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A Large Sieve Density Estimate near $\sigma = 1$

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The proof of Linnik's result on the least prime in an arithmetic progression has been simplified by Rodosskii, Turán and Knapowski, and Fogels has extended the result to give primes in short intervals of arithmetic progressions. There are still two main lemmas on the distribution of zeros of the Dirichlet L-functions. The first of these, a density theorem, has been essentially extended by Fogels [5], who used Turan's power sum method and an ingenious additional argument to prove

$$\sum_{\chi \bmod q} N_\chi(\alpha, T) \ll T^{c(1-\alpha)}, \quad (T \geqq q). \tag{1}$$

Here $N_\chi(\alpha, T)$ denotes the number of zeros of $L(s, \chi)$ in the rectangle $\alpha \leqq \sigma \leqq 1$, $|t| \leqq T$, and c is a positive constant.

Recently Jutila [7] and Montgomery [12, 13] have found ways to prove "hybrid" density theorems, of the form

$$\sum_{q \leqq Q} \sum_{\chi}{}^{*} N_\chi(\alpha, T) \ll (Q^2 T)^{c(1-\alpha)} \log^b(QT). \tag{2}$$

These are simultaneous generalisations of the classical density theorems for the ζ-function and the "large sieve" density theorem of Bombieri [1], which has an extra factor of T on the right. Earlier methods had given estimates of the form

$$\sum_{\chi \bmod q} N_\chi(\alpha, T) \ll (qT)^{c(1-\alpha)} \log^b(qT). \tag{3}$$

Using additional new methods, Montgomery has also shown that c in (2) and (3) may be taken as $\frac{5}{2}$.

The purpose of this paper is to give a common basis for Fogels' device and the hybrid mean estimates of Montgomery and Jutila. This leads to a generalisation[1] of (1):

$$\sum_{q \leqq T} \sum_{\chi}{}^{*} N_\chi(\alpha, T) \ll T^{c(1-\alpha)} \quad (T \geqq 1). \tag{4}$$

[1] This was predicted by Turán [17, p. 150]. Using Turán's power-sum inequality and (14) below, Jutila proved (unpublished) that the number of zeros of all primitive L-functions to moduli $\leqq Q$ in a disc $|s-1| \leqq r$ is $\ll Q^{cr}$ $(r \leqq 1)$.

As with (1), the cost of no logarithms is a large value of c. Explicit values of c in an earlier version of (1) have been given by Chen [3] and Jutila [8].

As an application of (4), we show that, assuming there is no Siegel zero,

$$\sum_{1<q\leqq Q}\sum_{\chi}{}^{*}\left|\sum_{x}^{x+h}\chi(p)\log p\right|\ll h\cdot\exp\left(-a\frac{\log x}{\log Q}\right) \tag{5}$$

provided

$$x/Q\leqq h\leqq x, \quad \text{and} \quad \exp(\log^{\frac{1}{2}} x)\leqq Q\leqq x^{b}. \tag{6}$$

Here a and b are certain positive constants. If there is a Siegel zero, then one of the terms on the left must be modified, and the bound on the right may be reduced.

The methods used in the proof of (4) are a general mean value estimate for exponential sums, a large sieve estimate, due to Bombieri and Davenport, for character sums with prime argument, and an application of Turán's power sum lemma.

1. Mean Values of Exponential Sums

Let

$$S(t)=\sum c(v)\cdot e(v t) \qquad (e(x)=e^{2\pi i x}) \tag{7}$$

be an absolutely convergent exponential sum. Here the frequencies v run over an arbitrary sequence of real numbers and the coefficients are complex.

