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# Functors between Tensored Categories

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This paper is a sequel to "Natural Associativity and Commutativity" by MACLANE [1]. We shall consider two distinct ways of generalising MACLANE'S work.

We first investigate what it means for a functor $T\colon \mathscr{A}\to\mathscr{B}$ and a natural transformation

$$p\colon TA\otimes TB\to T(A\otimes B)$$

to be coherent with respect to coherent commutative and associative tensor products in $\mathscr{A}$ and $\mathscr{B}$. This type of coherence is necessary when introducing cup products and Steenrod operations into the derived functors of $T$. As a typical example, let $\mathscr{A}$ be the category of sheaves of abelian groups over a topological space $X$, let $\mathscr{B}$ be the category of abelian groups, let $Z$ be the constant sheaf whose stalk at every $x\in X$ is the integers and let

$$TA=\mathrm{Hom}(Z,A).$$

In the second part of the paper we solve the coherence problem, when, in addition to a coherent commutative and associative tensor product in $\mathscr{A}$, we have a functor

$$H\colon \mathscr{A}\times\mathscr{A}\to\mathscr{A}$$

and a natural isomorphism

$$H(A_1\otimes A_2,A_3)\cong H(A_1,H(A_2,A_3)).$$

The example we have in mind here is $H=\mathrm{Hom}$, when $\mathscr{A}$ is the category of abelian groups.

## §1. Coherence between two Categories

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories each with a coherent commutative and associative tensor product. For clarity we denote the tensor product in $\mathscr{A}$ by $\otimes$ and that in $\mathscr{B}$ by $\hat{\otimes}$. We denote the associativity and commutativity isomorphisms by

$$a\colon A_1\otimes(A_2\otimes A_3)\to(A_1\otimes A_2)\otimes A_3$$

and

$$c\colon A_1\otimes A_2\to A_2\otimes A_1 \quad \text{in } \mathscr{A}$$

and by

$$\alpha\colon B_1\hat{\otimes}(B_2\hat{\otimes}B_3)\to(B_1\hat{\otimes}B_2)\hat{\otimes}B_3$$

and

$$\gamma\colon B_1\hat{\otimes}B_2\to B_2\hat{\otimes}B_1 \quad \text{in } \mathscr{B}.$$

Let $T$: $\mathscr{A} \to \mathscr{B}$ be a functor and let

$$p\colon\ TA_1 \hat{\otimes} TA_2 \to T(A_1 \otimes A_2) \tag{1.1}$$

be a natural transformation.

We define the set of *T-iterates* to be the smallest set of functors from $\mathscr{A}^n$ to $\mathscr{B}$ ($n=1, 2, 3, \ldots$) such that it contains:

i) $TF$ where $F$ is any iterate of $\otimes$ [*1*] p. 32;

ii) $U \hat{\otimes} V$ where $U$ and $V$ are any $T$-iterates;

iii) $\rho U$ where $U$: $\mathscr{A}^n \to \mathscr{B}$ is any $T$-iterate and $\rho \in S(n)$, the symmetric group.

An *instance of p* is any natural transformation of the form

$$p\colon\ TF(A_1, \ldots, A_r) \hat{\otimes} TG(A_{r+1}, \ldots, A_n) \to T\big(F(A_1, \ldots, A_r) \otimes G(A_{r+1}, \ldots, A_n)\big)$$

where $F$ and $G$ are iterates of $\otimes$. Let $f$ be any morphism in $\mathscr{A}$. By a *T-expansion* of $f$ we mean an expansion in $\mathscr{B}$ of $Tg$, where $g$ is an expansion in $\mathscr{A}$ of $f$. By an *occurrence* of $a$, $c$, $\alpha$, $\gamma$ or $p$ we shall mean a permuted expansion of $\alpha, \alpha^{-1}, \gamma$ or $p$ or a permuted $T$-expansion of $a, a^{-1}$ or $c$.

We now form the category whose objects are $T$-iterates and whose morphisms are all natural transformations of these functors. We consider the subcategory $\mathscr{T}$ generated (in the sense of [*1*] p. 31) by occurrences of $a$, $c$, $\alpha$, $\gamma$ or $p$. We say that $\mathscr{A}$ and $\mathscr{B}$ are *T-coherent* if there is at most one morphism in $\mathscr{T}$ between any two objects.

