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HOMOTOPY OF PROJECTIONS AND FACTORIZATION IN THE UNITARY GROUP OF CERTAIN C^*-ALGEBRAS

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Abstract.

We prove that every unitary element in the matrix algebra $M_n(\mathscr{A})$ and the multiplier algebra $M(\mathscr{A})$ can be factored as a product of a diagonal unitary and some symmetries, where $\mathscr{A}$ is a C^*-algebra such that $RR(\mathscr{A}) = 0$ and $\mathrm{tsr}(\mathscr{A}) = 1$. In addition, we discuss the relations among homotopy of projections, the topological stable rank $\mathrm{tsr}(\mathscr{A})$, and some factorizations of unitary elements in the unitization of $\mathscr{A} \otimes \mathscr{K}$.

Introduction.

Assume that $\mathscr{A}$ is a unital C^*-algebra, $U(\mathscr{A})$ is the unitary group of $\mathscr{A}$, and $\mathscr{P}(\mathscr{A})$ is the space of all non-trivial projections (i.e., $\neq 0, 1$) which is equipped with the relative norm topology and called the *Grassmann space of* $\mathscr{A}$. Despite it is known that $U(\mathscr{A})$ and $\mathscr{P}(\mathscr{A})$ carry a great deal of information about $\mathscr{A}$, their detailed structure, both topological and algebraic, remains mysterious.

The purpose of this article is, first, to consider factorizations of unitaries in a C^*-algebra $\mathscr{A}$ with $\mathrm{RR}(\mathscr{A}) = 0$ and $\mathrm{tsr}(\mathscr{A}) = 1$, in the multiplier algebra $M(\mathscr{A})$ of $\mathscr{A}$ (if $\mathscr{A}$ is non-unital), and in the matrix algebras $M_n(\mathscr{A})$ where $n \geqq 1$. It turns out that every unitary element in $M_n(\mathscr{A})$ or in $M(\mathscr{A})$ can be factored as a product of two types of *simplest* unitaries, non-trivial symmetries (self-adjoint unitaries not equal to $+1, -1$) and diagonal unitaries. Secondly, we give necessary and sufficient conditions for the surjectivity of the natural map $U_n(\mathscr{A})/U_n^0(\mathscr{A}) \xrightarrow{\phi_{n,n+1}} U_{n+1}(\mathscr{A})/U_{n+1}^0(\mathscr{A})$ in terms of homotopy of projections, and consider the same type of factorizations of unitaries in the unitization of $\mathscr{A} \otimes \mathscr{K}$, where $U_n(\mathscr{A})$ is the unitary group of $M_n(\mathscr{A})$ and $U_n^0(\mathscr{A})$ is the identity path component of $U_n(\mathscr{A})$. All these two goals are reached by considering some intimate relations between the homotopy of projections and factorizations of unitaries.

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The factorizations mentioned above provide a better understanding about the unitary groups of $M_n(\mathscr{A})$ and the unitization of $\mathscr{A} \otimes \mathscr{K}$ for certain C^*-algebras. In fact, every symmetry can be written as $2r - 1$ for some projection $r \in \mathscr{P}(\mathscr{A})$ and presumbably well understood, and diagonal unitaries in $M_n(\mathscr{A})$ are explicitly created by all unitaries in $U(\mathscr{A})$. However, as one knows, the matrix algebras $M_n(\mathscr{A})$ ($n \geqq 1$) are technically difficult to deal with in general due to the high complexity of the manipulations of matrices with non-commutative entries; in turn it is unknown how $U(\mathscr{A})$ is related to $U_n(\mathscr{A})$ if at all.

The text is arranged as follows. In the first section, we prove that two Murray-von Neumann equivalent projections in $\mathscr{A}$ can be conjugated by either one symmetry and a unitary close to the identity in norm or by two symmetries. Based on this result, in the second section we factor each unitary in $U_n(\mathscr{A})$ ($n > 2$) as at most $2\left\{\frac{\ln n}{\ln 2}\right\}$ symmetries and a diagonal unitary, where $\{r\}$ is the smallest integer greater or equal to the number r. Alternatively, every unitary in $U_n(\mathscr{A})$ can be approximated in norm by products of $\left\{\frac{\ln n}{\ln 2}\right\}$ symmetries and a diagonal unitary. Again using the result above, we consider similar factorizations in the unitary group of $M(\mathscr{A})$, for which we need the aids from G. A. Elliott's technique for matroid algebras and our recent result in [59]. It turns out that every unitary in $M(\mathscr{A})$ can be approximated in norm by a product of a symmetry and a diagonal unitary (with respect to a decomposition of the identity); and in case $\mathscr{A}$ is simple, any unitary of $M(\mathscr{A})$ can be factored as a product of n symmetries and a unitary of the form $(1 - p_n) \oplus (p_n u_0 p_n)$ for some projection $p_n \in M(\mathscr{A})$ such that $2^n[p_n] = [1]$ for any $n \geqq 2$.

In the last section, assuming that $\mathscr{A}$ is an arbitrary unital C^*-algebra, we discuss the relation between the homotopy of projections and factoring unitaries in $U_n(\mathscr{A})$ as a product of symmetries and a diagonal unitary. In particular, the natural map

$$U_n(\mathscr{A})/U_n^0(\mathscr{A}) \xrightarrow{\phi_{n,\infty}} K_1(\mathscr{A})$$

is surjective if for each $m > n$ the path component of the space of projections in $M_m(\mathscr{A})$ containing the projection $1 \otimes e_{11}$ is connected, where $1 \otimes e_{11}$ is the $m \times m$ matrix whose only nonzero entry 1 is at the $(1, 1)$-place. In case the topological stable rank $n := \mathrm{tsr}(\mathscr{A})$ of $\mathscr{A}$ is finite, every unitary in the unitization of $\mathscr{A} \otimes \mathscr{K}$, whose unitary group is denoted by $U_\infty(\mathscr{A})$, can be approximated in norm by some products of a number of symmetries and a unitary of $U_n(\mathscr{A})$ (embedded into $U_\infty(\mathscr{A})$), where $\mathscr{K}$ denotes the algebra of all compact operators

on a separable Hilbert space; consequently, $\mathscr{A} \otimes \mathscr{K}$ is generated by projections and unitaries in $U_m(\mathscr{A})$ for some $m > n$.

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1. Factorizations in $U(\mathscr{A})$.

1.1. *Matrix decomposition of a C*-algebra.*

Let $\{p_i\}$ be a set of mutually orthogonal nonzero projections of $\mathscr{A}$ such that

$$\sum_i \oplus p_i = 1;$$

here we assume that such projections exist in $\mathscr{A}$. The above sum is called a *decomposition of the identity with respect to* $\{p_i\}$. Suppose further that the above decomposition contains a finite number of terms, if $\mathscr{A}$ is unital, and it contains infinitely many terms converging to the identity of $M(\mathscr{A})$ in the strict topology, if $\mathscr{A}$ is non-unital. We will often use the following *-isomorphism between a C^*-algebra $\mathscr{A}$ (or $M(\mathscr{A})$) and a *matrix decomposition of* $\mathscr{A}$ (*or* $M(\mathscr{A})$, *respectively*) *with respect to the decomposition* $\sum_i p_i = 1$:

$$x \mapsto \begin{pmatrix} p_1xp_1 & p_1xp_2 & p_1xp_3 & \cdots \\ p_2xp_1 & p_2xp_2 & p_2xp_3 & \cdots \\ p_3xp_1 & p_3xp_2 & p_3xp_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Under this *-isomorphism, one can easily check that the multiplication, the addition, the scalar multiplication and the involution follow the standard rules of the matrix manipulations. The above matrix is said to be the *matrix form of* x *with respect to the decomposition* $\sum_i p_i = 1$.

