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Critical Points of an Algebraic Function

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§ 1. Introduction

Let k denote an algebraically closed field of any characteristic and $f: X \to C$ a morphism between smooth, connected and complete varieties over k, where X has dimension n and C is a curve. Assume the generic fiber of f is smooth and connected. The problem to be studied in this paper is to relate numerical invariants of the critical points of f to absolute invariants of X, C and the generic fiber F (more precisely, let F denote any smooth fiber of f).

Notation. For $Y=X$, C, F let $c^j(Y)$ denote the j'th Chern class of the cotangent bundle of Y. $c^j(Y)$ may be thought of as an element of the Chow ring, $C^\bullet(Y)$ of Y in case Y is a projective variety, otherwise $c^j(Y)$ is considered an element of the Grothendieck ring $Gr^\bullet(Y)$, which is obtained from $K(Y)$ by taking the graded ring associated with the λ-filtration of $K(Y)$, see [SGA 6][1] expose X. When no confusion is possible and $j=\dim Y$, $c^j(Y)$ will denote an integer "degree of the top Chern class". — D denotes the divisor on X for which a local equation of $x \in X$ is obtained as follows: Let s be a local parameter for C at $f(x)$, $t_1, \ldots, t_n$ local parameters for X at x, then the greatest common divisor in $\mathcal{O}_{X,x}$ of

$$\partial(s\circ f)/\partial t_1, \ldots, \partial(s\circ f)/\partial t_n \tag{1.1}$$

is a local equation for D at $x \in X$.

(1.2) $\mathscr{K}_\bullet(f)$ denotes a finite complex of locally free coherent sheaves on X (defined below) which depends only on $f: X \to C$ (see (3.4)).

[1] Square brackets contain references to the bibliography at the end of the paper.

With this notation we prove

$$\sum_{i=0}^{n}(-1)^i\big(c^{n-i}(X)_{\bullet}D^i\big)=c^1(C)\,c^{n-1}(F)$$
$$+\sum_{i=0}^{n}(-1)^i\dim_k \mathrm{Ext}^i\big(\mathcal{O}_X,\mathscr{K}_{\bullet}(f)\big) \tag{1.3}$$

the Ext being the global hyperext on X.

(1.4) In the following special case the last term in the above formula becomes simple. Namely, say that $x\in X$ is a moderate critical point for f if for a parameter s for C at $f(x)$ and parameters $t_1,\dots,t_n$ for X at x the $\mathcal{O}_{X,x}$-ideal generated by

$$(\partial(s\circ f)/\partial t_i)/d,\qquad i=1,\dots,n$$

where d is a local equation for D (i.e. a greatest common divisor for the $\partial(s\circ f)/\partial t_i$'s) has finite k-codimension in $\mathcal{O}_{X,x}$. – For a moderate critical point $x\in X$, we put (with the notation above)

$$\mu_x(f)=\dim_k \mathcal{O}_{X,x}/\{\partial(s\circ f)/\partial t_i/d\}_{i=1,\dots,n}. \tag{1.5}$$

If all critical points are moderate critical points, then

$$\dim_k \mathrm{Ext}^i\big(\mathcal{O}_X,\mathscr{K}_{\bullet}(f)\big)=\begin{cases}\sum_{x\in X}\mu_x(f) & \text{for } i=0\\ 0 & \text{for } i\geqq 1.\end{cases} \tag{1.6}$$

In case $k=\mathbf{C}$ and all critical points of f are isolated critical points, $\mu_x(f)$ coincides with the $\mu_x(f)$ studied by Milnor in [M], and our formula reads

$$\mathbf{X}_X=\mathbf{X}_F\,\mathbf{X}_C+(-1)^n\sum_{x\in X}\mu_x(f)$$

where the $\mathbf{X}$ stands for the alternating sum of the Betti numbers, see (5.5).

If X is a surface then of course all critical points are moderate critical points. If furthermore char $k=0$ then $\mu_x(f)$ may be expressed as follows: Let the fiber of $f\colon X\to C$ through x have the inreducible components $F_1,\dots,F_r$, then

$$\mu_x(f)=\sum_i\big(2\,\delta_x(F_i)-\nu_x(F_i)\big)+\sum_{i\neq j}(F_i,F_j)_x+1 \tag{1.7}$$

where $\delta_x(F_i)$ denotes the k-codimension of $\mathcal{O}_{F_i,x}$ in its normalization in the function field of F_i, $\nu_x(F_i)$ denotes the number of (analytic) branches of F_i through x and $(F_i,F_j)_x$ the local intersection number.

In case $k=\mathbf{C}$ and X a surface, the above formula alows us to give an algebraic proof of a topological formula due to Zariski as generalized by Šafarevič and thus to compare the singularities of differentiable nature to the topological singularities.

It should be pointed out that this paper owes much to works of Grothendieck [SGA 6], Igusa [Ig], Jung [J], Šafarevič [Ša] and Zariski [Z].

