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Research Papers

Functional equations associated with triangle geometry

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Summary. Triangle geometry is treated in the context of functional equations of three variables a, b, c which may be regarded as the sidelengths of a variable triangle. Triangle *centers* (e.g., incenter, circumcenter, centroid), and *central lines* (e.g., the Euler line) are defined and partitioned into classes: *0-centers, 1-centers, 2-centers* and *0-lines, 1-lines, and 2-lines.* Criteria for parallelism, perpendicularity, and other geometric relations are proved in terms of these classes. The Euler line and central lines parallel or perpendicular to the Euler line serve as examples.

1. Introduction

This work springs from a computer-assisted search for two perpendicular lines in the plane of an arbitrary triangle. Thousands of cases were checked without success. The explanation for this surprise was found to be that a certain functional equation has no solution of the general form that was being sampled. More importantly, the problem led to others of its kind, thereby opening up an area of application of functional equations and a new setting in which to view triangle geometry.

Before turning to general considerations, we note that the perpendiculars being sought were not just any lines, but rather, lines that pass through pairs of notable points of the triangle, such as the centroid, incenter, circumcenter, orthocenter, and Fermat's point. The definition of *center* given below includes these five points and infinitely many others. The nonperpendicularity property can be stated as follows: if P, Q, R, S are centers for which lines PQ and RS are perpendicular for *all* triangles, then the two lines are algebraically quite dissimilar (Theorem 8, part v, and Theorem 9). This dissimilarity leads to a distinction between the centers at opposite ends of a diameter of the circumcircle; specifically, in Theorem 12, if one of these is a "0-center" then the other is a "2-center."

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Actually, the real projective plane is contained as a "small" substructure of the structure in which we shall work, namely a projective plane over a field of functions in three variables, these being the sidelengths a, b, c of a variable reference triangle $\triangle ABC$. Within this framework, a "line", for example, is a set of equivalence classes of functions. It is convenient and helpful to call these classes *points*, since, for any numerical sidelengths a, b, c the value of such a class is indeed a point in a real projective plane.

Although the setting for this work is a projective plane, no reference is made to theorems of projective geometry. Indeed, the methods belong entirely to elementary algebra.

2. Definitions: point, line, parallel, perpendicular

Let $\mathbb{T}$ be the set of ordered triples of sidelengths of triangles in the Euclidean plane, so that

$$\mathbb{T} = \{(a, b, c)\colon 0 < a < b + c, 0 < b < c + a, 0 < c < a + b\}.$$

Let $\sqrt{\mathscr{P}} = (1/4)\sqrt{(a + b + c)(b + c - a)(c + a - b)(a + b - c)}$, the area of the triangle ABC having sidelengths a, b, c. Let $(\mathbb{R}, +, \cdot)$ be the ring of polynomial functions in $a, b, c, \sqrt{\mathscr{P}}$ over the real number field, and let $(\mathbb{F}, +, \cdot)$ be the quotient field of $(\mathbb{R}, +, \cdot)$. A *point*, P, is an equivalence class of ordered triples (f_1, f_2, f_3) of functions f_i in $\mathbb{F}$, at least one of which is not the zero function, where two such triples (f_1, f_2, f_3) and (g_1, g_2, g_3) are *equivalent* if the following two conditions hold:

$$g_i = 0 \quad \text{iff } f_i = 0 \quad \text{for } i = 1, 2, 3; \quad \text{and} \quad f_1/g_1 = f_2/g_2 = f_3/g_3$$

on all of $\mathbb{T}$ except the zero-set of $g_1 g_2 g_3$. Note that P has infinitely many representatives (f, g, h), not only in $\mathbb{F}^3$, but also in $\mathbb{R}^3$. For any such (f, g, h) in $\mathbb{F}^3$, we write P with colons instead of commas, like this:

$$P = f(a, b, c)\colon g(a, b, c)\colon h(a, b, c). \tag{1}$$

The expression on the right hand side of this equation will be called *coordinates of P*. This is short for *homogeneous trilinear coordinates*, which in triangle geometry means any triple α, β, γ of numbers proportional to the directed distances from P to the sides BC, CA, AB, respectively, of the reference triangle $\triangle ABC$. The actual directed distances are $k\alpha, k\beta, k\gamma$, where $k = 2\sqrt{\mathscr{P}}/(a\alpha + b\beta + c\gamma)$.

Representation (1) includes coordinates expressed in terms of the *angles* A, B, C, as these can always be expressed in terms of a, b, c. For example, from $\cos A = (b^2 + c^2 - a^2)/2bc$ follows

$$\cos A : \cos B : \cos C = a(b^2 + c^2 - a^2) : b(c^2 + a^2 - b^2) : c(a^2 + b^2 - c^2).$$

The function $f(a, b, c) = a(b^2 + c^2 - a^2)$ will appear so many times in this work that we shall abbreviate it as $\hat{a}$.

Now suppose $Q = \alpha_1 : \beta_1 : \gamma_1$ and $R = \alpha_2 : \beta_2 : \gamma_2$ are points. The *line of* Q *and* R is the set of all points $P = \alpha : \beta : \gamma$ satisfying

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0 \tag{2}$$

for all (a, b, c) in $\mathbb{T}$.

Lines $l\alpha + m\beta + n\gamma = 0$ and $l'\alpha + m'\beta + n'\gamma = 0$ are *parallel* if

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ a & b & c \end{vmatrix} = 0 \tag{3}$$

for all (a, b, c) in $\mathbb{T}$ and *perpendicular* if

$$2abc(ll' + mm' + nn') - (mn' + m'n)\hat{a} - (nl' + n'l)\hat{b} - (lm' + l'm)\hat{c} = 0 \tag{4}$$

for all (a, b, c) in $\mathbb{T}$. (Henceforth the condition "for all (a, b, c) in $\mathbb{T}$" will be tacitly understood for all equations.) These definitions match Articles 4618 and 4620 in Carr [1]. Here, of course, there is quite a different meaning assigned to the underlying symbols a, b, c. Nevertheless, equations (2), (3), (4) are appropriate for defining "line", "parallel", and "perpendicular", since for any Euclidean triangle $\triangle ABC$ with fixed sidelengths a, b, c, equation (2) represents a line in the plane of $\triangle ABC$, and so on. Figure 1 represents the Euclidean triangle with sidelengths $(a, b, c) = (6.02, 8.32, 11)$ and eight lines:

$\mathscr{L}_1$ = Euler line
= line of $bc : ca : ab$ (centroid) and $\hat{a} : \hat{b} : \hat{c}$ (circumcenter)

$\mathscr{L}_2$ = line of $1 : 1 : 1$ (incenter) and $1/(1 + 2\cos A) : 1/(1 + 2\cos B) : 1/(1 + 2\cos C)$

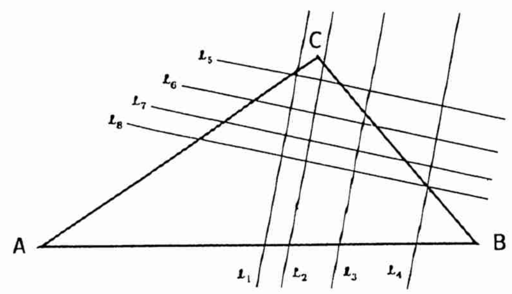

Figure 1. Line $\mathscr{L}_1$ is parallel to $\mathscr{L}_2$, $\mathscr{L}_3$, and $\mathscr{L}_4$, and perpendicular to $\mathscr{L}_5$, $\mathscr{L}_6$, $\mathscr{L}_7$, and $\mathscr{L}_8$.

$\mathscr{L}_3$ = line of $1-\cos(B-C): 1-\cos(C-A): 1-\cos(A-B)$ (Feuerbach point) and $1-2\cos A: 1-2\cos B: 1-2\cos C$

$\mathscr{L}_4$ = line of $\csc(A-\pi/3): \csc(B-\pi/3): \csc(C-\pi/3)$ (2nd isogonic center) and $\sin(A-\pi/3): \sin(B-\pi/3): \sin(C-\pi/3)$ (2nd isodynamic center)

$\mathscr{L}_5$ = the line $\alpha \cos A + \beta \cos B + \gamma \cos C = 0$

$\mathscr{L}_6$ = the line $\alpha \sin 2A \cos(B-C) + \beta \sin 2B \cos(C-A) + \gamma \sin 2C \cos (A-B) = 0$

$\mathscr{L}_7$ = the line $\alpha \sin 3A + \beta \sin 3B + \gamma \sin 3C = 0$

$\mathscr{L}_8$ = the line $\alpha \sin(A-\pi/3) + \beta \sin(B-\pi/3) + \gamma \sin(C-\pi/3) = 0$

3. Isogonal and associated conjugates on a line

Suppose $P = \alpha: \beta: \gamma$ is a point such that $\alpha\beta\gamma \neq 0$. The *isogonal conjugate* of P is the point defined by $P^{-1} = \beta\gamma: \gamma\alpha: \alpha\beta$, or equivalently, $1/\alpha: 1/\beta: 1/\gamma$. The *associated conjugate* of P is the point defined by $\hat{P}^{-1} = a\beta\gamma: b\gamma\alpha: c\alpha\beta$. In contrast to these algebraic definitions, a geometric definition of isogonal conjugate appears in many books without mention of coordinates. For an interesting property involving coordinates for various kinds of conjugates, see Eves and Kimberling [2].

THEOREM 1. *Let $\mathscr{L}$ be a line given by $l\alpha + m\beta + n\gamma = 0$, where $lmn \neq 0$. There exists one and only one point P such that $\mathscr{L}$ contains both the isogonal conjugate of P and the associated conjugate of P. If $P = p_\alpha: p_\beta: p_\gamma$, then an equation for $\mathscr{L}$ is*

$$(b-c)p_\alpha\alpha + (c-a)p_\beta\beta + (a-b)p_\gamma\gamma = 0,$$

and every point on $\mathscr{L}$ has the form

$$(u+av)p_\beta p_\gamma: (u+bv)p_\gamma p_\alpha: (u+cv)p_\alpha p_\beta,$$

where u and v are in $\mathbb{R}$ and satisfy $u(b, c, a) = u(a, b, c)$ and $v(b, c, a) = v(a, b, c)$.

Proof. Since $\mathscr{L}$ is given by $l\alpha + m\beta + n\gamma = 0$, another equation for $\mathscr{L}$ is $(b-c)l'\alpha + (c-a)m'\beta + (a-b)n'\gamma = 0$, where $l' = (c-a)(a-b)l$, $m' = (a-b)(b-c)m$, $n' = (b-c)(c-a)n$. Let $P = l' : m' : n'$. Clearly, the two conjugates, P^{-1} and $\hat{P}^{-1}$, lie on $\mathscr{L}$.

