Werk

Titel: 9. Linaer Systems: Some Examples.

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Proof. Suppose f and g are local equations at x of two members of $\mathscr{S}(\mathscr{L})$. Then (as above) $\exists\, n \ni$

$$f^{n-1}, g^{n-1} \in (\mathscr{B}_n)_x.$$

In the expansion of $(f+g)^{2n-1}$, each term contains a factor of f^{n-1} or g^{n-1}, the other factor being a product of n f's or g's. Hence

$$(f+g)^{2n-1} \in (\mathscr{B}_{2n})_x,$$

so that (by Corollary 8.5) the divisor corresponding to $f+g$ belongs to $\mathscr{S}(\mathscr{L})$. Hence $\mathscr{S}(\mathscr{L})$ is linear.

Next, suppose $D, D' \in \mathscr{C}(\mathscr{L})$, and suppose D'' is in the pencil spanned by them. If C is any curve on X, we must show C meets D''. Suppose not. Then

$$(D.C)=(D'.C)=(D''.C)=0,^5$$

so that by hypothesis we would have $C \subset D$ and $C \subset D'$. But then we would have $C \subset D''$ as well, contradiction.

Finally, the linearity of $\mathscr{A}(\mathscr{L})$ results from that of $\mathscr{S}(\mathscr{L})$ and $\mathscr{C}(\mathscr{L})$, by the use of Proposition 3.10, which says precisely that

$$\mathscr{A}(\mathscr{L}) = \mathscr{S}(\mathscr{L}) \cap \mathscr{C}(\mathscr{L}).$$

Remark 8.10. The linearity of $\mathscr{S}(\mathscr{L})$ is related, by virtue of Corollary 3.6, to the following fact, mentioned (without proof) in [14]: If $D_1, \ldots, D_r$ span $|D|$ on a complete variety X and $\Gamma(X - \operatorname{Supp} D_i)$ is a finite k-algebra $\forall\, i$, then so is $\Gamma(X - \operatorname{Supp} D')\ \forall\, D' \in |D|$.

9. Linear Systems: Some Examples

Proposition 9.1. *If X is a complete variety, D a divisor on X, $V = X - \operatorname{Supp} D$, and A a k-subalgebra of $\Gamma(V)$, then V proper over* $\operatorname{Spec} A \Rightarrow \forall\, x \in \operatorname{Supp} D\ \exists\, n \in \mathbb{Z}_+$ *and* $D' \in \mathscr{L}_n^A$ *(the linear subsystem of $|nD|$ determined by $A \cap L(nD)$) such that*

$$\operatorname{ord}_x D' < \operatorname{ord}_x (nD).$$

Proof. As in the proof of (ii) $\Rightarrow$ (iii) of Theorem 8.4, we have

$$f_x^{-1} = \sum_i h_i g_i,$$

where f_x is a local equation for D at x, $h_i \in \mathcal{O}_{X,x}$, and $g_i \in A$. For some $n > 0$, $g_i \in A_n\ \forall\, i$ (hence $g_i f_x^n \in \mathcal{O}_{X,x}$), so multiplying by f_x^n and noting that

[5] The symbol $(.)$ can be taken to be, for example, the Snapper-Kleiman intersection number; see [9] for a discussion of its properties.

$\operatorname{ord}_x h_i \geqq 0 \ \forall\, i$, we see that

$$\operatorname{ord}_x(g_{i_0} f_x^n) \leqq \operatorname{ord}_x(f_x^{n-1}) < \operatorname{ord}_x(f_x^n)$$

for some i_0. Thus if we let $D' = nD + (g_{i_0})$, the conclusion follows.

In order to see that the converse is false (even if we assume nD can be moved, for each n, to a *smooth* divisor), we first prove the following proposition, which exhibits a situation where semi-affineness implies base-point freeness:

Proposition 9.2. *Let Y be a complete variety, $\mathscr{L}$ an invertible sheaf on Y (possibly without sections), φ: $X \to Y$ the corresponding projectively completed line bundle (i.e.*

$$X = \operatorname{Proj}(\mathscr{O}_Y \oplus \mathscr{L}),$$

and Y_0 (resp. Y_∞) the divisor on X corresponding to the 0- (resp. ∞-) section. Then the following are equivalent:

i) *For some $n > 0$, $\mathscr{L}^{\otimes n}$ is generated by its sections;*

ii) *For some $n > 0$, $|n Y_0|$ is base-point free;*

iii) *$X - Y_0$ is semi-affine.*

Proof. (i) $\Rightarrow$ (ii) We may suppose $\mathscr{L} = \mathscr{O}_Y(D)$, with D a (possibly non-effective) divisor on Y. Then

$$Y_0 \sim Y_\infty + \varphi^*(D),$$

so that

$$n Y_0 \sim n Y_\infty + \varphi^*(nD).$$

If $x \in Y_0$, there is a divisor $D' \sim nD$ with $x \notin \operatorname{Supp} D'$. Hence

$$x \notin \operatorname{Supp}\big(n Y_\infty + \varphi^*(D')\big),$$

which shows that $|n Y_0|$ has no base points.

