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Autor: SMALE, S.

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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Topology and Mechanics. II

The Planar n-Body Problem

S. SMALE* (Pisa)

Introduction

Recall the setting of the planar n-body problem of celestial mechanics. One is given n positive numbers, the masses $m_1, \ldots, m_n$. Then the configuration space (with center of mass fixed at the origin, once and for all) is the linear space

$$M = \{x = (x_1, \ldots, x_n) \in (E^2)^n \mid \sum m_i x_i = 0\}.$$

The phase space is the tangent bundle $T(M) = M \times M$. We have the canonical product structure on $T(M)$ because M is a linear space. The kinetic energy $K: T(M) \to \mathbf{R}$ is the function $K(x, v) = K(v) = \frac{1}{2} \sum m_i \|v_i\|^2$ where $(x, v) \in M \times M = T(M)$, $v = (v_1, \ldots, v_n)$, $\sum m_i v_i = 0$, and $\|v_i\|$ is the norm of v_i in E^2.

Let $\Delta_{ij} = \{(x_1, \ldots, x_n) \in M \mid x_i = x_j\}$, $\Delta = \bigcup \Delta_{ij}$ and define the potential energy $V: M - \Delta \to \mathbf{R}$ by $V(x_1, \ldots, x_n) = -\sum \dfrac{m_i m_j}{\|x_i - x_j\|}$. Here the union, sum are taken over all pairs (i, j) with $i \neq j$. Via Newton's equations there is defined the dynamics, given by a vector field X_L on the tangent bundle $T = T(M - \Delta)$. Properly speaking the configuration space of the planar n-body problem is $M - \Delta$ and the phase space is T. We let $\pi: T(M) \to M$ denote the projection (and its various restrictions as well).

The total energy $E: T \to \mathbf{R}$, given by $E = K + V \circ \pi$ or $E(x, v) = K(v) + V(x)$, is an integral of X_L. That is to say E is constant on each orbit. A second integral, angular momentum $J: T \to \mathbf{R}$ is defined by $J(x, v) = J_x(v) = \sum m_i x_i \times v_i$ where for example $x_i \times v_i$ can be defined by $x_i^1 v_i^2 - x_i^2 v_i^1$ in terms of Cartesian coordinates on E^2.

The group of rotations of E^2, $G = S^1$ acts on M and thus T by the restriction of the diagonal action, leaving invariant Δ, V, K, E, J, and X_L.

Let $I: T \to \mathbf{R} \times \mathbf{R}$ be given by $I(u) = (E(u), J(u))$ for $u \in T$. Note that I is constant on orbits.

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For $(c, p) \in \mathbf{R} \times \mathbf{R}$ let $I_{c,p} = I^{-1}(c, p)$; that is to say $I_{c,p}$ consists of all states with angular momentum p and energy c. This space $I_{c,p}$ is invariant under both the dynamics and the symmetry group S^1. In fact on the quotient, $\tilde{I}_{c,p} = I_{c,p}/S^1$ there is an induced vector field. We call these spaces $I_{c,p}$, $\tilde{I}_{c,p}$, by a slight abuse of notation, the *integral manifolds* of the planar n-body problem.

The goal of this paper is to make a global or topological study of the map $I: T \to \mathbf{R} \times \mathbf{R}$. This will have two main aspects; first, the investigation of the topological type of the integral manifolds and second, finding the structure of the bifurcation set of I, to be described shortly. Our first theorem on the integral manifolds is the following.

Theorem A. *In the planar n-body problem for any given choice of masses, suppose* $p = 0$. *Then*

(a) *for each* $c \geqq 0$, *there are diffeomorphisms*

$$\varphi_c: I_{c,p} \to S(E(S_K - \Delta)) \times \mathbf{R}$$

and

$$\tilde{\varphi}_c: \tilde{I}_{c,p} \to S(\tilde{E}(P_{n-2}(\mathbf{C}) - \Delta)) \times \mathbf{R},$$

(b) *for each* $c < 0$, *there are diffeomorphisms*

$$\varphi_c: I_{c,p} \to E(S_K - \Delta)$$

and

$$\tilde{\varphi}_c: \tilde{I}_{c,p} \to \tilde{E}(P_{n-2}(\mathbf{C}) - \Delta).$$

All of these maps are smooth in c.

Some explanations and remarks are in order.

Here S_K is the unit sphere in M with the K inner product on M. That is to say $S_K = \{z \in M \mid K(z) = 1\}$. Next $S_K - \Delta$ is S_K with $\Delta \cap S_K$ removed. $E(S_K)$ is a Riemannian vector space bundle over S_K and $E(S_K - \Delta)$ is the part over $S_K - \Delta$. This bundle will be described in Section 3. If E is a Riemannian vector space bundle, then $S(E)$ denotes the associated unit sphere bundle. $\tilde{E}(P_{n-2}(\mathbf{C}))$ is a Riemannian vector space bundle over complex projective space $P_{n-2}(\mathbf{C}) = S_K | S^1$ and will also be described in Section 3. $\tilde{E}(P_{n-2}(\mathbf{C}) - \Delta)$ is the part of this bundle over $P_{n-2}(\mathbf{C})$ with certain projective subspaces corresponding to Δ removed. Finally since the $I_{c,p}$, $\tilde{I}_{c,p}$ are naturally sub-manifolds of T for the various values of c, one can define smoothness in c simply by saying that the induced map on the subset of $\mathbf{R} \times T$ is smooth.

In (a) of Theorem A, the bundles are clearly trivializable, thus we have the following corollary.

Corollary. *In the planar n-body problem for zero angular momentum, if the energy* $c \geqq 0$, *then the integral manifold* $I_{c,p}$ *is diffeomorphic to*

$S^{2n-4} \times (S_K - \Delta) \times \mathbf{R}$. *If the energy* c *is negative, and* $p=0$, *then* $I_{c,p}$ *is diffeomorphic to* $\mathbf{R}^{2n-3} \times (S_K - \Delta)$.

In this paper S^q is the q-sphere and $\mathbf{R}^q$ Cartesian q-space.

Our next theorem on the integral manifolds is:

Theorem B. *In the planar n-body problem for any given choice of the masses, suppose* $c \geqq 0$ *and* $p \neq 0$. *Then there exist diffeomorphisms smooth in* (c, p),

$$\varphi_{c,p}\colon\ I_{c,p} \to E(S_K - \Delta)$$

and

$$\tilde{\varphi}_{c,p}\colon\ \tilde{I}_{c,p} \to \tilde{E}(P_{n-2}(\mathbf{C}) - \Delta).$$

with these spaces defined just as in part (b) *of Theorem* A.

To understand our results on the integral manifolds and bifurcation in the negative energy half plane, $p \neq 0$, we define relative equilibria.

Given $(m_1, \ldots, m_n)$, a position, $x = (x_1, \ldots, x_n) \in M - \Delta$ of the n-bodies is said to be a *relative equilibrium* and $x \in R_e$ if there is a 1-parameter group of rotations of E^2 which induces a motion of the x_i, $\varphi_t(x) = (\varphi_t(x_1), \ldots, \varphi_t(x_n))$ satisfying the equations of motion (of Newton).

Then S^1, the group of rotations of E^2, acting on M, leaves R_e invariant; non-zero scalar multiplication on M also leaves R_e invariant. Let $\bar{R}_e$ be the set of equivalence classes of R_e, $x, x^1 \in R_e$ considered equivalent if they differ by a rotation followed by a scalar multiplication.

Then R_e, $\bar{R}_e$ depend on the given masses only. One could have started just as well with E^3, the 3-space problem to define R_e, $\bar{R}_e$. However the $\bar{R}_e$ are the same for E^2, E^3 if of course the masses are the same.

The following facts are known cf. [5].

(a) If $n=2$, $\bar{R}_e$ has one element.

