Also clearly $\sum_{i=0}^{n} w_1(\xi)^{n-i}\,w_i(M)=w_n\big(\mathrm{Hom}(\xi,TM)\big)$. Therefore, Remark 1.9 leads to another proof.

Remark 2.3. If n is even, then Proposition 1.3 implies the following additional identity

$$\theta(M,\xi)=w_n(M)\,[M]=\big(\chi(M),\ \text{taken mod } 2\big).$$

Hence the relation

$$0=w_1(\xi)^n+w_1(\xi)^{n-1}\,w_1(M)+\cdots+w_1(\xi)^{n-i}w_i(M)+\cdots+w_1(\xi)\,w_{n-1}(M) \qquad (1')$$

holds for all line bundles ξ over a closed even-dimensional C^∞-manifold M.

Example 2.4. Consider the canonical line bundle λ over the real projective space $RP(n)$. It is well known that $\alpha=w_1(\lambda)$ generates $H^*(RP(n);\mathbb{Z}_2)$, and that $w(RP(n))=(1+\alpha)^{n+1}$. Hence $\theta(RP(n),\lambda)=(1+\alpha)^{-1}(1+\alpha)^{n+1}[RP(n)]=\alpha^n[RP(n)]=1$. On the other hand, we can also see this easily from the geometry of $RP(n)$. Note that $T(RP(n))=\mathrm{Hom}(\lambda,\lambda^\perp)=\lambda\otimes\lambda^\perp$, where $\lambda^\perp$ is the orthogonal complement of λ in $RP(n)\times\mathbb{R}^{n+1}$. Hence $\mathrm{Hom}(\lambda, T(RP(n)))=\lambda\otimes\lambda\otimes\lambda^\perp=\lambda^\perp$. If we project a fixed element $x\in S^n$ orthogonally into $l^\perp$ at each line $l\in RP(n)$, we obtain a smooth section of $\mathrm{Hom}(\lambda, T(RP(n)))$ which has just one non-degenerate zero.

Next we want to classify line bundle imbeddings up to concordance.

Definition 2.5. Two imbeddings j_0, j_1 of the line bundle ξ into the tangent bundle of the closed manifold M are called concordant if they are homotopic through bundle imbeddings of ξ into $TM\oplus\mathbb{R}$.

Clearly this is the case if and only if there is a bundle imbedding $p^*(\xi)\hookrightarrow T(M\times I)$ which extends the imbedding of $p^*(\xi)|\partial(M\times I)$ into $T(M\times I)|\partial(M\times I)$ given by j_0 and j_1. (Here I is the unit interval $[0,1]$, and $p: M \times I \to M$ is the projection on the first factor.) Hence we can treat this problem with the methods developed so far.

Proposition 2.6. *For a line bundle ξ over an odd dimensional manifold M any two imbeddings of ξ into TM are concordant.*

Proof. This follows from Corollary 1.5 and the fact that $\chi(M' \times I) = \chi(M') = 0$ for every component M' of M.

Proposition 2.7. *Let ξ be a line bundle over a connected manifold M of even dimension n. If ξ is trivial, then all imbeddings of ξ into TM are concordant. However, if ξ is non-trivial and $\chi(M)=0$, then there are precisely two concordance classes of imbeddings of ξ into TM.*

Proof. To get the first conclusion, observe that all embeddings of a trivial line bundle into TM can be deformed, within $TM \oplus \mathbb{R}$, to an imbedding into the added trivial line bundle.

So assume that ξ is non-trivial. If $\chi(M)=0$, then we can find a nowhere vanishing section j_0 of $\mathrm{Hom}(\xi, TM)$ and split $\mathrm{Hom}(\xi, TM)$ into the trivial bundle τ generated by j_0, and an $(n-1)$-dimensional complementary sub-bundle η. Also there is an imbedded circle $S^1 \subset M$ over which ξ is still non-trivial. Now observe that the normal bundle ν of S^1 in M is isomorphic to $\eta|S^1$, because $w_1(\nu) = w_1(TM|S^1) = w_1(\mathrm{Hom}(\xi, TM)|S^1) = w_1(\eta|S^1)$. Hence, we can find a section s of η which, in a tubular neighborhood of S^1, has S^1 as its set of zeroes and is transversal to the zero section there. Choose a real-valued function α on M which is $+1$ outside of such tubular neighborhood and -1 around S^1. Then the section $j_1 = (\alpha \cdot j_0, s)$ of $\mathrm{Hom}(\xi, TM) = \tau \oplus \eta$ has no zeroes.

Next pick a section of $\xi|S^1$ with just one non-degenerate zero z; extend it in the obvious way to ξ over the whole tubular neighborhood of S^1 and then over all of M to get a section f of ξ. Now if, in the obvious way, we deform $j_1 = (\alpha \cdot j_0, s, 0)$ within $\tau \oplus \eta \oplus \xi$ first into $(\alpha \cdot j_0, s, f)$, then into (j_0, s, f), and finally into $(j_0, 0, 0)$, this homotopy will vanish precisely once, namely at the zero z of f, halfway thru the middle part of the deformation. Expressed in terms of imbeddings of ξ, this just means that $\theta(M \times I, p^*(\xi), j_0 \cup j_1) = 1$, and therefore the imbeddings $j_0, j_1: \xi \to TM$ represent two different concordance classes. From Proposition 1.8 it follows that there can be no other concordance classes than these.

Proposition 2.8. *An imbedding j of the line bundle ξ into the tangent bundle of the connected, even dimensional manifold M is concordant to its negative $-j$ if and only if the number*

$$w(\xi)^{-2} \cdot w(M)\,[M] \in \mathbb{Z}_2$$

vanishes.

Proof. Since this is clear for trivial ξ, we may assume ξ is non-trivial. Let η be a complement of $j(\xi)$ in $TM \oplus \mathbb{R}$, and let $h: \xi \to \eta$ be bundle map with only non-degenerate zeroes $z_1, \ldots, z_r$. As in the proof of Proposition 2.2 one can show that the number r of zeroes, taken mod 2, equals $(w(\xi)^{-1} w(\eta))[M] = (w(\xi)^{-2} w(M))[M]$. Now the bundle map

$$f: p^*(\xi) \to T(M \times I) \cong p^*(j(\xi)) \oplus p^*(\eta),$$

given by

$$f_{(x,t)} = \cos(\pi t) \cdot j_x + \sin(\pi t) \cdot h_x,$$

has its only, non-degenerate zeroes at $(z_1, 1/2), \ldots, (z_r, 1/2)$ and coincides with j on $M \times \{0\}$ and with $-j$ on $M \times \{1\}$. Therefore $\theta(M \times I, p^*(\xi), j \cup -j)$ equals r mod 2, which is $w(\xi)^{-2} w(M)[M]$. Proposition 2.8 then follows from Proposition 1.8.

Remark 2.9. A similar argument involving Proposition 1.3 can be used to show that for a line bundle ξ over an odd dimensional manifold M the characteristic number $w(\xi)^{-2} w(M)[M]$ vanishes.

Thus we have the relation

$$\begin{aligned} 0 = w_1(\xi)^{n-1} w_1(M) + \cdots + w_1(\xi)^{n-2i-1} \\ \cdot w_{2i+1}(M) + \cdots + w_1(\xi)^2 w_{n-2}(M) + w_n(M). \end{aligned} \tag{2'}$$

Definition 2.10. For a line bundle ξ over the n-manifold M define the invariant $\phi(M, \xi) \in \mathbb{Z}_2$ by

$$\phi(M, \xi) = \Big(\sum_{0 \leq 2i \leq n} w_1(\xi)^{n-2i} \cdot w_{2i}(M)\Big)[M].$$

If n is even, then

$$\begin{aligned} \phi(M, \xi) &= \big(w_1(\xi)^n + w_1(\xi)^{n-2} w_2(M) + \cdots + w_1(\xi)^2 w_{n-2}(M) + w_n(M)\big)[M] \\ &= w(\xi)^{-2} w(M)[M], \end{aligned}$$

and the significance of this invariant is explained by Proposition 2.8.

If n is odd, then

$$\phi(M, \xi) = \big(w_1(\xi)^n + w_1(\xi)^{n-2} \cdot w_2(M) + \cdots + w_1(\xi) w_{n-1}(M)\big)[M].$$

Adding Eq. (2′) above, and comparing with Proposition 2.2, we obtain:

Proposition 2.11. *For all line bundles ξ over an odd dimensional manifold M the invariants $\theta(M, \xi)$ and $\phi(M, \xi)$ are equal.*

Next we elaborate a little on the relations (1′) and (2′) in Remarks 2.3 and 2.9. This will enable us in various situations to simplify considerably the polynomial expressions which define $\phi(M, \xi)$ and $\theta(M, \xi)$.

Lemma 2.12. *The following relations hold for the Stiefel-Whitney classes $w_k = w_k(M)$ of an n-dimensional manifold M and for all $x \in H^1(M; \mathbb{Z}_2)$.*

If n and k are even and $0 \leq k < n$, then

$$x^{n-k} w_k = x^{n-k-1} w_{k+1}. \tag{1}$$

If n and k are odd and $0<k\leqq n$, *then*

$$x^{n-k}w_k=0 \tag{2}$$

Moreover, for arbitrary n, k with $0\leqq k\leqq n-2$ *we have*

$$\begin{aligned}&\binom{n-k-2}{2}x^{n-k}w_k+(n-k)(k+1)x^{n-k-1}w_{k+1}+\binom{k-1}{2}x^{n-k-2}w_{k+2}\\&\quad=(n-k)x^{n-k-1}w_1w_k+kx^{n-k-2}w_1w_{k+1}+x^{n-k-2}w_1^2w_k.\end{aligned} \tag{3}$$

Note that (1) and (2) refine the relations (1′) and (2′) obtained previously.

Proof. Using the well-known identities of Wu [23, 24] we get:

$$\begin{aligned}x^{n-k-1}w_1w_k&=Sq^1(x^{n-k-1}w_k)\\&=(n-k-1)x^{n-k}w_k+x^{n-k-1}(w_1w_k+(k+1)w_{k+1}).\end{aligned}$$

The resulting equation

$$(n-k-1)x^{n-k}w_k=(k+1)x^{n-k-1}w_{k+1}$$

implies (1) and (2). Furthermore we have

$$\begin{aligned}&x^{n-k-2}w_2w_k+x^{n-k-2}w_1^2w_k\\&\quad=Sq^2(x^{n-k-2}w_k)\\&\quad=\binom{n-k-2}{2}x^{n-k}w_k+(n-k)x^{n-k-1}(w_1w_k+(k+1)w_{k+1})\\&\qquad+x^{n-k-2}\left(w_2w_k+kw_1w_{k+1}+\binom{k-1}{2}w_{k+2}\right).\end{aligned}$$

This leads to relation (3).

Proposition 2.13. *For every line bundle* ξ *over an orientable manifold M of even dimension n we have*

$$\phi(M,\xi)=w_n(M)[M]=\theta(M,\xi).$$

In particular, if $n\equiv 2(4)$, *then* $\phi(M,\xi)=0$.

Proof. We write x for $w_1(\xi)$. At first we assume only that $w_1^2=w_1(M)^2=0$, rather than the stronger condition $w_1=0$ which is equivalent to M being orientable. Then the Eq. (3) in the last lemma implies the following identity for even k, $0\leqq k\leqq n-2$

$$\binom{n-k-2}{2}x^{n-k}w_k=\binom{k-1}{2}x^{n-k-2}w_{k+2}.$$

It follows that

$$x^{n-k}w_k=x^{n-k-2}w_{k+2}\qquad\text{if } n,k\equiv 0(4);$$

and

$$x^{n-k} w_k = 0 \quad \text{if } n, k \equiv 2(4) \text{ (even if } k = n).$$

This already proves the proposition in case $n \equiv 0(4)$.

Now let us assume that actually $w_1(M) = 0$, so that one side of Eq. (3) vanishes. Then if k is even, $0 \leqq k < n-2$, apply (3) to $k+1$ and use (1) to obtain the equation

$$\binom{n-k-3}{2} x^{n-k} w_k = \binom{k}{2} x^{n-k-2} w_{k+2}.$$

This implies that

$$x^{n-k} w_k = x^{n-k-2} w_{k+2} \quad \text{if } n \equiv 0(4),\ k \equiv 2(4);$$

and

$$x^{n-k} w_k = 0 \quad \text{if } n \equiv 2(4),\ k \equiv 0(4) \text{ (even if } k = n-2).$$

This completes the proof also in case $n \equiv 2(4)$.

Remark 2.14. In contrast it can be shown that in every positive even dimension n there exists a (non-orientable) connected n-manifold M with vanishing Euler characteristic and with a line bundle ξ over it such that $\phi(M, \xi) \neq 0$.

Remark 2.15. If M is orientable and of even dimension n, then the equations in the last proof can easily be combined with relation (1) in Lemma 2.12 to show that for all $x \in H^1(M, \mathbb{Z}_2)$ the following relations hold

$$0 = x^n = x^{n-1} w_1(M) = x^{n-2} w_2(M) = \cdots = x^{n-k} w_k(M) = \cdots = x\, w_{n-1}(M). \quad (4')$$

This implies and refines the theorem of Massey and Wu to the effect that $w_{n-1}(M) = 0$ (see e.g. [11]).

Proposition 2.16. *Let ξ be a line bundle over an n-dimensional manifold M satisfying $w_1(M)^2 = 0$. Then, if $n \equiv 1(4)$, we have the identity*

$$\theta(M, \xi) = w_1(\xi)\, w_{n-1}(M)[M].$$

If $n \equiv 3(4)$, then

$$\theta(M, \xi) = \big(w_1(\xi)^n + w_1(\xi)^{n-4} w_4(M) + \cdots + w_1(\xi)^{n-4i} \cdot w_{4i}(M) + \cdots + w_1(\xi)^3 w_{n-3}(M)\big)[M].$$

Proof. We write x for $w_1(\xi)$, and we use the identities (2) and (3) of Lemma 2.12. For even k, $0 \leqq k \leqq n-2$, we get

$$\binom{n-k-2}{2} x^{n-k} w_k + \binom{k-1}{2} x^{n-k-2} w_{k+2} = x^{n-k-1} w_1 w_k + x^{n-k-1} w_{k+1} = x^{n-k-1} w_1 w_k.$$

But this last expression vanishes also: if $k=0$, this follows from (2); if $k>0$, we get it by substituting $(k-1)$ of k in equation (3) and simplifying by means of (2).

If now $n\equiv 1(4)$ and $k\equiv 0(4)$, then

$$x^{n-k} w_k + x^{n-k-2} w_{k+2} = 0.$$

If $n\equiv 3(4)$ and $k\equiv 0(4)$, then

$$x^{n-k-2} w_{k+2} = 0.$$

Proposition 2.16 follows.

§ 3. The Case of the Orientation Bundle

Every n-manifold M has a naturally arising line bundle over it. Namely, the orientation bundle $\xi_M = \Lambda^n(TM)$. For later use we want to apply some of the results of § 2 to ξ_M. Observe that clearly $w_1(\xi_M) = w_1(M)$.

Theorem 3.1. *The orientation bundle ξ_M of a closed manifold M is isomorphic to a sub-bundle of TM if and only if the Euler characteristic of each connected component of M vanishes. (In other words, M admits an oriented hyperplane field if and only if M admits an oriented line field).*

In particular, if M is odd-dimensional, then ξ_M imbeds into TM.

Proof. We apply the Eqs. (2) and (3) of Lemma 2.12 to

$$x = w_1(\xi_M) = w_1(M)$$

and to odd k, and we get: for each closed manifold M of odd dimension n the following relations hold

$$\begin{aligned} 0 &= w_1(M)^n = \cdots = w_1(M)^{n-i} w_i(M) = \cdots = w_1(M)^2 w_{n-2}(M) \\ &= w_1(M) w_{n-1}(M) = w_n(M). \end{aligned} \tag{4}$$

In view of Theorem 2.1 and Proposition 2.2, this suffices to establish Theorem 3.1.

Proposition 3.2. *Let M be a closed manifold of dimension n. Then:*

If $n\equiv 2(4)$, then $\phi(M,\xi_M)=0$; if $n\equiv 0(4)$, then

$$\phi(M,\xi_M) = (1+w_1(M))^{-4} w(M)[M] = \sum_{0\leq 4i\leq n} w_1(M)^{n-4i} w_{4i}(M)[M].$$

Proof. Apply identity (3) of Lemma 2.12 to the case when $x=w_1(M)$, n is even and $k\equiv 0(4)$, to obtain

$$\left(\binom{n-k-2}{2}+1\right) w_1(M)^{n-k} \cdot w_k(M) = w_1(M)^{n-k-2} w_{k+2}(M).$$

The proposition follows.

Proposition 2.6, 2.8 and 3.2 imply that every imbedding j of ξ_M into TM is concordant to $-j$, provided $n \not\equiv 0(4)$. In contrast we have:

Proposition 3.3. *In each positive dimension n, $n \equiv 0(4)$, there exists a closed n-manifold M and an imbedding j: $\xi_M \to TM$ which is not concordant to its negative $-j$.*

Proof. In view of Theorem 2.1 and Proposition 2.8, it suffices to exhibit a connected n-manifold M with vanishing Euler number $\chi(M)$ and such that $\phi(M, \xi_M) \neq 0$. For this purpose, write $n = 4r$, and consider the projectification $M = RP(1, \ldots, 1, 1, 0)$ of the vector bundle $\lambda_1 \oplus \cdots \oplus \lambda_{2r} \oplus \mathbb{R}$ over $(S^1)^{2r}$, where λ_i is the pullback of the non-trivial line bundle over the i-th factor. Then using the notation of the proof of Lemma 3.4 in [20] (and putting $\sigma_i = w_i(\lambda_1 \oplus \cdots \oplus \lambda_{2r})$), we have

$$\begin{aligned} w(M) &= (1+c) \cdot \prod_{i=1}^{2r} ((1+c)+\alpha_i) \\ &= (1+c)((1+c)^{2r} + \cdots + (1+c)\,\sigma_{2r-1} + \sigma_{2r}). \end{aligned}$$

Also $w_1(\xi_M) = w_1(M) = c + \alpha_1 + \cdots + \alpha_{2r}$; and since $\alpha_1^2 = 0$, we get $w(\xi_M)^2 = (1+c)^2$.

Hence

$$\begin{aligned} \phi(M, \xi_M) &= ((1+c)^{-2}\, w(M))[M] \\ &= ((1+c)^{2r-1} + \cdots + \sigma_{2r-1} + (1+c+\cdots+c^n)\,\sigma_{2r})[M] \\ &= c^{2r}\,\sigma_{2r}[M] = c^{2r} \cdot \alpha_1 \ldots \alpha_{2r}[M] \\ &= 1. \end{aligned}$$

Remark 3.4. Here is a geometric clue to some of the analogies between the relations which we got at the end of §2 for line bundles over orientable manifolds, and the relations which occur in this section (compare e.g. (4′) in Remark 2.15 with (4) in the proof of Theorem 3.1).

For a closed manifold M consider the total space $M' = RP(\xi_M \oplus \mathbb{R})$ of the projective line bundle which belongs to $\xi_M \oplus \mathbb{R}$. M' is naturally equipped with a line bundle ξ over it. Also M' bounds an oriented manifold (it is diffeomorphic to the sphere bundle in $\xi_M \oplus \mathbb{R}$). Hence, assigning the bordism class of a classifying map of ξ to the bordism class of M, we obtain a homomorphism

$$\omega\colon \mathfrak{N}_n \to \tilde{\Omega}_{n+1}(BO(1)),$$

where $\tilde{\Omega}_{n+1}(BO(1))$ is the kernel of the obvious forgetful map

$$\Omega_{n+1}(BO(1)) \to \Omega_{n+1}.$$

It can be shown by elementary geometric arguments that ω is actually an isomorphism, the inverse being a "Smith homomorphism" (compare

P.E. Conner and E.E. Floyd, Torsion in SU bordism, Mem. AMS 60; and [18], p. 217). Obviously, ω extends to a canonical isomorphism $\mathfrak{N}_n \oplus \Omega_{n+1} \cong \Omega_{n+1}(BO(1))$.

Next we identify $H^*(M, \mathbb{Z}_2)$ with its image in $H^*(M', \mathbb{Z}_2)$ under the obvious homomorphism, and we write x for $w_1(\xi)$. Note that $x^2 = x \cdot w_1(M)$, and that

$$w(M') = (1+x)(1+x+w_1(M))\, w(M) = (1+w_1(M))\, w(M).$$

Now consider Whitney numbers of the form

$$w_1(\xi)^{n+1-k} \prod_{i=2}^{n} w_i(M')^{j_i} [M'], \tag{5}$$

where $0 \leqq j_2, \ldots, j_n$ and $\sum_i i \cdot j_i = k \leqq n$. We have

$$\begin{aligned} &\Big(x^{n+1-k} \prod_i w_i(M')^{j_i}\Big)[M'] \\ &\quad = \Big(x \cdot w_1(M)^{n-k} \prod_i (w_i(M) + w_1(M)\, w_{i-1}(M))^{j_i}\Big)[M'] \qquad (6) \\ &\quad = \Big(w_1(M)^{n-k} \prod_i (w_i(M) + w_1(M)\, w_{i-1}(M))^{j_i}\Big)[M] \end{aligned}$$

which is a sum of Stiefel-Whitney numbers of M. On the other hand, observe that the last expression in (6) equals $\big(w_1(M)^{n-k} \prod_i w_i(M)^{j_i}\big)[M]$ plus a sum of Stiefel-Whitney numbers containing higher powers of $w_1(M)$; so we can express the Stiefel-Whitney numbers of M recursively and in a natural way by Whitney numbers of (M', ξ) of the form (5). This way those relations among Whitney numbers of the form (5) which hold for all line bundles ξ over oriented $(n+1)$-manifolds M', correspond in a one-to-one fashion to relations among Stiefel-Whitney numbers of arbitrary n-manifolds. In particular, using Poincaré duality and the fact that every $x \in H^1(M', \mathbb{Z}_2)$ is the first Stiefel-Whitney class of some line bundle ξ over M', we get for fixed coefficients $a_{j_2, \ldots, j_n} \in \mathbb{Z}_2$:

$$\sum_{\substack{0 \leqq j_2, \ldots, j_n \\ \Sigma i \cdot j_i = n}} a_{j_2, \ldots, j_n} \cdot \prod_{i=2}^{n} w_i(M')^{j_i} \in H^n(M', \mathbb{Z}_2)$$

vanishes for every closed, smooth, oriented $(n+1)$-manifold M' if and only if

$$\sum_{\substack{0 \leqq j_2, \ldots, j_n \\ \Sigma i \cdot j_i = n}} a_{j_2, \ldots, j_n} \cdot \prod_{i=2}^{n} (w_i(M) + w_1(M)\, w_{i-1}(M))^{j_i} [M] = 0$$

for all closed, smooth n-manifolds M.

For a nonnegative integer $k \leqq n$ the Eq. (6) specializes to yield

$$\begin{aligned}(x^{n+1-k} \cdot w_k(M'))[M'] \\ = (w_1(M)^{n-k} w_k(M) + w_1(M)^{n-k+1} w_{k-1}(M))[M].\end{aligned} \tag{7}$$

This explains the analogy e.g. between the relations (4) and (4′) mentioned above. As another example, note that (7), together with the relation $w_1(M)^2 \cdot w_{n-2}(M) = w_1(M)\, w_{n-1}(M) = 0$ for $n \equiv 0(4)$ (as obtained in the proof of Proposition 3.2), suffices to prove the first part of Proposition 2.16 for orientable manifolds.

Finally observe that

$$\theta(M', \xi) = w_n(M)[M]$$

(this was already noticed in [19]), and that

$$\phi(M', \xi) = \theta(M, \xi_M).$$

§ 4. Counting Concordance Classes of Line Fields

By a line field on a manifold M we mean a smooth one-dimensional subvector-bundle of the tangent bundle TM. Thus a line field can also be viewed as a smooth section in the projective space bundle belonging to TM.

Definition 4.1. Two line fields $\xi_0, \xi_1 \subset TM$ on M are called concordant if there is a subline-bundle of $T(M \times I)$ restricting to

$$\xi_i \subset T(M \times \{i\}) \subset T(M \times I)$$

on $M \times \{i\}$ for $i = 0, 1$.

Equivalently, ξ_0 and ξ_1 are concordant if they are homotopic as sections in the projective space bundle of $TM \oplus \mathbb{R}$.

Theorem 4.2. *Let M be a closed connected manifold of dimension n, and let b denote the $\mathbb{Z}_2$-dimension of $H^1(M, \mathbb{Z}_2)$.*

Then there is only a finite number $a(M)$ of concordance classes of line fields on M.

If n is even and if the Euler characteristic $\chi(M)$ vanishes, then $a(M)$ equals $2^b - 1$ plus the number of those $x \in H^1(M, \mathbb{Z}_2)$ such that

$$(1+x)^{-2}\, w(M)[M] = 0.$$

(*However if $\chi(M) \neq 0$, then M admits no line field.*)

If n is odd, then there is a canonical one-to-one correspondence between concordance classes of line fields on M, and those cohomology classes $x \in H^1(M, \mathbb{Z}_2)$ which satisfy the condition $(1+x)^{-1}\, w(M)[M] = 0$.

Proof. First recall that $BO(1)$ is an Eilenberg-MacLane space of type $(\mathbb{Z}_2, 1)$; hence if we associate to each line bundle over M its first Stiefel-Whitney class, we obtain a bijection between isomorphism classes of line bundles over M, and elements in $H^1(M, \mathbb{Z}_2)$. Now two concordant line fields on M have isomorphic underlying line bundles; so we get a well defined map w_1 from the set of concordance classes of line fields on M into $H^1(M, \mathbb{Z}_2)$. We will see presently that w_1 maps at most two different concordance classes to a given $x \in H^1(M; \mathbb{Z}_2)$. Therefore $a(M)$ is finite, since $H^1(M, \mathbb{Z}_2)$ is.

If n is odd, then $x \in H^1(M, \mathbb{Z}_2)$ lies in the image of w_1 precisely if $(1+x)^{-1} w(M)[M]=0$. Indeed, according to Theorem 2.1 and Proposition 2.2 this mod 2 number is the obstruction to imbedding a line bundle ξ with $w_1(\xi)=x$ into TM. Moreover, Proposition 2.6 implies that w_1 is injective. Thus w_1 gives the one-to-one correspondence we claim in the last statement of the theorem.

Next assume that n is even and $\chi(M)=0$. For a fixed cohomology class $x \in H^1(M, \mathbb{Z}_2)$ let ξ be a line bundle with $w_1(\xi)=x$. If $x \neq 0$, but $(1+x)^{-2} w(M)[M]=0$, then we can conclude from Proposition 2.7 and 2.8 that there are imbeddings $j_0, j_1 \colon \xi \hookrightarrow TM$ such that $[j_0]=[-j_0]$ and $[j_1]=[-j_1]$ are the two concordance classes of imbeddings of ξ into TM. Clearly the image bundles $j_0(\xi)$ and $j_1(\xi)$ represent two different concordance classes of line fields, and these are all which get mapped into x under w_1. On the other hand, if $x=0$ or $(1+x)^{-2} w(M)[M] \neq 0$ and if $j \colon \xi \hookrightarrow TM$ is some imbedding, then j and $-j$ generate all concordance classes of imbeddings of ξ, but give rise to only one class of line fields. Hence if we count all $x \in H^1(M, \mathbb{Z}_2)$ with $x=0$ or $(1+x)^{-2} w(M)[M] \neq 0$ once, and all other x twice, we get $a(M)$. This completes the proof.

Corollary 4.3. *If M is even dimensional, orientable and connected and if $\chi(M)=0$, then*

$$a(M)=2^{b+1}-1.$$

Proof. According to Proposition 2.13 we know for every line bundle ξ over M that

$$\phi(M, \xi)=w_n(M)[M]=(\chi(M))_2=0.$$

Hence $(1+x)^{-2} w(M)[M]=0$ for all $x \in H^1(M, \mathbb{Z}_2)$, and the corollary follows.

Similarly Theorem 4.2, together with Proposition 2.16, implies the following.

Corollary 4.4. *If M is a connected manifold of dimension $n \equiv 1\,(4)$ and with $w_1(M)^2=0$, then*

$$a(M)=\begin{cases} 2^b & \text{if } w_{n-1}(M)=0 \\ 2^{b-1} & \text{if } w_{n-1}(M) \neq 0. \end{cases}$$

Example 4.5. The only closed connected surfaces which admit a line field, are the torus and the Klein bottle.

The torus carries seven concordance classes of line fields. In contrast, the homotopy classes of line fields over $S^1 \times S^1$ correspond to elements in the homotopy set $[S^1 \times S^1, RP(1)] \simeq [S^1 \times S^1, S^1] = \mathbb{Z} \oplus \mathbb{Z}$.

The Klein bottle K carries five concordance classes of line fields, which can be represented as in Fig. 1.

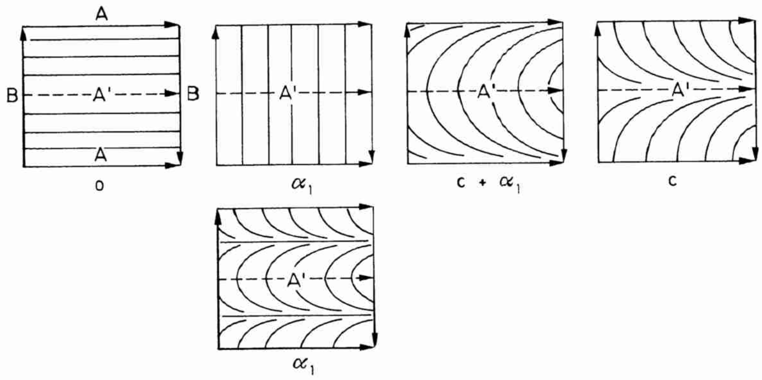

Fig. 1. The integral curves of line fields on the Klein bottle which represent the five concordance classes

Here we use the basis α_1, c of $H^1(K, \mathbb{Z}_2)$, which is dual to the basis $[A'], [B]$ of $H_1(K, \mathbb{Z}_2)$, to indicate under each line field its first Stiefel-Whitney class.

In contrast, K has infinitely many different homotopy classes of line fields; it follows from [17], 36.4, 36.6 and 37.2, that they correspond to the elements in $H^1(K; \tilde{\pi}_1) \simeq H_1(K; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$, since the coefficient bundle $\tilde{\pi}_1$ of the projectification of TK is associated with the orientation bundle of K.

§ 5. Bordism Groups of Manifolds with q-Plane Fields

Definition 5.1. Let $\Omega_n(q)$ (resp. $\Omega_n^{or}(q)$) be the bordism group of oriented n-dimensional manifolds with an arbitrary (resp. oriented) q-plane field; and let $\mathfrak{N}_n(q)$ (resp. $\mathfrak{N}_n^{or}(q)$) denote the corresponding bordism group based on unoriented manifolds. Finally let $\mathfrak{N}_n^{coor}(q)$ be the bordism group of (unoriented) n-manifolds with a co-oriented q-plane field.

Thus, to define $\Omega_n^{(or)}(q)$, we consider pairs (M, ξ), where M is a smooth oriented closed n-manifold, and ξ is an (oriented) q-dimensional subbundle of TM. Two such pairs (M_0, ξ_0) and (M_1, ξ_1) are called bordant if there is (i) a compact oriented $(n+1)$-manifold X whose boundary ∂X

is the disjoint union of M_1 (with its original orientation) and of M_0 (with the opposite of its original orientation), and (ii) an (oriented) q-dimensional sub-bundle ζ of TX which extends

$$\xi_0 \cup \xi_1 \subset T(\partial X) \subset TX\,|\,\partial X$$

(inducing the original orientation on both ξ_0 and ξ_1). Clearly bordism is an equivalence relation. Disjoint union provides an addition, which makes the resulting set of bordism classes $[M, \xi]$ into a group. Actually there is also a natural product operation which gives $\sum_{n,q} \Omega_n^{(or)}(q)$ the structure of a bigraded ring. In particular, $\Omega_*^{(or)}(q) = \sum_{n \geqq q} \Omega_n^{(or)}(q)$ (q fixed) is in a natural way a module over the classical bordism ring $\Omega_* \subseteq \Omega_*^{(or)}(0)$.

Similar remarks apply to the groups $\mathfrak{N}_n^{(or)}(q)$.

To define $\mathfrak{N}_n^{coor}(q)$ we use pairs (M, ξ) with an orientation of the quotient bundle TM/ξ. In the bordism X we identify $(TX/\zeta)|\partial X$ with $\bigcup_{l=0,1} \mathbb{R} \oplus (TM_l/\xi_l)$ by means of an outward pointing vector field, and we require an orientation on TX/ζ which restricts to the original orientation on TM_1/ζ_1 and to the opposite of the original orientation on TM_0/ξ_0. Observe that these co-orientation conditions can actually be expressed without resorting to quotient bundles. Indeed, the orientation bundle $\Lambda^{n-q}(TM/\xi)$ of TM/ξ is canonically isomorphic (via exterior multiplication on the left) to the bundle $\mathrm{Hom}(\Lambda^q \xi, \Lambda^n TM)$.

We can also introduce bordism groups $'\Omega_n^{(or)}(q)$, $'\mathfrak{N}_n^{(or)}(q)$ and $'\mathfrak{N}_n^{coor}(q)$ by calling two pairs (M_0, ξ_0) and (M_1, ξ_1) bordant if the $(q+1)$-plane field on ∂X, spanned (and possibly oriented) by an appropriate vector field normal to ∂X and by $\xi_0 \cup \xi_1$, extends to a $(q+1)$-plane field over all of the bordism X.

However, this gives nothing new:

Duality Lemma 5.2. *Taking complementary plane fields induces canonical isomorphisms*

$$'\Omega_n(q) \simeq \Omega_n(n-q); \qquad '\Omega_n^{or}(q) \simeq \Omega_n^{or}(n-q);$$

$$'\mathfrak{N}_n(q) \simeq \mathfrak{N}_n(n-q); \qquad '\mathfrak{N}_n^{or}(q) \simeq \mathfrak{N}_n^{coor}(n-q); \qquad '\mathfrak{N}_n^{coor}(q) \simeq \mathfrak{N}_n^{or}(n-q).$$

Thus we may restrict our attention to the groups introduced in Definition 5.1. To study them, let us consider forgetful maps such as the Ω_*-linear homomorphism

$$f\colon\ \Omega_*(q) \to \Omega_*(BO(q)) \qquad (\text{resp. } f\colon \Omega_*^{or}(q) \to \Omega_*(BSO(q))),$$

or the $\mathfrak{N}_*$-linear homomorphism

$$f\colon\ \mathfrak{N}_*(q) \to \mathfrak{N}_*(BO(q)) \qquad (\text{resp. } f\colon \mathfrak{N}_*^{or}(q) \to \mathfrak{N}_*(BSO(q))).$$

Here we identify e.g. $\Omega_*(BO(q))$ with the bordism module of oriented manifolds with arbitrary q-dimensional vector bundles over them, and f associates to q-plane fields their underlying vector bundles.