1.2. *Notations.*

Let $\mathscr{A}$ be a unital C^*-algebra and $U(\mathscr{A})$ be equipped with the relative norm topology. Two unitaries in $U(\mathscr{A})$ are said to be homotopic if they are in the same path component of $U(\mathscr{A})$. Slightly abusing a standard notation, we use $\tilde{\mathscr{A}}$ to denote the unitization of $\mathscr{A}$ in case $\mathscr{A}$ is non-unital and to denote $\mathscr{A}$ itself in case $\mathscr{A}$ is unital.

As usual, the notation '$p \sim q$' denotes the Murray-von Neumann equivalence in $\mathscr{P}(\mathscr{A})$, and $[p]$ denotes the equivalence class containing p. Under the natural ordering induced by the equivalence, $[p] \leqq [q]$ iff p is equivalent to a subprojection of q.

Two projections $p, q \in \mathscr{P}(\mathscr{A})$ are said to be unitarily equivalent, if there exists

a unitary u of $\tilde{\mathscr{A}}$ such that $upu^* = q$; and p and q are said to be homotopic if p and q are in the same path component of $\mathscr{P}(\mathscr{A})$. In case $\mathrm{RR}(\mathscr{A}) = 0$ and $\mathrm{tsr}(\mathscr{A}) = 1$, two projections $p, q \in \mathscr{P}(\mathscr{A})$ are equivalent if and only if p and q are unitarily equivalent, again if and only if p and q are homotopic [48, 3.4]). The same conclusions hold also for purely infinite simple C^*-algebras ([21] and [51]).

A C^*-algebra is said to have the cancellation property, if $p_1 \sim p_2$ whenever four projections of $\mathscr{A}$ satisfy $r_1 \oplus p_1 \sim r_2 \oplus p_2$ and $r_1 \sim r_2$. Under the condition $\mathrm{RR}(\mathscr{A}) = 0$, $\mathscr{A}$ has cancellation if and only if $\mathrm{tsr}(\mathscr{A}) = 1$ ([3]). Specific examples of such C^*-algebras include the following C^*-algebras and the tensor products with the algebra $\mathscr{K}$: all AF algebras [20], all Bunce-Deddens algebras ([7], [6]), all finite factors (von Neumannalgebras) [13], many inductive limit algebras ([4], [23], [5], among others), and certain irrational rotation algebras ([41], [14]).

1.3. LEMMA. *Assume that* $\mathrm{RR}(\mathscr{A}) = 0$ *and* $\mathrm{tsr}(\mathscr{A}) = 1$. *Then the following three statements are equivalent:*

(i) *Two projections* $p, q \in \mathscr{P}(\mathscr{A})$ *are equivalent.*

(ii) *For each positive number* ε *there exists a symmetry* s *and a unitary* u_ε *in* $\tilde{\mathscr{A}}$ *such that* $\|u_\varepsilon - 1\| < \varepsilon$ *and* $su_\varepsilon pu_\varepsilon^* s = q$.

(iii) *There are two symmetries* s *and* s' *in* $\tilde{\mathscr{A}}$ *such that* $ss'ps's = q$.

PROOF. It is trivial that (iii) implies (1). In order to prove that (i) implies (ii) and (ii) implies (iii), we need to use the construction in [48, 2.1] which is based on the idea in [11] with further analyses. First, write p as a 2×2 matrix form with respect to the decomposition $q \oplus (1 - q) = 1$, that is,

$$p = \begin{pmatrix} qpq & qp(1-q) \\ (1-q)pq & (1-q)p(1-q) \end{pmatrix}.$$

For each positive number $\delta < 1 - \dfrac{1}{\sqrt{2}}$ there exists [48, 2.1] a projection p_0 with a matrix form

$$p_0 = \begin{pmatrix} a' & b \\ b^* & c' \end{pmatrix}$$

such that

$$\|p - p_0\| < \delta,$$

where a' and c' are positive elements of $\mathscr{A}$ with finite spectra in $[0, 1]$. If δ is small enough, it is well known that there exists a unitary $u_\varepsilon \in U(\mathscr{A})$ such that

$$\|u_\varepsilon - 1\| < \varepsilon \quad \text{and} \quad u_\varepsilon p u_\varepsilon^* = p_0.$$

Set $f_0 = \chi_{\{1\}}(a')$ and $e_0 = \chi_{\{1\}}(c')$ (it is possible that $f_0 = 0$ or $e_0 = 0$, or both are

zero). Of course f_0 is a projection in $q\mathscr{A}q$ and e_0 is a projection in $(1-q)\mathscr{A}(1-q)$. With respect to the decomposition $f_0 \oplus (q - f_0) = q$ one can write

$$a' = \begin{pmatrix} f_0 & 0 \\ 0 & a \end{pmatrix},$$

where $\sigma(a)\backslash\{0\} := \{t_i\}_{i=1}^m \subset (0,1)$. Similarly, with respect to $(1-q-e_0) \oplus e_0 = 1-q$ one can write

$$c' = \begin{pmatrix} c & 0 \\ 0 & e_0 \end{pmatrix}.$$

Then $\sigma(c)\backslash\{0\} = \{1-t_i\}_{i=1}^m \subset (0,1)$, since

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

is a subprojection of p_0, where one notices that $f_0 b = 0$ and $be_0 = 0$ due to the equalities $a - a^2 = bb^*$ and $c - c^2 = b^*b$. Thus, with respect to the decomposition

$$f_0 \oplus (q - f_0) \oplus (1 - q - e_0) \oplus e_0 = 1$$

one can write p_0 as the following 4×4 matrix form:

$$p_0 = \begin{pmatrix} f_0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b^* & c & 0 \\ 0 & 0 & 0 & e_0 \end{pmatrix};$$

where $a = (q - f_0)p_0(q - f_0)$, $b = (q - f_0)p_0(1 - q - e_0)$, and $c = (1 - q - e_0)$ $p_0(1 - q - e_0)$. By the construction in [48, 2.1], if one writes

$$c = \sum_{i=1}^{m} (1 - t_i) r_i$$

for some mutually orthogonal projections $\{r_i\}_{i=1}^m \subset (1 - q - e_0)\mathscr{A}(1 - q - e_0)$, then

$$a = v\left(\sum_{i=1}^{m} r_i - c\right)v^* = \sum_{i=1}^{m} t_i v r_i v^*$$

for some partial isometry $v \in \mathscr{A}$ such that $v^*v = \sum_{i=1}^m r_i := e$. Furthermore, the relation among b, c and v is given by

$$b = v(e - c)^{\ddagger}. \quad .$$

Set $vv^* = f$. Clearly, $ef = 0$, f is the range projection of a, and e is the range

projection of c. Obviously, $f_0 f = 0$ and $e_0 e = 0$. It is due to the cancellation property that $q - f_0 - f \neq 0$ as long as $e_0 \neq 0$.