Lemma 1. *Let $\delta=\theta/T$, with $0<\theta<1$. Then*

$$\int_{-T}^{T}|S(t)|^2\,dt\ll_{\theta}\int_{-\infty}^{\infty}\left|\delta^{-1}\sum_{x}^{x+\delta}c(v)\right|^2 dx. \tag{8}$$

Proof. The integral on the right may be written as

$$\int_{-\infty}^{\infty}|C_\delta(x)|^2\,dx, \quad \text{with } C_\delta(x)=\delta^{-1}\sum_{|v-x|\leqq\frac{1}{2}\delta}c(v).$$

Put $F_\delta(x)=\delta^{-1}$ or 0 according as $|x|\leqq\frac{1}{2}\delta$ or not. Then

$$C_\delta(x)=\sum c(v)\,F_\delta(x-v).$$

Taking Fourier transforms, we get $\hat{C}_\delta=S\cdot\hat{F}_\delta$. Since the series (7) converges absolutely, C_δ is a bounded integrable function, and hence is square-integrable. By Plancherel's theorem,

$$\int_{-\infty}^{\infty}|C_\delta(x)|^2\,dx=\int_{-\infty}^{\infty}|S(t)\,\hat{F}_\delta(t)|^2\,dt.$$

Since

$$\hat{F}_\delta(t)=\frac{\sin\pi\delta t}{\pi\delta t}\gg 1, \quad \text{for } |t|\leqq T,$$

the result follows.

Now let S be an absolutely convergent Dirichlet series:

$$S(t)=\sum a_n n^{it}. \tag{9}$$

Making the substitution $\log y=2\pi x$, we get (with $\theta=1/2\pi$):

Theorem 1. *We have*

$$\int_{-T}^{T}|S(t)|^2\,dt\ll T^2\int_0^\infty\left|\sum_y^{y\tau}a_n\right|^2\frac{dy}{y}, \quad \textit{with } \tau=e^{1/T}. \tag{10}$$

We will apply (10) to sums of the form

$$S(\chi,t)=\sum a_n\chi(n)\,n^{it}, \tag{11}$$

where χ is a Dirichlet character.

Lemma 2. *We have*

$$\sum_{\chi \bmod q}\left|\sum_y^{y+z}a_n\chi(n)\right|^2\leqq(q+z)\sum_y^{y+z}|a_n|^2. \tag{12}$$

Proof. By the orthogonality relations and the Schwarz inequality, the left side of (12) is

$$\varphi(q)\sum_{\substack{h=1\\(h,q)=1}}^{q}\left|\sum_{\substack{y\\n\equiv h}}^{y+z}a_n\right|^2\leqq\varphi(q)\sum_{\substack{h=1\\(h,q)=1}}^{q}\left(\frac{z}{q}+1\right)\sum_{\substack{y\\n\equiv h}}^{y+z}|a_n|^2.$$

Theorem 2. *For $T\geqq 1$, we have*

$$\sum_{\chi \bmod q}\int_{-T}^{T}|S(\chi,t)|^2\,dt\ll\sum(qT+n)\,|a_n|^2. \tag{13}$$

Proof. Using (10) and (12), the left side of (13) is

$$\ll T^2\int_0^\infty\sum_{\chi \bmod q}\left|\sum_y^{y\tau}a_n\chi(n)\right|^2\frac{dy}{y}\ll T^2\int_0^\infty(q+y(\tau-1))\sum_y^{y\tau}|a_n|^2\frac{dy}{y}.$$

The coefficient of $|a_n|^2$ here is

$$T^2q\int_{n/\tau}^{n}\frac{dy}{y}+T^2(\tau-1)\int_{n/\tau}^{n}dy=Tq+T^2(\tau-1)(1-\tau^{-1})\,n,$$

which is $\ll qT+n$ provided $T\geqq 1$.

Lemma 3. *We have*

$$\sum_{q \leqq Q} \sum_{\chi}{}^{*} \left| \sum_{y}^{y+z} a_n \chi(n) \right|^2 \ll (Q^2+z) \sum_{y}^{y+z} |a_n|^2. \tag{14}$$

Proof. This follows from (3) and (5) of [6].