§ 2. Some *K*-Theory

The basic tool in this paper is K-theory [SGA 6]. Since that seminar is not orderly published in the moment of this writing, we first recall some basic results. Next is discussed "Chern classes and the Kozul complex" which may be considered a K-theoretical version of the classical Poincaré-Hopf Theorem. After that, "twisting a section by its zero-divisor", an elaboration on an idea I have learned from Igusa's paper [Ig]. To complete the picture we give a K-theoretical version of the connection between the self-intersection of the diagonal and the top Chern class of the tangent bundle and finally a version of the "principle of conservation of numbers".

K-Theory. We follow the notation of [SGA 6]. Recall that for Y a smooth, connected, complete variety over k, $K(Y)$ denotes the Grothendieck group of the category of locally free coherent sheaves on Y considered a ring via $\otimes$. If E is any coherent sheaf on Y, [E] denotes the class in $K(Y)$ obtained by taking a locally free resolution of E.

Exterior r'th power induces an operation λ^r on $K(Y)$ which gives rise to the operation γ^r as defined by

$$\gamma^r(x)=\lambda^r(x+r-1).$$

The rank function on the category of locally free sheaves induces ε: $K(Y)\to\mathbf{Z}$. $-\mathrm{Fil}^r K(Y)$ denotes the ideal in $K(Y)$ generated by

$$\gamma^{i_1}(x_1)\dots\gamma^{i_k}(x_k) \quad \text{where } i_1+\cdots+i_k\geqq r$$

and $\varepsilon(x_i)=0$. The graded ring associated with this filtration is denoted $Gr^\bullet(Y)$. If $\dim Y=m$, then $\mathrm{Fil}^{m+1}K(Y)=0$ [SGA 6]. Expose X, Theorem 1.3.2.

The Euler-Poincaré characteristic $E\mapsto\sum(-1)^i\dim_k H^i(X,E)$ induces χ: $K(Y)\to\mathbf{Z}$ whose restriction to $\mathrm{Fil}^m K(Y)$ defines the intersection number

$$Gr^{m-i}(Y)\times Gr^i(Y)\to\mathbf{Z}$$

denoted $x, y\mapsto(x\cdot y)$.

For $x \in K(Y)$, $c^i(x)$ denotes the image of $\gamma^i(x-\varepsilon(x)) \in \mathrm{Fil}^i K(Y)$ in $Gr^i(Y)$, [SGA 6], V.6.7.

Sheaf $\mapsto$ dual sheaf induces the operation $x \mapsto \check{x}$ in $K(Y)$. Recall that $c^i(x)=(-1)^i c^i(x)$, [SGA 6], VIII.3.5.

(2.1) **Lemma.** *Let E be a locally free coherent sheaf of rank r on Y, then*

$$\lambda_{-1}(\check{E})=\sum_{i\geqq 0}(-1)^i[\overset{i}{\wedge}(\check{E})]$$

belongs to $\mathrm{Fil}^r K(Y)$ *and its class in* $Gr^r(Y)$ *is* $c^r(E)$*—as it follows by a simple computation and the formula just quoted.*

(2.2) *Kozul Complex and Chern Classes*

Let E be a locally free coherent sheaf on Y and s a global section of E over Y. Then we recall the definition of the Kozul complex associated with (s, E)

$$\mathscr{K}_\bullet(E, s)$$

where $\mathscr{K}_r(E, s)=\overset{r}{\wedge}\check{E}$ and $d_r\colon \mathscr{K}_r \to \mathscr{K}_{r-1}$ is given by (let $e_1, e_2, \ldots$ denote trivializing sections of E over a small open affine)

$$d_r(e_1^* \wedge e_2^* \cdots \wedge e_r^*)=\sum_i (-1)^{i+1} e_i^*(s)\, e_1^* \wedge \cdots \hat{e}_i^* \cdots \wedge e_r^*.$$

(2.3) **Lemma.** *Let E be a locally free coherent sheaf on Y of rank $m=\dim Y$. Then*

$$c^m(E)=\sum_i (-1)^i \dim_k \mathrm{Ext}^i(\mathscr{O}_Y, \mathscr{K}_\bullet(E, s))$$

where Ext *is the global hyperext on Y and $c^m(E)$ is "degree of the top Chern class of E".*

Proof. According to Lemma (2.1) is

$$c^m(E)=\sum_i (-1)^i \chi(\overset{i}{\wedge}\check{E})=\sum (-1)^i \chi(\mathscr{K}_i(E, s))$$

$$=\sum_{i,j}(-1)^{i+j}\dim_k H^j(Y, \mathscr{H}_i \mathscr{K}_\bullet(E, s)).$$

Conclusion by the spectral sequence

$$H^p(Y, \mathscr{H}_q \mathscr{K}_\bullet(E, s)) \Rightarrow \mathrm{Ext}^\bullet(\mathscr{O}_Y, \mathscr{K}_\bullet(E, s)). \tag{2.4}$$

