Proof. The kernel of $\pi_*|\mathfrak{e}_i$ is $\mathfrak{e}_i \cap \mathfrak{d} = \mathfrak{e}_i \cap (\mathfrak{d}_1 + \mathfrak{d}_2)$. But since $\mathfrak{d}_1 \subseteq \mathfrak{e}_1$ the kernel of $\pi_*|\mathfrak{e}_1$ is $\mathfrak{d}_1 + \mathfrak{e}_1 \cap \mathfrak{d}_2$. But $\mathfrak{e}_1 \cap \mathfrak{d}_2 \subseteq \mathfrak{e}_1 \cap \mathfrak{e}_2 = \mathfrak{d}_1 \cap \mathfrak{d}_2 \subseteq \mathfrak{d}_1$ by Lemma I.6.7. Thus $\mathfrak{d}_1$ is the kernel of $\pi_*|\mathfrak{e}_1$. Similarly $\mathfrak{d}_2$ is the kernel of $\pi_*|\mathfrak{e}_2$. But now since $\mathfrak{e} = \mathfrak{e}_1 + \mathfrak{e}_2$ (see (I.6.12)) one has $\mathfrak{e}/\mathfrak{d} = \mathfrak{e}_1/\mathfrak{d}_1 + \mathfrak{e}_2/\mathfrak{d}_2$. But $\mathfrak{e}_1/\mathfrak{d}_1 = (\mathfrak{e}_1 + \mathfrak{d})/\mathfrak{d} = (\mathfrak{e}_1 + \mathfrak{d}_2)/\mathfrak{d}$. Similarly $\mathfrak{e}_2/\mathfrak{d}_2 = (\mathfrak{e}_2 + \mathfrak{d}_1)/\mathfrak{d}$. Thus $\mathfrak{e}_1/\mathfrak{d}_1 \cap \mathfrak{e}_2/\mathfrak{d}_2 = [(\mathfrak{e}_1 + \mathfrak{d}_2) \cap (\mathfrak{e}_2 + \mathfrak{d}_1)]/\mathfrak{d}$. However $(\mathfrak{e}_1 + \mathfrak{d}_2) \cap (\mathfrak{e}_2 + \mathfrak{d}_1) = \mathfrak{d}$ by Lemma I.6.4. Thus the sum is direct.

But now since $\mathfrak{e}_1 \subseteq \mathfrak{a}$ and $\mathfrak{e}_2 \subseteq \mathfrak{a}^\perp$ it follows that $\mathfrak{e}_1/\mathfrak{d}_1$ and $\mathfrak{e}_2/\mathfrak{d}_2$ are orthogonal relative to $\hat{B}_g$. On the other hand since $(\mathfrak{e}_i/\mathfrak{d}_i)_{\mathbf{C}} = \mathfrak{h}_i/(\mathfrak{d}_i)_{\mathbf{C}} \oplus \bar{\mathfrak{h}}_i/(\mathfrak{d}_i)_{\mathbf{C}}$ it follows that $\mathfrak{e}_i/\mathfrak{d}_i$ is stable under j for $i = 1, 2$ so that $\mathfrak{e}_1/\mathfrak{d}_1$ and $\mathfrak{e}_2/\mathfrak{d}_2$ are orthogonal with respect to S_g.

Chapter II
Polarizations of Solvable Lie Groups

II.1. Algebraic Preliminaries

Henceforth, unless stated otherwise, G will be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. We will use $\mathfrak{n}$ to denote the nil-radical of $\mathfrak{g}$, i.e., the maximal nilpotent ideal of $\mathfrak{g}$. If $\mathfrak{r}$ is any subalgebra of $\mathfrak{g}$ and R is the connected subgroup of G with Lie algebra $\mathfrak{r}$ then it is well-known that R is a closed, simply connected subgroup of G. We will use N to denote the subgroup corresponding to the nil-radical $\mathfrak{n}$. Let $g \in \mathfrak{g}'$ and let B_g denote the alternating bilinear form on $\mathfrak{g}$ defined by g. Similarly if $f \in \mathfrak{n}'$, $\mathfrak{n}'$ the dual vector space to $\mathfrak{n}$, B_f will denote the alternating bilinear form that f defines on $\mathfrak{n}$. Now let $f = g|\mathfrak{n}$. Then $f \in \mathfrak{n}'$. It is clear that then

$$B_g|\mathfrak{n} = B_f.$$

Since $\mathfrak{n}$ is an ideal in $\mathfrak{g}$, $\mathfrak{n}$ is stable under $\mathrm{Ad}\, G$. Hence the contragredient representation of G induces an action of G on $\mathfrak{n}'$ and so we may consider $\mathfrak{n}'$ as a G-module. Now $f \in \mathfrak{n}'$ is given by $f = g|\mathfrak{n}$. Thus if G_f is the isotropy group of G at f then

$$G_g \subseteq G_f. \tag{II.1.1}$$

Moreover, restricting the action of G on $\mathfrak{n}'$ to N is the coadjoint representation of N so that

$$G_f \cap N = N_f \tag{II.1.2}$$

where N_f is the isotropy group of N at f. Since N is simply connected it is easily seen that N_f is connected.

Let us return to the notation of Section I.6 and recall that if $\mathfrak{n}\subseteq\mathfrak{g}$ we define $\mathfrak{n}^0$ as the orthocomplement of $\mathfrak{n}$ in $\mathfrak{g}$ relative to B_g.

Proposition II.1.1. *Let $\mathfrak{g}_f$ be the Lie algebra of G_f and let $\mathfrak{n}^0$ be as defined above. Then $\mathfrak{g}_f=\mathfrak{n}^0$. Explicitly*

$$\mathfrak{g}_f=\{x\in\mathfrak{g}\,|\,\langle g,[y,x]\rangle=0 \text{ for all } y\in\mathfrak{n}\}.$$

Proof. Since $\mathfrak{n}$ is the nil-radical of $\mathfrak{g}$, $[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{n}$ and we have

$$\langle g,[y,x]\rangle=\langle f,[y,x]\rangle \qquad x, y\in\mathfrak{g}.$$

Thus, the above is true for $x\in\mathfrak{g}$, $y\in\mathfrak{n}$. Thus $\langle g,[y,x]\rangle=0$ for all $y\in\mathfrak{n}$ if and only if $x\cdot f=0$; that is, if and only if $x\in\mathfrak{g}_f$.

Note that Proposition II.1.1 implies that $\mathfrak{n}^0$ is a subalgebra of $\mathfrak{g}$. Now let $\mathfrak{a}=\mathfrak{n}^0+\mathfrak{n}=\mathfrak{g}_f+\mathfrak{n}$ as in Section I.6. Since $[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{n}$, $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, let $\tilde{a}\subseteq\mathfrak{g}'$ be the space of all linear functionals that vanish on $\mathfrak{a}$.

Lemma II.1.2. *Let all notation be as above. If $N_g=N\cap G_g\subseteq N_f$ then the correspondence $a\to a\cdot g$ induces a bijection*

$$N_f/N_g\to g+\tilde{\mathfrak{a}}.$$

Note that

(II.1.3) $$g+\tilde{a}=\{k\in\mathfrak{g}'\,|\,k|\mathfrak{a}=g|\mathfrak{a}\}.$$

Proof. Let $\mathfrak{n}_f$ and $\mathfrak{n}_g$ be the Lie algebras of N_f and N_g respectively. Then $\mathfrak{n}_f=\mathfrak{g}_f\cap\mathfrak{n}=\mathfrak{n}^0\cap\mathfrak{n}$ and $\mathfrak{n}_g=\mathfrak{g}_g\cap\mathfrak{n}=\mathfrak{g}^0\cap\mathfrak{n}$. We next observe that the correspondence $x\to x\cdot g$, $x\in\mathfrak{n}_f$, induces an exact sequence

(II.1.4) $$0\to\mathfrak{n}_g\to\mathfrak{n}_f\to\tilde{\mathfrak{a}}\to 0.$$

Using the notation of Section I.6 one has

$$\mathfrak{a}^0=(\mathfrak{n}+\mathfrak{n}^0)^0=\mathfrak{n}^0\cap(\mathfrak{n}^0)^0.$$