(b) For $n=3$, $\bar{R}_e$ has five elements (Euler and Lagrange). In the 3-body problem for given m_1, m_2, m_3, there are 3 collinear relative equilibrium classes (Euler). This means that the three x_i lie in a line. There are two relative equilibrium classes formed by the x_i at vertices of an equilateral triangle (Lagrange), the two differing by a permutation.

(c) There are $n!/2$ collinear relative equilibrium classes for each n (Moulton [3]). We give a proof of this fact in the appendix using critical point theory and Theorem C below.

(d) For the cases $n=4, 5$, there are some further results. See [5] for references.

The question as to whether $\bar{R}_e$ is finite for every choice of $(m_1, \ldots, m_n)$ seems to be a major open problem. In any case for $n>3$, the nature of relative equilibrium is not very well understood, and there are many unanswered questions in this area.

As far as the history of these things, Wintner [5] gives the background to the relative equilibria, although in the slightly more general context of central configurations. Also in general this is a good reference to the n-body problem from a very classical point of view.

The following theorem could be considered as implicit in the classical literature (see [5]). In our developement, it comes out as a corollary to a theorem on "rotating coordinate systems" (see (1.1) below). We use this Theorem C in the proof of Theorem D below.

Theorem C. *For any given choice of the masses in the (planar) n-body problem let* $S_K=\{z\in M | K(z)=1\}$ *and* $V_S: S_K-\Delta\to\mathbf{R}$ *be the restriction of the potential energy to* S_K *with* Δ *removed. Let* $z\in M-\Delta$ *be normalized so that* $K(z)=1$. *Then* z *is a relative equilibrium if and only if* z *is a critical point of* V_S.

We proceed now to the bifurcation question.

If $f: P\to Q$ is a smooth map from a manifold P to a manifold Q, then the *bifurcation set of* f, $\Sigma(f)$ is the set of points $y\in Q$ over which f fails to be locally trivial. That is to say $y\notin\Sigma(f)$ if and only there is an open set U containing y and a map $g: f^{-1}(U)\to f^{-1}(y)$ such that $h: f^{-1}(U)\to f^{-1}(y)\times U$ is a diffeomorphism where $h(x)=(g(x), f(x))$. The set of critical values of f (i.e. the points y, where $f(x)=y$ for some x with the derivative $Df(x): P_x\to Q_y$ not surjective) will be denoted by $\Sigma'(f)$. Clearly $\Sigma'(f)$ is a subset of $\Sigma(f)$.

Write $\Sigma=\Sigma(I)$ and $\Sigma'=\Sigma'(I)$.

Theorem D. *Given the masses of the planar n-body problem*

(a) *The set* Σ' *of critical values of* $I: T\to\mathbf{R}\times\mathbf{R}$ *is given by* $\Sigma'=\bigcup_{\alpha\in\bar{R}_e}\Sigma_\alpha$ *where*

$$\Sigma_\alpha=\{(-V(pz)^2, p)\in\mathbf{R}\times\mathbf{R} | p\neq 0, z\in\alpha, K(z)=1\}.$$

(b) *The bifurcation set* Σ *of* I *is precisely the union of* Σ' *with the* 2 *coordinate axes of* $\mathbf{R}\times\mathbf{R}$.

Note that $V(pz)=\dfrac{1}{p}V(z)$ so that for each α, Σ_α is a cubic curve in $\mathbf{R}\times\mathbf{R}$. Note also that $\bar{R}_e$ may be a finite set (see above) for every choice of the masses. Finally we remark that $V(pz)$ is constant on $z\in\alpha$ with $K(z)=1$.

I would expect that generically (i.e. for almost all values of $(m_1, \ldots, m_n)$, the Σ_α corresponding to the collinear relative equilibria to be distinct from each other and from the other relative equilibria.

We pass now to the integral manifolds for negative energy and non-zero angular momentum. If Q is a manifold with boundary ∂Q, a new smooth manifold $\beta_k(Q)$ is obtained by identifying for each $q\in\partial Q$ the

subset $S^{k-1} \times q$ to a point in $S^{k-1} \times Q$. More generally if E is a Riemannian vector space bundle over a manifold with boundary, $\beta(E)$ will denote the smooth manifold obtained by taking the associated unit sphere bundle and identifying to a point each fiber over the boundary. Details of these things are given in Section 5.

Let $l_p: S_K - \Delta \to \mathbf{R}$ be the function defined by $l_p(z) = -V(p\,z)^2$, V being the potential energy. Let $Q_{c,p} = l_p^{-1}(-\infty, c]$. If c is a regular value of l_p then $Q_{c,p}$ is a manifold with boundary and $\beta_k(Q_{c,p})$ is a smooth manifold. But for every c, $\beta_k(Q_{c,p})$ can be defined as a topological space in the same way, $l_p^{-1}(c)$ playing the role of the boundary.

One can also define $\tilde{Q}_{c,p}$ to be the subspace of $P_{n-2}(\mathbf{C}) - \Delta$ defined by $\tilde{Q}_{c,p} = \tilde{l}_p^{-1}(-\infty, c]$ where $\tilde{l}_p P_{n-2}(\mathbf{C}) - \Delta \to \mathbf{R}$ is induced on the quotient by l_p.

With the above and using the notation of Theorem A we have

Theorem E. *For any given choice of the masses in the planar n-body* problem, with $c<0$ and $p \neq 0$, $I_{c,p}$ *has the structure of* $\beta_k(Q_{c,p})$ *and* $\tilde{I}_{c,p}$ *is* $\beta(\tilde{E}_a(\tilde{Q}_{c,p}))$ *where* $k=2n-2$, *and* $\tilde{E}_a$ *is the bundle described in* (9.4) *of* [4]. *Furthermore there is a function* $\gamma(p)<0$ *for* $p \neq 0$ *(defined by the cubic* Σ_α *farthest to the right in* $\mathbf{R} \times \mathbf{R}$*) such that for* $\gamma(p)<c<0$, $I_{c,p}$ *is diffeomorphic to* $S^{k-1} \times (S_k - \Delta)$ *and* $\tilde{I}_{c,p}$ *is* $S(\tilde{E}_a(P_{n-2}(\mathbf{C}) - \Delta))$.

Theorem E reduces the study of the integral manifolds of the planar n-body problem $c<0$, $p \neq 0$ essentially to the study of the potential energy on the sphere $S_K - \Delta$ with the diagonal removed.

One sees easily that for the case $n=2$, the preceeding theorems yield:

1st Corollary. *For the 2-body problem,* $S_K - \Delta = S^1$, $\bar{R}_e$ *has one element,* $I_{c,p}$ *is* $\emptyset$, S^1, $S^1 \times S^1$ *or* $S^1 \times \mathbf{R}$ *for* $p \neq 0$ *and* $S^1 \times \mathbf{R}$ *or* $S^0 \times S^1 \times \mathbf{R}$ *for* $p=0$. *The bifurcation set consists of a single cubic and the coordinate axes.*

This checks for example with Section 1 of [4].

Now consider the case $n=3$.

Given 3 masses in the 3-body problem, the restriction of the associated potential function V_S: $S_K - \Delta \to \mathbf{R}$ induces a function $\tilde{V}_S$ on the quotient $(S_K - \Delta)/S^1 = P_1(\mathbf{C}) - \Delta = S^2 - \Delta$ where the quotient is a 2-sphere minus 3 points corresponding to Δ. $\tilde{V}_S$ is everywhere negative and tends to $-\infty$ at each of the 3 points of Δ and has 5 critical points. The values of the 2 corresponding to the Lagrange relative equilibria coincide.

Using our theorems above, one can obtain the following.