Now, as J. Alexander pointed out to me, it is convenient to introduce relative bordism groups. So consider triples (M, ξ, j) where M is an oriented, compact n-manifold, possibly with boundary, ξ is an (oriented) q-dimensional vector bundle over M, and j: $\xi|\partial M \to T(\partial M)$ is a vector bundle imbedding. Two such triples $(M_0, \xi_0, j_0)'$ and (M_1, ξ_1, j_1) are called bordant if there exist (i) an oriented compact smooth $(n+1)$-dimensional manifold X whose boundary contains the disjoint union $M_1 \cup -M_0$, (ii) an (oriented) q-bundle ζ over X which extends the (oriented) bundles ξ_0 and ξ_1, and (iii) a vector bundle injection $\tilde{j}$ from $\zeta|\partial X-(\mathring{M}_0 \cup \mathring{M}_1)$ into $T(\partial X)$ which restricts to j_l over ∂M_l, $l=0,1$. This defines an equivalence relation: to obtain reflexivity and transitivity we use the technique of straightening the angle. Clearly the bordism classes $[M, \xi, j]$ form a group which we denote by

$$\Omega_n(BO(q), q) \qquad (\text{resp. } \Omega_n(BSO(q), q)).$$

The relevance of these groups stems from the fact that they fit into sequences

$$\cdots \to \Omega_n(q) \xrightarrow{f} \Omega_n(BO(q)) \to \Omega_n(BO(q), q) \xrightarrow{\partial} \Omega_{n-1}(q) \to \cdots$$

and

$$\cdots \to \Omega_n^{or}(q) \xrightarrow{f} \Omega_n(BSO(q)) \to \Omega_n(BSO(q), q) \xrightarrow{\partial} \Omega_{n-1}^{or}(q) \to \cdots.$$

Here the occurring (forgetful) homomorphisms are obvious, and standard arguments show that the sequences are actually exact.

Similar relative bordism groups and long exact sequences are obtained if we use unoriented manifolds.

§6. Computation of $\Omega_n^{(or)}(1)$ and $\mathfrak{N}_n^{(or)}(1)$

The invariant $\theta(M, \xi) = w(\xi)^{-1} w(M)[M]$ induces homomorphisms

$$\theta\colon \Omega_n(B(S)O(1)) \to \mathbb{Z}_2, \quad \text{and} \quad \theta\colon \mathfrak{N}_n(B(S)O(1)) \to \mathbb{Z}_2.$$

Similarly, we have homomorphisms

$$\sigma\colon \Omega_n(B(S)O(1), 1) \to \mathbb{Z}_2, \quad \text{and} \quad \sigma\colon \mathfrak{N}_n(B(S)O(1), 1) \to \mathbb{Z}_2,$$

which associate $\theta(M, \xi, j)$ to a relative bordism class $[M, \xi, j]$. To see that σ is well-defined, consider a bordism $(X, \zeta, \tilde{j})$ between (M_0, ξ_0, j_0) and (M_1, ξ_1, j_1), and let j_l extend to a section v_l of $\mathrm{Hom}(\xi_l, TM_l)$ with only non-degenerate zeroes, $l=0,1$. Then v_0, v_1 and j, together with a vector field normal to ∂X, define a section in $\mathrm{Hom}(\zeta \oplus \mathbb{R}, TX)|\partial X$

which can be extended into a section s over all of X. Next, for $p=0$ or 1, consider the $p\cdot(n+p-1)$-codimensional submanifold $A_p=\bigcup_{x\in X} A_p(x)$ of the total space of $\mathrm{Hom}(\zeta\oplus\mathbb{R}, TX)$, where

$$A_p(x)=\{h\in \mathrm{Hom}(\zeta_x\oplus\mathbb{R}, T_xX)\,|\,\dim \mathrm{Ker}\, h=p\}$$

(compare [5], p. 120). Our section s can be required to map X into $A_0\cup A_1$ and to be transverse regular to A_1. Then $s^{-1}(A_1)$ is a one-dimensional compact submanifold of X whose boundary consists of the zeroes of j_0 and j_1. Thus $\theta(M_1,\xi_1,j_1)-\theta(M_0,\xi_0,j_0)$ is the mod 2 number of points in the boundary of $s^{-1}(A_1)$, and therefore vanishes.

Observe also that, if n is even, $\sigma([M,\xi,j])$ equals the mod 2 Euler number $\chi(M)_2$ of M (cf. Proposition 1.3 and Corollary 1.5).

Theorem 6.1. *Under all four orientedness assumptions σ is injective. More precisely, σ gives the following isomorphisms:*

$$\Omega_{2k}(BO(1),1)=\mathfrak{N}_{2k}(BO(1),1)=\mathbb{Z}_2,$$
$$\Omega_{2k}(BSO(1),1)=\mathfrak{N}_{2k}(BSO(1),1)=\mathbb{Z}_2;$$

and

$$\Omega_{2k-1}(BO(1),1)=\mathfrak{N}_{2k-1}(BO(1),1)=\mathbb{Z}_2,$$
$$\Omega_{2k-1}(BSO(1),1)=\mathfrak{N}_{2k-1}(BSO(1),1)=0,$$

for all positive n $(=2k$ or $=2k-1)$.

(Note the similarity of these groups with the (co)homology groups of $BO(1)$ with coefficients in $\mathbb{Z}_2$, resp. $\mathbb{Z}$.)

Proof. Let $[M,\xi,j]$ lie in the kernel of σ.

First let n be even. Then $\chi(M)\equiv 0(2)$. For $n\geq 4$ consider $M\times I$ with the lifted line bundle $p^*(\xi)$. If we attach a handle $D^1\times D^n$ (resp. $D^2\times D^{n-1}$) in small disks (over which ξ is trivial) in the interior of $M\times\{0\}$, the part of the boundary of $M\times I$ corresponding to $M\times\{0\}$ decreases (resp. increases) its Euler number by 2 [16]. This is due to the following well-known additivity property of the Euler characteristic: if we identify two compact, even-dimensional manifolds M_1 and M_2 along full components of their boundaries, then the Euler characteristic of the resulting manifold equals $\chi(M_1)+\chi(M_2)$. Thus we can extend $M\times I$ to obtain a manifold X such that $\partial X=M\times\{1\}\cup\partial M\times I\cup M'$, where M' is connected and has vanishing Euler number. Also $p^*(\xi)$ can be extended suitably to a line bundle ζ over X. According to Corollary 1.5, we can extend the vector bundle injection $j\colon \zeta|\partial M'\to T(\partial X)$ injectively over all of M', and of course also over $\partial M\times I$. Thus $[M,\xi,j]=0$.

If $n=2$, we have to do without attaching handles of the form $D^2\times D^{n-1}$, since this would in general disconnect the boundary of $M\times I$. Let ∂M

consist of the circles $S_1, \ldots, S_l$, and extend M to a closed surface $\tilde{M}$ by glueing in disks $D_1, \ldots, D_l$. Since $\xi|\partial M$ has to be trivial, we can also extend ξ to a line bundle $\tilde{\xi}$ over $\tilde{M}$. Now if M is oriented, $(\tilde{M}, \tilde{\xi})$ bounds because $\Omega_2(BO(1)) = \Omega_2 \oplus \mathfrak{N}_1 = 0$ (see Remark 3.4), resp.

$$\Omega_2(BSO(1)) = \Omega_2 = 0;$$

moreover, since $\chi(\tilde{M})$ and $\chi(M)$ are even, so is l. If M is not oriented, then $(\tilde{M}, \tilde{\xi})$ is bordant to $l \equiv \chi(\tilde{M})$ times $(RP(2), \mathbb{R})$ plus a multiple of $(RP(1,0), \lambda)$ (see [19], Lemma 3.2); now λ occurs as a line field on the Klein bottle $RP(1,0)$; and the complement of a disk in $RP(2)$ admits a nowhere vanishing vector field, tangent to the boundary circle. In any case, attaching an appropriate bordism to $\tilde{M} \times I$, we obtain a bordism from (M, ξ, j) to an even number of triples of the form (D^2, $\mathbb{R}$, vector field along S^1). But these can be cancelled pairwise by handles $D^1 \times D^2$. Therefore $[M, \xi, j] = 0$ also when $n = 2$.

Finally, if n is odd, we may again assume that M is connected. Then, applying Proposition 1.8 if ξ is non-trivial, or otherwise using the fact that $2\chi(M) = \chi(\partial M) = 0$, we can extend j to a line bundle imbedding over all of M. Hence again $[M, \xi, j] = 0$.

This establishes the injectivity of σ. To determine its image, observe that

$$\sigma([D^{2k}, \mathbb{R}, \text{vector field on } S^{2k-1}]) = \chi(D^{2k})_2 = 1;$$

and that

$$\theta([RP(2k-1), \lambda]) = 1 \qquad \text{(see Example 2.4)}.$$

Thus it only remains to show that $\sigma([M, \xi, j]) = 0$ if $[M, \xi, j]$ lies in $\Omega_{2k-1}(BSO(1), 1)$ or in $\mathfrak{N}_{2k-1}(BSO(1), 1)$. But this follows immediately from the discussion in §1 (preceding Remark 1.9) and the fact that $\chi(M') = (1/2)\chi(\partial M') = 0$ for each component M' of M, since the nowhere vanishing vector field j is tangent to ∂M. ‖

Next recall that $\Omega_*(BO(1)) \simeq \Omega_* \oplus \mathfrak{N}_{*-1}$ (cf. Remark 3.4), and $\mathfrak{N}_*(BO(1)) \simeq \mathfrak{N}_* \otimes H_*(BO(1), \mathbb{Z}_2)$ [3] are well known. Hence the following result determines the groups $\Omega_*^{(or)}(1)$ and $\mathfrak{N}_*^{(or)}(1)$ completely.

Theorem 6.2. *All of the following sequences are exact (for all $k \geq 0$ resp. $n > 0$).*

$$\text{(i)} \quad 0 \longrightarrow \Omega_{4k+4}(1) \xrightarrow{f} \Omega_{4k+4}(BO(1)) \xrightarrow{\theta = \chi_2} \mathbb{Z}_2 \to 0$$

$$0 \longrightarrow \Omega_{4k+3}(1) \xrightarrow{f} \Omega_{4k+3}(BO(1)) \xrightarrow{\theta} \mathbb{Z}_2 \to 0$$

$$0 \longrightarrow \Omega_{4k+2}(1) \xrightarrow{f} \Omega_{4k+2}(BO(1)) \longrightarrow 0$$

$$0 \to \mathbb{Z}_2 \xrightarrow{\varepsilon} \Omega_{4k+1}(1) \xrightarrow{f} \Omega_{4k+1}(BO(1)) \xrightarrow{\theta} \mathbb{Z}_2 \to 0$$

$$\begin{array}{llll}
\text{(ii)} & 0 \longrightarrow \Omega^{or}_{4k+4}(1) \xrightarrow{f} \Omega_{4k+4} & \xrightarrow{\chi_2} \mathbb{Z}_2 \to 0 \\
& 0 \longrightarrow \Omega^{or}_{4k+3}(1) \xrightarrow{f} \Omega_{4k+3} & \longrightarrow 0 \\
& 0 \longrightarrow \Omega^{or}_{4k+2}(1) \xrightarrow{f} \Omega_{4k+2} & \longrightarrow 0 \\
& 0 \to \mathbb{Z}_2 \xrightarrow{\varepsilon} \Omega^{or}_{4k+1}(1) \xrightarrow{f} \Omega_{4k+1} & \longrightarrow 0, \\
\text{(iii)} & 0 \longrightarrow \mathfrak{N}_n(1) \xrightarrow{f} \mathfrak{N}_n(BO(1)) & \xrightarrow{\theta} \mathbb{Z}_2 \to 0, \\
\text{(iv)} & 0 \longrightarrow \mathfrak{N}^{or}_{2k+2}(1) \xrightarrow{f} \mathfrak{N}_{2k+2} & \xrightarrow{\chi_2} \mathbb{Z}_2 \to 0 \\
& 0 \longrightarrow \mathfrak{N}^{or}_{2k+1}(1) \xrightarrow{f} \mathfrak{N}_{2k+1} & \longrightarrow 0.
\end{array}$$

Here χ_2 *assigns to* $[M, \xi]$ *or* $[M]$ *the mod* 2 *Euler number*

$$\chi(M)_2 = w_n(M)[M].$$

In (i) *and* (ii) $\varepsilon = \partial \cdot \sigma^{-1}$ *can be defined by* $\varepsilon(1) = [S^{4k+1}$, *trivial line field*$] = [S^1 \times CP(2k)$, *trivial line field*$]$.

Moreover, $\Omega_{4k+1}(1)$ *and* $\Omega^{or}_{4k+1}(1)$ *consist entirely of* 2*-torsion.*

(The image of f in $\Omega_n(BO(1))$ and $\mathfrak{N}_n(BO(1))$ was already determined in [19].)

Proof. Using Theorem 6.1 and Example 2.4, we can extract the (short) exact sequences here from the long exact sequences of §5.

To exclude the possibility of 4-torsion, consider $[M, \xi] \in \Omega^{(or)}_{4k+1}(1)$. Since $\Omega_{4k+1}(BO(1)) = \Omega_{4k+1} \oplus \mathfrak{N}_{4k}$ and Ω_{4k+1} consist entirely of 2-torsion, there exists $[X, \zeta, j] \in \Omega_{4k+2}(B(S)\,O(1), 1)$ with

$$\partial[X, \zeta, j] = 2[M, \xi].$$

To show that $[M, \xi]$ is of order 2, or, equivalently, that

$$\sigma([X, \zeta, j]) = \chi(X)_2 = 0,$$

we use the following

Proposition 6.3. *A class* α *in the bordism ring* Ω_* *can be represented by a manifold which admits an orientation reversing diffeomorphism, if and only if* $2\alpha = 0$.

As R.J. Rowlett pointed out to me, this follows easily from Proposition 5 in [1].

Thus let Y be an oriented bordism from M to a manifold M' with an orientation reversing diffeomorphism h. Identifying via h, we can make $Y \cup X \cup Y$ into a closed oriented $(4k+2)$-manifold which necessarily has even Euler number. The last statement of Theorem 6.2 follows.

Remark 6.4. The image of the forgetful map f in $\mathfrak{N}_n(BO(1))$ has already been computed previously by Stong [19], who uses a suitable explicit basis of the $\mathfrak{N}_*$-module $\mathfrak{N}_*(BO(1))$ to represent sufficiently many classes in $\mathfrak{N}_n(BO(1))$ by manifolds with line fields. This basis, and the determination of $\mathfrak{N}_n^{(or)}(1)$ in Theorem 6.2, allow one to prove: every element in $\mathfrak{N}_n^{(or)}(1)$ can be represented by a line field transversal to a foliation of codimension one. The corresponding statement for $\Omega_n^{(or)}(1)$ holds at least if the signature of the underlying manifold vanishes. For more details see [7].

§7. Computation of $\mathfrak{N}_n^{coor}(1)$

Let $\xi \subset TM$ be a line field over the closed n-manifold M. Then exterior multiplication to the left gives a bundle isomorphism

$$\Lambda^{n-1}(TM/\xi) \xrightarrow{\cong} \mathrm{Hom}(\xi, \xi_M),$$

where $\xi_M = \Lambda^n TM$ as in §3. Thus orientations on the normal bundle TM/ξ correspond to isomorphisms (given up to multiplication by positive functions) between ξ and ξ_M. Therefore we can identify $\mathfrak{N}_n^{coor}(1)$ with the bordism group of classes of pairs (M, j), where j is a vector bundle embedding of ξ_M into TM. Two such pairs (M_0, j_0) and (M_1, j_1) are bordant, if there is a bordism X between M_0 and M_1 with an injection $\tilde{j}: \xi_X \to TX$ such that $j_l = \tilde{j} \cdot i s_l$; here $i s_l: \xi_{M_l} \xrightarrow{\cong} \xi_X | M_l$ is the isomorphism given by left exterior multiplication with an outward pointing (if $l=1$), resp. an inward pointing (if $l=0$), vector field of X along M_l. Note that $-[M, j] = [M, -j]$.

To compute $\mathfrak{N}_n^{coor}(1)$, it will again be convenient to use relative bordism. Thus consider pairs (M, j^∂), where M is a compact n-manifold, and $j^\partial: \xi_{\partial M} \to T(\partial M)$ is an injective vector bundle map. We call two such pairs (M_0, j_0^∂) and (M_1, j_1^∂) bordant, if there is (i) a compact $(n+1)$-manifold X whose boundary ∂X contains the disjoint union of M_0 and M_1, and (ii) a bundle injection $\tilde{j}$ from $\xi_{\partial X} | \partial X - (\mathring{M}_0 \cup \mathring{M}_1)$ into $T(\partial X)$ such that $j_l^\partial = \tilde{j} \cdot i s_l$, $l = 0, 1$, where the isomorphisms $i s_l: \xi_{\partial M_l} \to \xi_{\partial X} | \partial M_l$ are given by vector fields pointing into M_1 and out of M_0. The group of bordism classes $[M, j^\partial]$ is denoted by $\mathfrak{N}_n^{coor}(BO(1), 1)$.

Again we have a well-defined homomorphism

$$\sigma: \mathfrak{N}_n^{coor}(BO(1), 1) \to \mathbb{Z}_2,$$

which associates $\theta(M, \xi_M, j^\partial)$ to $[M, j^\partial]$ (using an outward pointing vector field, we identify j^∂ here with an injection from $\xi_M | \partial M$ into $T(\partial M)$).

For even n we then have also $\sigma[M, j^\partial] = \chi(M)_2$.

Theorem 7.1. *σ is an isomorphism for $n > 1$.*

Proof. To establish injectivity, we proceed precisely as in the proof of Theorem 6.1, except for the discussion of the case $n = 2$, where we have to replace $(RP(2), \mathbb{R})$ by $(RP(2), \lambda)$ (compare Example 2.4).

For even n, the surjectivity of σ is clear (e.g. $\sigma[RP(2k)]=1$). Hence assume that $n=2k+1$, $k>0$. Consider the vector field on D^{2k}, which attaches to each point its locus vector, and homotop it, around S^{2k-1}, into a vector field tangent to S^{2k-1}. Identifying D^{2k} with $D^{2k}\times\{0\}$ in $D^{2k}\times D^1$, we can combine a suitable extension of the vector field above, with the locus vector field along D^1, to obtain a homomorphism v: $\xi_M \to TM$ over the solid Klein bottle

$$M=D^{2k}\times D^1/(x_1,x_2,\ldots,x_{2k};\,1)\sim(-x_1,x_2,\ldots,x_{2k};\,-1).$$

The only zero of v lies at $(0,\ldots,0;0)$, and v restricts to an embedding $j^\partial\colon \xi_{\partial M}\to T(\partial M)$. Clearly $\sigma([M,j^\partial])=1$.

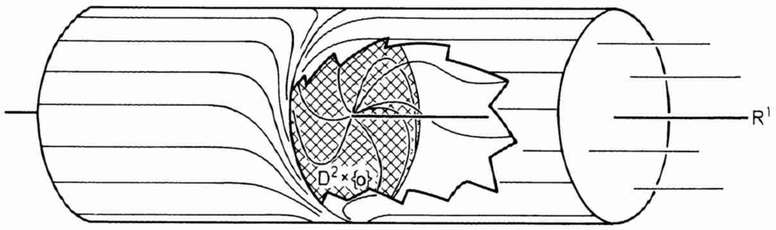

Fig. 2. The integral curves of the vector field on $D^{2k}\times D^1$, $k=1$, which defines v

Finally note that $\mathfrak{N}_1^{coor}(BO(1),1)=0$.

Theorem 7.2. *For all positive, odd n the natural forgetful map*

$$f\colon \mathfrak{N}_n^{coor}(1)\to\mathfrak{N}_n$$

is an isomorphism.

If n is even, then the sequence

$$0\to\mathbb{Z}_2\xrightarrow{\partial\circ\sigma^{-1}}\mathfrak{N}_n^{coor}(1)\xrightarrow{f}\mathfrak{N}_n\xrightarrow{\chi_2}\mathbb{Z}_2\to 0$$

is exact.

Furthermore, $\mathfrak{N}_n^{coor}(1)$ *has 4-torsion if and only if* $n\equiv 0(4)$.

Proof. For $n>1$ consider the diagram

$$\begin{array}{ccccccccc}\cdots\to\mathfrak{N}_n^{coor}(1) & \xrightarrow{f} & \mathfrak{N}_n & \to & \mathfrak{N}_n^{coor}(BO(1),1) & \xrightarrow{\partial} & \mathfrak{N}_{n-1}^{coor}(1)\to\cdots \\ & & & \searrow^{\chi_2} & \downarrow{\scriptstyle\sigma\,\cong} & & \\ & & & & \mathbb{Z}_2 & & \end{array}$$

The horizontal sequence is exact. Moreover commutativity follows easily from the relations in Remark 2.3 and in the proof of Theorem 3.1. Thus the vanishing behavior of χ_2 implies the first two statements of the theorem.

To check for possible 4-torsion, let $[M,j]\in\mathfrak{N}_{2k}^{coor}(1)$, and assume that M is connected. Consider $M\times I$ with

$$j^{\partial}=j\cup j\colon \xi_{\partial(M\times I)}=\xi_M\cup\xi_M\to T(\partial(M\times I))=TM\cup TM.$$

Then $\partial[M\times I,j^{\partial}]=2[M,j]$; furthermore, from the way we have to identify $\xi_{\partial(M\times I)}$ with $\xi_{M\times I}|\partial(M\times I)$, we see that

$$\sigma[M\times I,j^{\partial}]=\theta(M\times I,p^*(\xi_M),j\cup -j).$$

Thus, since σ and ∂ are injective here, $2[M,j]=0$ precisely if

$$\theta(M\times I,p^*(\xi_M),j\cup -j)=0.$$

But according to § 1 and 2, especially Proposition 1.8, this is equivalent to j being concordant to its negative $-j$. The results of § 3 now imply the remainder of Theorem 7.2.

It follows that for $n\equiv 2(4)$, there is still a (non-canonical) isomorphism $\mathfrak{N}_n^{coor}(1)\simeq\mathfrak{N}_n$. For $n\equiv 0(4)$, however, $\mathfrak{N}_n^{coor}(1)\simeq\mathbb{Z}_4+(\mathbb{Z}_2)^{d-2}$, where d is the dimension of $\mathfrak{N}_n$ over $\mathbb{Z}_2$.

§ 8. The Case of q-Plane Fields for Higher q

If $q>1$, and ξ is a q-dimensional vector bundle over a compact n-manifold M, then a homomorphism $v\colon\xi\to TM$ needs by no means be injective even if it has no zeroes. However, if $n>2q-3$, then we can always modify v so that it becomes injective outside of a closed $(q-1)$-dimensional submanifold S of M, and that the kernel of v is one-dimensional at every point x of S. Since $v|S$ has constant rank, the singularity S comes naturally equipped with the one-dimensional Kernel bundle and the $(n-q+1)$-dimensional cokernel bundle of $v|S$. From arguments as the ones in the beginning of § 6 we can see that this construction is compatible with bordism, and therefore it gives rise to well-defined homomorphisms such as

$$\mathfrak{N}_n(BO(q),q)\xrightarrow{\sigma}\mathfrak{N}_{q-1}(BO(1)\times BO(n-q+1))$$

(for the other orientedness cases, the range of σ has to be modified). This seems to be the appropriate generalization of the homomorphism σ defined in § 6. It can be shown [9] that these "singularity homomorphisms" σ are injective, and combined with the long exact sequences in § 5, they provide a powerful tool for the study of the bordism groups $\Omega_n^{(or)}(q)$ and $\mathfrak{N}_n^{(or)}(q)$.

Now we turn to the case $q=n$ which in general is excluded from the discussion above, but which still can be treated, via duality (cf.: Lemma 5.2), by line field methods. Indeed, our groups, identified with $'\Omega_n^{(or)}(0)$, $'\mathfrak{N}_n^{(coor)}(0)$ or $'\mathfrak{N}_n^{or}(0)$, occur near the end of long exact sequences where the relevant third term can be determined by invariants such as θ or $\tilde{\theta}$. We obtain:

Theorem 8.1. *None of the groups* $\Omega_n^{(or)}(n) = {}'\Omega_n^{(or)}(0)$ *and* $\mathfrak{N}_n^{(or)}(n) = {}'\mathfrak{N}_n^{(coor)}(0)$ *and* $\mathfrak{N}_n^{coor}(n) = {}'\mathfrak{N}_n^{or}(0)$ *has 4-torsion. They fit into the following exact sequences (for* $n \geqq 1$*):*

$$
\begin{array}{lllll}
\text{(i)} & 0 \longrightarrow \Omega_n(n) & \xrightarrow[\cong]{'f} \Omega_n \to 0, & & \text{if } n \not\equiv 1(4); \\
& 0 \to \mathbb{Z}_2 \xrightarrow{\varepsilon} \Omega_n(n) & \xrightarrow{'f} \Omega_n \to 0, & & \text{if } n \equiv 1(4). \\
\text{(ii)} & 0 \to \mathbb{Z} \xrightarrow{\varepsilon} \Omega_n^{or}(n) & \xrightarrow{'f} \Omega_n \to 0, & & \text{if } n \text{ is even;} \\
& 0 \longrightarrow \Omega_n^{or}(n) & \xrightarrow[\cong]{'f} \Omega_n \to 0, & & \text{if } n \equiv 3(4); \\
& 0 \to \mathbb{Z}_2 \xrightarrow{\varepsilon} \Omega_n^{or}(n) & \xrightarrow{'f} \Omega_n \to 0, & & \text{if } n \equiv 1(4). \\
\text{(iii)} & 0 \longrightarrow \mathfrak{N}_n(n) & \xrightarrow[\cong]{'f} \mathfrak{N}_n \to 0, & & \text{for all } n. \\
\text{(iv)} & 0 \to \mathbb{Z}_2 \xrightarrow{\varepsilon} \mathfrak{N}_n^{or}(n) & \xrightarrow{'f} Im_n \to 0, & & \text{if } n \text{ is even;} \\
& 0 \longrightarrow \mathfrak{N}_n^{or}(n) & \xrightarrow[\cong]{'f} Im_n \to 0, & & \text{if } n \text{ is odd}
\end{array}
$$

(here Im_n is the image of Ω_n in $\mathfrak{N}_n$ under the natural forgetful homomorphism).

$$
\begin{array}{lllll}
\text{(v)} & 0 \to \mathbb{Z} \xrightarrow{\varepsilon} \mathfrak{N}_n^{coor}(n) & \xrightarrow{'f} \mathfrak{N}_n \to 0, & & \text{if } n \text{ is even;} \\
& 0 \longrightarrow \mathfrak{N}_n^{coor}(n) & \xrightarrow[\cong]{'f} \mathfrak{N}_n \to 0, & & \text{if } n \text{ is odd.}
\end{array}
$$

Here we always define $\varepsilon(1)$ *to be the class of the sphere* S^n *with the obvious structure;* $'f$ *comes from taking the underlying manifold.*

Finally observe that classes in $'\Omega_n^{or}(0)$ and $'\mathfrak{N}_n^{or}(0)$ are formed by closed manifolds $M = M_+ \cup M_-$ where the trivial 0-plane field on M_+ and M_- is oriented according to the subscript. It is not hard to show that every class can actually be represented by some M_+ alone. Thus $\Omega_n^{or}(n) = {}'\Omega_n^{or}(0)$, resp. $\mathfrak{N}_n^{coor}(n) = {}'\mathfrak{N}_n^{or}(0)$, can be identified with the refined bordism group Ω_0^n, resp. $\mathfrak{M}_0^n$, of Reinhart [16], who considers two closed manifolds M_1 and M_2 as bordant if a bordism from M_1 to M_2 admits a non-singular vector field pointing outwards at M_2 and inwards at M_1. Furthermore the sequences (ii) and (v) are the ones which Reinhart uses to compute his groups.

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On Surfaces of Class VII_0

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§ 0. Introduction

In this paper, we study (compact complex analytic) surfaces S satisfying the following two conditions:

(A) *the first Betti number* $b_1(S)$ *equals one and the second Betti number* $b_2(S)$ *vanishes,*

(B) S *contains no curves.*

We shall construct three kinds of such surfaces S_M, $S^{(+)}_{N,p,q,r;t}$, $S^{(-)}_{N,p,q,r}$ in § 2–4, respectively, and, in § 5, shall give a theorem to the effect that *any surface satisfying* (A), (B) *and an additional condition* (C) *is one of them* (see § 5 for the condition (C)). The remaining sections § 6–9 will be devoted to the proof of this theorem.

We recall some results of Kodaira concerning the structure of surfaces. He defines surfaces S of Class VII_0 by the following condition:

$$(\mathrm{VII}_0) \qquad b_1(S)=1 \quad \textit{and} \quad S \textit{ is minimal}.$$

He has completely determined the structure of surfaces other than those of Class VII_0 and their quadric transforms in [1–3]. Moreover, he showed that some kind of elliptic surfaces and all of Hopf surfaces satisfy (A) and belong to the Class VII_0 but do not satisfy (B), and proved a theorem to the effect that *surfaces which satisfy* (A) *but do not satisfy* (B) *turn out to be elliptic surfaces or Hopf surfaces.* He showed that *any deformations of elliptic surfaces of Class* VII_0 *or Hopf surfaces are elliptic surfaces of Class* VII_0 *or Hopf surfaces.* In the preface of [2], he questioned whether there exist surfaces of Class VII_0 other than elliptic surfaces and Hopf surfaces. By the theorem of Kodaira quoted above, it suffices to consider the following two problems:

i) Find surfaces S of Class VII_0 with $b_2(S) \neq 0$.

ii) Find surfaces which satisfy both (A) and (B).

Our result may be regarded as an answer to the second problem.

I would like to express my hearty thanks to Professor Kodaira, who introduced me to this problem and whose suggestions throughout the writing of this paper were invariably helpful. Most important, he allowed me to read his unpublished note on surfaces of Class VII_0. I would also like to thank Professor E. Bombieri who kindly communicated that he discovered such surfaces independently and gave a characterization of them. Last but not least, I would like to thank Dr. S. Iitaka who read throughout my illegible manuscript and corrected many errors.

§ 1. Preliminaries and Some Lemmas

In this paper, by a *surface* we mean a connected compact complex manifold of dimension 2. We employ the notations used in Kodaira [1–3]. For example, we write $H^\nu(\mathcal{S})$ instead of $H^\nu(S, \mathcal{S})$ for a sheaf $\mathcal{S}$ on a surface S, and we denote by $c_1^2(S)$ and $c_2(S)$ the Chern numbers of S. The irregularity $\dim H^1(\mathcal{O})$ and the geometric genus $\dim H^2(\mathcal{O})$ are indicated by $q(S)$ and $p_g(S)$, respectively.

By using Theorem 3 of Kodaira [1] and the formula of Riemann-Roch, we can easily prove

Proposition 1. *Let S be a surface satisfying* (A). *Then* $c_1^2(S)=c_2(S)=0$, $q(S)=1$, $p_g(S)=0$, $h^{1,0}(S)=\dim H^0(\Omega^1)=0$ *and* $H^1(\mathbb{C}^*)\simeq H^1(\mathcal{O}^*)$ *canonically. Moreover*

(1) $\sum_{\nu=0}^{2}(-1)^\nu \dim H^\nu(\mathcal{O}(F))=0$ *for any line bundle F, and*

(2) $\sum_{\nu=0}^{2}(-1)^\nu \dim H^\nu(\Theta)=0$ *for the sheaf Θ of holomorphic tangent vector fields.*

In what follows we let S denote a surface satisfying conditions (A) and (B).

Lemma 1. (i) *Any unramified covering surface S' of S satisfies the same conditions* (A) *and* (B).

(ii) *Any deformation S'' of S satisfies* (A) *and* (B).

(iii) $mK \neq 0$ *for any* $m \neq 0$ *in* $\mathbb{Z}$, *where K denotes the canonical bundle of S.*

(iv) $H^\nu(\mathcal{O}(F))=0$, $\nu=0,1,2$, *for* $F\neq 0, K$.

Proof. (i) It is clear that S' satisfies (B). Hence, by the classification theory of Kodaira [1], we see that S' is a $K3$ surface, a complex torus or a surface of Class VII_0. Since $c_2(S')=c_2(S)=0$, S' is not a $K3$ surface. If S' is a complex torus, then S is a Kähler surface and, consequently, $b_1(S)$ is even. Thus $b_1(S')=1$. From $c_2(S')=0$ it follows that $b_2(S')=0$. Hence S' satisfies (A).

(ii) Evidently S'' satisfies (A). If S'' contains a curve, then S'' is an elliptic surface or a Hopf surface and therefore S is also an elliptic surface or a Hopf surface by theorems of Kodaira quoted in §0. Thus S'' satisfies (B).

(iii) If $mK=0$ for some $m \neq 0$, then there exists an m-fold unramified covering surface S' such that $p_g(S') \neq 0$. This contradicts the above statement (i) and Proposition 1.

(iv) The vanishing of $H^\nu(\mathcal{O}(F))$ follows from (1), the duality theorem of Serre and the fact that S contains no curves, q.e.d.