But $(\mathfrak{n}^0)^0=\mathfrak{g}^0+\mathfrak{n}$. Thus $\mathfrak{a}^0=\mathfrak{n}^0\cap(\mathfrak{g}^0+\mathfrak{n})$. Since $\mathfrak{g}^0\subseteq\mathfrak{n}^0$, one has $\mathfrak{a}^0=\mathfrak{g}^0+\mathfrak{n}\cap\mathfrak{n}^0$. Thus one has

(II.1.5) $$\mathfrak{a}^0=\mathfrak{g}_g+\mathfrak{n}_f.$$

Hence $\mathfrak{n}_f$ is orthogonal to $\mathfrak{a}$ relative to B_g. This implies, however, that $\mathfrak{n}_f\cdot g\subseteq\tilde{\mathfrak{a}}$ so that $x\cdot g\in\tilde{\mathfrak{a}}$ for all $x\in\mathfrak{n}_f$. But, since $x\cdot g=0$ if and only if $x\in\mathfrak{g}_g$, to prove (II.1.4) it suffices to show that the mapping $\mathfrak{n}_f\to\tilde{\mathfrak{a}}$ is surjective. We need only to show that the real dimensions of $\mathfrak{n}_f/\mathfrak{n}_g$ and $\tilde{\mathfrak{a}}$ agree. Now if dim( ) denotes here the real dimension of the vector space in the bracket, we have by (II.1.5) that

(II.1.6) $$\dim\mathfrak{a}^0=\dim\mathfrak{g}_g+\dim\mathfrak{n}_f/\mathfrak{n}_g.$$

On the other hand, since $\mathfrak{g}_g \subseteq \mathfrak{a}$ one has

$$\dim \mathfrak{a} + \dim \mathfrak{a}^0 = \dim \mathfrak{g} + \dim \mathfrak{g}^0.$$

Thus by (II.1.6) and the fact that $\mathfrak{g}^0 = \mathfrak{g}_g$ we have

$$\dim \mathfrak{g} - \dim \mathfrak{a} = \dim \mathfrak{n}_f/\mathfrak{n}_g.$$

However, $\dim \tilde{\mathfrak{a}} = \dim \mathfrak{g} - \dim \mathfrak{a}$ so that $\dim \mathfrak{n}_f/\mathfrak{n}_g = \dim \tilde{\mathfrak{a}}$. This proves (II.1.4).

But now since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n} \subseteq \mathfrak{a}$ it follows that the elements of $\tilde{\mathfrak{a}}$ vanish on $[\mathfrak{g}, \mathfrak{g}]$ so that the elements of $\tilde{\mathfrak{a}}$ are fixed under the coadjoint representation. Hence if $x \in \mathfrak{n}_f$ one has

$$\exp x \cdot g = g + x \cdot g. \tag{II.1.7}$$

Then, since the exponential mapping in N is a bijection, the correspondence $a \to a \cdot x$ for $a \in N_f$ induces an injection $N_f/N_g \to g + \tilde{\mathfrak{e}}$. However the mapping is a surjection, since if $k \in \tilde{\mathfrak{a}}$ there exists $x \in \mathfrak{n}_f$ such that $x \cdot g = k$. Hence $\exp x \cdot g = g + k$ by (II.1.7).

Now let $e = g | \mathfrak{g}_f \in \mathfrak{g}'_f$. We will now determine the isotropy group $(G_f)_e$ at e for the coadjoint representation of G_f. Begin by observing that N_f is normal in G_f since $[\mathfrak{g}_f, \mathfrak{g}_f] \subseteq \mathfrak{n} \cap \mathfrak{g}_f = \mathfrak{n}_f$. Thus $G_g N_f$ is a well-defined subgroup of G_f.

Proposition II.1.3. *Let all notation be as above. Then*

$$(G_f)_e = G_g N_f.$$

Proof. If $a \in G_f$ then $(a \cdot g) | \mathfrak{g}_f = a \cdot e$. Thus $G_g \subseteq (G_f)_e$. But if $a \in N_f$ then $a \cdot g - g$ is orthogonal to $\mathfrak{a}$ by Lemma II.1.2, and hence $a \cdot g - g$ is orthogonal to $\mathfrak{g}_f$. Thus

$$0 = (a \cdot g - g) | \mathfrak{g}_f = a \cdot e - e$$

so that $G_g N_f \subseteq (G_f)_e$. Conversely, let $b \in (G_f)_e$. Then

$$0 = b \cdot e - e = (b \cdot g - g) | \mathfrak{g}_f.$$

However $(b \cdot g - g) | \mathfrak{n} = b \cdot f - f = 0$ since $b \in G_f$. Thus

$$(b \cdot g - g) | \mathfrak{a} = 0.$$

But then by Lemma II.1.2 there exists $a \in N_f$ such that $a \cdot g = b \cdot g$. Hence $a^{-1} b = c \in G_g$. Then $b = a c \in N_f G_g = G_g N_f$.

We are now going to examine the three subalgebras $\mathfrak{n}$, $\mathfrak{g}_f$ and $\mathfrak{a} = \mathfrak{n} + \mathfrak{n}^0$ of $\mathfrak{g}$. The subgroup corresponding to $\mathfrak{g}_f$ is the identity component $(G_f)_0$ of G_f. As $G_g \subseteq G_f$ the subgroup $M = G_g (G_f)_0$ is closed because

$$(G_f)_0 \subseteq M \subseteq G_f. \tag{II.1.8}$$

Of course, if $\mathfrak{m}$ is the Lie algebra of M one has

(II.1.9)
$$\mathfrak{m}=\mathfrak{g}_f=\mathfrak{n}^0.$$

Now let A_0 be the connected (and hence closed) subgroup corresponding to the ideal $\mathfrak{a}$. Clearly $A_0=(G_f)_0 N$. But $G_f N$ is closed because the orbit $N\cdot f\subseteq\mathfrak{n}'$ is closed, since N is nilpotent, and $G_f N$ is just the stabilizer of $N\cdot f$ with respect to the action of G on $\mathfrak{n}'$. But A_0 is the identity component of the closed group $G_f N$. Thus $A=MN=G_g A_0$ is closed since

$$(G_f N)_0\subseteq A\subseteq G_f N.$$

We now set $g|\mathfrak{n}=f$, $g|\mathfrak{m}=e$, and $g|\mathfrak{a}=k$.

Proposition II.1.4. *In the above notation*

$$M_e=G_g N_f, \quad A_k=G_g N_f.$$

Proof. By (II.1.8) one has that $M_e\subseteq(G_f)_e$. But $(G_f)_e=G_g N_f$ by Proposition II.1.3. But $G_g N_f\subseteq M$. Thus $G_g N_f\subseteq M_e$ and hence $G_g N_f=M_e$ and we have our first assertion.

To prove our second assertion we begin by noting that since $k|\mathfrak{n}=f$, $A_k\subseteq G_f$. But, also $k|\mathfrak{m}=e$ and hence $A_k\subseteq(G_f)_e$. Thus $A_k\subseteq G_g N_f$ by Proposition II.1.3. However, since $G_g\subseteq A$, we have that $G_g\subseteq A_k$. Now, by Lemma II.1.2, $N_f\subseteq A_k$. Thus $A_k=G_g N_f$.

We will now assume that $g\in\mathfrak{g}'$ is integral as defined in Definition I.5.2. Then there exists a character $\eta_g\colon G_g\to\pi$, said to correspond to g, such that

$$\frac{d}{dt}\eta_g(\exp tx)|_{t=0}=2\pi i\langle g,x\rangle \quad \text{for all } x\in\mathfrak{g}_g.$$

Proposition II.1.5. *Let $g\in\mathfrak{g}'$ be integral then the linear functionals e, f and k are integral. Moreover, there exists characters η_e on M_e, η_f on N_f and η_k on A_k corresponding to the linear functions e, f and k. η_f is uniquely determined by f and η_e and η_k are uniquely determined by the condition that they extend η_g and η_f.*

Proof. Since N_f is connected and simply connected, η_f exists and is unique. But now G_g normalizes N_f and since f is invariant under $G_g\subseteq G_f$, one has $\eta_f(aba^{-1})=\eta_f(b)$ for all $b\in N_f$, and $a\in G_g$. Thus (η_g,η_f) defines a character on the semi-direct product $G_g\times N_f$. Now $N_g=G_g\cap N_f$ is connected. Since $f|\mathfrak{n}_g=g|\mathfrak{n}_g$ where $\mathfrak{n}_g=\mathfrak{g}_g\cap\mathfrak{n}_f$ we have that $\eta_f|N_g=\eta_g|N_g$. Thus (η_g,η_f) is trivial on the kernel of the surjection

$$\tau\colon\ G_g\times N_f\to G_g N_f$$

defined by

$$\tau(a,b)=ab.$$

Thus (η_g,η_f) is uniquely of the form $\eta\circ\tau$, where η is a character on $G_g N_f$. The result then follows from Proposition II.1.3.

