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Flux in Axiomatic Potential Theory

I. Cohomology

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0. Introduction

Suppose $(W, \mathscr{H})$ is a pair consisting of a locally compact Hausdorff space W and a complete presheaf $\mathscr{H}$ of vector spaces of continuous scalar-valued functions over W satisfying the axioms for "harmonic" functions of Brelot [4]. The purpose of this paper is the construction of a fine resolution for $\mathscr{H}$ which is well adapted for explicit analytic determination of the sheaf cohomology groups $H^q(W, \mathscr{H})$, together with the determination of those groups for an important class of such presheaves (loosely, those for which there is a globally defined positive superharmonic function) without any additional hypotheses on $\mathscr{H}$. Actually, we shall determine H^q for a somewhat more extensive class of presheaves, which is related to the axiomatic treatment of the normal operators and principal functions of Sario [see, e.g., 13]. Succeeding papers will relate the problem of determining the H^q to other areas of axiomatic harmonic-function theory, particularly the adjoint-sheaf theory of Hervé [9], in the presence of additional assumptions about $\mathscr{H}$.

Cohomological questions arise naturally in the classical problem of "extending singularities" of harmonic functions. The foundation of the theory of principal functions is essentially a Cousin problem: given an open set U containing a compact set A in a Riemann surface W with the usual harmonic functions $\mathscr{H}$, and a function s harmonic in $U \setminus A$ (which we may perhaps think of as defined in A also, although not harmonic there), a function h harmonic in U and a function p harmonic in $W \setminus A$ are to be determined, such that $s-h$ and p agree in $U \setminus A$ and thus together define a function on the whole surface (really only on $W \setminus A$, if s is not defined in A) which "has the same singularities in A" as s did. (Generally W is noncompact and "conditions at an ideal boundary" are placed on p; these conditions are given by the normal operator.) This is a Cousin problem for the open cover $\{U, W \setminus A\}$ of W with datum s in $U \cap (W \setminus A) = U \setminus A$: the problem is that of determining whether the Čech cocycle specified by s is a coboundary. The theorem that such a Čech one-cocycle with coefficients in $\mathscr{H}$ cobounds if and only if it has zero flux through a union of smooth curves in U that

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separates A from the "ideal boundary" is classical; the fact that only a single linear condition on s is involved suggests that the dimension of H^1 (which is clearly a vector space over the scalars) is one, and the form of that condition suggests that the flux functional somehow generates the dual of H^1. Supposing for the sake of example that W is closed (no "ideal boundary" conditions), one can easily confirm this suspicion. Indeed, the sequence of sheaves $0 \to \mathscr{H} \xrightarrow{\text{id}} \mathscr{E} \xrightarrow{\Delta} \mathscr{E}^2 \to 0$ (where $\mathscr{E}$ is the sheaf of C^∞ functions on W, $\mathscr{E}^2$ the C^∞ 2-forms, and the Laplacian is invariantly represented as $\bar{\partial}\partial$) is exact: exactness at $\mathscr{E}^2$ is all that needs checking, and the Newton-kernel solution of Poisson's equation in the plane tells us that. This is a fine resolution of $\mathscr{H}$ [8, p. 177], so we may read off the cohomology groups $H^q(W, \mathscr{H})$, all of which vanish except for $q=0, 1$. By the maximum principle, $H^0(W, \mathscr{H})$ consists only of constant functions; $H^1(W, \mathscr{H}) \cong \Gamma(W, \mathscr{E}^2)/\Delta[\Gamma(W, \mathscr{E})]$, so we can determine it by determining the codimension of the range of the Laplacian on the global C^∞ functions. Showing that this codimension is 1 is an exercise in the manipulation of distributions – or if W is a torus (and we only want an illustrative example), an easier exercise in looking at Fourier series. Identification of the dual of the space $\Gamma(W, \mathscr{E}^2)/\Delta[\Gamma(W, \mathscr{E})]$ with the flux functional comes about via the Gauss-Green relation

$$\int_R \Delta g \, dA = \int_{\partial R} \frac{\partial g}{\partial n} \, dS$$

for regions R with smooth boundary, as the reader will find it amusing to verify.

Unfortunately, this approach is not immediately applicable to normal operators and principal functions. If W is an open Riemann surface, $W \setminus A$ is a "bordered boundary neighborhood" of W and L a normal operator for A ([13, p. 48ff.] – but our boundary neighborhoods are the *complements* of those considered in [13]), then the range of L is a vector space of harmonic functions in a punctured neighborhood of ∞ in the one-point compactification W^* of W. If L was L_0 (say), and $W \setminus B$ is another bordered boundary neighborhood with $B \supseteq A$, then the action of L_0 on B is consistent with its action on A in the obvious sense, so the spaces of functions on punctured neighborhoods of ∞ given by the ranges of the operators L_0 have a presheaf property. A natural thing to do, then, in order to formulate the principal function problem as a Cousin problem like the one we considered on closed surfaces, is to introduce a new sheaf on W^* whose sections over neighborhoods of ∞ are harmonic functions in those neighborhoods (punctured at ∞) whose behavior in complements of compact sets is that forced on them by the normal operator. But it then becomes unclear how to modify the sheaf

resolution in order to approach the principal function problem cohomologically.

Another obvious generalization of the closed-Riemann-surface problem given above arises through axiomatization: replace the compact Riemann surface W by a compact, connected, locally connected Hausdorff space, and the harmonic functions on W by a complete presheaf $\mathscr{H}$ of vector spaces of "harmonic" functions which satisfy the axioms of Brelot [4]. Abstracting away the differential framework in which the original problem was set also abstracts away the sheaves of differentiable functions and differential forms which were used to resolve the sheaf $\mathscr{H}$ of "genuine" harmonic functions, and seems to leave nowhere to start. However, we propose to axiomatize the problem in this fashion, axiomatize normal operators similarly, and then construct a fine resolution of the sheaves that arise in the axiomatic setting. The results in this setting can then be applied not only to normal operators and principal functions on Riemann surfaces (where our cohomological results appear to be new) but also to similar objects on Riemannian manifolds or to the sheaves of solutions of second-order elliptic equations with unsmooth coefficients [9, Chapter VII, p. 560ff.].

The paper is organized as follows: § 1 sketches an approach to the axiomatization of Sario's normal operators – most of the details are omitted, since the approach parallels the elementary part of the Brelot theory very closely. An axiomatization of a similar subject has been carried out by Maeda [11]. In § 2 we define the sheaves that give a fine resolution of $\mathscr{H}$ (and its normal-operator variant), prove that they are fine, and construct the resolution. § 3 treats the axiomatized principal function problem, certain cases of which must be treated in detail as lemmas for the determination of $H^1(W, \mathscr{H})$ (or its normal-operator variant) since they lead to a definition of flux functionals, which represent the dual of $H^1(W, \mathscr{H})$. § 4 is concerned with determining $H^1(W, \mathscr{H})$, the only cohomology group of positive index not explicitly furnished (as zero) by the resolution. The concluding § 5 offers notes on the material of its predecessors.

The author wishes to acknowledge with thanks a number of stimulating discussions. Nakai first brought the case $1 \in \sigma(MH)$ of (3.2) below to the author's attention as a then-unsolved problem. Bungart suggested the problem of determining $H^q(W, \mathscr{H})$ for $q>0$. Brelot made several penetrating observations on some subjects treated here in the course of conversation with the author. Schaefer kindly furnished the author with a preliminary draft of [15], in addition to numerous insights into the subject of positive linear operators.

Throughout most of the discussion below we shall tacitly have chosen and fixed a pair $(W, \mathscr{H})$ consisting of a space and a complete

presheaf of functions satisfying Brelot's axioms. (We shall not reproduce these axioms in full; see [4, p. 61ff.] or [10, pp. 170–171].) It is known that the third axiom is equivalent to an equicontinuity condition; see, e.g., [16, p. 686]. There is no problem involved in complexifying a given "real" $\mathscr{H}$, and we shall do this whenever it may be convenient (e.g., for spectral theory). As a small modification, we shall drop the assumption that W is non-compact; this will not change things locally. We shall, however, assume that W has a countable base for its topology, so that it and its one-point compactification W^* are separable, metrizable, Lindelöf and hereditarily paracompact – and these last properties will be used without explicit reference throughout the sheaf-theoretic § 2. We shall retain the notation $\mathscr{H}_U^c$ of [16] for the space of continuous scalar-valued functions on $\overline{U}$ whose restrictions to U belong to $\mathscr{H}_U$, for open sets $U \subseteq W$. And we shall define a word to describe a well-known concept:

Definition (0.1). *If $\mathscr{H}$ is a (complete pre-) sheaf of "harmonic" functions on W and ϕ is a continuous nonzero-scalar-valued function on W, then the (pre-) sheaves $\mathscr{H}$ and $\phi^{-1} \cdot \mathscr{H}$ (cf.* [10, p. 193]*) will be called multiplicatively equivalent.*

It is quite obvious that multiplicative equivalence is an equivalence relation; if ϕ is positive one can find many of this relation's properties in [10, § 6, p. 193ff.], and extending those to negative ϕ is not difficult. If ϕ is complex-valued one must observe that $\phi^{-1}\mathscr{H}$ does not necessarily satisfy the Brelot axioms when $\mathscr{H}$ does, because the "reality" assumptions built into the axioms are not necessarily preserved.

For all questions of functional analysis in partially ordered vector spaces which we do not explicitly discuss below, we refer the reader to [14, Chapter V and Appendix].

Important Remark. The reader who is interested only in the cohomological properties of (pre-)sheaves of "harmonic" functions, or who is giving this paper its first reading, can avoid the technical complications induced by normal structures (Definition (1.3) below) by adopting the following devices: References to a normal structure $\mathfrak{L}$ (and thus to sheaves $\mathscr{H}^{\mathfrak{L}}$, $\mathscr{R}^{\mathfrak{L}}$, $\mathscr{Q}^{\mathfrak{L}}$, etc.) in the theorems and proofs of § 2 can simply be ignored; the construction given there of a fine resolution for $\mathscr{H}$ is independent of any normal-structure considerations, and will be unaffected. § 3 should be read (with one exception) as referring to a pair $(W, \mathscr{H})$ satisfying the Brelot axioms with W compact; the superscript "$\mathfrak{L}$" appearing on $\mathscr{H}$ then appears vacuously, as the introductory paragraph of that § indicates, and W^* means W. (The exception is Theorem (3.4), a proof of the Hervé extension theorem [9, Theorem 13.2, p. 458], in which the notion of the least Dirichlet operator from

Definition (1.2) below is required.) The operators L_A: $\mathscr{C}(\partial A) \to \mathscr{H}_{W\setminus A}$ are to be read as the operators H: $\mathscr{C}(\partial A) \to \mathscr{H}_{W\setminus A}$ that solve the Dirichlet problem for $\mathscr{H}$ on $W\setminus A$. Such statements as "**1** is (or is not) normal-harmonic" should simply be read as "**1** is (or is not) harmonic;" these statements occur, for example, in Theorems (3.6) and (3.8). Theorem (3.7) becomes redundant. The same procedure can be followed in § 4; the exceptional treatment accorded the point at infinity in that section then becomes superfluous.

1. "Normal" Operators and Normal Structures

We begin by making the only reasonable definition, in the axiomatic setting, of an object resembling the normal operators of [13] but related to the Brelot theory. ($(W, \mathscr{H})$ is fixed throughout this section.)

Definition (1.1). *Let $A \subseteq W$ be a proper compact set. A semi-normal operator on A is a nonnegative linear mapping L: $\mathscr{C}(\partial A) \to \mathscr{H}^c_{W\setminus A}$ for which $Lf|\partial A = f$. If $L\mathbf{1} \leqq \mathbf{1}$, L is a quasi-normal operator* [13, p. 302].

If there is a positive superharmonic function on W and $A \subseteq W$ is a proper outer-regular compact set [10, p. 184], then there is always at least one semi-normal operator on A, and it is quasi-normal if $\mathbf{1} \in \bar{\mathscr{H}}$. Indeed, if W is compact then $W\setminus A$ is a regular proper open set in W, and $f \to H(f, W\setminus A)$ is easily seen to be a semi-normal operator and in fact to be the only semi-normal operator on A; it is clearly quasi-normal if $\mathbf{1} \in \bar{\mathscr{H}}$. On the other hand, if W is noncompact let $\{W_i\}_{i=1}^{\infty}$ be an increasing exhaustion of W by regular regions [10, p. 183] all of which contain A. For each $f \in \mathscr{C}(\partial A)$ one may solve the Dirichlet problem for $W_i\setminus A$ with boundary values f on ∂A and **0** on ∂W_i; call the solution h_i. If $f \geqq \mathbf{0}$ and $j \geqq i$, it is easy to see that the function defined as h_i in W_i and **0** in $W\setminus W_i$ is subharmonic in W_j; consequently, the sequence $\{h_j|W_i\}_{j\geqq i}$ increases in W_i and thus converges to a function $L_* f$ in W_i. The values of the function $L_* f$ thus constructed are obviously independent of i, so we have a mapping $f \to L_* f$ from nonnegative elements of $\mathscr{C}(\partial A)$ to $\mathscr{H}_{W\setminus A}$ which can be extended to a linear operator in only one way; the operator L_* thus constructed is clearly nonnegative (since $L_* f \geqq h_i$ for all indices i). Suppose $\mathbf{1} \in \bar{\mathscr{H}}$; then it is clear that $L_* \mathbf{1} \leqq \mathbf{1}$ and it is quite easy to check that $\lim_{W\setminus A} L_* f(x) = f(x)$ for $f \in \mathscr{C}(\partial A)$ and $x \in \partial A$. Indeed, it suffices to check this for positive functions f which assume their maxima at x, since these generate $\mathscr{C}(\partial A)$. If $\alpha = f(x)$, then one must have $h_i \leqq \alpha$ for all i, since α is superharmonic; thus $L_* f \leqq \alpha$, which disposes of the limit superior. On the other hand, $\lim_{W_1\setminus A} h_1(x) = \alpha$ and $h_1 \leqq L_* f$ take care of the limit inferior. This shows that L_* is quasi-normal if $\mathbf{1} \in \bar{\mathscr{H}}$; since the positive superharmonic function on W whose

existence we assume can be taken to be continuous by modifying it using the technique of [4, Proposition 11, p. 95], the semi-normality of L_* for the general case can be deduced from the case where $\mathbf{1} \in \check{\mathscr{H}}$ by using the known relations between superharmonic functions of multiplicatively equivalent presheaves [10, § 6, p. 193ff.].

Definition (1.2). *The operator L_* constructed above is the least Dirichlet operator (for the outer-regular set A).*

It is not hard to see that if A is an outer-regular proper compact set and U is a proper regular region containing A (so that $\mathscr{H}|U$ satisfies the Brelot axioms with respect to U, and U is non-compact), then the operator L_* relative to U is just the linear operator from $\mathscr{C}(\partial A)$ to $\mathscr{H}^c_{U\setminus A}$ which sends $f \in \mathscr{C}(\partial A)$ to the solution of the Dirichlet problem on $U\setminus A$ with boundary values f on ∂A and $\mathbf{0}$ on ∂U. As a partial converse to the construction above we should observe that if there is a semi-normal operator L on A and no component of $W\setminus A$ has one boundary point, then A is outer-regular: for if $x_0 \in \partial A$ and $f \in \mathscr{C}(\partial A)$ is a nonnegative function whose unique zero is at x_0, then Lf is a barrier [10, p. 181] for $W\setminus A$ at x_0. Indeed, if this were not the case then Lf would have a zero in $W\setminus A$ and thus vanish identically in some component of $W\setminus A$; but under the hypothesis we made on W, the fact that $Lf \in \mathscr{H}^c_{W\setminus A}$ would imply that f had at least two distinct zeros on ∂A, contrary to its choice.

We have observed that if W is compact, in the presence of a positive superharmonic function on W there is exactly one semi-normal operator on any proper outer-regular A and that it is the operator solving the Dirichlet problem for the regular open set $W\setminus A$. If W is noncompact, there may be many semi-normal operators on A; we are guaranteed that at least one exists, namely L_*. Any fixed one of these semi-normal operators solves an exterior Dirichlet problem in the punctured neighborhood of ∞ whose complement is A. In the classical case of normal operators on a Riemann surface, the operators L_0 or L_1 are "consistent" when the neighborhoods of ∞ on whose boundaries they are defined are varied: if A and B are compact sets with $A \subseteq B$ and $L_{\alpha,A}$ ($L_{\alpha,B}$ respectively) are the L_α-operators defined on $\mathscr{C}(\partial A)$ and $\mathscr{C}(\partial B)$ respectively, then $L_{\alpha,A} = L_{\alpha,B} \circ (\cdot|\partial B) \circ L_{\alpha,A}$ (where $\alpha = 0$ or 1). In order to axiomatize the behavior of the classical normal operators – in other words, in order to specify consistent ways to solve exterior Dirichlet problems in the complements of "large" compact sets – one is thus led to the following definitions. (Note that the compact sets in question are not required to be outer-regular in the sense of [10].)

Definitions (1.3). *A basic family of semi-normal operators on a non-compact space W is a family $\mathfrak{L}$ of pairs (A, L_A), where each A is a proper*

compact subset of W and L_A is a semi-normal operator on A, such that for every compact $K \subseteq W$ there exists an $(A, L_A) \in \mathfrak{L}$ with $K \subseteq A^0$, and $L_A = L_B \circ (L_A(\cdot)|\partial B)$ whenever (A, L_A) and (B, L_B) are elements of $\mathfrak{L}$ with $A \subseteq B$. Such a basic family defines a complete presheaf $\mathscr{H}^{\mathfrak{L}}$ over W^* of vector spaces of continuous functions on W in the following way: if U is an open subset of W^*, then a function h on $U \setminus \{\infty\}$ belongs to $\mathscr{H}^{\mathfrak{L}}_U$ if it is $\mathscr{H}$-harmonic in $U \setminus \{\infty\}$ and either $\infty \notin U$ or h satisfies the relation $L_A(h|\partial A) = h|(U \setminus \{\infty\}) \setminus A$ for some $(A, L_A) \in \mathfrak{L}$ with $W \setminus A \subseteq U$. $\mathscr{H}^{\mathfrak{L}}$ will be called the normal structure subordinate to $\mathscr{H}$ determined by $\mathfrak{L}$.

Prescribing a normal structure subordinate to a given $\mathscr{H}$ is the axiomatic version of prescribing a "general" construction for a normal operator in the classical sense; cf. [13, p. 46 ff.], where constructions for the operators L_α are given. The basic family of semi-normal operators $\mathfrak{L}$ that determines the structure $\mathscr{H}^{\mathfrak{L}}$ can be viewed as a "neighborhood base at ∞ consisting of sets regular for $\mathscr{H}^{\mathfrak{L}}$;" looking at it in this way underlines the fact that a normal structure is an axiomatic object very similar to the Brelot presheaves. The material of (1.4) through (1.7) below further underscores that similarity.

Definition (1.4). *Let V be an open subset of W^*. An extended-real-valued function s defined on $V \setminus \{\infty\}$ is normal-superharmonic (with respect to $\mathscr{H}^{\mathfrak{L}}$) if $s \in \bar{\mathscr{H}}_{V \setminus \{\infty\}}$ and either $\infty \notin V$ or there exists a compact K with $W \setminus K \subseteq V$ such that for every pair $(A, L_A) \in \mathfrak{L}$ with $K \subseteq A$, the inequality $L_A f \leqq s|W \setminus A$ holds for every $f \in \mathscr{C}_{\mathbb{R}}(\partial A)$ with $f \leqq s|\partial A$. Such a function s is of potential type if $s \geqq \mathbf{0}$ and if whenever $h \in \mathscr{H}^{\mathfrak{L}}_V$ and $h \leqq s$, then $h \leqq \mathbf{0}$. Normal-subharmonic functions are defined dually.*

Since normal-superharmonic functions are $\mathscr{H}$-superharmonic, they have all the properties of such functions – e.g., lower-semicontinuity, finiteness at some point of every component of the domain, etc. – together with an additional property, that of a particular type of behavior at ∞. They satisfy the following important minimum principle:

Lemma (1.5). *If U is an open subset of W with $W \setminus U$ compact and there exists an $(A, L_A) \in \mathfrak{L}$ with $W \setminus U \subseteq A$, and if s is a normal-superharmonic function on U with $(\liminf_U s)(x) \geqq 0$ for every $x \in \partial U$, then $s \geqq \mathbf{0}$ throughout U.*

Proof. This is a variant of [10, Proposition 1.2, p. 171]. It is easy to see that $L_A \mathbf{1} > \mathbf{0}$ throughout $W \setminus A$: if $L_A \mathbf{1}$ had any zeros in $W \setminus A$ it would vanish identically in a component of that set and so have zeros on ∂A, which it doesn't. Assuming s takes negative values, take a pair $(B, L_B) \in \mathfrak{L}$ with B sufficiently large that $B^0 \supseteq W \setminus U$, that B can serve as the set "K" of the definition of normal-superharmonicity for s on U, and that s takes negative values on B. By the minimum principle for super-

harmonic functions [10, Proposition 2.9, p. 177] s takes negative values on ∂B, so we can set $\alpha_0 = \inf\{\alpha: (\alpha L_A \mathbf{1} + s)|\partial B > 0\}$ and have $\alpha_0 > 0$ and $(\alpha_0 L_A \mathbf{1} + s)|\partial B \geqq \mathbf{0}$. This last inequality implies that $\alpha_0 L_A \mathbf{1} + s$ is nonnegative everywhere on U, because $-\alpha_0 L_A \mathbf{1} \leqq s$ on ∂B implies that $L_B[-\alpha_0 L_A \mathbf{1}] = -\alpha_0 L_A \mathbf{1} \leqq s$, or $\mathbf{0} \leqq \alpha_0 L_A \mathbf{1} + s$, throughout $\overline{W \setminus B}$; on the other hand, the minimum principle for superharmonic functions on the relatively compact set $U \cap B^0$ insures that $\mathbf{0} \leqq \alpha_0 L_A \mathbf{1} + s$ on that set. For $0 < \beta < \alpha_0$, let $K_\beta = \{x \in \partial B: (\beta L_A \mathbf{1} + s)(x) \leqq 0\}$. The intersection of the K_β's is nonempty, and $(\alpha_0 L_A \mathbf{1} + s)(x_0) = 0$ must hold for any x_0 in that intersection. By [10, Proposition 2.1, p. 175], $\alpha_0 L_A \mathbf{1} + s = \mathbf{0}$ must hold throughout the component of x_0 in U, and so at any boundary point of that component, say x_1, we must have $(\liminf_U s)(x_1) \leqq -\alpha_0 L_A \mathbf{1}(x_1) < 0$, contrary to hypothesis. Q.E.D.