Theorem 3. *For $T \geqq 1$, we have*

$$\sum_{q \leqq Q} \sum_{\chi}{}^{*} \int_{-T}^{T} |S(\chi, t)|^2 \, dt \ll \sum (Q^2 T + n) |a_n|^2. \tag{15}$$

Proof. Same as (13), except that (14) is used instead of (12).

Estimate (13) may be used in the proof of mean value and density estimates for L-functions to a single modulus. Estimate (15) is a continuous analogue of Montgomery's hybrid sieve inequality [12, Theorem 2]. Estimates of the same strength have also been used by Jutila [7, §3].

If the coefficients a_n are irregular, then (10) is more precise than (13) or (15), since in (10) the coefficients are first smoothed, then squared. Fogels' proof of (1) may be simplified by the use of (10) in [5, §5–7, 9] in combination with the Brun-Titchmarsh inequality and Turán's method. For the proof of (4), we use a variant of (15) which applies with more precision to sums with prime argument.

Theorem 4. *Assume $a_n = 0$ if n has any prime factor $\leqq Q$. Then for $T \geqq 1$, we have*

$$\sum_{q \leqq Q} \log \frac{Q}{q} \sum_{\chi}{}^{*} \int_{-T}^{T} |S(\chi, t)|^2 \, dt \ll \sum (Q^2 T + n) |a_n|^2. \tag{16}$$

The proof is the same as (15) except that instead of (14) the following inequality, due to Bombieri and Davenport [2, Theorem 4] is used. For the convenience of the reader, we repeat their proof.

Lemma 4. *With the same assumption as in Theorem 4,*

$$\sum_{q \leqq Q} \log \frac{Q}{q} \sum_{\chi}{}^{*} \left| \sum_{y}^{y+z} a_n \chi(n) \right|^2 \ll (Q^2+z) \sum_{y}^{y+z} |a_n|^2. \tag{17}$$

Proof. For each character χ to modulus q, we have

$$\tau(\bar{\chi}) \chi(n) = \sum_{a=1}^{q} \bar{\chi}(a) \, e\left(\frac{a n}{q}\right), \quad \text{for } (n, q) = 1,$$

where $\tau(\bar{\chi})$ is defined by putting $n = 1$. Let

$$S(\chi) = \sum_{y}^{y+z} a_n \chi(n), \quad \text{and} \quad S(\alpha) = \sum_{y}^{y+z} a_n e(n\alpha).$$

Using the assumption, we get for $q \leqq Q$

$$\tau(\bar{\chi})\, S(\chi)=\sum_{a=1}^{q} \chi(a)\, S\left(\frac{a}{q}\right).$$

By the orthogonality relations,

$$\sum_{\chi} |\tau(\bar{\chi})\, S(\chi)|^2=\varphi(q) \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|S\left(\frac{a}{q}\right)\right|^2.$$

Using Bombieri's large sieve inequality for exponential sums [6, (3)], we get

$$\sum_{q \leqq Q} \frac{1}{\varphi(q)} \sum_{\chi \bmod q} |\tau(\bar{\chi})\, S(\chi)|^2 \ll (Q^2+z) \sum_{y}^{y+z} |a_n|^2. \tag{18}$$

If f is the conductor of χ, then $q=fr$, and $|\tau(\bar{\chi})|^2=f$ or 0 according as r is square-free and prime to f or not [4, p. 148]. Also, by the assumption, $S(\chi)=S(\psi)$, where ψ is the (primitive) character to modulus f which induces χ. Hence the left side of (18) is

$$\sum_{f \leqq Q} \frac{f}{\varphi(f)} \Bigg(\sum_{\substack{r \leqq Q/f\\(r,f)=1}} \frac{\mu^2(r)}{\varphi(r)}\Bigg) \Big(\sum_{\psi \bmod f}^{*} |S(\psi)|^2\Big). \tag{19}$$

Since [11]

$$\sum_{\substack{r \leqq x\\(r,f)=1}} \frac{\mu^2(r)}{\varphi(r)} \geqq \frac{\varphi(f)}{f} \log x,$$

we get that (19) is not smaller than the left side of (17).