(ii) $\Rightarrow$ (iii) Immediate.

(iii) $\Rightarrow$ (i) Note first that

$$\Gamma(X - Y_0) \simeq \coprod_{n \geqq 0} \Gamma(Y, \mathscr{L}^{\otimes n}),$$

the latter having a natural structure of graded ring. Now we must show that some $\mathscr{L}^{\otimes n}$ has no base points. Suppose $y \in Y$ and U is an affine neighborhood of y on which $\mathscr{L}$ is trivial. Then

$$f^{-1}(U) \simeq U \times \mathbb{P}^1,$$

and for an appropriate choice of (inhomogeneous) coordinate t on $\mathbb{P}^1$ (extended to $U \times \mathbb{P}^1$) we may assume Y_0 and Y_∞ are defined by $t = 0$ and

$t=\infty$ (resp.) in $\varphi^{-1}(U)$. Then every element of $\Gamma(X-Y_0)$ has the form

$$g=\sum_{i=0}^{m} f_i t^{-i}$$

in $\varphi^{-1}(U)-Y_0$, with $f_i \in \mathscr{B}_i$ (the base locus ideal of $\mathscr{L}^{\otimes i}$ in U). If we had $y \in \mathscr{B}_i\ \forall\, i>0$, then the restriction of g to the fibre $\{y\}\times\{t\neq 0\}$ would be constant for each g, which (by Proposition 1.10) would contradict the assumption that $X-Y_0$ is semi-affine. Hence $\forall\, y\in Y\ \exists\, n(y) \ni y \notin$ base locus of $\mathscr{L}^{\otimes n(y)}$. It follows, by quasi-compactness, that there is such an n independent of y.

Remark 9.3. The completeness of Y is not needed in (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i), and can be replaced by "Y semi-affine" in (ii) $\Rightarrow$ (iii).

Corollary 9.4. *Let Y be a semi-affine variety and $\mathscr{L}$ an invertible sheaf on Y. If $\mathscr{L}^{\otimes n}$ is generated by its sections on Y for some $n>0$, then the ring*

$$\coprod_{n\geqq 0} \Gamma(Y, \mathscr{L}^{\otimes n})$$

is a finitely generated k-algebra.

Proof. With the notation of Proposition 9.2, this ring is $\Gamma(X-Y_0)$. The result then follows from that proposition (and Remark 9.3), together with Corollary 3.6.

Remark 9.5. If $\mathscr{L}=\mathscr{O}(D)$ for a divisor D on Y, then the above ring is precisely the ring $R^*[D]$ of [15], where Zariski proves the corresponding result for Y a complete nonsingular surface (actually his proof works in higher dimensions as well)—cf. [15, proof of Theorem 6.5].

Now let us recall Zariski's example, worked out in [2], of a complete nonsingular variety Y and an open affine subset of Y whose complement is the support of a prime divisor D' with the property that the base scheme of $|nD'|$ $(n=1, 2, \ldots)$ is an integral curve, and each $|nD'|$ contains a smooth divisor. (Y is obtained by blowing up 12 points in general position on a nonsingular cubic lying in a plane contained in $\mathbb{P}^3$, and D' is the proper transform of a general quartic surface passing through the cubic curve.) Applying the above proposition to this example, we get:

Corollary 9.6. *There exists a smooth, complete variety X, and a divisor D on X, such that each $|nD|$ contains a smooth divisor, yet $X-\operatorname{Supp} D$ is not semi-affine. (Hence the converse of Proposition 9.1 is false.)*

Proof. Let Y and D' be as above, let $\mathscr{L}=\mathscr{O}_Y(D')$, φ, X, Y_0, and Y_∞ be as in Proposition 9.2, and let $D=Y_0$. Then since $|nD'|$ has base points for each n, $X-\operatorname{Supp} D$ cannot be semi-affine, by the proposition. On the

other hand, X is certainly smooth and complete (since Y is), and since

$$D \sim \varphi^*(D') + Y_\infty,$$

it follows (by the second theorem of Bertini and the fact that each nD' can be moved so as to pass smoothly through its base locus) that each $|nD|$ contains a smooth divisor.