For any $F \in H^1(\mathbb{C}^*) \simeq H^1(\mathcal{O}^*)$, we write $\Omega^1(F) = \Omega^1 \otimes \mathcal{O}(F)$. By $\mathbb{C}(F)$ and $d\mathcal{O}(F)$ we denote the sheaf of germs of locally constant sections of F and the sheaf of germs of d-closed holomorphic 1-forms with coefficients in F, respectively (cf. Section 11 of Kodaira [2]). Then we have the following exact sequences:

$$0 \to \mathbb{C}(F) \to \mathcal{O}(F) \xrightarrow{d} d\mathcal{O}(F) \to 0,$$

$$0 \to d\mathcal{O}(F) \to \Omega^1(F) \xrightarrow{d} \Omega^2(F) \to 0.$$

From these sequences, we can derive the exact sequences:

$$\begin{aligned} 0 &\longrightarrow H^0(\mathbb{C}(F)) \to H^0(\mathcal{O}(F)) \longrightarrow H^0(d\mathcal{O}(F)) \\ &\xrightarrow{\vartheta} H^1(\mathbb{C}(F)) \to H^1(\mathcal{O}(F)) \longrightarrow H^1(d\mathcal{O}(F)) \\ &\longrightarrow H^2(\mathbb{C}(F)) \to H^2(\mathcal{O}(F)) \longrightarrow H^2(d\mathcal{O}(F)) \\ &\longrightarrow H^3(\mathbb{C}(F)) \to 0, \end{aligned} \tag{3}$$

$$\begin{aligned} 0 &\longrightarrow H^0(d\mathcal{O}(F)) \to H^0(\Omega^1(F)) \xrightarrow{d} H^0(\Omega^2(F)) \\ &\longrightarrow H^1(d\mathcal{O}(F)) \to H^1(\Omega^1(F)) \xrightarrow{d} H^1(\Omega^2(F)) \\ &\longrightarrow H^2(d\mathcal{O}(F)) \to H^2(\Omega^1(F)) \xrightarrow{d} H^2(\Omega^2(F)) \\ &\longrightarrow H^3(d\mathcal{O}(F)) \to 0. \end{aligned} \tag{4}$$

Hence, in view of Lemma 1 (iv), we obtain, by using $\Omega^2(F) \simeq \mathcal{O}(K+F)$,

$$\vartheta\colon H^\nu(d\mathcal{O}(F)) \simeq H^{\nu+1}(\mathbb{C}(F)) \quad \text{for } F \neq 0, K, \tag{5}$$

$$H^\nu(d\mathcal{O}(F)) \simeq H^\nu(\Omega^1(F)) \quad \text{for } F \neq 0, -K. \tag{6}$$

Let $\tilde{S}$ be the universal covering surface of S and G the covering transformation group of $\tilde{S}$ over S. Defining $j(\mu)$ to be $\tilde{S} \times \mathbb{C}/\sim$ for any $\mu \in \mathrm{Hom}(G, \mathbb{C}^*)$, where $(p, \zeta) \sim (g(p), \mu(g)^{-1} \cdot \zeta)$, $p \in \tilde{S}, \zeta \in \mathbb{C}$, for $g \in G$, we have a canonical isomorphism $j\colon \mathrm{Hom}(G, \mathbb{C}^*) \simeq H^1(\mathbb{C}^*)$. For simplicity, we write $F_\mu = j(\mu)$ and $\mu_F = j^{-1}(F)$. It is easy to check that the Chern class $c(F)=0$ if and only if $\mu_F \,|\, \mathrm{Tor}(G/[G,G]) = 1$. (For any abelian group H, we denote by $\mathrm{Tor}\, H$ the torsion group of H.) Let $\mathscr{P} = \mathrm{Ker}(c\colon H^1(\mathcal{O}^*) \to H^2(\mathbb{Z}))$ and let $\mathrm{Hom}(G, \mathbb{C}^*)_0$ denote the subgroup

$$\{\mu \in \mathrm{Hom}(G, \mathbb{C}^*) \,|\, \mu \,|\, \mathrm{Tor}(G/[G,G]) = 1\}.$$

Let $\bar{G}=\{g\in G \mid g \bmod [G,G]$ is of finite order$\}$. Then we have the exact sequence:

(7) $$0\to\bar{G}\to G\to\mathbb{Z}\to 0.$$

From the above remark, it follows that

$$j\mid\mathrm{Hom}(G,\mathbb{C}^*)_0\colon \mathrm{Hom}(G,\mathbb{C}^*)_0\simeq\mathscr{P}.$$

We fix an element $g_0\in G$ representing a generator of $G/\bar{G}\simeq\mathbb{Z}$. For a given $\mu\in\mathrm{Hom}(G,\mathbb{C}^*)$, $\mathbb{C}$ can be considered as a G-module by defining $g\cdot\zeta=\mu(g)\zeta$ for any $g\in G$ and $\zeta\in\mathbb{C}$. We denote by $H_\mu^\nu(G)$ the ν-th cohomology group of G with respect to μ. Regarding $\mathbb{C}$ as a trivial $\bar{G}$-module, we form the ν-th cohomology group $H^\nu(\bar{G})$.

Let g_0^* be the automorphism of $H^\nu(\bar{G})$ induced from the inner automorphism of $\bar{G}$: $g\to g_0 g g_0^{-1}$, $g\in\bar{G}$. We define $H^\nu(\bar{G})^\lambda$ to be

$$\{\gamma\in H(\bar{G})\mid g_0^*\gamma=\lambda\gamma\} \quad \text{for } \lambda\in\mathbb{C}^*.$$

Using exact sequences of cohomology groups derived from (7), we obtain

(8) $$H_\mu^1(G)\simeq H^1(\bar{G})^{\mu(g_0)}, \quad \text{for any non-trivial}$$

$\mu\in\mathrm{Hom}(G,\mathbb{C}^*)_0$, and

(9) $$H^1(\bar{G})^1=0.$$

Refering to [4], we have

(10) $$H^1(\mathbb{C}(F))\simeq H_{\mu_F}^1(G) \quad \text{for any } F\in H^1(\mathbb{C}^*).$$

Hence it follows from (8) that

(11) $$H^1(\mathbb{C}(F))\simeq H^1(\bar{G})^{\mu_F(g_0)} \quad \text{for any non-trivial } F\in\mathscr{P}.$$

Kodaira establishes the following duality theorem and the stability theorem:

(12) $$\dim H^\nu(\mathbb{C}(F))=\dim H^{4-\nu}(\mathbb{C}(-F)) \quad \text{for } F\in H^1(\mathbb{C}^*),$$

(13) $$\sum_{\nu=0}^{4}(-1)^\nu\dim H^\nu(\mathbb{C}(F))=c_2(S)=0 \quad \text{for } F\in\mathscr{P}.$$

(See § 13 of Kodaira [3].)

We define $\bar{\mu}\in\mathrm{Hom}(G,\mathbb{C}^*)$ for $\mu\in\mathrm{Hom}(G,\mathbb{C}^*)$ by putting $\bar{\mu}(g)=\overline{\mu(g)}$ for any $g\in G$ and define $\bar{F}\in H^1(\mathbb{C}^*)$ for $F\in H^1(\mathbb{C}^*)$ by putting $\bar{F}=j(\bar{\mu}_F)$. In case $\mu=\bar{\mu}$ and $F=\bar{F}$, we say that μ is *real* and F is *real*, respectively.

For later use, we give a lemma.

Lemma 2. *If* $\omega \in H^0(\Omega^1(F))$ *and* $\eta \in H^0(\Omega^1(E))$, $\omega \neq 0$, $\eta \neq 0$, *then either*

i) $F=E$ *and* $\omega = c\eta$ *for a constant* $c \neq 0$, *or*

ii) $F+E+K=0$ *and* $\omega \wedge \eta \neq 0$, $\omega \wedge \eta \in H(\Omega^2(F+E))$, *and moreover* $\omega \in H^0(d\mathcal{O}(F))$ *and* $\eta \in H^0(d\mathcal{O}(E))$.

Especially if $2F+K \neq 0$, *then* $\dim H^0(\Omega^1(F)) \leqq 1$.

Note that $\omega \in H^0(\Omega^1(F))$ and $\eta \in H^0(\Omega^1(E))$ are written, respectively, in the form $\{\omega_i\}$ and $\{\eta_i\}$ by using local holomorphic 1-forms ω_i, η_i where $\omega_i = f_{ij}\omega_j$, $\eta_i = e_{ij}\eta_j$, $F=\{f_{ij}\}$, $E=\{e_{ij}\} \in H^1(\mathbb{C}^*) \simeq H^1(\mathcal{O}^*)$. $\omega \wedge \eta$ is then defined to be $\{\omega_i \wedge \eta_i\}$.

Proof of Lemma 2. Assume $\omega \wedge \eta = 0$. Then $\omega_i = f_i \eta_i$ for some non-zero local meromorphic function f_i. Evidently $\{f_i\}$ gives a meromorphic section of $F-E$. Since S contains no curves, $\{f_i\}$ turns out to be a non-vanishing holomorphic section of $F-E$. Hence we obtain $F-E=0$. If we choose $f_{ij}=e_{ij}$, then $f_i = f_j = c$ is constant and $\omega_i = c\eta_i$, i.e., $\omega = c\eta$.

Assume $\omega \wedge \eta \neq 0$. Then the corresponding divisor $(\omega \wedge \eta)$ is 0, because S contains no curves. Since $\omega \wedge \eta \in H^0(\Omega^2(F+E)) = H^0(\mathcal{O}(F+E+K))$, we obtain $F+E+K=0$. Moreover if $d\omega = \{d\omega_i\} \neq 0$, then $d\omega$ gives a nonzero element of $H^0(\Omega^2(F))$. Hence from Lemma 1(iv) we see that $K+F=0$ and therefore $E=0$. This implies that $\eta \in H^0(\Omega^1)$ and $h^{1,0}(S) \geqq 1$. Thus, by (4), we obtain $\omega \in H^0(d\mathcal{O}(F))$ and, similarly, $\eta \in H^0(d\mathcal{O}(E))$, q.e.d.

§ 2. Surfaces S_M

Let $M=(m_{ij}) \in SL(3,\mathbb{Z})$ be a unimodular matrix with eigen-values $\alpha, \beta, \bar{\beta}$ such that $\alpha > 1$ and $\beta \neq \bar{\beta}$. We choose a *real* eigen-vector (a_1, a_2, a_3) and an eigen-vector (b_1, b_2, b_3) of M corresponding to α and β, respectively. Clearly α is an irrational number and

(14) $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ are linearly independent over $\mathbb{R}$,

$$(\alpha a_j, \beta b_j) = \sum_{k=1}^{3} m_{jk}(a_k, b_k) \quad \text{for } j=1,2,3. \tag{15}$$

By $\mathbb{H}$ we denote the upper half of the complex plane. Let G_M be the group of analytic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0&: (w,z) \to (\alpha w, \beta z), \\ g_i&: (w,z) \to (w+a_i, z+b_i) \quad \text{for } i=1,2,3. \end{aligned} \tag{16}$$

(14) and (15) imply that the action of G_M on $\mathbb{H} \times \mathbb{C}$ is properly discontinuous and has no fixed points. We define S_M to be the quotient surface $\mathbb{H} \times \mathbb{C}/G_M$. Then it follows from (14), (15), (16) that S_M is differentiably a 3-torus bundle over a circle and the relations between the generators

g_0, g_1, g_2, g_3 of G_M are as follows:

$$g_i g_j = g_j g_i \qquad \text{for } i, j = 1, 2, 3,$$
$$g_0 g_i g_0^{-1} = g_1^{m_{i1}} g_2^{m_{i2}} g_3^{m_{i3}} \qquad \text{for } \quad i = 1, 2, 3.$$

From this we derive

$$G_M/[G_M, G_M] \simeq \mathbb{Z} \oplus \mathbb{Z}_{e_1} \oplus \mathbb{Z}_{e_2} \oplus \mathbb{Z}_{e_3}$$

where $e_1, e_2, e_3 \neq 0$ are the elementary divisors of $M - I$. Hence $c_2(S_M) = 0$, $b_1(S_M) = 1$ and, consequently, $b_2(S_M) = 0$.

Let $\Gamma = \langle g_1, g_2, g_3 \rangle$ be the subgroup of G_M generated by g_1, g_2, g_3.

Lemma 3 (Kodaira). *Any Γ-invariant holomorphic function on $\mathbb{H} \times \mathbb{C}$ is constant.*

Proof. Let $f(w, z)$ be a Γ-invariant holomorphic function on $\mathbb{H} \times \mathbb{C}$. Fix $\operatorname{Im} w$. Then f is bounded as a function of $(\operatorname{Re} w, z)$ by (14). Hence $f(w, z)$ is bounded as a function of z for each fixed w and therefore f does not depend on z, i.e., $f(w, z) = f(w, 0)$. Moreover,

$$f(w, 0) = f\left(w + \sum_{j=1}^{3} n_j a_j, 0\right)$$

for any $(n_1, n_2, n_3) \in \mathbb{Z}^3$. On the other hand,

$$\left\{ \sum_{j=1}^{3} n_j a_j \,\middle|\, (n_1, n_2, n_3) \in \mathbb{Z}^3 \right\}$$

is everywhere dence in $\mathbb{R}$, because

$$\alpha a_j \in \left\{ \sum_{j=1}^{3} n_j a_j \,\middle|\, (n_1, n_2, n_3) \in \mathbb{Z}^3 \right\}$$

by (15) and α is irrational. Hence $f(w, z) = f(w, 0)$ is constant, q.e.d.

Proposition 2. i) *S_M contains no curves.*

ii) $\dim H^\nu(S_M, \Theta) = 0$ *for* $\nu = 0, 1, 2$.

Thus S_M satisfies (A) *and* (B), *and is rigid.*

Proof. i) By Proposition 1, any complex line bundle F on S_M is defined by $\mu_F \in \operatorname{Hom}(G_M, \mathbb{C}^*)$, and any holomorphic section

$$\psi \in H^0(S_M, \mathcal{O}(F))$$

is regarded as a holomorphic function $f = f(w, z)$ on $\mathbb{H} \times \mathbb{C}$ satisfying

$$g^* f = \mu_F(g) f$$

for any $g \in G_M$. Since $\Gamma/[G_M, G_M]$ is a finite group, there exists a positive integer m such that $\mu_F(g)^m = 1$ for $g \in \Gamma$. Hence $g^* f^m = \mu_F(g)^m f^m = f^m$ for any $g \in \Gamma$. In view of Lemma 3, we see that f^m is constant and therefore f is constant. Thus $H^0(S_M, \mathcal{O}(F)) = 0$ for any non-trivial F and, consequently, S_M contains no curves.

ii) Each $\theta \in H^0(S_M, \Theta)$ can be written in the form:

$$\theta = a(w, z)\frac{\partial}{\partial w} + b(w, z)\frac{\partial}{\partial z}$$

where $a(w, z)$ and $b(w, z)$ are holomorphic functions on $\mathbb{H} \times \mathbb{C}$. (16) implies that $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial z}$ are Γ-invariant. Hence $a(w, z)$ and $b(w, z)$ are also Γ-invariant. From Lemma 3, we infer that $a(w, z)$ and $b(w, z)$ are both constant, while by (16)

$$g_0^* \frac{\partial}{\partial w} = \frac{1}{\alpha}\frac{\partial}{\partial w}, \qquad g_0^* \frac{\partial}{\partial z} = \frac{1}{\beta}\frac{\partial}{\partial z}.$$

Hence $g_0^* a(w, z) = \alpha\, a(w, z)$ and $g_0^* b(w, z) = \beta\, b(w, z)$, so $a(w, z) = b(w, z) = 0$. Therefore $\theta = 0$, and consequently $H^0(S_M, \Theta) = 0$.

Each $\eta \in H^0(S_M, \Omega^1 \otimes \Omega^2)$ can be written in the form:

$$\eta = a(w, z)\, dw \otimes (dw \wedge dz) + b(w, z)\, dz \otimes (dw \wedge dz)$$

where $a(w, z)$ and $b(w, z)$ are holomorphic functions on $\mathbb{H} \times \mathbb{C}$. From (16) we infer that $dw \otimes (dw \wedge dz)$ and $dz \otimes (dw \wedge dz)$ are Γ-invariant. Hence $a(w, z)$ and $b(w, z)$ are also Γ-invariant. Again, by Lemma 3, $a(w, z)$ and $b(w, z)$ are constant. (16) implies

$$g_0^*\, dw \otimes (dw \wedge dz) = \alpha^2 \beta\, dw \otimes (dw \wedge dz),$$

$$g_0^*\, dz \otimes (dw \wedge dz) = \alpha \beta^2\, dz \otimes (dw \wedge dz).$$

Hence $g_0^* a(w, z) = (\alpha^2 \beta)^{-1} a(w, z)$ and $g_0^* b(w, z) = (\alpha \beta^2)^{-1} b(w, z)$, so $a(w, z) = b(w, z) = 0$ since $\alpha^2 \beta, \alpha \beta^2 \neq 1$. Therefore $\eta = 0$ and, consequently, $H^0(S_M, \Omega^1 \otimes \Omega^2) = 0$. By the Serre duality we have

$$H^2(S_M, \Theta) = H^0(S_M, \Omega^1 \otimes \Omega^2) = 0.$$

Combined with (2), this proves that $H^1(S_M, \Theta) = 0$, q.e.d.

§ 3. Surfaces $S^{(+)}_{N,p,q,r;t}$

Let $N = (n_{ij}) \in SL(2, \mathbb{Z})$ be a unimodular matrix with two real eigenvalues α, $1/\alpha$, $\alpha > 1$. We choose *real* eigenvectors (a_1, a_2) and (b_1, b_2) of N corresponding to α and $1/\alpha$, respectively, and we fix integers p, q, r $(r \neq 0)$

and a complex number t. We define (c_1, c_2) to be the solution of the following equation:

$$(17)\qquad (c_1, c_2) = (c_1, c_2)\cdot {}^tN + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q)$$

where $e_i = \frac{1}{2} n_{i1}(n_{i1}-1) a_1 b_1 + \frac{1}{2} n_{i2}(n_{i2}-1) a_2 b_2 + n_{i1} n_{i2} b_1 a_2$ $(i=1, 2)$. Let $G^{(+)}_{N,p,q,r;t}$ be the group of analytic automorphisms of $\mathbb{H}\times\mathbb{C}$ generated by

$$(18)\qquad \begin{aligned} &g_0\colon (w, z) \to (\alpha w, z+t) \\ &g_i\colon (w, z) \to (w+a_i, z+b_i w + c_i) \qquad i=1, 2 \\ &g_3\colon (w, z) \to \left(w, z+\frac{b_1 a_2 - b_2 a_1}{r}\right). \end{aligned}$$

Let Γ be the subgroup $\langle g_1, g_2, g_3\rangle$ of $G^{(+)}_{N,p,q,r;t}$. For each fixed $y = \operatorname{Im} w$, Γ defines an automorphism group Γ_y of $\mathbb{R}\times\mathbb{C}\simeq\mathbb{R}^3$ in an obvious manner. Then g_3 commutes with every element of $G^{(+)}_{N,p,q,r;t}$, $g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r$ and g_0 normalizes Γ. Moreover,

(19) the action of Γ_y on $\mathbb{R}^3$ is properly discontinuous and has no fixed points.

This follows from the fact that (a_1, b_1) and (a_2, b_2) are linearly independent over $\mathbb{R}$. (17), (18) and (19) imply that the action of $G^{(+)}_{N,p,q,r;t}$ on $\mathbb{H}\times\mathbb{C}$ is properly discontinuous and has no fixed points. We define $S^{(+)}_{N,p,q,r;t}$ to be the quotient surface $\mathbb{H}\times\mathbb{C}/G^{(+)}_{N,p,q,r;t}$. By (18), $S^{(+)}_{N,p,q,r;t}$ is differentiably a fibre bundle over a circle with typical fibre $S_y = \mathbb{R}^3/\Gamma_y$, where S_y is differentiably a circle bundle over a 2-torus. In view of (17) and (18), we have the following relations between the generators g_0, g_1, g_2, g_3 of $G^{(+)}_{N,p,q,r;t}$:

$$g_3 g_i = g_i g_3 \quad \text{for } i=0, 1, 2, \qquad g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r,$$
$$g_0 g_1 g_0^{-1} = g_1^{n_{11}} g_2^{n_{12}} g_3^p, \qquad g_0 g_2 g_0^{-1} = g_1^{n_{21}} g_2^{n_{22}} g_3^q.$$

Note that as an abstract group $G^{(+)}_{N,p,q,r;t}$ does not depend on t, and that

$$G^{(+)}_{N,p,q,r;t}/[G^{(+)}_{N,p,q,r;t}, G^{(+)}_{N,p,q,r;t}] \simeq \mathbb{Z}\oplus\mathbb{Z}_{e_1}\oplus\mathbb{Z}_{e_2}\oplus\mathbb{Z}_r,$$

where $e_1, e_2 \neq 0$ are the elementary divisors of $N - I$. Hence

$$b_1(S^{(+)}_{N,p,q,r;t}) = 1 \quad \text{and} \quad b_2(S^{(+)}_{N,p,q,r;t}) = 0.$$

Lemma 4. *Any Γ-invariant holomorphic function on $\mathbb{H}\times\mathbb{C}$ is constant.*

Proof is similar to that of Lemma 3.

Proposition 3. i) *$S^{(+)}_{N,p,q,r;t}$ contains no curves.*

ii) $\dim H^0(S^{(+)}_{N,p,q,r;t}, \Theta) = \dim H^1(S^{(+)}_{N,p,q,r;t}, \Theta) = 1$,

$\dim H^2(S^{(+)}_{N,p,q,r;t}, \Theta) = 0$.

Thus $S^{(+)}_{N,p,q,r;t}$ satisfies (A) *and* (B).

Proof. The proof of i) is similar to that of Proposition 2.

Each $\theta \in H^0(S^{(+)}_{N,p,q,r;t}, \Theta)$ can be written in the form

$$\theta = a(w,z)\frac{\partial}{\partial z} + b(w,z)\frac{\partial}{\partial w},$$

where $a(w,z)$ and $b(w,z)$ are holomorphic functions on $\mathbb{H} \times \mathbb{C}$. Since

$$g_i^* \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \qquad \text{for } i = 0, 1, 2, 3,$$

$$g_0^* \frac{\partial}{\partial w} = \frac{1}{\alpha}\frac{\partial}{\partial w}, \qquad g_i^* \frac{\partial}{\partial w} = \frac{\partial}{\partial w} - b_i \frac{\partial}{\partial z} \qquad \text{for } i = 1, 2,$$

$$g_3^* \frac{\partial}{\partial w} = \frac{\partial}{\partial w},$$

we get

$$g_0^* a(w,z) = a(w,z), \qquad g_3^* a(w,z) = a(w,z),$$

$$g_i^* a(w,z) = a(w,z) + b_i b(w,z) \qquad \text{for } i = 1, 2,$$

$$g_0^* b(w,z) = \alpha b(w,z), \qquad g_i^* b(w,z) = b(w,z) \qquad \text{for } i = 1, 2, 3.$$

Hence, by Lemma 4, we have $b(w,z) \equiv 0$ and $a(w,z) = \text{constant}$, i.e., $\theta = a\frac{\partial}{\partial z}$. Therefore $\dim H^0(S^{(+)}_{N,p,q,r;t}, \Theta) = 1$.

Each $\eta \in H^0(S^{(+)}_{N,p,q,r;t}, \Omega^1 \otimes \Omega^2)$ can be written in the form

$$\eta = a(w,z)\, dw \otimes (dw \wedge dz) + b(w,z)\, dz \otimes (dw \wedge dz)$$

where $a(w,z)$ and $b(w,z)$ are holomorphic functions on $\mathbb{H} \times \mathbb{C}$. Since

$$g_0^*\, dw \otimes (dw \wedge dz) = \alpha^2\, dw \otimes (dw \wedge dz),$$

$$g_i^*\, dw \otimes (dw \wedge dz) = dw \otimes (dw \wedge dz) \qquad \text{for } i = 1, 2, 3,$$

$$g_0^*\, dz \otimes (dw \wedge dz) = \alpha\, dz \otimes (dw \wedge dz),$$

$$g_i^*\, dz \otimes (dw \wedge dz) = dz \otimes (dw \wedge dz) + b_i\, dw \otimes (dw \wedge dz) \qquad \text{for } i = 1, 2,$$

$$g_3^*\, dz \otimes (dw \wedge dz) = dz \otimes (dw \wedge dz),$$

we get

$$g_0^* a(w,z) = \alpha^{-2} a(w,z),$$
$$g_i^* a(w,z) = a(w,z) - b_i b(w,z) \quad \text{for } i=1,2,$$
$$g_3^* a(w,z) = a(w,z),$$
$$g_0^* b(w,z) = \alpha^{-1} b(w,z),$$
$$g_i^* b(w,z) = b(w,z) \quad \text{for } i=1,2,3.$$

Hence, by Lemma 4, we have $a(w,z) = b(w,z) \equiv 0$, i.e., $\eta = 0$. Therefore $H^2(S^{(+)}_{N,p,q,r;t}, \Theta) \simeq H^0(S^{(+)}_{N,p,q,r;t}, \Omega^1 \otimes \Omega^2) = 0$.

By (2), we conclude that $\dim H^1(S^{(+)}_{N,p,q,r;t}, \Theta) = 1$, q.e.d.

It follows from Proposition 3 that $S^{(+)}_{N,p,q,r;t}$ has a 1-dimensional locally complete family of deformations. In fact, the family is explicitly given in the following:

Proposition 4. *$\mathscr{S}_{N,p,q,r} = \{S^{(+)}_{N,p,q,r;t}\}_{t \in \mathbb{C}}$ is a locally complete family of deformations.*

Proof. It is evident that $\mathscr{S}_{N,p,q,r}$ is an analytic family of deformations. We calculate the infinitesimal deformation

$$\partial S^{(+)}_{N,p,q,r;t}/\partial t|_{t=t_0} \in H^1(S^{(+)}_{N,p,q,r;t_0}, \Theta)$$

for each t_0. By (18) we find local coordinates (w_i, z_i) on $S^{(+)}_{N,p,q,r;t}$ such that

$$w_i = \alpha^{m_{ij}} w_j + a_{ij},$$
$$z_i = z_j + b_{ij} w_j + c_{ij} + t\, m_{ij},$$

where $\{m_{ij}\}$ is a Betti base of 1-cocycles on $S^{(+)}_{N,p,q,r;t}$ and a_{ij}, b_{ij}, c_{ij} are constants not depending on t. We define θ_{ij} as follows:

$$\theta_{ij} = \left.\frac{\partial w_i}{\partial t}\right|_{t=t_0} \cdot \frac{\partial}{\partial w_i} + \left.\frac{\partial z_i}{\partial t}\right|_{t=t_0} \cdot \frac{\partial}{\partial z_i} = m_{ij} \frac{\partial}{\partial z_i}.$$

Then $\{\theta_{ij}\}$ is a cocycle which represents $\partial S^{(+)}_{N,p,q,r;t}/\partial t|_{t=t_0}$. We assume that $\{\theta_{ij}\}$ is cohomologous to zero, i.e., $\theta_{ij} = \theta_i - \theta_j$ where the

$$\theta_i = f_i \frac{\partial}{\partial z_i} + g_i \frac{\partial}{\partial w_i}$$

are local holomorphic vector fields. Then

$$m_{ij} \frac{\partial}{\partial z_j} = \left(f_i \frac{\partial}{\partial z_i} + g_i \frac{\partial}{\partial w_i}\right) - \left(f_j \frac{\partial}{\partial z_j} + g_j \frac{\partial}{\partial w_j}\right),$$

and

$$\frac{\partial}{\partial w_j} = b_{ij} \frac{\partial}{\partial z_i} + \alpha^{m_{ij}} \frac{\partial}{\partial w_i}, \qquad \frac{\partial}{\partial z_i} = \frac{\partial}{\partial z_j},$$

and therefore

$$m_{ij} = f_i - f_j - b_{ij} g_j, \qquad 0 = g_i - \alpha^{m_{ij}} g_j.$$

From this it follows that $\{g_i\} \in H^0(S^{(+)}_{N,p,q,r;t_0}, \mathcal{O}(-K))$, while

$$H^0(S^{(+)}_{N,p,q,r;t_0}, \mathcal{O}(-K)) = 0$$

by Proposition 3. Therefore $g_i = 0$, and $m_{ij} = f_i - f_j$. Hence $\{df_i\} \in H^0(S^{(+)}_{N,p,q,r;t_0}, \Omega^1)$. Since $h^{1,0}(S^{(+)}_{N,p,q,r;t_0}) = 0$ by Proposition 1, $df_i = 0$ and therefore f_i is constant. This implies that $\{m_{ij}\}$ is zero in $H^1(S^{(+)}_{N,p,q,r;t_0}, \mathbb{C})$. Thus we derive a contradiction. Therefore $\partial S^{(+)}_{N,p,q,r;t}/\partial t|_{t=t_0} \neq 0$ for each t_0, q.e.d.

§ 4. Surfaces $S^{(-)}_{N,p,q,r}$

Let $N = (n_{ij}) \in GL(2, \mathbb{Z})$ be a 2×2 matrix with $\det N = -1$ having two real eigen-values α, $-1/\alpha$ such that $\alpha > 1$. We choose *real* eigen-vectors (a_1, a_2) and (b_1, b_2) of N corresponding to α and $-1/\alpha$, respectively, and we fix integers p, q, r $(r \neq 0)$. We define (c_1, c_2) to be the solution of the following equation:

$$-(c_1, c_2) = (c_1, c_2)\,{}^tN + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q), \tag{20}$$

where $e_i = \frac{1}{2} n_{i1}(n_{i1} - 1) a_1 b_1 + \frac{1}{2} n_{i2}(n_{i2} - 1) a_2 b_2 + n_{i1} n_{i2} b_1 a_2$ $(i = 1, 2)$. Let $G^{(-)}_{N,p,q,r}$ be the group of analytic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} &g_0\colon (w, z) \to (\alpha w, -z),\\ &g_i\colon (w, z) \to (w + a_i, z + b_i w + c_i) \quad \text{for } i = 1, 2,\\ &g_3\colon (w, z) \to \left(w, z + \frac{b_1 a_2 - b_2 a_1}{r}\right). \end{aligned} \tag{21}$$

Then the subgroup $\langle g_0^2, g_1, g_2, g_3 \rangle$ coincides with $G^{(+)}_{N^2,p_1,q_1,r;0}$ for certain $(p_1, q_1) \in \mathbb{Z}^2$ of which index in $G^{(-)}_{N,p,q,r}$ equals 2. g_0 defines an involution of $S^{(+)}_{N^2,p_1,q_1,r;0}$ free from fixed points. Thus the action of $G^{(-)}_{N,p,q,r}$ on $\mathbb{H} \times \mathbb{C}$ is properly discontinuous and has no fixed points. We define a surface $S^{(-)}_{N,p,q,r}$ to be the quotient surface $\mathbb{H} \times \mathbb{C}/G^{(-)}_{N,p,q,r}$. It is clear that $S^{(-)}_{N,p,q,r}$ has $S^{(+)}_{N^2,p_1,q_1,r;0}$ as its unramified double covering surface. Hence $S^{(-)}_{N,p,q,r}$ contains no curves and $c_2(S^{(-)}_{N,p,q,r}) = 0$. In view of (20) and (21), we have the following relations between the generators g_0, g_1, g_2, g_3 of $G^{(-)}_{N,p,q,r}$:

$$g_3 g_i = g_i g_3 \quad \text{for } i = 1, 2, \qquad g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r,$$

$$g_0 g_1 g_0^{-1} = g_1^{n_{11}} g_2^{n_{12}} g_3^p, \qquad g_0 g_2 g_0^{-1} = g_1^{n_{21}} g_2^{n_{22}} g_3^q, \qquad g_0 g_3 g_0^{-1} = g_3^{-1}.$$

From this we derive

$$G^{(-)}_{N,p,q,r}/[G^{(-)}_{N,p,q,r}, G^{(-)}_{N,p,q,r}] \simeq \mathbb{Z} \otimes \mathbb{Z}_{e_1} \otimes \mathbb{Z}_{e_2} \otimes \mathbb{Z}_{e_3},$$

where $e_1, e_2 \neq 0$ are the elementary divisors of $N-I$ and $e_3=2$ or 1 according as r is even or odd. Hence $b_1(S^{(-)}_{N,p,q,r})=1$, $b_2(S^{(-)}_{N,p,q,r})=0$. Therefore $S^{(-)}_{N,p,q,r}$ *satisfies* (A) *and* (B).

Although $S^{(-)}_{N,p,q,r}$ is very similar to $S^{(+)}_{N,p,q,r;t}$, it is a rigid surface as will be shown in the following

Proposition 5. $\dim H^{\nu}(S^{(-)}_{N,p,q,r}, \Theta)=0$ *for* $\nu=0, 1, 2$.

Proof. Since $S^{(+)}_{N^2,p_1,q_1,r;0} \to S^{(-)}_{N,p,q,r}$ is an unramified covering map, we have

$$\dim H^2(S^{(-)}_{N,p,q,r}, \Theta) \leqq \dim H^2(S^{(+)}_{N^2,p_1,q_1,r;0}, \Theta)=0$$

and

$$\dim H^0(S^{(-)}_{N,p,q,r}, \Theta) \leqq \dim H^0(S^{(+)}_{N^2,p_1,q_1,r;0}, \Theta)=1,$$

while the base $\frac{\partial}{\partial z}$ of $H^0(S^{(+)}_{N^2,p_1,q_1,r;0}, \Theta)$ is transformed by g_0 into $-\frac{\partial}{\partial z}$. Hence $\dim H^0(S^{(-)}_{N,p,q,r}, \Theta)=0$. By (2), this implies that $\dim H^1(S^{(-)}_{N,p,q,r}, \Theta) =0$, q.e.d.

§ 5. Statement of the Theorem

First we remark that all the surfaces constructed in previous sections satisfy the following condition:

(C) *There exists a line bundle* F_0 *such that*

$$\dim H^0(\Omega^1(F_0)) \neq 0.$$

In fact, this follows from (16), (18) and (21). The following theorem is a converse to this fact.

Theorem. *If a surface S satisfies the conditions* (A), (B) *and* (C), then S *is (complex analytically homeomorphic to)* S_M, $S^{(+)}_{N,p,q,r;t}$ *or* $S^{(-)}_{N,p,q,r}$.

The rest of this paper will be devoted to the proof of this theorem. First, $H^0(d\mathcal{O}(F_0)) \neq 0$ will be proven in § 6 and after a preliminary study of a line bundle L such that $H^0(d\mathcal{O}(L)) \neq 0$, it will be established in § 7 that two cases I and II occur. Then in § 8 and 9 it will be proven that surfaces in Case I turn out to be S_M of § 2 and that surfaces in Case II turn out to be either $S^{(+)}_{N,p,q,r;t}$ of § 3 or $S^{(-)}_{N,p,q,r}$ of § 4. In the proof, we shall use the methods and ideas of Kodaira [2, 3; § 11, 13].

Remark. Bombieri informed us that he obtained the following

Theorem. *If* S *satisfies* (A) *and* (B) *and if the canonical bundle* K *is not real, then* S *is* S_M.

This theorem is obtained as a corollary to our theorem. In fact, if $\dim H^1(\mathbb{C}(K)) \neq 0$, we obtain, by (5),

$$\dim H^0(d\mathcal{O}(\bar{K})) = \dim H^1(\mathbb{C}(\bar{K})) = \dim H^1(\mathbb{C}(K)) \neq 0,$$

since $K \neq \bar{K}$. If $\dim H^1(\mathbb{C}(K))=0$, we obtain, by (3) and (4),

$$\dim H^1(\Omega^1(K)) \neq 0.$$

Hence, by the Serre duality and the formula of Riemann-Roch,

$$\dim H^0(\Omega^1(K)) \quad \text{or} \quad \dim H^0(\Omega^1(-K)) \neq 0.$$

Thus, in any case, S satisfies (C), and our theorem implies that S is S_M, $S^{(+)}_{N,p,q,r;t}$ or $S^{(-)}_{N,p,q,r}$. Since the canonical bundles of $S^{(+)}_{N,p,q,r;t}$ and $S^{(-)}_{N,p,q,r}$ are real, S is S_M.

§ 6. Proof of $H^0(d\mathcal{O}(F_0)) \neq 0$

In this section we prove that $H^0(d\mathcal{O}(F_0)) \neq 0$ for any F_0 such that $H^0(\Omega^1(F_0)) \neq 0$. In order to prove this, we assume $H^0(d\mathcal{O}(F_0))=0$ and derive a contradiction.