To prove uniqueness, let $\mathscr{L}_P$ be the line of P^{-1} and $\hat{P}^{-1}$ and suppose Q is a point such that both Q^{-1} and $\hat{Q}^{-1}$ lie on $\mathscr{L}_P$. Let $\mathscr{L}^*_{PQ}$ be the line of P^{-1} and Q^{-1}, and let $\mathscr{L}^{**}_{PQ}$ be the line of $\hat{P}^{-1}$ and $\hat{Q}^{-1}$. Obviously, $\mathscr{L}^{**}_{PQ} = \mathscr{L}^*_{PQ}$. Write Q as $q_\alpha : q_\beta : q_\gamma$. Then an equation for $\mathscr{L}^*_{PQ}$ is

$$\left(\frac{1}{p_\beta q_\gamma} - \frac{1}{p_\gamma q_\beta}\right)\alpha + \left(\frac{1}{p_\gamma q_\alpha} - \frac{1}{p_\alpha q_\gamma}\right)\beta + \left(\frac{1}{p_\alpha q_\beta} - \frac{1}{p_\beta q_\alpha}\right)\gamma = 0,$$

and for $\mathscr{L}^{**}_{PQ}$,

$$\frac{1}{a}\left(\frac{1}{p_\beta q_\gamma} - \frac{1}{p_\gamma q_\beta}\right)\alpha + \frac{1}{b}\left(\frac{1}{p_\gamma q_\alpha} - \frac{1}{p_\alpha q_\gamma}\right)\beta + \frac{1}{c}\left(\frac{1}{p_\alpha q_\beta} - \frac{1}{p_\beta q_\alpha}\right)\gamma = 0.$$

Since $\mathscr{L}^{**}_{PQ} = \mathscr{L}^*_{PQ}$, we have

$$\frac{1}{a}\left(\frac{1}{p_\beta q_\gamma} - \frac{1}{p_\gamma q_\beta}\right) : \frac{1}{b}\left(\frac{1}{p_\gamma q_\alpha} - \frac{1}{p_\alpha q_\gamma}\right) : \frac{1}{c}\left(\frac{1}{p_\alpha q_\beta} - \frac{1}{p_\beta q_\alpha}\right)$$
$$= \left(\frac{1}{p_\beta q_\gamma} - \frac{1}{p_\gamma q_\beta}\right) : \left(\frac{1}{p_\gamma q_\alpha} - \frac{1}{p_\alpha q_\gamma}\right) : \left(\frac{1}{p_\alpha q_\beta} - \frac{1}{p_\beta q_\alpha}\right),$$

which implies $Q = P$.

Now suppose R is a point on $\mathscr{L}$. By (2), there exist $\hat{u}$ and $\hat{v}$ in $\mathbb{F}$ such that $\hat{u}(b,c,a) = \hat{u}(a,b,c)$ and $\hat{v}(b,c,a) = \hat{v}(a,b,c)$, and R has first coordinate $(\hat{u} + a\hat{v})p_a$. Write $\hat{u} = u_1/u_2$, $\hat{v} = v_1/v_2$, where u_1, u_2, v_1, v_2 are in $\mathbb{R}$. Then with $u = u_1 v_2$ and $v = u_2 v_1$, we have R expressed as in the final sentence of the theorem. □

4. Centers and standard forms

A point $\hat{f}(a,b,c) : \hat{g}(a,b,c) : \hat{h}(a,b,c)$ is a *center* if there exists a function $f(a,b,c)$ in $\mathbb{R}$ such that the following conditions hold:

(F1) $\hat{f}(a,b,c) : \hat{g}(a,b,c) : \hat{h}(a,b,c) = f(a,b,c) : f(b,c,a) : f(c,a,b)$;

(F2) $f(a,c,b) = f(a,b,c)$;

(F3) $f(a,b,c)$ is homogeneous in a, b, c; that is, $f(ax,bx,cx) = x^n f(a,b,c)$ for some nonnegative integer n.

Note, in (F1), that the arguments of f, namely a, b, c, followed by b, c, a, followed by c, a, b, are the cyclic permutations of a, b, c. In particular, c, a, b is obtained from b, c, a exactly as b, c, a is obtained from a, b, c. These permutations match Euclidean constructions in which one first constructs an object (e.g., a median) relative to vertices A, B, C, then constructs in the same way the same kind of object relative to vertices B, C, A, and then repeats the construction relative to vertices C, A, B.

Property (F2), the symmetry of $f(a, b, c)$ in b and c, also matches a property of constructions of many notable triangle points: an object constructed relative to vertices A, B, C remains invariant if the construction is carried out relative to vertices A, C, B. For example, the median from A to the point halfway from B to C is the same as the median from A to the point halfway from C to B.

Let $f(a, b, c)$ be an element of $\mathbb{R}$ for which (F2) and (F3) hold and for which $f(a, b, c)$, $f(b, c, a)$, and $f(c, a, b)$ have no common divisor of degree $\geqslant 1$ in a, b, or c. (Here and in the sequel, all references to division will be understood to mean division in $\mathbb{R}$.) Then $f(a, b, c)$ has a factorization

$$f(a, b, c) = (b - c)^p (c - a)^q (a - b)^q s(a, b, c) \tag{5}$$

for some nonnegative integers p and q, such that the following conditions hold:

(S1) $s(a, b, c)$ is not divisible by $b - c$, $c - a$, or $a - b$;

(S2) $s(a, b, c)$, $s(b, c, a)$, $s(c, a, b)$ have no common divisor of degree $\geqslant 1$ in a, b, or c.

Let X denote the center $f(a, b, c) : f(b, c, a) : f(c, a, b)$. If in (5) p and q are even then X is a *0-center*; if p is odd then X is a *1-center*; if q is odd then X is a *2-center*. If X is a 0-center or a 2-center, we call X an *even center*. Representation (5) and conditions (S1) and (S2) imply the existence of a function $g(a, b, c)$ in $\mathbb{R}$, unique except for multiplication by real numbers, where

$$f(a, b, c) = \begin{cases} g(a, b, c) & \text{if } X \text{ is a 0-center} \\ (b - c) g(a, b, c) & \text{if } X \text{ is a 1-center} \\ (c - a)(a - b) g(a, b, c) & \text{if } X \text{ is a 2-center,} \end{cases} \tag{6}$$

and the following conditions hold:

(SF0) if X is a 0-center, then $g(a, b, c)$ has one of the forms $(b - c)^{2m} s(a, b, c)$ or $(c - a)^{2m} (a - b)^{2m} s(a, b, c)$;

(SF1) if X is a 1-center, then $g(a, b, c)$ has the form $(b-c)^{2m}s(a, b, c)$;

(SF2) if X is a 2-center, then $g(a, b, c)$ has the form $(c-a)^{2m}(a-b)^{2m}s(a, b, c)$;

in all three cases, m is a nonnegative integer, and $s(a, b, c)$ satisfies (S1) and (S2).

We call (6) a *standard form* for X and introduce the notation $\mathbb{C}\{f(a, b, c)\}$ to represent the center with coordinates $f(a, b, c):f(b, c, a):f(c, a, b)$. This notation is sometimes extended to $\mathbb{C}_{SF}\{f(a, b, c)\}$, where the added subscript SF means that the indicated coordinates are in standard form; that is, $f(a, b, c)$ is of one of the forms $g(a, b, c)$, $(b-c)g(a, b, c)$, or $(c-a)(a-b)g(a, b, c)$ in (6). Whenever this latter notation occurs, it will be tacitly understood that $g(a, b, c)$ has the properties associated with (6).

For example, the 1-center

$$\frac{1}{(c-a)(a-b)}:\frac{1}{(a-b)(b-c)}:\frac{1}{(b-c)(c-a)}$$

can be written as $\mathbb{C}\{(b-c)^2(c-a)(a-b)\}$ or as $\mathbb{C}_{SF}\{b-c\}$. Other examples follow:

1. The incenter, centroid, circumcenter, and orthocenter are $\mathbb{C}_{SF}\{1\}$, $\mathbb{C}_{SF}\{bc\}$, $\mathbb{C}_{SF}\{\hat{a}\}$, and $\mathbb{C}_{SF}\{\hat{b}\hat{c}\}$, respectively.
2. The first isodynamic point, $\sin(A+\pi/3):\sin(B+\pi/3):\sin(C+\pi/3)$, or $\mathbb{C}_{SF}\{a\sqrt{\mathscr{P}}+(\sqrt{3}/4)a(b^2+c^2-a^2)\}$, is a 0-center. Other 0-centers $\mathbb{C}_{SF}\{t(a, b, c)\}$ for which $t(a, b, c)$ invovles $\sqrt{\mathscr{P}}$ are the isogonic centers, the Napolean points, the isoperimetric point, and the point of equal detour. (These and many others are listed in Kimberling [3] with coordinates and references.)
3. The Steiner point, $\mathbb{C}\{1/[a(b^2-c^2)]\}$, given by $\mathbb{C}_{SF}\{(c-a)(a-b)bc(a+b)\times(a+c)\}$, is a 2-center.

Suppose X is a center. Then the isogonal conjugate of X is a 0-, 1-, or 2-center according as X is a 0-, 2-, or 1-center, respectively. Similarly, the associated conjugate of X is a 0-, 1-, or 2-center according as X is a 0-, 2-, or 1-center, respectively.

THEOREM 2. *Suppose $t(a, b, c)$ is a nonzero element of $\mathbb{R}$ satisfying the homogeneity property* (F3). *There exist polynomials $h_{il}(b, c)$ such that*

$$t(a, b, c) = \sum_{l=0}^{1} (\sqrt{\mathscr{P}})^l \sum_{i=0}^{n} e_{il}a^i h_{il}(b, c), \tag{7}$$

where for each i, l, the coefficient e_{il} equals 0 or 1, and if $e_{il} = 1$ then $h_{il}(b, c)$ is homogeneous of degree $n - i - 2l$ in b and c. If $t(a, c, b) = t(a, b, c)$, then $h_{il}(c, b) = h_{il}(b, c)$ for all i, l.

If $t(a, c, b) = -t(a, b, c)$, then $h_{il}(c, b) = -h_{il}(b, c)$, and $h_{il}(b, c)$ is divisible by $b - c$ for all i, l. Consequently, $t(a, b, c)$ has the form $(b - c)g(a, b, c)$ where $g(a, b, c)$ lies in $\mathbb{R}$ and satisfies $g(a, c, b) = g(a, b, c)$.