Since $\sigma(a) \cup \sigma(c) \subset [0, 1)$, it is obvious that

$$\|a\| < 1, \quad \|f - a\| < 1,$$
$$\|c\| < 1, \quad \|e - c\| < 1.$$

It follows from the well known fact (we cannot locate a reference)

$$\|q_1 - q_2\| = \max\{\|q_1(1 - q_2)\|, \|q_2(1 - q_1)\|\}$$

that

$$\left\|\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} - f\right\| < 1, \quad \left\|\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} - e\right\| < 1.$$

Set

$$s_0 = \left\{e \oplus f - \left(\left(\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} - f\right)^2\right\}^{-\frac{1}{2}} \left\{\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} + f - (e \oplus f)\right\}.$$

Then s_0 is a symmetry of $(e \oplus f)\mathscr{A}(e \oplus f)$ such that

$$s_0 f s_0 = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}.$$

One can also similarly define a symmetry conjugating $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and e, but we do not need it here.

Let us show now that (i) implies (ii). If $q \sim p$, it follows from the cancellation property that $q - f_0 - f \sim e_0$, for $p \sim p_0$. Let v_0 be a partial isometry in $\mathscr{A}$ such that $v_0 v_0^* = e_0$ and $v_0^* v_0 = q - f_0 - f$. Since $e_0(q - f_0 - f) = 0$, then $s_1 := v_0 + v_0^*$ is a symmetry in the corner

$$(e_0 \oplus (q - f_0 - f))\mathscr{A}(e_0 \oplus (q - f_0 - f))$$

such that $s_1(q - f_0 - f)s_1 = e_0$. Set

$$s := s_0 \oplus s_1 \oplus f_0 \oplus (1 - q - e - e_0).$$

Then s is a symmetry of $\tilde{\mathscr{A}}$ such that $sp_0 s = q$. Therefore, $su_\varepsilon p u_\varepsilon^* s = q$, and hence (ii) holds.

For (ii) implying (iii), we need to find a symmetry $s' \in \tilde{\mathscr{A}}$ such that $s'ps' = u_\varepsilon p u_\varepsilon^*$. The existence of s' follows immediately from the estimate $\|p - u_\varepsilon p u_\varepsilon^*\| \leqq 2\|u_\varepsilon - 1\| < 1$ (if $\varepsilon < \frac{1}{2}$) and a well known fact that two close projections are equivalent by a symmetry (for example, see [61, Lemma 2]).

We now have the following factorization result in $U(\tilde{\mathscr{A}})$.

1.4. THEOREM. *Suppose that* $\mathrm{RR}(\mathscr{A}) = 0$, $\mathrm{tsr}(\mathscr{A}) = 1$, *$p$ is any nonzero projection of $\mathscr{A}$, u is an arbitrary unitary in $\tilde{\mathscr{A}}$, and ε is any positive number.*

(i) *There exist two symmetries s, s' and a unitary u_ε in $\tilde{\mathscr{A}}$ such that*

$$\|u_\varepsilon - 1\| < \varepsilon \quad \text{and} \quad u = u_\varepsilon s \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} = ss' \begin{pmatrix} u_1' & 0 \\ 0 & u_2' \end{pmatrix},$$

where u_1 and u_1' are unitaries of $p\mathscr{A}p$, u_2 and u_2' are unitaries of $(1-p)\tilde{\mathscr{A}}(1-p)$.

(ii) *If $q \in \mathscr{P}(\mathscr{A})$ with $p \sim q \leqq 1 - p$, then there exist two symmetries s, s' and a unitary u_ε of $\tilde{\mathscr{A}}$ such that $\|u_\varepsilon - 1\| < \varepsilon$ and*

$$u = u_\varepsilon s \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} = u_\varepsilon s \begin{pmatrix} 0 & b^* \\ b & cc^* \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & ba + c \end{pmatrix}$$

$$= ss' \begin{pmatrix} 0 & a' \\ b' & c' \end{pmatrix} = ss' \begin{pmatrix} 0 & b'^* \\ b' & c'c'^* \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & b'a' + c' \end{pmatrix},$$

where the matrices above are unitaries of $\tilde{\mathscr{A}}$ (with respect to the decomposition $p \oplus (1-p) = 1$), and

$$\begin{pmatrix} 0 & b^* \\ b & cc^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & b'^* \\ b' & c'c'^* \end{pmatrix}$$

are symmetries of $\tilde{\mathscr{A}}$.

PROOF. (i) Since upu^* is unitarily equivalent to p, by Lemma (1.3) there exist two symmetries s, s' and a unitary u_ε close to the identity (in norm) within ε such that

$$su_\varepsilon^* up = psu_\varepsilon^* u \quad \text{and} \quad s'sup = ps's\,u.$$

It follows that

$$u = u_\varepsilon s \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} = ss' \begin{pmatrix} u_1' & 0 \\ 0 & u_2' \end{pmatrix}.$$

(ii) If $p \sim q \leqq 1 - p$, then $upu^* \sim q$. It follows from Lemma (1.3) again that

$$su_\varepsilon^* upu^* u_\varepsilon s = s'supu^* ss' = q,$$

where s and s' are symmetries and u_ε is a unitary close to 1 in norm within ε. It is then easily checked by calculation (see [55, 2,1], if not clear) that

$$su_\varepsilon^* u = \begin{pmatrix} 0 & a \\ b & b \end{pmatrix} \quad \text{and} \quad s'su = \begin{pmatrix} 0 & a' \\ b' & c' \end{pmatrix},$$

while by matrix multiplication ([55, 2.1]) one sees

$$\begin{pmatrix} 0 & a \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & b^* \\ b & cc^* \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & ba + c \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} 0 & a' \\ b' & c' \end{pmatrix} = \begin{pmatrix} 0 & b'^* \\ b' & c'c'^* \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & b'a' + c' \end{pmatrix}.$$

Thus, we have completed the proof.

1.5. REMARKS.

(i) Some conclusions similar to Theorem 1.4 also hold in case $\mathscr{A}$ is a purely infinite, simple C^*-algebra; namely, there exist two symmetries s, s' of $\tilde{\mathscr{A}}$ such that

$$u = s's\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

where u_1 is a unitary of $(1-p)\tilde{\mathscr{A}}(1-p)$, and u_2 is a unitary of $p\mathscr{A}p$; alternatively, there exist symmetries s_1 and s_2 of $\tilde{\mathscr{A}}$ such that

$$u = s_2 s_1\begin{pmatrix} 0 & a \\ b & c \end{pmatrix} = s_2 s_1\begin{pmatrix} 0 & b^* \\ b & cc^* \end{pmatrix}\begin{pmatrix} 1-p & 0 \\ 0 & ba + c \end{pmatrix},$$

the above matrix decompositions are with respect to $p \oplus (1-p) = 1$. Since purely infinite simple C^*-algebras are not the target of this article, we leave a proof for the above claim elsewhere ([61, 1.5]).