In view of (4), the assumption implies that $H^0(\mathcal{O}(F_0+K)) \neq 0$. Hence $F_0=-K$ by Lemma 1 (iv). Moreover, recalling Lemma 2 and using $\Theta \simeq \Omega^1(-K)$, we have

$$\begin{aligned} &\dim H^0(\Omega^1(-K))=1, && \dim H^0(\Theta)=1 \\ &H^0(\Omega^1(F))=0 && \text{for any } F \neq -K \in H^1(\mathcal{O}^*) \\ &H^0(d\mathcal{O}(F))=0 && \text{for every } F \in H(\mathcal{O}^*). \end{aligned} \tag{22}$$

Combining these with (4) and (5), we obtain

$$\begin{aligned} &\dim H^1(\mathbb{C}(F))=0 \qquad \text{for any } F \neq 0, K \\ &\dim H^1(\mathbb{C}(K)) \leqq 1. \end{aligned} \tag{23}$$

Next we prove the following assertion:

There exists a surface S satisfying the conditions (A), (B) *and* (22) *of which canonical bundle K is defined by $\{\kappa^{-m_{ij}}\}$, where $\kappa>0$ and $\{m_{ij}\}$ is a Betti base of* 1*-cocycles on S.*

First of all, we note that any unramified covering surface S' of S satisfies (A) and (B) by Lemma 1 and

$$H^0(S', d\mathcal{O}(-K))=0 \quad \text{and} \quad H^0(S', \Omega^1(-K)) \neq 0.$$

In fact, we let θ be a non-zero element of $H^0(S, \Omega^1(-K))$. Denote by θ' the pull back of θ on S'. Then, since $d\theta \neq 0$, it is clear that $d\theta' \neq 0$. Hence, by (4) and Lemma 2, $\dim H^0(S', d\mathcal{O}(-K)) < \dim H^0(S', \Omega^1(-K)) \leqq 1$. This implies that $H^0(S', d\mathcal{O}(-K))=0$ and $H^0(S', \Omega^1(-K))=1$. Thus S' satisfies (22).

Therefore, replacing S by a suitable finite unramified covering surface we can assume that $c(K)=0$, since $H^2(\mathbb{Z})$ is a finite group (see Kodaira [2], Theorem 33). Hence $K=\{\kappa^{-m_{ij}}\}$, where $\kappa\in\mathbb{C}^*$ and $\{m_{ij}\}$ is a Betti base of 1-cocycles on S. By (23), we have two cases: $\dim H^1(\mathbb{C}(K))=1$ and 0.

If $\dim H^1(\mathbb{C}(K))=1$, then $\dim H^1(\mathbb{C}(\bar{K}))=1$. Hence it follows from (23) that $K=\bar{K}$ and $\kappa=\bar{\kappa}$. Replacing S by a double unramified covering surface if necessary, we can assume that $\kappa>0$.

If $\dim H^1(\mathbb{C}(K))=0$, then, by (23), $\dim H^1(\mathbb{C}(F))=0$ for any $F\in H^1(\mathbb{C}^*)$. On the other hand, (2) and (22) yield

$$\text{(24)}\qquad \dim H^0(\Theta)=\dim H^1(\Theta)=1,\qquad \dim H^2(\Theta)=0.$$

Since $H^2(d\mathcal{O}(-K))\simeq H^3(\mathbb{C}(-K))\simeq H^1(\mathbb{C}(K))=0$ by (5) and (12), we obtain from (4) and (24) the isomorphism

$$d\colon H^1(\Omega^1(-K))=H^1(\Theta)\to H^1(\Omega^2(-K))=H^1(\mathcal{O}).$$

Since $H^2(\Theta)=0$ by (24), there exists an analytic family $\mathcal{S}=\{S_t\}_{|t|<\varepsilon}$ of surfaces such that $S_0=S$ and the infinitesimal deformation $\left.\frac{\partial S_t}{\partial t}\right|_{t=0}\in H^1(\Theta)$ does not vanish. Since $c_1(S_t)=c_1(S)=0$, the canonical bundle K_t of S_t is defined by a 1-cocycle $\{\kappa_t^{-m_{ij}}\}$, where $\kappa_t\in\mathbb{C}^*$ depends holomorphically on t. Moreover, we can choose local coordinates (z_i, w_i, t) on $\mathcal{S}$ such that $dz_i\wedge dw_i=\kappa_t^{m_{ij}}\,dz_j\wedge dw_j$. We have

$$d\left(\left.\frac{\partial S_t}{\partial t}\right|_{t=0}\right)=\left\{m_{ij}\,\kappa_t^{-1}\left.\frac{d\kappa_t}{dt}\right|_{t=0}\right\}$$

(see Kodaira [2], pp. 715–716). Since d is injective and $\left.\frac{\partial S_t}{\partial t}\right|_{t=0}\neq 0$, we obtain $\left.\frac{d\kappa_t}{dt}\right|_{t=0}\neq 0$. Hence there exists a point t_0, $|t_0|<\varepsilon$, such that $\kappa_{t_0}=r\exp\left(2\pi\sqrt{-1}\,\frac{n}{m}\right)$, m, n being integers and $r>0$. By Lemma 1, S_{t_0} also satisfies (A) and (B). Since $H^1(S,\mathbb{C}(F))=0$ for any $F\neq 0$ in $H^1(S,\mathbb{C}^*)$, we obtain $H^1(S_{t_0},\mathbb{C}(F))=0$ for any $F\neq 0$ in $H^1(S_{t_0},\mathbb{C}^*)$. Combining this with (12) and (13), we obtain $H^2(S_{t_0},\mathbb{C}(-K_{t_0}))=0$. Hence it follows from (5) that $H^0(S_{t_0},d\mathcal{O}(-K_{t_0}))=0$ and $H^1(S_{t_0},d\mathcal{O}(-K_{t_0}))=0$. In view of (4), we infer that

$$\dim H^0(S_{t_0},\Omega^1(-K_{t_0}))=\dim H^0(S_{t_0},\Omega^2(-K_{t_0}))=1.$$

Choose a suitable m-fold unramified covering surface S'_{t_0} of S_{t_0}. Then the canonical bundle K'_{t_0} of S'_{t_0} is defined by $\{(r^m)^{-m_{ij}}\}$ where $r^m>0$. S'_{t_0} also satisfies (A), (B) and (22) by the previous argument. Thus it suffices to consider S'_{t_0} instead of S.

In what follows in §6 we assume that *S has the canonical bundle $K=\{\kappa^{-m_{ij}}\}$ with $\kappa>0$.*

Now, by (22), we can take a non-zero holomorphic vector field θ on S. Since S contains no curves and $c_2(S)=0$, θ does not vanish anywhere. Hence we can choose local coordinates (z_i, w_i) such that $\theta=\frac{\partial}{\partial z_i}$. By

$$\frac{\partial}{\partial z_i}=\theta=\frac{\partial}{\partial z_j}=\frac{\partial z_i}{\partial z_j}\frac{\partial}{\partial z_i}+\frac{\partial w_i}{\partial z_j}\frac{\partial}{\partial w_i},$$

we obtain

$$\frac{\partial z_i}{\partial z_j}=1 \quad \text{and} \quad \frac{\partial w_i}{\partial z_j}=0.$$

This implies that

$$w_i=g_{ij}(w_j),$$
$$z_i=z_j+f_{ij}(w_j),$$

where g_{ij} and f_{ij} are holomorphic functions of w_j. Moreover

$$dw_i\wedge dz_i=g'_{ij}(w_j)\,dw_j\wedge dz_j,$$

where $g'_{ij}(w_j)=\frac{dg_{ij}}{dw_j}(w_j)$. This shows that the 1-cocycle $\{g'_{ij}\}$ defines the anti-canonical bundle $-K$. Thus $\{g'_{ij}\}$ is cohomologous to $\{\kappa^{m_{ij}}\}$, i.e.,

$$g'_{ij}(w_j)=g_i^{-1}(w_i, z_i)\cdot\kappa^{m_{ij}}\,g_j(w_j, z_j),$$

where g_i is a non-vanishing holomorphic function. Applying $\theta=\frac{\partial}{\partial z_i}=\frac{\partial}{\partial z_j}$ to this equation, we obtain

$$\partial\log g_j/\partial z_j=\partial\log g_i/\partial z_i.$$

Hence $\gamma=\partial\log g_j/\partial z_j$ is constant and

$$g_i=g_i(w_i, z_i)=e^{\gamma z_i}\,h_i(w_i),$$

where $h_i(w_i)$ is a non-vanishing holomorphic function of w_i. Consequently, we obtain

$$e^{\gamma z_i}\,h_i(w_i)\,dw_i=\kappa^{m_{ij}}\,e^{\gamma z_j}\,h_j(w_j)\,dw_j.$$

Comparing this with (22), we see that $\gamma\neq 0$. In fact, if $\gamma=0$, then

$$\{h_i(w_i)\,dw_i\}\in H^0(d\mathcal{O}(-K)).$$

By replacing $\int^{w_i} h_i(w_i)\,dw_i$ and γz_i by w_i and z_i, respectively, we obtain local coordinates (w_i, z_i) such that

$$e^{z_i}\,dw_i=\kappa^{m_{ij}}\,e^{z_j}\,dw_j, \qquad \frac{\partial}{\partial z_i}=\frac{\partial}{\partial z_j}=\frac{1}{\gamma}\theta \tag{25}$$

and $w_i=g_{ij}(w_j)$, where g_{ij} is a holomorphic function of w_j.

For any holomorphic function $f=f(w)$, we write $f'=\dfrac{df}{dw}$ and denote by $\delta(f)$ the Schwarzian derivative of f, i.e.,

$$\delta(f)=2(f''/f')'-(f''/f')^2.$$

Applying this to $g_{ik}=g_{ij}\circ g_{jk}$, we obtain

$$\delta(g_{ik})=\delta(g_{ij})\cdot(g'_{jk})^2+\delta(g_{jk}). \tag{26}$$

Since $g'_{ik}=g'_{ij}\cdot g'_{jk}$, $\{g'_{ij}\}$ defines a line bundle E which is analytically equivalent to $-K$ in view of (25). Let m be an arbitrary integer. Any local holomorphic section ϕ of mE is written in the form $\{\phi_i\}$, where holomorphic functions ϕ_i satisfy $\phi_i=(g'_{ij})^m\phi_j$. We define $\theta\phi$ to be $\{\theta\phi_i\}$. Since $\theta g'_{ij}=0$, we have $\theta\phi_i=(g'_{ij})^m\theta\phi_j$. Thus $\theta\phi$ is also a local holomorphic section of mE. Let $\mathcal{O}_w(mE)$ denote the subsheaf of $\mathcal{O}(mE)$ consisting of germs of holomorphic sections ϕ such that $\theta\phi=0$. It is clear that $\theta\phi=0$ if and only if ϕ_i are holomorphic functions of w_i only. We have the exact sequence

$$0\to\mathcal{O}_w(mE)\to\mathcal{O}(mE)\xrightarrow{\theta}\mathcal{O}(mE)\to 0. \tag{27}$$

The formula (26) shows that $\{\delta(g_{ij})\}\in H^1(\mathcal{O}_w(-2E))$. Since $\mathcal{O}(-2E)=\mathcal{O}(2K)$, we have $H^\nu(\mathcal{O}(-2E))=H^\nu(\mathcal{O}(2K))=0$ for each ν by Lemma 1. Combining this with (27), we infer that $H^1(\mathcal{O}_w(-2E))=0$. Consequently, there exist holomorphic functions $\phi_i=\phi_i(w_i)$ such that

$$\delta(g_{ij})=\phi_j-(g'_{ij})^2\,\phi_i. \tag{28}$$

The differential equation $\delta(x_i)=\phi_i$ has a solution $x_i=x_i(w_i)$ with non-vanishing derivative $x'_i(w_i)$ defined on some neighbourhood. Hence, by an appropriate choice of the local coordinates, we may assume that $w_i\to x_i(w_i)$ is a biholomorphic map. We define a holomorphic function $l_{ij}=l_{ij}(x_j)$ of x_j by $l_{ij}\circ x_j=x_i\circ g_{ij}$. Then we obtain

$$\delta(l_{ij})(x'_j)^2+\delta(x_j)=\delta(x_i)(g'_{ij})^2+\delta(g_{ij}).$$

Comparing this with (28), we obtain $\delta(l_{ij})=0$. This implies that $l_{ij}(x_j)$ is a linear function of x_j, i.e.,

$$x_i(w_i)=(a_{ij}\,x_j(w_j)+b_{ij})/(c_{ij}\,x_j(w_j)+d_{ij}),$$

where $a_{ij},b_{ij},c_{ij},d_{ij}$ are constant and $a_{ij}d_{ij}-b_{ij}c_{ij}=1$. Replacing $x_i(w_i)$ and $\log(e^{z_i}/x'_i(w_i))$ by w_i and z_i, respectively, we obtain by (25) local coordinates (w_i,z_i) such that

$$\begin{aligned} &e^{z_i}\,dw_i=\kappa^{m_{ij}}\,e^{z_j}\,dw_j, \qquad \frac{\partial}{\partial z_i}=\frac{\partial}{\partial z_j}\\ &w_i=(a_{ij}\,w_j+b_{ij})/(c_{ij}\,w_j+d_{ij}). \end{aligned} \tag{29}$$

This formula shows in particular that any one of the holomorphic functions w_i, say w_{i_0}, can be continued analytically along any continuous curve on S and the totality of analytic continuations of w_{i_0} forms a multi-valued meromorphic function W on S. Consider W as a single-valued meromorphic function on the universal covering surface $\tilde{S}$. Evidently W has no points of indeterminacy on $\tilde{S}$. Hence W is a holomorphic mappings of $\tilde{S}$ into $\mathbb{P}^1$.

By means of (29), we obtain

$$e^{z_i/2} = \pm(\kappa^{m_{ij}/2} \cdot c_{ij} w_j e^{z_j/2} + \kappa^{m_{ij}/2} \cdot d_{ij} e^{z_j/2}),$$
$$w_i e^{z_i/2} = \pm(\kappa^{m_{ij}/2} \cdot a_{ij} w_j e^{z_j/2} + \kappa^{m_{ij}/2} \cdot b_{ij} e^{z_j/2}).$$

By replacing $w_i e^{z_i/2}$ and $e^{z_i/2}$ by ξ_i and η_i, respectively, we obtain local coordinates (ξ_i, η_i) such that

$$\begin{aligned} \xi_i &= \alpha_{ij} \xi_j + \beta_{ij} \eta_j, \\ \eta_i &= \gamma_{ij} \xi_j + \delta_{ij} \eta_j, \end{aligned} \tag{30}$$

and $\xi_i/\eta_i = w_i$, where α_{ij}, β_{ij}, γ_{ij}, δ_{ij} are constant. The totalities of analytic continuations of ξ_{i_0} and η_{i_0} form single-valued holomorphic functions ξ and η on $\tilde{S}$ such that

$$\begin{aligned} g^* \xi &= \alpha(g) \xi + \beta(g) \eta, \\ g^* \eta &= \gamma(g) \xi + \delta(g) \eta, \end{aligned} \tag{31}$$

for any $g \in G$ (the covering transformation group of $\tilde{S}$ over S), and $\xi/\eta = W$, where $\alpha(g)$, $\beta(g)$, $\gamma(g)$, $\delta(g)$ are constant. It is evident that ξ and η cannot vanish simultaneously, and that $d\xi \wedge d\eta$ does not vanish anywhere. Hence $\Phi = (\xi, \eta)$ defines a holomorphic non-degenerate mapping of $\tilde{S}$ into $\mathbb{C}^2 - 0$.

Proposition 6. *We may assume that*

$$\operatorname{Im} W = \frac{1}{2\sqrt{-1}} (\xi \bar{\eta} - \bar{\xi} \eta) > 0 \tag{32}$$

and

$$\begin{pmatrix} \alpha(g) & \beta(g) \\ \gamma(g) & \delta(g) \end{pmatrix} \in GL^+(2, \mathbb{R}) \tag{33}$$

in the formula (31).

Proof of Proposition 6 (due to Kodaira). Let $\theta = \xi_i \frac{\delta}{\partial \xi_i} + \eta_i \frac{\partial}{\partial \eta_i}$ and $\tilde{\theta}$ = the pull back of θ on $\tilde{S}$. We denote by $p(P, \zeta, t)$ the integral curve of $\zeta \tilde{\theta}$ passing through $P \in \tilde{S}$ where $\zeta \in \mathbb{C}$. Then $p(P, \zeta, t)$ is defined for

$-\infty < t < +\infty$, since $\tilde{\theta}$ is complete on $\tilde{S}$. Let $f(P,\zeta)=p(P,\zeta,1)$. Then f is a holomorphic mapping of $\tilde{S}\times\mathbb{C}$ onto $\tilde{S}$, and

$$\xi\big(f(P,\zeta)\big)=\xi(P)\cdot e^{\zeta},$$

$$\eta\big(f(P,\zeta)\big)=\eta(P)\cdot e^{\zeta}.$$

Let π be the canonical mapping of $\mathbb{C}^2-0$ onto $\mathbb{P}^1$. Then

$$\pi\circ\Phi\big(f(P,\zeta)\big)=W(P).$$

We denote by C_P the orbit $f(P,\mathbb{C})$ passing through P. Then, $W(C_P)=W(P)$. If $C_P\cap C_Q\neq\phi$, then $C_P=C_Q$.

Now, take a sufficiently small disk Δ_P transversally to C_P so that W is biholomorphic on Δ_P. Since $\Phi\big(f(P,\zeta)\big)=\big(\xi(P)\,e^{\zeta},\ \eta(P)\,e^{\zeta}\big)$, $\Phi\circ f$: $\Delta_P\times\mathbb{C}\to\pi^{-1}\big(W(\Delta_P)\big)\simeq W(\Delta_P)\times\mathbb{C}^*$ is the universal covering. Let $\mathcal{U}_P=f(\Delta_P\times\mathbb{C})$. Then, f: $\Delta_P\times\mathbb{C}\to\mathcal{U}_P$ is the universal covering and Φ: $\mathcal{U}_P\to\pi^{-1}\big(W(\Delta_P)\big)$ is an unlimited covering. Let $\bar{\tilde{S}}$ denote the set of orbits C_P in $\tilde{S}$, and let $\tilde{W}(P)=C_P\in\bar{\tilde{S}}$. Then $\tilde{W}$ is a mapping of $\tilde{S}$ onto $\bar{\tilde{S}}$. We define a subset $\bar{\mathcal{U}}\subset\bar{\tilde{S}}$ to be *open* if and only if $\tilde{W}^{-1}(\bar{\mathcal{U}})$ is open in $\tilde{S}$. Then $\tilde{W}$ is continuous and $\bar{\tilde{S}}$ is connected. Since $\tilde{W}^{-1}\big(\tilde{W}(\mathcal{U}_P)\big)=\mathcal{U}_P$ is open in $\tilde{S}$, $\bar{\mathcal{U}}_P=\tilde{W}(\mathcal{U}_P)$ is open in $\bar{\tilde{S}}$. Thus, $\bar{\tilde{S}}=\bigcup_P\bar{\mathcal{U}}_P$ is an open covering of $\bar{\tilde{S}}$ and $\bar{\mathcal{U}}_P$ is homeomorphic to Δ_P.

Next, we prove that $\bar{\tilde{S}}$ is a Hausdorff space. Suppose $C_P\neq C_Q$. If $W(C_P)=W(P)\neq W(Q)=W(C_Q)$, we choose Δ_P and Δ_Q such that

$$W(\Delta_P)\cap W(\Delta_Q)=\emptyset\,.$$

Then $\mathcal{U}_P\cap\mathcal{U}_Q=\emptyset$ and hence $\bar{\mathcal{U}}_P\cap\bar{\mathcal{U}}_Q=\emptyset$. If $W(C_P)=W(C_Q)$, we choose Δ_P and Δ_Q such that $W(\Delta_P)=W(\Delta_Q)$. Then, $\Phi(\mathcal{U}_P)=\Phi(\mathcal{U}_Q)$. If $\mathcal{U}_P\cap\mathcal{U}_Q\neq\emptyset$, then $\mathcal{U}_P=\mathcal{U}_Q$ and $C_P=C_Q$, since $\mathcal{U}_P$ and $\mathcal{U}_Q$ are unlimited covering spaces of $\Phi(\mathcal{U}_P)=\Phi(\mathcal{U}_Q)$. Hence $\mathcal{U}_P\cap\mathcal{U}_Q=\emptyset$ and $\bar{\mathcal{U}}_P\cap\bar{\mathcal{U}}_Q=\emptyset$. Thus $\bar{\tilde{S}}$ is a Hausdorff space.

Now, let $\phi(C_P)=W(P)$. Then ϕ is a continuous mapping of $\bar{\tilde{S}}$ into $\mathbb{P}^1$ and $\phi\circ\tilde{W}(P)=W(P)$. Moreover, ϕ is a homeomorphism of $\bar{\mathcal{U}}_P$ onto $W(\Delta_P)\simeq\Delta_P$. Hence ϕ is a local homeomorphism of $\bar{\tilde{S}}$ into $\mathbb{P}^1$. This implies that $\bar{\tilde{S}}$ has a structure of Riemann surface. We denote this Riemann surface by C. Then $\tilde{W}$: $\tilde{S}\to C=\bar{\tilde{S}}$ and ϕ: $C=\bar{\tilde{S}}\to\mathbb{P}^1$ are holomorphic. Moreover, $\tilde{W}$: $\tilde{S}\to C$ is a $\mathbb{C}$ or $\mathbb{C}^*$-bundle over C. Since any $g\in G$ maps orbits to orbits, g induces a biholomorphic mapping $\bar{g}$: $C\to C$.

Let W_C be the $\mathbb{C}^*$-bundle over C induced by ϕ from π: $\mathbb{C}^2-0\to\mathbb{P}^1$. Then $\tilde{S}$ is the universal covering of W_C. Since $\pi_1(\tilde{S})=0$, $\pi_1(C)=0$ and C is $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{H}$.

If $C=\mathbb{P}^1$, then ϕ: $C\to\mathbb{P}^1$ is an isomorphism. Hence $W_C=\mathbb{C}^2-0$ and $\tilde{S}=\mathbb{C}^2-0$. Thus S is a Hopf surface. This contradicts (B).

If $C=\mathbb{C}$, then

$$g^* \tilde{W} = \bar{g}(\tilde{W}) = a(g)\,\tilde{W} + b(g),$$

where $a \in \mathrm{Hom}(G, \mathbb{C}^*)$ and $b(g) \in \mathbb{C}$. Hence $d\tilde{W}$ is an element of $H^0(d\mathcal{O}(F_a))$. This contradicts (22).

Hence $C=\mathbb{H}$ and

$$g^* \tilde{W} = \bar{g}(\tilde{W}) = \frac{a(g)\,\tilde{W} + b(g)}{c(g)\,\tilde{W} + d(g)},$$

where

$$\begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix} \in SL(2, \mathbb{R}).$$

Since $dW \neq 0$ everywhere on $\tilde{S}$, this implies that there exist local holomorphic functions $\tilde{w}_i$ on S such that $w_i = \phi_i(\tilde{w}_i)$, where $d\phi_i \neq 0$, $\mathrm{Im}\,\tilde{w}_i > 0$ and

$$\tilde{w}_i = \frac{a_{ij}\tilde{w}_j + b_{ij}}{c_{ij}\tilde{w}_j + d_{ij}},$$

where

$$\begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \in SL(2, \mathbb{R}).$$

Replacing $\log(e^{z_i} \cdot \phi_i')$ and $\tilde{w}_i$ by z_i and w_i, respectively, we may assume that

$$\mathrm{Im}\, w_i > 0,$$

and

$$\begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \in SL(2, \mathbb{R}),$$

in (29). Hence

$$\mathrm{Im}\, w_i = \frac{1}{2\sqrt{-1}} (\xi_i \bar{\eta}_i - \bar{\xi}_i \eta_i) > 0,$$

and

$$\begin{pmatrix} \alpha_{ij} & \beta_{ij} \\ \gamma_{ij} & \delta_{ij} \end{pmatrix} \in GL^+(2, \mathbb{R}),$$

in (30). By analytically continuating ξ_{i_0} and η_{i_0}, we obtain (32) and (33), q.e.d.

We denote by $\mathcal{D}$ the image of Φ in $\mathbb{C}^2$ and denote by $(X+\sqrt{-1}\,Y, U+\sqrt{-1}\,V)$ the global coordinate of $\mathbb{C}^2$ such that $(X+\sqrt{-1}\,Y)\circ\Phi = \xi$ and $(U+\sqrt{-1}\,V)\circ\Phi = \eta$. The formula (32) leads to

$$\mathcal{D} \subset \mathcal{G} = \left\{ (X+\sqrt{-1}\,Y, U+\sqrt{-1}\,V) \,\middle|\, \det \begin{pmatrix} Y & X \\ V & U \end{pmatrix} > 0 \right\}.$$

We identify $\mathscr{G}$ with $GL^+(2, \mathbb{R})$ as a differentiable manifold, and consider $\Phi(P)$ as a matrix of $GL^+(2, \mathbb{R})$ for $P \in \tilde{S}$. By (33) we know that

$$\theta_1 = \Phi^* \left(X \frac{\partial}{\partial X} + U \frac{\partial}{\partial U} \right), \quad \theta_2 = \Phi^* \left(Y \frac{\partial}{\partial Y} + V \frac{\partial}{\partial V} \right),$$

$$\theta_3 = \Phi^* \left(X \frac{\partial}{\partial Y} + U \frac{\partial}{\partial V} \right) \quad \text{and} \quad \theta_4 = \Phi^* \left(Y \frac{\partial}{\partial X} + V \frac{\partial}{\partial U} \right)$$

define global vector fields on S. Hence any linear combination of $\theta_1, \theta_2, \theta_3, \theta_4$ is complete on $\tilde{S}$. Let $p(P, a, b, c, d, t)$ be the integral curve of $a\theta_1 + b\theta_2 + c\theta_3 + d\theta_4$ passing through $P \in \tilde{S}$ where $a, b, c, d \in \mathbb{R}$. Then $p(P, a, b, c, d, t)$ is defined for $-\infty < t < +\infty$. Let

$$f_P(a, b, c, d) = p(P, a, b, c, d, 1).$$

Then f_P is a differentiable mapping of $\mathbb{R}^4$ into $\tilde{S}$, and

$$\Phi(f_P(a, b, c, d)) = \Phi(P) \cdot \exp \begin{bmatrix} b & d \\ c & a \end{bmatrix}. \tag{34}$$

This implies that $\mathscr{D}$ coincides with $\mathscr{G}$. Let

$$V_r = \{(a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 < r\},$$

$\tilde{U}_P = f_P(V_r)$ and $U_P = \Phi \circ f_P(V_r)$. If we take a sufficiently small $r > 0$, then, by (34), $f_P\colon V_r \to \tilde{U}_P$ and $\Phi\colon \tilde{U}_P \to U_P$ are homeomorphisms for any $P \in \tilde{S}$. We take an $r' < r$ such that

$$\exp V_{r'} \cdot \exp V_{r'} \subset \exp V_r.$$

Let $\tilde{U}'_P = f_P(V_{r'})$ and $U'_P = \Phi \circ f_P(V_{r'})$. Then we obtain that $U_Q \supset U'_P$ for any $Q \in \Phi^{-1}(U'_P)$. Let $\tilde{U}$ be a connected component of $\Phi^{-1}(U'_P)$. Then

$$U_Q \supset U'_P \supset \Phi(\tilde{U})$$

for any $Q \in \tilde{U}$. Since $\tilde{U}_Q \cap \tilde{U} \neq \phi$ and $\Phi\colon \tilde{U}_Q \to U_Q$ is a homeomorphism, we obtain that $\tilde{U}_Q \supset \tilde{U}$. Hence $\tilde{U} = \Phi^{-1}(\tilde{U}'_P) \cap \tilde{U}_Q$ and $\Phi\colon \tilde{U} \to U'_P$ is a homeomorphism. Thus $\Phi\colon \tilde{S} \to \mathscr{D} = \mathscr{G}$ is evenly covered. Namely, $\Phi\colon \tilde{S} \to \mathscr{G}$ is the universal covering.

Let Γ denote the subgroup $\left\{ \begin{pmatrix} \alpha(g) & \beta(g) \\ \gamma(g) & \delta(g) \end{pmatrix} \middle| g \in G \right\}$ of $GL^+(2, \mathbb{R})$ and let $p\colon \tilde{G}L^+(2, \mathbb{R}) \to GL^+(2, \mathbb{R})$ be the universal covering group of $GL^+(2, \mathbb{R})$. Then by (31) we establish that S is diffeomorphic to $\tilde{\Gamma} \backslash \tilde{G}L^+(2, \mathbb{R})$ where $\tilde{\Gamma}$ is a discrete subgroup of $\tilde{G}L^+(2, \mathbb{R})$ such that $p(\tilde{\Gamma}) = \Gamma$.

Let $\Gamma_0 = \{A \mid A \in \Gamma, \det A = 1\}$ and $\Gamma_1 = \left\{ \frac{1}{\sqrt{\det A}} A \mid A \in \Gamma \right\}$. It is easy to see $[\Gamma_1, \Gamma_1] = [\Gamma, \Gamma] \subset \Gamma_0$. Note that $\tilde{\Gamma}, \Gamma$ and Γ_1 are finitely generated.

Now we prove that

(35) *Γ is irreducible over $\mathbb{C}$. Γ_1 is also irreducible over $\mathbb{C}$.*

Assume that Γ is reducible over $\mathbb{C}$, then we can find a matrix $A \in GL(2, \mathbb{C})$ such that

$$A \begin{pmatrix} \alpha(g) & \beta(g) \\ \gamma(g) & \delta(g) \end{pmatrix} A^{-1} = \begin{pmatrix} a(g) & b(g) \\ 0 & d(g) \end{pmatrix}$$

for any $g \in G$. Let $\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta \end{pmatrix}$. Then, by (31),

$$g^* \eta_1 = d(g)\, \eta_1 .$$

Thus $d\eta_1$ defines a non-zero element of $H^0(d\mathcal{O}(F_d))$. This contradicts (22).

Furthermore, we prove that

(36) *Γ_0 is discrete in $SL(2, \mathbb{R})$.*

$\operatorname{Ker} p \subset \tilde{G}L^+(2, \mathbb{R})$ is an infinite cyclic subgroup of the center of $\tilde{G}L^+(2, \mathbb{R})$. Let $\tilde{g}$ be a generator of $\operatorname{Ker} p$ and let $\tilde{\Gamma}_0 = \tilde{\Gamma} \cap p^{-1}(\Gamma_0)$. We write $\Delta_0 = \langle \tilde{g}, \tilde{\Gamma}_0 \rangle = p^{-1}(\Gamma_0)$ and $\Delta = \langle \tilde{g}, \tilde{\Gamma} \rangle = p^{-1}(\Gamma)$. We assume that Γ_0 is not discrete in $SL(2, \mathbb{R})$. This implies that Δ_0 is not discrete in $\tilde{G}L^+(2, \mathbb{R})$. Thus there exists a sequence $\{g_n\}_{n \geqq 1}$ of elements g_n of Δ_0 such that

$$g_n \to \mathrm{id} \quad \text{in} \quad \tilde{G}L^+(2, \mathbb{R}) \quad \text{as } n \to \infty$$

and $g_n \notin \langle \tilde{g} \rangle$ for any $n \geqq 1$. For any $g \in \tilde{\Gamma}$,

$$g_n\, g\, g_n^{-1} g^{-1} \to \mathrm{id} \quad \text{in} \quad \tilde{G}L^+(2, \mathbb{R}) \quad \text{as } n \to \infty .$$

Since $\tilde{g}$ belongs to the center of $\tilde{G}L^+(2, \mathbb{R})$, we have

$$[\Delta, \Delta] \subset [\tilde{\Gamma}, \tilde{\Gamma}] \subset \tilde{\Gamma} .$$

Note that $g_n\, g\, g_n^{-1} g^{-1} \in [\Delta, \Delta]$ for any $n \geqq 1$ and that $\tilde{\Gamma}$ is discrete in $\tilde{G}L^+(2, \mathbb{R})$. Hence

$$g_n\, g\, g_n^{-1} g^{-1} = \mathrm{id} \quad \text{for any sufficiently large } n.$$

Recalling that $\tilde{\Gamma}$ is finitely generated, we obtain

$$g_n\, g = g\, g_n \quad \text{for any } g \in \tilde{\Gamma} \quad \text{and for } n \gg 0 .$$

Let $A_n = p(g)$. Then it follows that $A_n \neq I$ and

$$A_n A = A A_n \quad \text{for any } A \in \Gamma \quad \text{and for } n \gg 0$$

and $\det A_n = 1$. Since Γ is irreducible by (35), this implies that $A_n = -I$ for $n \gg 0$ by virtue of Shur's lemma. This contradicts that $A_n \neq I$ and

$$A_n \to I \quad \text{in} \quad GL^+(2, \mathbb{R}) \quad \text{as } n \to \infty .$$

Finally we prove that

(37) *Γ_1 is discrete in $SL(2, \mathbb{R})$.*

If Γ_1 is not discrete in $SL(2, \mathbb{R})$, there exists a sequence $\{A_n\}_{n\geqq 1}$ of elements of Γ_1 such that

(38) $$A_n \to I \quad \text{in} \quad SL(2, \mathbb{R}) \quad \text{as } n\to\infty, \ A_n \neq I.$$

For any $A \in \Gamma_1$,

$$A_n A A_n^{-1} A^{-1} \to I \quad \text{in} \quad SL(2, \mathbb{R}) \quad \text{as } n\to\infty.$$

Since $A_n A A_n^{-1} A^{-1} \in [\Gamma_1, \Gamma_1] \subset \Gamma_0$ and Γ_0 is discrete in $SL(2, \mathbb{R})$ by (36), we obtain

$$A_n A A_n^{-1} A^{-1} = I \quad \text{for sufficiently large } n.$$

Recalling that Γ_1 is finitely generated, we obtain

$$A_n A = A A_n \quad \text{for any } A \in \Gamma_1 \quad \text{and for } n \gg 0$$

and $\det A_n = 1$. Since Γ_1 is irreducible by (35), this implies that $A_n = -I$ for $n \gg 0$ by virtue of Shur's lemma. This contradicts (38).

Now let Γ_P denote the linear fractional transformation group of $\mathbb{H}$ associated with Γ. Then (37) implies that Γ_P is properly discontinuous. From (31) we see that $W = \xi/\eta$ gives a holomorphic mapping of S onto $\mathbb{H}/\Gamma_P$. Hence there are many curves on S. Thus we arrive at a contradiction. This establishes $H^0(d\mathcal{O}(F_0)) \neq 0$.

§ 7. Two Cases

Throughout this section we denote by L a line bundle such that $H^0(d\mathcal{O}(L)) \neq 0$. After exhibiting certain properties of the line bundle L, we shall prove a key proposition in which it will be shown that two cases occur.