2. Turán's Zero-Detecting Method

Let $L(s,\chi)$ be an L-function to modulus $\leqq T$. Let $w=1+iv$ with $|v| \leqq T$. If $\chi=\chi_0$, we also assume that $|v| \geqq 2$. Put $\mathscr{L}=\log T$. According to Linnik's "density lemma" [14, p. 331], L has $\ll r\mathscr{L}$ zeros in any disc $|s-w| \leqq r$, provided $\mathscr{L}^{-1} \leqq r \leqq 1$. In this section, we show by Turán's method that if L has a zero near w, then for suitable x, y the sum

$$S_{x,y}(\chi, v)=\sum_{x}^{y} \frac{\chi(p) \log p}{p^w}$$

is large. In what follows $r_0, A, B, C, D, \ldots$ are certain positive constants.

Theorem 5. *If $L(s,\chi)$ has a zero in the disc $|s-w| \leqq r$, with $\mathscr{L}^{-1} \leqq r \leqq r_0$, then for each $x \geqq T^A$, we have*

$$\int_{x}^{x^B} |S_{x,y}(\chi, v)| \frac{dy}{y} \gg x^{-Cr} \log^2 x. \tag{20}$$

Proof. We have

$$\frac{L'}{L}(s,\chi)=\sum\frac{1}{s-\rho}+O(\mathcal{L}),\qquad |s-w|\leqq\tfrac{1}{2},$$

where ρ runs over the zeros in the disc $|s-w|\leqq 1$. By Cauchy's inequality, it follows that

$$\frac{\mathcal{D}^k}{k!}\frac{L'}{L}(s,\chi)=(-1)^k\sum\frac{1}{(s-\rho)^{k+1}}+O(4^k\mathcal{L}),\qquad |s-w|\leqq\tfrac{1}{4}.$$

We choose $s=w+r$ and estimate the contribution of the terms with $|\rho-w|>\lambda$, where $r\leqq\lambda\leqq 1$. There are $\ll 2^j\lambda\mathcal{L}$ terms with $2^j\lambda<|\rho-w|\leqq 2^{j+1}\lambda$, and each of these terms is $\ll(2^j\lambda)^{-(k+1)}$. Hence for $k\geqq 1$, the contribution is

$$\ll\sum_{j\geqq 0}(2^j\lambda)^{-k}\mathcal{L}\ll\lambda^{-k}\mathcal{L}.$$

Thus for $\mathcal{L}^{-1}\leqq r\leqq\lambda\leqq\frac{1}{4}$, we have

$$\frac{\mathcal{D}^k}{k!}\frac{L'}{L}(s,\chi)=(-1)^k{\sum}'\frac{1}{(s-\rho)^{k+1}}+O(\lambda^{-k}\mathcal{L}),\tag{21}$$

where ρ now runs over the zeros with $|\rho-w|\leqq\lambda$. By the density lemma, there are $\leqq A_1\lambda\mathcal{L}$ such zeros, and by the assumption, $\min|s-\rho|\leqq 2r$. It follows by Turán's second power-sum theorem [15] that

$$\left|{\sum}'\frac{1}{(s-\rho)^{k+1}}\right|\geqq(Dr)^{-(k+1)}$$

for some integer $k\in[K,2K]$, provided $K\geqq A_1\lambda\mathcal{L}$. Choosing $\lambda=A_2Dr$, with A_2 sufficiently large, the sum in (21) then dominates the remainder, since for any constant C_0, we have $(Dr)^{-(k+1)}\geqq C_0(A_2Dr)^{-k}\mathcal{L}$ provided $A_2^k\geqq C_0Dr\mathcal{L}$; and this holds for $k\geqq A_1A_2Dr\mathcal{L}$ for large A_2, since $r\mathcal{L}\geqq 1$. Thus we get

$$\frac{\mathcal{D}^k}{k!}\frac{L'}{L}(s,\chi)\gg(Dr)^{-(k+1)}$$

for some integer $k\in[K,2K]$, provided $K\geqq Er\mathcal{L}$. Here we must assume $r\leqq r_0$, since we had $\lambda\leqq\frac{1}{4}$. Rewriting this gives

$$\sum\frac{\Lambda(n)\chi(n)}{n^w}p_k(r\cdot\log n)\gg D^{-k}/r,\qquad\text{with }p_k(u)=e^{-u}u^k/k!.\tag{22}$$