2nd Corollary. *For any given choice of the masses in the planar* 3*-body problem*

(a) *The set of critical values* Σ' *of* I: $T \to \mathbf{R} \times \mathbf{R}$ *consists of* 4 *cubic curves in the energy-angular momentum plane given by the equations*

$\{(c, p) \in \mathbf{R} \times \mathbf{R} | c p^2 = d_i\}$ $d_i < 0$, $i = 1, 2, 3, 4$ *with the* d_i *not necessarily distinct.*

(b) *The bifurcation set* Σ *of* I *consists of* Σ' *together with the coordinate axes.*

(c) *For* $p = 0$, $c < 0$ *or for* $p \neq 0$, $c \geqq 0$, $I_{c,p} =$ *(i.e. is diffeomorphic to)* $\mathbf{R}^3 \times (S^3 - \Delta)$ *and* $\tilde{I}_{c,p} = \mathbf{R}^3 \times (S^2 - 3 \text{ pts.})$. *Note that* $S^3 - \Delta = S^1 \times (S^2 - 3 \text{ pts.})$.

(d) *For* $p = 0$, $c \geqq 0$, $I_{c,p}$ *is diffeomorphic to* $S^2 \times \mathbf{R} \times (S^3 - \Delta)$ *and* $\tilde{I}_{c,p} = S^2 \times \mathbf{R} \times (S^2 - 3 \text{ pts.})$.

(e) *For each* $p \neq 0$, *there is* $c_1(p) < 0$ *(in fact a cubic) such that for* $c < c_1(p)$, $I_{c,p} =$ *the disjoint union of* 3 *copies of* $S^1 \times (S^5 - S^3)$ *and* $\tilde{I}_{c,p}$ *is the disjoint union of* 3 *copies of* $S^5 - S^3$. *Here* $S^3 \subset S^5$ *is the standard sphere so that* $S^5 - S^3 = S^1 \times \mathbf{R}^4$.

(f) *For each* $p \neq 0$, *there is a constant* $c_4(p) < 0$ *(another cubic) such that for* $c_4(p) < c < 0$, $I_{c,p}$ *is diffeomorphic to* $\beta_4(S^3 - \Delta)$ *or* $S^3 \times (S^3 - \Delta)$ *and* $\tilde{I}_{c,p} = S^3 \times (S^2 - 3 \text{ pts.})$.

Now after this corollary there still remains the question of the structure of $I_{c,p}$, $\tilde{I}_{c,p}$ for $p \neq 0$ and $c_1(p) < c < c_4(p)$ for $n = 3$. For this consider the following hypothesis which would seem to be true.

Hypothesis. $\tilde{V}_S$: $S^2 - 3 \text{ pts.} \to \mathbf{R}$ has 3 non-degenerate saddle points (the Euler points) and 2 non-degenerate maximums (the Lagrange points).

With this hypothesis and our above theorems it is an easy exercise to write down the topology of the integral manifolds for the remaining cases. We leave this to the reader.

Birkhoff at the end of his book "Dynamical Systems" [2] discusses these questions here for the problem in 3-space. He restricts his discussion to $n = 3$, negative energy, non-zero angular momentum and the result he states (in our language) is that $\tilde{I}_{c,p}$ changes just at the Lagrange and Euler relative equilibria (without studying the structure of Σ or Σ' themselves). He gives some of the argument for this without attempting to make a complete proof. Birkhoff also raises the question as to what is the topology of the integral manifolds (for the 3-body problem in E^3 where he relates his question to regularization). Wintner [5] raises a number of similar questions.

For background material of this paper, [4] is recommended. Also motivation is given in [4] for the study of the bifurcation set. Smooth means C^∞, and if ψ_c: $X \to Y$ depends on $c \in \mathbf{C}$, then smooth in c means that the induced map $\mathbf{C} \times X \to Y$ is smooth.

Section 1

The goal of this section is to give a proof of the following theorem, as well as background generalities on mechanical systems.

(1.1) Theorem. *Suppose* (M, K, V, G) *is a mechanical system with symmetry in the sense of* (4.1) *of* [4] *so that* M *is configuration space,* K *kinetic energy,* V *potential energy and the Lie group* G *acts on* M *leaving* K *and* V *invariant. Let* $X \in \mathfrak{G}$, *the Lie algebra of* G. *Then* X *can also be considered as a vector field on* M *generating local solution curves, say* $\psi_t(x)$, $x \in M$. *Denote by* $\varphi_t(X(x))$ *the solutions in* M *of the second order equation giving the dynamics of this mechanical system. Then* $\psi_t(x) = \varphi_t(X(x))$ *all* t *if and only if* x *is a critical point of the real valued function on* M *which sends* x *into* $V(x) - K(X(x), X(x))$.

Remark. In the case of the classical n-body problem, the projected orbits described in (1.1) are the relative equilibria. Thus (1.1) gives a critical point criterion for finding the relative equilibria which is exploited in the following section to prove Theorem C.

Theorem (1.1) of course is also valid when $V \equiv 0$ and one is dealing with problems of geodesics on a Riemannian manifold possessing a group of isometries. The reader may for example test (1.1) for geodesics on surfaces of revolution, or homogeneous spaces with an invariant metric. (See R. Hermann, Indag. Math. **24**, 291–298 (1962).)

For the proof of (1.1), we make some excursion into a certain class of dynamical systems, putting some classical facts into the context of manifolds. We first take the Lagrangian point of view, then the Hamiltonian.

To Birkhoff and Wintner, "*dynamical system*" had the following meaning. (I take the liberty of extending their context to manifolds, mainly for clarity.) One has as configuration space a C^∞ manifold M with phase space as its tangent bundle $T = T(M)$, and the projection $\pi\colon T \to M$. Moreover one has given a smooth function L called the Lagrangian on T satisfying the following:

(1.2) *Condition.* L on each fiber of T is a polynomial of degree two and in the second order terms is positive definite.

As a consequence of (1.2) one can write L as a sum of its homogeneous parts, $L = K + F - V \circ \pi$. Thus K corresponds to a Riemannian metric on M as the kinetic energy, V is a function on M like the potential energy while F being linear on each T_x can be thought of simply as a 1-form on M.

The equations of motion correspond to a vector field X_L on T defined by L by Lagrange's equations via Hamilton's principle. The "integral of Jacobi" is the function $E = K + V \circ \pi$ on T. So E is an integral which is the usual total energy in the case $F = 0$. See [1] for a reference for this

development in a more general context, or say [5] from a classical point of view.

One has a similar situation from the Hamiltonian point of view. In this case configuration space M is given together with a function H: $T^* \to \mathbf{R}$ on the cotangent space $T^* = T^*(M)$ which satisfies the analogous condition to (1.2).

(1.2′) *Condition.* H on each fiber of T^* is a polynomial of degree two and in the 2nd order terms is positive definite.

Again as a consequence of (1.2′), H can be written as a sum of its homogeneous parts $H = K_1 + f_X + V_1 \circ \pi$ where K_1 is given by a Riemannian metric on $T^* \to M$, X is a tangent vector field on M, $f_X(w_x) = (w_x, X(x))$, $w_x \in T_x^*$ and V_1 is a function on M. Via the canonical symplectic form on T^*, H defines a vector field X_H on T^* which gives the dynamics of the system.

In this case H itself is an integral. Note the following aspect which can lead to much confusion if one is not careful. In the Lagrangian point of view, the integral E is independent of the linear terms in L, while in the Hamiltonian point of view, the integral depends directly on the linear terms of H. Again one can see [1] for a developement in a more general context.

Now one can pass from the Lagrangian point of view to the Hamiltonian (and back again) by a map Φ_L: $T \leftrightarrow T^*$ called the Legendre transformation. One can compute directly the Φ_L in this case using [1], [5] for background, obtaining

$$\Phi_L(v_x)(w_x) = 2K(v_x, w_x) + F(w_x),\ v_x, w_x \in T_x,$$

and that Φ_L is a diffeomorphism.

If one takes $H = E \circ \Phi_L^{-1}$, given L satisfying (1.2), then H satisfies (1.2′) and Φ_L maps the dynamical system defined by L from the Lagrangian point of view into that defined on T^* by H via the Hamiltonian point of view.