Property (α). *If $2L + K = 0$, then $L \neq K, \bar{K}$ and $L = \bar{L}$.*

If $L = K$ or $\bar{K}$, then $2L + K = 0$ implies that $2K + K = 0$ or $2\bar{K} + K = 0$. In the latter case, $K = -2\bar{K} = 4K$. Thus, in any case, $3K = 0$. This contradicts Lemma 1.

Next we assume that $L \neq \bar{L}$. Since $L \neq K, \bar{K}$, it follows from (5) that

$$\dim H^0(d\mathcal{O}(L)) = \dim H^1(\mathbb{C}(L)) = \dim H^1(\mathbb{C}(\bar{L})) = \dim H^0(d\mathcal{O}(\bar{L})).$$

Combining this with Lemma 2, we obtain $L + \bar{L} + K = 0$, while $2L + K = 0$. Hence $L = \bar{L}$.

Property (β). $\dim H^0(d\mathcal{O}(L)) \leqq 1$.

We assume that $\dim H^0(d\mathcal{O}(L)) \geqq 2$ and derive a contradiction. By Lemma 2, we have $2L+K=0$. Thus, by Property (α), $L \neq K$ and $L=\bar{L}$. Hence L is represented by $\{l_{ij}\}$, where $l_{ij} \in \mathbb{R}-0$ and, by (5), we can choose two linearly independent elements $\{a_{ij}\}$ and $\{b_{ij}\}$ from $H^1(\mathbb{C}(L))$ such that $a_{ij}, b_{ij} \in \mathbb{R}$. Write $\xi = \vartheta^{-1}(\{a_{ij}\})$ and $\eta = \vartheta^{-1}(\{b_{ij}\})$. Then, since ξ and η belong to $H^0(d\mathcal{O}(L))$, we have $\xi = \{d\xi_i\}$ and $\eta = \{d\eta_i\}$, where local holomorphic functions ξ_i and η_i satisfy

$$\xi_i = l_{ij}\xi_j + a_{ij}, \qquad (39)$$
$$\eta_i = l_{ij}\eta_j + b_{ij}.$$

We note that, by Lemma 2, $\xi \wedge \eta = \{d\xi_i \wedge d\eta_i\}$ does not vanish anywhere.

Let $w_i = \operatorname{Re}\xi_i + \sqrt{-1}\operatorname{Re}\eta_i$ and $z_i = \operatorname{Im}\xi_i + \sqrt{-1}\operatorname{Im}\eta_i$. Then, by (39), we obtain

$$w_i = l_{ij}, w_j + (a_{ij} + \sqrt{-1}\, b_{ij}), \qquad (40)$$
$$z_i = l_{ij} z_j,$$

where $dw_i \wedge dz_i \wedge d\bar{w}_i \wedge d\bar{z}_i$ does not vanish anywhere. Hence, by taking (w_i, z_i) as local coordinates, we can introduce a complex structure S^* on the underlying differentiable manifold of S. Clearly S^* satisfies the condition (A). Let L^* be a line bundle on S^* defined by $\{l_{ij}\}$. Then $L^* \neq 0$ in $H^1(S^*, \mathcal{O}^*)$. In fact, if $L^*=0$ in $H^1(S^*, \mathcal{O}^*)$, then z_i does not vanish anywhere. Hence, by (40), $d\log z_i = d\log z_j$ is a holomorphic 1-form on S^*. This contradicts Proposition 1. Since $\{z_i\}$ defines a holomorphic section of L^*, S^* contains a curve. Thus, by the result of Kodaira quoted in § 0, S^* is a Hopf surface or an elliptic surface with $P_{12}(S^*) = \dim H^0(S^*, \mathcal{O}(12K^*)) > 0$, where K^* is the canonical bundle of S^*. If S^* is a Hopf surface, then $\pi_1(S) = \pi_1(S^*)$ contains $\mathbb{Z}$ as a subgroup of finite index. Hence S is itself a Hopf surface by Kodaira [3]. This is absurd. If S^* is an elliptic surface with $P_{12}(S^*) > 0$, then $\dim H^0(S^*, \mathcal{O}(-24L^*)) > 0$ since $2L^* = -K^*$ by (40). This contradicts that $\dim H^0(S^*, \mathcal{O}(L^*)) > 0$ and $L^* \neq 0$.

In view of (5), Property (β) implies

$$\dim H^1(\mathbb{C}(F)) \leqq 1 \quad \text{for } F \neq K. \qquad (41)$$

Property (γ). $mL \neq 0$ *for any non-zero integer* m.

We assume $mL=0$ for some non-zero m and take a non-zero $\xi \in H(d\mathcal{O}(L))$. Construct a finite unramified covering $p: S' \to S$ such that $p^*L=0$. Then $p^*\xi \neq 0 \in H^0(S', d\mathcal{O})$, which contradicts $h^{1,0}(S')=0$.

Writing $L=\{l_{ij}\}$, $l_{ij} \in \mathbb{C}^*$, we can describe a nonzero $\xi \in H^0(d\mathcal{O}(L))$ as $\{dw_i\}$, and hence we have

$$w_i = l_{ij} w_j + a_{ij}, \quad a_{ij} \in \mathbb{C}.$$

Property (δ). $\{a_{ij}\} \neq 0$ *in* $H^1(\mathbb{C}(L))$.

Otherwise, we have $a_i \in \mathbb{C}$ such that $a_{ij} = -a_i + l_{ij} a_j$. Hence $w_i + a_i = l_{ij}(w_j + a_j)$. Thus we obtain a non-zero element $\{w_i + a_i\} \in H^0(\mathcal{O}(L))$. Then $L=0$ by Lemma 1 (iv). This contradicts Property (γ).

Now we take a non-zero element ξ of $H^0(d\mathcal{O}(F_0))$ whose existence is assured in § 6. We write $F_0 = \{f_{ij}\}$, $K = \{\kappa_{ij}^{-1}\}$, $f_{ij}, \kappa_{ij} \in \mathbb{C}^*$ and $\xi = \{dw_i\}$, where $dw_i = f_{ij}\, dw_j$. Since $c_2(S)=0$ and S contains no curves, $\xi = \{dw_i\}$ does not vanish anywhere. Hence, choosing appropriate holomorphic functions z_i, we obtain local coordinates (w_i, z_i). We have

$$dw_i \wedge dz_i = (g_i^{-1} \cdot \kappa_{ij} \cdot g_j)\, dw_j \wedge dz_j,$$

where the g_i are non-vanishing local holomorphic functions. Replace $\int^{z_i} g_i\, dz_i$ by z_i. Then we have

$$(dz_i - f_{ij}^{-1}\, \kappa_{ij}\, dz_j) \wedge dw_j = 0.$$

Hence we obtain local coordinates (w_i, z_i) such that

$$\begin{aligned} w_i &= f_{ij}\, w_j + a_{ij} \\ z_i &= f_{ij}^{-1}\, \kappa_{ij}\, z_j + g_{ij}(w_i), \end{aligned} \tag{42}$$

where the a_{ij} are constants and the $g_{ij} = g_{ij}(w_i)$ are local holomorphic functions of the w_i. Hence we obtain

$$\frac{\partial}{\partial z_i} = f_{ij}\, \kappa_{ij}^{-1}\, \frac{\partial}{\partial z_j}.$$

Let m be an arbitrary integer. Any local holomorphic section ϕ of $mF_0 - K$ is written in the form $\{\phi_i\}$, where holomorphic functions ϕ_i satisfy $\phi_i = f_{ij}^m\, \kappa_{ij}\, \phi_j$. Since $\frac{\partial \phi_i}{\partial z_i} = f_{ij}^{m+1} \cdot \frac{\partial \phi_j}{\partial z_j}$, $\frac{\partial \phi}{\partial z} = \left\{ \frac{\partial \phi_i}{\partial z_i} \right\}$ gives a local holomorphic section of $(m+1) F_0$. Let $\mathcal{O}_w(mF_0 - K)$ denote the subsheaf of $\mathcal{O}(mF_0 - K)$ consisting of germs of those holomorphic sections ϕ which satisfy $\frac{\partial \phi}{\partial z} = 0$. Note that ϕ satisfies $\frac{\partial \phi}{\partial z} = 0$ if and only if ϕ_i are holomorphic functions of w_i. We have the exact sequence

$$0 \to \mathcal{O}_w(mF_0 - K) \to \mathcal{O}(mF_0 - K) \to \mathcal{O}((m+1) F_0) \to 0. \tag{43}$$

From (42) it follows that

$$g_{ik}(w_i)=g_{ij}(w_i)+f_{ij}^{-1}\kappa_{ij}g_{jk}(w_j),$$
$$g'_{ik}(w_i)=g'_{ij}(w_i)+f_{ij}^{-2}\kappa_{ij}g'_{jk}(w_j),$$
$$g''_{ik}(w_i)=g''_{ij}(w_i)+f_{ij}^{-3}\kappa_{ij}g''_{jk}(w_j),$$

namely

$$\{g_{ij}\}\in H^1(\mathcal{O}_w(-F_0-K)),$$
$$(44)\qquad \{g'_{ij}\}\in H^1(\mathcal{O}_w(-2F_0-K)),$$
$$\{g''_{ij}\}\in H^1(\mathcal{O}_w(-3F_0-K)).$$

Proposition 7. *Only the following two cases occur:*

Case (I). $H^1(\mathcal{O}_w(-2F_0-K))=0$.

Case (II). $H^1(\mathcal{O}_w(-3F_0-K))=0$, *and either* $2F_0+K=0$ *or* $2F_0+2K=0$.

Proof. If $\dim H^1(\mathcal{O}_w(-2F_0-K))\neq 0$, then, by Lemma 1 (iv), (43) and Property (γ), we have $\dim H^1(\mathcal{O}(-2F_0-K))\neq 0$. Hence, from Lemma 1 (iv), it follows that $-2F_0-K=0$ or K. If moreover $\dim H^1(\mathcal{O}_w(-3F_0-K))\neq 0$, then, by the same reasoning as above, we have $-3F_0-K=0$ or K. Hence K, $2K$ or $4K=0$. This contradicts Lemma 1, q.e.d.

Proposition 8. *In Case* (I), *there exist local coordinates* (w_i, z_i) *such that*

$$(45)\qquad \begin{aligned} w_i&=f_{ij}w_j+a_{ij}\\ z_i&=f_{ij}^{-1}\kappa_{ij}z_j+b_{ij},\end{aligned}$$

where a_{ij}, $b_{ij}\in\mathbb{C}$, $\{a_{ij}\}\neq 0$ *in* $H^1(\mathbb{C}(F_0))$ *and* $\{b_{ij}\}\neq 0$ *in* $H_1(\mathbb{C}(-F_0-K))$.

Proof. By hypothesis and (44), $\{g'_{ij}\}\in H^1(\mathcal{O}_w(-2F_0-K))=0$. Hence there exist holomorphic functions $h_i=h_i(w_i)$ of w_i such that

$$g'_{ij}(w_i)=f_{ij}^{-2}\kappa_{ij}h'_j(w_j)-h'_i(w_i).$$

Thus we obtain

$$g_{ij}(w_i)=f_{ij}^{-1}\kappa_{ij}h_j(w_j)-h_i(w_i)+b_{ij},$$

where $b_{ij}\in\mathbb{C}$. Replacing $z_i+h_i(w_i)$ by z_i, we obtain, in view of (42), local coordinates (w_i, z_i) satisfying (45). By Property (δ), $\{a_{ij}\}\neq 0$ and $\{b_{ij}\}\neq 0$, q.e.d.

Proposition 9. *In Case* (II), *there exist local coordinates* (w_i, z_i) *such that*

$$(46)\qquad \begin{aligned} w_i&=f_{ij}w_j+a_{ij}\\ z_i&=f_{ij}^{-1}\kappa_{ij}z_j+b_{ij}w_j+c_{ij},\end{aligned}$$

where $a_{ij}, b_{ij}, c_{ij} \in \mathbb{C}$ *and* $\{a_{ij}\} \neq 0$ *in* $H^1(\mathbb{C}(F_0))$ *and* $\{b_{ij}/f_{ij}\} \neq 0$ *in* $H^1(\mathbb{C}(-2F_0-K))$.

Proof. By hypothesis and (44), $\{g''_{ij}\} \in H^1(\mathcal{O}_w(-3F_0-K))=0$. Hence there exist holomorphic functions $h_i = h_i(w_i)$ of w_i such that

$$g''_{ij}(w_i) = f_{ij}^{-3}\, \kappa_{ij}\, h''_j(w_j) - h''_i(w_i).$$

Thus we obtain

$$g_{ij}(w_i) = f_{ij}^{-1}\, \kappa_{ij}\, h_j(w_j) - h_i(w_i) + b_{ij}\, w_j + c_{ij}$$

where $b_{ij}, c_{ij} \in \mathbb{C}$. Replacing $z_i + h_i(w_i)$ by z_i, we obtain, in view of (42), local coordinates (w_i, z_i) satisfying (46). By Property (δ), $\{a_{ij}\} \neq 0$. Now we assume that $\{b_{ij}/f_{ij}\} = 0$ in $H^1(\mathbb{C}(-2F_0-K))$. Then there exist constants b_i such that

$$b_{ij}/f_{ij} = f_{ij}^{-2}\, \kappa_{ij}\, b_j - b_i.$$

Hence

$$z_i + b_i w_i = f_{ij}^{-1}\, \kappa_{ij}(z_j + b_j w_j) + b_i a_{ij} + c_{ij},$$

and therefore $\{d(z_i + b_i w_i)\} \in H(d\mathcal{O}(-F_0-K))$. Hence $2F_0 + 2K \neq 0$ by Property (γ) and therefore $2F_0 + K = 0$. Thus $\{dw_i\}$ and $\{d(z_i + b_i w_i)\}$ give two linearly independent elements of $H^0(d\mathcal{O}(F_0))$. This contradicts Property (β), q.e.d.

Proposition 10. $2F_0 + K \neq 0$.

Proof. We assume $2F_0 + K = 0$ and derive a contradiction. By Property (α), F_0 is real, and thus we can write $F_0 = \{f_{ij}\}$, $K = \{\kappa_{ij}^{-1}\}$, where $f_{ij}, \kappa_{ij} \in \mathbb{R} - 0$, $f_{ij}^2 = \kappa_{ij}$.

In Case (I), $\{(w_i, z_i)\}$ in Proposition 8 defines two linearly independent elements $\{dw_i\}$ and $\{dz_i\}$ of $H^0(d\mathcal{O}(F_0))$, since $F_0 = -F_0 - K$. This contradicts Property (β).

In Case (II), we use $w_i, z_i, a_{ij}, b_{ij}, c_{ij}$ in Proposition 9. In view of (41) and Proposition 9, we obtain $\dim H^1(\mathbb{C}(F_0)) = 1$. We can take non-zero element $\{r_{ij}\}$ of $H^1(\mathbb{C}(F_0))$ such that $r_{ij} \in \mathbb{R}$, since F_0 is real. Then

$$a_{ij} = a\, r_{ij} + f_{ij}\, a_j - a_i,$$

where $a \in \mathbb{C}^*$ and $a_i \in \mathbb{C}$. Similarly if we take non-zero element $\{s_{ij}\}$ of $H^1(\mathbb{C}(-2F_0-K)) = H^1(\mathbb{C})$ such that $s_{ij} \in \mathbb{R}$, then

$$b_{ij}/f_{ij} = b\, s_{ij} + b_j - b_i,$$

where $b \in \mathbb{C}^*$ and $b_i \in \mathbb{C}$. Hence, replacing $(w_i + a_i)/a$, $z_i + b_i(w_i + a_i)/a\, b$, $r_{ij}, f_{ij}\, s_{ij}$ by w_i, z_i, a_{ij}, b_{ij}, respectively, we can assume that a_{ij} and b_{ij} are real in (46). Then it is evident that

$$\{\operatorname{Im} c_{ij}\} \in H^1(\mathbb{C}(-F_0-K)) = H^1(\mathbb{C}(F_0)).$$

Thus

$$\operatorname{Im} c_{ij} = c\, a_{ij} + f_{ij}\, c_j - c_i,$$

where $c, c_i \in \mathbb{R}$, and hence

$$z_i - \sqrt{-1}(c\, w_i - c_i) = f_{ij}\{z_j - \sqrt{-1}(c\, w_j - c_j)\} + b_{ij}\, w_j + c_{ij} - \sqrt{-1} \cdot \operatorname{Im} c_{ij}.$$

Replacing $z_i - \sqrt{-1}(c\, w_i - c_i)$ and $c_{ij} - \sqrt{-1} \cdot \operatorname{Im} c_{ij}$ by z_i and c_{ij}, we can assume that $a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$ in (46).

The analytic continuations of w_i and z_i produce holomorphic functions W and Z on S such that

$$\begin{aligned} g^* W &= \mu_{F_0}(g)\, W + a(g) \\ g^* Z &= \mu_{F_0}(g)\, Z + b(g)\, W + c(g), \qquad \text{for } g \in G, \end{aligned} \tag{47}$$

where $\mu_{F_0}(g), a(g), b(g), c(g) \in \mathbb{R}$ and $dW \wedge dZ$ does not vanish anywhere. Let $U = \operatorname{Re} W$, $V = \operatorname{Im} W$, $X = \operatorname{Re} Z$ and $Y = \operatorname{Im} Z$. Then, by (47), we obtain

$$\begin{aligned} g^* U &= \mu_{F_0}(g)\, U + a(g) \\ g^* V &= \mu_{F_0}(g)\, V \\ g^* X &= \mu_{F_0}(g)\, X + b(g)\, U + c(g) \\ g^* Y &= \mu_{F_0}(g)\, Y + b(g)\, V \qquad \text{for } g \in G, \end{aligned} \tag{48}$$

where $dU \wedge dV \wedge dX \wedge dY$ does not vanish anywhere.

We take a normal subgroup G' of finite index in G such that

$$\mu_{F_0}(g) > 0 \qquad \text{for } g \in G',$$

and take the finite unramified covering surface S' of S corresponding to G'. We consider U, V, X, Y as multi-valued functions on S.

Let Σ be $\{P \in S' \mid V(P) = 0\}$. If Σ is empty, V does not vanish anywhere. We may assume that $V > 0$ everywhere. Hence V gives a differentiable non-degenerate fibering of S' over $T^1 = \mathbb{R}^+ / \langle \mu_{F_0}(g) \mid g \in G' \rangle$ with connected fibre S_r over $r \in T^1$. The restriction $H^1(S', \mathbb{C}) \to H^1(S_r, \mathbb{C})$ is a zero-mapping and the formula (48) implies that $\mu_{F_0}(g)^{-1} b(g)$ is an element of $H^1_1(G') \simeq H^1(S', \mathbb{C})$. Hence the restriction $Y \mid S_r$ of Y to S_r is a single-valued function on S_r, while $d(Y \mid S_r) \neq 0$ everywhere on S_r. This contradicts the compactness of S_r. Thus Σ is non-empty.

Let $S^+ = \{P \in S' \mid V(P) \geqq 0\}$. Then S^+ is a compact manifold whose boundary is Σ. Let $\bar{M}$ be a connected component of S^+ and M the interior of $\bar{M}$. Then M is connected and the inclusion of M into $\bar{M}$ is a homotopy equivalence. Let $p\colon \tilde{S} \to S'$ be the universal covering and $\tilde{M}$ a connected component of $p^{-1}(M)$. We denote by Φ the differentiable non-degenerate mapping of $\tilde{S}$ into $\mathbb{R}^4$ defined by $\Phi(P) = (U(P), X(P), Y(P), V(P))$ for $P \in \tilde{S}$. Then Φ sends $\tilde{M}$ into $\mathbb{R}^3 \times \mathbb{R}^+$.

Let (u, x, y, v) be the global coordinate of $\mathbb{R}^4$. By (48) we know that

$$\theta_1 = \Phi^* \left(y \frac{\partial}{\partial x} + v \frac{\partial}{\partial u}\right), \quad \theta_2 = \Phi^* \left(y \frac{\partial}{\partial y} + v \frac{\partial}{\partial v}\right),$$

$$\theta_3 = \Phi^* \left(v \frac{\partial}{\partial x}\right) \quad \text{and} \quad \theta_4 = \Phi^* \left(v \frac{\partial}{\partial y}\right)$$

define global vector fields on S'. Hence any linear combination $\theta(\alpha, \beta, \gamma, \delta) = \alpha\theta_1 + \beta\theta_2 + \gamma\theta_3 + \delta\theta_4$ of them is complete on $\tilde{S}$. We denote by $p(P, \alpha, \beta, \gamma, \delta, t)$ the integral curve of $\theta(\alpha, \beta, \gamma, \delta)$ passing through $P \in \tilde{S}$. Then $p(P, \alpha, \beta, \gamma, \delta, t)$ is defined for $-\infty < t < +\infty$. Moreover, by the definition of $\tilde{M}$, we obtain that $p(P, \alpha, \beta, \gamma, \delta, t)$ lies in $\tilde{M}$ for each $P \in \tilde{M}$. Let $f_P(\alpha, \beta, \gamma, \delta) = p(P, \alpha, \beta, \gamma, \delta, 1)$. Then f_P is a differentiable mapping of $\mathbb{R}^4$ into $\tilde{M}$ for $P \in \tilde{M}$. We can deduce from the definition of f_P that

$$\begin{aligned}
U \circ f_P(\alpha, \beta, \gamma, \delta) &= U(P) + \alpha V(P) \cdot (\exp\beta - 1)/\beta, \\
X \circ f_P(\alpha, \beta, \gamma, \delta) &= X(P) + (\alpha Y(P) + \gamma V(P)) \cdot (\exp\beta - 1)/\beta \\
&\quad + \alpha\delta V(P) \cdot (\beta \exp\beta - \exp\beta + 1)/\beta^2, \\
Y \circ f_P(\alpha, \beta, \gamma, \delta) &= Y(P) \cdot \exp\beta + \delta V(P) \cdot \exp\beta, \\
V \circ f_P(\alpha, \beta, \gamma, \delta) &= V(P) \cdot \exp\beta.
\end{aligned}$$

This implies that $\Phi \circ f_P$: $\mathbb{R}^4 \to \mathbb{R}^3 \times \mathbb{R}^+$ is a homeomorphism. Hence f_P: $\mathbb{R}^4 \to \mathscr{U}_P = f_P(\mathbb{R}^4) \subset \tilde{M}$ and Φ: $\mathscr{U}_P \to \mathbb{R}^3 \times \mathbb{R}^+$ are homeomorphisms. From this we can easily derive that $\mathscr{U}_P = \tilde{M}$. Thus Φ: $\tilde{M} \to \mathbb{R}^3 \times \mathbb{R}^+$ is a homeomorphism.

Let Γ be the subgroup of G' formed by topological transformations which send each point of $\tilde{M}$ into $\tilde{M}$. The formula (48) defines a representation R: $\Gamma \to R(\Gamma)$ of Γ by affine transformations of $\mathbb{R}^3 \times \mathbb{R}^+$. Φ induces a homeomorphism of M onto $\mathbb{R}^3 \times \mathbb{R}^+/R(\Gamma)$. We note that $\mu_{F_0}(g_1) \neq 1$ for certain $g_1 \in \Gamma$. In fact, if $\mu_{F_0}(g) = 1$ for any $g \in \Gamma$, then (48) implies that the restriction $V|\overline{M}$ of V to $\overline{M}$ is a single-valued function on $\overline{M}$ since the inclusion $M \to \overline{M}$ is a homotopy equivalence, while $V|\overline{M} = 0$ on $\partial\overline{M}$ and $dV \neq 0$ everywhere in M. This contradicts the compactness of $\overline{M}$. Replacing $W - (a(g_1)/(1 - \mu_{F_0}(g_1)))$ and

$$Z - \{c(g_1) + b(g_1)(a(g_1)/(1 - \mu_{F_0}(g_1)))\}/(1 - \mu_{F_0}(g_1))$$

by W and Z, respectively, we can assume that $a(g_1) = c(g_1) = 0$ in (47) and (48). Hence there exists an element $g_1 \in \Gamma$ such that $R(\Gamma)$ is generated by

$$R(g_1)\colon (u, x, y, v) \to (\mu_{F_0}(g_1)u,\ \mu_{F_0}(g_1)x + b(g_1)u,\ \mu_{F_0}(g_1)y + b(g_1)v,\ \mu_{F_0}(g_1)v).$$

Thus we obtain that $\pi_1(M) = \langle g_1 \rangle$. Since the inclusion $M \to \overline{M}$ is a homotopy equivalence, we obtain that $\pi_1(\overline{M}) = \langle g_1 \rangle$. Hence, by (48),

Φ induces a homeomorphism of $\bar{M}$ onto

$$\{(u, x, y, v) \in \mathbb{R}^4 \mid v \geqq 0, (u, x, y, v) \neq 0\} / \langle R(g_1) \rangle .$$

Let $C = \{P \in \bar{M} \mid U(P) = V(P) = 0\}$. Then C is a compact real submanifold of S' and is defined locally by $W = 0$. Hence C is a curve in S' and therefore S contains a curve. This contradicts (B), q.e.d.

Remark. It is Kodaira who observed that one can choose such local coordinates as in Proposition 8 and 9.

§ 8. Case (I)

By Lemma 2 we obtain

$$\dim H^0(d\mathcal{O}(F)) = 0 \quad \text{for any } F \neq F_0, -F_0 - K .$$

From this and (5) it follows that

(49) $$\dim H^1(\mathbb{C}(F)) = 0 \quad \text{for any } F \neq 0, K, F_0, -F_0 - K .$$

In view of Proposition 8

(50) $$\dim H^1(\mathbb{C}(F_0)) \neq 0 \quad \text{and} \quad \dim H^1(\mathbb{C}(-F_0 - K)) \neq 0,$$

where $F_0 \neq -F_0 - K$ by Proposition 10, and $F_0, -F_0 - K \neq 0$.

Proposition 11. *If F_0 is not real, then $\bar{F}_0 = K$ and $-F_0 - K$ is real.*

Proof. Suppose that F_0 is not real. Then, by (50), $\dim H^1(\mathbb{C}(F_0)) = \dim H^1(\mathbb{C}(\bar{F})) \neq 0$. Hence, in view of (49), we obtain

$$\bar{F}_0 = K \quad \text{or} \quad -F_0 - K .$$

If $\bar{F}_0 = K$, then $-F_0 - K = -\bar{K} - K$ is real. We assume $\bar{F}_0 = -F_0 - K \neq K$ and derive a contradiction. If $F_0 = K$, then $\bar{K} = \bar{F}_0 = -F_0 - K = -2K$. This implies that $3K = 0$. This contradicts Lemma 1. Thus $F_0 \neq K$. (41) and (50) lead to

$$\dim H^1(\mathbb{C}(F_0)) = \dim H^1(\mathbb{C}(-F_0 - K)) = 1 .$$

We use w_i, z_i, a_{ij}, b_{ij} in Proposition 8. Since $\{a_{ij}\} \neq 0$ in $H^1(\mathbb{C}(F_0))$, we see that $\{\bar{a}_{ij}\} \neq 0$ in $H^1(\mathbb{C}(\bar{F}_0)) = H^1(\mathbb{C}(-F_0 - K))$. Since $\{b_{ij}\} \neq 0$ in $H^1(\mathbb{C}(-F_0 - K))$, we obtain

$$\bar{a}_{ij} = a\, b_{ij} + f_{ij}^{-1} \kappa_{ij} a_j - a_i ,$$

where $a \in \mathbb{C}^*$ and $a_i \in \mathbb{C}$. Replacing $(a z_i - a_i)$ by z_i, we may assume in (45) that

$$w_i = f_{ij} w_j + a_{ij} ,$$

$$z_i = \bar{f}_{ij} z_j + \bar{a}_{ij} .$$

Let $\xi_i = w_i$ and $\eta_i = \bar{z}_i$. Then

$$(51) \qquad \begin{aligned} \xi_i &= f_{ij}\,\xi_j + a_{ij} \\ \eta_i &= f_{ij}\,\eta_j + a_{ij}, \end{aligned}$$

and $d\xi_i \wedge d\eta_i \wedge d\bar{\xi}_i \wedge d\bar{\eta}_i$ does not vanish anywhere. Hence, by taking (ξ_i, η_i) as local coordinates, we can introduce a complex structure S^* on the underlying differentiable manifold of S. Clearly S^* satisfies the condition (A). Let F_0^* be a line bundle on S^* defined by $\{f_{ij}\}$. Then $F_0^* \neq 0$ in $H^1(S^*, \mathcal{O}^*)$. In fact, if $F_0^* = 0$ in $H^1(S^*, \mathcal{O}^*)$, then $F_0^* = 0$ in $H^1(S^*, \mathbb{C}^*)$ by Proposition 1, and therefore $F_0 = 0$. This contradicts Property (γ). By (51) $\{(\xi_i - \eta_i)\}$ defines a holomorphic section of F_0^*. Thus S^* contains a curve, and, by the result of Kodaira quoted in § 0, S^* is a Hopf surface or an elliptic surface with

$$P_{12}(S^*) = \dim H^0(S^*, \mathcal{O}(12K^*)) > 0,$$

where K^* is the canonical bundle of S^*. If S^* is a Hopf surface, then $\pi_1(S) = \pi_1(S^*)$ contains $\mathbb{Z}$ as a subgroup of finite index. Hence S is itself a Hopf surface by Kodaira [3]. This is absurd. If S^* is an elliptic surface with $P_{12}(S^*) > 0$, then $\dim H^0(S^*, \mathcal{O}(-24F_0^*)) > 0$ since $2F_0^* = -K^*$ by (51). This contradicts that $\dim H^0(S^*, \mathcal{O}(F_0^*)) > 0$ and $F_0^* \neq 0$, q.e.d.

By (45) and Proposition 11, we may assume, without loss of generality, that F_0 *is real.*

Proposition 12. *F_0 and $-F_0 - K$ do not coincide with K.*

Proof. We assume that F_0 or $-F_0 - K$ coincides with K, and derive a contradiction. If $F_0 = K$, then $-F_0 - K = -2F_0$ is real. If $-F_0 - K = K$, then $-\bar{F}_0 - \bar{K} = K = -F_0 - K$ is real by Proposition 11 (where we consider $-F_0 - K$ instead of F_0). In any case, $-F_0 - K$ is also real. Hence we may assume, without loss of generality, that $F_0 = K$. We can take transition functions as $F_0 = \{\kappa_{ij}^{-1}\} = K$, where $\kappa_{ij} \in \mathbb{R} - 0$. Then (45) turns out to be

$$\begin{aligned} w_i &= \kappa_{ij}^{-1}\, w_j + a_{ij}, \\ z_i &= \kappa_{ij}^2\, z_j + b_{ij}. \end{aligned}$$

It follows from (41) and (50) that $\dim H^1(\mathbb{C}(-2K)) = 1$. Let $\{r_{ij}\} \neq 0$ in $H^1(\mathbb{C}(-2K))$, where $r_{ij} \in \mathbb{R}$. Then

$$b_{ij} = b\, r_{ij} + \kappa_{ij}^2\, b_j - b_i$$

where $b \in \mathbb{C}^*$ and $b_i \in \mathbb{C}$. By replacing $(z_i + b_i)/b$ and r_{ij} by z_i and b_{ij}, respectively, we may assume in the above formula that $b_{ij} \in \mathbb{R}$.

The analytic continuations of w_i and z_i produce holomorphic functions W and Z on $\tilde{S}$ such that

$$g^* W = \kappa(g)^{-1} W + a(g),$$
$$g^* Z = \kappa(g)^2 Z + b(g) \quad \text{for } g \in G,$$

where $\kappa = \mu_{-K}$, $\kappa(g)$, $b(g) \in \mathbb{R}$ and $dW \wedge dZ$ does not vanish anywhere. Let $U = \operatorname{Re} W$, $V = \operatorname{Im} W$, $X = \operatorname{Re} Z$, $Y = \operatorname{Im} Z$. Then

(52)
$$\begin{aligned} g^* U &= \kappa(g)^{-1} U + a_1(g) \\ g^* V &= \kappa(g)^{-1} V + a_2(g) \\ g^* X &= \kappa(g)^2 X + b(g) \\ g^* Y &= \kappa(g)^2 Y \quad \text{for } g \in G, \end{aligned}$$

where $a_1(g) = \operatorname{Re} a(g)$ and $a_2(g) = \operatorname{Im} a(g)$. Consider Y as a multi-valued differentiable function on S, and let Σ be $\{P \in S \mid Y(P) = 0\}$ and Σ_0 a connected component of Σ. If Σ is not empty, Σ_0 is a compact 3-dimensional real submanifold of S. (52) implies that $\omega = dU \wedge dV \wedge dX$ gives a closed real 3-form on S. Since $\omega_0 = \omega \mid \Sigma_0$ gives a volume form on Σ_0, we obtain

$$\int_{\Sigma_0} \omega \neq 0.$$

This implies that Σ_0 is not homologous to zero on S. Let S^+ be

$$\{P \in S \mid Y(P) \geqq 0\}.$$

Then S^+ is a compact manifold whose boundary is Σ. Each 1-cycle γ representing an element of $H_1(S^+, \mathbb{C})$ can be taken from the interior of S^+. Hence $\gamma \cdot \Sigma_0 = 0$. In view of the duality this implies that each 1-cycle in S^+ is homologous to zero in S, namely the inclusion $H_1(S^+, \mathbb{C}) \to H_1(S, \mathbb{C})$ is a zero mapping. Therefore the restriction $Y \mid S^+$ of Y to S^+ is a single-valued function on S^+, while $Y \mid S^+ = 0$ on $\partial S^+ = \Sigma$ and $d(Y \mid S^+) \neq 0$ everywhere in the interior of S^+. This contradicts the compactness of S^+. Hence Σ is empty, i.e., Y does not vanish anywhere. We may assume that $Y > 0$ everywhere. Since $\kappa(g_0)^2 \neq 1$, Y gives a differentiable non-degenerate fibering of S over $T^1 = \mathbb{R}^+ / \langle \kappa(g_0)^2 \rangle$ with connected fibre S_r over $r \in T^1$. The subgroup $\bar{G}$ of G defined in § 1 is isomorphic to $\pi_1(S_r)$.