Let us consider further the structure of M.

Proposition II.1.6. *Assume* $g \in \mathfrak{g}'$ *and* $g \neq 0$. *Then* M_e *and* $\operatorname{Ker} \eta_e$ *are normal subgroups of* M, *so that* $\mathfrak{m}_e$ *and* $\mathfrak{p} = \operatorname{Ker} e | \mathfrak{m}_e$ *are ideals in* $\mathfrak{m}$. *(Here, of course,* $\mathfrak{m}_e = \mathfrak{g}_g + \mathfrak{n}_f$ *is the algebra of* M_e.*) Moreover* $M/\operatorname{Ker} \eta_e$ *is a connected Lie group with Lie algebra* $\mathfrak{m}/\mathfrak{p}$. *Furthermore* $\mathfrak{m}/\mathfrak{p}$ *is a Heisenberg Lie algebra with* $\mathfrak{m}_e/\mathfrak{p}$ *as the* 1*-dimensional center.*

Proof. Now since $G_f \cap N = N_f$ and since N_f is connected, one has $(G_f)_0 \cap N = N_f$. Thus

$$M \cap N = N_f.$$

On the other hand, if a prime denotes the commutator subgroup, one has $G' \subseteq N$ since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}$. Thus

$$M' \subseteq N_f$$

since $M' \subseteq M \cap N = N_f$. Thus if $a \in M_e$ and $b \in M$ one has $bab^{-1}a^{-1} = c \in N_f \subseteq M_e$. Hence $bab^{-1} = ca \in M_e$ so that M_e is normal. Now if $\mathfrak{q} = \operatorname{Ker} f | \mathfrak{n}_f$ then since $\mathfrak{g}_f \cap \mathfrak{n} = \mathfrak{n}_f$ is stable under $\operatorname{Ad} G_f$ and f is fixed by G_f, it follows that $\mathfrak{q}$ is stable under $\operatorname{Ad} G_f$ and hence under $\operatorname{Ad} M$. Thus $\mathfrak{q}$ is an ideal in $\mathfrak{m}$ and hence $Q \subseteq N_f$, the corresponding connected Lie group, is normal in M. Hence if M_0 is the identity component of M, then M_0/Q is a Lie group with Lie algebra $\mathfrak{m}/\mathfrak{q}$. Now let $a \in M_e$ and let $x \in \mathfrak{m}$. Since $\operatorname{Ad} a$ induces the identity operator on $\mathfrak{g}/\mathfrak{n}$ one has $a \cdot x - x \in \mathfrak{n} \cap \mathfrak{m} = \mathfrak{n}_f$. But $\langle f, a \cdot x - x \rangle = \langle e, a \cdot x - x \rangle = \langle e, a \cdot x \rangle - \langle e, x \rangle$. However $\langle e, a \cdot x \rangle = \langle a^{-1} \cdot e, x \rangle = \langle e, x \rangle$ since $a \in M_e$. Thus $a \cdot x - x \in \mathfrak{q}$. Hence $\operatorname{Ad} a$ induces the identity operator on $\mathfrak{m}/\mathfrak{q}$. But then conjugation by a induces the identity operator on M_0/Q so that for any $b \in M_0$ one has

$$bab^{-1}a^{-1} \in Q.$$

However, $Q \subseteq \operatorname{Ker} \eta_e$ so that $\eta_e(bab^{-1}) = \eta_e(a)$. But clearly this also holds if $b \in G_g$ since $G_g \subseteq M_e$. Since $M = G_g M_0$ one therefore has

$$\eta_e(bab^{-1}) = \eta_e(a)$$

for all $a \in M_e$, $b \in M$. Hence $\operatorname{Ker} \eta_e$ is normal in M.

Next note that $e | \mathfrak{m}_e \neq 0$. Indeed if not, then $e | \mathfrak{n}_f = f | \mathfrak{n}_f = 0$. However this implies $f = 0$ since $\mathfrak{n}$ is nilpotent. (One knows that if $0 \neq h \in \mathfrak{n}'$ then $h | \mathfrak{n}_h \neq 0$ (see [7]).) But if $f = 0$ then $g | [\mathfrak{g}, \mathfrak{g}] = 0$ so that $G_g = G$ and hence $\mathfrak{m}_e = \mathfrak{g}$. Consequently $e | \mathfrak{m}_e = g$. But $g \neq 0$ which is a contradiction. Thus $e | \mathfrak{m}_e \neq 0$. But since $e | \mathfrak{m}_e \neq 0$ the character η_e maps the identity component $(M_e)_0$ of M_e onto the unit circle $\mathbf{T}$. Thus given any $a \in G_g$ there exists $b \in (M_e)_0 \subseteq M_0$ such that $b^{-1} a \in \operatorname{Ker} \eta_e$. Thus $a \in b \operatorname{Ker} \eta_e$ or $G_g \subseteq M_0 \operatorname{Ker} \eta_e$. But then $M = G_g M_0 \subseteq M_0 \operatorname{Ker} \eta_e$ so that M_0 maps onto $M/\operatorname{Ker} \eta_e$ proving that $M/\operatorname{Ker} \eta_e$ is connected.

Now since $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{n} \cap \mathfrak{m} = \mathfrak{n}_f \subseteq \mathfrak{m}_e$, it follows that

$$\mathfrak{m}/\mathfrak{m}_e = (\mathfrak{m}/\mathfrak{p})/(\mathfrak{m}_e/\mathfrak{p})$$

is abelian. Now $\mathfrak{m}_e/\mathfrak{p}$ is 1-dimensional since $e|\mathfrak{m}_e \neq 0$. But by definition of $\mathfrak{m}_e$ one has $[\mathfrak{m}, \mathfrak{m}_e] \subseteq \operatorname{Ker} \mathfrak{e}$. Thus $[\mathfrak{m}, \mathfrak{m}_e] \subseteq \mathfrak{m}_e \cap \operatorname{Ker} e = \mathfrak{p}$ and hence $\mathfrak{m}_e/\mathfrak{p}$ is central in $\mathfrak{m}/\mathfrak{p}$. To prove the proposition therefore we have only to show that $\mathfrak{m}_e/\mathfrak{p}$ is exactly the center of $\mathfrak{m}/\mathfrak{p}$. However this follows immediately from the general fact that the alternating form B_e is non-singular on $\mathfrak{m}/\mathfrak{m}_e$. QED.

II.2. Admissible Polarizations

Recall that a polarization $\mathfrak{h}$ at g is, amongst other things, a maximal isotropic subspace (m.i.s.) of $\mathfrak{g}_{\mathbf{C}}$ relative to B_g.

Definition II.2.1. Let $g \in \mathfrak{g}'$ and let $\mathfrak{n}_{\mathbf{C}} = \mathfrak{n} + i\mathfrak{n} \subseteq \mathfrak{g}_{\mathbf{C}}$. A polarization $\mathfrak{h}$ at g is said to be admissible if $\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}$ is a m.i.s. of $\mathfrak{n}_{\mathbf{C}}$ relative to the alternating bilinear form B_g induces on $\mathfrak{n}_{\mathbf{C}}$.

Theorem II.2.1. *If $\mathfrak{h}$ is an admissible polarization at g then*

1) $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}$ *is a polarization at f for N;*

2) $\mathfrak{h}_2 = \mathfrak{h} \cap \mathfrak{m}_{\mathbf{C}}$ *is a polarization at e for M and*

$$\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2 \subseteq \mathfrak{a}_{\mathbf{C}};$$

3) $\mathfrak{h}$ *is a polarization at k for A.*

Further, $\mathfrak{h}$ is positive if and only if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are positive.