Corollary (1.6). *Let U be an open subset of W with $W \setminus U$ compact. Suppose there is a $(B, L_B) \in \mathfrak{L}$ with $W \setminus U \subseteq B$. Then if h_1 and h_2 are elements of $\mathscr{H}_U^c$ which satisfy $L_{A_i}(h_i|\partial A_i) = h_i|U \setminus A_i$ for some $(A_i, L_{A_i}) \in \mathfrak{L}$ with $W \setminus U \subseteq A_i$, $i = 1, 2$, then $h_1|\partial U = h_2|\partial U \Rightarrow h_1 = h_2$ throughout U. If $s \in \mathscr{H}_U$ satisfies $L_A(f|\partial A) \leqq s|W \setminus A$ for all $f \in \mathscr{C}_{\mathbb{R}}(\partial A)$ with $f \leqq s$ for all* ***sufficiently large*** *$(A, L_A) \in \mathfrak{L}$ with $W \setminus A \subseteq U$, then it satisfies this relation for* ***all*** *such (A, L_A); if $s \in \mathscr{H}$ and $L_A(s|\partial A) = s|W \setminus A$ for* ***some*** *$(A, L_A) \in \mathfrak{L}$ with $W \setminus A \subseteq U$, then $L_A(s) = s|W \setminus A$ for* ***all*** *$(A, L_A) \in \mathfrak{L}$ with $W \setminus A \subseteq U$.*

The proofs of the assertions of the corollary are entirely standard.

One might observe at this point that there is always at least one normal structure subordinate to $\mathscr{H}$ for any pair $(W, \mathscr{H})$ which possesses a positive superharmonic function defined on all of W, since it is easy to verify that the least Dirichlet operators on two outer-regular sets A and B, $A \subseteq B$, are consistent. This important structure, which is canonically associated with $\mathscr{H}$, will be denoted by ${}_*\mathscr{H}$.

Numerous elementary properties of superharmonic functions carry over directly to normal-superharmonic functions. For example, the Perron process is valid with virtually no modification (cf. [11, Theorem 3.2, p. 205]), and every nonnegative normal-superharmonic function has a greatest normal-harmonic minorant – thus every such function can be decomposed in a unique way as the sum of a normal-harmonic function and a normal-superharmonic function of potential type.

If $\mathbf{1} \in \mathscr{H}_W$, then the condition $L\mathbf{1} \leqq \mathbf{1}$ imposed on quasi-normal operators says precisely that $\mathfrak{L}$ is a basic family of quasi-normal operators if and only if $\mathbf{1}$ is normal-superharmonic with respect to $\mathscr{H}^{\mathfrak{L}}$. It will be very useful in some of the following material to know the following proposition, which makes it possible to solve the exterior Dirichlet problem in "large" punctured neighborhoods of ∞.

Proposition (1.7). *If $\mathscr{H}^{\mathfrak{Q}}$ is a normal structure subordinate to $\mathscr{H}$ for which* **1** *is normal-superharmonic, and A is an outer-regular compact subset of W, then for each $f \in \mathscr{C}(\partial A)$ there is exactly one function Lf on $W \setminus A$ which belongs to $\mathscr{H}^{\mathfrak{Q}}_{W\setminus A}$ and satisfies $\lim_{W\setminus A} Lf = f$ on ∂A, and which is positive if f is positive.*

Proof sketch. This can be treated exactly as the interior Dirichlet problem is treated in [10, § 3], with only the most minor modifications. One defines upper and lower competing classes of normal super- and subharmonic functions on $W \setminus A$ in exactly the same way as [10, p. 179]. The minimum principle of (1.5) above is valid for $W \setminus A$ – the proof we gave goes through with **1** playing the rôle previously assigned to $L_A \mathbf{1}$ – and so every upper function dominates every lower function and the infimum of the upper and supremum of the lower classes are normal-harmonic, i.e. belong to $\mathscr{H}^{\mathfrak{Q}}_{W\setminus A}$. The proof of the comparison theorem (see, e.g., [10, Theorem 3.2, p. 180]) requires virtually no modification if the set on which comparison is made (the ω of [10]) is relatively compact, as it is in the proof of [10, Theorem 3.3] which shows that every point of $\partial(W \setminus A)$ which possesses a barrier is a regular point for the "exterior Dirichlet problem" which $\mathfrak{Q}$ gives. Since A was taken to be outer-regular, it has a barrier at each of its points by definition.

Thus a normal structure for which **1** is normal-superharmonic gives us a very ample supply of semi-normal operators, all of which must clearly be consistent.

We conclude this section with an observation which will allow us to reduce the study of harmonic sheaves over compact spaces W to the study of normal structures on non-compact spaces W whenever it may be convenient (though we shall actually use this only in part of § 4 below). The following lemma will be useful.

Lemma (1.8). *Let X be a connected, locally connected compact Hausdorff space. There is at least one point $x_0 \in X$ for which $X \setminus \{x_0\}$ is connected.*

Proof. Excluding the possibility that X is a one-point space, we see that there are connected proper open subsets of X. The (Hausdorff-Kuratowski-Zorn) maximal principle produces a maximal chain (under inclusion) $\mathfrak{U}$ of such sets; $W = \bigcup \mathfrak{U}$ is a maximal such set, since the facts that $\{X \setminus U : U \in \mathfrak{U}\}$ is a chain under containment and that X is compact imply the existence of a point $x_0 \in \bigcap \{X \setminus U : U \in \mathfrak{U}\}$. We may take $x_0 \in \partial W$ with no loss of generality; but then we see that x_0 is the only point in $X \setminus W$, because if V were an open connected neighborhood of x_0 excluding some point $x_1 \in X \setminus W$, the set $W \cup V$ would be a connected proper open set whose existence would contradict the maximality of W. Q.E.D.

Now if $(X, \mathscr{H})$ is a pair satisfying the Brelot axioms with X compact, we may find $x_0 \in X$ for which $W = X \setminus \{x_0\}$ is connected. $(W, \mathscr{H}|W)$

therefore satisfies the Brelot axioms and is non-compact. X is the one-point compactification of W with x_0 the point at infinity. A normal structure can now be defined on W by letting $\mathfrak{Q}$ be the basic family of semi-normal operators given by the pairs (A, L_A) where A is the complement of a neighborhood U of x_0 in X which is regular for $\mathscr{H}$ and L_A is the harmonic-extension operator $f \to H(f, U)$ on $\mathscr{C}(\partial U) = \mathscr{C}(\partial A)$ (followed, if precision seems necessary, by restriction to $U \setminus \{x_0\}$). The sheaves $\mathscr{H}$ and $(\mathscr{H}|W)^{\mathfrak{Q}}$ are isomorphic under the most natural isomorphism possible: every function in $(\mathscr{H}|W)^{\mathfrak{Q}}$ in a neighborhood of x_0 has a continuous extension to $X = W \cup \{x_0\}$ which belongs to $\mathscr{H}$ in that neighborhood, and every function in $\mathscr{H}$ in a neighborhood of x_0 becomes a function of $\mathscr{H}^{\mathfrak{Q}}$ when restricted to the neighborhood punctured at x_0.

2. Fine Resolutions of the Sheaves of Harmonic and Normal-Harmonic Functions

Throughout this section we shall again fix a pair $(W, \mathscr{H})$ satisfying the Brelot axioms (with W possibly compact); we shall also fix a normal structure subordinate to $\mathscr{H}$, calling it $\mathfrak{Q}$. There will be no assumption that a global (super-)harmonic or ([normal-]super-)harmonic function exists on W.

We begin by constructing a fine sheaf in which $\mathscr{H}$ (or $\mathscr{H}^{\mathfrak{Q}}$ respectively) is a subsheaf.

Definition (2.1). *$\mathscr{R}$ is the presheaf over W of vector spaces of scalar-valued functions on open subsets of W determined by the following condition: if U is an open subset of W, $f \in \mathscr{R}_U$ iff every $x \in U$ has a neighborhood V in U such that $f|V$ is a linear combination of nonnegative continuous $\mathscr{H}$-subharmonic functions with domain V. $\mathscr{R}^{\mathfrak{Q}}$ is the presheaf over W^* of vector spaces of scalar-valued functions on open subsets of W satisfying: $f \in \mathscr{R}^{\mathfrak{Q}}_U$ iff f is defined in $U \setminus \{\infty\}$, every point $x \in U \cap W$ has a neighborhood $V \subseteq U \cap W$ such that $f|V$ is a linear combination of nonnegative continuous $\mathscr{H}$-subharmonic functions with domain V, and if $\infty \in U$ then ∞ has a neighborhood V in U such that $f|V \setminus \{\infty\} = \sum \alpha_i s_i$, where the $\{\alpha_i\}$ are scalars and the $\{s_i\}$ nonnegative $\mathscr{H}^{\mathfrak{Q}}$-normal-subharmonic functions on $V \setminus \{\infty\}$ which are continuous on $V \setminus \{\infty\}$. $\mathscr{R}^{\mathfrak{Q}^*}$ is the presheaf over W^* of vector spaces of scalar-valued functions on open subsets of W^* which satisfy the defining conditions of $\mathscr{R}^{\mathfrak{Q}}$ together with the condition that the functions $\{s_i\}$ occurring in the linear combination $f|V = \sum \alpha_i s_i$ have continuous (obviously uniquely determined) extensions to V which are continuous at ∞.*

All these presheaves are easily seen to be complete, and thus we can identify them with their associated sheaves in the usual way and identify $\mathscr{R}_U$, etc., with $\Gamma(U, \mathscr{R})$, etc.

Proposition (2.2). *If Z is an open subset of W or W^* such that $\mathbf{1}\in\mathscr{H}|Z$ ($\mathscr{H}^{\Omega^*}|Z$ if $\infty\in Z$), then the restrictions to Z of the sheaves defined above are sheaves of algebras over their scalar fields. Action by pointwise multiplication makes $\mathscr{R}^{\Omega}|Z$ a sheaf of $\mathscr{R}^{\Omega^*}|Z$-modules. $\mathscr{R}|Z$ and $\mathscr{R}^{\Omega^*}|Z$ are inverse-closed in the sense that an element of $\mathscr{R}_U$ or $\mathscr{R}_U^{\Omega^*}$ ($U\subseteq Z$) respectively which has no zeros in U has an inverse in $\mathscr{R}_U$ or $\mathscr{R}_U^{\Omega^*}$ respectively (necessarily, of course, the pointwise inverse).*

Proof. We shall drop the "$|Z$" throughout the proof for the sake of a manageable notation.

To prove the assertions of the proposition which do not involve inversion, it will suffice to show that the product of two nonnegative $\mathscr{H}$-subharmonic functions on an open set $V\subseteq Z$ can be written as a linear combination of nonnegative $\mathscr{H}$-subharmonic functions there (together with a similar statement for nonnegative $\mathscr{H}^{\Omega}$-normal-subharmonic functions in punctured neighborhoods of ∞). If X is a regular neighborhood of x_0 with $x_0\in X\subseteq V$ and ρ_x denotes the representing measure for the point $x\in X$ relative to X, then ρ_x has total mass 1 because $\mathbf{1}$ is harmonic. The convexity properties of the functions $(t\to t^p)$ on $\overline{\mathbb{R}}^+$ for real p give us the inequalities $\int s^p\,d\rho_x\geqq[\int s\,d\rho_x]^p$ for $p\geqq 1$ or $p<0$, where s is a nonnegative and continuous function on ∂X. Consequently if f is a nonnegative continuous subharmonic function on V and $p\geqq 1$, then $H(f^p, X)\geqq[H(f, X)]^p\geqq f^p$ on X, showing that f^p is also a subharmonic function. If f and g are both nonnegative continuous subharmonic functions on V, the trivial identity $fg=(1/2)\cdot[(f+g)^2-f^2-g^2]$ thus represents fg as a linear combination of nonnegative continuous subharmonic functions. The same proof works for $\mathscr{H}^{\Omega}$ in a punctured neighborhood of ∞; one simply uses the "outside" representing measures derived from the semi-normal operators L_A, $(A, L_A)\in\Omega$, in punctured neighborhoods $W\setminus A$ of ∞. Extension of this to $\mathscr{R}^{\Omega^*}$ is completely trivial.

To handle the inverse-closure assertion, begin by considering the problem of showing that if $x_0\in V\subseteq Z$ and if f and g are continuous subharmonic functions in V such that $f(x_0)\neq g(x_0)$, then $\mathbf{1}/(f-g)$ is a difference of subharmonic functions in some neighborhood of x_0. It is equivalent and slightly more convenient to assume that f and g are superharmonic. By multiplying everything in sight by an appropriate constant we can arrange to have $(f-g)(x_0)>5/2$, and by replacing f and g by $f-\alpha$ and $g-\alpha$ for properly chosen α, we can arrange to have $f(x_0)>2$ and $0>g(x_0)>-1/2$; shrinking V if necessary, we can assume that $f>\mathbf{2}$ and $\mathbf{0}>g>-\mathbf{1/2}$ throughout V. We can write $\mathbf{1}/(f-g)$ as $(\mathbf{1}/f)\cdot(\mathbf{1}-(g/f))^{-1}$, and since we know that $\mathscr{R}_V$ is a ring it will suffice to show that the factors belong to $\mathscr{R}_V$ separately. The first factor is easy to handle,

since the convexity inequality above and the fact that f is superharmonic and positive give $H(f^{-p}, X) \geqq H(f, X)^{-p} \geqq f^{-p}$ for any $p>0$ and any regular region $X \subseteq V$; thus $\mathbf{1}/f$ is continuous, nonnegative and subharmonic in V. For the factor $(\mathbf{1}-(g/f))^{-1}$ we may write, formally,

$$\begin{aligned}\frac{\mathbf{1}}{\mathbf{1}-(g/f)} &= \sum_{n=0}^{\infty} (g/f)^n \\ &= \sum_{n=0}^{\infty} (-1)^n [(\mathbf{1}/f)^n \cdot (-g)^n] \\ &= (1/2) \cdot \sum_{n=0}^{\infty} (-1)^n \left([f^{-n} + (-g)^n]^2 - f^{-2n} - (-g)^{2n}\right) \\ &= (1/2) \cdot \left\{ \sum_{k=0}^{\infty} \left([f^{-2k} + (-g)^{2k}]^2 + f^{-4k-2} + (-g)^{4k+2}\right) \right. \\ &\quad \left. - \sum_{k=0}^{\infty} \left([f^{-2k-1} + (-g)^{2k+1}]^2 + f^{-4k} + (-g)^{4k}\right) \right\}.\end{aligned}$$

Using the estimates $|f^{-1}| < \mathbf{1/2}$ and $|g| < \mathbf{1/2}$ which we have available uniformly on V, the reader can easily justify these formal manipulations. The two series given within the braces on the last line above converge uniformly throughout V, necessarily to continuous functions; since each term of each series is a nonnegative subharmonic function in V, the sums are nonnegative subharmonic functions on V and $(\mathbf{1}-(g/f))^{-1}$ is exhibited as an element of $\mathscr{R}_V$. This verifies the inverse-closure assertion for $\mathscr{R}$ and at all finite points of Z for sections of $\mathscr{R}^{\Omega^*}$, and it remains to check that the argument goes through if $x_0 \in Z$ is replaced by ∞. The assumption that $\mathbf{1}$ is harmonic and normal-harmonic in Z insures that any nonnegative normal-subharmonic function defined and continuous in a punctured neighborhood of ∞ and possessing an extension continuous at ∞ can be written as a difference of normal-superharmonic functions with the same continuity property, and that the sum of such a function and a constant is again such a function. In taking V smaller, if necessary, so that the inequalities $f > \mathbf{2}$ and $\mathbf{0} > g > -\mathbf{1/2}$ would be satisfied throughout V, we used the continuity of f and g at x_0, and if we are working with an element of $\mathscr{R}^{\Omega^*}$ we have continuity at ∞ available for the same purpose. Thus the same argument, with only trivial modifications, disposes of $\mathscr{R}^{\Omega^*}$ at ∞. Q.E.D.

Proposition (2.3). *If Z is an open subset of W or W^* for which $\mathbf{1} \in \Gamma(Z, \mathscr{H})$ (or $\mathbf{1} \in \Gamma(Z, \mathscr{H}^{\Omega^*})$ if $\infty \in Z$), then for every $x_0 \in Z$ and open neighborhood U of x_0 in Z, there exists an element f of $\Gamma(W, \mathscr{R})$ or $\Gamma(W^*, \mathscr{R}^{\Omega^*})$ respectively for which $\mathbf{0} \leqq f \leqq \mathbf{1}$, $f = \mathbf{1}$ in a neighborhood of x_0, and $f = \mathbf{0}$ outside a closed subset of U.*

Proof. If $x_0 \neq \infty$, let V be a connected regular neighborhood of x_0 with $\overline{V} \subseteq W \cap U$, let A be an outer-regular compact set with $x_0 \in A^0 \subseteq A \subseteq V$ [10, Proposition 4.1, p. 184], and define f by setting $f = \mathbf{0}$ on $\overline{W^* \setminus V}$, $f = H(\chi_{\partial A}, V \setminus A)$ on $V \setminus A$, and $f = \mathbf{1}$ on A. It is easily verified that f is normal-subharmonic at every point of $W^* \setminus A$ and superharmonic at every point of V. If $x_0 = \infty$, let Y be a neighborhood of ∞ with $\infty \in Y \subseteq \overline{Y} \subseteq U$ and for every $x \in \partial Y$ find a connected regular neighborhood V_x of x with $\overline{V}_x \subseteq U \setminus \{\infty\}$ and an outer-regular compact set A_x with $x \in A_x^0 \subseteq A_x \subseteq V_x$, and define a function $f_x \in \Gamma(W, \mathscr{R}^{\Omega^*})$ by applying the process just described. A finite family $\{A_{x_i}^0\}_{i=1}^n$ will cover ∂Y, and the finite supremum $g = \bigvee_{i=1}^n f_{x_i}$, which obviously belongs to $\Gamma(W^*, \mathscr{R}^{\Omega^*})$, is zero at ∞ and 1 in a neighborhood of ∂Y. Thus the function $f = \mathbf{1} - g \mid Z \in \Gamma(Z, \mathscr{R}^{\Omega^*})$ is 1 at ∞ and zero in a neighborhood of ∂Y, and thus can be extended to an element of $\Gamma(W^*, \mathscr{R}^{\Omega^*})$ by setting it equal to $\mathbf{0}$ outside Y. The extension is the desired f. Q.E.D.

Corollary (2.4). *Under the hypotheses of the proposition above, $\Gamma(Z, \mathscr{R})$ or $\Gamma(Z, \mathscr{R}^{\Omega^*})$ contain partitions of unity subordinate to any given locally finite covering of Z.*

Proof. The usual construction of a partition of unity (for C^∞ functions, see, e.g., [8, pp. 288 – 289]) works perfectly well in the present situation: we have an ample supply of functions with small supports by the Proposition, and it is easy to see that $\Gamma(W, \mathscr{R})$ and $\Gamma(W^*, \mathscr{R}^{\Omega^*})$ are closed under the formation of infinite sums all but finitely many of whose summands vanish in some neighborhood of each point ("locally finite sums"). The construction requires that one be able to divide by a nonzero element of $\Gamma(Z, \mathscr{R})$ in $\Gamma(Z, \mathscr{R})$ (or by such an element of $\Gamma(Z, \mathscr{R}^{\Omega^*})$ in $\Gamma(Z, \mathscr{R}^{\Omega^*})$ respectively, and (2.2) insures that one is indeed able to do this.