For example, if $\mathscr{A}$ and $\mathscr{B}$ are $T$-coherent, the following diagrams of natural transformations will be commutative

$$\begin{array}{ccc} TA \hat{\otimes} TB & \xrightarrow{p} & T(A \otimes B) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle Tc} \\ TB \hat{\otimes} TA & \xrightarrow{p} & T(B \otimes A), \end{array} \tag{1.2}$$

$$\begin{array}{ccccc} TA \hat{\otimes} (TB \hat{\otimes} TC) & \xrightarrow{1 \hat{\otimes} p} & TA \hat{\otimes} T(B \otimes C) & \xrightarrow{p} & T\big(A \otimes (B \otimes C)\big) \\ \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle Ta} \\ (TA \hat{\otimes} TB) \hat{\otimes} TC & \xrightarrow{p \hat{\otimes} 1} & T(A \otimes B) \hat{\otimes} TC & \xrightarrow{p} & T\big((A \otimes B) \otimes C\big), \end{array} \tag{1.3}$$

$$\begin{array}{ccccc} (TA \otimes TB) \hat{\otimes} (TC \hat{\otimes} TD) & & \xleftrightarrow{\cong} & & (TA \hat{\otimes} TC) \hat{\otimes} (TB \hat{\otimes} TD) \\ \downarrow{\scriptstyle p \hat{\otimes} p} & & & & \downarrow{\scriptstyle p \hat{\otimes} p} \\ T(A \otimes B) \hat{\otimes} T(C \otimes D) & & & & T(A \otimes C) \hat{\otimes} T(B \otimes D) \\ \downarrow{\scriptstyle p} & & & & \downarrow{\scriptstyle p} \\ T\big((A \otimes B) \otimes (C \otimes D)\big) & & \xleftrightarrow{\cong} & & T\big((A \otimes C) \otimes (B \otimes D)\big) \end{array} \tag{1.4}$$

where the upper horizontal isomorphism is a composition of occurrences of $\alpha, \alpha^{-1}$ and $\gamma$ and the lower horizontal isomorphism is a composition of occurrences of $a, a^{-1}$ and $c$. These horizontal isomorphisms are unique because of the coherence conditions in $\mathscr{A}$ and $\mathscr{B}$.

**1.5. Lemma.** *If* (1.2) *and* (1.3) *are commutative, then so is* (1.4).

*Proof.* The proof consists of the commutative diagram on page 224. The bottom part of the diagram is symmetrical with the top part, and so we omit details there. Those squares, where commutativity does not follow from (1.2) or (1.3), commute because the morphisms are natural transformations.

**1.6. Theorem.** *$\mathscr{A}$ and $\mathscr{B}$ are T-coherent if and only if the diagrams of natural transformations* (1.2), (1.3) *and* (1.4) *are commutative.*

*Proof.* After omitting some brackets, every $T$-iterate has the form

$$T(A_{\delta_1} \otimes \cdots \otimes A_{\delta i_1}) \hat{\otimes} \cdots \hat{\otimes} T(A_{\delta(i_r+1)} \otimes \cdots \otimes A_{\delta n})$$

where $\delta$ is a permutation of $(1, \ldots, n)$. This gives us a partition of $(1, \ldots, n)$ into $(r+1)$ disjoint non-empty subsets. We introduce a partial ordering into the set of partitions of $(1, \ldots, n)$ into disjoint non-empty subsets. We say $(I_1, \ldots, I_r) \geqq (J_1, \ldots, J_s)$ if for each pair $(i, j)$ with $1 \leqq i \leqq r$, $1 \leqq j \leqq s$, we have $I_i \subseteq J_j$ or $I_i \cap J_j = \emptyset$. Let $U$ and $V$ be $T$-iterates corresponding to partitions $(I_1, \ldots, I_r)$ and $(J_1, \ldots, J_s)$ of $(1, \ldots, n)$ respectively. It is easy to see that there is a morphism from $U$ to $V$ in our subcategory $\mathscr{T}$, if and only if $(I_1, \ldots, I_r) \geqq (J_1, \ldots, J_s)$. Moreover $r-s$ is the number of occurrences of $p$ in the morphism.