There are constants B_1, B_2 such that

$$p_k(u) \leqq (2D)^{-k}, \qquad \text{for } u \leqq B_1 k;$$
$$p_k(u) \leqq (2D)^{-k} e^{-\frac{1}{2}u}, \qquad \text{for } u \geqq B_2 k.$$

In fact, putting $u = v k$ and using $k! \geqq (k/e)^k$, we get $p_k(u) \leqq (v\, e^{1-v})^k$, from which these inequalities follow easily.

Given $x \geqq T^A$, with $A = B_1 E$, put $K = B_1^{-1} r \cdot \log x$, so that $K \geqq E r \mathscr{L}$. Let $k \in [K, 2K]$ satisfy (22). We have, with $B = 2B_2/B_1$,

$$\sum_{n \leqq x} \frac{\Lambda(n)\,\chi(n)}{n^w} p_k(r \cdot \log n) \ll (2D)^{-k} \sum_{n \leqq x} \frac{\Lambda(n)}{n} \ll (2D)^{-k}\, k/r;$$

$$\sum_{n > x^B} \frac{\Lambda(n)\,\chi(n)}{n^w} p_k(r \cdot \log n) \ll (2D)^{-k} \sum \frac{\Lambda(n)}{n^{1+\frac{1}{2}r}} \ll (2D)^{-k}/r.$$

It follows that

$$\sum_{x}^{x^B} \frac{\Lambda(n)\,\chi(n)}{n^w} p_k(r \cdot \log n) \gg D^{-k}/r \tag{23}$$

provided k is large enough; this is ensured by large enough A. Since $p_k \leqq 1$, the contribution to (23) of the prime powers $n = p^\nu$ with $\nu \geqq 2$ is $\ll x^{-\frac{1}{3}}$, and therefore may be ignored. Putting $S(y) = S_{x,y}(\chi, v)$, the remaining sum is

$$\int_x^{x^B} p_k(r \cdot \log y)\, dS(y) = p_k(r \cdot \log x^B)\, S(x^B) - \int_x^{x^B} S(y)\, p_k'(r \cdot \log y)\, r \frac{dy}{y}.$$

The first term on the right is

$$\ll (2D)^{-k} \sum_{n \leqq x^B} \frac{\Lambda(n)}{n} \ll (2D)^{-k}\, k/r,$$

and therefore may be ignored. Since $p_k' = p_{k-1} - p_k \ll 1$, it follows that

$$\int_x^{x^B} |S(y)| \frac{dy}{y} \gg D^{-k}/r^2 \gg x^{-Cr} \log^2 x.$$

3. Proof of the Density Estimate

Theorem 6. *We have*

$$\sum_{q \leqq T} \sum_{\chi}{}^{*} N_\chi(\alpha, T) \ll T^{c(1-\alpha)}. \tag{24}$$

Proof. Since $N_\chi(\alpha, T) \ll T \cdot \log T$ for $q \leqq T$, it suffices to prove (24) for $1-\alpha$ sufficiently small. Since the left side is a decreasing function of α

and the right side is essentially constant for $1-\alpha \ll \mathcal{L}^{-1}$, it suffices to prove (24) for $1-\alpha \gg \mathcal{L}^{-1}$. Since the right side is $\geqq 1$, we may ignore the boundedly many zeros of $\zeta(s)$ $(q=1)$ in $0<\sigma<1$, $|t|\leqq 2$.