(2.5) *Twisting a Section by Its Zero Divisor*

Let E denote a locally free coherent sheaf on Y and s a non-zero global section of E over Y. We are going to define a divisor $D=D(E, s)$ and a section s_0 of $L(D)^{-1} \otimes E$ over Y, where $L(D)$ is the sheaf dual to the sheaf

of ideals defining D, such that s is the image of s_0 by

$$L(D)^{-1} \otimes E \to E \quad \text{(induced by } L(D)^{-1} \to \mathcal{O}_Y\text{)}.$$

Let $y \in Y$ and let $e_1, e_2, \ldots$ be trivializing sections of E over a neighbourhood U of x, write

$$s = \sum a_i e_i, \qquad a_i \in \Gamma(U, \mathcal{O}_Y).$$

Shrinking U, we may write $a_i = b_i d$ with $b_i, d \in \Gamma(U, \mathcal{O}_X)$ and such that the stalk of d at y is a greatest common divisor for the stalks of the a_i's. Let D denote the divisor whose local equation at $y \in Y$ is d. s_0 denotes the global section of $L(D)^{-1} \otimes E$ whose restriction to U is $\sum d \otimes b_i e_i$. s_0 will be called the section obtained by twisting s by its zero divisor.

(2.6) **Lemma.** *Let E be a locally free coherent sheaf of rank m and L an invertible sheaf. Then*

$$c^m(E \otimes L) = \sum_{i=0}^{m} c^{m-i}(E)\, c^1(L)^i.$$

Proof. By induction and the "splitting principle" in the form [SGA 6] VI.5.3 one proves the more general

$$c^r(E \otimes L) = \sum \binom{n-r-i}{i} c^{r-i}(E)\, c^1(L)^i.$$

(2.7) *Self-Intersection of the Diagonal*

Let Y be a smooth, connected m-dimensional variety over k. Then

$$c^m(T_Y) = (\Delta_Y \cdot \Delta_Y)$$

where T_Y denotes the tangent bundle on Y and Δ_Y the class of the diagonal in $K(Y \times Y)$.

Proof. This results by applying $\mathbf{X}$ to the following formula: Let i: $Y \to Y \times Y$ denote the diagonal, with the notation of [B-S] we have

$$\Delta_Y \cdot \Delta_Y = i_!(1) \cdot i_!(1) = i_!\big(i^! i_!(1)\big)$$

by the projection formula

$$= i_!\big(\lambda_{-1}(\Omega^1_Y)\big)$$

by [SGA 6], VII.2.7 and the fact that Ω^1_Y is the conormal bundle for i. Conclusion by (2.1).

(2.8) "*Conservation of Numbers*"

Let $f: X \to C$ be a morphism as described in § 1. For $c \in C$ let $\mathcal{O}_{X,c}$ denote the structure sheaf of the fiber of f at c considered as a coherent sheaf on X. Obviously $[\mathcal{O}_{X,c}] \in \mathrm{Fil}^1 K(X)$. Put $F_c =$ the class of $[\mathcal{O}_{X,c}]$ in $Gr^1(X)$. Then for any $y \in Gr^{n-1}(X)$ the number $(y \cdot F_c)$ is independent of $c \in C$.

Proof. More generally it is true that for any $\eta \in K(X)$ $c \mapsto ([\mathcal{O}_{X,c}] \cdot \eta)$ is constant. It suffices to prove this for $\eta = [E]$ where E is a locally free coherent sheaf on X. But now

$$([\mathcal{O}_{X,c}] \cdot \eta) = \sum (-1)^i \dim_k H^i(X_c, E_c)$$

where E_c denotes the sheaf induced by E in the fiber X_c of f at c. But, the last expression is constant since $f: X \to C$ is flat and C is connected [EGA III] 7.9.4.

§ 3. The Divisor $D(f)$ and the Complex $\mathcal{K}_\bullet(f)$

X and C denote smooth, connected and proper varieties over the algebraically closed field k; $\dim X = n$ and $\dim C = 1$. We consider the surjective morphism $f: X \to C$ whose generic fiber is smooth and connected. – More precisely, we require that there exists a closed point $c_0 \in C$ such that $f^{-1}(c_0) = X_0$ is smooth and connected. This implies by the Jacobian criterion that all but finitely many fibers are smooth. Moreover, all fibers of f are connected by Zariski's Connectedness Theorem [EGA III] 4.3.2, which we may apply since $f_*(\mathcal{O}_X) \simeq \mathcal{O}_C$ ($f_* \mathcal{O}_X$ is coherent by [EGA III] 4.3.1 and torsionfree, hence locally free since C is a smooth curve. It suffices to prove that $f_* \mathcal{O}_X$ has rank 1. But $f_* \mathcal{O}_X \otimes k(c_0) \to H^0(X_0, \mathcal{O}_{X_0})$ is surjective, hence an isomorphism by [EGA III] § 7).

$$T_X,\ T_C \quad \text{respectively } \Omega^1_X,\ \Omega^1_C$$

denotes the tangent respectively cotangent bundles. f induces the map

$$0 \to f^* \Omega^1_C \to \Omega^1_X$$

and hence the map

$$0 \to \mathcal{O}_X \to \Omega^1_X \otimes f^* T_C$$

i.e. we have a canonical section global section $s_1(f)$ of $\Omega^1_X \otimes f^* T_C$ over X.