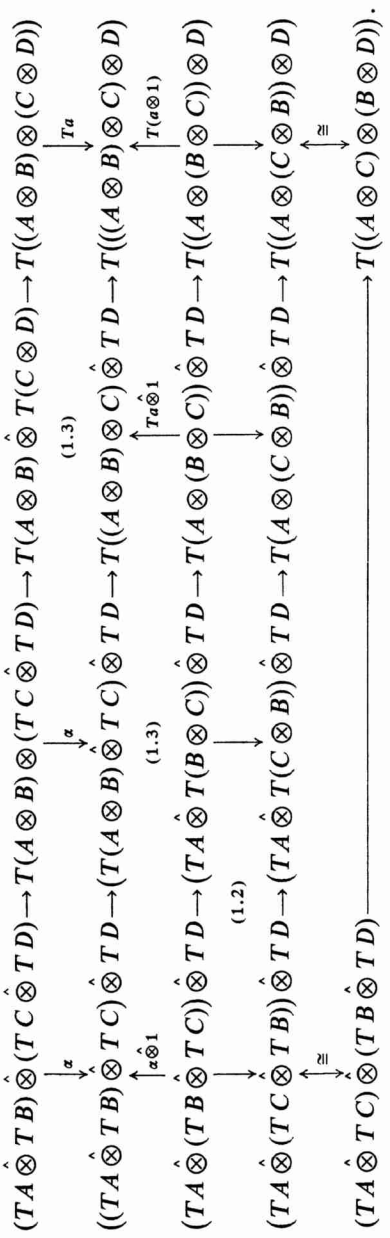
If $r=s$ (which is the same as having $(I_1, \ldots, I_r) = (J_1, \ldots, J_s)$) then there is exactly one morphism from $U$ to $V$. To see this we note first that an occurrence of $a$ or $c$ will commute with an occurrence of $\alpha$ or $\gamma$, and then apply the coherence conditions in $\mathscr{A}$ and $\mathscr{B}$.

We now show that if $r=s+1$ there is at most one morphism from $U$ to $V$. Without loss of generality we may suppose $I_{r-1} \cup I_r = J_s$ and $I_i = J_i$ if $i < r-1$. Without loss of generality $I_{r-1} = (1, \ldots, t)$ and $I_r = (t+1, \ldots, u)$. The relevant instance of $p$ is

$$TF(A_{\rho 1}, \ldots, A_{\rho t}) \hat{\otimes} TG(A_{\tau(t+1)}, \ldots, A_{\tau u}) \ldots$$
$$\cdots \longrightarrow T\big(F(A_{\rho 1}, \ldots, A_{\rho t}) \otimes G(A_{\tau(t+1)}, \ldots, A_{\tau u})\big)$$

where $\rho$ is a permutation of $(1, \ldots, t)$, $\tau$ is a permutation of $(t+1, \ldots, u)$ and $F$ and $G$ are iterates of $\otimes$.

Given any iterates $F'$ and $G'$, of multiplicity $t$ and $u-t$ respectively, we easily see that the above instance of $p$ is equal to the composition




of a number of occurrences of $a$ and $c$ followed by

$$p\colon\ TF'(A_1, \dots, A_t) \hat{\otimes} \hat{\otimes} TG'(A_{t+1}, \dots, A_u) \dots \dots \to T\big(F'(A_1, \dots, A_t) \otimes \otimes_i G'(A_{t+1}, \dots, A_u)\big)$$

followed by an equal number of occurrences of $a$ and $c$. In trying to prove that the two morphisms from $U$ to $V$ are equal, we can therefore assume (after using 1.2 if necessary) that the instances of $p$ occurring in each morphism are the same. Now using occurrences of $\alpha$ and $\gamma$, we can assume that the occurrences of $p$ are equal. This reduces the problem to the case $r=s$ which is already solved.

Now suppose $(r-s) \geqq 2$. We use induction on $r-s$. A morphism from $U$ to $V$ gives rise to a chain of partitions each greater than the next and containing exactly one more subset. Suppose we have two morphisms from $U$ to $V$ and $V$ corresponds to $(J_1, \dots, J_s)$. Let $(K_1, \dots, K_{s+1})$ be the penultimate partition for one morphism and $(L_1, \dots, L_{s+1})$ the penultimate partition for the other morphism.

We have four different possible types of situation, which without loss of generality can be represented by:

a) $(K_1, \dots, K_{s+1}) = (L_1, \dots, L_{s+1})$ or

b) $K_s \cup K_{s+1} = J_s = L_s \cup L_{s+1}$ and $K_i = J_i = L_i$ for $i < s$ and

i) $L_{s+1} \subset K_{s+1}$ or

ii) none of the four sets $L_s \cap K_s, L_{s+1} \cap K_s, L_s \cap K_{s+1}, L_{s+1} \cap K_{s+1}$ is empty or

c) $K_s \cup K_{s+1} = J_s = L_{s-1}$, $K_{s-1} = J_{s-1} = L_s \cup L_{s+1}$ and $K_i = J_i = L_i$ for $i < s-1$.