Let $w=1+iv$ with $|v|\leqq T$ (and $|v|\geqq 2$ if $\chi=\chi_0$). Put $r=2(1-\alpha)$. We may assume that $\mathcal{L}^{-1}\leqq r\leqq r_0$. By Theorem 5, if $L(s,\chi)$ has a zero in the disc $|s-w|\leqq r$, then for $x\geqq T^A$, (20) holds. We choose $x=T^{\max(A,5)}$. By the Schwarz inequality, we get (with $c=4C\cdot\max(A,5)$)

$$T^{c(1-\alpha)}\,\mathcal{L}^{-3}\int_x^{x^B}|S_{x,y}(\chi,v)|^2\,\frac{dy}{y}\gg 1.$$

Since there are $\ll r\mathcal{L}$ zeros in the disc $|s-w|\leqq r$, and each zero $\beta=\beta+i\gamma$ with $\beta\geqq\alpha$ and $|\gamma|\leqq T$ is detected in this way over a v-interval of length $\gg r$ in $|v|\leqq T$, we get

$$N_\chi(\alpha,T)\ll T^{c(1-\alpha)}\,\mathcal{L}^{-2}\int_x^{x^B}\int_{-T}^{T}|S_{x,y}(\chi,v)|^2\,dv\,\frac{dy}{y},$$

and hence, for some $y\in[x,x^B]$,

$$\sum_{q\leqq T}\sum_\chi{}^{*} N_\chi(\alpha,T)\ll T^{c(1-\alpha)}\,\mathcal{L}^{-1}\sum_{q\leqq T}\sum_\chi{}^{*}\int_{-T}^{T}|S_{x,y}(\chi,v)|^2\,dv. \qquad (25)$$

We may estimate the double sum on the right by Theorem 4. Since $x\geqq T^5$, we have

$$\sum_{q\leqq T^2}\log(T^2/q)\sum_\chi{}^{*}\int_{-T}^{T}|S_{x,y}(\chi,v)|^2\,dv\ll\sum_x^y(T^5+p)\frac{\log^2 p}{p^2}\ll\mathcal{L}^2.$$

It follows the double sum on the right of (25) is $\ll\mathcal{L}$, and hence the left side is $\ll T^{c(1-\alpha)}$.

4. Application of the Density Estimate

There is a positive constant c_1 such that at most one primitive L-function to modulus $\leqq T$ has a zero in the region

$$\sigma>1-\frac{c_1}{\log T},\qquad |t|\leqq T, \qquad (26)$$

and if there is such an exception, the exceptional zero is real, simple and unique [4, §13, 14]. We denote the exceptional zero by $1-\delta$. As $\delta\log T\to 0$, the zero-free region (26) may be widened to

$$\sigma>1-\frac{c_2}{\log T}\log\left(\frac{e\,c_1}{\delta\log T}\right),\qquad |t|\leqq T, \qquad (27)$$

with $1-\delta$ still the only exception. The proof of (27) in [9] applies only to all L-functions to a given modulus. However, if χ_1 and χ_2 are different primitive characters to moduli q_1 and q_2, then we may apply the result of [9] to $L(s, \chi_0\chi_1)$ and $L(s, \chi_0\chi_2)$, where χ_0 is the principal character to modulus $q_1 q_2$, and get (27) with c_1 and c_2 halved. We will also use Siegel's estimate [4, § 21]

$$\delta \gg T^{-\varepsilon}, \qquad \text{for each } \varepsilon>0. \tag{28}$$

Theorem 7. *We have*

$$\sum_{q\leqq Q}\sum_{\chi}{}^{*}\left|\sum_{x}^{x+h}\chi(p)\log p\right| \ll h\cdot\exp\left(-a\frac{\log x}{\log Q}\right) \tag{29}$$

provided $x/Q\leqq h\leqq x$ *and* $\exp(\log^{\frac{1}{2}} x)\leqq Q\leqq x^b$. *Here the term with* $q=1$ *must be read as*

$$\sum_{x}^{x+h}\log p-h,$$

and if there is an exceptional zero $1-\delta$ *of* $L(s,\chi')$, *with* $\delta\log Q\leqq d$, *then the corresponding term must be read as*

$$\sum_{x}^{x+h}\chi'(p)\log p+h\,\xi^{-\delta}, \qquad \textit{for some } \xi\in[x, x+h].$$