Example 9.7. Let $X = \mathbb{P}_k^3$, Q = a nonsingular quadric in X, L = a line in Q, and $P_1, \ldots, P_r$ = distinct points of L. Let $\varphi: \bar{X} \to X$ be the blow-up of $P_1, \ldots, P_r$, let $E_i = \varphi^*(P_i)$ $(i = 1, \ldots, r)$, and let $\bar{Q}$ and $\bar{L}$ be the proper transforms of Q and L (resp.). Then

i) $(\bar{Q}.\bar{L}) = 2 - r$,

ii) $(\bar{Q}^3) = 8 - r$,

iii) $\bar{X} - \bar{Q}$ is affine.

Proof. (i) and (ii) follow from the fact that the intersection number drops by 1 for each point that is blown up, together with the observation that $(Q.L) = 2$ and $(Q^3) = 8$.

(iii) We first show that $\bar{X} - \bar{Q}$ contains no complete curve. Suppose $\bar{C}$ were a complete (intergral) curve in $\bar{X} - \bar{Q}$. Since each E_i is a projective plane which meets $\bar{Q}$, $E_i - \bar{Q}$ is affine, hence $\bar{C}$ cannot lie entirely in any E_i. Hence

$$(\bar{C}.E_i) = s_i \geqq 0 \qquad \forall\, i,$$

and if $C = \varphi(\bar{C})$ then

$$C.Q = \sum s_i P_i,$$

since C meets Q transversally at each P_i. It follows that

$$(C.Q) = \sum s_i.$$

Since $C \neq L$, there is a plane H in X containing L but not C. Then we have

$$\sum s_i = (C.Q) = 2(C.H) \geqq 2\left(\sum s_i\right),$$

or $\sum s_i \leqq 0$, so that $s_i = 0\ \forall\, i$. Consequently $C \subset X - Q$, which is impossible. Hence $\bar{X} - \bar{Q}$ contains no complete curve.

It now remains to show that $\bar{X} - \bar{Q}$ is semi-affine, for which we use the criterion of Corollary 8.7. More precisely, we will show that if $\mathscr{B}_n$ is the base locus ideal sheaf of the system $|n\bar{Q}|$, for $n \geqq 1$, then

a) $r \leqq 2 \Rightarrow \mathscr{B}_1 = \mathcal{O}_{\bar{X}}$, i.e. $\bar{Q}$ is base-point free,

b) $r \geqq 3 \Rightarrow \mathscr{B}_1 =$ ideal sheaf of $\bar{L}$,

c) $r \geqq 3 \Rightarrow \mathscr{B}_{r-1} \supset \mathscr{B}_1^{r-2}$.[6]

[6] It is not difficult to show that $\mathscr{B}_{r-1} = \mathscr{B}_1^{r-2}$, but this fact is not needed in the present argument.

In particular, then, $\bar{Q}$ has an affine complement by Corollary 8.7, taking $\mathscr{L}=|\bar{Q}|$ and

$$n=\begin{cases} 1 & \text{if } r\leqq 2 \\ r-1 & \text{if } r\geqq 3 \end{cases}$$

in the statement of the corollary.

a) First observe that for any value of r the system $|n\bar{Q}|$ corresponds, on X, to the system of all $2n$-ics S which have multiplicity (at least) n at each P_i. (The correspondence carries S on X into $\varphi^* S-\sum n E_i$ on $\bar{X}$.) Now suppose $r=2$, and $\bar{P}\in\bar{X}$. If $\bar{P}\notin E_1\cup E_2$ then $\varphi(\bar{P})=P$, with $P\neq P_i$ $(i=1,2)$. We can therefore find planes A_1 and A_2 through P_1 and P_2 (resp.) with $P\notin A_1\cup A_2$. Then the proper transform $\overline{A_1+A_2}\in|\bar{Q}|$, so that $\bar{P}$ is not a base point of $|\bar{Q}|$. If, on the other hand, $\bar{P}\in E_1$ (say), then it corresponds to some direction (or line) through P_1. Then choosing A_1 to contain P_1 but not that line, and A_2 to contain P_2 but not P_1, shows that again $\overline{A_1+A_2}$ works. If $r=1$ the argument is similar.