(3.1) $D(f)$ denotes the divisor of zero's of $s_1(f)$ (see (2.5)).

(3.2) $L(f)$ denotes the dual sheaf to the sheaf of ideals defining $D(f)$.

(3.3) $s_0(f)$ denotes the global section of $\Omega^1_X \otimes f^* T_C \otimes L(f)^{-1}$ obtained by twisting $s_1(f)$ by $D(f)$ (see (2.5)).

(3.4) The Kozul complex associated with the global section $s_0(f)$ of $\Omega^1_X \otimes f^* T_C \otimes L(f)^{-1}$ will be denoted $\mathcal{K}_\bullet(f)$ (see (2.2)).

According to (3.1) and (2.5) a local equation for $D(f)$ at the point $x \in X$ may be obtained as follows: Let s be a local parameter for C at $f(x)$ and let $t_1, \ldots, t_n$ be local parameters for X at x; in the stalk of $\Omega^1_X \otimes f^* T_C$ at x we have

$$s_1(f) = \sum \partial(s \circ f)/\partial t_i \, dt_i \otimes \partial/\partial s \circ f \tag{3.5}$$

whence

(3.6) The greatest common divisor d_x in $\mathcal{O}_{X,x}$ of the $\partial(s \circ f)/\partial t_i$'s is a local equation for $D(f)$ at x.

Another consequence of (3.5) is

(3.7) The stalk of $\mathscr{H}_0(\mathscr{K}_\bullet(f))$ at x is isomorphic to $\mathcal{O}_{X,x}/\{\partial(s \circ f)/\partial t_i/d_x\}_{i=1,\ldots,n}$.

It follows from [Se] IV-6 Proposition 4 that

$$\operatorname{Supp} \mathscr{H}_i(\mathscr{K}_\bullet(f)) \subseteq \operatorname{Supp} \mathscr{H}_0(\mathscr{K}_\bullet(f))$$

thus

$$\operatorname{codim} \operatorname{Supp} \mathscr{H}_i(\mathscr{K}_\bullet(f)) \geqq 2. \tag{3.8}$$

(3.9) **Lemma.** i) *Any irreducible component of $D(f)$ is component of one of the fibers of f.*

ii) *For $c \in C$ let $F_c = \sum_i n_{ci} D_{ci}$ denote the fiber of f at c in the sense of intersection theory (if t is a local parameter for C at c, then $t \circ f$ has an n_{ci}-fold zero along D_{ci}) and suppose the n_{ci}'s are not divisible by* $\operatorname{char}(k)$. *Then*

$$D(f) = \sum_c \partial F_c$$

where $\partial F_c = \sum_i (n_{ci} - 1) D_{ci}$.

Proof. i) Let E be a component of $D(f)$ and suppose E is not a component of any fiber of f. Then f maps E onto C, but since the map $X \to C$ is flat it follows that all fibers of f are singular which contradicts the assumption made on the generic fiber.

ii) Let $x \in X$ be a point f the fiber of f at c at which D_{ci} is non-singular and such that the D_{cj}'s for $j \neq i$ do not pass x. Choose a parameter s for C at c and parameters $t_1, \ldots, t_n$ for X at x such that t_1 is a local equation for D_{ci} at x. Then $(s \circ f)_x = v t_1^{n_{ci}}$, where v is a unit in $\mathcal{O}_{X,x}$. Now

$$\partial(s \circ f)/\partial t_1 = t_1^{n_{ci}-1}(n_{ci} v + t_1 \, \partial v/\partial t_1)$$

$$(i \geqq 2) \qquad \partial(s \circ f)/\partial t_i = t_1^{n_{ci}} \, \partial v/\partial t_i. \quad \text{Q.E.D.}$$

§ 4. The General Formula

We keep the notation of § 3. Recall that $c^j(X)$ denotes $c^j(\Omega^1_X)$, that F denotes any non-singular fiber of f and that $c^{n-1}(F)$, $c^1(C)$ denotes the degree of the top Chern class of the cotangent bundle on F, C ($c^1(C)=2g-2$, where g is the genus of C).

(4.1) **Theorem.**

$$\sum_{i=0}^{n}(-1)^i\big(c^{n-i}(X)\cdot D(f)^i\big)=c^1(C)\,c^{n-1}(F)+\sum_{i=0}^{n}(-1)^i\dim_k \mathrm{Ext}^i\big(\mathcal{O}_X,\mathscr{K}_\bullet(f)\big).$$

Proof. Applying (2.3) and the Definitions (3.1)–(3.4) it suffices to prove

$$(4.2)\quad c^n\big(\Omega^1_X\otimes f^*T_C\otimes L(f)^{-1}\big)=\sum(-1)^i\big(c^{n-i}(X)\cdot D(f)^i\big)-c^1(C)\,c^{n-1}(F).$$

From (2.6) we get

$$(4.3)\quad c^n\big(\Omega^1_X\otimes f^*T_C\otimes L(f)^{-1}\big)=\sum c^{n-i}(\Omega^1_X)\big(f^\bullet(c^1(T_C))-c^1(L(f))\big)^i.$$