Consequently, we have the following proposition.

Proposition (2.5). *If Z is an open subset of W or W^* such that $\mathbf{1} \in \Gamma(Z, \mathscr{H})$ (or $\mathbf{1} \in \Gamma(Z, \mathscr{H}^{\Omega^*})$ if $\infty \in Z$), then $\mathscr{R} \mid Z$ or $\mathscr{R}^{\Omega} \mid Z$ is a fine sheaf.*

Proof. This is entirely standard, and resembles, e.g., [8, p. 175]; we simply use the partitions of unity in $\Gamma(Z, \mathscr{R})$ or $\Gamma(Z, \mathscr{R}^{\Omega^*})$ to induce sheaf homomorphisms of $\mathscr{R} \mid Z$ or $\mathscr{R}^{\Omega} \mid Z$.

Theorem (2.6). *For any $\mathscr{H}$ and normal-harmonic structure Ω subordinate to $\mathscr{H}$, the sheaves $\mathscr{R}$ and $\mathscr{R}^{\Omega}$ are fine.*

Proof. Suppose we can show that every $x_0 \in W$ (or W^* respectively) has a neighborhood Z for which $\mathscr{R} \mid Z$ (or $\mathscr{R}^{\Omega} \mid Z$, respectively) is fine. Then if U is open in W (or W^*, respectively) and V is a neighborhood

of $x_0 \in U$ with $x_0 \in V \subseteq \overline{V} \subseteq Z \cap U$, we shall have $(\mathscr{R}|Z)|\overline{V}$ (or $(\mathscr{R}^{\mathfrak{Q}}|Z)|\overline{V}$, respectively) fine, since the restriction of a fine sheaf to a closed set is fine [5, p. 97]. The evident transitivity of restriction then implies that $\mathscr{R}|\overline{V}$ (or $\mathscr{R}^{\mathfrak{Q}}|\overline{V}$, respectively) is fine, and since a locally fine sheaf is fine [5, p. 97], we shall have shown that $\mathscr{R}$ (or $\mathscr{R}^{\mathfrak{Q}}$ respectively) is fine.

Now suppose $\mathscr{H}$ and $\phi^{-1} \cdot \mathscr{H}$ are multiplicatively equivalent (see the end of § 0 above), and $\phi > \mathbf{0}$. Since the super- and subharmonic functions associated with $\phi^{-1} \cdot \mathscr{H}$ are just the quotients by ϕ of those associated with $\mathscr{H}$, the presheaf which would be produced by replacing $\mathscr{H}$ with $\phi^{-1} \cdot \mathscr{H}$ in the construction of $\mathscr{R}$ above is just the presheaf $\phi^{-1} \cdot \mathscr{R}$ whose spaces $\phi^{-1} \cdot \mathscr{R}_U$ are given by $\{\phi^{-1} \cdot f : f \in \mathscr{R}_U\}$. Identical considerations are valid for $\mathscr{H}^{\mathfrak{Q}}$ and $\mathscr{R}^{\mathfrak{Q}}$ (and [although we shall not make use of this fact] would be valid for $\mathscr{R}^{\mathfrak{Q}^*}$ if ϕ were assumed to have a limit at ∞, i.e., a continuous extension to W^*). The operation inverse to taking quotients by ϕ is multiplication by ϕ, and clearly these operations are sheaf isomorphisms over W (or W^*) which are each other's inverses. Thus in order to show that $\mathscr{R}|Z$ is fine, it suffices to find a continuous positive function ϕ on W for which $\phi^{-1} \cdot \mathscr{R}|Z$ is fine (where $Z \subseteq W$) and similarly for $\mathscr{R}^{\mathfrak{Q}}|Z$ (where $Z \subseteq W^*$). But we can do this for a suitable neighborhood Z of any $x_0 \in W$ or W^*. Indeed, if $x_0 \in W$ let Z be a regular region with compact closure in W and set $\phi = \mathbf{1}$ in $W \setminus Z$, $\phi = H(\mathbf{1}, Z)$ in Z. ϕ is continuous on W, and the function $\mathbf{1} \in \phi^{-1} \cdot \mathscr{H}_Z$. By Proposition (2.5) above, $\mathscr{R}|Z$ is a fine sheaf. Similarly, let Z be $W^* \setminus A$ where $(A, L_A) \in \mathfrak{Q}$ and set $\phi = \mathbf{1}$ in $W \setminus Z$, $\phi = L_A \mathbf{1}$ in Z. Again ϕ is continuous on W, and $\mathbf{1} \in \phi^{-1} \cdot \mathscr{H}_Z^{\mathfrak{Q}^*}$, so $\mathscr{R}^{\mathfrak{Q}}|Z$ is fine. Q.E.D.

The construction of the next sheaf in the resolution is a little more involved, and we begin with some preliminary considerations. Let U be an open region in W, and let $\mathfrak{P}_U$ denote the cone of continuous potentials of $(U, \mathscr{H}|U)$ (which may consists only of the element $\mathbf{0}$). Hervé [9, § 15, pp. 461–467] has developed a theory of measures associated with functions by means of specific restriction which we can apply to each $\mathsf{p} \in \mathfrak{P}_U$. Namely, given such a potential p, for every point $x \in U$ there is a Borel measure $E \to \mathsf{p}(x; E)$ on W^* characterized by the condition that for every *open* subset $E \subseteq W^*$, $\mathsf{p}(x; E)$ is the value of the specific restriction of p to $E \cap U$ at x. (In the language of [9], we are taking the measure μ^x on the one-point compactification of U characterized by the condition that $\mu^x(\omega) = \mathsf{p}_\omega(x)$ for every open $\omega \subseteq U$, observing that [since p is a potential] this measure places no mass at the point at infinity, extending μ^x to all of W^* by the trivial procedure of letting $\mu^x(E)$ be $\mu^x(E \cap U)$ for $E \subseteq W^*$, and then writing $\mathsf{p}(x; E)$ for $\mu^x(E)$.) The number $\mathsf{p}(x; E)$ can be thought of as depending on x for fixed E, and it turns out that $\mathsf{p}(\cdot; E)$ is a continuous potential as a function of the first argument

for each fixed Borel set $E \subseteq W^*$. It is not difficult to prove this assertion directly; however, this is just the fact that $f \to \int f(y)\, d\mathsf{p}(\cdot; y)$ is a continuous kernel, a result which forms part of [12, Theorem 2, pp. 362–363]. If p_1 and p_2 belong to $\mathfrak{P}_U$ and $x \in U$, $E \subseteq W^*$, then also

$$(\alpha_1 \mathsf{p}_1 + \alpha_2 \mathsf{p}_2)(x; E) = \alpha_1 \mathsf{p}_1(x; E) + \alpha_2 \mathsf{p}_2(x; E)$$

for nonnegative scalars α_1, α_2. This is proved for open E in [9, Proposition 15.2, p. 466], and the extension to arbitrary Borel sets E is immediate. Since $\mathsf{p} \to \mathsf{p}(\cdot; E)$ is additive and positively homogeneous, it can be extended to a linear mapping of the space $\mathfrak{Q}_U$ of linear combinations of elements of $\mathfrak{P}_U$ in exactly one way, and if real scalars are employed then $\mathsf{q} \to \mathsf{q}(\cdot; E)$ will be positive in what one might call the sense of specific order: if $\mathsf{q}_1, \mathsf{q}_2 \in \mathfrak{Q}_U$ and $\mathsf{q}_1 - \mathsf{q}_2 \in \mathfrak{P}_U$, then also $\mathsf{q}_1(\cdot, E) - \mathsf{q}_2(\cdot, E) \in \mathfrak{P}_U$ for any Borel set $E \subseteq W^*$. We will denote the linear mapping $\mathsf{q} \to \mathsf{q}(\cdot, E)$ by λ_E; it is completely specified by the relation $\lambda_E \mathsf{p} = \mathsf{p}(\cdot; E)$ for $\mathsf{p} \in \mathfrak{P}_U$.

It is simply a matter of straightforward verification, which we unkindly leave to the reader, to show that if $\mathfrak{L}$ is a normal structure subordinate to $\mathscr{H}$ and $(A, L_A) \in \mathfrak{L}$, then for any region U in W^* with $\infty \in U$ and $U \setminus \{\infty\} \subseteq W \setminus A$ the entire specific restriction theory of [9, §§ 12 and 15] – particularly in the "kernel" formulation of [12, § 1] – carries over to the (finite-valued) normal-superharmonic functions on $U \setminus \{\infty\}$, and particularly to the continuous normal-harmonic functions of potential type on $U \setminus \{\infty\}$ in the sense of (1.4) above. That is, to any normal-superharmonic function p of potential type on $U \setminus \{\infty\}$ and any open subset E of W^* we can associate the specific restriction $\mathsf{p}(\cdot; E)$ of p to $E \cap U$, that is, the smallest (in the sense of the partial order generated by the cone of normal-superharmonic functions of potential type) normal-superharmonic function of potential type that differs from p on $E \cap U$ by a normal-superharmonic function on $E \cap U$ (not necessarily a nonnegative one); again $(\mathsf{p}, E) \to \mathsf{p}(\cdot; E)$ is additive in E for fixed p and additive in p for fixed E, etc., as above. Now let $\mathfrak{P}_U^{\mathfrak{L}}$ denote the cone of continuous normal-superharmonic functions of potential type for such a set U; however, if U is an open region in W let $\mathfrak{P}_U^{\mathfrak{L}}$ denote $\mathfrak{P}_U$ (vacuously in $\mathfrak{L}$). By repeating the steps we followed above in defining the transformations λ_E: $\mathfrak{P}_U \to \mathfrak{P}_U$ for each Borel set $E \subseteq W^*$ where U was a region in W, we can define an additive and positively homogeneous operator λ_E: $\mathfrak{P}_U^{\mathfrak{L}} \to \mathfrak{P}_U^{\mathfrak{L}}$ in the case where $\infty \in U$; the set function $E \to [\lambda_E \mathsf{p}](x)$ will be a Borel measure for each $\mathsf{p} \in \mathfrak{P}_U^{\mathfrak{L}}$ and $x \in U \setminus \{\infty\}$, and defining $\mathfrak{Q}_U^{\mathfrak{L}}$ as the space of functions on $U \setminus \{\infty\}$ spanned by $\mathfrak{P}_U^{\mathfrak{L}}$ we shall be able uniquely to extend λ_E to a linear transformation of $\mathfrak{Q}_U^{\mathfrak{L}}$, just as we did on "finite" regions U above. (Note that, strictly speaking, we should distinguish the λ_E's operating on different spaces –

such a distinction, however, will turn out shortly to be almost superfluous.)

We now have assigned a vector space $\mathfrak{Q}_U$ (or $\mathfrak{Q}_U^{\mathfrak{L}}$ respectively) to each element U of a basis for the topology of W (or W^* respectively); namely, we have assigned a space to each region in W (and to each region in W^* which contains ∞ and is contained [except for ∞] in $W \setminus A$ for some $(A, L_A) \in \mathfrak{L}$). For future reference we observe that $\mathfrak{Q}_U \subseteq \mathscr{R}_U$ as a vector subspace (and similarly $\mathfrak{Q}_U^{\mathfrak{L}} \subseteq \mathscr{R}_U^{\mathfrak{L}}$). If $V \subseteq U$ and V also belongs to the basis for W (or W^*) described above, then there is a natural linking map $r_{VU}: \mathfrak{P}_U \to \mathfrak{P}_V$, namely the map which assigns to $\mathsf{p} \in \mathfrak{P}_U$ the potential $\mathsf{p}|V - h$, where h is the greatest harmonic minorant of $\mathsf{p}|V$ in V; similar maps are available for $r_{VU}: \mathfrak{P}_U^{\mathfrak{L}} \to \mathfrak{P}_V^{\mathfrak{L}}$, etc., using normal-harmonic minorants and normal-superharmonic functions of potential type in appropriate punctured neighborhoods of ∞. The transitivity relation $r_{YU} = r_{YV} \circ r_{VU}$ for $Y \subseteq V \subseteq U$ is easily verified. These linking maps are positively homogeneous and additive on the cones $\mathfrak{P}_U$ and $\mathfrak{P}_U^{\mathfrak{L}}$, and consequently can be extended in a unique way to linear maps r_{VU}: $\mathfrak{Q}_U \to \mathfrak{Q}_V$, etc. We shall denote the presheaf formed by the $\mathfrak{Q}_U$ and r_{VU} on W by Q and its associated sheaf by $\mathscr{Q}$; similarly, the presheaf formed by the $\mathfrak{Q}_U^{\mathfrak{L}}$ and r_{VU} on W^* will be denoted by $Q^{\mathfrak{L}}$ and its associated sheaf by $\mathscr{Q}^{\mathfrak{L}}$.

We summarize these considerations in the following definition:

Definition (2.7). *For each open region U in W (or W^* respectively) let $\mathfrak{P}_U$ (or $\mathfrak{P}_U^{\mathfrak{L}}$ respectively) denote the cone of continuous potentials of $(U, \mathscr{H}|U)$ (or normal-superharmonic functions of potential type on U, respectively); let $\mathfrak{Q}_U = \mathfrak{Q}_U - \mathfrak{P}_U$ (or $\mathfrak{Q}_U^{\mathfrak{L}} = \mathfrak{P}_U^{\mathfrak{L}} - \mathfrak{P}_U^{\mathfrak{L}}$ respectively). For open regions U and V with $V \subseteq U \subseteq W$, let r_{VU} denote the unique linear extension to $\mathfrak{Q}_U$ of the mapping $r_{VU}: \mathfrak{P}_U \to \mathfrak{P}_V$ which assigns to $\mathsf{p} \in \mathfrak{P}_U$ the potential $\mathsf{p}|V - h$ in $\mathfrak{P}_V$, where h is the greatest harmonic minorant in V of $\mathsf{p}|V$; if a normal structure $\mathscr{H}^{\mathfrak{L}}$ is present and $V \subseteq U \subseteq W^*$ are regions in W^*, let r_{VU} denote the unique linear extension to $\mathfrak{Q}_U^{\mathfrak{L}}$ of the map r_{VU}: $\mathfrak{P}_U^{\mathfrak{L}} \to \mathfrak{P}_U^{\mathfrak{L}}$ which assigns to $\mathsf{p} \in \mathfrak{P}_U^{\mathfrak{L}}$ the potential (or normal-superharmonic function of potential type) $\mathsf{p}|V - h$ in $\mathfrak{P}_V^{\mathfrak{L}}$, where h is the greatest (normal-) harmonic minorant in V of $\mathsf{p}|V$. Let Q denote the presheaf on W formed by the $\mathfrak{Q}_U$ and the r_{VU}; let $Q^{\mathfrak{L}}$ denote the presheaf on W^* formed by the $\mathfrak{Q}_U^{\mathfrak{L}}$ and the r_{VU}, and let $\mathscr{Q}$ and $\mathscr{Q}^{\mathfrak{L}}$ respectively denote their respective associated sheaves over W and W^* respectively.*

Definition (2.8). *For a given open region $U \subseteq W$ (or W^* respectively) and a Borel set $E \subseteq W$ (or W^*), let λ_E denote the operator on $\mathfrak{Q}_U$ (or $\mathfrak{Q}_U^{\mathfrak{L}}$ respectively) given by the unique linear extension of the mapping $\mathsf{p} \to \mathsf{p}(\cdot; E)$ of $\mathfrak{P}_U$ ($\mathfrak{P}_U^{\mathfrak{L}}$) into itself that sends each potential (or normal-superharmonic function of potential type) to its specific restriction to $E \cap U$.*

Remark. This is a minor point, but the reader should observe that Q and $Q^{\mathfrak{Q}}$ are presheaves whose groups are indexed only by a basis for the topology of W (or W^* respectively), as in [8, p. 120], rather than by the family of all open subsets of W (or W^*), as in [3, p. 1] or [5, p. 32]. This causes no trouble in associating a sheaf with the presheaf, as in [3, p. 4]. However, in order to be able to define homomorphisms of presheaves indexed only by a basis for the topology of their space, one has to require that the same basis index both sheaves. Now all the presheaves occurring in this paper except Q and $Q^{\mathfrak{Q}}$ are indexed by all open sets of W or W^* respectively, and it is easy to see that their associated sheaves do not change if we agree to consider only those groups which are indexed by elements of the basis used to index Q (or $Q^{\mathfrak{Q}}$ respectively). Consequently, we can think of all presheaves occurring below as being indexed by the basis that indexes Q (or $Q^{\mathfrak{Q}}$ respectively), whenever it is convenient. (It is easy to see how to modify the definitions of Q and $Q^{\mathfrak{Q}}$ in order to have the $\mathfrak{Q}_U$ and $\mathfrak{Q}_U^{\mathfrak{Q}}$ defined for all open sets U; we refrained from doing this in order to be able to cite [9] and [12] directly.)

Proposition (2.9). *The linear transformations λ_E defined above commute with the linking maps r_{VU}, and consequently induce endomorphisms of $\mathcal{Q}$ and $\mathcal{Q}^{\mathfrak{Q}}$, which will also be denoted by λ_E. The support of λ_E is contained in $\bar{E}$.*

Proof. Let $V \subseteq U$ be regions, and $\mathsf{p} \in \mathfrak{P}_U$. If $Z \subseteq W^*$ is an open set, then for any $\mathsf{p} \in \mathfrak{P}_U$ we have

$$\mathsf{p} = \lambda_Z \mathsf{p} + \lambda_{W^* \setminus Z} \mathsf{p} \qquad \text{on } U. \tag{1}$$

Both summands on the right are potentials on U, and the second of them is the specifically largest minorant of p which is harmonic on $Z \cap U$. Now $r_{VU}\mathsf{p}$ is a potential on V, and just as with (1) above we have

$$r_{VU}\mathsf{p} = \lambda_Z r_{VU}\mathsf{p} + \lambda_{W^* \setminus Z} r_{VU}\mathsf{p} \qquad \text{on } V \tag{2}$$

where the second term of the right is the specifically largest minorant of $r_{VU}\mathsf{p}$ which is harmonic on $Z \cap V$. Applying r_{VU} to (1) gives

$$r_{VU}\mathsf{p} = r_{VU}\lambda_Z \mathsf{p} + r_{VU}\lambda_{W^* \setminus Z}\mathsf{p} \qquad \text{on } V \tag{3}$$

and comparing this with (2) and using what is known about the second term on the right of (2) gives

$$r_{VU}\lambda_Z \mathsf{p} \geqq \lambda_Z r_{VU}\mathsf{p} \qquad \text{for any open } Z. \tag{4}$$

Now suppose that $\{s_\alpha\}$ is a downward-directed family of nonnegative superharmonic functions whose infimum s happens to be superharmonic. If for each α the function h_α is the greatest harmonic minorant of s_α,

then each h_α is nonnegative and $\{h_\alpha\}$ is also downward-directed; consequently, $h=\lim_\alpha h_\alpha$ exists, equals the infimum of $\{h_\alpha\}$, is a harmonic function, and is even attained uniformly on compacta. It is easy to see that h is the greatest harmonic minorant of s. The same considerations apply, *mutatis mutandis*, to normal-superharmonic functions. Letting E be a Borel set in W^* and applying the general considerations we just made to the family $\{(\lambda_Z \mathsf{p})|V\colon Z \text{ open in } W^*, Z\supseteq E\}$, we see that

$$\begin{aligned} r_{VU}\lambda_E \mathsf{p} &= r_{VU}[\lim\{\lambda_Z \mathsf{p}\colon Z \text{ open}, Z\supseteq E\}] \\ &= \lim\{r_{VU}\lambda_Z \mathsf{p}\colon Z \text{ open}, Z\supseteq E\} \\ &\geqq \lim\{\lambda_Z r_{VU}\mathsf{p}\colon Z \text{ open}, Z\supseteq E\} \\ &= \lambda_E r_{VU}\mathsf{p}. \end{aligned}$$

On the other hand, we could have done the same things with $W^*\backslash E$, and since $\lambda_E+\lambda_{W^*\backslash E}$ is the identity operator both on $\mathfrak{P}_U$ and on $\mathfrak{P}_V$, we must have

$$r_{VU}\lambda_E \mathsf{p}=\lambda_E r_{VU}\mathsf{p}$$

for every Borel set E. The same arguments work, with only the most trivial modifications, in case one of the other of V or U contains ∞ and λ_E involves the normal structure. We now know that $r_{VU}\lambda_E=\lambda_E r_{VU}$ holds on a set of generators of $\mathfrak{Q}_U$ or $\mathfrak{Q}_U^{\mathfrak{o}}$, so it must hold in general. Checking directly that the λ_E's induce endomorphisms of $\mathcal{Q}$ and $\mathcal{Q}^{\mathfrak{o}}$ is fairly easy — obviously they induce endomorphisms of the stalks at the points of W or W^*, and one only has to check definitions to verify continuity. A less direct but more perspicuous way to look at this question is to observe that if $\mathfrak{S}$ denotes the algebra of simple Borel functions on W (or W^* respectively) with the same scalars as the $\mathfrak{Q}_U$'s (or $\mathfrak{Q}_U^{\mathfrak{o}}$'s respectively), then letting $f=\sum\alpha_i\chi_{E_i}\in\mathfrak{S}$ act on $\mathfrak{Q}_U$ (or $\mathfrak{Q}_U^{\mathfrak{o}}$ respectively) by

$$\Lambda_f \mathsf{q}=\sum\alpha_i\lambda_{E_i}\mathsf{q} \qquad \text{for } \mathsf{q}\in\mathfrak{Q}_U \text{ (or } \mathfrak{Q}_U^{\mathfrak{o}})$$

makes $\mathfrak{Q}_U$ (or $\mathfrak{Q}_U^{\mathfrak{o}}$) an $\mathfrak{S}$-module. This fact follows readily from the relation $\lambda_E[\lambda_F \mathsf{p}]=\lambda_{(E\cap F)}\mathsf{p}$, valid for all Borel sets E and F and potentials $\mathsf{p}\in\mathfrak{P}_U$ (or normal-superharmonic functions of potential type $\mathsf{p}\in\mathfrak{P}_U^{\mathfrak{o}}$ respectively) because it is valid for E and F compact or open subsets of U [12, § 2(9), p. 361]. Thus we are dealing with a presheaf of modules (over a constant presheaf of rings), so the continuity of the λ_E's is simply the fact that the sheaf associated with a presheaf of modules is a sheaf of modules [5, p. 35]. (Actually, the homomorphism Λ that makes the $\mathfrak{Q}_U$'s [or $\mathfrak{Q}_U^{\mathfrak{o}}$'s respectively] $\mathfrak{S}$-modules can be extended to the whole algebra $\mathfrak{B}$ of bounded Borel functions on W^* with very little

trouble – e.g., just take uniform limits from $\mathfrak{S}$. The map $f \to \Lambda_f \mathsf{p}$ for a fixed p is just the "kernel" of [12, Theorem 2, p. 362]. For another look at the same thing, cf. [2, Chapter II, p. 294ff.].)