Finally, if $t(b, c, a) = t(a, b, c) = -t(a, c, b)$, then there exists $\hat{t}(a, b, c)$ in $\mathbb{R}$ such that

$$t(a, b, c) = (b - c)(c - a)(a - b)\hat{t}(a, b, c).$$

and $\hat{t}(a, b, c)$ is symmetric in a, b, c.

Proof. A function $t(a, b, c)$ as described in the first sentence has the form $\sum d(i, j, k, l)a^i b^j c^k(\sqrt{\mathscr{P}})^l$, the sum extending over all (i, j, k, l) for which $i + j + k + 2l = n$, where n is the degree of homogeneity of $t(a, b, c)$. It follows, since $\mathscr{P}$ is a polynomial in a, b, c, and $\sqrt{\mathscr{P}}$ is homogeneous of degree 2, that $t(a, b, c)$ has the form

$$t(a, b, c) = \sum e(i, j, k, l)a^i b^j b^k(\sqrt{\mathscr{P}})^l,$$

the sum extending over all (i, j, k, l) for which $i + j + k + 2l = n$ and $l \in \{0, 1\}$. Then

$$t(a, b, c) = \sum_{l=0}^{1} (\sqrt{\mathscr{P}})^l \sum_{i=0}^{n} a^i \sum_{j,k} e(i, j, k, l)b^j c^k.$$

For each i, l, if $e(i, j, k, l) = 0$ for all j, k satisfying $j + k = n - i - 2l$, then put $e_{il} = 0$; otherwise put $e_{il} = 1$, and put

$$h_{il}(b, c) = \sum_{j,k} e(i, j, k, l)b^j c^k.$$

Equation (7) is now established, and $h_{il}(b, c)$ is clearly homogeneous as asserted.

Suppose $t(a, c, b) = t(a, b, c)$. Then

$$0 = t(a, c, b) - t(a, b, c) = \sum_{l=0}^{1} (\sqrt{\mathscr{P}})^l \sum_{i=0}^{n} e_{il}a^i(h_{il}(c, b) - h_{il}(b, c)).$$

If $h_{il}(c, b) \neq h_{il}(b, c)$ for some i, l then the sum on the right is a nonzero linear combination of linearly independent functions, but this is impossible, since the sum is identically zero. Therefore, $h_{il}(c, b) = h_{il}(b, c)$ for all i, l.

Now suppose $t(a, c, b) = -t(a, b, c)$. Then $e(i, k, j, l) = -e(i, j, k, l)$ for all j, k, so that

$$h_{il}(b, c) = \sum_{j,k} e(i, k, j, l)(b^j c^k - b^k c^j),$$

the sum extending over all j, k for which $j \leqslant k$. Each difference $b^j c^k - b^k c^j$ factors as $(b - c)r_{jk}(b, c)$, where $r_{jk}(c, b) = r_{jk}(b, c)$. Thus,

$$h_{il}(b, c) = (b - c) \sum_{j,k} e(i, k, j, l)r_{jk}(b, c),$$

which implies by (7) that $t(a, b, c)$ has the asserted form.

Finally, suppose $t(b, c, a) = t(a, b, c) = -t(a, c, b)$. Then, as already shown, $b - c$ divides $t(a, b, c)$. Consequently, $c - a$ divides $t(b, c, a)$ and $a - b$ divides $t(c, a, b)$, so that $t(a, b, c)$ is divisible by all three; that is, $t(a, b, c) = (b - c)(c - a)(a - b)\hat{t}(a, b, c)$ for some $\hat{t}(a, b, c)$ in $\mathbb{R}$. We have $\hat{t}(a, c, b) = \hat{t}(a, b, c)$ and $\hat{t}(b, c, a) = \hat{t}(a, b, c)$, so that $\hat{t}(a, b, c)$ is symmetric in a, b, c. □

5. Central lines — a first look

A *central line* is a line joining two centers. As an example, the line joining the centroid and the circumcenter is a central line. The name of this particular line in triangle geometry is the *Euler line*, and as in triangle geometry, it contains the orthocenter, the center of the nine-point circle, and other centers which over the years have acquired names. Another central line of special interest is the *line at infinity*, $\mathscr{L}^\infty$, with equation $a\alpha + b\beta + c\gamma = 0$. See, for example, Carr [1, Article 4612].

It is of interest to compare various centers on the Euler line, along with symmetric polynomials $u(a, b, c)$ and $v(a, b, c)$ which, in the sense of Theorem 1, define them. In the notation of Theorem 1, we may take P to be the center having first coordinate $(b + c)\hat{a}$, so that P^{-1} and $\hat{P}^{-1}$ have first coordinates $(\sec A)/(b + c)$ and $(a \sec A)/(b + c)$, respectively.

Not every point on a central line is a center. To see this, note that a point $\alpha : \beta : \gamma$ is on the line of two centers $\alpha_1 : \beta_1 : \gamma_1$ and $\alpha_2 : \beta_2 : \gamma_2$ if and only if (2) holds, which is equivalent to the existence of functions $x = x(a, b, c)$ and $y = y(a, b, c)$ such that $(\alpha, \beta, \gamma) = (\alpha_1, \beta_1, \gamma_1)x + (\alpha_2, \beta_2, \gamma_2)y$. Clearly, there are many choices of x and y for which (F1) fails. Note, however, that if x and y are each symmetric in a, b, c, chosen so that $\alpha : \beta : \gamma$ satisfies (F3), then $\alpha : \beta : \gamma$ is a center.

Table 1. Centers on the Euler line

center	$u(a, b, c)$	$v(a, b, c)$
$\mathbb{C}\left\{\frac{\sec A}{b+c}\right\}$	1	0
$\mathbb{C}\left\{\frac{\tan A}{b+c}\right\}$	0	1
$\mathbb{C}\{\sec A\}$ (orthocenter)	$a+b+c$	-1
$\mathbb{C}\{\csc A\}$ (centroid)	$(3)+(21)+2(1^3)$ (See note below.)	$-(2)-2(1^2)$
$\mathbb{C}\{\cos A\}$ (circumcenter)	$(1)[-(14)+2(2^2)+4(1^3)(1)]$	$(1^2)(4)-2(321)-2(3^2)-4(2^3)$
$\mathbb{C}\{\sin A \tan A\}$	abc	$-bc-ca-ab$

Symmetric polynomials are abbreviated as follows: $(3)=a^3+b^3+c^3$; $(2^2)=b^2c^2+c^2a^2+a^2b^2$; $(21)=a^2b+ab^2+b^2c+bc^2+c^2a+a^2c$; and so on.

DEFINITIONS. A central line $\mathscr{L}$ with equation $f(a, b, c)\alpha + f(b, c, a)\beta + f(c, a, b)\gamma = 0$ is a *0-line*, *1-line*, or *2-line* according as the associated center $X = f(a, b, c):f(b, c, a):f(c, a, b)$ is a 0-center, 1-center, or 2-center, respectively. An *even line* is a line that is a 0-line or a 2-line. We abbreviate $\mathscr{L}$ as $\mathbb{L}\{f(a, b, c)\}$ and say that $\mathscr{L}$ is in *standard form* if $X = \mathbb{C}_{SF}\{f(a, b, c)\}$. To indicate that $\mathscr{L}$ is in standard form, we write $\mathbb{L}_{SF}\{f(a, b, c)\}$.

The Euler line is an example of a 1-line, whereas $\mathscr{L}^\infty$ is a 0-line having the distinctive property, easily proved using equations (3) and (4), that every line is both parallel to $\mathscr{L}^\infty$ and also perpendicular to $\mathscr{L}^\infty$.

A *zero-sum function* is an element $t(a, b, c)$ of $\mathbb{R}$ such that

$$t(a, b, c) + t(b, c, a) + t(c, a, b) = 0. \tag{8}$$

A *zero-sum center* is a center $\mathbb{C}_{SF}\{t(a, b, c)\}$ for which $t(a, b, c)$ is a zero-sum function. For any 0-center $\mathbb{C}_{SF}\{k(a, b, c)\}$, for example, $\mathbb{C}\{2k(a, b, c) - k(b, c, a) - k(c, a, b)\}$ is a zero-sum center.

THEOREM 3. *A line is a central line if and only if it has an equation* $L\alpha + M\beta + N\gamma = 0$ *where* $L: M: N$ *is a center.*

Proof. First suppose $\mathscr{L}$ is a central line. This means that there are distinct centers $\mathbb{C}\{f_1(a, b, c)\}$ and $\mathbb{C}\{f_2(a, b, c)\}$ such that $\mathscr{L} = \mathbb{L}\{l(a, b, c)\}$, where, by (2),

$$l(a, b, c) = f_1(b, c, a)f_2(c, a, b) - f_1(c, a, b)f_2(b, c, a).$$

Now

$$l(a, c, b) = f_1(c, b, a)f_2(b, a, c) - f_1(b, a, c)f_2(c, b, a)$$
$$= f_1(c, a, b)f_2(b, c, a) - f_1(b, c, a)f_2(c, a, b) = -l(a, b, c).$$

Define $m(a, b, c) = l(b, c, a)$ and $n(a, b, c) = l(c, a, b)$, and let $L = 1/mn$, $M = 1/nl$, $N = 1/lm$. Clearly, $L: M: N$ is a center, and $\mathscr{L}$ is given by the equation $L\alpha + M\beta + N\gamma = 0$.

For the converse, suppose $L: M: N$ is a center. Let $f_1(a, b, c) = (2a - b - c)MN$ and $f_2(a, b, c) = (2a^2 - b^2 - c^2)MN$. The centers $\mathbb{C}\{f_1(a, b, c)\}$ and $\mathbb{C}\{f_2(a, b, c)\}$ both lie on the central line $L\alpha + M\beta + N\gamma = 0$. □

THEOREM 4A. *Suppose* $\mathscr{L} = \mathbb{L}_{SF}\{(b - c)g(a, b, c)\}$ *is a 1-line. Then every center on* $\mathscr{L}$ *is an even center of the form* $\mathbb{C}\{(u + av)w(a, b, c)\}$, *where* $w(a, b, c) = g(b, c, a)g(c, a, b)$, *and* u *and* v *are elements of* $\mathbb{R}$ *that are symmetric in* a, b, c.