Proof. Since $B_f = B_g|\mathfrak{n}$, $\mathfrak{h}_1$ is a m.i.s. of $\mathfrak{n}_{\mathbf{C}}$ relative to B_f. Thus $\mathfrak{n}_f \subseteq \mathfrak{h}_1$. Also $\mathfrak{h}_1$ is a subalgebra of $\mathfrak{n}_{\mathbf{C}}$. Hence, since N_f is connected, $\mathfrak{h}_1$ is stable under $\operatorname{Ad} N_f$. But, also by Corollary I.6.8, if we let $e = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$ we obtain $\mathfrak{e}_1 = (\mathfrak{h}_1 + \bar{\mathfrak{h}}_1) \cap \mathfrak{n} = \mathfrak{e} \cap \mathfrak{n}$. Hence $\mathfrak{e}_1$ is a subalgebra of $\mathfrak{n}$ and hence $\mathfrak{h}_1$ is a polarization at f.

We again have that $B_e = B_g|\mathfrak{m}$ and that

$$\mathfrak{m}_{\mathbf{C}} = (\mathfrak{g}_f)_{\mathbf{C}} = (\mathfrak{n}^0)_{\mathbf{C}} = \mathfrak{n}_{\mathbf{C}}^{\perp}.$$

Thus $\mathfrak{h}_2 = \mathfrak{h} \cap \mathfrak{m}_{\mathbf{C}}$ is a subalgebra. Now by Proposition I.6.5, $\mathfrak{h}_2 = \mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}^{\perp}$ is a m.i.s. of $\mathfrak{n}_{\mathbf{C}}^{\perp}$. Further, since $\mathfrak{h}$ is a polarization at g, $\mathfrak{h}$ is stable under $\operatorname{Ad} G_g$. Thus $\mathfrak{h}_2$ is stable under $\operatorname{Ad} G_g$. Now, since $\mathfrak{n} \cap \mathfrak{n}^0 = \mathfrak{n}_f$ by Lemma I.6.7 we have that

$$(\mathfrak{h}_1 \cap \mathfrak{h}_2) \cap \mathfrak{g} = \mathfrak{n}_f. \tag{II.2.1}$$

Thus $\mathfrak{n}_f \subseteq \mathfrak{h}_2$ and hence $\mathfrak{h}_2$ is also stable under $\operatorname{Ad} N_f$. We may now apply Proposition II.1.3 to conclude that $\mathfrak{h}_2$ is stable under $\operatorname{Ad} M_e$. Finally, by Corollary I.6.8 (with $\mathfrak{m} = \mathfrak{n}^0$) $\mathfrak{e}_2 = (\mathfrak{h}_2 + \bar{\mathfrak{h}}_2) \cap \mathfrak{m}$ is a subalgebra, since $\mathfrak{e}_2 = \mathfrak{e} \cap \mathfrak{m}$. Hence $\mathfrak{h}_2$ is a polarization at $\mathfrak{e}$.

We will now apply Proposition I.6.10 with $\mathfrak{d}=\mathfrak{h}\cap\mathfrak{g}$, $\mathfrak{d}_1=\mathfrak{h}_1\cap\mathfrak{g}=\mathfrak{h}_1\cap\mathfrak{n}$, $\mathfrak{d}_2=\mathfrak{h}_2\cap\mathfrak{g}=\mathfrak{h}_2\cap\mathfrak{m}$ to conclude that

$$\mathfrak{e}/\mathfrak{d}=\mathfrak{e}_1/\mathfrak{d}_1\oplus\mathfrak{e}_2/\mathfrak{d}_2.$$

But $\hat{B}_f$ and S_f are just the restrictions of $\hat{B}_g$ and S_g to $\mathfrak{e}_1/\mathfrak{d}_1$. Similarly $\hat{B}_e$ and S_e are just the restrictions of $\hat{B}_g$ and S_g to $\mathfrak{e}_2/\mathfrak{d}_2$. We again apply Proposition I.6.10 to conclude that $\mathfrak{h}$ is positive if and only if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are positive.

Finally, since $\mathfrak{h}=\mathfrak{h}_1+\mathfrak{h}_2$, one has by Proposition I.6.5 that $\mathfrak{h}\subseteq\mathfrak{a}_{\mathbf{C}}=\mathfrak{n}_{\mathbf{C}}+\mathfrak{n}_{\mathbf{C}}^{\perp}$. Further, $\mathfrak{h}$ is a m.i.s. of $\mathfrak{a}_{\mathbf{C}}$ since it is a m.i.s. of $\mathfrak{g}_{\mathbf{C}}$. To prove assertion (3) of our theorem it remains only to show that $\mathfrak{h}$ is stable under $\operatorname{Ad} A_k$. Now, because $\mathfrak{h}$ is a polarization at g, $\mathfrak{h}$ is stable under $\operatorname{Ad} G_g$. Further, $\mathfrak{h}$ is stable under $\operatorname{Ad} N_f$, since $\mathfrak{n}_f\subseteq\mathfrak{h}$. Thus Proposition II.1.3 implies that $\mathfrak{h}$ is stable under $\operatorname{Ad} A_k$.

Remark II.2.2. It is a consequence of Theorem II.2.1 that an admissible polarization $\mathfrak{h}$ at g defines a polarization for the four groups G, N, M and $A=MN$. We will henceforth speak of the four polarizations defined by an admissible polarization at g.

First recall that the relationships between the corresponding four isotropy groups G_g, N_f, M_e and A_k is given by Proposition II.1.4. Let $(E_1)_0$ and $(E_2)_0$ be the connected subgroups of G corresponding to $\mathfrak{e}_1$ and $\mathfrak{e}_2$. Since $\mathfrak{e}=\mathfrak{e}_1+\mathfrak{e}_2$ and $[\mathfrak{e}_1,\mathfrak{e}_2]\subseteq\mathfrak{e}_1$ (because $[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{n}$; see Corollary 1.6.8) one clearly has $E_0=(E_1)_0(E_2)_0$ returning to the notation of § 1.4. But now the "E" groups corresponding to the four polarizations mentioned above, in the order mentioned above, are E, E_1, E_2, and E where $E=G_g E_0$, $E_1=N_f(E_1)_0$ and $E_2=M_e(E_2)_0$. Of course $E_1=(E_1)_0$ since $N_f\subseteq(E_1)_0$. (For the case of $k\in\mathfrak{a}'$ one notes that $A_k E_0$ is the "E" group. But $A_k=G_g N_f$ and $N_f\subseteq E_0$ so that $A_k E_0=E$.) But now E_2 normalizes E_1 since $A_k=M_e=G_g N_f$ normalizes $\mathfrak{e}$ ($\mathfrak{h}$ is a polarization at k) and hence M_e normalizes $\mathfrak{e}\cap\mathfrak{n}=\mathfrak{e}_1$. Now one clearly has

$$E=E_1 E_2.$$

We wish to show all four polarizations satisfy the Pukansky condition. For that we need the following general proposition about closed orbits. If π is a representation of a nilpotent Lie algebra $\mathfrak{n}$ on a finite dimensional vector space V then a vector $v\in V$ is called a zero weight vector for $\mathfrak{n}$ if for any $x\in\mathfrak{n}$, $\pi(x)^k v=0$ for some k.

Proposition II.2.3. *Let G be a connected solvable Lie group and let $\mathfrak{g}$ be its Lie algebra. Assume V is a real finite dimensional G (and hence $\mathfrak{g}$) module. Let $v\in V$ and assume $\mathfrak{g}$ can be written $\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2$ (not necessarily direct) where $\mathfrak{g}_1,\mathfrak{g}_2$ are nilpotent subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$,* *in such a way that v is a zero weight vector for both $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Then the orbit $G \cdot v \subseteq V$ is closed.*

Proof. Let V_1 be the space of all zero weight vectors for $\mathfrak{g}_1$ in V. Since $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$ it is clear that V_1 is stable under G and hence $G \cdot v \subseteq V_1$. Thus we may assume $V_1 = V$ or that every element of $\mathfrak{g}_1$ operates as a nilpotent operator.