Finally, we have to check that λ_E vanishes outside $\bar{E}$. If $x \notin \bar{E}$, $x \neq \infty$ and U is a region containing x, then there is a region V with $x \in V \subseteq W \cap (U \setminus \bar{E})$, and if $\mathsf{p} \in \mathfrak{P}_U$ then $\lambda_E \mathsf{p}$ is harmonic in $U \setminus \bar{E}$ [9, Theorem 12.2, p. 457], so $r_{VU} \lambda_E \mathsf{p} = \lambda_E r_{VU} \mathsf{p} = \mathbf{0}$ and the element determined by p in the stalk $\mathcal{Q}_x$ (or $\mathcal{Q}_x^{\mathfrak{Q}}$ if $x \neq \infty$) is annihilated by λ_E. Thus λ_E acts as 0 on those stalks which lie over points of the complement of $\bar{E}$. The modification necessary to handle $\mathcal{Q}_\infty^{\mathfrak{Q}}$ is trivial. Q.E.D.

Theorem (2.10). *The sheaves $\mathcal{Q}$ and $\mathcal{Q}^{\mathfrak{Q}}$ are fine.*

Proof. Let $\{\Omega_i\}_{i=1}^{\infty}$ be an open locally finite cover of W or W^* respectively, and let $\{\omega_i\}_{i=1}^{\infty}$ be an open locally finite cover of W or W^* respectively with $\bar{\omega}_i \subseteq \Omega_i$ for each index i. For $i = 1, 2, \ldots$, set

$$E_i = \omega_i \setminus \bigcup_{j=1}^{i-1} \omega_j;$$

then it is claimed that the $\{\lambda_{E_i}\}_{i=1}^{\infty}$ form a partition of unity subordinate to $\{\Omega_i\}_{i=1}^{\infty}$. Clearly each λ_{E_i} is the zero endomorphism in $W^* \setminus \bar{\omega}_i \supseteq W^* \setminus \Omega_i$, and from a "countably additive" viewpoint we have $\mathsf{p} = \sum_{i=1}^{\infty} \lambda_{E_i} \mathsf{p}$ for any potential $\mathsf{p} \in \mathfrak{P}_U$ for any region U (or the corresponding statement for $\mathsf{p} \in \mathfrak{P}_U^{\mathfrak{Q}}$). But no convergence question is really involved in $\mathcal{Q}$ or $\mathcal{Q}^{\mathfrak{Q}}$: at any y in W or W^* we may select a relatively compact neighborhood U, and any element of the stalk $\mathcal{Q}_y$ or $\mathcal{Q}_y^{\mathfrak{Q}}$ respectively is a linear combination of canonical images of potentials $\mathsf{p} \in \mathfrak{P}_V$ (or elements $\mathsf{p} \in \mathfrak{P}_V^{\mathfrak{Q}}$ respectively) with $V \subseteq U$. Since only finitely many Ω_i meet $\bar{U}$, only finitely many $\bar{\omega}_i$ meet U, and λ_{E_i} acts as 0 on $\mathcal{Q}_y$ or $\mathcal{Q}_y^{\mathfrak{Q}}$ whenever $\bar{\omega}_i$ does not meet U. Thus the canonical image of p in $\mathcal{Q}_y$ or $\mathcal{Q}_y^{\mathfrak{Q}}$ is the sum of the canonical images of $\{\lambda_{E_i} \mathsf{p}\}_{i=1}^{n}$ where n is taken so large that $i > n$ implies that Ω_i does not meet U, and so in the sheaf $\mathcal{Q}$ or $\mathcal{Q}^{\mathfrak{Q}}$ the sum is really a locally finite one. The assertion that the $\{\lambda_{E_i}\}_{i=1}^{\infty}$ provide a partition of unity subordinate to $\{\Omega_i\}_{i=1}^{\infty}$ is thus demonstrated, and since that locally finite cover was arbitrary, we have shown that $\mathcal{Q}$ and $\mathcal{Q}^{\mathfrak{Q}}$ are fine. Q.E.D.

Theorem (2.11). *For any $\mathcal{H}$ and any normal structure $\mathfrak{Q}$ subordinate to $\mathcal{H}$, there is an exact sequence of sheaves*

$$0 \to \mathcal{H} \to \mathcal{R} \xrightarrow{\Delta} \mathcal{Q} \to 0,$$

or

$$0 \to \mathcal{H}^{\mathfrak{Q}} \to \mathcal{R}^{\mathfrak{Q}} \xrightarrow{\Delta} \mathcal{Q}^{\mathfrak{Q}} \to 0$$

respectively. Consequently, $H^q(W, \mathscr{H})$ *and* $H^q(W^*, \mathscr{H}^{\mathfrak{Q}})$ *are the zero groups for* $q \geqq 2$.

Proof. We obviously want the injection of $\mathscr{H}$ into $\mathscr{R}$ or $\mathscr{H}^{\mathfrak{Q}}$ into $\mathscr{R}^{\mathfrak{Q}}$ to be the identity mapping, so we may concentrate on the next mapping in the sequence. We may also allow ourselves the convenience of working over the real scalars, since the complex scalars may always be introduced by tensoring over the reals. Let $x \in W$ or W^* respectively, let $f \in \mathscr{R}_U$ or $\mathscr{R}_U^{\mathfrak{Q}}$ respectively for some neighborhood U of x, and suppose $V \subseteq U$ is a connected open set and s_1 and s_2 two continuous nonnegative superharmonic functions on V with $f|V = s_1 - s_2$ (or the corresponding continuous nonnegative normal-superharmonic functions on $V \setminus \{\infty\}$ where $V \supseteq W^* \setminus A$ for some $(A, L_A) \in \mathfrak{L}$, respectively). Now $f|V$ determines an element of $\mathfrak{Q}_V$ or $\mathfrak{Q}_V^{\mathfrak{Q}}$ in an obvious way: if h_i is the greatest (normal-)harmonic minorant of s_i in V, $i = 1, 2$, then $(s_1 - h_1) - (s_2 - h_2)$ is an element of $\mathfrak{Q}_V$ or $\mathfrak{Q}_V^{\mathfrak{Q}}$. Suppose Y is another open connected subset of U and $f|Y = s_3 - s_4$ with s_3 and s_4 nonnegative, (normal-)superharmonic and continuous on Y, and h_i is the greatest (normal-)superharmonic minorant of s_i in Y, $i = 3, 4$. Then for open connected $Z \subseteq V \cap Y$ we have

$$s_1 - s_2 = s_3 - s_4,$$

or

$$s_1 + s_4 = s_2 + s_3$$

in Z, and thus if g_i is the greatest (normal-)harmonic minorant of s_i on Z, $i = 1, 2, 3, 4$, we must have

$$g_1 + g_4 = g_2 + g_3.$$

It is obvious that the greatest (normal-)harmonic minorant of $(s_i - h_i)|Z$ is just $g_i - (h_i|Z)$, and in consequence the relations

$$r_{ZV}[(s_1 - h_1) - (s_2 - h_2)] = (s_1 - g_1) - (s_2 - g_2)$$

and

$$r_{ZY}[(s_3 - h_3) - (s_4 - h_4)] = (s_3 - g_3) - (s_4 - g_4)$$

must hold. Consequently, f has associated with it a well-determined element $[\Delta f](y)$ of the stalk $\mathscr{Q}_y$ or $\mathscr{Q}_y^{\mathfrak{Q}}$ at each $y \in U$, and Δf is a serration of the sheaf $\mathscr{Q}$ or $\mathscr{Q}^{\mathfrak{Q}}$ over U. It is obvious that this serration is continuous, since $\mathscr{Q}$ is defined (cf. [3, pp. 4–5]) as a quotient of the disjoint union

$$\bigcup \{(Y \times \mathfrak{Q}_Y) : Y \text{ open and connected in } W\}$$

and the restriction to V of the serration Δf can be factored through this disjoint union by simply sending

$$V \to V \times \mathfrak{Q}_V,$$

by

$$y \to (y, (s_1 - h_1) - (s_2 - h_2)).$$

Similar considerations apply for $\mathscr{Q}^{\mathfrak{Q}}$. So we now have a continuous mapping $\Delta: \mathscr{R} \to \mathscr{Q}$, or $\Delta: \mathscr{R}^{\mathfrak{Q}} \to \mathscr{Q}^{\mathfrak{Q}}$, which is obviously a homomorphism on stalks; it remains to show that its kernel and image are what we need.

Checking the image is easy. Indeed, if $x \in W$ (or W^*) and m is an element in the stalk $\mathscr{Q}_x$ (or $\mathscr{Q}^{\mathfrak{Q}}_x$, respectively), then there is a basic regular region U containing x for which m is the canonical image of some $\mathsf{q} \in \mathfrak{Q}_U$. But $\mathfrak{Q}_U \subseteq \mathscr{R}_U$, and r_{UU} is the identity mapping, so $\Delta \mathsf{q} = m$ when q is construed as an element of $\mathscr{R}_U$. The argument for $\Delta: \mathscr{R}^{\mathfrak{Q}} \to \mathscr{Q}^{\mathfrak{Q}}$ is the same.

Checking the kernel is no harder: we need only observe that every section f of $\mathscr{R}$ or $\mathscr{R}^{\mathfrak{Q}}$ on an open set U can be written in a connected open neighborhood V of each point $x \in U$ as $f|V = s_1 - s_2$, where the s_i are continuous, nonnegative and normal-superharmonic as above, and that $f|Y$ is harmonic in Y for some connected open Y, $x \in Y \subseteq V$, if and only if the equation $(s_1 - g_1) - (s_2 - g_2) = \mathbf{0}$ holds in Y, where g_i is the greatest harmonic minorant of s_i in Y. Indeed, if that equation holds, then $f|Y = (s_1 - s_2)|Y = g_1 - g_2 \in \mathscr{H}_Y$, while if $f|Y = (s_1 - s_2)|Y \in \mathscr{H}_Y$ then s_1 and s_2 differ by a harmonic function on Y, whence their greatest harmonic minorants differ by the same harmonic function and so

$$(s_1 - g_1) - (s_2 - g_2) = (s_1 - s_2) - (g_1 - g_2) = \mathbf{0}.$$

But that equation holds for some Y with $x \in Y \subseteq V$ if and only if the section Δf of $\mathscr{Q}$ determined by the section f of $\mathscr{R}$ hits the zero element of the stalk $\mathscr{Q}_x$; cf. the discussion above which defined Δ. Hence f is harmonic in U if and only if the section of $\mathscr{Q}$ that f determines, i.e., Δf, lies in the kernel of $\Delta: \mathscr{R} \to \mathscr{Q}$. The argument for $\Delta: \mathscr{R}^{\mathfrak{Q}} \to \mathscr{Q}^{\mathfrak{Q}}$ is the same. Q.E.D.

3. The First Cousin Problem for Two-Element Covers; Flux

Again we shall fix a pair $(W, \mathscr{H})$ satisfying the Brelot axioms. If W is non-compact we shall also fix a normal structure $\mathfrak{Q}$ subordinate to $\mathscr{H}$ and assume that $\mathbf{1} \in \bar{\mathscr{H}}^{\mathfrak{Q}}$ globally. By Proposition (1.7) above, this assumption implies that the "exterior Dirichlet problem" can be solved, in the manner that $\mathfrak{Q}$ specifies, for any outer-regular proper compact $A \subseteq W$, and for any such A we shall use L to denote the quasi-normal operator that specifies that solution (i.e., we shall cease in general to index the operators by the A's). If W is compact and A is proper, compact and outer-regular — meaning simply that $W \setminus A$ is regular — then we shall use L to denote $H(\cdot, W \setminus A): \mathscr{C}(\partial A) \to \mathscr{H}^c_{W \setminus A}$,

W^* will mean W, and $\mathscr{H}^{\Omega}$ will mean, vacuously in Ω, simply $\mathscr{H}$. We now assume that $\mathbf{1}\in\bar{\mathscr{H}}$. (We could probably reduce the case in which W is compact to the case in which it is non-compact by employing Lemma (1.8) and the discussion following it; but we would then have always to consider outer-regular sets A with $x_0\notin A$, and the amount of care required to do this does not appear to justify itself.)

Let us now consider a class of very simple Čech cocycles on W^* with coefficients in $\mathscr{H}^{\Omega}$, namely those cocycles that belong to covers of W^* which possess only two elements. It is easier to think about pairs of sets, one of which is open and one of which is closed and contained in the first set, particularly for analytical purposes; so we define a *Cousin pair* on W^* to be a pair (A, U) where A and U are subsets of W^* with A a compact proper subset of W, U an open subset of W^*, and $A\subseteq U$. The pair will be said to be *proper* if U is a subset of W with compact proper closure. A *Cousin datum* for the pair (A, U) is an element of $\mathscr{H}^{\Omega}_{U\setminus A}$ – in other words, a cocycle in $Z^1(N(\{W^*\setminus A, U\}), \mathscr{H}^{\Omega})$. A pair is *special* if A is outer-regular and either $U=W$ (in which case the pair is *improper*) or the pair is proper and U is inner-regular. A datum is *special* if it belongs to $\mathscr{H}^{c}_{U\setminus A}$. *Coboundaries* and the property of being *cohomologous* are defined for (A, U) just as they are for the Čech cover $\{W^*\setminus A, U\}$ of W^*. If a special datum is a coboundary, so that it can be written as $s=p-h$ where $p\in\mathscr{H}^{\Omega}_{W^*\setminus A}$ and $h\in\mathscr{H}_U$ (or $\mathscr{H}^{\Omega}_W$ if the pair is improper), then it is easy to see that p and h have continuous extensions to ∂A and ∂U (if it is $\neq\emptyset$) respectively, so that in fact $s=Lp-Hh$ where $p\in\mathscr{C}(\partial A)$, $h\in\mathscr{C}(\partial U)$, H is $H(\cdot, U)$, and L is the quasi-normal operator on A which $\mathscr{H}^{\Omega}$ generates. If W is non-compact, the notion of cohomology is thus defined relative to a choice of $\mathscr{H}^{\Omega}$; but that choice is irrelevant in giving data on proper pairs. We shall say that two data on a proper pair are *absolutely cohomologous* if they are cohomologous for any choice of $\mathscr{H}^{\Omega}$. This will certainly be the case if their difference has a harmonic extension to a neighborhood of A, which would say precisely that the cocycles given by the data were cohomologous in the sense of cohomology with compact supports in W and coefficients in $\mathscr{H}$. A pair (B, V) is of course a *refinement* of (A, U) if $A\subseteq B$ and $V\subseteq U$; it is a refinement *from above* or *below* respectively if $A=B$ or $V=U$ respectively.

Recall that it is obvious that if (B, V) is a refinement of (A, U) and s_1 and s_2 are Cousin data for (A, U), then the restrictions of s_1 and s_2 are cohomologous whenever s_1 and s_2 are – equivalently, restrictions of coboundaries are coboundaries. Less obvious but still well known is the fact that if s is a datum for (A, U) which has the property that its restriction to (B, V) is a coboundary, then s is a coboundary for (A, U) – equivalently, one can determine whether s_1 and s_2 are cohomologous by

looking at a refinement. This can easily be verified directly but is just the theorem that if $\mathfrak{U}$ and $\mathfrak{V}$ are coverings of a space and $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, then the restriction mapping of the Čech cohomology groups $H^1(N(\mathfrak{U}), \mathscr{S}) \to H^1(N(\mathfrak{V}), \mathscr{S})$ is an injection for any sheaf $\mathscr{S}$ [7, pp. 46–47]. This fact enables us to determine when a datum is a coboundary in a particularly explicit way.

Lemma (3.1). *Every Cousin pair (A, U) has a proper special refinement such that U has finitely many components, no boundaries of distinct components of U have points in common, and every component of U contains of A.*

Proof. Obviously U has a proper refinement with these properties (except perhaps the disjoint-boundaries restriction); one can shrink U slightly to give it compact proper closure and then, since A is compact, only finitely many components of U are required to cover A and any that do not meet A may be discarded. Assuming that U has these properties, then, the lemma follows immediately from the facts that every component of U has an exhaustion by regular inner regions and that every compact subset of a noncompact space can be expanded slightly to an outer-regular compact set [10, Theorem 4.3 and Proposition 4.1, pp. 184–185]. Q.E.D.

The following theorem is the basic tool for investigating the properties of Cousin pairs.

Theorem (3.2). *Let (A, U) be a proper special Cousin pair satisfying the conditions of* (3.1) *above. Let $V \supseteq \overline{U}$ be a connected open subset of W. Suppose $M: \mathscr{C}(\partial A) \to \mathscr{H}_{V \setminus A}$ is a nonnegative linear operator with $\lim_{V \setminus A} Mf = f$ at ∂A for all $f \in \mathscr{C}(\partial A)$, and such that $M\mathbf{1} \leqq \mathbf{1}$. Let H denote the operator $H: g \to H(g, U)|\partial A$, and by abuse of language identify M with $f \to Mf|\partial U$. Then exactly one of the following alternatives holds:*

(A) *$1 \notin \sigma(MH)$, i.e., 1 is not an element of the spectrum of the operator $MH \in \mathscr{L}(\mathscr{C}(\partial U))$. For every $s \in \mathscr{H}^c_{U \setminus A}$ there exist unique functions $h \in \mathscr{C}(\partial U)$ and $p \in \mathscr{C}(\partial A)$ such that $s = H(h, U) + Mp$ in $U \setminus A$. The functions h and p depend linearly and continuously on s. There is a continuous linear operator $S: \mathscr{H}^c_{U \setminus A} \to \mathscr{H}_{V \setminus A}$ such that*

(1) *The range of S is contained in the range of M, and for every $s \in \mathscr{H}^c_{U \setminus A}$ the restriction $Ss|U \setminus A$ is cohomologous to s;*

(2) *If $s|\partial A \geqq \mathbf{0}$, then Ss is positive whenever $s|\partial U \leqq M[s|\partial A]|\partial U$; in particular, Ss is positive whenever $s|\partial U = \mathbf{0}$;*

(3) *$Ss = M\{(I - HM)^{-1}[s|\partial A - H[s|\partial U]]\}$, so Ss is positive whenever s has an extension to all of U which is superharmonic.*

(B) $1 \in \sigma(MH)$. $\mathbf{1}$ *is harmonic in* U *and* $M\mathbf{1}=\mathbf{1}$ *in* $V \setminus A$, *so* $\mathbf{1} \in \mathscr{H}_V$. *There exist a strictly positive measure* μ *and a nonnegative measure* ν *on* ∂U *and* ∂A *respectively, uniquely determined up to a multiple, such that there exists* $h \in \mathscr{C}(\partial U)$ *for which* $s - H(h, U)$ *and* $M\big[(s-H(h, U))\big|\partial A\big]$ *agree on their common domain when and only when* $\int s\, d\mu = \int s\, d\nu$. *(When this happens, of course,* $M\big[(s-H(h, U))\big|\partial A\big]$ *is cohomologous to* s.*) Moreover,* μ *is absolutely continuous with bounded derivative with respect to harmonic measure on* U, *and the support of* ν *contains the boundary of every non-relatively-compact component of* $V \setminus A$.