The 2-centers on $\mathscr{L}$ *are of the form* $\mathbb{C}\{(c - a)(a - b)w(a, b, c)t(a, b, c)\}$ *where* $\mathbb{C}\{t(a, b, c)\}$ *is a zero-sum center. Every center of this form lies on* $\mathscr{L}$.

Proof. Let $\mathscr{L} = \mathbb{L}_{SF}\{(b - c)g(a, b, c)\}$ be a 1-line, and let $X = \mathbb{C}\{g(a, b, c)\}$. Then $X^{-1} = \mathbb{C}\{w(a, b, c)\}$ and $\hat{X}^{-1} = \mathbb{C}\{\hat{w}(a, b, c)\}$, where $\hat{w}(a, b, c) = aw(a, b, c)$. Both X^{-1} and $\hat{X}^{-1}$ lie on $\mathscr{L}$, and by Theorem 1, so does every point

$$(u + av)w(a, b, c): (u + bv)w(b, c, a): (u + cv)w(c, a, b) \tag{9}$$

for which $u(b, c, a) = u(a, b, c)$, $v(b, c, a) = v(a, b, c)$, and u and v are in $\mathbb{R}$. In fact, by Theorem 1, *every center* on $\mathscr{L}$ has the form (9) with the attendant conditions on u and v.

We show next that if Y is a center of the form (9), then u and v may be assumed to be symmetric in a, b, c. If Y is a center, then one of the following conditions holds:

(i) $u(a, c, b) = u(a, b, c)$ and $v(a, c, b) = v(a, b, c)$;

(ii) $u(a, c, b) = -u(a, b, c)$ and $v(a, c, b) = -v(a, b, c)$.

In the second case, u and v can, by Theorem 2, be replaced in (9) by symmetric functions $\hat{u}$ and $\hat{v}$. This proves that there is no loss in assuming that u and v satisfy (i) and are therefore symmetric.

To see that Y must be an even center, suppose to the contrary that Y is a 1-center. Since u and v are symmetric, $u + av$ cannot have standard form $(b - c)h(a, b, c)$ and so must be of the form

$$(b - c)^p(c - a)^q(a - b)^q h(a, b, c),$$

where $p - q$ is an odd positive integer and $h(a, b, c)$ is not divisible by any of $b - c$, $c - a$, and $a - b$. Thus $Y = \mathbb{C}_{SF}\{(b - c)^{p-q}h(a, b, c)\}$, and since Y is on $\mathscr{L}$, we have

$$(b - c)^e Q(a, b, c) + (c - a)^e Q(b, c, a) + (a - b)^e Q(c, a, b) = 0,$$

where e is even and $Q(a, b, c) = h(a, b, c)g(a, b, c)$. Put $c = a$ to get $Q(a, a, b) = 0$. Now by the division algorithm, $Q(a, b, c) = (a - b)q(a, b, c) + r(b, c)$ for some q and r in $\mathbb{R}$, so that $Q(a, a, c) = r(a, c)$. The condition $Q(a, a, b) = 0$ (for *all* a and b) implies $r(a, c) = 0$, so that $a - b$ divides $Q(a, b, c)$. However, as $a - b$ divides neither $h(a, b, c)$ nor $g(a, b, c)$, we have reached a contradiction. Therefore, Y is an even center.

Clearly, a 2-center lies on $\mathscr{L}$ if and only if (8) applies as asserted. □

It is helpful to examine examples in connection with Theorem 4A. In addition to those in Table 1, we mention one other. The point of intersection of a 1-line $\mathbb{L}\{(b - c)g(a, b, c)\}$ with $\mathscr{L}^\infty$ is $\mathbb{C}\{b(a - b)g(c, a, b) + c(a - c)g(b, c, a)\}$, for which, in (9),

$$u(a, b, c) = \frac{a^2}{g(a, b, c)} + \frac{b^2}{g(b, c, a)} + \frac{c^2}{g(c, a, b)}$$

and

$$v(a, b, c) = -\frac{a}{g(a, b, c)} - \frac{b}{g(b, c, a)} - \frac{c}{g(c, a, b)}.$$

THEOREM 4B. *Suppose* $\mathscr{L} = \mathbb{L}\{g(a, b, c)\}$ *is an even line.*

Case 1. $\mathscr{L}$ *a 0-line. In this case every center* X *on* $\mathscr{L}$ *has the form* $\mathbb{C}\{(b - c)(u + av)w(a, b, c)\}$, *where* $w(a, b, c) = g(b, c, a)g(c, a, b)$, *and* u *and* v *are elements of* $\mathbb{R}$ *that satisfy*

$$u(b, c, a) = u(a, b, c) \quad \text{and} \quad v(b, c, a) = v(a, b, c). \tag{10}$$

X *is a 1-center if and only if* u *and* v *are symmetric, i.e., they satisfy not only* (10), *but also*

$$u(a, c, b) = u(a, b, c) \quad \text{and} \quad v(a, c, b) = v(a, b, c).$$

An even center $\mathbb{C}_{SF}\{e(a, b, c)\}$ *lies on* $\mathscr{L}$ *if and only if* $\mathbb{C}\{e(a, b, c)g(a, b, c)\}$ *is a zero-sum center.*

Case 2. $\mathscr{L}$ *a 2-line,* $\mathbb{L}_{SF}\{(c - a)(a - b)h(a, b, c)\}$. *Here, every center on* $\mathscr{L}$ *has the form* $\mathbb{C}\{(b - c)t(a, b, c)h(b, c, a)h(c, a, b)\}$ *for some zero-sum function* $t(a, b, c)$. *Every center of this form is on* $\mathscr{L}$.

Proof. Suppose X is a center on an even line $\mathbb{L}_{SF}\{g(a, b, c)\}$. For case 1, we apply Theorem 4A to $\mathbb{L}_{SF}\{(b-c)g(a, b, c)\}$ to obtain

$$X = (b-c)(u+av)w(a, b, c)\colon (c-a)(u+bv)w(b, c, a)\colon (a-b)(u+cv)w(c, a, b), \tag{11}$$

which is clearly a 1-center if and only if u and v are symmetric.

The remaining assertions of the theorem obviously hold. □

Theorems 4A and 4B translate into dual theorems, in which the roles of central lines and centers are reversed. For example, if $X = \mathbb{C}_{SF}\{(b-c)g(a, b, c)\}$ is a 1-center, then every central line containing X is an even line of the form $\mathbb{L}\{(u+av)w(a, b, c)\alpha\}$, where $w(a, b, c) = g(b, c, a)g(c, a, b)$, and u and v are elements of $\mathbb{R}$ that are symmetric in a, b, c.

Equation (8) plays a prominent role in Theorems 4A and 4B. We solve this functional equation in the next theorem.

THEOREM 4C. *Suppose*

$$t(a, b, c) = \sum_{l=0}^{1} (\sqrt{\mathscr{P}})^l \sum_{i=0}^{n} a^i \sum_{j,k} e(i, j, k, l) b^j c^k$$

is an element of $\mathbb{R}$; *as in Theorem 2, the inner summation is over the set* $\mathscr{S}$ *of all* j, k *satisfying* $j \geqslant 0$, $k \geqslant 0$, $i+j+k+2l = n$.

Case 1. $t(a, c, b) = t(a, b, c)$. *Here,* $t(a, b, c)$ *is a zero-sum function if and only if*

$$e(i, j, k, l) + e(j, k, i, l) + e(k, i, j, l) = 0$$

for every j, k *in* $\mathscr{S}$, *for* $i = 0, 1, \ldots, n$, *for* $l = 0, 1$.

Case 2. $t(a, c, b) = -t(a, b, c)$. *Here,* $t(a, b, c)$ *is a zero-sum function if and only if there exist constants* d_{il} *such that*

$$t(a, b, c) = \sum_{l=0}^{1} (\sqrt{\mathscr{P}})^l \sum_{i=0}^{[n/2]} d_{il} a^i (b^i c^{n-2i-l} - c^i b^{n-2i-l})$$

Proof. Case 1. As a symmetric function of a, b, c, the sum in (8) is a sum of symmetric functions of the form

$$(e(i, j, k, l) + e(j, k, i, l) + e(k, i, j, l)) \sum a^i b^j c^k (\sqrt{\mathscr{P}})^l.$$

Since the functions $\sum a^i b^j c^k (\sqrt{\mathscr{P}})^l$ are linearly independent, (8) holds if and only if the coefficients are all zero, as was to be proved.

Case 2. By Theorem 2, $t(a, b, c)$ is a sum of functions of the form $p(a, b, c) = da^i(\sqrt{\mathscr{P}})^l(b^jc^k - c^jb^k)$, so that the sum in (8) is a sum of functions $S(a, b, c)$ of the form

$$S(a, b, c) = p(a, b, c) + p(b, c, a) + p(c, a, b).$$

Suppose (8) holds. Then for any choice of (i, j, k, l), we have $S(a, b, c) = 0$. If $i \neq j$ and $i \neq k$, then $j = k$, so that $p(a, b, c) = 0$. If $i = j$ and $k = j$, then again $p(a, b, c) = 0$. If $i = j$ and $k \neq j$, or if $i \neq j$ and $i = k$, then $p(a, b, c) \neq 0$, so that each $p(a, b, c)$ appearing in $t(a, b, c)$ has the form $da^i(\sqrt{\mathscr{P}})^l(b^ic^k - c^ib^k)$. Moreover, each such $p(a, b, c)$ does in fact clearly imply $S(a, b, c) = 0$. Therefore, the general linear combination given in the statement of the theorem is the solution of (8) in Case 2. □

Among the simplest zero-sum functions $t(a, b, c)$ satisfying the conditions of Theorem 4C are the following.