Let π denote the representation of G (and also $\mathfrak{g}$) on V defining the module structure. Let $H \subseteq \operatorname{Aut} V$ denote the algebraic closure of $\pi(G_2)$ where G_i are the subgroups of G corresponding to $\mathfrak{g}_i$, $i = 1, 2$. Let $\mathfrak{h}$ be the Lie algebra of H so that one can write $\mathfrak{h}$ as a semi-direct sum $\mathfrak{h} = \mathfrak{s} + \mathfrak{n}$ where $\mathfrak{n}$, an ideal in $\mathfrak{h}$, is the set of all nilpotent operators in $\mathfrak{h}$ and $\mathfrak{s}$ is an abelian Lie algebra all of whose elements are semi-simple operators.

Of course $\pi(\mathfrak{g}_2) \subseteq \mathfrak{h}$ and for any $x \in \mathfrak{g}_2$ let $\pi(x)_s$ and $\pi(x)_n$ be the components of $\pi(x)$ in $\mathfrak{s}$ and $\mathfrak{n}$ respectively. Next observe that

$$\mathfrak{n} = \{\pi(x)_n \mid x \in \mathfrak{g}_2\}. \tag{II.2.2}$$

Indeed if $\mathfrak{n}_1$ is the subspace of $\mathfrak{n}$ given by the right side above then we assert $\mathfrak{n}_1$ is a Lie algebra. Indeed if $x, y \in \mathfrak{g}_2$ then

$$\begin{aligned}[\pi(x)_n, \pi(y)_n] &= [\pi(x) - \pi(x)_s, \pi(y) - \pi(y)_s] \\ &= \pi[x, y] - [\pi(x), \pi(y)_s] - [\pi(x)_s, \pi(y)].\end{aligned}$$

But $\pi[x, y] = \pi[x, y]_n$ since the elements in $\pi[\mathfrak{g}_2, \mathfrak{g}_2]$ are nilpotent. However since $\pi(\mathfrak{g}_2)$ is a Lie algebra it is stable under ad $\pi(\mathfrak{g}_2)$. But then since H is the algebra closure of $\pi(G_2)$ it follows that $\pi(\mathfrak{g}_2)$ is stable under ad $\mathfrak{h} \supseteq$ ad $\mathfrak{s}$. Thus $[\pi(x), \pi(y)_s] = \pi(z)$ for some $z \in \mathfrak{g}_2$. But $\pi(z) \in \mathfrak{n}$. That is, $\pi(z) = \pi(z)_n$ since $[\pi(x)_s, \pi(y)_s] = 0$ since $\mathfrak{s}$ is abelian. Thus $[\pi(x), \pi(y)_s] \in \mathfrak{n}_1$. Similarly $[\pi(x)_s, \pi(y)] \in \mathfrak{n}_1$. Thus $\mathfrak{n}_1$ is a Lie algebra and the argument above proves that $\mathfrak{n}_1$ is stable under ad $\mathfrak{s}$ so that $\mathfrak{s} + \mathfrak{n}_1$ is an algebraic Lie algebra. But clearly $\pi(\mathfrak{g}_2) \subseteq \mathfrak{s} + \mathfrak{n}_1$. Hence $\mathfrak{s} + \mathfrak{n}_1 = \mathfrak{s} + \mathfrak{n}$ so that $\mathfrak{n} = \mathfrak{n}_1$ proving (II.2.2). Now since $\pi(\mathfrak{g}_2)$ normalizes $\pi(\mathfrak{g}_1)$ it follows that $\mathfrak{h}$ and in particular $\mathfrak{n}$ normalizes $\pi(\mathfrak{g}_1)$ and hence if $\mathfrak{m} = \pi(\mathfrak{g}_1) + \mathfrak{n}$ then $\mathfrak{m}$ is a Lie subalgebra of End V. But if M is the corresponding connected Lie group then observe M is unipotent. Indeed first of all $\mathfrak{m}$ is clearly solvable. But since all the operators in $\pi(\mathfrak{g}_1)$ and in $\mathfrak{n}$ are nilpotent it follows from Lie's theorem that all elements in $\mathfrak{m}$ are nilpotent. Hence M is unipotent. But then $M \cdot v$ is closed. It suffices therefore to show that $M \cdot v = G \cdot v$. But now clearly $M = \pi(G_1)\, N$ where N is the subgroup of G corresponding to $\mathfrak{n}$. On the other hand $\pi(G) = \pi(G_1)\, \pi(G_2)$. Hence it suffices to show that $G_2 \cdot v = N \cdot v$. For this let $V_2 \subseteq V$ be zero weight space for $\mathfrak{g}_2$ so that $v \in V_2$. But since V_2 is stable under $\pi(\mathfrak{g}_2)$ it is stable under $\mathfrak{h}$. However $\mathfrak{s}$ vanishes on V_2 since $\pi(G_2)$ is clearly unipotent on V_2. But then the restriction $\pi(\mathfrak{g}_2)|V_2$ equals $\mathfrak{n}|V_2$ by (II.2.2). But then $\pi(G_2)|V_2=N|V_2$. Hence $G_2 \cdot V=N\cdot v$. QED.

For our applications it is convenient to know that we may slightly weaken the hypothesis in Proposition II.2.3.

Corollary II.2.4. *Let the assumptions be as in Proposition* II.2.3 *except that instead of assuming* $\mathfrak{g}_2$ *is nilpotent we assume that for any* $x\in\mathfrak{g}_2$ *one has* $\pi(x)^k v=0$ *for some* k. *Then the orbit* $G\cdot v$ *is closed.*

Proof. Let $\mathfrak{g}_2'$ be a Cartan subalgebra of $\mathfrak{g}_2$. Then $\mathfrak{g}_2'$ is nilpotent and clearly v is a zero weight vector for $\mathfrak{g}_2'$. But one has $\mathfrak{g}_2=\mathfrak{g}_2'+[\mathfrak{g}_2,\mathfrak{g}_2]$. Put $\mathfrak{g}_1'=\mathfrak{g}_1+[\mathfrak{g}_2,\mathfrak{g}_2]$.

But since all the elements in $\operatorname{ad}[\mathfrak{g}_2,\mathfrak{g}_2]$ are nilpotent on $\mathfrak{g}$ it follows that $\mathfrak{g}_1'$ is a nilpotent ideal in $\mathfrak{g}$ and v is a zero weight vector for $\mathfrak{g}_1'$. However $\mathfrak{g}=\mathfrak{g}_1'+\mathfrak{g}_2'$ so that by Proposition II.2.3 one has $G\cdot v$ is closed. QED.

The machinery we have discussed now applies in the case at hand because of

Proposition II.2.5. *If* $\mathfrak{h}$ *is an admissible polarization at* g *then the four polarizations it defines satisfy the Pukansky condition. That is* $E\cdot g$ *is closed in* $\mathfrak{g}'$, $E_1\cdot f$ *is closed in* $\mathfrak{n}'$, $E_2\cdot e$ *is closed in* $\mathfrak{m}'$ *and* $E\cdot k$ *is closed in* $\mathfrak{a}'$.

Proof. We recall $\mathfrak{e}=\mathfrak{e}_1+\mathfrak{e}_2$ where $\mathfrak{e}_1=\mathfrak{e}\cap\mathfrak{n}$ is clearly an ideal in $\mathfrak{e}$. Since $\mathfrak{e}_1\subseteq\mathfrak{n}$ it follows that $\mathfrak{e}_1$ is a nilpotent ideal in $\mathfrak{e}$ and g (resp. f) is a zero weight vector for $\mathfrak{e}_1$. Now observe that for $x\in\mathfrak{e}_2$ one has $x\cdot x\cdot g=0$ (respectively $x\cdot x\cdot e=0$, $x\cdot x\cdot k=0$). Indeed if $y\in\mathfrak{g}$ one has $\langle x\cdot x\cdot g, y\rangle=\langle g,[x,[x,y]]\rangle=\langle f,[x,[x,y]]\rangle$ since $[x,[x,y]]\in\mathfrak{n}$. But $\langle f,[x,[x,y]]\rangle=\langle x\cdot f,[x,y]\rangle$ since also $[x,y]\in\mathfrak{n}$. However since $x\in\mathfrak{e}_2\subseteq\mathfrak{g}_f$ one has $x\cdot f=0$. Thus $x\cdot x\cdot g=0$. Similarly for e and k replacing g. Thus Corollary II.2.4 applies. Note that the possible disconnectedness of the groups E, E_2 play no role by Remark I.5.1. QED.