Now it follows from (41), (49) and (50) that

(53) $$\dim H^1(\mathbb{C}) = \dim H^1(\mathbb{C}(-2K)) = 1, \quad \dim H^1(\mathbb{C}(K)) \neq 0$$

and

(54) $$H^1(\mathbb{C}(F)) = 0 \quad \text{for any } F \neq 0, K, -2K.$$

Since $-2K\in\mathscr{P}$, we infer from (9), (11), (53), (54) that

(55) $\dim H^1(\bar{G})^{\kappa(g_0)^2}=1, \quad \dim H^1(\bar{G})^{\mu}=0 \quad \text{for } \mu\neq\kappa(g_0)^{-1},\ \kappa(g_0)^2.$

Since S_r is compact, $\bar{G}$ is finitely generated and, consequently, the automorphism g_0^* of $H^1(\bar{G})$ is represented by an integral matrix M with $\det M=\pm 1$. (55) implies that $\kappa(g_0)^2$ is an eigen-value of M. If $\kappa(g_0)^2$ is the only eigen-value, then $\kappa(g_0)^2=1$ and therefore $2K=0$. This contradicts Lemma 1. Thus we infer from (55) that the eigen-values of M are $\kappa(g_0)^2$ and $\kappa(g_0)^{-1}$. In particular we obtain $\dim H^1(\bar{G})^{\kappa(g_0)^{-1}}\neq 0$. Combining this with (11), (53), (54), we obtain that $K\in\mathscr{P}$, namely, $\kappa=\mu_{-K}$ is trivial on $\bar{G}$. Thus we infer from (52) that there exist on S_r multi-valued functions U_r, V_r, X_r such that

$$g^*U_r=U_r+a_1(g),$$
$$g^*V_r=V_r+a_2(g),$$
$$g^*X_r=X_r+b(g) \quad \text{for } g\in\bar{G},$$

where $\bar{G}$ is considered as the covering transformation group of the universal covering space $\tilde{S}_r$ over S_r and $dU_r\wedge dV_r\wedge dX_r$ does not vanish anywhere. This implies that S_r is a real 3-torus and $\bar{G}\simeq\mathbb{Z}^3$. Thus M is an integral 3×3 matrix with $\det M=\pm 1$. Let m, n be the multiplicities of the eigen-values $\kappa(g_0)^2$, $\kappa(g_0)^{-1}$ of M, respectively. Then

$$\kappa(g_0)^{2m-n}=\kappa(g_0)^{2m}\cdot\kappa(g_0)^{-n}=\det M=\pm 1.$$

Since $|\kappa(g_0)|\neq 1$, we obtain that $2m-n=0$, while $m+n=3$. Therefore $n=2$, namely, $\kappa(g_0)^{-1}$ is an eigen-value of M with multiplicity 2. Hence $\kappa(g_0)^{-1}=\pm 1$ and $2K=0$. This contradicts Lemma 1, q.e.d.

Proposition 13. *$-F_0-K$ is not real.*

Proof. We assume that $-F_0-K$ is real and derive a contradiction. By (41), (50) and Proposition 12,

$$\dim H^1(\mathbb{C}(F_0))=\dim H^1(\mathbb{C}(-F_0-K))=1.$$

Thus, as in the proof of Proposition 12, we may assume that $f_{ij}, \kappa_{ij}, a_{ij}, b_{ij}$ are real numbers in (45). The analytic continuations of w_i and z_i produce holomorphic functions W and Z on $\tilde{S}$ such that

$$g^*W=\mu_{F_0}(g)\,W+a(g),$$
$$g^*Z=\mu_{F_0}(g)^{-1}\,\kappa(g)\,Z+b(g), \quad \text{for } g\in G,$$

where $\mu_{F_0}(g)$, $\kappa(g)=\mu_{-K}(g)$, $a(g)$, $b(g)\in\mathbb{R}$. Let $U=\operatorname{Re} W$, $V=\operatorname{Im} W$, $X=\operatorname{Re} Z$ and $Y=\operatorname{Im} Z$. Then

$$\begin{aligned} g^* U &= \mu_{F_0}(g)\, U + a(g),\\ g^* V &= \mu_{F_0}(g)\, V,\\ g^* X &= \mu_{F_0}(g)^{-1}\,\kappa(g)\, X + b(g),\\ g^* Y &= \mu_{F_0}(g)^{-1}\,\kappa(g)\, Y \quad \text{for } g\in G, \end{aligned}$$

where $dU\wedge dV\wedge dX\wedge dY$ does not vanish anywhere. We take a normal subgroup G' of finite index in G such that $\mu_{F_0}(g)>0$ and $\kappa(g)>0$ for $g\in G'$, and take the unramified covering surface S' of S corresponding to G'. We denote by F_0' the line bundle on S' induced from F_0 on S. We consider U, V, X, Y as multi-valued functions on S'.

Let Σ be $\{P\in S' \mid V(P)=0\}$ and Σ_0 a connected component of Σ. As in the proof of Proposition 10, we obtain that Σ is not empty. Let S^+ be $\{P\in S' \mid V(P)\geqq 0\}$ and S^- be $\{P\in S' \mid V(P)\leqq 0\}$. Then S^+ and S^- are compact manifolds with Σ as their common boundary and $S^+\cap S^-=\Sigma$. Let Ξ be $\{P\in S' \mid V(P)\geqq 0, Y(P)=0\}$ and Ξ_0 a connected component of Ξ. Then Ξ is not empty. In fact, if Ξ is empty, Y does not vanish anywhere in S^+. Let $f=V/|Y|^c$ where $c=\log\mu_{F_0}(g_0)/(\log\kappa(g_0)-\log\mu_{F_0}(g_0))$. Then f is a single-valued continuous function on S^+. It is evident that $f=0$ on $\partial S^+=\Sigma$ and $df\neq 0$ everywhere in $S^+-\partial S^+$. This contradicts the compactness of S^+. Hence Ξ_0 is a compact manifold with boundary $\partial\Xi_0=\{P\in\Xi_0 \mid V(P)=0\}$.

We note that the restriction $H^1(S,\mathbb{C})\to H^1(\Xi_0,\mathbb{C})$ is not a zero mapping. In fact, otherwise, the restriction $V|\Xi_0$ of V to Ξ_0 is a single-valued function on Ξ_0, while $V|\Xi_0=0$ on $\partial\Xi_0$ and $d(V|\Xi_0)\neq 0$ everywhere in $\Xi_0-\partial\Xi_0$. This contradicts the compactness of Ξ_0. Similarly we can prove that the restriction $H^1(S,\mathbb{C})\to H^1(\Sigma_0,\mathbb{C})$ is not a zero mapping. Hence we obtain that

$$H^0(\Xi,\mathbb{C}(F_0'))=0$$

and

$$H^0(\Sigma,\mathbb{C}(F_0'))=0.$$

Now let S^{++} be $\{P\in S^+ \mid Y(P)\geqq 0\}$ and S^{+-} be $\{P\in S^+ \mid Y(P)\leqq 0\}$. Then $S^{++}\cap S^{+-}=\Xi$. We take a connected component $(S^{++})_0$ of S^{++}. Then, similarly to the proof of Proposition 10, we can prove that $\pi_1((S^{++})_0)\simeq\mathbb{Z}$ and the restriction $H^1(S,\mathbb{C})\to H^1((S^{++})_0,\mathbb{C})$ is not a zero mapping. Hence we obtain that $H^1((S^{++})_0,\mathbb{C}(F_0'))=0$. Thus we obtain that

$$H^1(S^{++},\mathbb{C}(F_0'))=0$$

and, similarly,

$$H^1(S^{+-}, \mathbb{C}(F_0')) = 0.$$

From this and the Mayer-Vietoris sequence

$$H^0(\Xi, \mathbb{C}(F_0')) \to H^1(S^+, \mathbb{C}(F_0')) \to H^1(S^{++}, \mathbb{C}(F_0')) \oplus H^1(S^{+-}, \mathbb{C}(F_0')),$$

it follows that

$$H^1(S^+, \mathbb{C}(F_0')) = 0.$$

Similarly we obtain that

$$H^1(S^-, \mathbb{C}(F_0')) = 0.$$

From the above and the Mayer-Vietoris sequence

$$H^0(\Sigma, \mathbb{C}(F_0')) \to H^1(S', \mathbb{C}(F_0')) \to H^1(S^+, \mathbb{C}(F_0')) \oplus H^1(S^-, \mathbb{C}(F_0')),$$

we derive that

$$H^1(S', \mathbb{C}(F_0')) = 0.$$

By this and (10) we can find a constant c such that

$$a(g) = c - \mu_{F_0}(g)\, c, \qquad \text{for } g \in G'.$$

Hence we obtain that

$$g^*(W - c) = \mu_{F_0}(g)(W - c), \qquad \text{for } g \in G'.$$

This implies that S' contains a curve and, therefore, S contains a curve. This contradicts (B), q.e.d.

This proposition together with Proposition 11 imply $-\bar{F}_0 - \bar{K} = K$. From this we derive

$$\dim H^1(\mathbb{C}(K)) = \dim H^1(\mathbb{C}(-\bar{F}_0 - \bar{K})) = \dim H^1(\mathbb{C}(-F_0 - K)).$$

Thus, from (41), (49), (50), Proposition 12 and the above, it follows that

$$\begin{aligned} \dim H^1(\mathbb{C}) &= \dim H^1(\mathbb{C}(K)) = \dim H^1(\mathbb{C}(F_0)) \\ &= \dim H^1(\mathbb{C}(-F_0 - K)) = 1, \\ \dim H^1(\mathbb{C}(F)) &= 0 \qquad \text{for } F \neq 0, K, F_0, -F_0 - K. \end{aligned} \tag{56}$$

If we write $K = \{\kappa_{ij}^{-1}\}$, then, by $-\bar{F}_0 - \bar{K} = K$, we infer that

$$F_0 = \{|\kappa_{ij}|^2\}, \quad -F_0 - K = \{\bar{\kappa}_{ij}^{-1}\}.$$

Hence (45) turns out to be

$$\begin{aligned} w_i &= |\kappa_{ij}|^2\, w_j + a_{ij}, \\ z_i &= \bar{\kappa}_{ij}^{-1}\, z_j + b_{ij}, \end{aligned}$$

where we may assume $a_{ij} \in \mathbb{R}$ as in the proof of Proposition 12. The analytic continuations of w_i and z_i produce holomorphic functions W and Z on $\tilde{S}$ such that

$$(57)\qquad \begin{aligned} g^* W &= |\kappa(g)|^2 W + a(g), \\ g^* Z &= \overline{\kappa(g)}^{-1} Z + b(g) \quad \text{for } g \in G, \end{aligned}$$

where $\kappa = \mu_{-K}$, $a(g) \in \mathbb{R}$, $b(g) \in \mathbb{C}$ and $dW \wedge dZ$ does not vanish anywhere. Let $U = \operatorname{Re} W$ and $V = \operatorname{Im} W$. Then we have $g^* V = |\kappa(g)|^2 V$ for any $g \in G$. Consider V as a multi-valued differentiable function on S, and let Σ be $\{P \in S \mid V(P) = 0\}$ and Σ_0 a connected component of Σ. If Σ is not empty, Σ_0 is a compact 3-dimensional real submanifold of S. (57) implies that $\omega = \dfrac{1}{2\sqrt{-1}} dU \wedge dZ \wedge d\bar{Z}$ gives a closed real 3-form on S. Since $\omega_0 = \omega \mid \Sigma_0$ gives a volume form on Σ_0, we obtain

$$\int_{\Sigma_0} \omega \neq 0.$$

This implies that Σ_0 is not homologous to zero on S. Hence, similarly to the proof of Proposition 12, we can derive a contradiction. Thus Σ is empty, i.e., V does not vanish anywhere. We may assume that $V > 0$ everywhere. Since $|\kappa(g_0)|^2 \neq 1$, V gives a differentiable non-degenerate fibering of S over $T^1 = \mathbb{R}^+ / \langle |\kappa(g_0)|^2 \rangle$ with connected fibre S_r over $r \in T^1$. The subgroup $\bar{G}$ of G defines in § 1 is isomorphic to $\pi_1(S_r)$.

Since $F_0 = F_{|\kappa|^2} \in \mathscr{P}$, we infer from (9), (11), (56) that

$$(58)\qquad \begin{aligned} &\dim H^1(\bar{G})^{|\kappa(g_0)|^2} = 1, \\ &\dim H^1(\bar{G})^{\mu} = 0 \quad \text{for } \mu \neq |\kappa(g_0)|^2,\ \overline{\kappa(g_0)}^{-1},\ \kappa(g_0)^{-1}. \end{aligned}$$

Since S_r is compact, $\bar{G}$ is finitely generated and therefore the automorphism g_0^* of $H(\bar{G})$ is represented by an integral matrix M with $\det M = \pm 1$. (58) implies that $|\kappa(g_0)|^2$ is an eigen-value of M. If $|\kappa(g_0)|^2$ is the only eigen-value, then $|\kappa(g_0)|^2 = 1$ and, consequently, $F_0 = 0$. This contradicts Property (γ). Thus, by (58), we infer that $\overline{\kappa(g_0)}^{-1}$ or $\kappa(g_0)^{-1}$ is also an eigen-values of M, namely $\dim H^1(\bar{G})^{\mu} \neq 0$ for $\mu = \overline{\kappa(g_0)}^{-1}$ or $\kappa(g_0)^{-1}$. Hence, by (11) and (56), we obtain $\bar{K} \in \mathscr{P}$ or $K \in \mathscr{P}$. Thus $K \in \mathscr{P}$ and therefore κ is trivial on $\bar{G}$ and the eigen-values of M are $|\kappa(g_0)|^2$, $\overline{\kappa(g_0)}^{-1}$ and $\kappa(g_0)^{-1}$, where $\overline{\kappa(g_0)} \neq \kappa(g_0)$. From this and (57) we infer that there exist on S_r multi-valued functions U_r, X_r, Y_r such that

$$\begin{aligned} g^* U_r &= U_r + a(g), \\ g^* X_r &= X_r + b_1(g), \\ g^* Y_r &= Y_r + b_2(g) \quad \text{for any } g \in \bar{G}, \end{aligned}$$

where $\bar{G}$ is considered as the covering transformation group of the universal covering $\tilde{S}_r$ over S_r, $b_1(g)=\operatorname{Re} b(g)$, $b_2(g)=\operatorname{Im} b(g)$ and $dU_r \wedge dX_r \wedge dY_r$ does not vanish anywhere. This implies that S_r is a real 3-torus and $\bar{G} \simeq \mathbb{Z}^3$. Thus M is an integral 3×3 matrix with eigen-values $|\kappa(g_0)|^2$, $\overline{\kappa(g_0)}^{-1}$, $\kappa(g_0)^{-1}$. In particular, $M \in SL(3, \mathbb{Z})$.

Now we take generators g_1, g_2, g_3 of $\bar{G}$ such that

$$g_0 g_i g_0^{-1} = g_1^{m_{i1}} g_2^{m_{i2}} g_3^{m_{i3}} \quad \text{for } i=1,2,3, \tag{59}$$

where $M=(m_{ij})$. Then g_0, g_1, g_2, g_3 generate G. Replacing $W-(a(g_0)/1-|\kappa(g_0)|^2)$ and $Z-(b(g_0)/1-\overline{\kappa(g_0)}^{-1})$ by W and Z, respectively, we can assume that $a(g_0)=b(g_0)=0$ in (57). Then the restrictions of a and b to $\bar{G}$ give non-trivial elements of $H^1(\bar{G})^{|\kappa(g_0)|^2}$ and $H^1(\bar{G})^{\overline{\kappa(g_0)}^{-1}}$, respectively, namely

$$\begin{aligned} a(g_0 g g_0^{-1}) &= |\kappa(g_0)|^2 a(g), \\ b(g_0 g g_0^{-1}) &= \overline{\kappa(g_0)}^{-1} b(g) \quad \text{for } g\in\bar{G}. \end{aligned} \tag{60}$$

Let $a_i=a(g_i)$, $b_i=b(g_i)$ for $i=1,2,3$. Then, by (59) and (60), we obtain

$$\begin{aligned} (a_1, a_2, a_3)\cdot {}^tM &= |\kappa(g_0)|^2 \cdot (a_1, a_2, a_3), \\ (b_1, b_2, b_3)\cdot {}^tM &= \overline{\kappa(g_0)}^{-1} \cdot (b_1, b_2, b_3). \end{aligned} \tag{61}$$

We can assume that $|\kappa(g_0)|^2>1$ by writing, if necessary, g_0 and M for g_0^{-1} and M^{-1}, respectively. Now we define automorphisms h_0, h_1, h_2, h_3 of $\mathbb{H}\times\mathbb{C}$ as follows:

$$\begin{aligned} h_0 &: (w,z) \to (|\kappa(g_0)|^2 w, \overline{\kappa(g_0)}^{-1} z), \\ h_i &: (w,z) \to (w+a_i, z+b_i) \quad \text{for } i=1,2,3. \end{aligned}$$

Then (61) implies that the automorphism group of $\mathbb{H}\times\mathbb{C}$ generated by h_0, h_1, h_2, h_3 coincides with G_M defined in § 2. Thus, by (57), we see that (W, Z) defines an isomorphism of S onto $S_M=\mathbb{H}\times\mathbb{C}/G_M$.

§ 9. Case (II)

We infer from Proposition 10 that

$$2F_0+2K=0 \quad \text{and} \quad F_0 \neq K. \tag{62}$$

If $H^0(d\mathcal{O}(L)) \neq 0$, then $L=F_0$ or $-F_0-K$ by Lemma 2, while, if $L=-F_0-K$, then $2L=0$ by (62). This contradicts Property (γ). Thus, by Property (β),

$$\begin{aligned} \dim H^0(d\mathcal{O}(F_0)) &= 1, \\ \dim H^0(d\mathcal{O}(F)) &= 0 \quad \text{for } F \neq F_0. \end{aligned}$$

Hence, by (5), we obtain

$$\begin{aligned}&\dim H^1(\mathbb{C})=\dim H^1(\mathbb{C}(K))=\dim H^1(\mathbb{C}(F_0))=1,\\ &\dim H^1(\mathbb{C}(F))=0 \qquad \text{for } F\neq 0, K, F_0.\end{aligned} \tag{63}$$

In fact, $\dim H^1(\mathbb{C}(K))=\dim H^1(\mathbb{C}(-2F_0-K))\geqq 1$ by Proposition 9 and, since $H^0(d\mathcal{O}(K))=0$ by $K\neq F_0$, $\dim H^1(\mathbb{C}(K))\leqq \dim H^1(\mathcal{O}(K))= q=1$.

Proposition 14. *K and F_0 are real.*

Proof. Assume that K is not real. Then $\dim H^1(\mathbb{C}(\bar{K}))=\dim H^1(\mathbb{C}(K)) =1$. Hence $\bar{K}=F_0$ by (63), and therefore $2(K+\bar{K})=0$ by (62). Thus, letting $K=\{\kappa_{ij}^{-1}\}$, $\kappa_{ij}\in\mathbb{C}^*$, we have $|\kappa_{ij}|^4=\kappa_i/\kappa_j$ for $\kappa_i\in\mathbb{R}^+$. Replacing $\kappa_j^{\frac{1}{4}}\cdot\kappa_{ij}\cdot\kappa_i^{-\frac{1}{4}}$ by κ_{ij} we can assume that $|\kappa_{ij}|=1$. Then, by using this expression of K in (46), we have a volume preserving structure on S, since $dw_i\wedge dz_i=\kappa_{ij}\,dw_j\wedge dz_j$. Hence, by Kodaira [2; the proof of Theorem 38], we know that $12K=0$. This contradicts Lemma 1. Thus K is real, and therefore F_0 is also real by (63), q.e.d.

By this proposition and (63) we may assume, as in the proof of Proposition 10, that $f_{ij}, \kappa_{ij}, a_{ij}, b_{ij}\in\mathbb{R}$ in (46).

Proposition 15. *If $F_0+K\neq 0$, we may assume that c_{ij} in (46) is also real.*

Proof. Since $a_{ij}, b_{ij}\in\mathbb{R}$, we have $\{\operatorname{Im} c_{ij}\}\in H^1(\mathbb{C}(-F_0-K))$. If $\{\operatorname{Im} c_{ij}\}\neq 0$, we infer from (63) that $-F_0-K=0$, K or F_0. By hypothesis, $-F_0-K\neq 0$, and by Proposition 10, $-F_0-K\neq F_0$. Hence $-F_0-K=K$ and therefore $2K=0$. This contradicts Lemma 1. Thus we can find $c_i\in\mathbb{C}$ such that $\operatorname{Im} c_{ij}=f_{ij}^{-1}\kappa_{ij}c_j-c_i$. By replacing $z_i+\sqrt{-1}\,c_i$ by z_i, we may assume that c_{ij} is real, q.e.d.

The analytic continuations of w_i and z_i produce holomorphic functions W and Z on $\tilde{S}$ such that

$$\begin{aligned}&g^*W=\mu_{F_0}(g)\,W+a(g),\\ &g^*Z=\mu_{F_0}(g)^{-1}\kappa(g)\,Z+b(g)\,W+c(g), \qquad \text{for } g\in G,\end{aligned} \tag{64}$$

where $\mu_{F_0}(g)$, $\kappa(g)=\mu_{-K}(g)$, $a(g)$, $b(g)\in\mathbb{R}$ and $c(g)\in\mathbb{C}$. Moreover, if $F_0+K\neq 0$, then $c(g)\in\mathbb{R}$. We may assume, as in the previous section, that $a(g_0)=b(g_0)=0$. Let $U=\operatorname{Re} W$, $V=\operatorname{Im} W$, $X=\operatorname{Re} Z$, $Y=\operatorname{Im} Z$, $c'(g)=\operatorname{Re} c(g)$, $c''(g)=\operatorname{Im} c(g)$. Then we obtain:

Subcase II_a: $F_0+K=0$,

$$\begin{aligned}&g^*U=\kappa(g)\,U+a(g),\\ &g^*V=\kappa(g)\,V,\\ &g^*X=X+b(g)\,U+c'(g),\\ &g^*Y=Y+b(g)\,V+c''(g).\end{aligned} \tag{65}$$

Subcase $\mathrm{II_b}$: $F_0+K\neq 0$.

$$
(66)\qquad
\begin{aligned}
g^* U &= \mu_{F_0}(g)\, U + a(g),\\
g^* V &= \mu_{F_0}(g)\, V,\\
g^* X &= \mu_{F_0}(g)^{-1}\, \kappa(g)\, X + b(g)\, U + c(g),\\
g^* Y &= \mu_{F_0}(g)^{-1}\, \kappa(g)\, Y + b(g)\, V.
\end{aligned}
$$

Moreover $dU\wedge dV\wedge dX\wedge dY$ does not vanish anywhere in either cases. We consider U, V, X, Y as multi-valued functions on S. Let Σ be $\{P\in S \mid V(P)=0\}$ and Σ_0 a connected component of Σ. We prove that Σ is empty. We assume that Σ is not empty. Then Σ_0 is a real differentiable submanifold of dimension 3. First in Subcase $\mathrm{II_a}$, $c''\neq 0$ in $\mathrm{Hom}(G,\mathbb{R})$. In fact, if $c''=0$, the restriction $Y|\Sigma_0$ of Y to Σ_0 is a single-valued function on Σ_0, while $d(Y|\Sigma_0)\neq 0$ everywhere on Σ_0. This contradicts the compactness of Σ_0. Hence, by (65), $\eta=\exp r\,Y\cdot dU\wedge dX$ is a differentiable 2-form on Σ_0 where $r=-\log\kappa(g_0)/c''(g_0)$, while

$$\omega = d\eta = r\exp r\,Y\cdot dY\wedge dU\wedge dX$$

is a volume form on Σ_0. Hence

$$0\neq \int_{\Sigma_0}\omega = \int_{\Sigma_0} d\eta = 0.$$

This is a contradiction. Next in Subcase $\mathrm{II_b}$, (66) implies that the restriction $Y^2|\Sigma_0$ of Y^2 to Σ_0 is a single-valued function on Σ_0, while $d(Y^2|\Sigma_0)\neq 0$ everywhere in $\{P\in\Sigma_0 \mid Y(P)\neq 0\}$. This contradicts the compactness of Σ_0. Thus Σ is empty and, therefore, we may assume that $V>0$ everywhere. Hence $\mu_{F_0}(g)>0$ for any $g\in G$ and therefore $F_0\in\mathscr{P}$. Since $\mu_{F_0}(g_0)\neq 1$, V gives a differentiable non-degenerate fibering of S over $T^1=\mathbb{R}^+/\langle\mu_{F_0}(g_0)\rangle$ with connected fibre S_r over $r\in T^1$. The subgroup $\bar{G}$ of G defined in § 1 is isomorphic to $\pi_1(S_r)$. Hence $\bar{G}$ is finitely generated. Thus the automorphism g_0^* of $H^1(\bar{G})$ is represented by an integral matrix N with $\det N=\pm 1$. Recalling (9), (11), we infer from (63) that

$$
(67)\qquad
\begin{aligned}
&\dim H^1(\bar{G})^{\mu}=0 \qquad \text{for } \mu\neq\mu_{F_0}(g_0),\ \kappa(g_0)^{-1},\\
&\dim H^1(\bar{G})^{\mu_{F_0}(g_0)}=1.
\end{aligned}
$$

Thus $\mu_{F_0}(g_0)$ is an eigen-value of N. If $\mu_{F_0}(g_0)$ is the only eigen-value, then $\mu_{F_0}(g_0)=1$ and, consequently, $F_0=0$. This contradicts Property (γ). Hence, by (67), $\kappa(g_0)^{-1}$ is also an eigen-value of N and $\dim H^1(\bar{G})^{\kappa(g_0)^{-1}}\neq 0$. Therefore, by (11) and (63), the homomorphism $\mu\in\mathrm{Hom}(G,\mathbb{C}^*)_0$ defined by $\mu(g_0)=\kappa(g_0)^{-1}$ equals to κ^{-1}. Hence $K\in\mathscr{P}$, in other words, κ is trivial on $\bar{G}$ and $\dim H^1(\bar{G})^{\kappa(g_0)^{-1}}=\dim H^1(\mathbb{C}(K))=1$. Thus the eigen-values of N are $\mu_{F_0}(g_0)$ and $\kappa(g_0)^{-1}$. Since $\mu_{F_0}(g_0)$ is a positive real number $\neq 1$, $(\mu_{F_0}(g_0)\cdot\kappa(g_0)^{-1})^2=1$ and $\det N=\pm 1$, the multiplicity n of $\mu_{F_0}(g_0)$ is equal to the multiplicity of $\kappa(g_0)^{-1}$.

Proposition 16. *The multiplicity n of $\mu_{F_0}(g_0)$ equals* 1, *and N is an integral* 2×2 *matrix with* $\det N=\pm 1$.

Proof. Assume that $n\geqq 2$. Then there exists a non-trivial element e of $H^1(\bar{G})$ such that

$$e(g_0 h g_0^{-1})=\mu_{F_0}(g_0)(e(h)+a(h)) \qquad \text{for } h\in\bar{G}.$$

Let m be a homomorphism of G into $\mathbb{R}$ defined by $m(g_0)=1$. We define

$$B(g)=\begin{bmatrix}\mu_{F_0}(g) & \mu_{F_0}(g)\cdot m(g)\\ 0 & \mu_{F_0}(g)\end{bmatrix} \qquad \text{for } g\in G.$$

For any $g=g_0^m h\in G$ where $h\in\bar{G}$, we define

$$\begin{aligned}\tilde{e}(g)&=\tilde{e}(g_0^m h)\\ &=\mu_{F_0}(g_0^m)\,e(h)+\mu_{F_0}(g_0^m)\,m(g_0^m)\,a(h).\end{aligned}$$

Then we obtain easily that

$$\begin{bmatrix}\tilde{e}(g_1 g_2)\\ a(g_1 g_2)\end{bmatrix}=\begin{bmatrix}\tilde{e}(g_1)\\ a(g_1)\end{bmatrix}+B(g_1)\cdot\begin{bmatrix}\tilde{e}(g_2)\\ a(g_2)\end{bmatrix}, \tag{68}$$

for $g_1, g_2\in G$. We consider $\mathbb{C}^2$ as a G-module by defining $g\cdot\mathfrak{a}=B(g)\cdot\mathfrak{a}$ where $\mathfrak{a}\in\mathbb{C}^2$ and $g\in G$. We denote by $H_B^1(G)$ the first cohomology group with respect to this G-module structure. Then (68) implies that $\begin{bmatrix}\tilde{e}\\ a\end{bmatrix}$ represents a non-trivial element of $H_B^1(G)$. Let $\{m_{ij}\}$ be a base of $H^1(\mathbb{R})$ and let

$$B=\left\{\begin{bmatrix}f_{ij} & f_{ij}m_{ij}\\ 0 & f_{ij}\end{bmatrix}\right\}.$$

Then B defines a flat vector bundle over S of rank 2. We denote by $\mathbb{C}(B)$ the sheaf of germs of locally constant sections of B. Then we have the following isomorphism

$$H^1(S,\mathbb{C}(B))\simeq H_B^1(G)$$

(see MacLane [4]).

This implies that there exists a 1-cochain $\{e_{ij}\}$ composed of constants e_{ij} such that

$$e_{ik}=e_{ij}+f_{ij}e_{jk}+f_{ij}m_{ij}a_{jk},$$

where we use the a_{ij} in (46). Thus $\left\{\begin{pmatrix} e_{ij} \\ a_{ij} \end{pmatrix}\right\}$ gives a non-trivial element of $H^1(\mathbb{C}(B))$.

On the other hand, we have $H^1(\mathcal{O}(B))=0$. In fact, this follows from Lemma 1 (iv) and the exact sequence:

$$H^1(\mathcal{O}(F_0)) \to H^1(\mathcal{O}(B)) \to H^1(\mathcal{O}(F_0)),$$

which is derived from the following exact sequence

$$0 \to \mathcal{O}(F_0) \to \mathcal{O}(B) \to \mathcal{O}(F_0) \to 0.$$

Thus $\left\{\begin{pmatrix} e_{ij} \\ a_{ij} \end{pmatrix}\right\}$ is zero in $H^1(\mathcal{O}(B))$, namely

$$\begin{pmatrix} e_{ij} \\ a_{ij} \end{pmatrix} = \begin{pmatrix} f_{ij} & f_{ij} m_{ij} \\ 0 & f_{ij} \end{pmatrix} \cdot \begin{pmatrix} e_j \\ a_j \end{pmatrix} - \begin{pmatrix} e_i \\ a_i \end{pmatrix},$$

where e_i and a_i are holomorphic functions. From this and (46) we obtain that $w_i + a_i = f_{ij}(w_j + a_j)$. Since S contains no curves, $w_i + a_i = 0$ and therefore $da_i = -dw_i$. Thus we obtain $de_i = f_{ij} de_j - f_{ij} m_{ij} dw_j$. Let $de_i = \alpha_i dw_i + \beta_i dz_i$, where α_i, β_i are holomorphic functions. Then $\beta_i = f_{ij}^2 \kappa_{ij}^{-1} \beta_j$ by (46). This implies $\beta_i = 0$ by Proposition 10, hence $-m_{ij} = \alpha_i - \alpha_j$. Thus $d\alpha_i = d\alpha_j$ is a holomorphic 1-form on S, hence $d\alpha_i = 0$ by $h^{1,0}(S)=0$. This implies that $\{m_{ij}\}=0$ in $H^1(\mathbb{C})$. This is absurd, q.e.d.

From this proposition it follows that $\bar{G}/[\bar{G}, \bar{G}] = H \oplus (\text{torsion})$, where $H \simeq \mathbb{Z}^2$. Let $\bar{\bar{G}} = \{g \in \bar{G} \mid g \bmod [\bar{G}, \bar{G}] \text{ is of finite order}\}$. We take two elements g_1, g_2 of $\bar{G}$ which represent a set of generators of H such that

$$g_0 g_i g_0^{-1} = g_1^{n_{i1}} g_2^{n_{i2}} \quad \text{modulo } \bar{\bar{G}}, \qquad \text{for } i=1,2, \tag{69}$$

where $N=(n_{ij})$. Since $a(g_0)=b(g_0)=0$, (8) implies that the restrictions of a and b to $\bar{G}$ give non-trivial elements of $H^1(\bar{G})^{\mu_{F_0}(g_0)}$ and $H^1(\bar{G})^{\kappa(g_0)^{-1}}$, respectively, namely

$$\begin{aligned} a(g_0 g g_0^{-1}) &= \mu_{F_0}(g_0) \cdot a(g), \\ b(g_0 g g_0^{-1}) &= \kappa(g_0)^{-1} \cdot b(g), \qquad \text{for } g \in \bar{G}. \end{aligned} \tag{70}$$

Let $a_i = a(g_i)$, $b_i = b(g_i)$ for $i=1,2$. Then by (69) and (70) we obtain

$$(a_1, a_2) \cdot {}^tN = \mu_{F_0}(g_0) \cdot (a_1, a_2), \qquad (b_1, b_2) \cdot {}^tN = \kappa(g_0)^{-1} \cdot (b_1, b_2). \tag{71}$$

Note that (a_1, a_2) and (b_1, b_2) are linearly independent since $F_0 \neq K$. Hence $\bar{\bar{G}} = \{g \in G \mid a(g) = b(g) = 0\}$. Let U_r, X_r, Y_r be the restrictions of the multi-valued functions U, X, Y to a fibre S_r. Since κ, μ_{F_0} are trivial on

$\bar{G}$ and $c''(g)$ in (65) vanishes on $\bar{G}$, we obtain

$$(72)\qquad \begin{aligned} g^* U_r &= U_r + a(g), \\ g^* X_r &= X_r + b(g)\, U_r + c(g), \\ g^* Y_r &= Y_r + b(g)\, V_0 \quad \text{for } g \in \bar{G}, \end{aligned}$$

where $\bar{G}$ is considered as the covering transformation group of the universal covering space $\tilde{S}_r$ over S_r, and V_0 is a non-zero constant. Obviously we may assume that $V_0 = 1$. Since $dU_r \wedge dX_r \wedge dY_r$ does not vanish anywhere and (a_1, b_1), (a_2, b_2) are linearly independent, it follows from (72) that (U_r, Y_r) gives a differentiable non-degenerate fibering of S_r over $T^2 = \mathbb{R}^2/\langle (a_1, b_1), (a_2, b_2) \rangle$ with fibre a circle T^1. This implies that $\bar{\bar{G}} \simeq \pi_1(T^1) \simeq \mathbb{Z}$. Let g_3 be a generator of $\bar{\bar{G}} \simeq \mathbb{Z}$. Then we obtain $\bar{G} = \langle g_1, g_2, g_3 \rangle$ and $G = \langle g_0, g_1, g_2, g_3 \rangle$. It follows from (69) that

$$(73)\qquad \begin{aligned} g_0 g_1 g_0^{-1} &= g_1^{n_{11}} g_2^{n_{12}} g_3^{p}, \\ g_0 g_2 g_0^{-1} &= g_1^{n_{21}} g_2^{n_{22}} g_3^{q}, \end{aligned}$$

for certain integers p, q. Since $[\bar{G}, \bar{G}]$ is contained in $\bar{\bar{G}} \simeq \mathbb{Z}$ with finite index and g_3 commutes with g_1, g_2 by (72), $[\bar{G}, \bar{G}]$ is generated by $g_1^{-1} g_2^{-1} g_1 g_2$ and

$$(74)\qquad g_1^{-1} g_2^{-1} g_1 g_2 = g_3^{r}$$

for certain non-zero integer r.