Zero-sum polynomials of degree 2 of homogeneity:

$$t(a, b, c) = \begin{cases} d_0(-2a^2 + b^2 + c^2) + d_1(-2bc + ab + ac) & \text{if } t(a, c, b) = t(a, b, c) \\ d_0(b^2 - c^2) + d_1 a(b - c) & \text{if } t(a, c, b) = -t(a, b, c). \end{cases}$$

Zero-sum polynomials of degree 3 of homogeneity:

$$t(a, b, c) = \begin{cases} d_0(-2a^3 + b^3 + c^3) + d_1(b^2c + bc^2 - a^2b - a^2c) \\ \quad + d_2(ab^2 + ac^2 - a^2b - a^2c) & \text{if } t(a, c, b) = t(a, b, c) \\ d_0(b^3 - c^3) + d_1 a(b^2 - c^2) + d_2a^2(b - c) & \text{if } t(a, c, b) = -t(a, b, c). \end{cases}$$

Zero-sum polynomials of degree 4 of homogeneity:

$$t(a, b, c) = \begin{cases} d_0(-2a^4 + b^4 + c^4) + d_1(-2b^2c^2 + a^2b^2 + a^2c^2) \\ \quad + d_2abc(-2a + b + c) + d_3a^3(b + c) \\ \quad + d_4a(b^3 + c^3) + (-d_3 - d_4)bc(b^2 + c^2) & \text{if } t(a, c, b) = t(a, b, c) \\ d_0(b^4 - c^4) + d_1 abc(b - c) + d_2a^2(b^2 - c^2) & \text{if } t(a, c, b) = -t(a, b, c). \end{cases}$$

THEOREM 5. *The point X of intersection of two central lines $\mathscr{L}_1$ and $\mathscr{L}_2$ is a center. If $\mathscr{L}_1$ or $\mathscr{L}_2$ is a 1-line, then X is an even center.*

If $\mathscr{L}_1 = \mathbb{L}_{SF}\{g(a, b, c)\}$ and $\mathscr{L}_2 = \mathbb{L}_{SF}\{h(a, b, c)\}$ are even lines satisfying

$$g(b, a, a)h(a, a, b) \neq g(a, a, b)h(b, a, a), \tag{12}$$

then X is a 1-center.

Proof. Suppose $\mathscr{L}_1$ and $\mathscr{L}_2$ are given by $l\alpha + m\beta + n\gamma = 0$ and $L\alpha + M\beta + N\gamma = 0$, respectively. It is easy to check using (2) that the point $\alpha : \beta : \gamma = mN - nM : nL - lN : lM - mL$ lies on both lines. We have

$$\begin{aligned}\alpha(a, c, b) &= m(a, c, b)N(a, c, b) - n(a, c, b)M(a, c, b)\\ &= m(a, b, c)N(a, b, c) - n(a, b, c)M(a, b, c)\\ &= \alpha(a, b, c).\end{aligned}$$

Clearly $\beta(a, b, c) = \alpha(b, c, a)$ and $\gamma(a, b, c) = \alpha(c, a, b)$, and homogeneity of α follows easily from that of l and L. Thus requirements (F1)–(F3) are satisfied, so that $\alpha : \beta : \gamma$ is a center.

If $\mathscr{L}_1$ or $\mathscr{L}_2$ is a 1-line, then by Theorem 4A, X is an even center (but not necessarily a 0-center; to see that X can be a 2-center, note that $\mathbb{C}_{SF}\{(c-a)(a-b)\}$ lies on both of the 1-lines $\mathbb{L}_{SF}\{(b-c)(2a^2 - b^2 - c^2)\}$ and $\mathbb{L}_{SF}\{(b-c)(2bc - ca - ab)\}$.

If $\mathscr{L}_1$ and $\mathscr{L}_2$ are both even as described, then X has first coordinate $t(a, b, c)$ given by

$$t(a, b, c) = g(b, c, a)h(c, a, b) - g(c, a, b)h(b, c, a).$$

By Theorem 2, $t(a, b, c)$ has the form $(b-c)^{2m+1}z(a, b, c)$ for some z in $\mathbb{R}$ satisfying $z(a, c, b) = z(a, b, c)$. If (10) holds, then $c - a$ does not divide $t(a, b, c)$, so that X must be a 1-center. □

6. Central lines: parallels, perpendiculars, and regular lines

DEFINITION. An even line $\mathscr{L} = \mathbb{L}_{SF}\{t(a, b, c)\}$ is a *regular* line if the center $\mathbb{C}\{t(c, a, b)b - t(b, c, a)c\}$ is a 1-center; otherwise, $\mathscr{L}$ is an *irregular* line.

THEOREM 6. *Suppose $\mathscr{L}$ and $\mathscr{L}^*$ are parallel central lines. Write $\mathscr{L}$ as $\mathbb{L}_{SF}\{f(a, b, c)\}$. Then there exist g, u, v in $\mathbb{R}$ such that the following conditions hold:*

(*i*) $\mathscr{L}^* = \mathbb{L}\{g(a, b, c)\}$

(*ii*) $g(a, b, c) = u(a, b, c)f(a, b, c) + v(a, b, c)a$

(*iii*) $u(b, c, a) = u(a, b, c)$

(*iv*) $v(b, c, a) = v(a, b, c)$

Suppose $\mathscr{L} \neq \mathscr{L}^*$.

Case 1. $\mathscr{L}$ a 1-line. In this case, there are two possibilities: If $g(a, c, b) = -g(a, b, c)$, then $\mathscr{L}^*$ is a 1-line, and $u(a, c, b) = u(a, b, c)$ and

$v(a, c, b) = -v(a, b, c)$. On the other hand, if $g(a, c, b) = g(a, b, c)$, then $\mathscr{L}^*$ is an irregular line, and $u(a, c, b) = -u(a, b, c)$ and $v(a, c, b) = v(a, b, c)$.

In the remaining two cases, $u(a, b, c)$ and $v(a, b, c)$ are symmetric functions in a, b, c.

Case 2. $\mathscr{L}$ a regular line. In this case, $\mathscr{L}^*$ is a regular line.

Case 3. $\mathscr{L}$ an irregular line. In this case, $\mathscr{L}^*$ is a 1-line or an irregular line.

Proof. By (3), there exist $\hat{g}, \hat{u}, \hat{v}$ in $\mathbb{F}$ such that $\hat{g} = f\hat{u} + a\hat{v}$, $\mathscr{L}^* = \mathbb{L}\{\hat{g}(a, b, c)\}$, and $\hat{u}$ and $\hat{v}$ satisfy (iii) and (iv). Write $\hat{u}(a, b, c) = u_1(a, b, c)/u_2(a, b, c)$ and $\hat{v}(a, b, c) = v_1(a, b, c)/v_2(a, b, c)$ to obtain $\hat{g}u_2v_2 = fu_1v_2 + av_1u_2$. Put $g = \hat{g}u_2v_2$, $u = u_1v_2$, and $v = v_1u_2$. It is easy to see that (i)–(iv) hold. Since $g(a, b, c)$ is a center, either $g(a, c, b) = -g(a, b, c)$ or else $g(a, c, b) = g(a, b, c)$.

Case 1. $\mathscr{L}$ a 1-line. Here, $f(a, c, b) = -f(a, b, c)$. First, suppose $g(a, c, b) = -g(a, b, c)$, so that $\mathscr{L}^*$ is a 1-line. Then (ii) gives

$$g(a, c, b) = f(a, c, b)u(a, c, b) + v(a, c, b)a,$$

so that

$$-g(a, b, c) = -f(a, b, c)u(a, c, b) + v(a, c, b)a.$$

Adding this to equation (ii) gives

$$f(a, b, c)(u(a, b, c) - u(a, c, b)) + (v(a, b, c) + v(a, c, b))a = 0. \tag{13}$$

If $u(a, c, b) \neq u(a, b, c)$, then (13) together with (iii) and (iv) imply that $f(a, b, c): f(b, c, a) = a: b$, contrary to $\mathscr{L} \neq \mathscr{L}^\infty$. Therefore, $u(a, c, b) = u(a, b, c)$, and by (13), $v(a, c, b) = -v(a, b, c)$.

Next, suppose $g(a, c, b) = g(a, b, c)$. Then

$$f(a, c, b)u(a, c, b) + v(a, c, b)a = f(a, b, c)u(a, b, c) + v(a, b, c)a,$$

so that

$$f(a, b, c)(u(a, c, b) + u(a, b, c)) = (v(a, c, b) - v(a, b, c))a.$$

Similarly,

$$f(b, c, a)(u(b, a, c) + u(b, c, a)) = (v(b, a, c) - v(b, c, a))b.$$

Unless $u(a, c, b) = -u(a, b, c)$, these equations, by (iii) and (iv), imply $f(a, b, c)b = f(b, c, a)a$, contrary to the assumption that $\mathscr{L} \neq \mathscr{L}^{\infty}$. Therefore, $u(a, c, b) = u(a, b, c)$, from which follows $v(a, c, b) = -v(a, b, c)$.

Applying Theorem 2 to $u(a, b, c)$, we now have

$$g(a, b, c) = (b - c)^{m}(c - a)^{m}(a - b)^{m}\hat{u}(a, b, c)f(a, b, c) + v(a, b, c)a,$$

where m is an odd positive integer and $\hat{u} \in \mathbb{R}$, so that, by Theorem 2,

$$g(a, b, c) = (b - c)^{m+1}(c - a)^{m}(a - b)^{m}\hat{u}(a, b, c)\hat{f}(a, b, c) + v(a, b, c)a,$$

where $\hat{f}$ in $\mathbb{R}$ is not divisible by $c - a$. This equation yields

$$\begin{aligned} g(c, a, b)b - g(b, c, a)c = {} & b(a - b)^{m+1}(b - c)^{m}(c - a)^{m}\hat{u}(a, b, c)\hat{f}(c, a, b) \\ & - c(c - a)^{m+1}(a - b)^{m}(b - c)^{m}\hat{u}(a, b, c)\hat{f}(b, c, a) \end{aligned}$$

so that the center $\mathbb{C}\{g(c, a, b)b - g(b, c, a)c\}$ can be written as $\mathbb{C}\{w(a, b, c)\}$ where

$$w(a, b, c) = c(a - c)\hat{u}(a, b, c)\hat{f}(b, a, c) - b(b - a)\hat{u}(a, b, c)\hat{f}(c, b, a).$$

Clearly, $w(a, c, b) = w(a, b, c)$. Before we can conclude that $\mathbb{C}\{w(a, b, c)\}$ is not a 1-center, we must, and do, recognize that $c - a$ does not divide $w(a, b, c)$; otherwise, conceivably, $w(a, b, c)$ could be of the form $(b - c)^2(c - a)(a - b)j(a, b, c)$ where $j(a, c, b) = j(a, b, c)$, so that $\mathbb{C}\{w(a, b, c)\}$ would be a 1-center, after all. Thus, $\mathbb{C}\{w(a, b, c)\}$ is not a 1-center. That is, $\mathscr{L}^*$ is an irregular line.