II.3. Existence of Admissible Polarizations

This section will be devoted to showing that admissible polarizations exist. We begin with the following lemma.

Lemma II.3.1. *Let* N *be a simply connected nilpotent Lie group and let* $\mathfrak{n}$ *be its Lie algebra. Let* $\operatorname{Aut}\mathfrak{n}$ *be the group of all Lie algebra automorphisms of* $\mathfrak{n}$ *so that* $\operatorname{Ad}N$ *is a subgroup of* $\operatorname{Aut}\mathfrak{n}$. *Regard* $\operatorname{Aut}\mathfrak{n}$ *as operating by contragredience on the dual* $\mathfrak{n}'$. *Let* $f\in\mathfrak{n}'$. *Assume* F *is a group and a homomorphism* $F\to\operatorname{Aut}\mathfrak{n}$ *is given (so that* F *operates on* $\mathfrak{n}$ *and* $\mathfrak{n}'$*) such that,* (1) *the commutator subgroup* F' *maps into* $\operatorname{Ad}N$ *and,* (2) $F\cdot f=f$. *Then there exists a positive polarization* $\mathfrak{h}_1$ *at* f *which is stable under* F.

Proof. We assume inductively that the result is true for all simply connected nilpotent Lie groups of dimension smaller than $\dim\mathfrak{n}$.

Let $\mathfrak{k}=\operatorname{Ker} f\,|\,\text{center}\ \mathfrak{n}$ when this space has positive dimension. Clearly $\mathfrak{k}$ is an ideal in $\mathfrak{n}$ which is stable under F. Thus F operates on $\mathfrak{n}/\mathfrak{k}$ inducing a map $F\to \operatorname{Aut}\mathfrak{n}/\mathfrak{k}$ where $F'\to \operatorname{Ad} N/K$ if K is the subgroup corresponding to $\mathfrak{k}$. Moreover if $f_0\in(\mathfrak{n}/\mathfrak{k})'$ is induced by f then f_0 is fixed by F. Now by induction there exists $\mathfrak{h}_0\subseteq(\mathfrak{n}/\mathfrak{k})_{\mathbf{C}}$, a positive polarization at f_0 stable under F. But then $\pi^{-1}\mathfrak{h}_0=\mathfrak{h}$ is clearly a positive polarization at f stable under F, where $\pi:\mathfrak{n}\to\mathfrak{n}/\mathfrak{k}$ is the quotient map (indeed $\mathfrak{e}=\pi^{-1}\mathfrak{e}_0, \mathfrak{d}=\pi^{-1}\mathfrak{d}_0$ and $\mathfrak{e}/\mathfrak{d}\cong\mathfrak{e}_0/\mathfrak{d}_0$). Thus we are done in this case so that we may assume $\dim\mathfrak{k}=0$ and hence center $\mathfrak{n}$ is one-dimensional, spanned by an element z where $\langle f, z\rangle=1$. Since f is fixed by F clearly z is also fixed under the action of F. Let (z) denote linear span of z.

Now consider $\mathfrak{k}=\text{center}\ \mathfrak{n}/(z)$ so that $\mathfrak{k}=\mathfrak{k}_1/(z)$ where $\mathfrak{k}_1\subseteq\mathfrak{n}$ is an ideal. Clearly $\operatorname{Aut}\mathfrak{n}$ operates on $\mathfrak{n}/(z)$ and $\mathfrak{k}$ is clearly stable under the action of this group. However $\operatorname{Ad} N$ operates trivially on $\mathfrak{k}$ since $[\mathfrak{n},\mathfrak{k}_1]\subseteq\mathbf{R}z$. Thus the abelian group F/F' operates on $\mathfrak{k}$. Let $\mathfrak{p}\subseteq\mathfrak{k}$ be an irreducible subspace under the action of F/F' so that $\dim\mathfrak{p}$ is either 1 or 2. Now since $\langle f, z\rangle=1$ we may write $\mathfrak{k}_1=\mathfrak{k}_0\oplus\mathbf{R}z$ where $\mathfrak{k}_0=\operatorname{Ker} f\,|\,\mathfrak{k}_1$. Since f is fixed under F and $\mathfrak{k}_1$ is stable under F it follows that $\mathfrak{k}_0$ is stable under F and that if $\pi:\mathfrak{n}\to\mathfrak{n}/(z)$ is the quotient map then π induces an F-isomorphism $\mathfrak{k}_0\to\mathfrak{k}$. Let $\mathfrak{p}_0\subseteq\mathfrak{k}_0$ be the F-irreducible subspace corresponding to $\mathfrak{p}\subseteq\mathfrak{k}$. Note then that F' must operate trivially on $\mathfrak{k}_0$.

Case 1. Assume $\dim\mathfrak{p}_0=1$ so that $\mathfrak{p}_0=\mathbf{R}w$. In this case we proceed along the lines used by Kirillov [7]. That is, let $g\in\mathfrak{n}'$ be the linear functional defined by the relation $[y,w]=\langle g, y\rangle z$. One has $g\neq 0$ since otherwise w would be central in $\mathfrak{n}$ contradicting the fact that center $\mathfrak{n}=\mathbf{R}z$. Thus there exists $x\in\mathfrak{g}$ such that $[x,w]=z$ and hence

$$\mathfrak{n}=\mathbf{R}x\oplus\mathfrak{n}_0$$

where $\mathfrak{n}_0=\operatorname{Ker} g$. But then $\mathfrak{n}_0$ is the centralizer of w and hence $\mathfrak{n}_0$ is a subalgebra stable under F. However, since $\mathfrak{n}_0$ has codimension 1 in $\mathfrak{n}$ and $\mathfrak{n}$ is nilpotent, $\mathfrak{n}_0$ is an ideal in $\mathfrak{n}$. In particular, $N=XN_0$ where X and N_0 are the subgroups corresponding to $\mathbf{R}x$ and $\mathfrak{n}_0$.

Now the action of F on $\mathfrak{n}_0$ induces an epimorphism $F\to F_0\subseteq\operatorname{Aut}\mathfrak{n}_0$ where $F'\to F_0'$. However $F'\to\operatorname{Ad}_n N=\operatorname{Ad}_n X\operatorname{Ad}_n N_0$. But $\operatorname{Ad}_n N_0$ operates trivially on $\mathbf{R}w$ since clearly $w\in\text{center}\ \mathfrak{n}_0$. On the other hand, F' operates trivially on $w\in\mathfrak{p}_0$ as observed above. But since $[x,w]=z$ no non-trivial element of $\operatorname{Ad}_n X$ operates trivially on w so we must have $F'\to\operatorname{Ad}_n N_0$ which implies $F_0'\subseteq\operatorname{Ad}_{n_0} N_0$.

Now clearly $f_0=f\,|\,\mathfrak{n}_0$ is invariant under F_0. Furthermore we assert that

$$(\mathfrak{n}_0)_{f_0}=\mathfrak{n}_f\oplus\mathbf{R}w. \tag{II.3.1}$$

Indeed $w\in(\mathfrak{n}_0)_{f_0}$ since $w_0\in$center $\mathfrak{n}_0$. To see that $\mathfrak{n}_f\subseteq(\mathfrak{n}_0)_{f_0}$ we have only to observe that $\mathfrak{n}_f\subseteq\mathfrak{n}_0$. But this is clear since otherwise there exists $y\in\mathfrak{n}_f$ such that $[y,w]=z$. But then

$$1=\langle f,[y,w]\rangle=-\langle y\cdot f,w\rangle$$

contradicting the fact that $y\cdot f=0$. Also one has $\mathfrak{n}_f\cap\mathbf{R}w=0$ since $\langle w\cdot f,x\rangle=\langle f,[x,w]\rangle=\langle f,z\rangle=1$. Finally if $y\in(\mathfrak{n}_0)_{f_0}$ let $c=\langle y\cdot f,x\rangle=\langle f,[x,y]\rangle$. But $\langle c\,w\cdot f,x\rangle=\langle f,c\,z\rangle=c$. Thus $\langle(y-c\,w)\cdot f,x\rangle=0$. But $(y-c\,w)\cdot f\,|\,\mathfrak{n}_0=(y-c\,w)\cdot f_0=0$ since $w\in(\mathfrak{n}_0)_{f_0}$. But then $y-c\,w=y_1\in\mathfrak{n}_f$ so that $y\in\mathfrak{n}_f+\mathbf{R}w$. This establishes (II.3.1).