First we consider Subcase II_a: $F_0 + K = 0$. Then $\mu_{F_0}(g_0) = \kappa(g_0)$ and therefore $\det N = 1$, i.e., $N \in SL(2, \mathbb{Z})$. We can assume $\mu_{F_0}(g_0) > 1$ by writing, if necessary, g_0, N for g_0^{-1}, N^{-1}, respectively. Let $c_1 = c'(g_1)$, $c_2 = c'(g_2)$, $c_3 = c'(g_3)$ and $t = c(g_0)$. Then, by (65), (73), (74), we obtain

$$(75)\qquad (c_1, c_2) = (c_1, c_2) \cdot {}^t N + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r} \cdot (p, q),$$

where $e_i = \frac{1}{2} n_{i1}(n_{i1} - 1)\, a_1 b_1 + \frac{1}{2} n_{i2}(n_{i2} - 1)\, a_2 b_2 + n_{i1} n_{i2}\, b_1 a_2$ for $i = 1, 2$, and

$$(76)\qquad c_3 = (b_1 a_2 - b_2 a_1)/r.$$

Now we define automorphisms h_0, h_1, h_2, h_3 of $\mathbb{H} \times \mathbb{C}$ as follows:

$$\begin{aligned} h_0 &: (w, z) \to (\mu_{F_0}(g_0)\, w, z + t), \\ h_i &: (w, z) \to (w + a_i, z + b_i w + c_i) \quad \text{for } i = 1, 2, \\ h_3 &: (w, z) \to (w, z + c_3). \end{aligned}$$

Then (71), (75) and (76) imply that the automorphism group of $\mathbb{H}\times\mathbb{C}$ generated by h_0, h_1, h_2, h_3 coincides with $G^{(+)}_{N,p,q,r;t}$ defined in § 3. Since $c''(g)=\operatorname{Im} c(g)=0$ for any $g\in\bar{G}$, we see by (64) that (W, Z) defines an isomorphism of S onto $S^{(+)}_{N,p,q,r;t}=\mathbb{H}\times\mathbb{C}/G^{(+)}_{N,p,q,r;t}$.

Next we consider Subcase II_b: $F_0+K\neq 0$. Since $2F_0+2K=0$, we obtain $\mu_{F_0}(g_0)=-\kappa(g_0)$ and therefore $\det N=-1$. We can assume $\mu_{F_0}(g_0)>1$ by writing, if necessary, g_0, N for g_0^{-1}, N^{-1}, respectively. Since $\mu_{F_0}(g_0)^{-1}\cdot\kappa(g_0)=-1\neq 1$, we may assume that $c(g_0)=0$ in (64). Let $c_1=c(g_1)$, $c_2=c(g_2)$, $c_3=c(g_3)$. Then by (65), (73), (74) we obtain

$$-(c_1, c_2)=(c_1, c_2)\cdot{}^tN+(e_1, e_2)+\frac{b_1 a_2-b_2 a_1}{r}\cdot(p, q), \tag{77}$$

where $e_i=\frac{1}{2}n_{i1}(n_{i1}-1)a_1 b_1+\frac{1}{2}n_{i2}(n_{i2}-1)a_2 b_2+n_{i1}n_{i2}b_1 a_2$ for $i=1,2$, and

$$c_3=(b_1 a_2-b_2 a_1)/r. \tag{78}$$

Note that $c_1, c_2, c_3\in\mathbb{R}$. Now we define automorphisms h_0, h_1, h_2, h_3 of $\mathbb{H}\times\mathbb{C}$ as follows:

$$h_0:\ (w, z)\to(\mu_{F_0}(g_0)\,w,\ -z),$$

$$h_i:\ (w, z)\to(w+a_i, z+b_i w+c_i)\quad \text{for } 1=1,2,$$

$$h_3:\ (w, z)\to(w, z+c_3).$$

Then (71), (77) and (78) imply that the automorphism group of $\mathbb{H}\times\mathbb{C}$ generated by h_0, h_1, h_2, h_3 coincides with $G^{(-)}_{N,p,q,r}$ defined in § 4. Thus, by (64), we see that (W, Z) defines an isomorphism of S onto $S^{(-)}_{N,p,q,r}=\mathbb{H}\times\mathbb{C}/G^{(-)}_{N,p,q,r}$.

Thus we complete the proof of Theorem in § 5.

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General Position in the Poincaré Duality Category

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§ 0. Introduction

The object of this paper is to establish a theory of general position for the Poincaré duality category. This is done by showing the following key lemma.

Lemma. (General Position.) *Let (X, Y) be a finite Poincaré pair of formal dimension n and suppose given a map $f\colon (D^k, S^{k-1}) \to (X, Y)$ where $k \leqq n-3$ and $2n \geqq 3k+4$. Then there exists a CW-pair (K, L) of dimension k, obtained from (D^k, S^{k-1}) by adding cells of dimension $\leqq (2k-n+2, 2k-n+1)$, and a map $g\colon (K, L) \to (X, Y)$ homotopic to an embedding, such that the following diagram commutes.*

$$\begin{array}{ccc} (D^k, S^{k-1}) & \xrightarrow{\ f\ } & (X, Y) \\ \cap & \nearrow_{g} & \\ (K, L) & & \end{array}$$

As we shall see the additional cells attached to (D^k, S^{k-1}) to form (K, L) can be thought of as the "singularity set" of the map f.

Using this lemma we obtain a number of results parallel to known results for maps into manifolds, in particular it can be used as a tool to prove embedding theorems, such as

Proposition. *Let $f\colon (D^k, S^{k-1}) \to (N^m, \partial N)$ be a map into the finite Poincaré pair $(N, \partial N)$ of formal dimension m, suppose $(N, \partial N)$ is $(2k-m+2)$-connected, and $m-k \geqq 4$, $2m \geqq 3k+4$, then f is homotopic to an embedding.*

Applying this proposition we get a result analogous to the Stalling's embedding theorem, and also the induced thickening theorem in the author's paper [2]. Further special cases of the above proposition give the following result.

Theorem. *Every finite Poincaré complex of dimension $\geqq 7$ has a handle decomposition.*

We also show how to use the structure of Poincaré pairs of low geometric dimension to get the following theorems.

Theorem. *Any map $f: S^n \to P^{2n+1}$ of the n-sphere into a finite Poincaré complex of formal dimension $2n+1$ is homotopic to an embedding if $n \geqq 3$.*

Theorem. *Any map $f: S^n \to P^{2n}$ of the n-sphere into a simply-connected finite Poincaré complex of formal dimension $2n$ is homotopic to an embedding if $n \geqq 4$.*

The methods used to prove the general position lemma are an extension of the original method of Levitt [3] and an extensive use of a trick akin to Zeeman's piping, which the author introduced in [1].

The arrangement of the paper is as follows. § 1 gives the necessary definitions and notations, § 2 outlines the proof of the general position lemma. §§ 3–5 contain the proof of the general position lemma. In § 6 we give the handle decomposition theorem and §§ 7–8 give the applications to embeddings.

§ 1. Definitions and Notations

We recall first some definitions. Since our orientation is geometric we will use Levitt's definition of a Poincaré complex which is as follows.

Definition 1. A Poincaré complex of formal dimension n, is a finite connected CW-complex X such that if N is a regular neighbourhood of X in some embedding $j: X \to S^{n+q}$ q large, then the map $\partial N \subseteq N$ is homotopy equivalent to a spherical fibration with fibre S^{q-1}.

A Poincaré pair, of formal dimension n, is a finite CW-pair (X, Y) with X connected, such that if (N, M) is a regular neighbourhood of an embedding $j: (X, Y) \subset D^{n+q}, S^{n+q-1}$ q large, then the map $\partial N - M \hookrightarrow N$ is homotopy equivalent to a spherical fibration with fibre S^{q-1}.

That spaces of the above type satisfy Poincaré duality follows from the Gysin Sequence for the spherical fibration.

Definition 2. Given a Poincaré pair (P, Q) of formal dimension n, and a CW-pair (K, L), with (K, L) finite CW-complexes; a map $f: (K, L) \to (P, Q)$ is said to be homotopic to an embedding if there exists a splitting of P mod boundary, that is

$$P = (N_1, \partial N_1) \cup_{N_0} (N_2, \partial N_2)$$

with Q split as $\partial N_1 - \operatorname{Int} N_0 \cup_{N_0} \partial N_2 - \operatorname{Int} N_0$ and a homotopy equivalence

$$k: (K, L) \to (N_1, \partial N_1 - \operatorname{Int} N_0)$$

such that the following diagram commutes up to homotopy.

$$\begin{array}{ccc} (K, L) & \xrightarrow{\quad f \quad} & (P, Q) \\ \Big\downarrow{\scriptstyle k} & & \Big\uparrow \\ (N_1, \partial N_1 - \operatorname{Int} N_0) & \xrightarrow{\ \subset\ } & (N_1 \cup N_2, \partial N_1 \cup \partial N_2) \end{array}$$

We shall use the following notational conventiones

1. If (K, L) is a CW-pair and $\dim K = k$, $\dim L = l$ we shall say (K, L) is of dimension (k, l).

2. Suppose we have a map $g\colon (K, L) \to (P, Q)$ and a spherical fibre space ξ over (P, Q) then we shall write $(K(\xi), L(\xi))$ for the total space of the fibre space induced by g over (K, L), and $(K(c\,\xi), L(c\,\xi))$ for the mapping cylinder pair of the map $\pi\colon (K(\xi), L(\xi)) \to (K, L)$, where π is the projection for the fibration induced by g.

§ 2. The General Position Lemma

This section is devoted to an outline of the proof of the following lemma.

Lemma 2.1. (The General Position Lemma.) *Let (X, Y) be a finite Poincaré pair of formal dimension n, and suppose given a map $f\colon (D^k, S^{k-1}) \to (X, Y)$ where $k \leqq n-3$, and $2n \geqq 3k+4$. Then there exists a CW-pair (K, L) of dimension k, obtained from (D^k, S^{k-1}) by adding cells of dimension $(2k-n+2,\ 2k-n+1)$ and a map $g\colon (K, L) \to (X, Y)$ homotopic to an embedding such that the following diagram commutes up to homotopy*

$$\begin{array}{ccc} (D^k, S^{k-1}) & \xrightarrow{\ f\ } & (X, Y) \\ \cap & \nearrow{\scriptstyle g} & \\ (K, L) & & \end{array}$$

Remark. The metastability condition $2n \geqq 3k+4$ is perhaps unfortunate and can probably be removed by a somewhat lengthy extension of the proof. Fortunately in our applications the condition will always be satisfied.

Construction of the Stable Model

Let $i\colon (X, Y) \to (P^{n+j}, Q^{n+j-1})$ be a codimension j PL-thickening of the pair (X, Y). Denote $\partial P - \operatorname{Int} Q$ by R so that $\partial R = \partial Q$. Then for sufficiently large j, we have the following:

(a) $R \subset P \to X$ is a spherical fibration ξ of dimension j. (That is with fibre S^{j-1}.)

(b) By general position we may suppose that the composition $i \circ f: (D^k, S^{k-1}) \to (P, Q)$ is an embedding.

In the notation of § 1 we can therefore define (essentially by restricting ξ to (D^k, S^{k-1})) a map

$$F_0(\xi, D): \big(D^k(c\xi), S^{k-1}(c\xi); D^k(\xi), S^{k-1}(\xi)\big) \to (P, Q; R, \partial R).$$

Now let $(M(D), N(S))$ be a regular neighborhood of (D^k, S^{k-1}) in (P, Q) and let $(\mathcal{M}(\xi, D), \mathcal{N}(\xi, S))$ be the mapping cylinder of the composition

$$\bar{p}(D): \big(D^k(\xi), S^{k-1}(\xi)\big) \to (D^k, S^{k-1}) \subset \big(M(D), N(S)\big)$$

then $F_0(\xi, D)$ extends to

$$F(\xi, D): \big(\mathcal{M}(\xi, D), \mathcal{N}(\xi, S)\big) \to (P, Q)$$

with $F(\xi, D)|(M(D), N(S))$ the inclusion.

This completes the construction of the stable model. See Figs. 1 and 2.

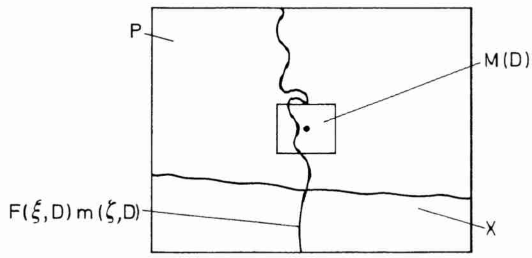

Fig. 1

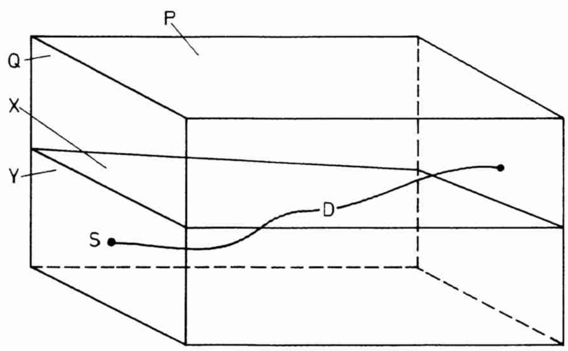

Fig. 2

Fig. 1 should be thought of as an end-on view of Fig. 2, but with more detail.

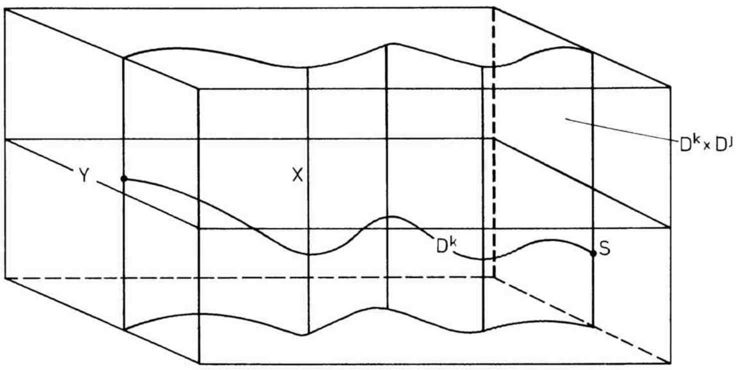

Fig. 3. The splitting of (P, Q) induced by the embedding of $D^k \times D^j$, retracts to give one of (X, Y)

Remark on Notation

We shall use the notations $(M(K), N(L))$, $\mathscr{M}(\xi, K)$ etc., will be used for the constructions obtained when (K, L) replace (D^k, S^{k-1}) in $(M(D), N(S))$ etc.

Outline of the Remainder of the Proof

We recapitulate briefly the strategy of Levitt's proof. In essence the idea was to "compress" the given embedding of (D^k, S^{k-1}) into (X, Y) by constructing an embedding of $(D^k \times D^j, S^{k-1} \times D^{j-1})$ in (P, Q). Fig. 3 indicates why this is an embedding of (D^k, S^{k-1}) in (X, Y).

The required embedding was achieved in 3 steps.

Step 1. The "homotopy" $F(\xi, D)|(\mathscr{M}(\xi, D) - M(\xi, D), \mathscr{N}(\xi, S) - N(\xi, S))$ was shifted off the core of $(M(D^k), N(S^{k-1}))$, so that we could suppose that the homotopy took place in $(P - \text{Int}, Q - \text{Int}\, N)$.

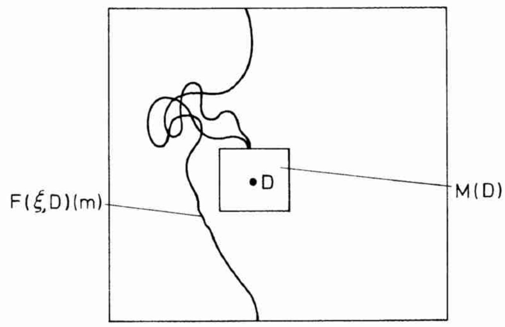

Fig. 4. Position after Step 1, end view

Step 2. The map

$$\bar{p}(D)\colon\ (D^k \times S^{j-1}, S^{k-1} \times S^{j-1}) \to (\partial M - \operatorname{Int} N, \partial N)$$

corresponding to the initial map in the homotopy of Step 1, was made an embedding.

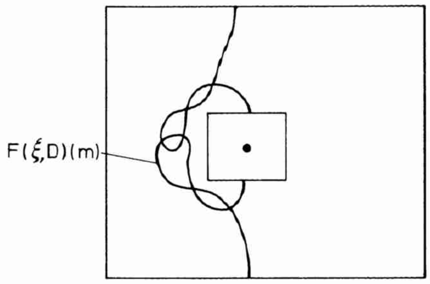

Fig. 5. Position after Step 2, end view

Step 3. The embedded image of

$$(D^k \times S^{j-1}, S^{k-1} \times S^{j-1})$$

in $(\partial M - \operatorname{Int} N, \partial N)$, provided by Step 2 was engulfed in a collar on $(R, \partial R)$. The construction of the required splitting is then straightforward.

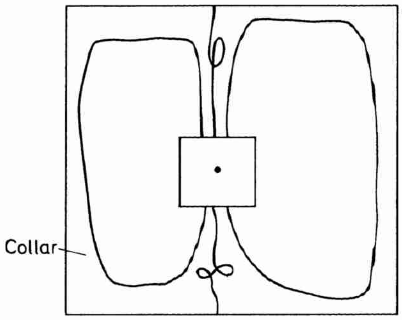

Fig. 6. Position after Step 3, end view

The general position lemma will be proved by trying to perform each of these steps, and seeing what can be salvaged under our conditions. As a guide to the proof we outline the steps below.

Step 1. We construct a *CW*-complex (K_α, L_α) by adding cells of dimension $\leqq (2k-n, 2k-n-1)$ to (D^k, S^{k-1}), and an embedding f_α:

$(K_\alpha, L_\alpha) \to (P, Q)$ such that the diagram

$$\begin{array}{ccc} (D^k, S^{k-1}) & \xrightarrow{i \circ f} & (P, Q) \\ {\scriptstyle j}\bigcap & \nearrow {\scriptstyle f_\alpha} & \\ (K_\alpha, L_\alpha) & & \end{array}$$

commutes up to homotopy. Further there is a map

$$F(\xi, K_\alpha)\colon \big(\mathcal{M}(\xi, K_\alpha), \mathcal{N}(\xi, L_\alpha)\big) \to (P, Q)$$

such that $F(\xi, K_\alpha) | [M(K_\alpha), N(L_\alpha)]$ is the inclusion and

$$F(\xi, K_\alpha)\big(\mathcal{M}(\xi, K_\alpha) - M(K_\alpha), \mathcal{N}(\xi, L_\alpha) - N(L_\alpha)\big) \cap f_\alpha(K_\alpha, L_\alpha)$$

is empty. So that we can suppose that

$$F(\xi, K_\alpha)\big(\mathcal{M}(\xi, K_\alpha) - M(K_\alpha), \mathcal{N}(\xi, L_\alpha) - N(L_\alpha)\big)$$

is disjoint from the interior of $\big(M(K_\alpha), N(L_\alpha)\big)$.

Step 2. We construct a CW-complex (K_β, L_β) by adding cells of dimension $\leqq (2k-n+2, 2k-n+1)$ to (K_α, L_α) and an embedding $f_\beta\colon (K_\beta, L_\beta) \to (P, Q)$ satisfying the conclusion of Step 1, and such that the map

$$\bar{p}(K_\beta)\colon \big(K_\beta(\xi), L_\beta(\xi)\big) \to \big(\partial M(K_\beta) - \operatorname{Int} N(L_\beta), \partial N(L_\beta)\big)$$

of which $\big(\mathcal{M}(K_\beta, \xi), \mathcal{N}(L_\beta, \xi)\big)$ is the mapping cylinder, is homotopic to an embedding.

Step 3. We construct a CW-complex (K_γ, L_γ) by adding cells of dimension $\leqq (2k-n+2, 2k-n+1)$ to (K_β, L_β) and an embedding $f_\gamma\colon (K_\gamma, L_\gamma) \to (P, Q)$ satisfying the conclusion of Step 2 and such that in $\big(P - M(K_\gamma), Q - N(L_\gamma)\big)$ we can engulf the $\big(K_\gamma(\xi), L_\gamma(\xi)\big)$ embedded in $\big(\partial M(K_\gamma) - \operatorname{Int} N(L_\gamma), \partial N(L_\gamma)\big)$ in a collar on $(R, \partial R)$. Exactly as in Levitt's original proof we can now produce an embedding of (K_γ, L_γ) in (X, Y) and this is the embedding whose existence is asserted by the lemma.

A Note on the Technique of the Proof

The technique of the proof is to put things in general position and then add cells to (K_i, L_i) so as to "pierce" the top dimensional cells of the double point set as in the author's paper [1]. For example in the case of the embedding of S^1 in D^3 – two lines illustrated.

We cannot deform the homotopy of the induced S^0 bundle into the boundary without crossing the embedded circle. If however we shift the image of the circle so that it becomes a figure 8, and thus (up to homotopy) an embedding of a $\theta = S^1 \cup a$ 1-cell then it becomes possible to do the

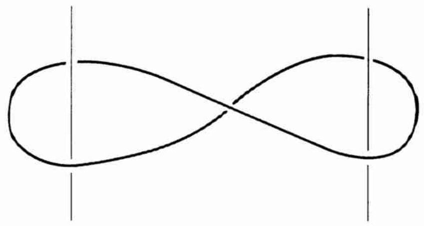

Fig. 7

required homotopy. (The author found this example most enlightening in thinking about the proof of the lemma, and recommends it to the reader's attention.) The 1-cell which we have added serves to identify the two points in S^1, whose image is the cross over in the figure 8.

Significance of the Condition $2n \geqq 3k+4$

The function of this condition in the proof is to ensure that each time a new cell is added it is possible to define the map $F(\xi, K)$ on this new cell so as not to introduce singularities of dimension equal to or greater than that of the one we are disposing of.

§ 3. Proof of the General Position Lemma Step 1

In this step we seek to deform the image of

$$(\mathscr{M}(\xi, D^k), \mathscr{N}(\xi, S^{k-1})) - (M(D_k), N(S^{k-1}))$$

off the interior of $(M(D^k), N(S^{k-1}))$. For brevity we will refer to this portion of the mapping cylinder as $(\mathscr{M}-M, \mathscr{N}-N)$. [The D^k, S^{k-1} are to be understood; this notation will be used in the more general situation where (D^k, S^{k-1}) is replaced by a CW-pair (K, L), where $\mathscr{M}-M$ will mean $\mathscr{M}(\xi, K) - M(K)$ as is appropriate in the context.]

We note first that we can homotop the image of

$$\bar{p}(D)\colon\ (D^k(\xi), S^{k-1}(\xi)) \to (M(D), N(S))$$

into $(\partial M(D) - \operatorname{Int} N(S), \partial N(S))$ since $(M(D), N(S))$ is a product with the unit interval for large j. We view $F(\xi, D)|\mathscr{M}-M, \mathscr{N}-N$ as a homotopy of this map into $(R, \partial R)$ and we seek to shift this homotopy off (D^k, S^{k-1}).

If we take a simplicial approximation and put the map in general position in (P, Q) [keeping the image of $\bar{p}(D)$ in $(\partial M - \operatorname{Int} N, \partial N)$], we may assume that the intersection $F(\xi, D)(\mathscr{M}-M, \mathscr{N}-N) \cap (D^k, S^{k-1})$ is a pair of dimension $(k+j+k-(n+j), (k+j-1)+(k-1)+(n+j-1)) = (2k-n, 2k-n+1)$. Denote this pair by (Σ, Σ'), it has a cell-complex structure because we can suppose (P, Q), PL-embedded in some large dimensional Euclidian space.

We note that the condition $2n \geqq 3k+4$ ensures that the double points of the homotopy are disjoint from (D^k, S^{k-1}).

Elimination of (Σ, Σ')

Consider the subcomplex of $F(\xi, D)(\mathcal{M}-M, \mathcal{N}-N)$ consisting of points such that if they are pushed along the homotopy (towards ∂P) they meet (Σ, Σ'). (This is the track of (Σ, Σ') under the homotopy run backwards and hence is $(\Sigma, \Sigma') \times I$ since $3k+4 \leqq 2n$.) The required complex (K_α, L_α) will be obtained as $(D^k, S^{k-1}) \cup_g (\Sigma, \Sigma') \times I$ where $g|(\Sigma, \Sigma') \times 1$ is the inclusion $(\Sigma, \Sigma') \subset (D^k, S^{k-1})$ and $g|(\Sigma, \Sigma') \times 0$ is obtained by pushing (Σ, Σ') back along the homotopy in $(\partial M - \operatorname{Int} N, \partial N)$ and then projecting. This also describes the extension of $f\colon (D^k, S^{k-1}) \to (P, Q)$ to $f_\alpha\colon (K_\alpha, L_\alpha) \to (P, Q)$. By a small shift we can assume f_α is an embedding and we take a regular neighbourhood of $f_\alpha(K_\alpha, L_\alpha)$ which is formed by glueing to (M, N) a relative regular neighbourhood of $((\Sigma, \Sigma') \times I, (\Sigma, \Sigma') \times S^0)$ in $(P - \operatorname{Int} M, Q - \operatorname{Int} N)$.

We now describe how to construct $(K_\alpha(\xi), L_\alpha(\xi))$.

Let $(M(\Sigma \times I), N(\Sigma' \times I))$ be the relative regular neighbourhood that we have added to (M, N); corresponding to the I factor in $(\Sigma \times I, \Sigma' \times I)$ we can perform an ambient isotopy of (P, Q) that shrinks down the I-factor. In particular this shows that $(\Sigma \times 1)(\xi)$ and $(\Sigma \times 0)(\xi)$ are homotopic, and therefore the ambient isotopy can be used to glue $(\Sigma \times 1)(\xi)$

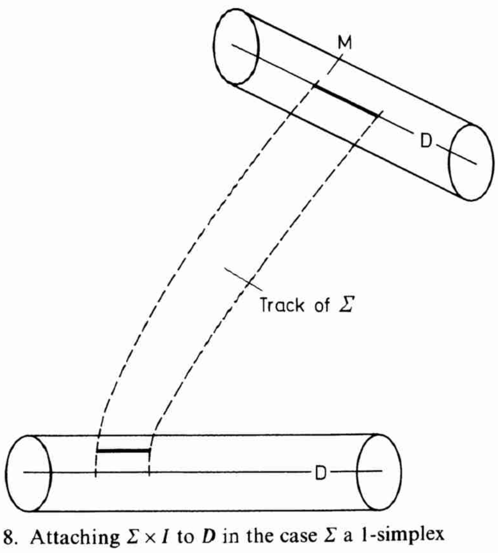

Fig. 8. Attaching $\Sigma \times I$ to D in the case Σ a 1-simplex

to $(\Sigma \times 0)(\xi)$. (The example of an S^1-bundle over S^1, and an identification of two of the fibres should be born in mind as an illustration.) This gives us $(K_\alpha(\xi), L_\alpha(\xi))$ and a map

$$\bar{p}(K_\alpha)\colon\ (K_\alpha(\xi), L_\alpha(\xi)) \to (\partial M(K_\alpha) - \operatorname{Int} N(L_\alpha), \partial N(L_\alpha)).$$

It remains to see that the associated intersection

$$F(\xi, K_\alpha)(\mathcal{M} - M, \mathcal{N} - N) \cap f_\alpha(K_\alpha, L_\alpha)$$

is empty. It is clear that (Σ, Σ') has disappeared, further our assumption $3k+4 \leqq 2n$ implies that $F(\xi, K_\alpha)(\mathcal{M} - M, \mathcal{N} - N)$ does not meet $(\Sigma \times I, \Sigma' \times I)$ by general position. Thus (K_α, L_α) is our required complex and Step 1 is complete.

§ 4. Proof of the General Position Lemma Step 2

In Step 1, we constructed a pair (K_α, L_α) and a map

$$F(\xi, K_\alpha)\ L(\mathcal{M}(K_\alpha, \xi), \mathcal{N}(L_\alpha, \xi)) \to (P, Q)$$

with $F(\xi, K_\alpha)(\mathcal{M} - M, \mathcal{N} - N) \subset (P - \operatorname{Int} M(K_\alpha), Q - \operatorname{Int} N(L_\alpha))$.

In this section we consider the problem of deforming the map

$$\bar{p}(K_\alpha)\colon\ (K_\alpha(\xi), L_\alpha(\xi)) \to (\partial M(K_\alpha) - \operatorname{Int} N(L_\alpha), \partial N(L_\alpha))$$

into an embedding. We recall that $\bar{p}(K_\alpha)$ can be thought of as the initial map of the homotopy $F(\xi, K_\alpha)|(\mathcal{M} - M)$. In Levitt's paper [3], this was achieved by using a relative version of Stalling's Theorem on embedding homotopy types (Theorem 5.2 in [3]). Unfortunately the map $\bar{p}(K_\alpha)$ is not sufficiently connected for us to be able to use the theorem as it stands, however we recall that the method of Stalling's proof [5] is such that if the dimension of the singular set is sufficiently low then the iterative procedure he describes yields an embedding. [In fact Stallings himself remarks this in [5].]

Now if we put $\bar{p}(K_\alpha)$ in general position the singular set (Σ_p, Σ'_p) (i.e. the self intersection of $\bar{p}(K_\alpha)(K_\alpha(\xi))$) has dimension

$$\begin{aligned}(2(k+j-1)-(n+j-1),\ 2(k+j-2)-(n+j-2))\\ = ((j-2)+2k-n+1,\ (j-2)+2k-n).\end{aligned}$$

Also the map $\bar{p}(K_\alpha)$ is $(j-1)$-connected. Thus the singular set is $(2k-n+1)$ dimensions too large for application of the Stallings Theorem, our procedure therefore will be to modify (K_α, L_α) by adding cells, so as to get a pair (K_β, L_β) for which the associated singular set has dimension $j-2$.

Description of the Inductive Hypothesis for Reduction of the Dimension of (Σ_p, Σ'_p)

We begin by remarking that by general position we can assume that the portion of $(K_\alpha(\xi), L_\alpha(\xi))$ contributed by the $(n-k-1)$-skeleton of (K, L) does not give rise to cells of (Σ_p, Σ'_p) of dimension $> j-2$, so that in what follows (K_α, L_α) remains unchanged on a neighbourhood of this skeleton.

We order the cells of (Σ_p, Σ'_p) in decreasing order of dimension (as before by supposing we are working in an ambient Euclidean space we can endow (Σ_p, Σ'_p) with a cell structure). We shall show that there exists a sequence of pairs (K_r, L_r) with $(K_0, L_0) = (K_\alpha, L_\alpha)$ and $(K_r - K_{r-i}, L_r - L_{r-1})$ of dimension r. Further for each r, we construct a map $f_r\colon (K_r, L_r) \to (P, Q)$ such that for the associated map

$$F(K_r, \xi)\colon \big(\mathcal{M}(K_r, \xi), \mathcal{N}(L_r, \xi)\big) \to (P, Q)$$

we have

$$F(K_r, \xi)[\mathcal{M} - M, \mathcal{N} - N] \subset \big(P - \operatorname{Int} M(K_r), Q - \operatorname{Int} N(L_r)\big)$$

and the singular set of the map

$$\bar{p}(K_r)\colon \big(K_r(\xi), L_r(\xi)\big) \to \big(\partial M(K_r) - \operatorname{Int} N(L_r), \partial N(L_r)\big)$$

is the s-skeleton of $(\Sigma_p, \Sigma'_p) \cup$ cells of dimension $< s$, where $s = (j-2) + (2k-n+1) - r$.

The method of Stallings will then apply to embed

$$\big(K_{2k-n+2}(\xi), L_{2k-n+2}(\xi)\big),$$

so that our required (K_β, L_β) will be (K_{2k-n+2}, L_{2k-n+2}).

Suppose therefore we have constructed (K_r, L_r). We will show how to construct an intermediate complex (K'_r, L'_r) so that there is one less s-cell in the singular set of $\bar{p}(K'_r)$ than the singular set of $\bar{p}(K_r)$.

Before giving the construction for general r, it will clarify the argument if we describe the case $r=1$ first. Let us suppose therefore that ρ is a cell of highest dimension in $(\Sigma_\rho, \Sigma'_\rho)$. Then we can find A_ρ, B_ρ in $\big(K_0(\xi), L_0(\xi)\big)$ such that $\bar{p}(K_0)(A_\rho) \cap \bar{p}(K_0)(B_\rho) = \rho$. Let Δ_ρ (resp. Δ'_ρ) be the barycentres of $q(A_\rho)$ (resp. $q(B_\rho)$) where q is the "projection"

$$q\colon \big(K_0(\xi), L_0(\xi)\big) \to (K_0, L_0).$$

Then $\Delta_\rho \cup \Delta'_\rho$ can be thought of as the image of an S^0 and hence gives an attaching map for a 1-cell. Now we suppose ρ is contained in a disc $D_0^{n+j-1} \subset \partial M(K_0) - \operatorname{Int} N(L_0)$. (If in fact ρ meets $\partial N(L_0)$ the argument requires some modifications which we shall omit.) Furthermore $\partial\rho \subset$

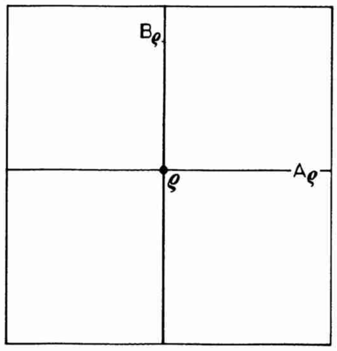

Fig. 9

∂D_0^{n+j-1}. Also ρ lies in the images of the cells A_ρ, B_ρ, "lying over" Δ_ρ and Δ'_ρ respectively. See Fig. 9, where we have written A_ρ, B_ρ for the images.

Now we can find an embedding $I \subset \operatorname{Int} D_0^{n+j-1}$ such that $I \cap A_\rho = I \cap B_\rho \subset \rho$, and the composition

$$S^0 \subset I \subset D_0^{n+j-1} \subset \partial M(K_0) \subset M(K_0)$$

is homotopic to

$$S^0 \to \Delta_\rho \cup \Delta'_\rho \subset K_0.$$

The idea now is to "lift" the interior of I out of $M(K_0)$ by adding a 1-handle, and then modify $K_0(\xi)$ inside this 1-handle to get $K_0 \cup I(\xi)$. We proceed as follows. Let D_1^{n+j-1} be an $(n+j-1)$-disc containing $\operatorname{Int} A_\rho$ and $\operatorname{Int} B_\rho$ in its interior and with $\partial I \subset \partial D_1^{n+j-1}$. Shrink D_1 slightly so as to give D'_1 as in the picture below.