We will use $\tilde{}$ to denote dual objects in the following way. $\tilde{K}$ is the Riemannian metric on T^* dual to K on T; if X is a vector field on M then $\tilde{X}$ denotes the dual 1-form defined via K, and if F is a 1-form $\tilde{F}$ will denote the dual vector field defined by K. The following proposition relates the homogeneous components of L and H in this situation.

(1.3) Proposition. *Let M be a manifold with Lagrangian satisfying* (1.2). *If $E = K + V \circ \pi$ and Φ_L is the Legendre transformation, then $H = E \circ \Phi_L^{-1}$ satisfies* (1.2′) *and Φ_L maps orbits of X_L onto orbits of X_H. Furthermore, continuing to use the notation just after* (1.2) *and* (1.2′), $K_1 = \frac{1}{4}\tilde{K}$, $X = -\frac{1}{2}\tilde{F}$ *and* $V_1 = V + K(X, X)$.

Proof. Apply the above formula for Φ_L.

We remark that the *action* $A: T \to R$ in this context is defined simply by $A(v_x) = \Phi_L(v_x)(v_x) = 2K(v_x) + F(v_x)$.

For the proof of (1.1) we will also need

(1.4) Proposition. *Given L, H as in* (1.3), *the singular points of the equations of motion X_L on T or X_H on T^* are precisely of the form on T, the* 0*-vector at x, 0_x where the derivative, $DV(x) = 0$; and on T^*, $\Phi_L(0_x) = F_x$ where again $DV(x) = 0$. All the above uses the notation of* (1.3).

Proof. The singular points of X_H on T^* are precisely the points w where $DH(w) = 0$ by Hamilton equations. Since Φ_L is a diffeomorphism and $H = E \circ \Phi_L^{-1}$, this means that X_L has zeros precisely at the points v where $DE(v) = 0$. But $DE(v) = 0$ if and only if $v = 0_x$ and $DV(x) = 0$. The rest follows.

Let X be a vector field on M. Now let $\hat{X}$ be the vector field on T^* induced by X and ψ_t the (local) 1-parameter group on T^* generated by $\hat{X}$ (see [1] p. 97). Thus ψ_t is obtained by taking the derivatives of the (local) 1-parameter group generated by X. Thus ψ_t preserves the canonical 2-form on T^*. Now let $\psi: \mathbf{R} \times T^* \to \mathbf{R} \times T^*$ be defined by $\psi(t, w) = (t, \psi_t(w))$. Strictly speaking since ψ_t may be only defined for small t, ψ has as domain some open set of $\mathbf{R} \times T^*$ containing $0 \times T^*$. Following the notation of [1] let ψ_* be the map on vector fields of $\mathbf{R} \times T^*$ defined by $(\psi^{-1})^*$.

(1.5) Proposition. *Let X be a vector field on M, $f_X: \mathbf{R} \times T^* \to \mathbf{R}$ defined by $f_X(t, \eta_x) = (\eta_x, X(x))$ and ψ_* defined using X as in the previous paragraph. Suppose $H: R \times T^* \to \mathbf{R}$ is some function. Then $\psi_* \tilde{X}_H = \tilde{X}_K$ where $K = H \circ \psi + f_X$.*

Here $\tilde{X}_H, \tilde{X}_K$ are the time dependent Hamiltonian vector fields defined from H, K respectively. See [1]. We have used $\tilde{}$ in two different ways, but from the context, there will be no ambiguity. Essentially for each t, $\tilde{X}_H$ on $t \times T^*(M)$ is the usual Hamiltonian vector field corresponding to the restriction $H: t \times T^*(M) \to \mathbf{R}$. All of this data is defined in at least some open set in $\mathbf{R} \times T^*$ containing $0 \times T^*$.

The proof of (1.5) is a consequence of Hamilton-Jacobi theory. For example use 21.8 and 21.9 in [1].

(1.6) Proposition. *Let H be a Hamiltonian (i.e. a function) on T^*, X a vector field on M such that the induced vector field $\hat{X}$ on T^* applied to H is zero. Define $H' = H + f_X$, where $f_X: T^* \to \mathbf{R}$ is $f_X(\eta_x) = (\eta_x, X(x))$. Then the zeros of $X_{H'}$ are precisely the solution curves of X_H which project into M into solution curves of X.*

Proof of (1.6). We use (1.5), with $K=H'$ and the function H, K of (1.5) defined from the H, H' of (1.6) by making them constant in t. So from (1.5) one obtains $\psi_* \tilde{X}_H = \tilde{X}_{H'}$, or $\tilde{X}_H = \psi^* \tilde{X}_{H'}$. Let φ_t be the local 1-parameter group generated by X_H, φ'_t by $X_{H'}$. Then the last equation translates into $\varphi_t = \psi_t \varphi'_t$. Thus $\varphi'_t(w) = w$ is the condition that $\varphi_t(w) = \psi_t(w)$ and we obtain the conclusion of (1.6).

Now we prove (1.1). Let $H = E \circ \Phi_L^{-1}$ where $L = K - V \circ \pi$ $E = K + V \circ \pi$. Thus $H = \frac{1}{4}\tilde{K} + V \circ \pi$ by (1.3). Let $H' = H + f_X$, X the vector field on M as in (1.1). By (1.6) we are looking for the zeros of $X_{H'}$. Using (1.3) again, the L' corresponding to H' is given by $L' = K - 2\tilde{X} - (V - K(X, X))$ and the corresponding integral of Jacobi, $E^1 = K + V - K(X, X)$. Then (1.1) follows from (1.4).

Section 2

The goal of this section is to give the proofs of Theorem C and part (a) of Theorem D. We use the function V_p of Section 6 of [4] and we refer the reader to this reference for a background in what follows.

Recall from this reference that given a mechanical system with symmetry (M_0, K, V, G), there is associated a function $V_p: M_0 - \Lambda \to \mathbf{R}$ where in our case, $\Lambda = \emptyset$, $M_0 = M - \Delta$, angular momentum p takes values in R and simply $V_p(x) = V(x) + \dfrac{p^2}{4K(x)}$.

Write $S_K = \{z \in M \mid K(z) = 1\}$ so that $M - 0 = \mathbf{R}^+ \times S_K$, the diffeomorphism $M - 0 \to \mathbf{R}^+ \times S_K$ being given by $x \to \left(\|x\|_K, \dfrac{x}{\|x\|_K} \right)$.

Under this same map $M - \Delta$ is carried diffeomorphically onto $\mathbf{R}^+ \times (S_K - \Delta)$. Let $V_S: S_K - \Delta \to \mathbf{R}$ be the restriction of $V: M - \Delta \to \mathbf{R}$. Let $\sigma(f)$ denote the set of critical points of a map f.

(2.1) Proposition.

(a) $\sigma(V_p) = \left\{ (t, z) \in M - \Delta \,\middle|\, z \in \sigma(V_S), t = \dfrac{-p^2}{2V(z)} \right\}$.

(b) $\sigma(V - \|X\|_K^2) = \left\{ (t, z) \in M - \Delta \,\middle|\, z \in \sigma(V_S), t = \sqrt{\dfrac{-V(z)}{2\|X(z)\|_K^2}} \right\}$.

The function in (b) is that from $M - \Delta \to R$ of (1.1) where $G = S'$ and X can be characterized by a real number. Note that in (a) the case $p = 0$ is included and so there are no critical points of V_p in this case.

Proof of (2.1). By calculus (t, z) is a critical point of V_p when the two partial derivatives are zero or $D_t V_p(t, z) = 0$ and $D_z V_p(t, z) = 0$. Now

$$D_t V_p(t, z) = D_t \left(\frac{V(z)}{t} + \frac{p^2}{4t^2} \right) = \frac{-V(z)}{t^2} - \frac{p^2}{2t^3}.$$

Setting this to 0 gives us $t=\dfrac{-p^2}{2V(z)}$. On the other hand $D_z V_p(t,z)=\dfrac{1}{t}D_z V(z)$ and the first half of (2.1) follows.