(ii) We can write the unitary u in Theorem 1.4 as a product in the following form

$$u = (u_\varepsilon s u_\varepsilon^*)u_\varepsilon\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad u = (u_\varepsilon s u_\varepsilon^*)u_\varepsilon\begin{pmatrix} 0 & a \\ b & c \end{pmatrix},$$

where $u_\varepsilon s u_\varepsilon^*$ is also a symmetry. Hence, the order of s and u_ε in the above factorizations is up to one's choice. This observation will be freely used later without further comments.

2. Factorization of unitaries in $M_n(\mathscr{A})$.

Applying the factorizations theorem (1.4) in the last section, we will factor unitaries in $M_n(\tilde{\mathscr{A}})$ as a product of some symmetries and a diagonal unitary in this section. Using the elementary fact that each symmetry can be written as $2r - 1$ for some projections, one sees that the construction of $U_n(\mathscr{A})$ is completely determined by the Grassmann space of $M_n(\mathscr{A})$ and unitary of $\mathscr{A}$ itself.

2.1. *Further Notations.*

From now on, the identity of $M_n(\tilde{\mathscr{A}})$ is denoted by I_n which is often written as

$$\sum_{i=1}^{n} 1 \otimes e_{ii} = I_n,$$

where $\{1 \otimes e_{ij}\}$, denotes the set of matrices whose only nonzero entry 1 is at the (i,j)-place.

2.2. THEOREM. *Suppose that $\mathscr{A}$ is a C*-algebra such that $\mathrm{RR}(\mathscr{A}) = 0$ and $\mathrm{tsr}(\mathscr{A}) = 1$. If $p = p_1 \oplus p_2 \oplus \ldots \oplus p_n$ for some projections $\{p_i\} \subset \mathscr{P}(\mathscr{A})$ such that $[p_1] \leqq [p_2] \leqq \ldots \leqq [p_n]$ (i.e., $p_i \sim q_i \leqq p_{i+1}$ for $1 \leqq i \leqq n-1$) and u is a unitary of the corner $p\mathscr{A}p$, then there exist some symmetries $\{s_i\}_{i=1}^{m}$, $\{s'_j\}_{j=1}^{2m}$, $\{x_i\}_{i=1}^{2m}$ and $\{y_j\}_{j=1}^{3m}$ of $p\mathscr{A}p$ such that u can be approximated in norm by unitaries of $p\mathscr{A}p$ with both the following matrix forms*

$$s_1 s_2 \ldots s_m \begin{pmatrix} u_1 & 0 & \ldots & 0 \\ 0 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & u_n \end{pmatrix}, \quad s'_1 s'_2 \ldots s'_{2m} \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & p_{n-1} & 0 \\ 0 & \ldots & 0 & v \end{pmatrix}$$

and u can be exactly factored as the following products

$$u = x_1 x_2 \ldots x_{2m} \begin{pmatrix} u'_1 & 0 & \ldots & 0 \\ 0 & u'_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & u'_n \end{pmatrix} = y_1 y_2 \ldots y_{3m} \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & p_{n-1} & 0 \\ 0 & \ldots & 0 & v' \end{pmatrix},$$

where u_i and u'_i are some unitaries of the corner $p_i \mathscr{A} p_i$ $(1 \leqq i \leqq n)$, v and v' are some unitaries of $p_n \mathscr{A} p_n$, and m is the smallest integer such that $2^m \geqq n$.

As a consequence, if $\mathscr{A}$ is replaced by $M_n(\tilde{\mathscr{A}})$, p by I_n, and p_i by $1 \otimes e_{ii}$, then every unitary u of $M_n(\tilde{\mathscr{A}})$ can be approximated by products of some (m or $2m$) symmetries and a diagonal unitary (in either of the above two forms) and can be exactly factored as products of some ($2m$ or $3m$) symmetries and a diagonal unitary (in either of the above two forms).

PROOF. The proof is given by repeatedly using the factorizations in Theorem 1.4.

First, for any $\varepsilon > 0$ we carry out the following two factorizations

$$u = s_1 s_2 \ldots s_m u_\varepsilon \begin{pmatrix} u_1 & 0 & \ldots & 0 \\ 0 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & u_n \end{pmatrix} = x_1 x_2 \ldots x_{2m} \begin{pmatrix} u'_1 & 0 & \ldots & 0 \\ 0 & u'_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & u'_n \end{pmatrix},$$

where $\|u_\varepsilon - p\| < \varepsilon$.

Case 1. Assume that $n = 2^m$ for some integer m.

This is an elementary induction argument on m. Let $q = p_1 \oplus p_2 \oplus \ldots \oplus p_{2^{m-1}}$. Then $p - q = p_{2^{m-1}+1} \oplus \ldots \oplus p_n$. Since uqu^* and q are unitarily equivalent, by

Theorem 1.4. (i) there exists some symmetries s_1 $(= x_1)$, s_2, and a unitary v_1 in $p\mathscr{A}p$ such that $\|v_1 - p\| < \varepsilon$, and

$$s_1 v_1 u = \begin{pmatrix} u_1 & 0 \\ 0 & w_1 \end{pmatrix} \quad \text{and} \quad s_2 s_1 u = \begin{pmatrix} u_1' & 0 \\ 0 & w_1' \end{pmatrix},$$

where u_1, u_1' are unitaries of $q\mathscr{A}q$ and w_1, w_1' are unitaries of $(p-q)\mathscr{A}(p-q)$. Now the induction applies to the corners of both unitaries

$$\begin{pmatrix} u_1 & 0 \\ 0 & w_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_1' & 0 \\ 0 & w_1' \end{pmatrix}.$$

There are some symmetries $s_2', s_3', \ldots, s_m', x_3', x_4', \ldots, x_{2m}'$ and a unitary v_2' of $q\mathscr{A}q$ close to q in norm such that

$$u_1 = s_2' s_3' \ldots s_m' v_2' \begin{pmatrix} u_1 & 0 & \ldots & 0 \\ 0 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & u_{2^{m-1}} \end{pmatrix} \text{ and}$$

$$u_1' = x_3' x_4' \ldots x_{2m}' \begin{pmatrix} u_1' & 0 & \ldots & 0 \\ 0 & u_2' & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & u_{2^{m-1}}' \end{pmatrix};$$

and there are some symmetries $s_2'', s_3'', \ldots, s_m'', x_3'', x_4'', \ldots, x_{2m}''$ and a unitary v_2' of $(p-q)\mathscr{A}(p-q)$ close to $p-q$ in norm such that

$$w_1 = s_2'' s_3'' \ldots s_m'' v_2' \begin{pmatrix} u_{2^{m-1}+1} & 0 & \ldots & 0 \\ 0 & u_{2^{m-1}+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & u_{2^m} \end{pmatrix} \text{ and}$$

$$w_1' = x_1 x_2 \ldots x_{2m} \begin{pmatrix} u_{2^{m-1}+1}' & 0 & \ldots & 0 \\ 0 & u_{2^{m-1}+2}' & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & u_{2^m}' \end{pmatrix}.$$

For each $i \geqq 2$ set

$$s_i = \begin{pmatrix} s_i' & 0 \\ 0 & s_i'' \end{pmatrix}, \quad x_i = \begin{pmatrix} x_i' & 0 \\ 0 & x_i'' \end{pmatrix}, \quad \text{and} \quad v_2 = \begin{pmatrix} v_2' & 0 \\ 0 & v_2'' \end{pmatrix}.$$

Thus, for the case $n = 2^m$ the two factorizations follow clearly, where one needs to notice the observation in (ii) of (1.5).