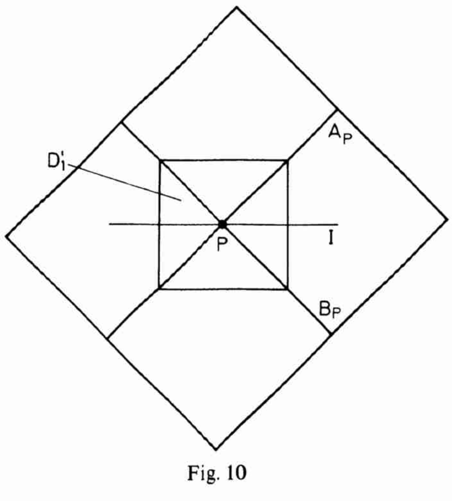

Fig. 10

Choose a piecewise-linear function $\varphi\colon D_1' \to [0, \varepsilon]$ with the following properties

1) $\varphi = 0$ on a collar on a regular neighbourhood of $I \cap \partial D_1'$.

2) $\varphi > 0$ elsewhere.

We now push D_1' out of $M(K_0)$ in the normal direction by an amount given by φ. (This is possible because there is a collar on $\partial M(K_0)$ in $P - \operatorname{Int} M(k_0)$, which is PL-homeomorphic to $\partial M(K_0) \times [0, 2\varepsilon]$.) Thus taking a small regular neighbourhood of the pushed D_1' adds a 1-handle to $M(K_0)$. This has the effect of adding a 1-cell to K, call this new complex K'. We must now show how to choose a complex of the homotopy type of $(K'(\xi), L'(\xi))$ and describe how to map it into $(M(K'), N(K'))$ with one less top dimensional cell in the self-intersection; as in Step 1 this will be done by embedding an $S^{j-1} \times I$ so as to be able to isotop the S^{j-1}'s over Δ_ρ and Δ_ρ' to a single S^{j-1}. Our dimensional conditions are such that we can find an embedding of $S^{j-1} \times I$ in the boundary of the attached handle, so that $S^{j-1} \times 0$ is homotopic to $S^{j-1} \simeq \Delta_\rho(\xi)$ and $S^{j-1} \times 1$ is homotopic to $S^{j-1} \simeq \Delta_\rho'(\xi)$, and we can shift the images of $S^{j-1} \times 0$ and $S^{j-1} \times 1$ so that they meet in a single point at the barycentre of ρ. Now if we use this isotopy to "lift" $S^{j-1} \times 0$ and $S^{j-1} \times 1$ into the handle we have our required $K'(\xi)$ mapped into $M(K')$. (Up to homotopy type the barycentre of ρ has been extended to a self-intersection along an S^{j-1} justified by the 1-cell added to K.)

We now describe briefly the general case, we have found (K_r, L_r) and we seek to construct (K_r', L_r') with one less s-cell in the singular set of $\bar{p}(K_r')$ than in that of $\bar{p}(K_r)$. Let ρ be the first s-cell in the singular set of $\bar{p}(K_r)(K_r(\xi), L_r(\xi))$ for the chosen ordering. Then we can find $(r-1)$-cells Δ_ρ and Δ_ρ' in the dual decomposition of (K_r, L_r) and a subcomplex Γ_ρ isomorphic to $\partial \Delta_\rho \times I$ with $\Delta_\rho \cup \Gamma_\rho \cup \Delta_\rho'$ homeomorphic to S^r and the cells of Γ_ρ in $(K_r - K_0, L_r - L_0)$ (so that in the case $r=2$, Γ_ρ is the union of 1-cells added at the first inductive level) so that we have a map

$$l\colon \partial D^{r+1} = S^r \to (K_r, L_r)$$

which will be the attaching map for an $(r+1)$-cell. These cells are related to ρ in the following manner. $\Delta_\rho, \Delta_\rho'$ are chosen so that ρ lies in the intersection of the images under $\bar{p}(K_r)$ of $N(\Delta_\rho, \partial\Delta_\rho)(\xi)$ and $N(\Delta_\rho', \partial\Delta_\rho')(\xi)$ where $N(\Delta_\rho, \partial\Delta_\rho)$ is a relative simplicial neighbourhood of $(\Delta_\rho, \partial\Delta_\rho)$ in the first derived of K_r, and $N(\Delta_\rho, \partial\Delta_\rho)(\xi)$ is the part of $K_r(\xi)$ "lying over" $N(\Delta_\rho, \partial\Delta_\rho)$. Γ_ρ is the union of cells added along $\partial\Delta_\rho$ and $\partial\Delta_\rho'$ at earlier stages in the induction.

As in the case $r=0$, ρ is contained in a disc D_0^{n+j-1}, and we can find cells A_ρ, B_ρ (of maximum dimensional (locally) in $K_r(\xi)$) with $A_\rho \cap B_\rho = \rho$, $A_\rho \in \operatorname{Image} N(\Delta_\rho, \partial\Delta_\rho)(\xi)$ and $B_\rho \in \operatorname{Image} N(\Delta_\rho', \partial\Delta_\rho')(\xi)$. (See Fig. 9.)

This time we can find an embedding $k\colon D^{r+1} \subset \operatorname{Int} D_0^{n+j-1}$ which has the following properties.

(1) If ∂D^{r+1} is written as $D_0^r \cup S^{r-1} \times I \cup D_1^{r+1}$ then $k(D_0^r)$, $k(D_1^r)$ correspond to Δ_ρ, Δ'_ρ and $S^{r-1} \times I$ corresponds to Γ_r, in the sense that there is a homotopy $F_t\colon \partial D^{r+1} \to M(K_r)$ with $F_0 = k$, $F_1 = l$.

(2) $D^{r+1} \cap A_\rho = D^{r+1} \cap B^\rho \cong D^r \subset \rho$.

Exactly as before let $D_1^{n+j-1} \subset D_0^{n+j-1}$ be an $(n+j-1)$-disc containing $\operatorname{Int} A_\rho$ and $\operatorname{Int} B_\rho$ in its interior with $\partial D^{r+1} \subset \partial D_1$, shrink D_1 slightly to D'_1 (see Fig 10). Choose a PL function $\varphi_{r+1}\colon D^{r+1} \to [0, \varepsilon]$ with the following properties.

(1) $\varphi_{r+1} = 0$ on a collar on a regular neighbourhood of $k(D^{r+1}) \cap \partial D_1$ in ∂D_1.

(2) $\varphi_{r+1} > 0$ elsewhere.

(3) $\varphi_{r+1} = \varphi_r$ on a neighbourhood of $S^{r-1} \times I$ in $\partial D'_1$.

Now push D'_1 out of $M(K_0)$ in the normal direction by an amount given by φ_{r+1}. As before taking a small regular neighbourhood of the pushed D'_1 in $P - \operatorname{Int} M(K_r)$ adds an $(r+1)$-handle to $(M(K_r), N(L_r))$, since condition (3) ensures that φ_{r+1} and φ_r push the cores of the r-handles to the same places. This has the effect of adding an $(r+1)$-cell to K_r along l, set $K'_r = K_r \cup_l e^{r+1}$.

We now show how to choose a complex of the homotopy type of $(K'_r(\xi), L'_r(\xi))$, and describe how to map it into $(M(K'_r), N(L'_r))$ with one less singular s-cell. As before our model for $(K'_r(\xi), L'_r(\xi))$ is obtained by adding $S^{j-1} \times D^r \times I$ to $(K_r(\xi), L_r(\xi))$ and squeezing down the I-factor to identify $S^{j-1} \times D^r \times 0$ and $S^{j-1} \times D^r \times 1$. Our dimensional conditions (including $3k+4 \leqq 2n$) allow us to embed the $S^{j-1} \times D^r \times I$ in $\partial M(K'_r)$ so that $S^{j-1} \times D^r \times 0$ is homotopic to $\Delta_\rho(\xi)$ and $S^{j-1} \times D^r \times 1$ is homotopic to $\Delta'_\rho(\xi)$, and so as to extend the given embedding of $S^{j-1} \times S^{r-1} \times I$. As before we shift these images so that they meet along $D^r \subset \rho$, and then "lift" the $S^{j-1} \times D^r \times \{0, 1\}$ into the added handle. This gives the required map

$$\bar{p}(K'_r)\colon\ (K'_r(\xi), L'_r(\xi)) \to (\partial M(K'_r) - \operatorname{Int} N(L'_r), \partial N'_r)$$

with one less s-cell in the singular set. Further the associated homotopy of $(K'_r(\xi), L'_r(\xi))$ to $(R, \partial R)$ will not introduce any singularities of the type removed in Step 1, since the assumption $3k+4 \leqq 2n$, implies that the $D^{r+1} \times S^{j-1}$ that we added (in homotopy!) cannot introduce any such singularities. Repeating the argument constructs (K_{r+1}, K_{r+1}) and hence (K_β, L_β).

In conclusion we apply the arguments of Stallings in [5] to embed (K_β, L_β) up to homotopy type in $(\partial M(K_\beta) - \operatorname{Int} N(L_\beta), \partial N(L_\beta))$.

§ 5. The Proof of the General Position Lemma. Step 3 and Conclusion

By the arguments of Steps 1 and 2 we have achieved the following: we have a pair (K_β, L_β) obtained from (D^k, S^{k-1}) by adding cells of dimension $\leqq (2k-n+2, 2k-n+1)$ and an embedding of (K_β, L_β) in (P, Q) with $(K_\beta(\xi), L_\beta(\xi))$ embedded up to homotopy type in

$$(\partial M(K_\beta) - \operatorname{Int} N(L_\beta), \partial N(L_\beta)).$$

Furthermore we have a map

$$F(K_\beta, \xi)\colon (\mathcal{M}(\xi, K_\beta), \mathcal{N}(\xi, L_\beta)) \to (P, Q)$$

extending this embedding and with the image of $(\mathcal{M} - M, \mathcal{N} - N)$ contained in $(P - \operatorname{Int} M(K_\beta), Q - \operatorname{Int} N(L_\beta))$. In Step 3 we propose to try and engulf $F(K_\beta, \xi)[\mathcal{M} - M, \mathcal{N} - N]$ in a collar on $(R, \partial R)$. This map can be thought of as a homotopy of $(K_\beta(\xi), L_\beta(\xi))$ from

$$(\partial M(K_\beta) - \operatorname{Int} N(L_\beta), \partial N(L_\beta))$$

into $(R, \partial R)$, and we begin by putting this homotopy in general position, keeping $(K_\beta(\xi), L_\beta(\xi))$ embedded in $(\partial M(K_\beta) - \operatorname{Int} N(L_\beta), \partial N(L_\beta))$.

The singular set of this homotopy is of dimension

$$[2(k+j) - n + j, 2(k+j-1) - (n+j-1)] = (2k-n+j, 2k-n-1+j).$$

Thus the homotopy collapses to a subcomplex of dimension

$$(2k-n+j+1, 2k-n+j).$$

Unfortunately the inclusions $\partial R \subset Q - \operatorname{Int} N(L_\beta)$ and $R \subset P - \operatorname{Int} M(K_\beta)$ are only $(j-1)$-connected so that we cannot engulf unless we can first reduce the dimension of the singular set to $j-2$, thus we must eliminate "$2k-n+2$-dimensions" of singularities. Since the method is essentially a repeat of that of Steps 1 and 2 we content ourselves with a brief sketch. As before we produce inductively a sequence of complexes (K_r, L_r) so that the singularity set of $F(K_r, \xi)[\mathcal{M} - M, \mathcal{N} - N]$ is the $(2k-n+j-r)$-skeleton of the singularity set of $F(K_\beta, \xi)[\mathcal{M} - M, \mathcal{N} - N]$ [union (possibly) cells of dimension less than $2k-n+j-r$] with dimension $(K_r - K_{r-1}, L_r - L_{r-1}) = r$ and $(K_0, L_0) = (K_\beta, L_\beta)$ and each (K_r, L_r) satisfying the conclusions of Step 2. The complex (K_γ, L_γ) of Step 3 is then (K_{2n-n+2}, L_{2k-n+2}).

Suppose we have constructed (K_r, L_r), an intermediate complex is constructed as follows. Let ρ be a $(2k-n+j-r)$-cell of the singularity set of $F(K_r, \xi)$, then we can find cells σ_0 and σ_1 of $(K_r(\xi), L_r(\xi))$, whose tracks under the homotopy give rise to ρ, and lying over r-cells Δ_0, Δ_1 in the dual decomposition of (K_r, L_r). Furthermore the boundaries $\partial \Delta_0, \partial \Delta_1$ correspond to cells of $(K_r - K_{r-1}, L_r - L_{r-1})$. Now we can attach an $(r+1)$-

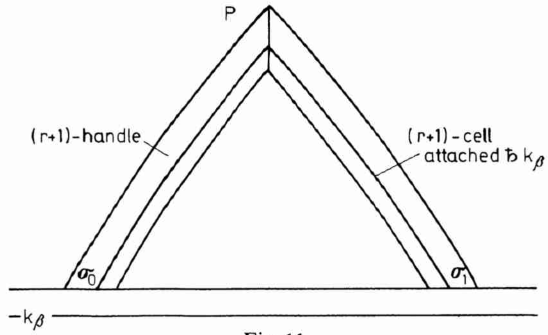

Fig. 11

handle to $(M(K_r), N(K_r))$ so as to pierce ρ. This $(r+1)$-handle can be thought of as a neighbourhood of the tracks of σ_0 and σ_1 up to ρ. (Cf. the construction in [1], and Fig. 11.)

As in Steps 1 and 2, a $D^{r+1} \times S^{j-1}$ is embedded in the boundary of this handle, then by collapsing the I-factor in $D^r \times I \times S^{j-1}$ we perform an isotopy that identifies $D^r \times 0 \times S^{j-1}$ and $D^r \times 1 \times S^{j-1}$ in $(K_r(\xi), L_r(\xi))$ to get $(K_r'(\xi), L_r'(\xi))$ embedded in $\partial M(K_r')$. The homotopy of the "added" $D^{r+1} \times S^{j-1}$ to the boundary $(R, \partial R)$ introduces no new singularities of a dimension equal to or greater than that of the one we are getting rid of, since we have imposed $3k+4 \leqq 2n$. The relevant inclusion of $R \subset P - \operatorname{Int} M(K_r')$ remains $(j-1)$-connected and so the induction can proceed to the stage where engulfing is possible using Lemma 5.3 of [3].

The proof is now completed exactly as in Levitt's original paper. If the collar engulfing $(\mathcal{M} - M, \mathcal{N} - N)$ is called (C, C') then we have a splitting of (X, Y) mod boundary given by

$$(X, Y) \simeq (X_0, Y_0) \cup_{Y_2} (X_1, Y_1)$$

where

$$Y_2 = \text{Closure}\,[(\overline{P - C} - M(K_\gamma)] \cap M(K_\gamma)$$

$$X_1 = M(K_\gamma) \qquad Y_1 = Y_2 \cup N(L_\gamma)$$

$$X_0 = \text{Closure}\,[(\overline{P - C}) - M(K_\gamma)] \qquad Y_0 = Y_2 \cup \text{Closure}\,[\overline{Q - C'} - N(L_\gamma)].$$

Thus we have a Poincaré embedding of (K_γ, L_γ) in (X, Y) such that the diagram below commutes up to homotopy

$$\begin{array}{ccc} (D^k, S^{k-1}) & \xrightarrow{\;f\;} & (X, Y) \\ \cap & \nearrow_{j} \text{ Poincaré embedding} & \\ (K_\gamma, L_\gamma) & & \end{array}$$

so that (K_γ, L_γ) is the complex whose existence is asserted by the lemma.

§ 6. Applications to Handle Decompositions

In this paragraph we prove the following result.

Theorem 6.1. *Every finite Poincaré complex P of dimension* $\geqq 7$ *has a handle decomposition.*

Remark. The author understands that Jones [9] can obtain the above result for dimension $\geqq 5$.

The proof follows from the two propositions below, which are of independent interest.

Proposition 6.2. *Let* $f: (D^n, S^{n-1}) \to (P^{2n+1}, \partial P)$ *be a map into a Poincaré pair* $(P, \partial P)$ *1-connected then f is homotopic to an embedding if* $n \geqq 3$.

Proposition 6.3. *Let* $f: (D^n, S^{n-1}) \to (P^{2n}, \partial P)$ *be a map into a Poincaré pair with* $(P, \partial P)$ *2-connected then f is homotopic to an embedding if* $n \geqq 4$.

Proof of Proposition 6.2. According to Lemma 2.1, there is a *CW*-pair (K, S^{n-1}) where K is obtained by adding 1-cells to D^n and a Poincaré embedding $g: (K, S^{n-1}) \to (P, \partial P)$ such that the composition

$$(D^n, S^{n-1}) \to (K, S^{n-1}) \to (P, \partial P)$$

is homotopic to f. So we have a splitting of $(P, \partial P)$ mod boundary

$$(P, \partial P) = (N_1, \partial N_1) \cup_{N_0} (N_2, \partial N_2)$$

where ∂P is split by $(\partial N_2 - N_0) \cup_{\partial N_0} (\partial N_1 - N_0)$ and $(N_1, \partial N_1 - N_0) \sim (K, S^{n-1})$. (See Fig. 12.)

The 1-cells attached to D^n give generators of $\pi_1(K, S^{n-1})$ and hence elements of $\pi_1(P, \partial P)$. Thus since $2n+1 \geqq 7$, we can suppose by Levitt's Theorem [3], that the generators are represented by arcs (D^1, S^0) embedded in $(N_0, \partial N_0)$. Since $\pi_1(P, \partial P)$ is zero there is a homotopy of

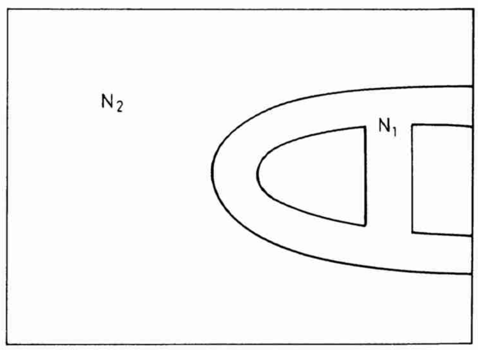

Fig. 12

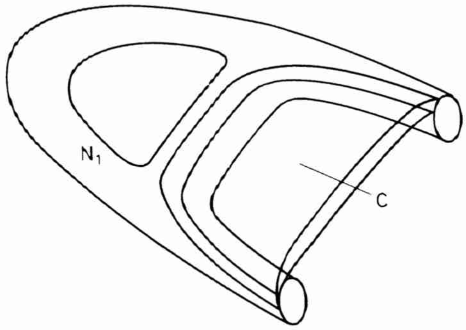

Fig. 13. Adding a $(2n+1)$-cell to N_1

each of these arcs into ∂P, and we can suppose that the homotopies lie in N_2. So that we have for each generator of $\pi_1(K, S^{n-1})$ a map

$$f\colon (D^2, D^1_+, D^1_-, S^0) \to (N_2, N_0, \partial N_2 - N_0, \partial N_0).$$

Levitt's theorem allows us to replace these maps by disjoint embeddings. We can then use these embedded discs to add $(2n+1)$-cells C to N, along $D^1 \times D^{2n-1}$ and subtract them from N_2. See Fig. 13.

Since $N_1 = D^{2n+1} \cup$ 1-handles, the above arguments show that we can suppose N_1 is embedded in $(P, \partial P)$ in the following way. If we write

$$N_1 = D^n \times D^{n+1} \cup S^{n-1} \times I \times D^{n+1} \cup \text{1-handles},$$

where the 1-handles are disjoint from $D^n \times D^{n+1}$ as in Fig. 14.

So we can write $N_1 = D^n \times D^{n+1} \cup Q$ and $Q = S^{n-1} \times I \times D^{n+1} \cup$ 1-handles is a product $Q_0 \times I$. Then the existence of the cells C allow us to suppose

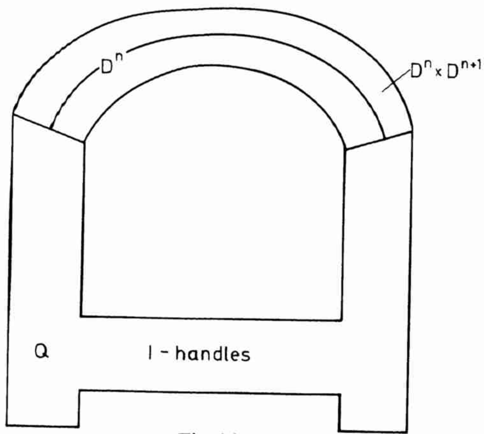

Fig. 14

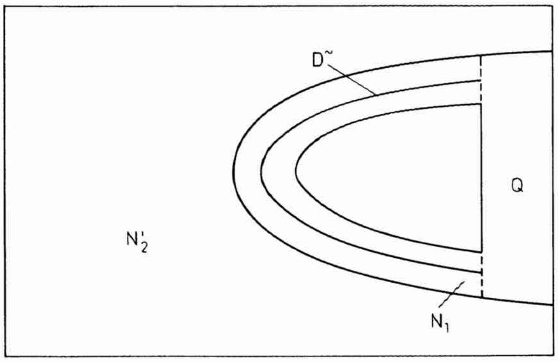

Fig. 15

that N_1 is embedded in $(P, \partial P)$ so that Q is embedded as a collar on $Q_0 \subset \partial P$. (See Fig. 15.)

Then we can delete Q from $(P, \partial P)$ by a simple homotopy equivalence and we will have $(D^n, S^{n-1}) \subset (P, \partial P)$ where D^n is the $D^n \times 0$ in N_1.

Proposition 6.3 is proved in a similar way. $n \geqq 4$ is needed to enable us to embed the 3-cells required to push 2-handles to the boundary.

Construction of Handle Decompositons

We shall prove Theorem 6.1 by showing that we can split P as the union of two Poincaré complexes each having a handle decomposition. The method is essentially the same as that used in Levitt's paper, except that our stronger embedding theorems enable us to get the full result. Consider the following situation, suppose P is split as

$$P \simeq (N_1, \partial N) \cup_{\partial N} (N_2, \partial N),$$

with N_1 homotopy equivalent to a finite CW-complex of dimension k, and $i\colon N_1 \to P$ k-connected, then $(N_1, \partial N)$ is $(n-k-1)$-connected, where n is the formal dimension of P, provided $n-k \geqq 3$. Now if $\tilde{N}$ denotes the universal cover, we have

$$H_i(\tilde{N}_2, \partial\tilde{N}) \cong H_i(\tilde{P}, \tilde{N}_1)$$

by excision, so that $(N_2, \partial N)$ is k-connected, hence by duality the homology of N_2 vanishes in dimensions $\geqq n-k$, the fact that N_2 is of the homotopy type of a finite CW-complex will follow from the construction of N_1 and N_2 (in fact our definition of embedding requires this) and thus we have N_2 homotopy equivalent to a CW-complex of dimension $\leqq n-k-1$.

We have the following proposition

Proposition 6.4. *Let $(N, \partial N)$ be a Poincaré pair of formal dimension $n \geqq 7$, with N homotopy equivalent to a CW-complex K, of dimension k, $2k \leqq n$ and i_*: $\pi_1(\partial N) \to \pi_1(N)$ an isomorphism then N has a handle decomposition, with handles of index $\leqq k$.*

Proof. Let L be the $(k-1)$-skeleton of K, then $(N, \partial N)$ has a splitting as

$$N \cong (P, \partial P) \cup_{\partial P} (Q, \partial P \cup \partial N)$$

with P homotopy equivalent to L. Now the homotopy equivalence f: $K \to N$ gives via the characteristic maps for the top cells, homotopy classes $[\alpha_i] \in \pi_k(N, P)$ i in some index set. By the Blakers-Massey Theorem {see for example [6]} each such class can be represented by an element $[\alpha_i] \in \pi_k(Q, \partial P)$. Now by using Proposition 6.2 or 6.3 as is appropriate to the parity of n, we can get disjoint embeddings, so that we have an embedding of K in N, since N is homotopy equivalent to K this is exactly a handle decomposition for N.

To complete the proof of Theorem 6.1 we proceed as follows.

$n = 2k+1$. Using Proposition 6.2 we can split N as $(N_1, \partial N_1) \cup (N_2, \partial N_2)$ where N_1 is homotopy equivalent to a CW-complex K of dimension $\leqq k$, hence N_2 is also homotopy equivalent to a CW-complex of dimension $\leqq k$ by the argument before Proposition 6.4. The handle decomposition now follows from Proposition 6.4.

$n = 2k$. Using Proposition 6.3 we can split N as $(N_1, \partial N_1) \cup (N_2, \partial N_2)$ where N_1 is homotopy equivalent to a CW-complex of dimension $\leqq k$, as before N_2 is also homotopy equivalent to a CW-complex of dimension $\leqq k$ so that Proposition 6.4 again gives us the result.

§ 7. Embeddings in the Metastable Range

The object of this paragraph is to obtain results analogous to the Stalling's Theorem [5] and Wall's Induced Thickening Theorem [10], the principal result is the following

Theorem 7.1. *Let f: $K^k \to P^m$ be a $(2k-m+2)$-connected map from the finite connected CW-complex K of dimension k, to the finite Poincaré Complex P of formal dimension m. Then f is homotopic to an embedding if $m-k \geqq 4$ and $2m \geqq 3k+4$.*

Remark. A similar result for K Poincaré is shown by Quinn in [8].

We have the following corollary of Theorem 7.1 which is a general position theorem for maps of CW-complexes into Poincaré complexes.

Corollary 7.2. *Let $f\colon K^k \to P^m$ be a map of the finite CW-complex K of dimension k, to the finite Poincaré complex P of formal dimension m, then for $m-k \geqq 4$ and $2m \geqq 3k+4$ there is a CW-complex L^k obtained by adding cells of dimension $\leqq (2k-m+2)$ to K, and an embedding $g\colon L \subset P^m$ such that the composition of the inclusion $i\colon K \to L$ and the embedding $g\colon L \to P$ is homotopic to f.*

This corollary follows immediately from Theorem 7.1 by observing that one can add cells of dimension $\leqq (2k-m+2)$ to K so as to make f $(2k-m+2)$-connected.

The proof of Theorem 7.1 is by induction. Suppose that $K = K_0 \cup e^k$ and that we have shown that $f|K_0$ is homotopic to an embedding, so that we have a splitting

$$P = (N_1, \partial N_1) \cup (N_2, \partial N_1)$$

with a homotopy equivalence $q\colon K_0 \to N_1$ such that the diagram below commutes.

$$\begin{array}{ccc} K_0 & \longrightarrow & N_1 \\ & \searrow^{f|K_0} & \cap \\ & & P \end{array}$$

Then the map $f\colon K \to P$ defines an element $[\alpha] \in \pi_k(P, N)$, and this element is in the image of $\pi_k(N_2, \partial N_1) \to \pi_k(P, N)$ since $(N_1, \partial N_1)$ is $(m-k-1)$-connected and $(N_2, \partial N_1)$ is $(2k-m+2)$-connected (at least if $k>2$, otherwise Theorem 7.2 follows from Levitt's theorem), so that by Toda's extension of the Blakers-Massey Theorem [6], $(P; N_1, N_2)$ is k-connected and $[\alpha]$ comes from $\pi_k(N_2, \partial N_1)$. Thus Theorem 7.1 will follow if we show: –

Proposition 7.3. *Let $f\colon (D^k, S^{k-1}) \to (N^m, \partial N)$ be a map into the finite Poincaré pair $(N, \partial N)$ of formal dimension m, with $(N, \partial N)$ $(2k-m+2)$-connected, then f is homotopic to an embedding provided also $m-k \geqq 4$.*

Proof. The proof uses an elaboration of the technique of Proposition 6.2. By Lemma 2.1, we can find an embedding $g\colon (K, L) \subset (N, \partial N)$ of the CW-pair (K, L) obtained from (D^k, S^{k-1}) by adding cells of dimension $\leqq (2k-m+2,\ 2k-m+1)$ to (D^k, S^{k-1}). We now want to push the cells attached to D^k into the boundary as we did in Proposition 6.2. Let the splitting of $(N, \partial N)$ mod boundary be given by

$$(N, \partial N) = (M_1, \partial M_1) \cup_{M_0} (M_2, \partial M_2) \qquad \text{(see Fig. 16)}$$

so that M_1 is homotopy equivalent to K and $\partial M_1 - M_0$ is homotopy equivalent to L. We want to attach cells to K so that K/L is homotopy equivalent to S^k. We proceed by induction. Suppose $\pi_r(K, L)$ is the first

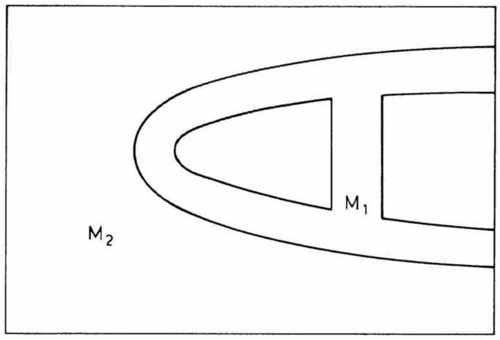

Fig. 16

non-vanishing relative homotopy group of (K, L). Then if $r > (2k - m + 2)$, $r = k$ and there is nothing to prove, so suppose $r < (2k - m + 2)$, let

$$\alpha\colon\ (D^r, S^{r-1}) \to (K, L)$$

be a generator of $\pi_r(K, L)$. Then by composing with the embedding of $(K, L) \subset (N, \partial N)$ we have a map $\alpha\colon (D^r, S^{r-1}) \to (N, \partial N)$ which is homotopic to a map $D^r \to \partial N$ as a map of pairs by the connectedness condition on $(N, \partial N)$. Let

$$H\colon\ (D^{r+1};\ D^r_+, D^r) \to (N;\ N, \partial N)$$

be the homotopy, then by the version of the Blakers-Massey theorem used earlier H is homotopic to a map

$$H\colon\ (D^{r+1}, D^r_+, D^r_-) \to \big(M_2;\ M_0, \partial N - (\partial M_1 \cap \partial N)\big).$$

See Fig. 17.

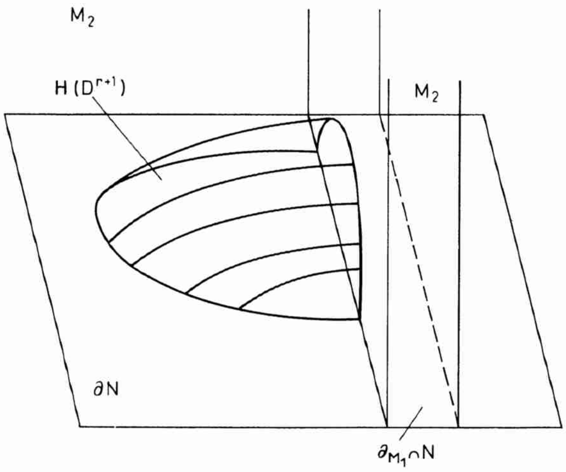

Fig. 17

If we can make H an embedding then the argument of Proposition 6.2 will apply to give an embedding

$$f\colon (D^k, S^{k-1}) \to (N, \partial N)$$

serving to complete the induction step.

But since $m-k \geqq 4$, $r \leqq 2k-m+2$, so $r+1 \leqq k$ and hence since $(M_2, \partial M_2)$ is $r-1 \geqq (2r-m+2)$-connected by excision and duality. So we can apply the inductive hypothesis on dimension to deform H to an embedding, proving the result.

§ 8. Embeddings on the Edge of the Stable Range

By analogy with the manifold case we call the range in which Levitt's embedding theorem holds the stable range, thus if we are trying to embed a k-dimensional complex, the stable range for the dimension of the target space is $n \geqq 2k+2$. In this paragraph we consider the cases $n = 2k+1$, and $n = 2k$ for simply-connected targets.

Theorem 8.1. *Let $f\colon S^n \to P^{2n+1}$ be a map into a finite Poincaré complex of formal dimension $2n+1$, then for $n \geqq 3$, f is homotopic to an embedding.*

Proof. By choosing maps $\alpha_i\colon S^1 \to P$, $i = 1, \ldots, r$, representing generators of $\pi_1(P)$, we can use Levitt's theorem to embed $\bigvee_{i=1}^{r} S^1$ and split P as $(Q, \partial R) \cup (R, \partial R)$ with R of the homotopy type of a wedge of circles, so that $(Q, \partial Q)$ is 1-connected. Now the homotopy class $[f] \in \pi_n(P^{2n+1})$ can be represented by a map $f_0\colon D^n, S^{n-1} \to (Q, \partial R)$ by shifting the image of S^n off the core of R. So by Lemma 2.1 there is a CW-pair (K, S^{n-1}) where K is obtained from D^n by adding cells of dimension 1 and an embedding $g\colon (K, S^{n-1}) \to (Q, \partial R)$ such that the diagram below commutes

$$\begin{array}{ccc} (D^n, S^{n-1}) & \xrightarrow{f_0} & (Q, \partial R) \\ \cap & \nearrow_{g} & \\ (K, S^{n-1}) & & \end{array}$$

but now we can apply Proposition 6.2 to show that f_0 is homotopic to an embedding. To complete the proof we will show that $f_0 | S^{n-1} \to \partial R$ extends to an embedding $f_1\colon (D^n, S^{n-1}) \to (R, \partial R)$. In fact $(R, \partial R)$ is homotopy equivalent to a manifold since R is of the homotopy type of a wedge of circles, furthermore the inclusion $S^{n-1} \subset \partial R$ is null-homotopic in R so that f_0 extends to an embedding in R by general position (in the manifold category). The union $f_0 \cup f_1\colon S^n \to P$ is our required embedding.

We now consider the case of embedding an n-sphere into a Poincaré complex of dimension $2n$. In this case one has the following result.

Theorem 8.2. *Let $f\colon S^n \to P^{2n}$ be a map of the n-sphere into the finite simply-connected Poincaré complex P. Then f is homotopic to an embedding if $n \geqq 4$.*

Proof. Choose a map $g\colon K \to P^{2n}$, where K is a wedge of 2-spheres and such that g is 2-connected. Since $2n \geqq 8$, g is homotopic to an embedding so that we have a splitting of P as

$$P = (R, \partial Q) \cup_{\partial Q} (Q, \partial Q)$$

with R homotopy equivalent to K so that $(Q, \partial Q)$ is 2-connected and the class $[f] \in \pi_n(P)$ can be represented by an element of $\pi_n(Q, \partial Q)$. Let this element be represented by a map $f_0\colon (D^n, S^{n-1}) \to (Q, \partial Q)$, then by Proposition 6.3 f_0 is homotopic to an embedding. As in Theorem 8.1 we wish to extend $f_0 | S^{n-1} \to \partial Q$ to an embedding $f_1\colon (D^n, S^{n-1}) \to (R, \partial Q)$, certainly we can find a map $f_1\colon (D^n, S^{n-1}) \to (R, \partial Q)$ such that the union $f_0 \cup f_1\colon S^n \to P$ is homotopic to f, so it suffices to show that we can make this f_1 an embedding. Now R is homotopy equivalent to K which is a wedge of 2-spheres so that $(R, \partial Q)$ has the homotopy type of a smooth manifold with boundary since it consists of 2-handles attached to a disk. Thus we can apply the Whitney Theorem to make f_1 an embedding.

Remark. One can use these theorems to develop surgery for Poincaré complexes. Details of a slightly different approach will appear elsewhere [11].

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