For the second half first note that

$$D_t(V(t,z))-K(X(t,z),X(t,z))=\frac{-V(z)}{t^2}-2t\,K(X(z),X(z))=0$$

and since $t>0$ one obtains that $t=\sqrt{\dfrac{-V(z)}{2\|X(z)\|_K^2}}$. The rest of (2.1) follows from the fact that the map $M\to M$ defined by $x\to X(x)$ is an isometry for the K metric, times a constant factor.

From the second half of (2.1), using (1.1) we obtain immediately this corollary.

(2.2) Corollary. *Let $z\in M$ be normalized so that $K(z)=1$. Then z is a relative equilibrium if and only if z is a critical point of the potential energy $V_S\colon S_K-\Delta\to\mathbf{R}$.*

Here as elsewhere K becomes naturally a function on M itself, since M is a linear space, and K is flat (see Section 4 of [4]). This (2.2) is Theorem C of the Introduction.

Using the first part of (2.1) we obtain

(2.3) Proposition. $\Sigma'(V_p)=\{-V(pz)^2 \mid z\in\sigma(V_S)\}$.

Proof.
$$\begin{aligned}\Sigma'(V_p)&=\left\{V_p(tz)\,\middle|\,z\in\sigma(V_S),\ t=\frac{-p^2}{2V(z)}\right\}\\&=\left\{V(tz)+\frac{p^2}{4t^2K(z)}\,\middle|\,z\in\sigma(V_S),\ t=\frac{-p^2}{2V(z)}\right\}\\&=\left\{-\frac{V(z)^2}{p^2}\,\middle|\,z\in\sigma(V_S)\right\}.\end{aligned}$$

The following proposition is easily checked noting that $J\colon T\to\mathbf{R}$ has no critical points.

(2.4) Proposition. *$u\in T$ is a critical point of $I\colon T\to\mathbf{R}\times\mathbf{R}$ if and only if $J(u)=p$ and $DE_p(u)=0$ where $E_p\colon J^{-1}(p)\to\mathbf{R}$ is the restriction of $E\colon T\to\mathbf{R}$. Furthermore $(c,p)\in\Sigma'(I)$ if and only if $c\in\Sigma'(E_p)$.*

Now it was shown in Section 6 of [4] that $\Sigma'(E_p)=\Sigma'(V_p)$. Putting this together with (2.2), (2.3) and (2.4) we obtain:

(2.5) Proposition. *In the planar n-body problem, $\Sigma'(I)$ is the set of $(c,p)\in\mathbf{R}\times\mathbf{R}$ with $c=-V(pz)^2$, where z is a relative equilibrium with $K(z)=1$.*

This is part (a) of Theorem D of the Introduction.

Section 3

The goal of this section is to give a proof of Theorem A. We first give a description of the bundles involved.

Here $E=E(S_K)$ is the vector space bundle over S_K with fiber E_z for $z\in S_K$ defined by $E_z=J_z^{-1}(0)$. The union of the E_z over $z\in S_K$ has the structure of a vector space bundle. For each z, the inner product K on M induces an inner product on $E_z\subset M$ and thus E has the structure of a Riemannian vector space bundle. The group S^1 acts naturally on E as a group of bundle automorphisms preserving K, and the quotient is a Riemannian vector space bundle over $P_{n-2}(\mathbf{C})$ which we denote by $\tilde{E}$ or $\tilde{E}(P_{n-2}(\mathbf{C}))$. As in the introduction we denote the restrictions of these bundles to $S_K-\Delta$, $P_{n-2}(\mathbf{C})-\Delta$ respectively by $E(S_K-\Delta)$, $\tilde{E}(P_{n-2}(\mathbf{C})-\Delta)$.

Now let $c\geqq 0$, $p=0$. Using the identification $M-\Delta$ with $\mathbf{R}^+\times(S_K-\Delta)$, we are able to write

$$I_{c,0}=\{(t,z,v)\in(M-\Delta)\times M \mid v\in J_z^{-1}(0),\ K(v)=c-V(tz)\}.$$

Note here that $J_{tz}^{-1}(0)=J_z^{-1}(0)$, $\|v\|^2=K(v)$, and $c-V(tz)>0$. Define a map $\varphi_c\colon I_{c,0}\to S(E(S_K-\Delta))\times\mathbf{R}^+$ by $\varphi_c(t,z,v)=\left(\frac{v}{\|v\|}\in E_z, t\right)$. It is seen immediately that φ_c is a diffeomorphism depending smoothly on c. The group S^1 acts on $I_{c,0}$ naturally, and on $S(E(S_K-\Delta))\times\mathbf{R}^+$ by the natural action on the first factor, trivial action on the second. The diffeomorphism φ_c is equivariant and by passage to the quotient we obtain the completion of the proof of part (a) of Theorem A.

Now consider $c<0$, $p=0$. Let $t_1\colon(-\infty,0)\times(S_K-\Delta)\to\mathbf{R}$ be defined by $t_1(c,z)=V(cz)$ so that $t_1(c,z)$ is characterized by $V(t_1(c,z)\,z)=c$.

Then we may write

$$I_{c,0}=\{(t,z,v)\mid 0<t\leqq t_1(c,z),\ v\in J_z^{-1}(0),\ K(v)=c-V(tz)\}.$$

Define

$$\varphi_c\colon I_{c,0}\to E(S_K-\Delta)\quad\text{by}\quad\varphi_c(t,z,v)=v\in E_z.$$

The map φ_c is easily seen to be a homeomorphism and a little extra work shows that φ_c is a diffeomorphism, smooth in c.

Section 4

The goal of this section is to prove Theorem B of the Introduction. Thus we consider $c\geqq 0$, $p\neq 0$.

Define $t_0\colon\mathbf{R}^+\times(S_K-\Delta)\times(\mathbf{R}-0)\to\mathbf{R}$ so that $V_p(t_0(c,z,p),z)=c$; i.e. let $t_0(c,z,p)$ be the unique positive solution of the quadratic equation $\frac{V(z)}{t_0}+\frac{p^2}{4t_0^2}=c$ or $t_0^2c-t_0V(z)-\frac{p^2}{4}=0$. Thus $t_0=\frac{V(z)+\sqrt{V(z)^2+p^2c}}{2c}$

if $c>0$ and $t_0=\dfrac{-p^2}{4V(z)}$ if $c=0$. Note that this defines a smooth function of (c, z, p) and that here $\mathbf{R}^+=[0, \infty)$ contrary to our previous usage $\mathbf{R}^+=(0, \infty)$. We hope this last causes no confusion.

It is easily seen that we may write

$$I_{c,p}=\{(t, z, v)\in(M-\Delta)\times M \mid t_0(c, z, p)\leqq t<\infty, v\in J_x^{-1}(p), K(v)=c-V(t, z)\}.$$

Let D be the subset of $\mathbf{R}^+\times(S_K-\Delta)\times(\mathbf{R}-0)\times\mathbf{R}^+$ of (c, z, p, t) such that $t\geqq t_0(c, z, p)$ and let λ be a smooth function $\lambda\colon D\to\mathbf{R}^+$ which has these properties,

(a) λ is 1 on some neighborhood of the boundary of D (i.e. $t=t_0$).

(b) $\lambda^2(c-V_p(t\,z))$ is a strictly monotonically increasing function of t and tends to ∞ as t tends to ∞.

(c) λ is S' invariant, where S' acts on $S_K-\Delta$ in the usual way, and trivial action elsewhere.

It is not difficult to show that such a λ exists.