In the latter case, the bound on the right of (29) *may be reduced, e.g. by a factor of* $(\delta\log Q)^2$.

Proof. For $q\leqq T\leqq x^{\frac{1}{2}}$, we have by [4, § 17, 19]

$$\sum_{n\leqq x}\chi(n)\Lambda(n)=\delta_\chi x-\sum\frac{x^\rho}{\rho}+O\left(\frac{x\log^2 x}{T}\right),$$

where $\delta_\chi=1$ or 0 according as $\chi=\chi_0$ or not, and the sum on the right is over the zeros of $L(s,\chi)$ in $0\leqq\sigma\leqq 1$, $|t|\leqq T$. The terms with $n=p^\nu$, $\nu\geqq 2$ contribute $\ll x^{\frac{1}{2}}$ to the sum, and therefore may be absorbed into the remainder. Since

$$\frac{(x+h)^\rho}{\rho}-\frac{x^\rho}{\rho}=\int_x^{x+h} y^{\rho-1}\,dy\ll h\,x^{\beta-1},$$

for $Q\leqq T$ we get, using $x/h\leqq Q$ and $\log x\leqq\log^2 Q$,

$$\sum_{x}^{x+h}\chi(p)\log p\ll h\left(\sum x^{\beta-1}+Q^2/T\right),$$

where, if $q=1$ or $\chi=\chi'$, the left side must be read as above, and the sum on the right is over non-exceptional zeros. Hence

$$\sum_{q\leq Q}\sum_{\chi}{}^{*}\left|\sum_{x}^{x+h}\chi(p)\log p\right|\ll h\Big(\sum_{q\leq Q}\sum_{\chi}{}^{*}\sum x^{\beta-1}+Q^{4}/T\Big). \tag{30}$$

The triple sum on the right is

$$-\int_{0}^{1} x^{\alpha-1}\,d_{\alpha}\Big(\sum_{q\leq Q}\sum_{\chi}{}^{*} N_{\chi}(\alpha,T)\Big)=\int_{0}^{1} x^{\alpha-1}\log x\sum\sum d\alpha+x^{-1}\sum\sum N_{\chi}(0,T).$$

Using Theorem 6, and assuming $T^{c}\leq x^{\frac{1}{2}}$, this is

$$\int_{0}^{1-\eta(T)} x^{\frac{1}{2}(\alpha-1)}\log x\,d\alpha+x^{-\frac{1}{2}}\ll x^{-\frac{1}{2}\eta(T)},$$

where $1-\eta(T)$ is either the bound (27) or the bound (26) according as there is an exceptional zero or not.

We choose $T=Q^{5}$. If there is no exceptional zero, then the parentheses on the right of (30) is

$$\ll\exp\left(-c_{3}\frac{\log x}{\log Q}\right)+Q^{-1}\ll\exp\left(-a\frac{\log x}{\log Q}\right).$$

If there is an exceptional zero, the parentheses is

$$\ll\exp\left(-c_{4}\frac{\log x}{\log Q}\log\left(\frac{c_{5}}{\delta\log Q}\right)\right)+Q^{-1},$$

$$\ll(\delta\log Q)^{c_{6}\frac{\log x}{\log Q}}+(\delta\log Q)^{2}\,Q^{-\frac{1}{2}}\qquad\text{(by (28))},$$

$$\ll(\delta\log Q)^{2}\exp\left(-a\frac{\log x}{\log Q}\right).$$

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