To verify the symmetry of $u(a, b, c)$ and $v(a, b, c)$ for the remaining cases, observe that the equation $g(a, c, b) = g(a, b, c)$ for these cases implies $g(a, c, b) = u(a, c, b)f(a, b, c) + v(a, c, b)a$. This and (ii) give $(u(a, c, b) - u(a, b, c))f(a, b, c) + (v(a, c, b) - v(a, b, c))a = 0$, so that if $u(a, c, b) \neq u(a, b, c)$ then $\mathscr{L} = \mathscr{L}^{\infty}$, a contradiction. Therefore, $u(a, c, b) = u(a, b, c)$, so that also $v(a, c, b) = v(a, b, c)$. These equations and (iii) and (iv) imply that $u(a, b, c)$ and $v(a, b, c)$ are symmetric.

Case 2. $\mathscr{L}$ a regular line. If $\mathscr{L}^*$ is not also regular then it is a 1-line or an irregular line. If $\mathscr{L}^*$ is a 1-line, then by Case 1, $\mathscr{L}$, as an even line parallel to $\mathscr{L}^*$, is irregular. This contradiction shows that $\mathscr{L}^*$ is even. Now suppose $\mathscr{L}^*$ is irregular. Since $\mathbb{L}\{f(a, b, c)\}$ is a 1-line, every even line parallel to it, including $\mathscr{L}$, is, by Case 1, irregular. This contradiction shows that $\mathscr{L}^*$ is a regular line.

Case 3. $\mathscr{L}$ an irregular line. If $\mathscr{L}^*$ is regular, then by Case 2, so is $\mathscr{L}$. This contradiction shows that $\mathscr{L}$ is irregular. □

Concerning case 2 of Theorem 6, it is possible for $\mathscr{L}^*$ to be a 2-line even though $\mathscr{L}$ is a 0-line. For example, put $f(a, b, c) = 2a^2 + bc$, $u(a, b, c) = 1$, and $v(a, b, c) = -a - b - c$.

THEOREM 7. *If lines $\mathscr{L}_1$ and $\mathscr{L}_2$ are both perpendicular to a line other than $\mathscr{L}^\infty$, then $\mathscr{L}_1$ and $\mathscr{L}_2$ are parallel.*

Proof. Suppose $\mathscr{L}_1$ and $\mathscr{L}_2$ are both perpendicular to $\mathscr{L}$, where $\mathscr{L} \neq \mathscr{L}^\infty$. Let

$$l'\alpha + m'\beta + n'\gamma = 0, \qquad l''\alpha + m''\beta + n''\gamma = 0, \quad \text{and} \quad l\alpha + m\beta + n\gamma = 0,$$

be equations for $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}$ respectively. By (4)

$$l(2abcl' - n'\hat{b} - m'\hat{c}) + m(2abcm' - l'\hat{c} - n'\hat{a}) = -n(2abcn' - m'\hat{a} - l'\hat{b})$$

$$l(2abcl'' - n''\hat{b} - m''\hat{c}) + m(2abcm'' - l''\hat{c} - n''\hat{a}) = -n(2abcn'' - m''\hat{a} - l''\hat{b}).$$

The determinant of the coefficient matrix of this system is

$$\begin{aligned}
&(l'm'' - m'l'')(4a^2b^2c^2 - \hat{c}^2) + (n'l'' - n''l')(2abc\hat{a} + \hat{b}\hat{c}) \\
&\quad + (m'n'' - n'm'')(2abc\hat{b} + \hat{c}\hat{a}) \\
&= c(2T_2 - T_4)[(m'n'' - n'm'')a + (n'l'' - n''l')b + (l'm'' - m'l'')c],
\end{aligned}$$

where

$$T_2 = b^2c^2 + c^2a^2 + a^2b^2 \quad \text{and} \quad T_4 = a^4 + b^4 + c^4.$$

If this determinant is zero for all a, b, c then (3) holds, so that $\mathscr{L}_1$ and $\mathscr{L}_2$ are parallel. Indeed, this case must prevail, else Cramer's rule yields $l/n = a/c$ so that $l\colon m\colon n = a\colon b\colon c$, which is to say that $\mathscr{L} = \mathscr{L}^\infty$, contrary to the hypothesis. □

THEOREM 8. *Suppose $\mathscr{L} = \mathbb{L}\{t(a, b, c)\}$ is an even line. Then the following statements are equivalent:*

(*i*) *$\mathscr{L}$ is a regular line.*
(*ii*) *$c - a$ does not divide $t(c, a, b)b - t(b, c, a)c$.*
(*iii*) *$t(a, a, b)b \neq t(b, a, a)a$.*
(*iv*) *Every central line parallel to $\mathscr{L}$ is a regular line.*
(*v*) *No central line perpendicular to $\mathscr{L}$ is a regular line.*

Proof. Define $w(a, b, c) = t(c, a, b)b - t(b, c, a)c$. Then $\mathscr{L}$ is regular if and only if the point $X = \mathbb{C}\{w(a, b, c)\}$ is a 1-center. Since $t(a, c, b) = t(a, b, c)$, we have

$w(a, c, b) = -w(a, b, c)$, so that $w(a, b, c) = (b - c)w_1(a, b, c)$ for some w_1 in $\mathbb{R}$, by Theorem 2.

Part 1. (*i*) $\Rightarrow$ (*ii*). Suppose (i) holds and (ii) fails. Then $w(a, b, c) = (c - a)w_2(a, b, c)$ for some w_2 in $\mathbb{R}$, so that $w(b, c, a) = (a - b)w_2(b, c, a)$. Thus $c - a$ and $a - b$ both divide $w(a, b, c)$, since $w(a, b, c) = w(b, c, a)$. It follows that

$$w(a, b, c) = [(b - c)(c - a)(a - b)]^{2p+1}\hat{w}(a, b, c) \tag{14}$$

for some $\hat{w}$ in $\mathbb{R}$ and $p \geqslant 0$, where $(b - c)(c - a)(a - b)$ does not divide $\hat{w}(a, b, c)$. Thus $X = \mathbb{C}\{\hat{w}(a, b, c)\}$, and $\hat{w}(a, c, b) = \hat{w}(a, b, c)$. If $b - c$ does not divide $\hat{w}(a, b, c)$, then X is an even center, contrary to (i). Therefore, $b - c$ divides $\hat{w}(a, b, c)$. But, since $\hat{w}(a, c, b) = \hat{w}(a, b, c)$, we have $\hat{w}(a, b, c) = (b - c)^{2m}\hat{w}_1(a, b, c)$ for some $\hat{w}_1$ in $\mathbb{R}$ and $m \geqslant 1$, where $c - a$ does not divide $\hat{w}_1(a, b, c)$ (for if it did, then $a - b$ would also divide $\hat{w}_1(a, b, c)$, contrary to the maximality of p in (14)). Therefore X is an even center, contrary to (i). We conclude that (i) $\Rightarrow$ (ii).

Part 2. (*ii*) $\Rightarrow$ (*i*). The identity $w(a, b, c) = (b - c)w_1(a, b, c)$, already proved, shows that, if $c - a$ does not divide $w(a, b, c)$, then X is a 1-center.

Part 3. (*ii*) $\Leftrightarrow$ (*iii*). By the division algorithm in $\mathbb{R}$, there exist $q(a, b, c)$ and $r(b, c)$ in $\mathbb{R}$ such that $w(a, b, c) = (c - a)q(a, b, c) + r(b, c)$. If $c - a$ divides $w(a, b, c)$ then $r(b, c) = 0$, so that $w(a, b, a) = 0$. Therefore (iii) $\Rightarrow$ (ii). For the converse, suppose (ii) holds, so that $r(b, c) \neq 0$. The equation $w(a, b, a) = r(b, a)$ then implies (iii).

Part 4. (*i*) $\Leftrightarrow$ (*iv*). If (iv) holds, then (i) holds since $\mathscr{L}$ is parallel to $\mathscr{L}$. Suppose now that (i) holds, and let $\mathscr{L}^*$ be a line parallel to $\mathscr{L}$. Now X is the point of intersection of $\mathscr{L}$ with $\mathscr{L}^\infty$, so it is also the point of intersection of $\mathscr{L}^*$ with $\mathscr{L}^\infty$. That is, if $\mathscr{L}^* = \mathbb{L}_{SF}\{\hat{t}(a, b, c)\}$, then $\mathbb{C}\{\hat{t}(c, a, b)b - \hat{t}(b, c, a)c\} = X$, which is a 1-center.

Part 5. (*i*) $\Leftrightarrow$ (*v*). Let $\mathscr{L}^\perp = \mathbb{L}\{a\hat{a}(t(b, c, a)b - t(b, c, a)c)\}$. By (4), $\mathscr{L}^\perp$ is perpendicular to $\mathscr{L}$. Obviously, $\mathscr{L}^\perp$ is a 1-line. By Theorem 6, every central line parallel to $\mathscr{L}^\perp$ is a 1-line or an irregular line. By Theorem 7, therefore, every central line $\mathscr{L}^*$ perpendicular to $\mathscr{L}$ is a 1-line or an irregular line. For the converse, assume that every central line perpendicular to $\mathscr{L}$ is a 1-line. One of these is $\mathscr{L}^\perp$, so that $\mathbb{C}\{a\hat{a}(t(c, a, b)b - t(b, c, a)c)\}$ is a 1-center, and (i) follows. □

THEOREM 9. *If $\mathscr{L}$ and $\mathscr{L}^*$ are perpendicular central lines and $\mathscr{L}$ is a 1-line, then $\mathscr{L}^*$ is an even line.*

Proof. $\mathscr{L} = \mathbb{L}_{SF}\{(b - c)\hat{l}(a, b, c)\}$, and suppose, contrary to the proposed conclusion, that $\mathscr{L}^* = \mathbb{L}_{SF}\{(b - c)\hat{l}'(a, b, c)\}$ is a 1-line that is perpendicular to $\mathscr{L}$. In the notation of equation (4), we have here $l = (b - c)\hat{l}(a, b, c)$, $m = (c - a)\hat{m}(a, b, c)$,

$n = (a - b)\hat{n}(a, b, c)$, and similarly for l', m', n'. Substitute these into (4) and put $c = b$ to obtain

$$0 = 2ab^2[(a-b)^2(\hat{m}\hat{m}' + \hat{n}\hat{n}')] - (a-b)^2\hat{a}(\hat{m}\hat{n}' + \hat{m}'\hat{n})$$
$$= (a-b)^2[2ab^2(\hat{m}\hat{m}' + \hat{n}\hat{n}') - a(b^2 + c^2 - a^2)(\hat{m}\hat{n}' + \hat{m}'\hat{n})]$$
$$= a(a-b)^2[2b^2(\hat{m} - \hat{n})(\hat{m}' - \hat{n}') + a^2(\hat{m}\hat{n}' + \hat{m}'\hat{n})]. \qquad (15)$$