Now, $f^\bullet\big(c^1(T_C)\big)\,c^1\big(L(f)\big)=0$ as it follows from the following argument:

$$c^1\big(L(f)\big)=[\mathcal{O}_D]\qquad\text{where } D=D(f)$$

and we can write $c^1(T_C)=\sum n_i[\kappa_{x(i)}]$ where the $x(i)$'s are points of C at which the fiber of f is smooth and $\kappa_{x(i)}$ denotes the residue field of C at $x(i)$ considered as a coherent sheaf on C. The flatness of f implies that $f^\bullet[\kappa_{x(i)}]=[\mathcal{O}_{F(i)}]$ where $F(i)=f^{-1}(x(i))$ considered as a subvariety of X. Finally, $\mathcal{O}_{F(i)}\otimes\mathcal{O}_D=0$ since D is supported by the critical points of f as it follows from (3.6). Hence (Lemma (3.9) i)) $\mathscr{T}or_i^{\mathcal{O}_X}(\mathcal{O}_{F(i)},\mathcal{O}_D)=0$ for all i and thus $f^\bullet\big(c^1(T_C)\big)\,c^1\big(L(f)\big)=0$.

Using this and $f^\bullet\big(c^1(T_C)\big)^i=0$ for $i\geqq 2$ we get from (4.3) that

$$c^n\big(\Omega^1_X\otimes f^*(T_C)\otimes L(f)^{-1}\big)=\sum_i(-1)^i\,c^{n-i}(\Omega^1_X)\,c^1\big(L(f)\big)^i+c^{n-1}(\Omega^1_X)\,f^\bullet\big(c^1(T_C)\big).$$

Comparing with (4.2) it remains to prove

$$\big(c^{n-1}(\Omega^1_X)\bullet f^\bullet(c^1(\Omega^1_C))\big)=c^{n-1}(F)\,c^1(C)$$

which by an argument similar to the one above and (2.8) is equivalent to

$$(4.4)\qquad \big(c^{n-1}(\Omega^1_X)\bullet[\mathcal{O}_F]\big)=c^{n-1}(F).$$

Proof of (4.4). Let $i\colon F \to X$ denote the inclusion and consider the short exact sequence

$$0 \to \mathcal{N}_{F/X} \to i^* \Omega^1_X \to \Omega^1_F \to 0$$

where $\mathcal{N}_{F/X}$ denotes the conormal bundle which is easily seen to be the trivial bundle on F. Hence

$$c^{n-1}(\Omega^1_F) = c^{n-1}(i^* \Omega^1_X) = i^\bullet c^{n-1}(\Omega^1_X);$$

according to the projection formula we have

$$i_\bullet(i^\bullet c^{n-i}(\Omega^1_X)) = c^{n-1}(\Omega^1_X)_\bullet[\mathcal{O}_F].$$

Taking Euler-Poincaré characteristics $\mathbf{X}$ on both sides and using that $\mathbf{X}$ commutes with $i_\bullet$, we get formula (4.4).

§ 5. Moderate Critical Points

$$f\colon X \to C, \quad \dim X = n.$$

(5.1) **Definition.** $x \in X$ is called a moderate critical point if $\operatorname{codim}_x \mathcal{H}_0(\mathcal{K}_\bullet(f)) = 0$.

Pick a parameter s for C at $f(x)$ and parameter system $t_1, \ldots, t_n$ for X at x and let d_x denote the greatest common divisor of the stalks of the $\partial(s \circ f)/\partial t_i$'s. According to (3.7), x is a moderated-critical point if and only if the ideal $\{\partial(s \circ f)/\partial t_j / d_i\}_{i=1,\ldots,n}$ has finite (linear) k-codimension in $\mathcal{O}_{X,x}$.

(5.2) **Definition.** Let x be a moderate critical point for f. Put

$$\begin{aligned}\mu_x(f) &= \dim_k(\mathcal{H}_0(\mathcal{K}_\bullet(f)))_x \\ &= \dim_k \mathcal{O}_{X,x}/\{\partial(s \circ f)/\partial t_j/d_i\}_{i=1,\ldots,n}.\end{aligned}$$

(5.3) **Proposition.** *Suppose $f\colon X \to C$ has only moderate critical points. Then*

$$\dim_k \operatorname{Ext}^i(\mathcal{O}_X, \mathcal{K}_\bullet(f)) = \begin{cases} \sum_{x \in X} \mu_x(f) & \text{for } i = 0 \\ 0 & \text{for } i \geqq 1. \end{cases}$$

Proof. It follows from [Se] IV-6, Proposition 4, that $\mathcal{H}_i(\mathcal{K}_\bullet(f)) = 0$ for $i \geqq 1$. $\mathcal{H}_0(\mathcal{K}_\bullet(f))$ is supported by a finite number of points so $H^i(X, \mathcal{H}_0(\mathcal{K}(f))) = 0$ for $i \geqq 1$. Conclusion by the spectral sequence (2.4).