Now by induction there exists a positive polarization $\mathfrak{h}_0\subseteq(\mathfrak{n}_0)_{\mathbf{C}}$ at f_0 which is stable under F_0. Clearly then one has

$$(\mathfrak{n}_f)_{\mathbf{C}}\subseteq((\mathfrak{n}_0)_{f_0})_{\mathbf{C}}\subseteq\mathfrak{h}_0\subseteq(\mathfrak{n}_0)_{\mathbf{C}}\subseteq\mathfrak{n}_{\mathbf{C}}.$$

But since $\mathfrak{h}_0$ is "half-way" between $((\mathfrak{n}_0)_{f_0})_{\mathbf{C}}$ and $(\mathfrak{n}_0)_{\mathbf{C}}$ it is also "half-way" between $(\mathfrak{n}_f)_{\mathbf{C}}$ and $\mathfrak{n}_{\mathbf{C}}$ because $\mathfrak{n}_f$ has codimensional 1 in $(\mathfrak{n}_0)_{f_0}$ and $\mathfrak{n}_0$ has codimension 1 in $\mathfrak{n}$. Thus if $\mathfrak{h}=\mathfrak{h}_0$ it follows that $\mathfrak{h}$ is a positive polarization at f which is stable under the action of F.

Now if dim $\mathfrak{p}_0=2$ we may write $\mathfrak{p}_0=\mathbf{R}w_1\oplus\mathbf{R}w_2$. If we define $g_j\in n'$, $j=1,2$ by the relation $[y,w_j]=\langle g_j,y\rangle\,z$ then g_1 and g_2 are linearly independent since otherwise $\mathfrak{p}_0\cap$center $\mathfrak{n}\neq 0$. But of course $\mathfrak{p}_0\cap$center $\mathfrak{n}=0$ since center $\mathfrak{n}=\mathbf{R}z$.

But then we may find elements $x_1,x_2\in\mathfrak{n}$ such that

(II.3.2) $$[x_i,w_j]=\delta_{ij}z.$$

Clearly then

(II.3.3) $$\mathfrak{n}=\mathbf{R}x_1\oplus\mathbf{R}x_2\oplus\mathfrak{n}_0$$

where $\mathfrak{n}_0=\operatorname{Ker}g_1\cap\operatorname{Ker}g_2$ is the centralizer of the subspace $\mathfrak{p}_0$. Since $\mathfrak{p}_0$ is stable under F it follows that $\mathfrak{n}_0$ is a subalgebra stable under F. In fact since $[\mathfrak{n},\mathfrak{n}]$ annihilates $\mathfrak{k}_1\supseteq\mathfrak{k}_0\supseteq\mathfrak{p}_0$, it follows that

(II.3.4) $$[\mathfrak{n},\mathfrak{n}]\subseteq\mathfrak{n}_0$$

and hence $\mathfrak{n}_0$ is an ideal in $\mathfrak{n}$. The action of F on $\mathfrak{n}_0$ induces an epimorphism $F\to F_0\subseteq\operatorname{Aut}\mathfrak{n}_0$, where F' maps onto F_0'. But the map $X_1\times X_2\times N_0\to N$ is bijective where $(a_1,a_2,b)\to a_1a_2b$ and where $N_0\subseteq N$ is the subgroup corresponding to $\mathfrak{n}_0$ and X_j is the subgroup corresponding to $\mathbf{R}x_j, j=1,2$. But now N_0 operates trivially on $\mathfrak{p}_0\subseteq\mathfrak{k}_0$. But since no non-trivial element of X_1X_2 operates trivially on $\mathfrak{p}_0$ by the relations (II.3.2) it follows that $F'\to\operatorname{Ad}_n N_0$ and hence $F_0'\subseteq\operatorname{Ad}_{n_0}N_0$.

Now let $f_0=f\,|\,\mathfrak{n}_0$. By induction there exists a positive polarization $\mathfrak{h}_0$ at f_0 which is stable under the action of F_0.

As in the case where $\dim p_0 = 1$ one has $[\mathfrak{n}_f, \mathfrak{p}_0] = 0$ so that $\mathfrak{n}_f \subseteq \mathfrak{n}_0$ and hence

(II.3.5) $$\mathfrak{n}_f \subseteq (\mathfrak{n}_0)_{f_0}.$$

Next observe that

(II.3.6) $$(\mathfrak{n}_0)_{f_0} \subseteq \mathfrak{n}_f + \mathfrak{p}_0 = \mathfrak{n}_f \oplus \mathfrak{p}_0.$$

Indeed if $y \in (\mathfrak{n}_0)_{f_0}$ and c_j, $j = 1, 2$ are defined by $c_j = \langle y \cdot f, x_j \rangle$ then $g = (y - c_1 w_1 - c_2 w_2) \cdot f$ is orthogonal to $\mathbf{R} x_1 + \mathbf{R} x_2$ by the relations (II.3.2). However clearly g is orthogonal to $\mathfrak{n}_0$ so that $g = 0$ which implies $y - c_1 w_1 - c_2 w_2 \in \mathfrak{n}_f$ and hence $y \in \mathfrak{n}_f + \mathfrak{p}_0$. Now $\mathfrak{n}_f \cap \mathfrak{p}_0 = 0$ since by the relations (II.3.2) any non-zero element $w \in \mathfrak{p}_0$ is such that $z \in \operatorname{Im} \operatorname{ad} w$. But since $\langle f, z \rangle \neq 0$ this implies $w \notin \mathfrak{n}_f$. Hence (II.3.6) is established.

Case 2. Assume $[w_1, w_2] = 0$. Then $\mathfrak{p}_0 \subseteq \mathfrak{n}_0$ and hence $\mathfrak{p}_0 \subseteq$ center $\mathfrak{n}_0$ which implies $\mathfrak{p}_0 \subseteq (\mathfrak{n}_0)_{f_0}$. Thus by (II.3.5) and (II.3.6) one has $(\mathfrak{n}_0)_{f_0} = \mathfrak{n}_f \oplus \mathfrak{p}_0$ so that $\mathfrak{n}_f$ has codimension 2 in $(\mathfrak{n}_0)_{f_0}$. Since $\mathfrak{n}_0$ has codimension 2 in $\mathfrak{n}$ this implies that $\mathfrak{h}_0$ is "half-way" between $(\mathfrak{n}_f)_{\mathbf{C}}$ and $\mathfrak{n}_{\mathbf{C}}$ and hence $\mathfrak{h} = \mathfrak{h}_0$ defines a positive polarization at f which is stable under F.

Case 3. Assume $[w_1, w_2] \neq 0$. Now since F' operates trivially on $\mathfrak{p}_0$ it follows that F operates, irreducibly, as an abelian group on the 2-dimensional space. The commuting ring in End $\mathfrak{p}_0$ is therefore isomorphic to $\mathbf{C}$ and hence w_1 and w_2 may be chosen in $\mathfrak{p}_0$ so that $\mathbf{C} u, \mathbf{C} \bar{u} \subseteq (\mathfrak{p}_0)_{\mathbf{C}}$ are stable under the action of F where $u = w_1 + i w_2$.