Case 2. Assume that $n < 2^m$, where m is the least integer satisfying this inequality.

Applying the same arguments as in the case 1 to the projection $q = \oplus_{i=1}^{2^{m-1}} p_i$ and any unitary u of $p\mathscr{A}p$, we find two symmetries s_0, s_0' and a unitary v_0 in $p\mathscr{A}p$ such that

$$u = s_0 v_0 \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} = s_0 s_0' \begin{pmatrix} w_1' & 0 \\ 0 & w_2' \end{pmatrix},$$

where the matrix forms are with respect to the decomposition

$$(\oplus_{i=1}^{2^{m-1}} p_i) \oplus (\oplus_{i=2^{m-1}+1}^{n} p_i) = p.$$

Notice that $1 < n - 2^{m-1} < 2^{m-1}$. The two factorizations of u clearly follow from the same arguments as for the case 1 applied to the corners corners w_1, w_2, w_1', and w_2'.

Secondly, we carry out the factorization

$$u = s_1' s_2' \dots s_{2m}' u_\varepsilon' \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & p_{n-1} & 0 \\ 0 & \dots & 0 & v \end{pmatrix} = y_1 y_2 \dots y_{3m} \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & p_{n-1} & 0 \\ 0 & \dots & 0 & v' \end{pmatrix}.$$

These are done by repeatedly using the factorizations in (ii) of Theorem 1.4.

Case 1. $n = 2^m$ for some integer m.

Since $[p_1] \leqq [p_2] \leqq \dots \leqq [p_n]$, it is clear that

$$p_1 \oplus p_2 \oplus \dots \oplus p_{2^{p-1}} \sim q \leqq p_{m^{m-1}+1} \oplus \dots \oplus p_{2^m}.$$

It is obvious that $q \sim u(p_1 \oplus p_2 \oplus \dots \oplus p_{2^{m-1}})u^*$. It follows from Theorem 1.4 (ii) that there exist symmetries z_0, z_1, z_2 and a unitary w_1 in $p\mathscr{A}p$ such that $\|w_1 - p\| < \dfrac{\varepsilon}{m}$ and

$$z_2 z_1 w_1 u = \begin{pmatrix} \oplus_{i=1}^{2^{m-1}} p_i & 0 \\ 0 & v_1 \end{pmatrix} = z_2 z_1 z_0 \begin{pmatrix} \oplus_{i=1}^{2^{m-1}} p_i & 0 \\ 0 & v_1' \end{pmatrix},$$

where the matrices are written with respect to the decomposition

$$(\oplus_{i=1}^{2^{m-1}} p_i) \oplus (p - \oplus_{i=1}^{2^{m-1}} p_i) = p.$$

Repeating this process recursively m times, we reach the following factorizations:

$$z_{2m} z_{2m-1} w_m z_{2m-2} z_{2m-3} w_{m-1} \dots z_2 z_1 w_1 u = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & p_{n-1} & 0 \\ 0 & \dots & 0 & v \end{pmatrix} \quad \text{and}$$

$$y_{3m}y_{3m-1}y_{3m-2}\cdots y_3y_2y_1u = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & p_{n-1} & 0 \\ 0 & \dots & 0 & v' \end{pmatrix},$$

where $z_1, z_2, \dots, z_{2m}, y_1, y_2, \dots, y_{3m}$ are some symmetries in $p\mathscr{A}p$ and $w_1, w_2, \dots, w_m$ are some unitaries in $p\mathscr{A}p$ close to the identity p in norm within $\frac{\varepsilon}{m}$. Set

$$\begin{aligned} s_1' &= w_1^* z_1 w_1, \\ s_2' &= w_1^* z_2 w_1, \\ s_3' &= w_1^* w_2^* z_3 w_2 w_1, \\ s_4' &= w_1^* w_2^* z_4 w_2 w_1, \\ s_5' &= w_1^* w_2^* w_3^* z_5 w_3 w_2 w_1, \dots, \\ s_{2m}' &= w_1^* w_2^* \dots w_m^* z_{2m} w_m \dots w_2 w_1, \\ u_\varepsilon &= w_1^* w_2^* \dots w_m^*. \end{aligned}$$

It is easy to check that $\|u_\varepsilon - p\| < \sum_{i=1}^m \frac{\varepsilon}{m} = \varepsilon$, and $\{s_j'\}_{j=1}^{2m}$ with u_ε give the factorizations wanted.

Case 2. $n < 2^m$, where m is the smallest integer satisfying the inequality.

Set $q = \oplus_{i=1}^{n-2^{m-1}} p_i$. Then $q \sim q' \leqq p - q$, since $0 \leqq n - 2^{m-1} < 2^{m-1}$. One can apply the same arguments as in the first step of the case 1 to q. The remaining job is the same as in the case 1.

2.3. REMARK. Under the same assumptions as in Theorem 2.2, one can obtain some upper bounds for the exponential rank and exponential length of $M_n(\mathscr{A})$. For example, if p and $\{p_i\}$ are as in Theorem 2.2, then

$$\operatorname{cer}(p\mathscr{A}p) \leqq \left\{\frac{m}{2}\right\} + \operatorname{cer}(p_n\mathscr{A}p_n) + \varepsilon$$

$$\operatorname{cel}(p\mathscr{A}p) \leqq \frac{m}{2}\pi + \operatorname{cel}(p_n\mathscr{A}p_n),$$

where $\{t\}$ is the smallest integer greater or equal to a number t.

After the original manuscript was written, H. Lin proved a break-through result which asserted that

$$\operatorname{cer}(\mathscr{A}) < 1 + \varepsilon, \quad \text{and hence } \operatorname{cel}(\mathscr{A}) \leqq \pi$$

for any C^*-algebra with $\mathrm{RR}(\mathscr{A}) = 0$ ($\operatorname{tsr}(\mathscr{A}) = 1$ or not). Thus, the above esti-

mates becomes worthless. We can not delete the above inequalities in this revision only because they are cited in some appeared or to be appeared articles of others.

2.4. EXAMPLE. The factorizations as in Theorem (2.2) are not always possible in an arbitrary C^*-algebra. For example, look at the commutative C^*-algebra $C(S^3)$ of all complex-valued continuous functions on the 3-sphere S^3. The homeomorphism $S^3 \cong \mathrm{SU}(2)$ gives a unitary u in $M_2(C(S^3))$ which is not connected to the identity I_2, where SU(2) is all unitary scalar matrices with determinant 1. Since each diagonal unitary of $M_2(C(S^3))$ and all symmetries are in the identity path component of the unitary group of $M_2(C(S^3))$, u cannot be factored into a product of some symmetries and a diagonal unitary.