(5.4) *Remark.* Suppose $f\colon X \to C$ has only isolated critical points (an isolated critical point is a moderate critical point) then $D(f) = 0$ and the

formula (4.1) takes the form

$$c^n(X)=c^{n-1}(F)\,c^1(C)+\sum_{x\in X}\mu_x(f)$$

$$\mu_x(f)=\dim_k \mathcal{O}_{X,x}/\{\partial(s\circ f)/\partial t_i\}_{i=1,\ldots,n}$$

where $s, t_1, \ldots, t_n$ are defined in (6.1).

In case $k=\mathbf{C}$ the formula may be written

$$\mathsf{X}_X=\mathsf{X}_F\mathsf{X}_C+(-1)^n\sum_{x\in X}\mu_x(f) \tag{5.5}$$

where X stands for the alternating sum of the Betti numbers. Also $\mu_x(f)$ has a topological interpretation in this case: according to Milnor [M], p. 115, 43, 57, 60, $\mu_x(f)$ can be obtained by considering a small sphere S_ε around x; $S_\varepsilon - f^{-1}(f(x))$ is a smooth fiber bundle over S^1 (projection induced by f, [M] p. 43), where each fiber has the homotopy type of a bouquet of n-spheres, $\mu_x(f)$ in number.

(5.5) *Remark.* The notion of moderate (respectively isolated) critical point depends only on the induced map

$$\hat{\mathcal{O}}_{C,f(x)}\to\hat{\mathcal{O}}_{X,x}$$

the same is true about the invariant $\mu_x(f)$.

(5.6) *Example.* Let x be an isolated critical point for f, s a local parameter for C at $f(x)$ and $t_1, \ldots, t_n$ parameters for X at x. Consider $s\circ f$ as an element of $k[\![t_1, \ldots, t_n]\!]$ and write

$$s\circ f=\sum_{i=1}^{\infty}H_i(t_1, \ldots, t_n)$$

where $H_i(t_1,\ldots,t_n)$ is a homogeneous polynomial of degree i. $H_1(t_1,\ldots,t_n)=0$ since x is a critical point for f. The Hessian $H_2(t_1, \ldots, t_n)$ of f at x is non-degenerate if and only if $\mu_x(f)=1$ provided char $k\neq 2$, as one verifies by a simple computation and Nakayama's Lemma.

(5.7) *Example.* Let k be an algebraically closed field of any characteristic, $w_1, \ldots, w_n$ positive rational numbers and $f\in k[T_1, \ldots, T_n]$ a weighted homogeneous polynomial of type $(w_1, \ldots, w_n)$, which means that f can be expressed as a linear combination of monomials $T_1^{i_1}\ldots T_n^{i_n}$ for which

$$i_1/w_1+\cdots i_n/w_n=1$$

(ex. $T_1^5+T_1T_2^6+T_3^7$ is a weighted homogeneous polynomial of type $(5, 15/2, 7)$).

If f has an isolated critical point at the origin then

$$\dim_k k[\![T_1, \ldots, T_n]\!]/\{\partial f/\partial T_i\}_{i=1,\ldots,n} = (w_1 - 1)(w_2 - 1)\ldots(w_n - 1).$$

In case $k = \mathbf{C}$, this is [M-O] Theorem 1. But, there are little difficulties in building an algebraic geometrical proof, valid in all characteristics, on the ideas in the proof of Theorem 1 of [M-O].

§ 6. Case of an Algebraic Surface (char $k = 0$)

k denotes an algebraically closed field of characteristic zero and X a smooth, connected projective surface over k.

Let ω denote a non-zero meromorphic 1-form on X and x a point of X. Choose parameters s and t for X at x and write $\omega = f(a\,ds + b\,dt)$ where $a, b \in \mathcal{O}_{X,x}$ such that a and b have no common divisors in $\mathcal{O}_{X,x}$. We follow Igusa [Ig] and write

$$\langle \omega \rangle_x = \dim_k \mathcal{O}_{X,x}/\{a, b\}.$$

Proposition (6.1) below studies this symbol in some details. The result is applied to give a second version of formula (4.1) in case of an irrational pencil on an algebraic surface.

(6.1) **Proposition.** *Let $f \in \mathcal{O}_{X,x}, f \neq 0$ write $f = u\,p_1^{n_1+1} \ldots p_r^{n_r+1}$ where $p_1, \ldots, p_r$ are irreducible elements of $\mathcal{O}_{X,x}$ with $(p_i) \neq (p_j)$ for $i \neq j$ and u is a unit in $\mathcal{O}_{X,x}$. Then*

(i) $$\langle df \rangle_x = \langle d(p_1 \ldots p_r) \rangle_x$$

(ii) $$\langle d(p_1 \ldots p_r) \rangle_x = \sum_i \left(2\delta(\mathcal{O}_{X,x}/p_i) - \nu(p_i)\right) + \sum_{i \neq j} (p_i \cdot p_j)_x + 1$$

where $\delta(\mathcal{O}_{X,x}/p_i) =$ the k-codimension of $\mathcal{O}_{X,x}/p_i$ in its normalization in the fraction field of $\mathcal{O}_{X,x}/p_i$. $\nu(p_i)$ denotes the number of factors in $\hat{p}_i \in \hat{\mathcal{O}}_{X,x}$ (number of analytic branches of p_i) and $(p_i \cdot p_j)_x = \dim_k \mathcal{O}_{X,x}/\{p_i, p_j\}$.