b) Suppose $r\geqq 3$. Then, by (i), $(\bar{Q}.\bar{L})<0$, so that $\bar{L}$ must lie in the base locus of $|n\bar{Q}|$ $\forall\, n\geqq 1$. On the other hand, reasoning such as in the proof of (a) (choosing one of the planes, however, to contain L) shows that no other point of $\bar{X}$ can be a base point of $|\bar{Q}|$ (hence of $|n\bar{Q}|$ for any $n\geqq 1$). We must show that the base locus is $\bar{L}$ as a *scheme*, however. Suppose first that $\bar{P}\in\bar{L}$, $\bar{P}\notin\bigcup E_i$. Then if x, y, z are inhomogeneous coordinates for X such that $L: x=y=0$ and $P: (0,0,0)$, we can take x, y, z as local coordinates at $\bar{P}$ as well, and the ideal of $\bar{L}$ at $\bar{P}$ is (x, y). Now $x(z-1)=0$ and $y(z-1)=0$ are quadrics in X containing L, hence giving elements of $|\bar{Q}|$. The local equations of the latter at $\bar{P}$ are also $x(z-1)$ and $y(z-1)$, and the ideal they generate in $\mathcal{O}_{\bar{X},\bar{P}}$ is (x, y), since $z-1$ is a unit at $\bar{P}$. Hence we are done in this case. On the other hand, suppose $\bar{P}=\bar{L}\cap E_1$ (say). This time, if x, y, z are coordinates for X such that $L: x=y=0$ and $P_1: (0,0,0)$, then x/z, y/z, z can be taken as coordinates at $\bar{P}$ on $\bar{X}$, and the ideal of $\bar{L}$ at $\bar{P}$ is $(x/z, y/z)$. But again $x(z-1)=0$ and $y(z-1)=0$ are quadrics in X through L, hence giving elements of $|\bar{Q}|$, and now the local equations of these quadrics at $\bar{P}$ are

$$\frac{x}{z}(z-1) \quad \text{and} \quad \frac{y}{z}(z-1),$$

which generate the ideal $(x/z, y/z)$ in $\mathcal{O}_{\bar{X},\bar{P}}$ (since $z-1$ is again a unit at $\bar{P}$). This completes the proof of (b).

c) Let us treat only the case $\bar{P}=\bar{L}\cap E_1$, since $\bar{P}\in\bar{L}-\bigcup E_i$ is similar and even simpler. Again choosing coordinates x, y, z such that $L: x=y=0$ and $P_1: (0,0,0)$, we may assume that $P_2, \ldots, P_r$ are at finite distance, and

have z-coordinates $t_2, \ldots, t_r$ (resp.). If $v \in \mathbb{Z}$, $0 \leqq v \leqq r-2$, the surface

$$x^v y^{r-2-v} z \prod_{i=2}^{r} (z - t_i) = 0$$

has degree $2r-2$ and has multiplicity $r-1$ at each P_i, hence gives an element of $|(r-1)\bar{Q}|$ whose local equation at $\bar{P}$ is

$$\left(\frac{x}{z}\right)^v \left(\frac{y}{z}\right)^{r-2-v} \prod_{i=2}^{r} (z - t_i) = 0$$

(since we must remove $r-1$ copies of E_1, whose local equation is $z=0$). But $\prod_{i=2}^{r} (z - t_i)$ is a unit at $\bar{P}$, so that $(\mathscr{B}_{r-1})_{\bar{P}}$ contains $(x/z)^v (y/z)^{r-2-v}$, and therefore contains $(\mathscr{B}_1^{r-2})_{\bar{P}}$, which is generated by these monomials for $v = 0, \ldots, r-2$.

Corollary 9.8. *There exists a nonsingular variety of some dimension n with an open affine subset whose complement supports no divisor D such that $(D^n) \geqq 0$,*[7] *in particular no pseudoample divisor.*

Proof. Take $r \geqq 9$, and use property (ii) of Example 9.7. (Of course, $r \geqq 3$ suffices for the non-pseudoample statement, by property (i).)

Remark 9.9. Recall that the following condition on a divisor D is called "almost base-point free" in [2, p. 177]:

(*) $\quad \forall\, x,\ \varepsilon > 0, \quad \exists\, n(x, \varepsilon) > 0,\ D_n \in |nD| \ni \operatorname{ord}_x(D_n) < n\varepsilon.$

Corollary 9.10. *The condition (*) is neither necessary nor sufficient for a divisor D on a smooth complete variety to have a semi-affine complement.*

Proof. The fact that (*) is not necessary is a consequence of Corollary 9.8, in view of the fact that (*) $\Rightarrow D$ is pseudoample (see [2], Corollary, p. 178). On the other hand, (*) is not sufficient by virtue of Corollary 9.6, which is an even stronger statement.

Corollary 9.11. *With the same notation as in Proposition 9.2, the following are equivalent:*

i) *$\mathscr{L}$ is torsion on Y (i.e., for some $n > 0$, $\mathscr{L}^{\otimes n} \simeq \mathscr{O}_Y$).*

ii) *For some $n > 0$, $n Y_0 \sim n Y_\infty$ on X.*

iii) *Both $X - Y_0$ and $X - Y_\infty$ are semi-affine.*

iv) *$X - Y_0 - Y_\infty$ is semi-affine.*

Proof. (i) $\Rightarrow$ (ii) As before,

$$n Y_0 \sim n Y_\infty + \varphi^*(nD)$$

for $\mathscr{L} = \mathscr{O}_Y(D)$. Since $nD \sim 0$, (ii) follows.

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[7] This answers a question of Mumford, transmitted to one of the authors by Kleiman.