Proof. Consider the problem of determining a function $h \in \mathscr{C}(\partial U)$ with the property that

$$M\big[(s-H(h, U))\big|\partial A\big]\big|\partial U = s|\partial U - h. \tag{J}$$

Defining the operator H as above, one may write this equation as

$$(I-MH)h = s\big|\partial U - M[s|\partial A]\big|\partial U. \tag{K}$$

It is clear that both H and M have operator norm at most 1 (since each is nonnegative and sends $\mathbf{1}$ to something smaller), so the operator norm of MH is at most 1 and its spectral radius is at most 1. Since the spectral radius of a nonnegative operator on $\mathscr{C}(\partial U)$ belongs to its spectrum [14, § 2.2, p. 263], if $1 \notin \sigma(MH)$ then the spectral radius of MH is strictly smaller than 1. Consequently $(I-MH)^{-1}$ exists and (K) has a unique solution. Setting $p = (s-H(h, U))\big|\partial A$, we see that $s = H(h, U) + Mp$ holds on ∂A by definition and on ∂U by (J); by the maximum principle, $s = H(h, U) + Mp$ throughout $\overline{U \setminus A}$. Thus we have found h and p as desired; since these steps are reversible and h is known to be unique, these are the only h and p with $s = H(h, U) + Mp$ throughout $\overline{U \setminus A}$.

For each $s \in \mathscr{H}^c_{U \setminus A}$, then, set $Ss = M\big[(s-H(h, U))\big|\partial A\big]$ where h is determined from (K); explicitly,

$$Ss = M\big[(s-H((I-MH)^{-1}[s|\partial U - M[s|\partial A]|\partial U], U))\big|\partial A\big]$$

from which the continuity of S is clear. That $Ss|U \setminus A$ is cohomologous to s is obvious. Since the Neumann series for $(I-MH)^{-1}$ converges and the terms of this series are nonnegative operators, $(I-MH)^{-1}$ is nonnegative. The explicit expression for S above thus shows that if $s|\partial A \geqq \mathbf{0}$ and $s|\partial U \leqq M[s|\partial A]\big|\partial U$, then Ss is nonnegative. The process of converting this expression into

$$Ss = M\{(I-HM)^{-1}[s|\partial A - H[s|\partial U]]\}$$

is a straightforward computation which we omit.

We now consider the case in which $1 \in \sigma(MH)$. Fortunately, MH has the property of being *irreducible* in the sense of [14, p. 269]: whenever

$\mathbf{0} \leqq f \in \mathscr{C}(\partial U)$, $f \neq \mathbf{0}$ and $\lambda > r(MH)$, the function

$$MH[\lambda I - MH]^{-1} f = \sum_{j=1}^{\infty} \lambda^{-j} (MH)^j f \in \mathscr{C}(\partial U)$$

is strictly positive. To see this, observe first that for any $g \in \mathscr{C}(\partial U)$, MHg can be construed as being defined on all of V: the way it is defined outside of A is clear, and it is natural to define it as being Hg on A. Defined in this way, the function MHg is clearly continuous on all of V and harmonic on $V \setminus \partial A$.

For any $f \in \mathscr{C}(\partial U)$ that is nonnegative and nonzero (as above) we claim that given any component U_1 of U, $MH[\lambda I - MH]^{-1} f$ is strictly positive throughout $\overline{U}_1$ whenever it is nonzero at some point of $\overline{U}_1$. Indeed, if $MH[\lambda I - MH]^{-1} f$ is nonzero at some point of $A \cap U_1$, this means precisely that $H[\lambda I - MH]^{-1} f$ is nonzero at some point of $A \cap U_1$ and thus is strictly positive throughout U_1. It is easy to see that $\overline{U}_1$ is contained in the union of $A \cap U_1$ with all those components of $V \setminus A$ that have boundary points on $A \cap U_1$; since $M(H[\lambda I - MH]^{-1} f)$ is strictly positive throughout each of those components of $V \setminus A$, we have what we wanted. It remains to dispose of the "second case" in which $MH[\lambda I - MH]^{-1} f$ might be zero on $A \cap U_1$ but be nonzero at some point $x_0 \in \overline{U}_1 \setminus A$. If V_0 is the component of $V \setminus A$ to which x_0 belongs, it is easy to see that $V_0 \nsubseteq \overline{U}_1$: if on the contrary $V_0 \subseteq \overline{U}_1$ held, we would have $\partial V_0 \subseteq U_1 \cap A$ and $MH[\lambda I - MH]^{-1} f$ would be identically zero on V_0. Thus $V_0 \cap \partial U_1 \neq \emptyset$, and since $MH[\lambda I - MH]^{-1} f$ is strictly positive throughout V_0, $MH[\lambda I - MH]^{-1} f$ is positive on a nonempty subset of ∂U_1. Consequently $H(MH[\lambda I - MH]^{-1} f)$ is strictly positive in U_1, and repeating the argument we just gave, we see that $MH(MH[\lambda I - MH]^{-1} f)$ is strictly positive in $\overline{U}_1$. Since

$$\begin{aligned} MH[\lambda I - MH]^{-1} f &= \lambda^{-1} MH\big[f + (MH)[\lambda I - MH]^{-1} f\big] \\ &\geqq \lambda^{-1} MH(MH[\lambda I - MH]^{-1} f), \end{aligned}$$

it is also true that $MH[\lambda I - MH]^{-1} f$ is strictly positive in $\overline{U}_1$. This establishes our claim.

Now let V_1 be the open subset of V consisting of all points at which $MH[\lambda I - MH]^{-1} f$ is positive. It is easy to see that V_1 is nonempty, since $f \neq \mathbf{0}$. Since $MH[\lambda I - MH]^{-1} f$ is harmonic in $V \setminus \partial A$ and thus cannot have a zero in $V \setminus \partial A$ without vanishing identically in a neighborhood of the zero, we must have $\partial V_1 \subseteq \partial A$. But if $x_1 \in \partial V_1$, then $x_1 \in A \cap U_1$ for some component U_1 of U, and since U_1 is a neighborhood of x_1, we see that $MH[\lambda I - MH]^{-1} f$ is positive at some point of U_1 and thus, as we have just seen, positive throughout $\overline{U}_1$. This contradicts the choice of x_1, so $\partial V_1 = \emptyset$, and $V = V_1$ since V is connected; we have established the strict positivity of $MH[\lambda I - MH]^{-1} f$.

(*Remark.* The following stronger statement is valid: if n is the number of components of U, then $\sum_{j=1}^{n} \lambda^{-j}(MH)^j f$ is strictly positive for every $\mathbf{0} \leqq f \in \mathscr{C}(\partial U)$ with $f \neq \mathbf{0}$. This fact can be established by an inductive argument, and the details are left to the reader. The result which we have fitted into the body of the proof is sufficient for the spectral-theoretic purpose for which we employ it.)

We now know that MH is irreducible. Moreover, it is obvious that the operator $H: \mathscr{C}(\partial U) \to \mathscr{C}(\partial A)$ can be factored through the space $\mathscr{H}_U$ (topologized by uniform convergence on compacta), which is a nuclear space [16, p. 686], so MH is a nuclear operator and thus *a fortiori* compact. We now need only apply [14, Theorem 3.2, p. 270] to conclude that the spectral subspaces of $\mathscr{C}(\partial U)$ and $\mathscr{M}(\partial U)$ associated with the subset $\{1\} \subseteq \sigma(MH)$ are one-dimensional and generated by a quasi-interior (thus strictly positive) element of $\mathscr{C}(\partial U)$ and a strictly positive measure $\mu \in \mathscr{M}^+(\partial U)$ respectively. Since $1 \geqq \|MH\| \geqq r(MH) = 1$, by [14, Theorem 3.3(i), p. 272] the eigenvector of MH corresponding to the eigenvalue 1 must be $\mathbf{1}$. The function $H(\mathbf{1}, U) \in \mathscr{H}_U^c$; $M[H(\mathbf{1}, U)|\partial A] \in \mathscr{H}_{V \setminus A}$ and $M[H(\mathbf{1}, U)|\partial A]|\partial U = MH\mathbf{1} = \mathbf{1}$. Consequently, the function g defined by $g = H(\mathbf{1}, U)$ in U and by $MH\mathbf{1}$ in $V \setminus A$ is well-defined — the two definitions coincide at $\partial(U \setminus A)$. Since $\mathbf{1} \in \bar{\mathscr{H}}_U$ and $M\mathbf{1} \leqq \mathbf{1}$, $g \leqq \mathbf{1}$ everywhere in V; but g attains the value 1 on ∂U, and therefore the superharmonic function $\mathbf{1} - g$ vanishes identically in the connected set V [10, Theorem 2.12(i), p. 178]. Thus $\mathbf{1} \in \mathscr{H}_V$; moreover,

$$M\mathbf{1} = M[H(\mathbf{1}, U)|\partial A] \quad \text{and} \quad M\mathbf{1}|\partial U = M[H(\mathbf{1}, U)|\partial A]|\partial U = MH\mathbf{1} = \mathbf{1},$$

so $\mathbf{1} - M\mathbf{1} \in \bar{\mathscr{H}}^+_{V \setminus A}$ has a zero in every component of its domain whose boundary is not contained in ∂A and thus vanishes identically there — which says precisely that $M\mathbf{1} = \mathbf{1}$, since $M\mathbf{1} = \mathbf{1}$ trivially in the other components.

We have already found that there is a unique (up to normalization) strictly positive measure μ on ∂U for which $(MH)'\mu = \mu$. Let $\nu = M'[\mu]$; since μ is an eigenvector of $(MH)'$ with eigenvalue 1, we also have $\mu = H'[\nu]$. Equation (K) will have a solution if and only if the integral of the right side of (K) with respect to μ vanishes [6, pp. 609–610]. This condition is

$$\int s\, d\mu - \int M[s|\partial A]\, d\mu = 0$$

or

$$\int s\, d\mu - \int s\, d[M'(\mu)] = 0$$

or

$$\int s\, d\mu = \int s\, d\nu$$

as desired.

Finally, to see that μ is absolutely continuous with respect to harmonic measure on U, let ρ be a harmonic measure on U (i.e., a sum $\rho=\sum_{k=1}^{n}\rho_{x_k}$, where an x_k is chosen from each of the components $\{U_k\}_{k=1}^n$ of the open set U). Since the linear mapping $g\to H(g,U)$ can be extended uniquely to map $L^1(\rho)$ continuously into $\mathscr{H}_U$, our linear transformation H can be extended similarly to a mapping $H\colon L^1(\rho)\to\mathscr{C}(\partial A)$ in a unique manner. H is still nuclear, since it is still factored through $\mathscr{H}_U$. One can then construe MH as a mapping of $L^1(\rho)$ into itself which takes its values in $\mathscr{C}(\partial U)$; it is still nuclear, so its nonzero spectrum consists of eigenvalues, and since it is clear that any eigenfunction of MH in $L^1(\rho)$ which belongs to a nonzero eigenvalue must in fact belong to $\mathscr{C}(\partial U)$ and thus be an eigenfunction of MH in $\mathscr{C}(\partial U)$, we see that the nonzero spectrum, eigenfunctions, and spectral multiplicities of MH on $L^1(\rho)$ are the same as they are on $\mathscr{C}(\partial U)$. Thus if $G\in L^\infty(\rho)=L^1(\rho)'$ is nonzero and such that $(MH)'[G]=G$, it is easy to see that the measure $G\cdot\rho$ is a nonzero eigenvector of $(MH)'$ on $\mathscr{M}(\partial U)$ corresponding to the eigenvalue 1. Thus $G\cdot\rho=\mu$ after suitable normalization, and μ is absolutely continuous with respect to ρ, with bounded derivative G. To see that the support of ν contains the boundary of every component of $V\setminus A$ which is not relatively compact in $V\setminus A$, let Z be such a component, let $x_0\in\partial Z\cap\partial A$, and suppose $\mathbf{0}\leqq q\in\mathscr{C}(\partial A)$ is such that $q(x_0)>0$. Then, since Mq is harmonic in Z and continuous at the points of ∂A, $Mq>\mathbf{0}$ at every point of Z. Z meets the neighborhood U of x_0; but $Z\cap U$ is proper in Z, since U is relatively compact in V, and therefore Z meets ∂U, necessarily in an open subset of ∂U. Since $Mq>\mathbf{0}$ on $Z\cap\partial U$ and μ is strictly positive, $0<\int Mq\,d\mu=\int q\,d[M'\mu]=\int q\,d\nu$.

We should note, perhaps, that this last fact implies that $\nu(\partial A\cap U_i)=\nu[\partial(A\cap U_i)]>0$ for every component U_i of U. (This will be useful later on.) Q.E.D.

Remark. It may not be superfluous to point out that the property that U had only finitely many components was not used in the proof of alternative (A) above. Hence the results established there are valid whenever $1\notin\sigma(MH)$ can be established, even if the finiteness condition is not satisfied.

Corollary (3.3). *Under the hypotheses of the Theorem, if alternative* (A) *holds then for any function s defined on $\overline{U}$ such that $s|\overline{U\setminus A}\in\mathscr{H}^c_{U\setminus A}$ there exists a function $h\in\mathscr{H}^c_U$ for which $s-h$ has an extension to V which is harmonic in $V\setminus A$ and belongs to the range of M. If alternative* (B) *holds, such an h exists if and only if $\int s\,d\mu=\int s\,d\nu$.*

Indeed, this is just a paraphrase of the theorem; the extension of $s-h$ which is harmonic in $V\setminus A$ and belongs to the range of M is just Ss.

The theorem leads to a nonelementary but "soft" proof of the Hervé extension theorem for potentials of compact support [9, Theorem 13.2, p. 458], which proof has the advantage of furnishing an explicit formula for the dependence of the extended potential on the given potential rather than simple existence and uniqueness. For the proof of the Hervé extension theorem as given in [9] there is no loss in assuming that $\mathbf{1}$ is superharmonic but not harmonic on W: for if $(W, \mathscr{H})$ satisfies the Brelot axioms, W is non-compact, and there exists a potential on W, then if W has a countable base there is a continuous potential p on W [4, pp. 95–96]. If we consider the sheaf $\mathsf{p}^{-1}\cdot\mathscr{H}$ we see that $\mathbf{1}$ is superharmonic but not harmonic in that sheaf, and the relations between the superharmonic functions for $\mathscr{H}$ and $\mathsf{p}^{-1}\cdot\mathscr{H}$ [10, § 6 *passim*, p. 193ff.] make it easy to verify that the Hervé theorem for $\mathscr{H}$ follows from the theorem for $\mathsf{p}^{-1}\cdot\mathscr{H}$ – indeed, one can write out an explicit extension formula for $\mathscr{H}$ in terms of the one for $\mathsf{p}^{-1}\cdot\mathscr{H}$.

Theorem (3.4) [Hervé]. *Let W be noncompact and $\mathbf{1}$ be ${}_*\mathscr{H}$-normal-superharmonic but not ${}_*\mathscr{H}$-normal-harmonic. Then given any superharmonic function v with compact support B on an open set $V\subseteq W$, there is a unique potential u on W whose support is B and for which $u-v$ is harmonic in V.*

Proof. (B, V) is a Cousin pair and $v|V\setminus B$ a Cousin datum for this pair. Taking a special refinement (A, U) with the properties enumerated in (3.1) above, and setting $M=L_*$ (the least Dirichlet operator on A) and taking W as the "V" of Theorem (3.2), we see that alternative (A) of the theorem must hold: otherwise $\mathbf{1}$ would be $\mathscr{H}$-harmonic in W and $L_*\mathbf{1}=\mathbf{1}$ so $\mathbf{1}$ would be normal-harmonic at ∞. Applying the theorem in the form given in Corollary (3.3) above, we see that there exists a harmonic function $h\in\mathscr{H}_U^c$ for which Sv and $v-h$ agree throughout their common domain $U\setminus A$, and we may define $u=Sv$ on $W\setminus A$ and $u=v-h$ in U. The uniqueness assertion of (3.2) insures that u is defined independently of the choice of (A, U), because the least Dirichlet operators are consistent. Since the property of being superharmonic is local, u is superharmonic. By the explicit expression for S in (3) of alternative (A) of (3.2), we have $u=Sv=L_*\{(I-HL_*)^{-1}[v|\partial A-H[v|\partial U]]\}$ in $W\setminus A$, and as pointed out in (3) the fact that v is superharmonic implies that Sv is nonnegative; since $u=Sv$ outside the compact set A, u is nonnegative outside a compact set and thus is nonnegative everywhere. Now if $\mathbf{1}\notin{}_*\mathscr{H}$, it is easy to see that the function defined as $\mathbf{1}$ on A and $L_*\mathbf{1}$ on $W\setminus A$ is a potential (cf. [10, § 5]). Consequently, since any harmonic minorant g of u on W is dominated by a suitable positive multiple of $L_*\mathbf{1}$ on $W\setminus A$ and a suitable

positive multiple of **1** on A, g is dominated by a potential (take the multiples to be the same) and so $g \leqq \mathbf{0}$. The function u is thus a potential. Q.E.D.

Corollary (3.5). *If in the setting of the preceding theorem we have a net $\{v_\iota\}_{\iota\in I}$ of superharmonic functions v as in the theorem, such that $\{v_\iota\}_{\iota\in I}$ converges uniformly on compacta in $V\setminus B$ and pointwise in B to a superharmonic function v_0, then the corresponding potentials $\{u_\iota\}_{\iota\in I}$ on W converge uniformly on compacta in $W\setminus B$ and pointwise on B to the potential u_0 corresponding to v_0.*

This follows immediately from the continuity of the operators L_* and H, provided that we choose A so that its interior contains B, as we may always do [10, Proposition 4.1, p. 184].

The content of the proof of (3.4) for Cousin data is the following:

Theorem (3.6). *If $\mathfrak{Q}$ is a normal structure subordinate to $\mathscr{H}$ for which* **1** *is normal-superharmonic but not normal-harmonic, then any Cousin datum on any Cousin pair is cohomologous to zero. If $(W, \mathscr{H})$ is any pair satisfying the Brelot axioms with $\mathbf{1}\in\mathscr{H}_W$, $Y\subseteq W$ is a regular open set with compact proper closure and (B, X) is a Cousin pair in W with $X\subseteq Y$, then there is a continuous linear transformation T: $\mathscr{H}_{X\setminus B}\to\mathscr{H}_{Y\setminus B}$ (where both spaces are given the topology of uniform convergence on compacta) such that $Ts|X\setminus B$ is absolutely cohomologous to s for any Cousin datum $s\in\mathscr{H}_{X\setminus B}$ and Ts has limit* **0** *at ∂Y.*

Proof. It suffices to prove the first statement for special Cousin pairs satisfying the conditions of (3.1) above, with special data. Let (A, U) be such a pair and s such a datum. Take $V=W$ and $M=L_A$ in (3.2), where (A, L_A) is related to $\mathfrak{Q}$ by the construction of (1.7) above. Alternative (A) of (3.2) holds, because $\mathbf{1}\in\mathscr{H}_W$ and $\mathbf{1}=L_A\mathbf{1}$ would imply $\mathbf{1}\in\mathscr{H}_W^{\mathfrak{Q}}$. Thus the restriction to $U\setminus A$ of the function Ss of (3.2) above is cohomologous to s; but since $Ss|W\setminus A$ is in the range of L_A, it is cohomologous to zero in the sense of cohomology in W^* with coefficients in $\mathscr{H}^{\mathfrak{Q}}$. Thus the same is true of s.

The second statement can easily be seen to be true for all Y if it is true for connected Y. Let us then assume Y connected, and let (A, U) be a special refinement of (B, X) satisfying the conditions of (3.1) above. The characterization of the L_*-operators given with (1.2) above insures that **1** is not in ${}_*(\mathscr{H}|Y)$. The statement we just proved is thus applicable to Y equipped with the normal structure ${}_*(\mathscr{H}|Y)$, and $s|U\setminus A$ is cohomologous to $Ss|U\setminus A$ (where S is the operator of (3.2)). If we set $Ts=Ss$ in $Y\setminus A$ and $Ts=H((Ss-s), U)+s$ in $U\setminus B$, it is easily verified that T has the properties we need. Q.E.D.