Proof. (ii) is proved in [Iv] III.5.

(i) Choose parameters s and t for X at x such that s *does not divide* f (in $\mathcal{O}_{X,x}$). Put $\omega = d(f)/p_1^{n_1} \ldots p_r^{n_r} = \omega_s\,ds + \omega_t\,dt = p_1 \ldots p_r\,du + \sum_i u\,p_1 \ldots \hat{p}_i \ldots p_r(n_i + 1)\,dp_i$.

Clearly, ω_s and $\omega_t \in \mathcal{O}_{X,x}$.

Notation. For $a, b \in \mathcal{O}_{X,x}$ we put $(a \cdot b) = \dim_k \mathcal{O}_{X,x}/\{a, b\}$. (6.1) will follow from (6.2)–(6.5) below.

(6.2) $(p_1 \ldots p_r \cdot \omega_t)$, $(\omega_t \cdot \omega_s)$ and $(\omega_t \cdot s)$ are all finite and
$(p_1 \ldots p_r \cdot \omega_t) = (\omega_t \cdot \omega_s) + (\omega_t \cdot s)$.

Proof. $(\omega_t \cdot s)$ is finite: If not $\omega \equiv 0$ on $s=0$ and hence $df \equiv 0$ on $s=0$, but f is not a constant on $s=0$.

$(\omega_s \cdot \omega_t)$ is finite:

Let us first establish that the only possible common divisors for ω_s and ω_t are the divisors of f. – Suppose q irreducible with $q|\omega_s$ and $q|\omega_t$ but $q \nmid f$; then $\omega \equiv 0$ on $q=0$ and hence $df \equiv 0$ on $q=0$ but f constant on $q=0$ is impossible.

Now suppose $p_j|\omega_s$ and $p_j|\omega_t$. By the expansion for ω above, this implies p_j divides as well $\partial p_j/\partial s$ as $\partial p_j/\partial t$ which is impossible since p_j is irreducible (Weierstrass Preparation Theorem, see [Iv] Ch. II). –

$(p_1 \dots p_r \cdot \omega_t)$ is finite:

Suppose $p_j|\omega_t$. Then $\omega \equiv \omega_s\, ds$ on $p_j=0$, which implies that $p_j|\omega_s$, but $(\omega_s \cdot \omega_t)$ is finite. – Let q be an irreducible element of $\mathcal{O}_{X,x}$ which divides ω_t and let τ be a parameter for a branch of $q=0$ at x. On $q=0$ we have

$$= d(f)/p_1^{n_1} \dots p_r^{n_r} \equiv \omega_s\, ds$$

expanding this after $d\tau$ we get ($v_\tau(-)$ denotes the order of zero's of a function on $q=0$ at the branch at which τ is a parameter)

$$-v_\tau(p_1^{n_1} \dots p_r^{n_r}) + v_\tau(f) - 1 = v_\tau(\omega_s) + v_\tau(s) - 1.$$

Sum this over all branches of $q=0$ above x to get

$$(p_1 \dots p_r \cdot q) = (\omega_s \cdot q) + (s \cdot q).$$

Sum over all divisors q of ω_t to get (6.2).

(6.3) $$(\omega_t \cdot s) = (p_1 \dots p_r \cdot s) - 1.$$

Proof. On $s=0$ we have

$$= d(f)/p_1^{n_1} \dots p_r^{n_r} = \omega_t\, dt$$

hence

$$(\omega_t \cdot s) = -(p_1^{n_1} \dots p_r^{n_r}) + (f \cdot s) - 1.$$

(6.4) Let $\bar{f} = p_1 \dots p_r$. Then

$$(\bar{f} \cdot \omega_t) = (\bar{f}, \partial \bar{f}/\partial t).$$

Proof. $\omega_t \equiv u(n_j+1)\, p_1 \dots \hat{p}_j \dots p_r\, \partial p_j/\partial t \pmod{p_j}$

$$\partial \bar{f}/\partial t \equiv p_1 \dots \hat{p}_j \dots p_r\, \partial p_j/\partial t \pmod{p_j}$$

(6.5) $(\partial \bar{f}/\partial s \cdot \partial \bar{f}/\partial t)$, $(\bar{f}, \partial \bar{f}/\partial t)$ and $(s \cdot \partial \bar{f}/\partial t)$ are all finite and

$$(\partial \bar{f}/\partial s \cdot \partial \bar{f}/\partial t) = (\bar{f} \cdot \partial \bar{f}/\partial t) - (s \cdot \partial \bar{f}/\partial t).$$

Proof. $(\partial\bar{f}/\partial s \cdot \partial f/\partial t)$ is finite:

Suppose $q \in \mathcal{O}_{X,x}$ is irreducible and that q divides $\partial\bar{f}/\partial s$ and $\partial\bar{f}/\partial t$ but not f. Then $df \equiv 0$ on $q=0$ but then also $\omega \equiv 0$ on $q=0$, hence $q|\omega_s$ and $q|\omega_t$ which is impossible by (6.2). Consequently the only possible common divisors of $\partial\bar{f}/\partial s$ and $\partial\bar{f}/\partial t$ are the divisors of $\bar{f}$. If p_j divides $\partial\bar{f}/\partial s$ and $\partial\bar{f}/\partial t$ then p_j divides ω_s and ω_t as follows by looking at the expansion of ω above.