Recall from Section 6 of [4] the smooth tangent vector field α_p defined on $M-\Delta$, which was characterized by the two conditions $\alpha_p(x)\in J_x^{-1}(p)$ and $\alpha_p(x)\perp J_x^{-1}(0)$ for each $x\in M-\Delta$.

Using α_p and λ we define our map

$$\varphi_{c,p}\colon I_{c,p}\to E(S_K-\Delta) \quad \text{by} \quad \varphi_{c,p}(t, z, v)=\lambda(c, z, p, t)\big(v-\alpha_p(t\,z)\big)\in E_z.$$

(4.1) Lemma. $K\big(\lambda(v-\alpha_p(t\,z))=\lambda^2(c-V_p(t\,z))\big)$.

Proof. Since K is homogeneous in λ of degree 2, we only note that

$$K\big(v-\alpha_p(t\,z)\big)=K(v)-K\big(\alpha_p(t\,z)\big)=K(v)-V_p(t\,z)=c-V_p(t\,z).$$

(Compare Section 6 of [4].)

Using (4.1) and the properties of λ one can now check that $\varphi_{c,p}$ satisfies the properties of Theorem B. Furthermore $\varphi_{c,p}$ is S' equivariant and so rest of Theorem 3 follows.

Section 5

In Section 8 of [4], we defined the notion of "reduced tangent bundle". Here we develope this idea though abandoning the name.

If E is a vector space bundle over a manifold M with possibly non-empty boundary ∂M, then $\alpha(E)$, the associated pinched cell bundle denotes the differentiable manifold obtained as follows. Take a Riemannian (or smooth Finsler) metric on E and let E_0 be the associated unit cell bundle. Then identify to a point each fiber over the boundary in E_0 to obtain $\alpha(E)$.

The following proposition gives an alternate description of $\alpha(E)$ and shows that as a differentiable manifold $\alpha(E)$ is well-defined.

(5.1) Proposition. *Let E be a vector space bundle equipped with a Riemannian metric over the manifold M, $\partial M=\emptyset$, and $f: M\to\mathbf{R}$ a smooth function with regular value $c\in R$. Let $g: E\to\mathbf{R}$ be the function $g(v_x)=\|v_x\|^2+f(x)$ for $v_x\in E_x$, the fiber over x. Then c is a regular value of g and so $g^{-1}(-\infty, c]=X(E, f, c)$ is a differentiable manifold with boundary. Furthermore if E, E' are two bundles over M as above with c a regular value for two functions f, f_1 with $f^{-1}(-\infty, c]=f_1^{-1}(-\infty, c]$ and E, E' are isomorphic as bundles over M_c (the isomorphism not necessarily preserving the metrics), then $X(E, f, c)$ is diffeomorphic to $X(E', f_1, c)$.*

Here $M_c=\{x\in M\mid f(x)\leqq c\}$ and E_c is E restricted to M_c. One sees that $X(E, f, c)$ is an alternate description of $\alpha(E_c)$. The proof of (5.1) is straightforward.

Remark. One may give a little more general proposition by considering in addition to $g: E\to\mathbf{R}$ of the form $g(v_x)=\|v_x\|^2+f(x)$, a smooth function say $h: E\to\mathbf{R}$ with this property: The restriction of h to each fiber E_x is proper and has only a single critical point, a non-degenerate minimum at the origin and $k: M\to\mathbf{R}$ defined by $k(x)=\min_{v\in E_x} h(x)$ is smooth. Then if c is a regular value of h, $h^{-1}(-\infty, c]=\alpha(E_c)$.

If M is a manifold perhaps with boundary, $\alpha_k(M)$ will denote the differentiable manifold $\alpha(E)$ where E is the trivial bundle $\mathbf{R}^k\times M$ over M. This is well-defined by proposition (5.1).

If E is a vector space bundle let $\partial\alpha(E)=\beta(E)$ and if M is a manifold with boundary let $\partial\alpha_k(M)=\beta_k(M)$. It is easily seen that $\alpha(E)$ has as deformation retract M and we have the following proposition.

(5.2) Proposition. *For any manifold with boundary M and any $k>0, j\geqq 0$, $H^j(\beta_k(M))=H^j(M)+H^{j+1-k}(M, \partial M)$.*

This proposition was given me by Browder but since we don't use it, we omit the (easy) proof.

(5.3) *Examples.*

(a) If $\partial M=\emptyset$, $\beta_k(M)=S^{k-1}\times M$.

(b) $\beta_1(M)=$ the double of M.

(c) $\beta_k(D^l)=S^{k+l-1}$.

(d) If $\partial M_1=\emptyset$, $\beta_k(M_1\times M_2)=M_1\times\beta_k(M_2)$.

(e) Example from mechanics. Let (M, K, V) be a simple mechanical system so that K is a Riemannian metric on M, $\partial M=\emptyset$, and $V: M\to\mathbf{R}$ is a function. Let c be a regular value of V, $M_c=\{x\in M\mid V(x)\leqq c\}$ and T_c the tangent bundle of M restricted to M_c. Then the energy level $E^{-1}(c)$ is $\beta(T_c)$.

Section 6

The goal of this section is to prove Theorem E. Thus we suppose always here that $c<0$, $p\neq 0$.

Recalling that $T=(M-\Delta)\times M=\mathbf{R}^+\times(S_K-\Delta)\times M$, one can write

$$I_{c,p}=\{(t,z,v)\in T\mid v\in J_{tz}^{-1}(p),\ K(v)=c-V(tz)\}.$$

Now let $t^-(c,p,z)$, $t^+(c,p,z)$ be the two real valued positive smooth functions defined on the domain in $(-\infty,0)\times(\mathbf{R}-0)\times(S_K-\Delta)$ of (c,p,z) with $z\in Q_{c,p}$, characterized by

$$V_p(t^-(c,p,z),z)=c,\qquad V_p(t^+(c,p,z),z)=c,\qquad t^-(c,p,z)\leqq t^+(c,p,z).$$

One can take these functions as solutions of

$$\frac{V(z)}{t}+\frac{p^2}{4t^2}=c\quad\text{or}\quad ct^2-V(z)\,t-\frac{p^2}{4}=0$$

or

$$t=\frac{-V(z)\pm\sqrt{V(z)^2+cp^2}}{2c}.$$

Note that t^-, t^+ coincide on $V(z)=-\sqrt{-cp^2}$ which is the boundary of $Q_{c,p}$ and for $V(z)>-\sqrt{-cp^2}$, t^-, t^+ fail to be defined. In fact one sees that the domain of definition for t^-, t^+ is just as stated, i.e., $\dfrac{-V(z)^2}{p^2}\leqq c$ or $l_p(z)\leqq c$ or $z\in Q_{c,p}$.

Recall from Section 4 the smooth tangent vector field α_p defined on $M-\Delta$, characterized by $\alpha_p(x)\in J_x^{-1}(p)$, and $\alpha_p(x)$ is normal to $J_x^{-1}(0)$ in T_x in the K inner product.

We may now alternately use the expression for $I_{c,p}$ given by the following:

(6.1) Lemma.

$$I_{c,p}=\{(t,z,v)\in T\mid v\in J_{tz}^{-1}(p),\ K(v)=c-V(tz),$$
$$z\in Q_{c,p}\ t^-(c,p,z)\leqq t\leqq t^+(c,p,z)\}.$$

Proof. If $K(v)=c-V(tz)$ then

$$K(v-\alpha_p(tz))=K(v)-K(\alpha_p(tz))=c-V(tz)-K(\alpha_p(tz))=c-V_p(tz)\geqq 0$$

so that $V_p(tz)\leqq c$. From the discussion preceeding the lemma it follows that $z\in Q_{c,p}$ and $t^-\leqq t\leqq t^+$. This proves the lemma.

A look at the graph of $V_p(tz)$ for fixed z, p as a function of t is helpful in understanding the previous part of this section and in fact the whole paper. For example $l_p(z)$ is the lowest point on this graph.