(Incidentally, it is clear that the same sort of continuous-dependence property present in the second assertion could also be proved in the setting of the first assertion – but we have no need of it in what follows.)

The Theorem (3.6) just given will be referred to as the "generalized Hervé extension theorem" – the Hervé theorem itself is a theorem on extension of a particular type of cocycle. We need another "generalized Hervé theorem" that can extend "singularities in neighborhoods of ∞." The proof is an easy adaptation of the proof of alternative (A) of (3.2) above, so we offer only a proof sketch.

Theorem (3.7). *Let $\mathfrak{Q}$ be a normal structure subordinate to $\mathscr{H}$. Let (A, U) be a special Cousin pair in W, with U connected, and let s be a special datum for (A, U). If $B \subseteq A$ is outer-regular, then there is a continuous linear operator S: $\mathscr{H}^c_{U\setminus A} \to \mathscr{H}^c_{U\setminus B}$ such that $S\,s|U\setminus A$ is cohomologous to s in the sense of cohomology with coefficients in $\mathscr{H}^{\mathfrak{Q}}$; indeed, $S\,s|U\setminus A$ differs from s by a restriction of an element of $\mathscr{H}^{\mathfrak{Q}}_{W\setminus A}$. If s is the restriction to $U\setminus A$ of a function which is normal-superharmonic in $W\setminus A$, then $S\,s$ extends to a normal-superharmonic function of potential type in $W\setminus B$ which differs from s in $W\setminus A$ by a normal-harmonic function.*

Proof Sketch. Let L be the operator that extends continuous functions on ∂A to elements of $\mathscr{H}^{\mathfrak{Q}}_{W\setminus A}$ as in (1.7) above, and let N: $\mathscr{C}(\partial U) \to \mathscr{H}^c_{U\setminus B}$ be the operator that sends $f \in \mathscr{C}(\partial U)$ to the solution of the Dirichlet problem on $W\setminus B$ with boundary values f on ∂U and $\mathbf{0}$ on ∂B. We are led as in (3.2) above to seek a function $f \in \mathscr{C}(\partial A)$ for which

$$N[(s - Lf)|\partial U]|\partial A = s|\partial A - f \tag{J'}$$

or

$$(I - NL)\,f = s|\partial A - N[s|\partial U]|\partial A\,. \tag{K'}$$

Now the fact that U is connected insures that $N\mathbf{1} < \mathbf{1}$ at every point of $U\setminus B^0$, since every component of $U\setminus B$ has boundary points on ∂B. Thus $\|NL\| < 1$, and the Neumann series for $(I - NL)^{-1}$ converges. Setting

$$\begin{aligned} S\,s &= N[(s - Lf)|\partial U] && \text{on } U\setminus B \\ &= s - Lf && \text{on } U\setminus A \quad \text{or} \quad W\setminus A \end{aligned}$$

where f is determined by solving (K'), we see that $S\,s$ is well-defined on $U\setminus B$; the relation

$$\begin{aligned} S\,s &= N\big[s|\partial U - L[(I - NL)^{-1}[s|\partial A - N[s|\partial U]|\partial A]]\big] \\ &= N\big[(I - LN)^{-1}[s|\partial U - L[s|\partial U]|\partial A]\big] \quad \text{on } U\setminus B \end{aligned}$$

shows that $S\,s$ tends to $\mathbf{0}$ at ∂B. If s is the restriction to $U\setminus A$ of a function normal-superharmonic in $W\setminus A$, this relation shows that $S\,s$ is non-negative in a neighborhood of ∂B; since $S\,s$ is clearly normal-super-

harmonic in $W\setminus B$ (being normal-superharmonic is a local property), we see that $S\,s$ is nonnegative throughout $W\setminus B$ by (1.5) above, and the fact that $\lim_{W\setminus B} S\,s=\mathbf{0}$ at every point of ∂B leads easily to the fact that $S\,s$ is normal-superharmonic of potential type (its greatest normal-harmonic minorant is forced to be nonpositive).

From the simplicity of the situation considered in (3.6) above, we turn to the situation in which $\mathbf{1}$ is harmonic and $\mathscr{H}^{\mathfrak{Q}}$-normal-harmonic, which is a little more involved.

Theorem (3.8). *Let $\mathfrak{Q}$ be a normal structure subordinate to $\mathscr{H}$ on W for which $\mathbf{1}$ is normal-harmonic. Then there is a correspondence*

$$(A, U)\to \Psi_{(A,U)}$$

which assigns to every Cousin pair in W^* a linear functional $\Psi_{(A,U)}$ on the linear space of Cousin data for the pair, such that

(1) *$s\in\mathscr{H}^{\mathfrak{Q}}_{U\setminus A}$ is cohomologous to zero iff $\Psi_{(A,U)}[s]=0$.*

(2) *If (B, V) is a refinement of (A, U), then for $s\in\mathscr{H}^{\mathfrak{Q}}_{U\setminus A}$, $\Psi_{(B,V)}[s|V\setminus B]=\Psi_{(A,U)}[s]$.*

(3) *If $p\in\mathscr{H}^{\mathfrak{Q}}_{U\setminus A}$ is a restriction of a normal-superharmonic function on U whose support is contained in A, then $\Psi_{(A,U)}[p]\geqq 0$, equality holding iff p is harmonic in U.*

If $(A, U)\to\Phi_{(A,U)}$ is another such correspondence, then there is a positive constant ρ for which $\Phi_{(A,U)}=\rho\cdot\Psi_{(A,U)}$ for all (A, U).

Proof. To prove existence, select once and for all a regular region Z with compact closure and a function p_Z on Z which is continuous on $Z\setminus\{z_1\}$ (where $z_1\in Z$ is chosen and henceforth fixed) and a nonzero potential with support z_1 relative to $(Z, \mathscr{H}\,|\,Z)$; such potentials exist, by [9, Theorem 16.1, p. 468]. Suppose Y is a regular region containing z_1 and having compact proper closure. By the Hervé extension theorem there is a unique superharmonic function p_Y on Y such that $\lim_{\partial Y}\mathsf{p}_Y=\mathbf{0}$ (this condition being equivalent to p_Y's being a potential relative to Y) and such that p_Y and p_Z differ by a harmonic function in $Y\cap Z$. We can drop the restriction that Y be connected by setting $\mathsf{p}_Y=\mathbf{0}$ on components of Y not containing z_1. Suppose (A, U) is a special refinement of $(\{z_1\}, Y)$ of the type of (3.1). We can apply Theorem (3.2) to (A, U) with $M=L_A\in\mathfrak{Q}$; evidently $L_A H\,\mathbf{1}=\mathbf{1}$, so $1\in\sigma(L_A H)$ and alternative (B) of the theorem holds. Let μ_U be a strictly positive eigenvector (for 1) of $(L_A H)'$, and let $\nu_A=L'_A\,\mu_U$. A necessary and sufficient condition that a datum $s\in\mathscr{H}^c_{U\setminus A}$ be a coboundary is then that

$$\int s\,d\mu_U-\int L_A[s|\partial A]\,d\mu_U=0,$$

or, equivalently, that

$$\int s\,dv_A - \int H[s|\partial U]\,dv_A = 0.$$

The left-hand side of this expression is strictly positive whenever s is the restriction to $U\setminus A$ of a superharmonic function defined in a neighborhood of $\overline{U}$ and having nonempty support contained in A: for if U_i is a component of U in which s is not harmonic, then $s(\cdot)-H(s, U)(\cdot)$ is positive throughout U_i and even bounded away from zero on $A\cap U_i$ – but, as we noted at the conclusion of the proof of (3.2), $\nu_A(A\cap U_i)>0$. In particular, the number $P(A, U)=\int\{\mathsf{p}_Y - H[\mathsf{p}_Y|\partial U]\}\,dv_A>0$, and we may define $\Psi^c_{(A,U)}$ by setting

$$\Psi^c_{(A,U)}[s]=\frac{1}{P(A,U)}\cdot\{\int s\,dv_A-\int H[s|\partial U]\,dv_A\}$$

$$=\frac{1}{P(A,U)}\cdot\{\int s\,dv_A-\int s\,d\mu_U\}$$

for all $s\in\mathscr{H}^c_{U\setminus A}$. This is obviously independent of the original choice of μ_U. It is almost as obvious that it is independent of the choice of Y which we began by making: indeed, if $X\supseteq Y$ was another open set satisfying the same specifications as Y and p_X was the potential on X furnished by the Hervé extension theorem, then $\mathsf{p}_X-\mathsf{p}_Y$ would be cohomologous to zero in $(\{z_1\}, X\cap Y)$ and consequently cohomologous to zero in (A, U) (assuming $U\subseteq X\cap Y$, as we would certainly want to do). Consequently $\int(\mathsf{p}_X-\mathsf{p}_Y)\,d\mu=\int(\mathsf{p}_X-\mathsf{p}_Y)\,dv$, which says precisely that $P(A, U)$ is independent of Y. More generally, if (A, U) is a strict refinement of some (B, V) and $s\in\mathscr{H}_{V\setminus B}$ is such that $s|U\setminus A$ is cohomologous to $t\in\mathscr{H}^c_{U\setminus A}$, then $\int(s-t)\,d\mu_U=\int(s-t)\,dv_A$, from which it follows that $\Psi^c_{(A,U)}[t]=\Psi^c_{(A,U)}[s|U\setminus A]$. If the pair (B, V) just introduced is special, and the datum $s\in\mathscr{H}^c_{V\setminus B}$, it is clear that the requirement that (A, U) refine (B, V) *strictly* can be dispensed with.

Suppose, then, that (B, V) is also special and has the properties of (3.1), and that the point $z_1\in B$ also. Repeating the procedure used on (A, U), we can define a linear functional $\Psi^c_{(B,V)}$ on $\mathscr{H}^c_{V\setminus B}$. It is easy to see that $\Psi^c_{(B,V)}[s]=\Psi^c_{(A,U)}[s|U\setminus A]$ for all $s\in\mathscr{H}^c_{V\setminus B}$. Indeed, these two linear functionals have the same null space, because s is cohomologous to zero in $\mathscr{H}^c_{V\setminus B}$ if and only if $s|U\setminus A$ is cohomologous to zero in $\mathscr{H}^c_{U\setminus A}$, and those conditions are equivalent to $\Psi^c_{(B,V)}[s]=0$ and $\Psi^c_{(A,U)}[s|U\setminus A]=0$ respectively. It thus suffices to show that they have the same value at some function, and assuming that the set Y chosen at the beginning of this procedure contains $\overline{V}$, a good choice is p_Y. Indeed, $\Psi^c_{(B,V)}[\mathsf{p}_Y]=1=\Psi^c_{(A,U)}[\mathsf{p}_Y|U\setminus A]$ by definition.

We can now define Ψ itself. Let (C, X) be a Cousin pair in W^*. There is a regular open set $X_1 \subseteq W$ with compact proper closure, such that $C_1 = C \cup \{z_1\} \subseteq X_1$. If $s \in \mathscr{H}^{\mathfrak{Q}}_{X \setminus C}$ is a Cousin datum, then by (3.6) there is a datum $t \in \mathscr{H}_{X_1 \setminus C}$ for which $t|(X \cap X_1) \setminus C = s|(X \cap X_1) \setminus C$ up to a coboundary, and any two such data t differ by a coboundary. We can now set $\Psi_{(C,X)}[s] = \Psi^c_{(B,V)}[t|B \setminus V]$, where (B, V) is a special refinement of (C_1, X_1) with the properties of (3.1), and it is now only a matter of verifying that this number does not depend on the choice of X_1 or (B, V). That follows quite readily from the relation $\Psi^c_{(B,V)}[\cdot] = \Psi^c_{(A,U)}[\cdot|U \setminus A]$ which we know holds when (A, U) is a refinement of (B, V) as in the preceding paragraph, and we leave the detailed checking to the reader.

It remains to show that Ψ is unique up to a positive multiplicative constant. Supposing that Φ also satisfies (1), (2) and (3) above, set $\rho = \Phi_{(\{z_1\},Z)}[\mathsf{p}_Z] > 0$. If (C, X) is a Cousin pair on W and $s \in \mathscr{H}^{\mathfrak{Q}}_{X \setminus C}$, find a regular open set $X_1 \subseteq W$ with compact proper closure for which $C_1 = C \cup \{z_1\} \subseteq X_1$. Let $t \in \mathscr{H}_{X_1 \setminus C}$ be a datum for which $t|(X \cap X_1) \setminus C = s|(X \cap X_1) \setminus C$ up to a coboundary, and let p_{X_1} be the extension of p_Z to X_1 given by the Hervé extension theorem. Then $t - \Psi_{(C_1,X_1)}[t|X_1 \setminus C_1] \cdot (\mathsf{p}_{X_1}|X_1 \setminus C_1)$ belongs to the null space of $\Psi_{(C_1,X_1)}$, so it is a coboundary. Consequently it belongs to the null space of $\Phi_{(C_1,X_1)}$, and now we have

$$\begin{aligned}
\Phi_{(C,X)}[s] &= \Phi_{(C,X\cap X_1)}[s|(X \cap X_1) \setminus C] \\
&= \Phi_{(C,X\cap X_1)}[t|(X \cap X_1) \setminus C] \\
&= \Phi_{(C_1,X_1)}[t|X_1 \setminus C_1] \\
&= \Phi_{(C_1,X_1)}[\Psi_{(C_1,X_1)}[t|X_1 \setminus C_1] \cdot (\mathsf{p}_{X_1}|X_1 \setminus C_1)] \\
&= \Psi_{(C_1,X_1)}[t|X_1 \setminus C_1] \cdot \Phi_{(C_1,X_1)}[\mathsf{p}_{X_1}|X_1 \setminus C_1] \\
&= \Psi_{(C_1,X_1)}[t|X_1 \setminus C_1] \cdot \Phi_{(\{z_1\},X_1)}[\mathsf{p}_{X_1}] \\
&= \Psi_{(C,X)}[s] \cdot \rho .
\end{aligned}$$

Since (C, X) and s were arbitrary, we are done. Q.E.D.

Definition (3.9). *A correspondence Ψ satisfying the conditions of* (3.8) *above is called a flux functional (determined by $\mathscr{H}$ and $\mathfrak{Q}$).*

Since $\mathscr{H}$ and $\mathfrak{Q}$ determine the flux functional up to a multiplicative constant, we can deduce the following properties of flux functionals from the construction in (3.8):

Corollary (3.10). *If Ψ is a flux functional determined by $\mathscr{H}$ and $\mathfrak{Q}$, and (A, U) is a Cousin pair in W, then*

(1) *The functional $s \to \Psi_{(A,U)}[s]$ is continuous on $\mathscr{H}_{U \setminus A}$ (equipped with the topology of uniform convergence on compacta); for any special (A, U) with $z_1 \in A$ satisfying the conditions of* (3.1), *the functional $s \to \Psi^c_{(A,U)}[s]$ is*

continuous on $\mathscr{H}^c_{U\setminus A}$ (equipped with the topology of uniform convergence [or, equivalently, uniform convergence on the boundary]).

(2) *If p is a normal-superharmonic function on $W\setminus A$ whose restriction to $U\setminus A$ is harmonic, then $\Psi_{(A,U)}[p|U\setminus A]\leqq 0$, with strict inequality iff p is normal-superharmonic but not normal-harmonic.*

(3) *If A_1 and A_2 are disjoint compact proper subsets of an open set U and $s\in\mathscr{H}_{U\setminus(A_1\cup A_2)}$, then*

$$\Psi_{(A_1,U\setminus A_2)}[s]+\Psi_{(A_2,U\setminus A_1)}[s]=\Psi_{(A_1\cup A_2,U)}[s].$$

In particular, if $U=W$ and $s\in\mathscr{H}^\Omega_{W\setminus(A_1\cup A_2)}$, then

$$\Psi_{(A_1,W\setminus A_2)}[s]=-\Psi_{(A_2,W\setminus A_1)}[s].$$

Proof. Assertion (1) can be read quite directly from the construction in (3.8) and the continuity of the operator T of (3.6). For assertion (2), we may assume with no loss of generality that $z_1\in A$, (A, U) is special and satisfies the requirements of (3.1); then

$$\begin{aligned}\Psi^c_{(A,U)}[p|U\setminus A]&=\frac{1}{P(A,U)}\left\{\int p\,dv_A-\int p\,d\mu_U\right\}\\&=\frac{1}{P(A,U)}\int\{L[p|\partial A]-p|\partial U\}\,d\mu_U\end{aligned}$$

and the integrand is everywhere nonpositive. Since μ_U is a strictly positive measure, if $\Psi^c_{(A,U)}[p|U\setminus A]=0$ then $p=L[p|\partial A]$ at all points of ∂U. Since $p-L[p|\partial A]$ is nonnegative and normal-superharmonic in $W\setminus A$ and each component of $W\setminus A$ that is not contained in U contains points of ∂U, $p=L[p|\partial A]$ and is normal-harmonic.

For assertion (3), since both sides of the equation are unchanged under refinement there is no loss of generality in assuming that $U=U_1\cup U_2$ where $U_i\supseteq A_i$ $(i=1,2)$, U has proper compact closure, and $U_1\cap U_2=\emptyset$. Each $s\in\mathscr{H}_{U\setminus(A_1\cup A_2)}$ can be written in a unique way as $s=s_1+s_2$ where $s_1|(U_2\setminus A_2)=\mathbf{0}=s_2|(U_1\setminus A_1)$. Consider an $s_1\in\mathscr{H}_{U\setminus(A_1\cup A_2)}$ for which $s_1|(U_2\setminus A_2)=\mathbf{0}$. If $t_1\in\mathscr{H}_{U\setminus A_1}$ is defined by $t_1|U_1\setminus A_1=s_1|U_1\setminus A_1$ and $t_1|U_2=\mathbf{0}$, then s_1 is clearly a restriction of t_1. Consequently,

$$\begin{aligned}\Psi_{(A_1,U\setminus A_2)}[s_1]&=\Psi_{(A_1,U\setminus A_2)}[t_1|U\setminus(A_1\cup A_2)]=\Psi_{(A_1,U)}[t_1]\\&=\Psi_{(A_1\cup A_2,U)}[t_1|U\setminus(A_1\cup A_2)]=\Psi_{(A_1\cup A_2,U)}[s_1].\end{aligned}$$

On the other hand, if

$$s_2\in\mathscr{H}_{U\setminus(A_1\cup A_2)}\quad\text{and}\quad s_2|(U_1\setminus A_1)=\mathbf{0},$$

then

$$\Psi_{(A_1,U\setminus A_2)}[s_2]=\Psi_{(A_1,U_1)}[s_2]=0.$$

The symmetry of the situation insures that the same relations hold with the 1's and 2's interchanged. Consequently if $s \in \mathscr{H}_{U\setminus(A_1\cup A_2)}$ and $s = s_1 + s_2$ with the s_i as above, we have

$$\begin{aligned}\Psi_{(A_1\cup A_2,U)}[s] &= \Psi_{(A_1\cup A_2,U)}[s_1] + \Psi_{(A_1\cup A_2,U)}[s_2] \\ &= \Psi_{(A_1,U\setminus A_2}[s_1] + \Psi_{(A_2,U\setminus A_1)}[s_2] \\ &= \Psi_{(A_1,U\setminus A_2)}[s_1+s_2] + \Psi_{(A_2,U\setminus A_1)}[s_2+s_1] \\ &= \Psi_{(A_1,U\setminus A_2)}[s] + \Psi_{(A_2,U\setminus A_1)}[s]. \qquad \text{Q.E.D.}\end{aligned}$$

4. $H^1(W^*, \mathscr{H}^{\mathfrak{L}})$ for the Case $\mathbf{1} \in \bar{\mathscr{H}}^{\mathfrak{L}}$

This section will be devoted to the material in its title, namely the determination of $H^1(W^*, \mathscr{H}^{\mathfrak{L}})$ in the case where $\mathbf{1}$ is globally normal-superharmonic. Again $(W, \mathscr{H})$ and $\mathfrak{L}$ will remain fixed throughout the proceedings, and since the blanket hypotheses of § 3 above are assumed its results can be applied. The proofs are essentially set up for non-compact W, and the reader can either observe that the modifications necessary to handle $(W, \mathscr{H})$ with W compact and $\mathbf{1} \in \bar{\mathscr{H}}$ are trivial, or else apply the considerations of (1.8) and the succeeding discussion above to deduce the results for the case of compact W (which we formulate as (4.4) below for the sake of completeness) from the case considered here.