$(\bar{f} \cdot \partial\bar{f}/\partial t)$ is finite:

Suppose p_j divides $\partial\bar{f}/\partial t$, then it follows from the expansion of ω that $p_j|\omega_t$, which is impossible by (6.2).

$(s \cdot \partial\bar{f}/\partial t)$ is finite:

Suppose s divides $\partial\bar{f}/\partial t$ then $d\bar{f} \equiv 0$ on $s=0$, which is impossible.

Let now $q \in \mathcal{O}_{X,x}$ be irreducible and let q divide $\partial\bar{f}/\partial t$. On $q=0$ is

$$d\bar{f} \equiv \partial\bar{f}/\partial s \, ds.$$

Taking a parameter τ for a branch of $q=0$ at x we get with the notation introduced in the proof of (6.2)

$$v_\tau(\bar{f}) - 1 = v_\tau(\partial\bar{f}/\partial s) + v_\tau(s) - 1.$$

Sum this over all branches of $q=0$ above x to get

$$(\bar{f} \cdot q) = (\partial\bar{f}/\partial s \cdot q) + (s \cdot q).$$

Sum this over all divisors of $\partial\bar{f}/\partial t$ to get (6.5).

Proof of (6.1).

$$\begin{aligned}
\langle d(p_1 \dots p_r)\rangle_x &= (\partial\bar{f}/\partial s \cdot \partial f/\partial t) && \text{by (6.5)} \\
&= (\bar{f} \cdot \partial\bar{f}/\partial t) - (\bar{f} \cdot s) + 1 && \text{by (6.5) and the triviality} \\
&\quad (\partial\bar{f}/\partial s \cdot s) = (\bar{f} \cdot s) - 1 && \\
&= (\bar{f} \cdot \omega_t) - (s \cdot \omega_t) && \text{by (6.3) and (6.4)} \\
&= \langle\omega\rangle_x = \langle df\rangle_x && \text{by definition. Q.E.D.}
\end{aligned}$$

Irrational Pencil on an Algebraic Surface

Let k be an algebraically closed field of characteristic 0, X (respectively C) a smooth, connected, projective surface (respectively curve) and $f: X \to C$ a surjective morphism whose generic fiber is smooth and connected. F denotes any smooth fiber of f. For $b \in C$, let $F_b = \sum_i n_{ib} F_{ib}$ denote

the intersection theoretical fiber of f at b and put

$$\mathsf{X}_b = \sum_i (2-2p_g(F_{ib})) - \sum_{x\in f^{-1}(b)} \left(\left(\sum_i \nu_x(F_{ib})\right)-1\right)$$

where $p_g(F_{ib})$ denotes the (geometric) genus of the curve F_{ib}.

Note, in case $k=\mathbf{C}$, X_b is the alternating sum of the Betti numbers of the fiber (in the sense of topology) of f at b. With this notation we have

$$\begin{aligned} &\mathsf{X}_X = \mathsf{X}_C\,\mathsf{X}_F + \sum_{b\in C}(\mathsf{X}_b - \mathsf{X}_F) \\ &(\mathsf{X}_X = c^2(X),\ \mathsf{X}_C = 2-2p_g(C),\ \mathsf{X}_F = 2-2p_g(F)). \end{aligned} \tag{6.6}$$

In case $k=\mathbf{C}$ (6.6) is proved by topological methods in [Ša] Chapter IV, Theorem 6.

Proof. According to (4.1) and (5.3)

$$\mathsf{X}_X = \mathsf{X}_F\,\mathsf{X}_C + (K\cdot D) - (D\cdot D) + \sum_{x\in X}\mu_x(f)$$

where $K=K_X$ denotes a canonical divisor on X, $D=\sum_{b\in C} D_b$, $D_b = \sum_i (n_{ib}-1)F_{ib}$ (see (3.9)).

Using $F^2=0$ and $F\cdot F_{ib}=0$ (see (2.8)) we get

$$(D_b\cdot D_b) - (K\cdot D_b) = \sum_{i,j}(F_{ib}\cdot F_{jb}) - (K_X\cdot F) + \sum_i (F_{ib}\cdot K)$$

using the genus formula and the adjunction formula:

$$2p_g(F_{ib}) - 2 = (F_{ib}\cdot K + F_{ib}) - \sum_{x\in F_{ib}} \delta_x(F_{ib})$$

and (6.1) we get (6.11) after a little computation. Q.E.D.

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