Noting that $J_{tz}^{-1}(0)=J_z^{-1}(0)$, let

$$I'_{c,p}=\{(t,z,u)\in T\,|\,t^-\leqq t\leqq t^+, u\in J_z^{-1}(0), K(u)=c-V_p(tz)\}.$$

Then from the previous considerations it follows that the map $\beta_{c,p}$: $I_{c,p}\to I'_{c,p}$ defined $\beta_{c,p}(t,z,v)=(t,z,v-\alpha_p(tz)=u)$ is a diffeomorphism between $I_{c,p}$ and $I'_{c,p}$ which is smooth in c, p.

Our goal is to represent $I'_{c,p}$ as $\beta_k(Q_{c,p})$.

Let E be the bundle over S_K we have seen in the Introduction and Section 3, and let ξ be the trivial line bundle over S_K. By thinking of the fiber ξ_z at $z\in S_K$ as the normal to E_z in M, one sees that the Whitney sum $E+\xi$ is a trivalizable vector space bundle over S_K. Using the notation of Section 5 and counting dimensions, it is then sufficient to represent $I'_{c,p}$ as $\beta((E+\xi)(Q_{c,p}))$ where of course $(E+\xi)(Q_{c,p})$ is the restriction of the bundle $E+\xi$ to $Q_{c,p}$.

Define for each $z\in S_K-\Delta$, $\tau(z)=\tau_p(z)$ to be the unique positive number such that $D_tV_p(tz)]_{t=\tau(z)}=0$. An easy calculation shows that $\tau(z)=\dfrac{-p^2}{2V(z)}$ and that $l_p(z)=V_p(\tau_p(z),z)$. Thus $l_p(z)$ is the minimum value of $V_p(tz)$ as t ranges over $\mathbf{R}^+$.

We construct a diffeomorphism g_p: $\mathbf{R}^+\times(S_K-0)\to\xi$ which is a diffeomorphism "on each fiber" $g_{p,z}$: $\mathbf{R}^+\times z\to\xi_z$ taking $\tau_p(z)$ into 0.

Then let f_p: $(E+\xi)\to\mathbf{R}$ be the function defined on the fiber over z by $f_{p,z}(u,g_{p,z}(t))=K(u)+V_p(tz)$. We note that $f_p^{-1}(c)=I'_{c,p}$ is a consequence of the construction of $I'_{c,p}$ and the map $\beta_{c,p}$. Furthermore $l_p(z)=\min_{u,t} f_{p,z}(u,g_{p,z}(t))$ is a smooth function on $S_K-\Delta$ and f_p satisfies the remarks after (5.1). Therefore $I_{c,p}=I'_{c,p}=\beta_k(l_p^{-1}(-\infty,c])=\beta_k(Q_{c,p})$.

The results about $\tilde{I}_{c,p}$ follow as well.

It is easily seen that l_p is proper and has a maximum value $\gamma(p)<0$ on $S_K-\Delta$. Then for $\gamma(p)<c<0$, $Q_{c,p}=S_K-\Delta$ and so $I_{c,p}$ is $S^{k-1}\times(S_K-\Delta)$. By Theorem D, $\gamma(p)$ is a cubic.

This finishes the proof of Theorem E.

Section 7

We give the proof of part (b) of Theorem D. From the definitions, this is a direct consequence of the following.

(7.1) Proposition. *For any given choice of masses in the planar n-body problem:*

(a) *The bifurcation set Σ of I contains the coordinate axes of the energy-angular momentum plane.*

(b) *Over the set* $\{(c,p)\in R\times R \mid c>0,\ p\neq 0\}$ *the map* $I\colon T\to \mathbf{R}\times\mathbf{R}$ *has a global differentiable product structure.*

(c) *Let* $(c_0,p_0)\in\mathbf{R}\times\mathbf{R}$ *be such that* $c_0<0$, $p_0\neq 0$, *and* $(c_0,p_0)\notin\Sigma'$. *Then there exists a neighborhood* U *of* (c_0,p_0) *in* $\mathbf{R}\times\mathbf{R}$ *and a diffeomorphism* $\varphi_{c,p}\colon I_{c,p}\to I_{c_0,p_0}$ *for each* $(c,p)\in U$, *smooth in* c,p.

Proof. By Theorem E, there is a region Q in $\mathbf{R}\times\mathbf{R}$, $c<0$, with boundary including the set $c=0$ and the set $c<0$, $p=0$ such that for $(c,p)\in Q$, $I_{c,p}=S^{2n-3}\times(S_K-\Delta)$. By Theorem A, Corollary, if $c<0$, $p=0$, $I_{c,p}=S^{2n-4}\times(S_K-\Delta)\times\mathbf{R}$. By Theorem B, if $c\geqq 0$, $p\neq 0$, $I_{c,p}=R^{2n-3}\times(S_K-\Delta)$. Putting these together we see that even the betti numbers of the $I_{c,p}$ change at the coordinate axes, so that surely these coordinate axes are in Σ. This proves (7.1) (a).

Furthermore (7.1)(b) follows directly from Theorem B. Thus it only remains to prove (c). For this we proceed as follows.

Recall that $Q_{c,p}=l_p^{-1}(-\infty,c]$ where $l_p\colon S_K-\Delta\to\mathbf{R}$ is defined by $l_p(z)=\dfrac{-V(z)^2}{p^2}$, and that l_p is proper with its critical values associated to Σ as in Section 2. Thus $Q_{c,p}=\{z\in S_K-\Delta \mid V(z)\leqq\sqrt{-c\,p^2}\}$.

It follows easily that one may choose a neighborhood U of (c_0,p_0) and construct diffeomorphisms $\sigma_{c,p}\colon Q_{c,p}\to Q_{c_0,p_0}$ smooth in $(c,p)\in U$. We will need such a family of diffeomorphisms with an additional growth property near $\partial Q_{c,p}$.

Denote by Y the bundle $E+\xi$ over $S_K-\Delta$ with fiber over $z\in S_K-\Delta$, Y_z and the restriction of Y to $Q_{c,p}$ by $Y_{c,p}$. Recall the function $f_p\colon Y\to\mathbf{R}$ defined in Section 6 with $\min_{y\in Y_z} f_p(y)=l_p(z)$ and satisfying the remark after (5.1). Thus on each Y_z, f_p has the topological structure of a norm in a finite dimensional Hilbert space. One may furthermore write

$$I'_{c,p}(z)=Y_z\cap f_p^{-1}(c)\quad\text{and}\quad I'_{c,p}=\bigcup I'_{c,p}(z).$$

We will define diffeomorphisms $\gamma_{c,p}\colon Y_{c,p}\to Y_{c_0,p_0}$ covering $\sigma_{c,p}\colon Q_{c,p}\to Q_{c_0,p_0}$ which will have the property that besides being smooth in (c,p), they induce by restriction diffeomorphisms $I'_{c,p}\to I'_{c_0,p_0}$. This will be sufficient to finish the proof of (7.1).

One can immediately obtain such a family of homeomorphisms $\gamma_{c,p}$ by simply asking that on each Y_z, $\gamma_{c,p}\colon Y_z\to Y_{\sigma_{c,p}}(z)$ maps the level surface (a sphere) $f_p^{-1}(c)$ into $f_{p_0}^{-1}(c_0)$. In fact this construction can be easily done so that it meets all smoothness conditions except for differentiability at 0 in Y_z when $z\in\partial Q_{c,p}$.

Furthermore if one modifys the diffeomorphisms $\sigma_{c,p}$ near $\partial Q_{c,p}$, one can construct the $\gamma_{c,p}$ so that they meet all the necessary smoothness requirements. We leave these last details to the reader.

Appendix on Moulton's Theorem

Here we give a proof of the following theorem of Moulton [3], a proof using critical point theory which fits into the methods of this paper.