2.5. COROLLARY. *Suppose that $\mathscr{A}$ is a C^*-algebra such that* $\mathrm{RR}(\mathscr{A}) = 0$ *and* $\mathrm{tsr}(\mathscr{A}) = 1$, *and suppose that p is a full projection of $\mathscr{A}$ (*i.e., *the closed ideal generated by p is $\mathscr{A}$) and u is any unitary of $\mathscr{A}$.*

(i) *If $\mathscr{A}$ is unital, then there exist some symmetries $s_1, s_2, \ldots, s_n$ in $\tilde{\mathscr{A}}$ (for some $n \geqq 1$) such that*

$$u = s_1 s_2 \ldots s_n \begin{pmatrix} 1-p & 0 \\ 0 & u_0 \end{pmatrix},$$

where u_0 is a unitary in the hereditary C^*-subalgebra $p\mathscr{A}p$.

(ii) *If $\mathscr{A}$ is non-unital, then there exist some symmetries $s_1, s_2, \ldots, s_n$ (for some $n \geqq 1$) and a unitary u_ε in $\tilde{\mathscr{A}}$ such that*

$$\|u_\varepsilon - 1\| < \varepsilon \quad \textit{and} \quad u = s_1 s_2 \ldots s_n u_\varepsilon \begin{pmatrix} 1-p & 0 \\ 0 & u_0 \end{pmatrix},$$

where u_0 is a unitary in the hereditary C^*-subalgebra $p\mathscr{A}p$.

PROOF. (i) Assume that $\mathscr{A}$ is unital. Since p is a full projection, it was proved [52,1.1] that

$$1 = q_1 \oplus q_2 \oplus \ldots \oplus q_l \quad \text{and} \quad [q_1] \leqq [q_2] \leqq \ldots \leqq [q_l] \leqq [p].$$

We may assume that $q_l \leqq p$. Then the conclusion follows from Theorem (2.2).

(ii) Assume that $\mathscr{A}$ is not unital. Since $\mathrm{RR}(\mathscr{A}) = 0$, there is an approximate identity of $\mathscr{A}$ consisting of projections. Then we can choose a projection $q \in \mathscr{A}$ and a unitary $u_{1\varepsilon}$ in $\tilde{\mathscr{A}}$ such that

$$\|u_{1\varepsilon} - 1\| < \frac{\varepsilon}{2}, \quad q > p, \quad \text{and} \quad u = u_{1\varepsilon} \begin{pmatrix} 1-q & 0 \\ 0 & u_1 \end{pmatrix},$$

where u_1 is a unitary in $q\mathscr{A}q$ (see the proof of [50, 4.1] or [54,2.6], for example). Now the problem has been reduced to the unital C^*-algebra $q\mathscr{A}q$.

3. Factorizations in $U(M(\mathscr{A}))$.

We now consider the same type of factorizations in the unitary group of the multiplier algebra $M(\mathscr{A})$ in case $\mathscr{A}$ is non-unital. G. A. Elliott proved [22,2.1] that every unitary algebra of a matroid algebra can be approximated in norm by the product of two block diagonal unitaries (with respect to two different decompositions of the identity). The same technique also applies to a C^*-algebra with $\mathrm{RR}(\mathscr{A}) = 0$ and $\mathrm{tsr}(\mathscr{A}) = 1$.

In this section we provide two different types of factorizations in the unitary group of $M(\mathscr{A})$; one is Proposition 3.1 which is proved by the combination of Elliott's technique in [22,2.1] and Theorem 1.4; the other is Theorem 3.2 which is proved by the combination of the technique of [22], [59,1.1] and Theorem 1.4.

3.1. PROPOSITION. *Assume that ε is any number, and $\mathscr{A}$ is a non-unital, σ-unital C^*-algebra such that* $\mathrm{RR}(\mathscr{A}) = 0$ *and* $\mathrm{tsr}(\mathscr{A}) = 1$. *Then every unitary u of $M(\mathscr{A})$ can be factored as $u = u_\varepsilon s u_0$, where $\|u_\varepsilon - 1\| < \varepsilon$, s is a symmetry, u_0 is a diagonal unitary $\oplus_{i=1}^{\infty} u_i'$, where u_i' is a unitary of the corner $(e_{n_i} - e_{n_{i-1}})\mathscr{A}(e_{n_i} - e_{n_{i-1}})$ $(\forall i \geqq 1)$, and $\{e_n\}$ is an approximate identity of $\mathscr{A}$ consisting of a sequence of increasing projections.*

PROOF. Using the technique of G. A. Elliott in [22,2.1], one can choose an approximate identity of $\mathscr{A}$ consisting of an increasing sequence of projections, say $\{e_n\}$. For any positive number ε there exists a unitary v in $M(\mathscr{A})$ with the following properties:

$$\|v - u\| < \varepsilon/2 \quad \text{and} \quad v = \left(\sum_{i=1}^{\infty} \oplus u_i\right)\left(\sum_{i=1}^{\infty} \oplus v_i\right),$$

where

$$u_i \in U((e_{2i-1} - e_{2i-1})\mathscr{A}(e_{2i+1} - e_{2i-1})),$$

$$v_i \in U((e_{2i} - e_{2i-2})\mathscr{A}(e_{2i} - e_{2i-2}) \quad (i \geqq 1, e_0 = 0).$$

Here one may need to change the notations after choosing a subsequence of $\{e_n\}$. Applying Theorem 1.4 to each u_i, one can write

$$u_i = s_i w_i \begin{pmatrix} u_{1i} & 0 \\ 0 & u_{2i} \end{pmatrix} \quad (i \geqq 1),$$

where the matrix decomposition is with respect to

$$(e_{2i+1} - e_{2i}) \oplus (e_{2i} - e_{2i-1}) = e_{2i+1} - e_{2i-1},$$

s_i is symmetry, w_i is a unitary of the corner $U((e_{2i+1} - e_{2i-1})\mathscr{A}(e_{2i+1} - e_{2i-1}))$ such that

$$\|w_i - (e_{2i+1} - e_{2i-1})\| < \varepsilon/2 \quad (i \geqq 1).$$

Set $s' = \sum_{i=1}^{\infty} \oplus s_i$, and $w = \sum_{i=1}^{\infty} \oplus w_i$. Then $\|w - 1\| < \varepsilon/2$ and

$$\sum_{i=1}^{\infty} \oplus u_i = s'w \sum_{i=1}^{\infty} (u_{1i} \oplus u_{2i}).$$

Clearly, $\sum_{i=1}^{\infty}(u_{1i} \oplus u_{2i}) \sum_{i=1}^{\infty} \oplus v_i := u_0$ is a diagonal unitary with respect to the decomposition $\sum_{i=1}^{\infty}(e_{2i} - e_{2i-2}) = 1$. Thus

$$u = (uv^*)v = (uv^*)swu_0 = (uv^*w)(w^*s'w)u_0$$

has the desired form, since

$$\|uv^*w - 1\| = \|u - w^*v\| \leqq \|u - v\| + \|w^* - 1\| < \varepsilon.$$

Set $u_\varepsilon = uv^*w$ and $s = w^*s'w$ as wanted.