Here, of course, $\hat{m}$ means $\hat{m}(a, b, b)$, which is $\hat{l}(b, b, a)$, and $\hat{n}$ means $\hat{n}(a, b, b)$, which is $\hat{l}(b, a, b)$. Since $\hat{l}(b, a, b) = \hat{l}(b, b, a)$, we have $\hat{n} = \hat{m}$. Similarly, $\hat{n}' = \hat{m}'$. Equation (15) therefore implies that $2a^2\hat{m}\hat{m}' = 0$, so that $\hat{m}(a, b, b) = 0$ or $\hat{m}'(a, b, b) = 0$. The first of these is equivalent to $\hat{m}(a, b, c)$ being divisible by $b - c$, hence to $\hat{l}(b, c, a)$ being divisible by $b - c$, hence to $\hat{l}(a, b, c)$ being divisible by $a - b$. However, this contradicts the assumption (SF1) made about $\hat{l}(a, b, c)$ in connection with standard form (6) for the 1-line $\mathscr{L}$. The equation $\hat{m}'(a, b, b) = 0$ leads to the same contradiction relative to the 1-line $\mathscr{L}^*$. Therefore $\mathscr{L}^*$ must be an even line. □

Examples corresponding to Theorem 9 include the following twelve regular 0-lines $t(a, b, c)\alpha + t(b, c, a)\beta + t(c, a, b)\gamma = 0$ which are perpendicular to the Euler line. It is sufficient to give a choice of $t(a, b, c)$ for each line:

$$a,\ a^3,\ a(b^2 + c^2),\ a(a+b)(a+c),\ \cos A,\ \sin 3A,\ \sin(A + \pi/6),\ \sin(A - \pi/6),$$
$$a\cos^2 A, \cos 2A\ \cos(B - C),\ \sin 2A\ \cos(B - C),\ \cos A(2\tan A - \tan B - \tan C).$$

7. Irregular lines

Irregular lines appear in Theorems 6 and 7 and are of some interest on their own. In this section, we consider these lines more specifically. Recall that, by Theorem 8, an even line $\mathscr{L}$ is irregular if there exists a 1-line parallel to $\mathscr{L}$. We show first how to construct infinitely many irregular lines parallel to *any* given 1-line $\mathscr{L} = \mathbb{L}_{SF}\{(b - c)g(a, b, c)\}$. To begin, note that an equation for $\mathscr{L}$ is

$$f(a, b, c)\alpha + f(b, c, a)\beta + f(c, a, b)\gamma = 0,$$

where

$$f(a, b, c) = (b - c)^2(c - a)(a - b)g(a, b, c).$$

Expand this last product and assemble the terms into the form $\phi(a, b, c) + \theta(a, b, c)a$. Then express $\theta(a, b, c)$ as $u(a, b, c) + r(a, b, c)$, where $u(a, b, c)$ is symmetric, so that

$$f(a, b, c) = \phi(a, b, c) + r(a, b, c)a + u(a, b, c)a.$$

This can be done in infinitely many ways in which the function $t(a, b, c) = \phi(a, b, c) + r(a, b, c)a$ is not divisible by $b - c$, so that the line $\mathscr{L}^*$ given by $t(a, b, c)\alpha + t(b, c, a)\beta + t(c, a, b)\gamma = 0$ is an even line, hence an irregular line. One of the simplest examples obtained in the manner just described is

$$t(a, b, c) = -bc(b - c)^2 + a^2(b^2 + c^2) + a^3(b + c) - a^4$$

and

$$u(a, b, c) = (3) + 2(1^3) - (21).$$

Here, the 0-line $\mathbb{L}_{SF}\{t(a, b, c)\}$ is parallel to the 1-line $\mathbb{L}_{SF}\{b - c\}$.

We turn now to the problem of describing irregular lines $\mathbb{L}_{SF}\{t(a, b, c)\}$ in terms of the coefficients of $t(a, b, c)$.

THEOREM 10. *Let* $\mathscr{L} = \mathbb{L}_{SF}\{t(a, b, c)\}$ *be an even line, where* $t(a, b, c) = \sum d(i, j, k)a^i b^j c^k$ *is a polynomial in* a, b, c. *Then* $\mathscr{L}$ *is an irregular line if and only if*

$$\sum_{j=0}^{n+1-l} d(n + 1 - l - j, j, l - 1) = \sum_{j=0}^{n-l} d(l, j, n - l - j) \qquad \text{for } l = 0, 1, \ldots, n + 1, \tag{16}$$

where $d(i, j, k) = 0$ *if* i, j, *or* k *is negative.*

Proof. Let $w(a, b, c) = t(c, a, b)b - t(b, c, a)c$. Then

$$w(a, b, c) = \sum_{\mathscr{S}} d(i, j, k)a^j b^{k+1} c^i - \sum_{\mathscr{S}} d(i, j, k)a^k b^i c^{j+1},$$

where the set $\mathscr{S}$ consists of all (i, j, k) satisfying $i \geqslant 0, j \geqslant 0, k \geqslant 0, i + j + k = n$, the degree of homogeneity of $t(a, b, c)$. Then

$$\begin{aligned} w(a, b, a) &= \sum_{\mathscr{S}} d(i, j, k)a^{i+j}b^{k+1} - \sum_{\mathscr{S}} d(i, j, k)a^{j+k+1}b^i \\ &= \sum_{l=0}^{n+1}\left(\sum_{j=0}^{n+1-l} d(n + 1 - l - j, j, l - 1) - \sum_{j=0}^{n-l} d(l, j, n - l - j)\right)a^{n+1-l}b^l. \end{aligned} \tag{17}$$

Now $c - a$ (and hence also $a - b$) divides $w(a, b, c)$ if and only if $w(a, b, a) = 0$, which occurs if and only if all the coefficients in (17) are 0, as asserted in (16). □

Theorem 10 generalizes to all $t(a, b, c)$ in $\mathbb{R}$ as follows: if t is expressed (in accord with Theorem 2) as $t(a, b, c) = t_0(a, b, c) + \sqrt{\mathcal{P}}t_1(a, b, c)$, where $t_m(a, b, c) = \sum d_m(i, j, k)a^i b^j c^k$ for $m = 0, 1$, then $\mathbb{L}_{SF}\{t(a, b, c)\}$ is an irregular line if and only if

$$\sum_{j=0}^{n_m+1-l} d_m(n+1-l-j, j, l-1) = \sum_{j=0}^{n_m-l} d_m(l, j, n-l-j) \text{ for } l = 0, 1, \ldots, n_m+1$$

for $m = 0, 1$. Here n_0 is the degree of homogeneity of $t(a, b, c)$ and $n_1 = n_0 - 2$.

We shall restrict our attention to polynomials in a, b, c, however, as characterized by (16). Let n be the degree of homogeneity of the polynomial $t(a, b, c)$. If $n \leqslant 3$, one finds from (16) that there are no such irregular lines. For $n = 4$, with the coefficients $d(i, j, k)$, written more compactly as d_{ijk}, the polynomial $w(a, b, a)$ in the proof of Theorem 10 is

$$\begin{aligned}
&(-d_{004} - d_{013} - d_{022} - d_{031} - d_{040})a^5 \\
&\quad + (d_{400} + d_{310} + d_{220} + d_{130} + d_{040} - d_{103} - d_{112} - d_{121} - d_{130})a^4 b \\
&\quad + (d_{301} + d_{211} + d_{121} + d_{031} - d_{202} - d_{211} - d_{220})a^3 b^2 \\
&\quad + (d_{202} + d_{112} + d_{022} - d_{301} - d_{310})a^2 b^3 + (d_{103} + d_{013} - d_{400})ab^4 + d_{004}b^5.
\end{aligned}$$

The system of equations obtained by putting the six coefficients equal to 0 is easily solved, with a general solution

$$t(a, b, c) = a^4 - a^3(b + c) + bc(b - c)^2 + a[d(21) + e(1^3)], \tag{18}$$

where d and e are arbitrary real numbers and (21) and (1^3) are symmetric polynomials as in Table 1. Equation (18) represents a family of irregular lines. It is notable that they are all parallel to the line $\mathbb{L}_{SF}\{a^4 - a^3(b + c) + bc(b - c)^2\}$. By Theorem 8, there exists a line $\mathcal{L}^*$ perpendicular to the lines given by (17), such that $\mathcal{L}^*$ is an irregular line. It is easy to check that $\mathbb{L}\{a\hat{a}(t(b, c, a)b - t(b, c, a)c)\}$, which is $\mathbb{L}_{SF}\{a\hat{a}(b^2 + c^2 - a(b + c))\}$, is such a line.

For $t(a, b, c)$ a polynomial with $n = 5$, the general irregular line $\mathbb{L}_{SF}\{t(a, b, c)\}$ is given by

$$\begin{aligned} t(a, b, c) &= da^5 + ea^4(b + c) + (d + 3e - 3f)a^3(b^2 + c^2) + (3f - 3e - 2d)a^3bc \\ &\quad + fa^2(b^3 + c^3) + (d + 2e - f)a(b^3c + c^3b) + d(b^4c + c^4b) \\ &\quad - d(b^3c^2 + c^3b^2) + a[g(2^2) + h(4) + i(21^2)], \end{aligned}$$

where d, e, f, g, h, i are arbitrary real numbers.

8. Centers on the circumcircle

We begin with several definitions. The *circumcircle*, Γ, is the set of isogonal conjugates of points on $\mathscr{L}^\infty$. Since $\mathscr{L}^\infty = \mathbb{L}\{a\}$, an equation for Γ is $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$.

Suppose $X = \alpha_1 : \beta_1 : \gamma_1$ is a point on Γ. Following [1, Article 4729], the *line tangent to* Γ *at* X is given by $(b\gamma_1 + c\beta_1)\alpha + (c\alpha_1 + a\gamma_1)\beta + (a\beta_1 + b\alpha_1)\gamma = 0$.

Suppose $X = \alpha_1 : \beta_1 : \gamma_1$ and $U = \alpha_2 : \beta_2 : \gamma_2$ are points. The *reflection of X about U* is the point $\alpha_3 : \beta_3 : \gamma_3$ with coordinates given by

$$\begin{aligned} \alpha_3 &= a\alpha_1\alpha_2 + b(2\alpha_2\beta_1 - \alpha_1\beta_2) + c(2\alpha_2\gamma_1 - \alpha_1\gamma_2) \\ \beta_3 &= a(2\beta_2\alpha_1 - \beta_1\alpha_2) + b\beta_1\beta_2 + c(2\beta_2\gamma_1 - \beta_1\gamma_2) \\ \gamma_3 &= a(2\gamma_2\alpha_1 - \gamma_1\alpha_2) + b(2\gamma_2\beta_1 - \gamma_1\beta_2) + c\gamma_1\gamma_2. \end{aligned}$$

Note that if X and U are centers, then the reflection of X about U is also a center. Two centers X and Y on Γ are *antipodal centers* if one is the reflection of the other about the circumcenter.