The fine resolution of $\mathscr{H}^{\mathfrak{L}}$ which § 2 above provides represents the cohomology groups $H^q(W^*, \mathscr{H}^{\mathfrak{L}})$ as quotients of groups derived from the resolution [8, p. 177], thus giving $H^q(W^*, \mathscr{H}^{\mathfrak{L}}) = 0$ for $q \geqq 2$ and $H^1(W^*, \mathscr{H}^{\mathfrak{L}}) \cong \Gamma(W^*, \mathscr{Q}^{\mathfrak{L}})/\Delta[\Gamma(W^*, \mathscr{R}^{\mathfrak{L}})]$; just as in the classical situation, therefore, we have to determine what global sections of $\mathscr{Q}^{\mathfrak{L}}$ are "global Laplacians." As § 3 suggests, the easy case of this problem is the one in which $\mathbf{1}$ is normal-superharmonic but not normal-harmonic, and the treatment of this case suggests a treatment of the case where $\mathbf{1} \in \mathscr{H}^{\mathfrak{L}}_W$.

Theorem (4.1). *If* $\mathbf{1} \in \bar{\mathscr{H}}^{\mathfrak{L}}$, *then* $H^1(W^*, \mathscr{H}^{\mathfrak{L}}) = 0$ *if and only if* $\mathbf{1} \notin \mathscr{H}^{\mathfrak{L}}$.

Proof. First it will be shown that if $\mathbf{1} \notin \mathscr{H}^{\mathfrak{L}}$, then every global section of $\mathscr{Q}^{\mathfrak{L}}$ is the Laplacian of a suitable global section of $\mathscr{R}^{\mathfrak{L}}$, which means precisely that $H^1(W^*, \mathscr{H}^{\mathfrak{L}}) = 0$. Take $M \in \Gamma(W^*, \mathscr{Q}^{\mathfrak{L}})$, and let $M(x)$ denote its value in the stalk $\mathscr{Q}^{\mathfrak{L}}_x$ for each $x \in W$. For each $x \in W$ there exist regions U_x containing x for which $M(x)$ is the canonical image of some $\mathsf{r}_x \in \mathfrak{Q}^{\mathfrak{L}}_{U_x}$. Since $\mathfrak{Q}^{\mathfrak{L}}_{U_x} \subseteq \mathscr{R}^{\mathfrak{L}}_{U_x}$, r_x can be construed as a section of $\mathscr{R}^{\mathfrak{L}}_{U_x}$ as in § 2; clearly $[\Delta \mathsf{r}_x](x) = M(x)$. Since two sections of a sheaf agree in a neighborhood of any point at which they agree, there is an open connected regular neighborhood V_x of x, with $x \in V_x \subseteq U_x$, such that $[\Delta \mathsf{r}_x](y) = M(y)$ for every $y \in V_x$. (If $x = \infty$, a regular neighborhood is of course the complement of some outer-regular set.) We can now select a finite set

$\{x_1, \ldots, x_n\} \subseteq W^*$ and corresponding regions $V_i = V_{x_i}$ and $U_i = U_{x_i}$ satisfying the specifications above, such that

$$W^* \subseteq \bigcup_{i=1}^{n} V_i.$$

With no loss of generality one may assume that $x_n = \infty$ and that $\infty \notin V_i$ for $i \neq n$. We can now replace the $\{\mathsf{r}_{x_i}\}_{i=1}^n$ by $\mathsf{q}_i = r_{V_i U_i}\, \mathsf{r}_{x_i} \in \mathfrak{Q}_{V_i}^{\mathfrak{Q}}$ and still have $M(y)$ be the canonical image of q_i for every $y \in V_i$. Now choose open sets Z_i with $\bar{Z}_i \subseteq V_i$, $i = 1, \ldots, n$, such that

$$W^* \subseteq \bigcup_{i=1}^{n} Z_i,$$

and find disjoint Borel sets $\{E_i\}_{i=1}^n$ with $E_i \subseteq Z_i$ and

$$\bigcup_{i=1}^{n} E_i = W^*$$

(for example, "disjointize" the Z_i as in the proof of (2.10) above). We shall then have

$$M = \sum_{i=1}^{n} \lambda_{E_i} M,$$

and the way that the action of the λ_{E_i} on $\mathfrak{Q}^{\mathfrak{Q}}$ is defined insures that $(\lambda_{E_i} M)(y)$ is the canonical image of $\lambda_{E_i} \mathsf{q}_i$ for every $y \in V_i$. It is clear that it will suffice to show that each $\lambda_{E_i} M \in \Gamma(W^*, \mathfrak{Q}^{\mathfrak{Q}})$ is the Laplacian of a suitable global section of $\mathfrak{R}^{\mathfrak{Q}}$.

For $i = 1, \ldots, n-1$, V_i is a proper regular region containing $\bar{Z}_i$; thus $\infty \notin \bar{Z}_i$ and $\bar{Z}_i$ is compact in W. As a result, $(\bar{Z}_i, V_i)$ is a Cousin pair in the sense of § 3 above. Since $\lambda_{E_i} \mathsf{q}_i = \lambda_{E_i}[\sum \alpha_j \mathsf{p}_j] = \sum \alpha_j \lambda_{E_i} \mathsf{p}_j$ where the α_j are scalars and the p_j are potentials in V_i, $\lambda_{E_i} \mathsf{q}_i$ is a linear combination of potentials in V_i which are harmonic outside $E_i \subseteq \bar{Z}_i$. Thus $\lambda_{E_i} \mathsf{q}_i | V_i \setminus \bar{Z}_i$ is a Cousin datum for the pair $(\bar{Z}_i, V_i)$. Theorem (3.6) insures that $\lambda_{E_i} \mathsf{q}_i$ is cohomologous to zero as a Cousin datum, which means precisely that there exist functions $u_i \in \mathscr{H}^{\mathfrak{Q}}_{W \setminus Z_i}$ and $h_i \in \mathscr{H}_{V_i}$ such that the function defined by

$$\begin{aligned} f_i &= \lambda_{E_i} \mathsf{q}_i - h_i \qquad \text{in } V_i \\ &= u_i \qquad\qquad\ \ \text{in } W^* \setminus \bar{Z}_i \end{aligned}$$

is well-defined. Clearly $f_i \in \Gamma(W^*, \mathfrak{R}^{\mathfrak{Q}})$ and $\Delta f_i = \lambda_{E_i} M$, since $\Delta f_i = 0$ in a neighborhood of the complement of V_i, while Δf_i equals the canonical image of $\lambda_{E_i} \mathsf{q}_i$ in V_i. The same argument goes through for $i = n$, except the Cousin pair in question becomes $(W \setminus V_i, W \setminus \bar{Z}_i)$.

For the converse, suppose that $H^1(W^*, \mathscr{H}^{\mathfrak{Q}}) = 0$ but that $\mathbf{1} \in \mathscr{H}^{\mathfrak{Q}}_{W^*}$. Let U be a proper regular region in W, and pick some p which is a non-

zero element of $\mathfrak{P}_U$ with support contained in a compact set $A \subseteq U$; then p defines a Cousin datum for the pair (A, U) by $\mathsf{p}|U\setminus A$, and since there is a natural *injection* $H^1(N(\{W^*\setminus A, U\}), \mathscr{H}^{\mathfrak{Q}}) \to H^1(W^*, \mathscr{H}^{\mathfrak{Q}}) = 0$, we can find $u \in \mathscr{H}^{\mathfrak{Q}}_{W\setminus A}$ and $h \in \mathscr{H}_U$ with

$$\begin{aligned} s &= \mathsf{p} - h \qquad \text{in } U \\ &= u \qquad \text{in } W\setminus A \end{aligned}$$

a well-defined function. Clearly s is normal-superharmonic throughout W, and not harmonic; by adding a suitable constant α to s if necessary we may assume that the minimum value attained by s on ∂A is zero (here we use the assumption that $\mathbf{1} \in \mathscr{H}^{\mathfrak{Q}}$, since α may be negative). By the minimum principle for normal-superharmonic functions of (1.5) above, $s \geqq \mathbf{0}$ throughout $W\setminus A$; but now we see that s is a nonnegative superharmonic function on W which takes a zero in W, contrary to the minimum principle for superharmonic functions [10, Theorem 2.12, p. 178]. Q.E.D.

The case where $\mathbf{1} \in \mathscr{H}^{\mathfrak{Q}}$ will essentially be treated by the same method we used above. Before dealing with it, however, it will be convenient to make some definitions which we would want to make eventually, and which help to phrase in an intuitive way the necessary and sufficient condition for a global section of $\mathscr{Q}^{\mathfrak{Q}}$ to be a global Laplacian.

Let $\mathscr{M}$ denote the (complete pre-)sheaf of Radon measures on open subsets of W^* with the obvious restriction mappings. We shall construct a homomorphism of presheaves from the presheaf $Q^{\mathfrak{Q}}$ whose associated sheaf is $\mathscr{Q}^{\mathfrak{Q}}$ to $\mathscr{M}$. (Strictly speaking we are only looking at those spaces $\mathscr{M}_U$ for which U is an open region in W^*, but that will cause no difficulty.) We assume that $\mathbf{1} \in \mathscr{H}^{\mathfrak{Q}}$, and we let Ψ be a flux functional for $\mathscr{H}^{\mathfrak{Q}}$ in the sense of (3.9). Ψ will be held fixed throughout what follows; but it will be obvious that since Ψ is unique up to a positive multiple, the ambiguity in the construction of the homomorphism will be that of a positive multiple also. Let U be a proper open region in W (i.e., suppose that $\infty \notin U$), and let $\mathsf{p} \in \mathfrak{P}_U$. Let $K \subseteq U$ be compact. For any Borel subset E of K, $\lambda_E\, \mathsf{p} \in \mathfrak{P}_U$ is a potential with support contained in K, and $(\lambda_E\, \mathsf{p})|U\setminus K$ is a Cousin datum for the pair (K, U). Consequently, we can form

$$T_U[\mathsf{p}](E) = \Psi_{(K,U)}[\lambda_E\, \mathsf{p}|U\setminus K]$$

and $T_U[\mathsf{p}](E) \geqq 0$ since $\lambda_E\, \mathsf{p}$ is a superharmonic function on U and Ψ satisfies (3) of Theorem (3.8) above. Now $[\lambda_E\, \mathsf{p}](x)$ is a finitely additive function of Borel sets E for each $x \in U$, so $T_U[\mathsf{p}](\cdot)$ is a finitely additive function of Borel subsets of K; however, $T_U[\mathsf{p}](\cdot)$ is also a countably additive function of Borel subsets of K, for if $\{E_k\}_{k=1}^{\infty}$ is an increasing

sequence of Borel subsets of K whose union is E_0, then $\{[\lambda_{E_k}\,\mathsf{p}](x)\}_{k=1}^{\infty}$ increases monotonely to $[\lambda_{E_0}\,\mathsf{p}](x)$ for each $x\in U$, and since all the $\lambda_{E_k}\,\mathsf{p}$ are harmonic in $U\setminus K$ this limit is attained uniformly on compacta there. By (1) of (3.10) above, $T_U[\mathsf{p}](E_0)=\lim_{k\to\infty} T_U[\mathsf{p}](E_k)$. Thus $T_U[\mathsf{p}](\cdot)$ is a Borel measure on K. It is easily verified that if J is another compact subset of U with $J\supseteq K$, then $T_U[\mathsf{p}](\cdot)$ assigns the same measures to sets $E\subseteq K$ when J is employed in the construction above; consequently, $T_U[\mathsf{p}](\cdot)$ is a well-defined function of relatively compact Borel subsets of U, finitely additive and countably additive on "bounded sequences" of Borel sets. It remains only to extend $T_U[\mathsf{p}](\cdot)$ to all Borel subsets of U by setting $T_U[\mathsf{p}](E)=\sup\{T_U[\mathsf{p}](E\cap K)\colon\ K\subseteq U \text{ compact}\}$, and verify that $T_U[\mathsf{p}](\cdot)$ is a Borel measure (possibly giving some sets infinite measure). $T_U[\mathsf{p}](\cdot)$ determines and is determined by a unique Radon measure in $\mathcal{M}_U$, which we can also denote by $T_U[\mathsf{p}](\cdot)$.

(*Remark.* If we use the $\mathfrak{B}$-module structure on $\mathfrak{Q}_U$ introduced parenthetically in the course of verifying that the λ_E's induced endomorphisms of *2* [§ 2 above, in the discussion preceding (2.10)], we can say this in a different way. Since $\mathcal{K}(U)\subseteq\mathfrak{B}$, and since it is easy to see that the support of the potential $\Lambda_f\,\mathsf{p}$ is contained in the support of f for $f\in\mathcal{K}(U)$, we can unambiguously set $T_U[\mathsf{p}](f)=\Psi_{(K,U)}[\Lambda_f\,\mathsf{p}|U\setminus K]$ where K contains the support of f. It is easy to see that this Radon measure is the same as the one induced by the set function $T_U[\mathsf{p}](\cdot)$ defined above.)

Since $\mathsf{p}\to\lambda_E\,\mathsf{p}$ is positively homogeneous and additive and Ψ is linear, we can extend the mapping $\mathsf{p}\to T_U[\mathsf{p}](\cdot)$ defined above to a linear transformation $T_U\colon\mathfrak{Q}_U^{\mathfrak{o}}\to\mathcal{M}_U$. (At this point it is necessary to insist that $T_U[\mathsf{p}]$ be thought of as a Radon measure, so that no questions of well-definition of $T_U[\mathsf{p}_1](E)-T_U[\mathsf{p}_2](E)$ can arise in situations where both $T_U[\mathsf{p}_j](E)=+\infty$.) If the restriction operator λ_F on $\mathcal{M}_U$ (for Borel sets $F\subseteq W^*$) is defined in the natural way (viz., $\lambda_F\mu$ is the measure $f\to\int f\cdot\chi_F\,d\mu$), then T_U "commutes" with λ_F. Indeed, for bounded Borel sets $E\subseteq U$ we have

$$\begin{aligned}T_U[\lambda_F\,\mathsf{p}](E)&=\Psi_{(K,U)}[\lambda_E(\lambda_F\,\mathsf{p})|U\setminus K]=\Psi_{(K,U)}[\lambda_{E\cap F}\,\mathsf{p}|U\setminus K]\\&=T_U[\mathsf{p}](E\cap F)=\lambda_F[T_U[\mathsf{p}]](E)\end{aligned}$$

for every potential $\mathsf{p}\in\mathfrak{P}_U$.

Essentially the same line of reasoning can be followed in case $\infty\in U$, except that two modifications must be made: we must insist that $\infty\in K$ when the compact set K is chosen, so that $(W\setminus U, W\setminus K)$ will be a Cousin pair. Since K can be taken as "large" as one wishes this causes no problems. However, in the definition of $T_U[\mathsf{p}](\cdot)$ one must take

$$T_U[\mathsf{p}](E)=-\Psi_{(W\setminus U,\,W\setminus K)}[\lambda_E\,\mathsf{p}|U\setminus K]$$

in order to get a positive measure out of a normal-superharmonic function of potential type; this is due to (2) of (3.10).

We now have to show that the T_U's taken together give a homomorphism of presheaves from $Q^{\mathfrak{Q}}$ to $\mathscr{M}$. Two slightly different cases arise. If $V \subseteq U$ are open regions in W, $\mathsf{p} \in \mathfrak{P}_U$, and r_{VU} is the linking map from $\mathfrak{Q}_U$ to $\mathfrak{Q}_V$ for the presheaf Q or $Q^{\mathfrak{Q}}$, then for any Borel subset of V with compact closure in V we have

$$\begin{aligned} T_U[\mathsf{p}](E) &= \Psi_{(K,U)}[\lambda_E \mathsf{p} | U \setminus K] && \text{where } E \subseteq K \subseteq V \text{ and } K \text{ is compact} \\ &= \Psi_{(K,V)}[\lambda_E \mathsf{p} | V \setminus K] && \text{since } \lambda_E \mathsf{p} \text{ is in } \mathscr{H}_{U \setminus K} \\ &= \Psi_{(K,V)}[r_{VU}(\lambda_E \mathsf{p}) | V \setminus K] && \text{because } r_{VU} \lambda_E \mathsf{p} \text{ and } \lambda_E \mathsf{p} \text{ differ by a harmonic function in } V \\ &= \Psi_{(K,V)}[\lambda_E(r_{VU} \mathsf{p}) | V \setminus K] = T_V[r_{VU} \mathsf{p}](E). \end{aligned}$$

The same argument works if $V \subseteq U$ are open regions in W^* with $\infty \in V$, $\mathsf{p} \in \mathfrak{P}_U^{\mathfrak{Q}}$ and r_{VU} the linking map from $\mathfrak{Q}_U$ to $\mathfrak{Q}_V$ for the presheaf $Q^{\mathfrak{Q}}$. However, in the case where U is a region in W^* containing ∞ but V is a region in W (not containing ∞), slight modifications are necessary: we have instead, letting $E \subseteq K \subseteq V$ with K compact and $J = K \cup \{\infty\}$,

$$\begin{aligned} T_U[\mathsf{p}](E) &= -\Psi_{(W \setminus U, W \setminus J)}[\lambda_E \mathsf{p} | U \setminus J] \\ &= -\Psi_{(W \setminus U, W \setminus K)}[\lambda_E \mathsf{p} | U \setminus K] && \text{since } \lambda_E \mathsf{p} \text{ is normal-harmonic in } U \setminus K \\ &= \Psi_{(K,U)}[\lambda_E \mathsf{p} | U \setminus K] && \text{by (3) of (3.10) above} \\ &= \cdots = T_V[r_{VU} \mathsf{p}](E) && \text{as above.} \end{aligned}$$

Since the arguments just given are valid for $\mathsf{p} \in \mathfrak{P}_U$ or $\mathfrak{P}_U^{\mathfrak{Q}}$ respectively, by linearity they are valid for $\mathsf{p} \in \mathfrak{Q}_U$ or $\mathfrak{Q}_U^{\mathfrak{Q}}$ respectively. Thus T is a homomorphism of $Q^{\mathfrak{Q}}$ into $\mathscr{M}$ (as presheaves of $\mathfrak{S}$-modules, also) and induces a homomorphism of the sheaf $\mathscr{Q}^{\mathfrak{Q}}$ into $\mathscr{M}$. We shall denote that induced homomorphism by τ, and again observe that τ is a homomorphism of $\mathfrak{S}$-modules – in other words, τ preserves the operation λ_E of restricting to a Borel set E. If $M \in \Gamma(W^*, \mathscr{Q}^{\mathfrak{Q}})$, then the measure τM is an element of $\Gamma(W^*, \mathscr{M}) = \mathscr{M}(W^*)$, and τM has finite total mass since W^* is compact. We may therefore form the mean $\int \mathbf{1}\, d[\tau M]$.

Definition (4.2). *If $M \in \Gamma(U, \mathscr{Q}^{\mathfrak{Q}})$ for open $U \subseteq W^*$, the measure $\tau M \in \mathscr{M}_U = \mathscr{M}(U)$ is the total charge distribution of M. If $M \in \Gamma(W^*, \mathscr{Q}^{\mathfrak{Q}})$, the mean $\int \mathbf{1}\, d[\tau M]$ is the total charge of M.*

We can now characterize $\Delta[\Gamma(W^*, \mathscr{R}^{\mathfrak{Q}})]$ in a manner which – since it reduces to the classical characterization in the classical situation considered in the introduction – is intuitively reasonable and natural.