3.2. In case $\mathscr{A}$ is a σ-unital, non-unital, simple C^*-algebra of real rank zero, we proved [59,1.1] that every projection p in $M(\mathscr{A})$ not in $\mathscr{A}$ can be written as

$$p = p_1 \oplus p_2 \oplus \ldots \oplus p_{2^n},$$

where $n \geqq 1$ can be any integer and $\{p_i\}_{i=1}^{2^n}$ are mutually equivalent orthogonal projections of $M(\mathscr{A})$ (of course not in $\mathscr{A}$). In particular, the identity of $M(\mathscr{A})$ can be written as

$$1 = p_1 \oplus p_2 \oplus \ldots. \oplus p_{2^n}.$$

Guaranteed by [56,1.2], we can write each p_i into a direct sum of a sequence of projections of $\mathscr{A}$, say

$$p_i = f_{i1} \oplus f_{i2} \oplus \ldots. \oplus f_{ij} \oplus \ldots.$$

Furthermore, we can assume that

$$f_{1j} \sim f_{2j} \sim \ldots \sim f_{2^n j} \quad \text{for all } j \geqq 1.$$

Now we have the following factorization of unitaries in $M(\mathscr{A})$:

3.3. THEOREM. *Suppose that $\mathscr{A}$ is a σ-unital, non-unital, simple C^*-algebra with* $\mathrm{RR}(\mathscr{A}) = 0$ *and* $\mathrm{tsr}(\mathscr{A}) = 1$. *Write* $1 = p_1 \oplus p_2 \oplus \ldots \oplus p_{2^n}$ *for some* $n \geqq 2$, *where* $p_i \sim p_j$ $(\forall 1 \leqq i,j \leqq 2^n)$. *If u is any unitary in $M(\mathscr{A})$ and ε is any positive number, then there exist a unitary u_ε in $M(\mathscr{A})$ with $\|u_\varepsilon - 1\| < \varepsilon$ and $2n$ symmetries $\{s_k\}_{k=0}^{2n}$ in $M(\mathscr{A})$ such that*

$$u = u_\varepsilon s_0 s_1 s_2 \ldots. s_{2n} \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & p_{2^n-1} & 0 \\ 0 & \ldots & 0 & v \end{pmatrix},$$

where v is a unitary of the corner $p_{2^n}M(\mathscr{A})p_{2^n}$ whose matrix form with respect to the decomposition $p_{2^n} = f_{2^n1} \oplus f_{2^n2} \oplus \ldots \oplus f_{2^nj} \oplus \ldots$ is block-diagonal; i.e., $v = \sum_{i=1}^{\infty} \oplus v_i$, *each v_i is a unitary in the corner $(f_{2^nm_i} - f_{2^nm_{i-1}})\mathscr{A}(f_{2^nm_i} - f_{2^nm_{i-1}}))$ for some sequence $\{m_i\}$.*

PROOF. Set $e_m = \sum_{j=1}^{m}(f_{1j} \oplus f_{2j} \oplus \ldots \oplus f_{2^nj})$ for all $m \geqq 1$. Then $\{e_m\}$ is an increasing approximate identity of $\mathscr{A}$ consisting of projections. It follows from Proposition 3.1 that there exist a unitary w_0 with $\|w_0 - 1\| < \frac{\varepsilon}{2}$ and a symmetry s_0' such that

$$s_0' w_0 u = \sum_{i=1}^{\infty} \oplus u_i,$$

where u_i is a unitary of $(e_{m_i} - e_{m_{i-1}})\mathscr{A}(e_{m_i} - e_{m_{i-1}})$ for all $i \geqq 1$ ($e_{m_0} = 0$). Clearly,

$$e_{m_i} - e_{m_{i-1}} = \sum_{j=m_{i-1}+1}^{m_i} (f_{1j} \oplus f_{2j} \oplus \ldots \oplus f_{2^nj})$$

$$= \left(\sum_{j=m_{i-1}+1}^{m_i} f_{1j}\right) \oplus \left(\sum_{j=m_{i-1}+1}^{m_i} f_{2j}\right) \oplus \ldots \oplus \left(\sum_{j=m_{i-1}+1}^{m_i} f_{2^nj}\right), \quad \text{and}$$

$$\sum_{j=m_{i-1}+1}^{m_i} f_{1j} \sim \left(\sum_{j=m_{i-1}+1}^{m_i} f_{2j}\right) \sim \ldots \sim \sum_{j=m_{i-1}+1}^{m_i} f_{2^nj}.$$

It then follows from Theorem 2.2 that there are a unitary u_i' and $2n$ symmetries in $(e_{m_i} - e_{m_{i-1}})\mathscr{A}e_{m_i} - e_{m_{i-1}})$, say $s_{i1}, s_{i2}, \ldots, s_{i,2n}$ (notice that n is the smallest integer such that $m \leqq 2^n$), such that $\|u_i' - (e_{m_i} - e_{m_{i-1}})\| < \frac{\varepsilon}{2}$ and

$$u_i = u_i' s_{i1} s_{i2} \ldots s_{i,2n} \begin{pmatrix} \sum_{j=m_{i-1}+1}^{m_i} f_{1j} & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \sum_{j=m_{i-1}+1}^{m_i} f_{2^n-1,j} & 0 \\ 0 & \ldots & 0 & v_i \end{pmatrix},$$

where v_i is a unitary in $(\sum_{j=m_{i-1}+1}^{m_i} \oplus f_{2^nj})\mathscr{A}(\sum_{j=m_{i-1}+1}^{m_i} \oplus f_{2^nj})$.

Set $v = \sum_{i=1}^{\infty} \oplus v_i$. Then v_i is a unitary of the corner $\sum_{j=m_{i-1}+1}^{m_i}(f_{2^nj} - f_{2^n,j-1})\mathscr{A}\sum_{j=m_{i-1}+1}^{m_i}(f_{2^nj} - f_{2^n,j-1}))$ for each $i \geqq 1$. Thus, v is a block-diagonal unitary of $p_{2^n}\mathscr{A}p_{2^n}$. Set $s_j = \sum_{i=1}^{\infty} \oplus s_{ij}$ for $1 \leqq j \leqq 2n$ and $w_1 = \sum_{i=1}^{\infty} \oplus u_i'$. It is a routine to check that s_j's are symmetries in $M(\mathscr{A})$ and w_1 is a unitary of $M(\mathscr{A})$ with $\|w_1 - 1\| < \frac{\varepsilon}{2}$. Now one can easily check that

$$u = (w_0^* w_1)(w_1^* s_0' w_1) s_1 s_2 \dots s_{2n} \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & p_{2n-1} & 0 \\ 0 & \dots & 0 & v \end{pmatrix}.$$

Set $u_\varepsilon = w_0^* w_1$ and $s_0 = w_1^* s_0' w_1$, as desired.

4. tsr($\mathscr{A}$) and factorizations in $\mathscr{A} \otimes \mathscr{K}$.