Case a) is solved immediately by our induction hypothesis.

In case b i) we use our induction hypothesis to show that each morphism factors through a $T$-iterate corresponding to the partition $(J_1, \ldots, J_{s-1}, K_s, L_{s+1}, K_{s+1} \cap L_s)$. We can now easily reduce the problem to the commutativity of 1.3, when $A$ is an iterate of $\otimes$ corresponding to $K_s$, $B$ an iterate corresponding to $K_{s+1} \cap L_s$ and $C$ an iterate corresponding to $L_{s+1}$.

In case b ii) both morphisms factor through a $T$-iterate corresponding to the partition

$$(J_1, \ldots, J_{s-1}, L_s \cap K_s, L_s \cap K_{s+1}, L_{s+1} \cap K_s, L_{s+1} \cap K_{s+1}).$$

We easily reduce the problem to the commutativity of 1.4, when $A$ is an iterate corresponding to $L_s \cap K_s$, $B$ to $L_s \cap K_{s+1}$, $C$ to $L_{s+1} \cap K_s$ and $D$ to $L_{s+1} \cap K_{s+1}$.

In case c) both morphisms factor through a $T$-iterate corresponding to the partition

$$(J_1, \ldots, J_{s-2}, L_s, L_{s+1}, K_s, K_{s+1}).$$

This case is solved by the commutative diagram

$$\begin{array}{ccc} (TA \hat{\otimes} TB) \hat{\otimes} (TC \hat{\otimes} TD) & \xrightarrow{p \hat{\otimes} 1} & T(A \otimes B) \hat{\otimes} (TC \hat{\otimes} TD) \\ \downarrow{\scriptstyle 1 \hat{\otimes} p} & & \downarrow{\scriptstyle 1 \hat{\otimes} p} \\ (TA \hat{\otimes} TB) \hat{\otimes} T(C \otimes D) & \xrightarrow{p \hat{\otimes} 1} & T(A \otimes B) \hat{\otimes} T(C \otimes D). \end{array}$$

This completes the proof of the theorem.

## § 2. The Coherence of Hom

Suppose $\mathscr{A}$ has a coherent commutative and associative tensor product and that we have a functor $H\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ and a natural isomorphism

$$b\colon H(A \otimes B, C) \xrightarrow{\cong} H(A, H(B, C)).$$

(For the sake of definiteness, we shall assume that $H$ is contravariant in the first variable and covariant in the second variable.)

We define the set of *H-iterates* to be the smallest set of functors from $\mathscr{A}^n$ to $\mathscr{A}$ $(n=1, 2, 3, \ldots)$ which contains

i) the identity functor;

ii) the functor $F \otimes G$ if $F$ and $G$ are $H$-iterates;

iii) the functor $H(F, G)$ if $F$ and $G$ are $H$-iterates;

iv) the functor $\alpha F$ if $F$: $\mathscr{A}^n \to \mathscr{A}$ is an $H$-iterate and $\alpha \in S(n)$, the symmetric group.

We define the set of *H-expansions* of a morphism $f$ in $\mathscr{A}$ to be the smallest set of morphisms of $\mathscr{A}$ which contains $f$ and for any $H$-expansion $e$ and any object $C$ in $\mathscr{A}$ the morphisms $e \otimes 1_C$, $1_C \otimes e$, $H(1_C, e)$, $H(e, 1_C)$. An *H-instance* of $c$ is a natural transformation

$$c\colon F(A_1, \ldots, A_n) \otimes G(A_{n+1}, \ldots, A_m) \ldots$$
$$\ldots \to G(A_{n+1}, \ldots, A_m) \otimes F(A_1, \ldots, A_n)$$

where $F$ and $G$ are $H$-iterates. We define instances of $a, a^{-1}, b, b^{-1}$ similarly. By an *occurrence* $a$, $c$ or $b$, we mean a permuted $H$-expansion of $a, a^{-1}, c, b$ or $b^{-1}$.

We consider the category whose objects are all $H$-iterates and whose morphisms are all natural transformations between there. Consider the subcategory $\mathscr{H}$ generated by occurrences of $a$, $c$ and $b$. We say that $a$, $b$ and $c$ are *coherent* if there is at most one morphism between any two objects in the subcategory.