Furthermore it is clear that since they are necessarily independent we may choose w_1, w_2 so that $[w_1, w_2] = z$. But then we may choose x_1 and x_2 so that $x_1 = w_1$, $x_2 = -w_2$ and hence (II.3.3) becomes

$$\mathbf{R} w_1 \oplus \mathbf{R} w_2 \oplus \mathfrak{n}_0 = \mathfrak{n}.$$

But then $\mathfrak{p}_0 \cap \mathfrak{n}_0 = 0$ so that, since $\mathfrak{n}_f \subseteq (\mathfrak{n}_0)_{f_0} \subseteq \mathfrak{n}_f + \mathfrak{p}_0$ by (II.3.5) and (II.3.6) one has $\mathfrak{n}_f = (\mathfrak{n}_0)_{f_0}$. But then since $\mathfrak{n}_0$ has codimension 2 in $\mathfrak{n}$, it follows that $\mathfrak{h}_0$ fails by one dimension of being a maximum isotropic subspace (m.i.s.) of $\mathfrak{n}_{\mathbf{C}}$ relative to B_f. That is, in the notation of Section I.6 one has $\dim \mathfrak{h}_0 = i(\mathfrak{n}_{\mathbf{C}}) - 1$.

Now put

$$\mathfrak{h} = \mathfrak{h}_0 + \mathbf{C} u.$$

Since $\mathfrak{h}_0 \subseteq (\mathfrak{n}_0)_{\mathbf{C}}$ and $u \in (\mathfrak{p}_0)_{\mathbf{C}}$ it follows that $[u, \mathfrak{h}_0] = 0$ so that not only $\mathfrak{h}$ is a m.i.s. of $\mathfrak{n}_{\mathbf{C}}$ but $\mathfrak{h}$ is a subalgebra stable under the action of F. Also since $\mathfrak{n}_f \subseteq \mathfrak{h}$ it follows that $\mathfrak{h}$ is stable under Ad N_f. But now $\mathfrak{h} + \bar{\mathfrak{h}} = \mathfrak{h}_0 + \bar{\mathfrak{h}}_0 + \mathbf{C} u + \mathbf{C} \bar{u} = (\mathfrak{h}_0 + \bar{\mathfrak{h}}_0) + (\mathfrak{p}_0)_{\mathbf{C}}$. However $\mathfrak{h}_0 + \bar{\mathfrak{h}}_0$ is a subalgebra since $\mathfrak{h}_0$ is a polarization at f_0. But $\mathfrak{h}_0 + \bar{\mathfrak{h}}_0 \subseteq (\mathfrak{n}_0)_{\mathbf{C}}$ and since $[\mathfrak{p}_0, \mathfrak{n}_0] = 0$

it follows that $\mathfrak{h}+\bar{\mathfrak{h}}$ is a subalgebra since $[(\mathfrak{p}_0)_{\mathbf{C}}, (\mathfrak{p}_0)_{\mathbf{C}}]=\mathbf{C}z$ and $z\in\mathfrak{n}_f=(\mathfrak{n}_0)_{f_0}\subseteq\mathfrak{h}$. Thus $\mathfrak{h}$ is a polarization at f. We have only to show that $\mathfrak{h}$ is positive.

But now since $\mathfrak{p}_0\cap\mathfrak{n}_0=0$ one has $\mathfrak{d}=\mathfrak{h}\cap\mathfrak{n}=\mathfrak{h}_0\cap\mathfrak{n}=\mathfrak{h}_0\cap\mathfrak{n}_0=\mathfrak{d}_0$. But if $\mathfrak{e}=(\mathfrak{h}+\bar{\mathfrak{h}})\cap\mathfrak{n}$ and $\mathfrak{e}_0=(\mathfrak{h}_0+\bar{\mathfrak{h}}_0)\cap\mathfrak{n}=(\mathfrak{h}_0+\bar{\mathfrak{h}}_0)\cap\mathfrak{n}_0$ then one has

$$\mathfrak{e}/\mathfrak{d}=\mathfrak{e}_0/\mathfrak{d}_0\oplus(\mathfrak{d}_0\oplus\mathfrak{p}_0)/\mathfrak{d}_0.$$

But this is an orthogonal direct sum relative to both $\hat{B}_f$ and S_f. Indeed this is clear since $\mathfrak{e}_0$ and $\mathfrak{d}_0$ are orthogonal relative to B_{f_0} and hence relative to B_f. But also $[\mathfrak{p}_0, \mathfrak{e}_0]=0$. Furthermore $(\mathfrak{d}_0+\mathfrak{p}_0)/\mathfrak{d}_0$ is stable under j since $(\mathfrak{p}_0)_{\mathbf{C}}=(\mathfrak{p}_0)_{\mathbf{C}}\cap\mathfrak{h}+(\mathfrak{p}_0)_{\mathbf{C}}\cap\bar{\mathfrak{h}}=\mathbf{C}u\oplus\mathbf{C}\bar{u}$. But by assumption S_f is positive definite on $\mathfrak{e}_0/\mathfrak{d}_0$. However it is positive definite on $(\mathfrak{d}_0+\mathfrak{p}_0)/\mathfrak{d}_0$ since if $[w_i]=w_i+\mathfrak{d}_0$, $i=1,2$ one has $j[w_1]=[w_2]$ and $j[w_2]=-[w_1]$. Thus $\{[w_1],[w_2]\}=0$ and

$$\{[w_1],[w_1]\}=(j[w_1],[w_1])=([w_2],[w_1])=\langle f,[w_1,w_2]\rangle=\langle f,z\rangle=1.$$

Similarly $\{[w_2],[w_2]\}=1$. Hence S_f is positive definite. QED.

We now return to the notation of Sections II.1 and II.2. It is clear that if $\mathfrak{h}$ is an admissible polarization at g, then $\mathfrak{h}_1=\mathfrak{h}\cap\mathfrak{n}_{\mathbf{C}}$ is stable under G_g. We will say that an admissible polarization $\mathfrak{h}$ is strongly admissible in case $\mathfrak{h}_1$ is stable under the action of $G_f\supseteq G_g$.

A polarization $\mathfrak{h}$ is called real if $\mathfrak{h}=\bar{\mathfrak{h}}$, that is, if $\mathfrak{e}=\mathfrak{d}$.

Theorem II.3.2. *Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $g\in\mathfrak{g}'$ be arbitrary. Let $\mathfrak{n}$ be the maximal nilpotent ideal in $\mathfrak{g}$ and let $f=g|\mathfrak{n}$. Then there exists a positive strongly admissible positive polarization $\mathfrak{h}$ at g (and hence in particular a positive admissible polarization at f).*

Proof. Now let $F=\mathrm{Ad}_n G_f$ so that $F\subseteq\mathrm{Aut}\,\mathfrak{n}$. Furthermore f is obviously fixed by F. Moreover, since $[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{n}$ one has for the commutator subgroups $G_f'\subseteq G'\subseteq N$ (using the notation of Lemma II.3.1). Thus $F'\subseteq\mathrm{Ad}\,N$ and hence the conditions of Lemma II.3.1 are satisfied. Thus there exists a positive polarization $\mathfrak{h}_1\subseteq\mathfrak{n}_{\mathbf{C}}$ at f which is stable under $\mathrm{Ad}\,G_f$.

Now if $g=0$ there is nothing to prove so that we may assume $g\neq0$ and use the notation of Proposition II.1.6. Now by Proposition II.1.6, $\mathfrak{m}/\mathfrak{p}$ is a Heisenberg Lie algebra and since $e\in\mathfrak{m}'$ vanishes on $\mathfrak{p}$ it defines an element $\hat{e}\in(\mathfrak{m}/\mathfrak{p})'$ which is non-zero on the center $\mathfrak{m}_e/\mathfrak{p}$ of $\mathfrak{m}/\mathfrak{p}$. But then clearly there exists a positive polarization $\hat{\mathfrak{h}}_2\subseteq(\mathfrak{m}/\mathfrak{p})_{\mathbf{C}}$ at $\hat{e}$ for the connected group $M/\mathrm{Ker}\,\eta_e$. In fact since $\mathfrak{m}/\mathfrak{p}$ is nilpotent, we may choose a real polarization by Kirillov's results. If $\pi:(\mathfrak{m})_{\mathbf{C}}\to(\mathfrak{m}/\mathfrak{p})_{\mathbf{C}}$ is the quotient map, let $\mathfrak{h}_2=\pi^{-1}(\hat{\mathfrak{h}}_2)\subseteq\mathfrak{m}_{\mathbf{C}}$ so that $\mathfrak{h}_2$ is a positive polarization at e for $M=G_g M_0$.