THEOREM 11. *Suppose X and W are points on the circumcircle,* Γ. *Let* $\mathscr{L}$ *be the line of X and W if* $W \neq X$; *let* $\mathscr{L}$ *be the line tangent to* Γ *at X if* $W = X$. *Let* $\mathscr{L}^*$ *be the line through W perpendicular to* $\mathscr{L}$. *Then the reflection of X about the circumcenter lies on* $\mathscr{L}^*$. *(See Fig. 2.)*

Proof. Write $X = \alpha_1 : \beta_1 : \gamma_1$ and $W = \alpha_2 : \beta_2 : \gamma_2$. If $W \neq X$, then an equation for $\mathscr{L}$ is $l\alpha + m\beta + n\gamma = 0$, where $l = \beta_1\gamma_2 - \beta_2\gamma_1$, $m = \gamma_1\alpha_2 - \gamma_2\alpha_1$, $n = \alpha_1\beta_2 - \alpha_2\beta_1$; on the other hand, if $W = X$, then $\mathscr{L}$ is given by $l = \beta_1\hat{C} - \gamma_1\hat{B}$, $m = \gamma_1\hat{A} - \alpha_1\hat{C}$, $n = \alpha_1\hat{B} - \beta_1\hat{A}$, where $\hat{A}$ abbreviates $(b^2 + c^2 - a^2)/(2bc)$, and similarly for $\hat{B}$ and

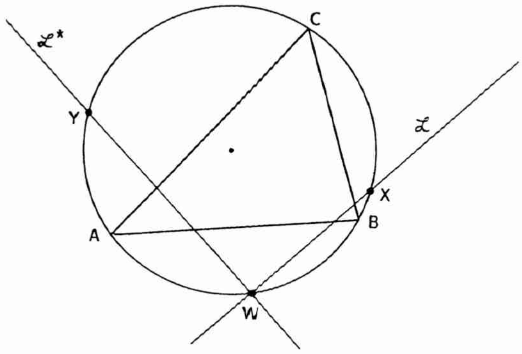

Figure 2. Antipodal centers X and Y.

$\hat{C}$. In both cases, the line $\mathscr{L}^*$ has equation $L\alpha + M\beta + N\gamma = 0$, where

$$L = \beta_2(n - l\hat{B} - m\hat{A}) - \gamma_2(m - n\hat{A} - l\hat{C})$$

$$M = \gamma_2(l - m\hat{C} - n\hat{B}) - \alpha_2(n - l\hat{B} - m\hat{A})$$

$$N = \alpha_2(m - n\hat{A} - l\hat{C}) - \beta_2(l - m\hat{C} - n\hat{B})$$

in accord with [1, Article 4625].

The circumcenter is $\hat{A}: \hat{B}: \hat{C}$, so that the reflection of X about the circumcenter is the point $\alpha_3: \beta_3: \gamma_3$ given by

$$\alpha_3 = (a\hat{A} - b\hat{B} - c\hat{C})\alpha_1 + 2b\hat{A}\beta_1 + 2c\hat{A}\gamma_1$$

$$\beta_3 = 2a\hat{B}\alpha_1 + (b\hat{B} - c\hat{C} - a\hat{A})\beta_1 + 2c\hat{B}\gamma_1$$

$$\alpha_3 = 2a\hat{C}\alpha_1 + 2b\hat{C}\beta_1 + (c\hat{C} - a\hat{A} - b\hat{B})\gamma_1.$$

It can be readily established by a computer algebra system that this point of reflection lies on $\mathscr{L}^*$. □

THEOREM 12. *Every center on the circumcircle* Γ *is an even center. If* X *is a 0-center on* Γ, *then its antipode is a 2-center.*

Proof. Suppose X is a center on Γ. To see that X must be even, assume to the contrary that X is a 1-center. Then its isogonal conjugate is a 2-center, for which we write $X^{-1} = \mathbb{C}_{SF}\{(c-a)(a-b)g(a,b,c)\}$. Since X^{-1} lies on $\mathscr{L}^{\infty}$, we have

$$(c-a)(a-b)g(a,b,c)a\alpha + (a-b)(b-c)g(b,c,a)b\beta + (b-c)(c-a)g(c,a,b)c\gamma = 0, \qquad \square$$

which, for $b = c$, implies $g(a,b,b) = 0$. However, this means that $b-c$ divides $g(a,b,c)$, a contradiction. Therefore, X is an even center.

Now suppose $X = \mathbb{L}_{SF}\{g(a,b,c)\}$ is a 0-center. Let $\mathscr{L}_X$ be the line of X and X^{-1}. Then

$$\mathscr{L}_X = \mathbb{L}\{g(a,b,c)(g(b,c,a) - g(c,a,b))((g(b,c,a) + g(c,a,b))\}.$$

By Theorem 2, $b-c$ divides $g(b,c,a) - g(c,a,b)$. So, to show that $\mathscr{L}_X$ is a 1-line, we must show that $c-a$ divides neither $g(b,c,a) - g(c,a,b)$ nor $g(b,c,a) + g(c,a,b)$. Suppose $c-a$ divides $g(b,c,a) - g(c,a,b)$. Then there exists k in $\mathbb{R}$ such that $g(b,c,a) - g(c,a,b) = (c-a)k(a,b,c)$. Put $c = a$ to get

$$g(b,a,a) = g(a,a,b). \tag{19}$$

Since X is on Γ, we also have $ag(b,c,a)g(c,a,b) + bg(c,a,b)g(a,b,c) + cg(a,b,c)g(b,c,a) = 0$. Put $c = a$ here to get $ag(b,a,a)g(a,a,b) + bg(a,a,b)g(a,b,a) + ag(a,b,a)g(b,a,a) = 0$, or equivalently, $g(a,a,b)(2ag(b,a,a) + bg(a,a,b)) = 0$. Together with (19), this implies $(2a+b)g^2(a,a,b) = 0$. But this means that $g(a,a,b) = 0$, so that $a-b$ divides $g(a,b,c)$, a contradiction. In the other case, i.e., if $c-a$ divides $g(b,c,a) + g(c,a,b)$, then in place of (19) we get $g(b,a,a) = -g(a,a,b)$, which leads to $(2a-b)g^2(a,a,b) = 0$, so that we reach the same contradiction. Therefore $\mathscr{L}_X$ is a 1-line.

Now, if $\mathscr{L}_X$ meets Γ in only one point W, let $W = X$; otherwise, let W be the point other than X at which $\mathscr{L}_X$ meets Γ. Let $\mathscr{L}_X^*$ be the line through W perpendicular to $\mathscr{L}_X$, so that $\mathscr{L}_X^*$ meets Γ at Y, the antipode of X, by Theorem 11. Since $\mathscr{L}_X$ is a 1-line, $\mathscr{L}_X^*$ is an even line, by Theorem 9. Write $\mathscr{L}_X^*$ as $\mathbb{L}\{t(a,b,c)\}$. Then $Y^{-1} = \mathbb{C}\{w(a,b,c)\}$, where $w(a,b,c) = bt(c,a,b) - ct(b,c,a)$. By Theorem 2, $b-c$ divides w. If $c-a$ divides w then, by Theorem 8, $\mathscr{L}_X^*$ is an irregular line, so that $\mathscr{L}_X$, being perpendicular to an irregular line, is an even line. This contradiction shows that $c-a$ does not divide w. Therefore Y^{-1} is a 1-center, so that Y is a 2-center. $\square$

Theorem 12 is exemplified by the Tarry point,

$$\mathbb{C}\left\{\frac{bc}{b^4 + c^4 - a^2b^2 - a^2c^2}\right\}$$

and the Steiner point, $\mathbb{C}\{bc(c^2-a^2)(a^2-b^2)\}$.

Other examples can be constructed in a manner suggested by the proof of Theorem 12. First, choose any 1-line; for purposes of illustration, we choose the Euler line. Determine its point of intersection with $\mathscr{L}^{\infty}$; for the Euler line, this point is $X^{-1} = \mathbb{C}_{SF}\{\cos A - 2\cos B \cos C\}$. Then create a line $\mathscr{L}^*$ perpendicular to that line; e.g., $\mathbb{L}_{SF}\{a^3\}$ is perpendicular to the Euler line. Find its point of intersection, Y^{-1}, with $\mathscr{L}^{\infty}$ (for the Euler case, $\mathbb{C}_{SF}\{bc^3 - cb^3\}$, so that Y is determined. To summarize the case at hand, the point having $\mathbb{C}\{(\cos A - 2\cos B \cos C)^{-1}\}$ is a 0-center on Γ, and its antipode, $\mathbb{C}\{a/(b^2 - c^2)\}$, is a 2-center.

9. Conclusion

In Sections 2–8, each (a, b, c) corresponds to an ordinary triangle. Here we wish to enlarge $\mathbb{T}$ to include triples that are not sidelengths of ordinary triangles. There is a price to be paid, namely the removal from $\mathbb{R}$ of all nonzero polynomial multiples of $\sqrt{\mathscr{P}}$. In the new setting, centers whose coordinates necessarily depend on $\sqrt{\mathscr{P}}$ are now left undefined. Specifically, let

$$(\mathscr{C}, +, \cdot) = \text{field of complex numbers}; \qquad \mathbb{T}' = \mathscr{C}^3;$$

$$(\mathbb{R}', +, \cdot) = \text{ring of polynomials over } \mathbb{T}.$$

The definitions, methods and results in Sections 2–8 readily extend to the new setting. For example, points are equivalence classes $f(a, b, c){:}f(b, c, a){:}f(c, a, b)$, where $f(a, b, c) \in \mathbb{R}'$. It is noteworthy that the many theorems about collinearity of points, concurrence of lines, parallelisms, and so on may now be viewed as theorems in functional equations. The familiar geometric facts are only special cases.

In particular, we note that Theorem 1, 7, 9, and 10 could have been proved by simply referring to well-known theorems in geometry, since these theorems hold for all triangles. However, the methods of proof used here show that these theorems and their consequences hold in the much more general setting in which a, b, c are arbitrary nonzero complex numbers.

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