Theorem (4.3). *If* $\mathbf{1} \in \mathscr{H}^{\mathfrak{Q}}_W$, *then a section* $M \in \Gamma(W^*, \mathscr{Q}^{\mathfrak{Q}})$ *belongs to* $\Delta[\Gamma(W^*, \mathscr{R}^{\mathfrak{Q}})]$ *if and only if* $\int \mathbf{1}\, d[\tau M] = 0$. *Consequently,*

$$\dim H^1(W^*, \mathscr{H}^{\mathfrak{Q}}) = 1.$$

Proof. As in (4.1) above, let $M \in \Gamma(W^*, \mathscr{Q}^{\mathfrak{Q}})$ and pick a finite set $\{x_1, \ldots, x_n\} \subseteq W^*$ with corresponding open regions $\{U_i\}_{i=1}^n$ and $\{V_i\}_{i=1}^n$, the V_i's regular, $x_i \in V_i \subseteq U_i$, and elements $\mathsf{r}_i \in \mathfrak{Q}_{U_i}$ $(1 \leqq i < n)$ and $\mathsf{r}_n \in \mathfrak{Q}^{\mathfrak{Q}}_{U_n}$, such that

$$W^* \subseteq \bigcup_{i=1}^{n} V_i$$

and such that $M(y)$ is the canonical image of r_i for each $y \in V_i$, $1 \leqq i < n$. Here we can be somewhat more careful than in (4.1), without loss of generality: we shall arrange that $x_n = \infty$, that $\infty \notin V_i$ for $i \neq n$, and that $\{V_i\}_{i=1}^n$ covers W^* irredundantly in the sense that no proper subfamily covers W^*. (Equivalently, every V_i possesses an element which belongs to no V_j, $j \neq i$.) Again choose open sets $\{Z_i\}_{i=1}^n$ with $\overline{Z}_i \subseteq V_i$, $1 \leqq i \leqq n$, such that $\{Z_i\}_{i=1}^n$ covers W^*, and find disjoint Borel sets $E_i \subseteq Z_i$, $1 \leqq i \leqq n$, whose union is W^*. Pick some $z_i \in V_i \setminus \bigcup_{j \neq i} V_j$ for $1 \leqq i \leqq n$, choosing $z_n = \infty$. Clearly $z_i \in Z_i \setminus \bigcup_{j \neq i} \overline{Z}_j$, so for $1 \leqq i < n$ there exists an outer-regular A_i with $z_i \in A_i$ such that A_i meets no $\overline{Z}_j$ for $j \neq i$; similarly, there exists a proper regular region Y_n which contains $\overline{Z}_j$ for all $j \neq n$ and whose complement is contained in Z_n. (For uniformity of notation we shall set $A_n = W \setminus Y_n$.) Again we can set $\mathsf{q}_i = r_{V_i U_i} \mathsf{r}_i$ and know that $M(y)$ is the canonical image of q_i for every $y \in V_i$, $1 \leqq i \leqq n$. We can also write

$$M = \sum_{i=1}^{n} \lambda_{E_i} M$$

again, and have $(\lambda_{E_i} M)(y)$ be the canonical image of $\lambda_{E_i} \mathsf{q}_i$ for every $y \in V_i$, $1 \leqq i \leqq n$; moreover, we still know that $\lambda_{E_i} \mathsf{q}_i$ is a linear combination of potentials in V_i which are harmonic outside $\overline{E}_i$, $1 \leqq i < n$, while $\lambda_{E_n} \mathsf{q}_n$ is a linear combination of normal-superharmonic functions of potential type in V_n, these functions being harmonic outside $\overline{E}_n$.

Since we have arranged that the V_i $(1 \leqq i < n)$ be regular regions, we can construct potentials $\mathsf{g}_i \in \mathfrak{P}_{V_i}$ by setting $\mathsf{g}_i = \mathbf{1}$ on A_i and g_i equal to the solution of the Dirichlet problem for $V_i \setminus A_i$ with boundary values $\mathbf{1}$

on ∂A_i and $\mathbf{0}$ on ∂V_i. Then $\mathsf{g}_i|V_i\setminus A_i$ is a Cousin datum for (A_i, V_i); its flux is necessarily positive, and we may set $\mathsf{w}_i=\beta_i\,\mathsf{g}_i$ where $\beta_i>0$ is so chosen that $\Psi_{(A_i,V_i)}[\mathsf{w}_i|V_i\setminus A_i]=1$. A similar construction for $i=n$ will produce $\mathsf{w}_n\in\mathfrak{P}^{\mathfrak{Q}}_{V_n}$ with $\Psi_{(A_n,Y_n)}[\mathsf{w}_n|Y_n\cap V_n]=-1$. For $1\leqq i\leqq n$, set

$$\begin{aligned}\alpha_i&=\int\mathbf{1}\,d[\tau(\lambda_{E_i}M)]\\&=\int\mathbf{1}\,d[\lambda_{E_i}(\tau M)]\\&=\int\chi_{E_i}\,d(\tau M)=\tau M(E_i).\end{aligned}$$

Since $\lambda_{E_i}M|V_i$ is the canonical image of $\lambda_{E_i}\,\mathsf{q}_i\in\mathfrak{Q}_{V_i}$, the definition of τ says that $(\lambda_{E_i}M|V_i)=T_{V_i}[\lambda_{E_i}\,\mathsf{q}_i]$, and since E_i is a relatively compact subset of V_i, this gives

$$\begin{aligned}\alpha_i&=\int_{V_i}\mathbf{1}\,d[\tau(\lambda_{E_i}M)]\\&=\int_{V_i}\mathbf{1}\,d[T(\lambda_{E_i}\,\mathsf{q}_i)]\\&=\int_{V_i}\chi_{E_i}\,d\big[T_{V_i}[\mathsf{q}_i]\big]\\&=\Psi_{(Z_i,V_i)}[\lambda_{E_i}\,\mathsf{q}_i|V_i\setminus\bar{Z}_i]\end{aligned}$$

since $\bar{E}_i\subseteq\bar{Z}_i$. Similarly,

$$\alpha_n=-\Psi_{(W\setminus V_n,\,W\setminus Z_n)}[\lambda_{E_n}\,\mathsf{q}_n|V_n\setminus\bar{Z}_n].$$

Consequently, the Cousin data on $V_i\setminus\bar{Z}_i$ $(1\leqq i\leqq n)$ given by the restrictions of the functions

$$\mathsf{z}_i=\lambda_{E_i}\,\mathsf{q}_i-\alpha_i\,\mathsf{w}_i\qquad(1\leqq i\leqq n)$$

have flux zero, and are therefore coboundaries: there are functions $h_i\in\mathscr{H}_{V_i}$ and $u_i\in\mathscr{H}^{\mathfrak{Q}}_{W\setminus Z_i}$ $(1\leqq i<n)$ and functions $h_n\in\mathscr{H}^{\mathfrak{Q}}_{V_n}$ and $u_n\in\mathscr{H}_{W\setminus Z_n}$ such that the functions defined by

$$\begin{aligned}f_i&=\mathsf{z}_i-h_i\quad\text{in }V_i\\&=u_i\qquad\text{in }W\setminus\bar{Z}_i\qquad(1\leqq i\leqq n)\end{aligned}$$

are well-defined. Clearly each $f_i\in\Gamma(W^*,\mathscr{R}^{\mathfrak{Q}})$. It is easy to verify that

$$\begin{aligned}\Delta f_i&=\text{the canonical image of }\mathsf{z}_i\text{ in }V_i\\&=0\text{ in }W\setminus\bar{Z}_i\qquad(1\leqq i\leqq n);\end{aligned}$$

this follows directly from the definition of Δ. It is also quite easy to compute $\int\mathbf{1}\,d[\tau(\Delta f_i)]$. Indeed, since Δf_i is zero outside $\bar{Z}_i$ and the sheaf

homomorphism τ can only shrink supports, we have

$$\begin{aligned}\int \mathbf{1}\, d[\tau(\Delta f_i)] &= \int_{V_i} \mathbf{1}\, d[\tau(\Delta f_i)] \\ &= \int_{V_i} \chi_{\mathbf{Z}_i}\, d[\tau(\Delta f_i)] \\ &= T_{V_i}[\mathbf{z}_i](\bar{Z}_i) \\ &= \Psi_{(\mathbf{Z}_i, V_i)}[\lambda_{\mathbf{Z}_i}\, \mathbf{z}_i | V_i \setminus \bar{Z}_i] \\ &= \Psi_{(\mathbf{Z}_i, V_i)}[\mathbf{z}_i | V_i \setminus \bar{Z}_i] \\ &= \alpha_i - \alpha_i = 0\end{aligned}$$

for $1 \leqq i < n$, and only trivial modifications are necessary to see that the same equation holds for $i = n$.

So the section

$$N = M - \sum_{i=1}^{n} \Delta f_i \in \Gamma(W^*, \mathscr{Q}^{\mathfrak{Q}})$$

shares two properties with M: it has the same total charge as M, and it belongs to the same coset of $\Delta[\Gamma(W^*, \mathscr{R}^{\mathfrak{Q}})]$ in $\Gamma(W^*, \mathscr{Q}^{\mathfrak{Q}})$ as M does. We now observe that for every point $y \in V_k$ $(1 \leqq k \leqq n)$, $N(y)$ is the canonical image of $\alpha_k\, \mathsf{w}_k$. This is most easily seen by writing

$$N = M - \sum_{i=1}^{n} \Delta f_i = \sum_{i=1}^{n} [\lambda_{E_i} M - \Delta f_i].$$

The i-th term of this sum is supported in V_i, and indeed in $\bar{Z}_i$; at any $y \in V_i$ the value

$$(\lambda_{E_i} M - \Delta f_i)(y)$$

is the canonical image of

$$[\lambda_{E_i}\, \mathsf{q}_i - (\lambda_{E_i}\, \mathsf{q}_i - \alpha_i\, \mathsf{w}_i)] = \alpha_i\, \mathsf{w}_i.$$

Thus the i-th term is actually supported in A_i $(1 \leqq i \leqq n)$. Since $\bar{Z}_k$ does not meet A_i unless $i = k$, we see that for a given $y \in V_k$ one of two things must happen to the i-th term of the sum: either $y \notin \bar{Z}_i$, so the i-th term is zero at y, or $y \in \bar{Z}_i$, so that the only A_j to which y can belong is A_i and that can happen only if $i = k$. Either way, therefore, $N(y) = [\lambda_{E_k} M - \Delta f_k](y)$ is the canonical image of $\alpha_k\, \mathsf{w}_k$.

Set

$$A = \bigcup_{i=1}^{n-1} A_i.$$

We have already observed that

$$Y_n \supseteq \bigcup_{i=1}^{n-1} \bar{Z}_i \supseteq A.$$

For each index $1 \leqq i < n$ we can construct by the Hervé extension theorem a potential v_i on Y_n whose restriction to Z_i differs from w_i by an element of $\mathscr{H}_{Z_i}$, and similarly we can construct by (3.7) above a normal-superharmonic function v_n of potential type on $W \setminus A$ which differs from w_n on Z_n by an element of $\mathscr{H}^{\Omega}_{Z_n}$. Clearly the fluxes of the v_i's are the same as those of the w_i's. Form

$$\sum_{i=1}^{n} \alpha_i \mathsf{v}_i | Y_n \setminus A .$$

Since the flux functional is linear, the flux of this datum is just

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} (\tau M)(E_i) = \int \mathbf{1}\, d[\tau M],$$

so this datum is a coboundary if and only if $\int \mathbf{1}\, d[\tau M] = 0$. If that condition is fulfilled, then there are functions $h \in \mathscr{H}_{Y_n}$ and $u \in \mathscr{H}^{\Omega}_{W \setminus A}$ such that

$$\begin{aligned} f &= \sum_{i=1}^{n} \alpha_i \mathsf{v}_i - h \quad \text{on } Y_n \\ &= u \qquad\qquad\quad \text{on } W \setminus A \end{aligned}$$

is a well-defined element of $\Gamma(W^*, \mathscr{R}^{\Omega})$. A fairly routine verification (of a type we should be getting used to by now) shows that the value of Δf at any point in Z_k is the canonical image of $\alpha_k \mathsf{w}_k$, $1 \leqq k \leqq n$. Since that is just the value of N at such a point, we have shown that $N = \Delta f$, or

$$M = \Delta \left[f + \sum_{i=1}^{n} f_i \right] \in \Delta[\Gamma(W^*, \mathscr{R}^{\Omega})].$$

Thus $\Delta[\Gamma(W^*, \mathscr{R}^{\Omega})]$ contains the hyperplane in $\Gamma(W^*, \mathscr{Q}^{\Omega})$ formed by $\{M : M \in \Gamma(W^*, \mathscr{Q}^{\Omega}), \int \mathbf{1}\, d[\tau M] = 0\}$. Consequently, either $\Delta[\Gamma(W^*, \mathscr{R}^{\Omega})] = \{M : M \in \Gamma(W^*, \mathscr{Q}^{\Omega}), \int \mathbf{1}\, d[\tau M] = 0\}$ or $H^1(W^*, \mathscr{H}^{\Omega}) = 0$. The latter eventuality, however would imply that $\mathbf{1} \notin \mathscr{H}^{\Omega}_W$, by (4.1) above, and so we have characterized the global Laplacians and shown that $\dim H^1(W^*, \mathscr{H}^{\Omega}) = 1$. Q.E.D.

The versions of these theorems for compact W can be summarized as follows:

Theorem (4.4). *If $(W, \mathscr{H})$ is a pair satisfying the Brelot axioms with W compact and $\mathbf{1} \in \mathscr{H}_W$, then $H^1(W, \mathscr{H}) = 0$ if and only if $\mathbf{1} \notin \mathscr{H}_W$, and if $\mathbf{1} \in \mathscr{H}_W$ then* $\dim H^1(W, \mathscr{H}) = 1$.

5. Notes

a) The parallel of the normal structure theory of § 1 with the axiomatic treatment of full-superharmonic functions (Kuramochi boundaries) of

Maeda [11] is obvious and deliberate – we have adopted the names normal-harmonic, normal-superharmonic, etc. to underline the parallel. It is easy to see that every full-harmonic structure determines a unique normal structure, because the full-harmonic structure prescribes solutions of the exterior Dirichlet problem in the complements of a cofinal family of compact sets by Maeda's axiom $\tilde{T}$ [11, p. 201]. However, the component-by-component definition of full-harmonic functions shows that not all normal structures come from full-harmonic ones. As an easy example, let $W=(0, 1)$ and $\mathscr{H}$ be the affine functions. For each closed interval $[a, b]\subseteq(0, 1)$ we assign to each function f on $\partial[a, b]$ the function in $W\setminus[a, b]$ determined by restricting to W the function on $[b-1, a]\cup[b, a+1]$ which is affine on each of these intervals and takes the value $f(a)$ at a and $a+1$ and takes the value $f(b)$ at $b-1$ and b. It is easy to see that this gives us a consistent family of quasi-normal operators and thus a normal structure; but it is not a full-harmonic structure, because a normal-harmonic function can vanish on the boundary of a component of its domain without being identically zero throughout the component.

b) Most of the considerations of § 2 are valid in axiomatic harmonic-function theories with much weaker convergence axioms than that of Brelot; the theorems will hold under the axioms of Bauer [1] or even the very weak assumptions of Boboc, Constantinescu and Cornea [2]. Some assumptions that insure the existence of plenty of superharmonic functions – at least locally – will have to be made. It is fairly easy to see, however, that if every point has a neighborhood in which there are sufficiently many continuous superharmonic functions to separate points – a local sort of Trennungsaxiom [1, p. 11] – then a Stone-Weierstrass argument will enable one to prove versions of (2.3) and (2.4), and consequently (2.5) and (2.6). The construction of the sheaf $\mathscr{Q}$ will similarly go through with weaker hypotheses, the specific restriction theory having been carried out under very weak assumptions by Boboc, Constantinescu and Cornea [2, Chapter II, p. 294ff.]. While their specific restriction theory only enables one to restrict specifically over continuous functions of compact support, that is all one really needs – our use of specific restriction over Borel sets has perhaps been a bit extravagant.

The fine resolution of $\mathscr{H}$ (or $\mathscr{H}^{\Omega}$) constructed in § 2 can also be applied to slightly more general situations. What a vector-valued harmonic function should be is clear, and therefore what a harmonic section of flat vector bundle should be is equally clear. Since all our considerations are local, they furnish a fine resolution of the sheaf of harmonic sections of any given flat vector bundle over W (or, *mutatis mutandis*, a fine resolution of the sheaf of normal-harmonic sections of a flat vector bundle over W^*).

A tempting generalization of the concept of (complete pre-) sheaf of functions satisfying the Brelot axioms might arise as follows: one is given a space W and a family of pairs $\{(W_\alpha, \mathscr{H}_\alpha)\}$ with $\{W_\alpha\}$ covering W, such that each $(W_\alpha, \mathscr{H}_\alpha)$ satisfies the Brelot axioms and pairs are related on $W_\alpha \cap W_\beta$ by multiplicative equivalence, i.e., for each (α, β) with $W_\alpha \cap W_\beta \neq \emptyset$ there is a continuous function $\phi_{\alpha\beta}$ with $\mathscr{H}_\beta = \phi_{\alpha\beta} \cdot \mathscr{H}_\alpha$. "Consistency" for the $\phi_{\alpha\beta}$'s leads to the requirement that they satisfy the 1-cocycle equation with multiplication as the operation. This almost immediately shows that nothing new arises here if the $\phi_{\alpha\beta}$'s are real-valued, since the sheaf of germs of real-valued positive continuous functions is fine, and representing the $|\phi_{\alpha\beta}|$'s as a coboundary gives a (complete pre-) sheaf $\mathscr{H}$ on W such that each $\mathscr{H}_\alpha$ is multiplicatively equivalent to $\mathscr{H}|W_\alpha$. The case in which the $\phi_{\alpha\beta}$'s are permitted to take complex values must be treated in a somewhat subtler way, but yields the same result.

c) The case of (3.2) in which U is connected and $1 \notin \sigma(MH)$ is due to M. Nakai (see [13, Appendix by Nakai, p. 293ff.]); the case in which U is not connected and in which $1 \in \sigma(MH)$ is due to the author, though a verification appears in [13, pp. 296–304] for the case in which U is connected. (From the present point of view this case is especially simple, since in this case MH is strictly positive – cf. [14, p. 268].)

An alternative approach to the theorem is the following: assume merely that U is proper and regular. MH is still compact. The norms of the powers of MH are uniformly bounded by 1; thus, by [15, § 3, ex. 1, p. 711], the operator MH is uniformly ergodic. The limit

$$P = \lim_{n \to \infty} \frac{1}{n} \cdot \left\{ \sum_{k=0}^{n-1} (MH)^k \right\}$$

exists in the norm topology of $\mathscr{L}(\mathscr{C}(\partial U))$ and is a nonnegative projection whose range on $\mathscr{C}_{\mathbb{R}}(\partial U)$ is a sublattice of $\mathscr{C}_{\mathbb{R}}(\partial U)$; it follows automatically from the definition of P that the range of P is the space of eigenvectors of MH belonging to $1 \in \sigma(MH)$. Suppose $\mathbf{0} \leqq f \in \mathscr{C}_{\mathbb{R}}(\partial U)$ and consider Pf, which is also nonnegative. From the fact that $MH[Pf] = Pf$ it follows that the function g defined by $H(Pf, U)$ in U and $M[H(Pf, U)|\partial A]$ in $V \setminus A$ is well-defined, and from $\mathbf{1} \in \mathscr{H}_U$ and $M\mathbf{1} \leqq \mathbf{1}$ it follows readily that $g \leqq \|Pf\| \cdot \mathbf{1}$ everywhere in V; but g attains the value $\|Pf\|$ at some point of U and is thus equal to $\|Pf\| \cdot \mathbf{1}$. Consequently the range of P contains only constant functions. Since P is idempotent and $P\mathbf{1}$ is constant, we must have either $P\mathbf{1} = \mathbf{0}$ or $P\mathbf{1} = \mathbf{1}$; the former alternative would imply that $P = 0$, contrary to the existence of at least one nonzero eigenvector for MH corresponding to the element $1 \in \sigma(MH)$. Taking $f = \mathbf{1}$ in the argument above, we see that $\mathbf{1} \in \mathscr{H}_V$ and $M\mathbf{1} = \mathbf{1}$ as in (3.2).

Since the spectral subspaces of $(MH)'$ and MH corresponding to $1 \in \sigma(MH)$ have the same dimension, the measure $\mu \in \mathcal{M}(\partial U)$ with $(MH)'\mu = \mu$ is unique up to a multiplicative constant, and μ can be chosen nonnegative [14, p. 267, Corollary]. The problem is to show that μ is strictly positive, as in (3.2). It is easy to see that it is *not* strictly positive if some component of U contains no points of A; thus in order that μ be strictly positive, U must have only finitely many components, each containing some points of A – the situation of (3.1). The strict positivity of μ is used strongly in § 3, in that division by the normalizing factor $P(A, U)$ in Theorem (3.8) requires the knowledge that this factor is not zero. It seems likely that the fact that $P(A, U) > 0$ can be established by *ad hoc* arguments not involving the strict positivity of μ; but (3.2) remains the strongest result available.

d) The concepts of total charge and total charge distribution introduced in (4.2) clearly do not require, for the constructions involved in their definition, the continuity of the elements of $\mathfrak{Q}_U$ or $\mathfrak{Q}_U^0$ respectively. The constructions are valid for arbitrary linear combinations of potentials, and the notions of total charge and total charge distribution can be defined in the same way for these more general functions. (The use of continuous potentials in the construction of the $\mathfrak{Q}_U$'s, we recall, was forced on us by the arguments required to show that $\mathcal{R}$ was fine – otherwise we could have used discontinuous potentials for the construction of $\mathcal{Q}$.) It is not difficult to see that if W is compact and the uniqueness condition of [9] holds (i.e., holds locally), then the sheaf homomorphism τ is an injection: the sections of the sheaf $\mathcal{Q}$ are completely determined by their total charge distributions, as they are in the classical situation. Question: Is τ an injection even if the uniqueness condition of [9] does not hold? This is a continuous version of the question of uniqueness, and might possibly have an affirmative answer.

e) The proofs of (4.1) and (4.3) are admittedly involved and direct. We feel that they are justified by their invoking no hypotheses other than the Brelot axioms and the (normal-[super-])harmonicity of **1**. Investigations to be reported in succeeding papers furnish proofs for compact W in the presence of assumptions that (among other things) imply that $\mathcal{H}$ admits an adjoint sheaf.

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