Theorem. *For any given choice of masses in the n-body problem, there are exactly $n!/2$ classes of collinear relative equilibria; that is to say, there are $n!/2$ classes of relative equilibria $x=(x_1, \ldots, x_n)$ where the x_i all belong to the same straight line through the origin.*

For the proof we use the notations, definitions, and Theorems of this paper, especially from the Introduction.

Choose some line $E^1 \subset E^2$. This defines a subset $M_r \subset M$ of $x=(x_1, \ldots, x_n)$ such that each $x_i \in E^1$. Let $S_r = S_K \cap M_r$, $S_r - \Delta = S_r - (S_r \cap \Delta)$.

Now while S' acts on S_K, only the rotation by π radians leaves S_r invariant. Thus $\mathbf{Z}_2$ the group of order 2 acts on S_r and on the quotient we have $(P_{n-2}(\mathbf{R}) - \Delta) \subset P_{n-2}(\mathbf{C}) - \Delta \xrightarrow{\tilde{V}} R$ where $P_{n-2}(\mathbf{R}) = S_r/\mathbf{Z}_2$ is real projective space, contained naturally in $P_{n-2}(\mathbf{C})$. Here $\tilde{V}$ is induced by the potential energy.

From Theorem C it follows immediately that the number of classes of relative equilibria is the number of critical points of $\tilde{V}: P_{n-2}(\mathbf{C}) - \Delta \to \mathbf{R}$ (the normalized relative equilibria themselves corresponding to critical points of V: $S_K - \Delta \to \mathbf{R}$). Now if a relative equilibrium $(x_1, \ldots, x_n)$ is collinear, then one may bring all the x_i into E^1 by a single rotation of E^2. From these considerations we obtain:

Proposition 1. *The classes of collinear relative equilibria are in a bijective correspondence with the critical points of $\tilde{V}$: $P_{n-2}(\mathbf{C}) - \Delta \to \mathbf{R}$ which lie in $P_{n-2}(\mathbf{R}) - \Delta \subset P_{n-2}(\mathbf{C}) - \Delta$.*

Next we reduce the problem to simply the study of the function $\tilde{V}: P_{n-2}(\mathbf{R}) - \Delta \to \mathbf{R}$ by:

Proposition 2. *If $x \in P_{n-2}(\mathbf{R}) - \Delta$ is a critical point of $\tilde{V}: P_{n-2}(\mathbf{R}) - \Delta \to \mathbf{R}$ then x is also a critical point of $\tilde{V}: P_{n-2}(\mathbf{C}) - \Delta \to \mathbf{R}$.*

For the proof of this and also later we will need to know the derivatives of the potential function.

Discussions with Shub were very helpful with this next proposition.

Proposition 3. *For given masses $m=(m_1, \ldots, m_n)$,*

$$V_m(x) = -\sum_{i \neq j} \frac{m_i m_j}{\|x_i - x_j\|},$$

(a) *The first derivative of V_m: $M - \Delta \to \mathbf{R}$ is given by*

$$DV_m(x)(v) = \sum_{i \neq j} \frac{m_i m_j}{\|x_i - x_j\|^3} (x_i - x_j, v_i - v_j)$$

for $v \in M$, *and the* 2nd *derivative*

$$D^2 V_m(x)(v, w) = Q_{x,m}(v, w) = -\sum \frac{m_i m_j}{\|x_i - x_j\|^3} \cdot \left(\frac{3}{\|x_i - x_j\|^2} (x_i - x_j, v_i - v_j)(x_i - x_j, w_i - w_j) - (v_i - v_j)(w_i - w_j) \right)$$

with, $v, w \in M$.

(b) *The second derivative of the restriction* $V_m : S_K - \Delta \to \mathbf{R}$ *is:*

$$D^2 V_m / (S_K - \Delta)(x)(v, w) = Q_{x,m}(v, w) + V_m(x)\, K_m(v, w).$$

Here (,) *denotes the inner product on* E^2, $\| \ \|$ *the norm on* E^2, K_m *the kinetic energy as an inner product on* M *for the given masses* $m = (m_1, \ldots, m_n)$. *The same formulae are valid for* E^1 *or* E^3 *instead of* E^2.

Proof of Proposition 3. For (a) one can compute on $(E^2)^n$ (and restrict). Then

$$DV_m(x)(v) = \sum_{k=1}^{n} D_k V_m(x)(v_k) = -\sum_{i \neq j} \sum_{k=1}^{n} D_k \left(\frac{m_i m_j}{\|x_i - x_j\|} \right)(v_k)$$

and for each i, j

$$\sum_{k=1}^{n} D_k \left(\frac{m_i m_j}{\|x_i - x_j\|} \right)(v_k) = \frac{(x_i - x_j, v_i - v_j)}{\|x_i - x_j\|^3}.$$

These together yield the first part of (a). For the second part of (b), one proceeds in the same way using

$$D^2 V_m(x)(v, w) = \sum D_k D_l V_m(x)(v_k, w_l).$$

For (b) one uses the formula

$$D^2 V_m / S_K - \Delta(x)(v, w) = D^2 V_m(x)\big(i(v), i(w)\big) + DV_m(x)\big(D^2 i(v, w)\big)$$

where $i : S_K \to M$ is the inclusion.

A standard calculation finishes then the proof of Proposition 3.

We now give the proof of Proposition 2. For $v_i \in E^2$, let $v_i = (v_i', v_i'')$ where $v_i' \in E^1$ and $v_i'' \in (E^1)^\perp$ in E^2. Then we can write $v = (v', v'')$ with $v' (= v_1', \ldots, v_n')$ etc. for each $v \in M$. Now if $x \in S_r \subset S_K$, $x \notin \Delta$, $T_x(S_K) = \{v \in M \mid v \perp x\}$ and $T_x(S_r) = \{v' \in M_r \mid v' \perp x\}$ where on M as always we use the K inner product. If $v \in T_x(S_K)$ and $v = (v', v'')$ then $v' \in T_x(S_r)$ since $K(v, x) = K(v', x)$.

By Proposition 3, if $x \in S_r - \Delta$, $v \in T_x(S_K)$, one sees that $DV_m(x)(v) = DV_m(x)(v')$. Then $DV_m(x)(v') = 0$ implies $DV(x)(v) = 0$. This yields Proposition 2.

Proposition 4. *$P_{n-2}(\mathbf{R})-\Delta$ has $n!/2$ components.*

Proof. Let $x=(x_1, \ldots, x_n)\in S_n-\Delta$, and associated to x, let $(i_1, i_2, \ldots, i_n)$ be an ordering of $(1, 2, 3, \ldots, n)$ defined by $i_j<i_k$ if and only if $x_{i_j}<x_{i_k}$ (note that the x_i are all distinct). With this correspondence, one sees that $S_r-\Delta$ has $n!$ components and that the quotient $P_{n-2}(\mathbf{R})-\Delta$ has $n!/2$ components.

Proposition 5. *If $x\in P_{n-2}(\mathbf{R})-\Delta$ is a critical point of $\tilde{V}_m: P_{n-2}(\mathbf{R})-\Delta\to\mathbf{R}$, then x is a non-degenerate maximum.*

Proof. Apply part (b) of Proposition 3 to see that for $V: S_r-\Delta\to\mathbf{R}$, $D^2V_m/(S_r-\Delta)(x)$ is a negative definite form in this case. The proposition follows.

We now obtain the proof of Moulton's Theorem itself. Toward Δ the function $\tilde{V}$ on $P_{n-2}(\mathbf{R})-\Delta$ tends to $-\infty$ and so has a maximum on each component of $P_{n-2}(\mathbf{R})-\Delta$. But elementary Morse theory excludes the possibility of 2 critical points on a single component, since each is a non-degenerate maximum. Thus one counts $n!/2$ critical points and hence $n!/2$ collinear relative equilibria.

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S. Smale
Math. Department
University of California
Berkeley, Calif. 94720, USA

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