For example, if $a$, $b$ and $c$ are coherent, the following diagram of natural isomorphisms will be commutative

$$\begin{array}{ccccc} H\big((A \otimes B) \otimes C, D\big) & \xrightarrow{H(a,\,1)} & H\big(A \otimes (B \otimes C), D\big) & \xrightarrow{b} & H\big(A, H(B \otimes C, D)\big) \\ \downarrow{\scriptstyle b} & & & & \downarrow{\scriptstyle H(1,\,b)} \\ H\big(A \otimes B, H(C, D)\big) & & \xrightarrow{\qquad\qquad b\qquad\qquad} & & H\big(A, H(B, H(C, D))\big). \end{array} \tag{2.1}$$

**2.2. Theorem.** *The diagram* 2.1 *is commutative if and only if* $a$, $b$ *and* $c$ *are coherent.*

*Proof.* We may represent the category $\mathscr{H}$ by a one-dimensional simplicial complex, in which an $H$-iterate is represented by a vertex and an occurrence is represented by an edge. We have to show that any circuit in this complex represents the identity morphism.

Let $T$ be an $H$-iterate and let $m(T)$ be the number of times $H$ appears in $T$. Formally, $m$ is defined by induction:

$$m(1)=0 \qquad \text{where } m \text{ is the identity functor;}$$
$$m(F \otimes G)=m(F)+m(G);$$
$$m\,H(F, G)=m\,F+m\,G+1;$$
$$m(\alpha F)=m\,F;$$

where $F$ and $G$ are $H$-iterates. A circuit in the one-dimensional complex is represented by an alternating sequence of vertices and edges. Let $n(m)$ be the number of vertices, corresponding to some $H$-iterate with $m(T)=m$. (We count a vertex $k$ times if the circuit passes through it $k$ times.) We assign to the circuit the integer $\Sigma_m 2^m n(m)$ and we shall prove that a circuit represents the identity morphism by induction on this integer.

If the circuit contains no edge corresponding to an occurrence of $b$, then the required result follows easily from the coherence of the tensor product. So we suppose that $b$ occurs at least once. We choose a vertex $T$ such that $m(T)$ is maximal and with an adjacent vertex $S$ such that $m(S)=m(T)-1$. We shall consider the various situations that can arise from the following part of the circuit

$$U \xrightarrow{\varphi} T \xrightarrow{\vartheta} S$$

where $\vartheta$ is of course an occurrence of $b$.

Assume first that $\varphi$ is an occurrence of $a$ or $c$, then we can easily construct a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & T \\ \downarrow{\scriptstyle \vartheta'} & & \downarrow{\scriptstyle \vartheta} \\ V & \xrightarrow{\varphi'} & S \end{array} \tag{2.3}$$

with $\vartheta'$ an occurrence of $b$ and $\varphi'$ an occurrence of $a$ or $c$. The commutativity results from the fact that $a, b$ or $c$ is a natural transformation. We have $m(U)=m(T)=m(S)+1=m(V)+1$. Replacing

$$U \xrightarrow{\varphi} T \xrightarrow{\vartheta} S$$

by

$$U \xrightarrow{\vartheta'} V \xrightarrow{\varphi'} S$$

we obtain a circuit with a smaller value of $\Sigma\, 2^m n(m)$.

Now suppose that $\varphi$ is an occurrence of $\beta$. Then it may or may not be possible to construct a commutative square as in (2.3). If it is possible, we have

$$m(T)=m(U)+1=m(S)+1=m(V)+2$$

and replacing $T$ by $V$ as above decreases $\Sigma 2^m n(m)$.

Finally, we suppose that $\varphi$ is an occurrence of $\beta$ and we cannot construct a commutative square like 2.3. Then the situation must be obtained from two adjacent occurrences of $b$ in 2.1 as follows. First we substitute $H$-iterates for $A$, $B$, $C$ and $D$. Then we take a permuted expansion of this situation. Since $m(T)=m(U)+1=m(S)+1$, $T$ must correspond to

$H(A, H(B, H(C, D)))$ in 2.1. By 2.1, we can replace

$$U \xrightarrow{\varphi} T \xrightarrow{\vartheta} S$$

by

$$U \xrightarrow{\vartheta'} V \xrightarrow{a'} W \xrightarrow{\varphi'} S$$

where $\vartheta'$ and $\varphi'$ are occurrences of $b$ and $a'$ is an occurrence of $a$. We have

$$m(T)=m(U)+1=m(S)+1=m(V)+2=m(W)+2\,.$$

Therefore $\Sigma 2^m\, n(m)$ is decreased by $2^{m(U)}$. This completes the proof.

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