4.1. *$U(\tilde{\mathscr{A}})$ and $\mathscr{P}(\mathscr{A})$.*

Let $\mathscr{A}$ be a C^*-algebra (may not be of real rank zero). A natural action of the unitary group $U(\tilde{\mathscr{A}})$ on the Grassmann space $\mathscr{P}(\mathscr{A})$ is defined by

$$(u, p) \mapsto upu^* \quad \forall u \in U(\tilde{\mathscr{A}}) \quad \text{and} \quad \forall p \in \mathscr{P}(\mathscr{A}).$$

If p is a fixed nontrivial projection, the orbit of p under this action, denoted by

$$\mathscr{G}_p := \{upu^* : u \in U(\tilde{\mathscr{A}})\},$$

is the union of those path components of $\mathscr{P}(\mathscr{A})$ consisting of projections unitarily equivalent to p. If $U(M(\mathscr{A}))$ is connected, then of course $\mathscr{G}_p$ is connected for any $p \in \mathscr{P}(\mathscr{A})$. On the other hand, $\mathscr{G}_p$ still can be connected even in case $U(M(\mathscr{A}))$ is not connected. For example, it was proved ([21, 2.4], [48, 3.4], [51]) that $\mathscr{G}_p$ consists of only one path component in cases either $\mathscr{A}$ is purely infinite and simple or $\mathscr{A}$ satisfies $\mathrm{RR}(\mathscr{A}) = 0$ and $\mathrm{tsr}(\mathscr{A}) = 1$ (not necessarily simple), no matter $U(M(\mathscr{A}))$ is connected or not. It is conceivable that $\mathscr{G}_p$ is connected for many other C^*-algebras. How the internal natures of a C^*-algebra relate to the connectedness of $\mathscr{G}_p$ is almost a total mystery. Our interest in this section is the connection between $\mathscr{G}_p$ is almost a total mystery. Our interest in this section is the connection between $\mathscr{G}_p$ and $U(\tilde{\mathscr{A}})$.

From now on, for any $n \geq 1$ the unitary group of $M_n(\tilde{\mathscr{A}})$ is denoted by $U_n(\mathscr{A})$ and the identity path component of $U_n(\mathscr{A})$ is denoted by $U_n^0(\mathscr{A})$. The unitary group of the unitization of $\mathscr{A} \otimes \mathscr{K}$ is denoted by $U_\infty(\mathscr{A})$.

If $m < n$, $U_m(\mathscr{A})$ is naturally embedded into $U_n(\mathscr{A})$ by

$$i_{m,n} : u_m \mapsto \begin{pmatrix} u_m & 0 \\ 0 & I_{n-m} \end{pmatrix}.$$

Of course, $U_m^0(\mathscr{A})$ is correspondingly embedded in $U_n^0(\mathscr{A})$. Clearly, $i_{m.n}$ induces a group homomorphism:

$$\phi_{m,n} : U_m(\mathscr{A})/U_m^0(\mathscr{A}) \to U_n(\mathscr{A})/U_n^0(\mathscr{A}).$$

However, the map $\phi_{m.n}$ is neither injective nor surjective in general ([3, 8.1.2]). Similarly, the natural embedding of $U_n(\mathscr{A})$ into $U_\infty(\mathscr{A})$ defined by

$$i_{n,\infty}: \mapsto \begin{pmatrix} u_n & 0 \\ 0 & 1 - I_n \end{pmatrix}$$

induces a group homomorphism

$$\phi_{n,\infty}: U_n(\mathscr{A})/U_n^0(\mathscr{A}) \to U_\infty(\mathscr{A})/U_\infty^0(\mathscr{A}) := K_1(\mathscr{A});$$

The homomorphism $\phi_{n,\infty}$ is again neither injective nor surjective in general. On the other hand, there are many interesting C^*-algebras such that $\phi_{n,\infty}$ is an isomorphism for an integer n. For example, M. A. Rieffel [40, 2.10] proved that if $\mathscr{A}$ is a unital C^*-algebra such that $\mathrm{tsr}(\mathscr{A}) < \infty$, then $\phi_{n,\infty}$ is an isomorphism whenever $n \geqq \mathrm{tsr}(\mathscr{A})$; J. Cuntz proved [17] that if $\mathscr{A}$ is a purely infinite, simple, unital C^*-algebra, then $\phi_{1,\infty}$ is an isomorphism (it holds also in case $\mathscr{A}$ is non-unital [50, §4]); but $\mathrm{tsr}(\mathscr{A}) = +\infty$. This example shows that *a finite topological stable rank* is not a necessary condition for the occurrence of an isomorphism $\phi_{n,\infty}$ for an integer n.

For any projection $p \in \mathscr{P}(\mathscr{A})$, we will consider the relation of the connectedness of $\mathscr{G}_p$ and the factorization in the unitary group $U(\mathscr{A})$, and we will then clarify when $\phi_{n,\infty}$ is surjective for an integer n.

4.2. LEMMA. *Assume that $\mathscr{A}$ is any C^*-algebra and $p \in \mathscr{P}(\mathscr{A})$). Then the following are equivalent:*

(i) *$\mathscr{G}_p$ is connected.*

(ii) *Every unitary u of $\tilde{\mathscr{A}}$ is homotopic (in $U(\tilde{\mathscr{A}})$) to a unitary with the form*

$$\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

where u_1 is in $U(p\mathscr{A}p)$ and u_2 is in $U((1-p)\tilde{\mathscr{A}}(1-p))$.

(iii) *Every unitary u of $\tilde{\mathscr{A}}$ can be written as*

$$u = s_1 s_2 \ldots s_l \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

where $\{s_i\}_{i=1}^l$ is a set of symmetries in $U(\tilde{\mathscr{A}})$, $u_1 \in U(p\mathscr{A}p)$ and $u_2 \in U((1-p)\tilde{\mathscr{A}}(1-p))$.

PROOF. If $\{u(t): t \in [0,1]\}$ is a path in $U(\tilde{\mathscr{A}})$, then $\{u(t)pu(t)^*: t \in [0,1]\}$ is a path in $\mathscr{P}(\mathscr{A})$. Hence, it is trivial that (ii) implies (i) and (iii) implies (i). Since each isometry is in the identity component of $U(\tilde{\mathscr{A}})$, then it is clear that (iii) implies (ii), We need only to show that (i) implies (iii).

By assumption upu^* is homotopic to p. One can choose a finite number of projections $p_1, p_2, \ldots, p_l$ such that $p_l = p$ and $\|upu^* - p_1\| < 1$, and $\|p_i - p_{i+1}\| < 1$ for $1 \leqq i \leqq l-1$. Set

$$s_1 = (1 - (upu^* - p_1)^2)^{-\frac{1}{2}}(upu^* + p_1 - 1), \quad \text{and}$$

$$s_{i+1} = (1 - (p_i - p_{i+1})^2)^{-\frac{1}{2}}(p_i + p_{i+1} - 1) \quad \forall 1 \leqq i \leqq l - 1.$$

Then $s_1, s_2, \ldots, s_l$ are symmetries such that $s_1 upu^* s_1 = p_1$ and $s_i p_i s_i = p_{i+1}$ for $1 \leqq i \leqq l - 1$. It follows that

$$s_l \ldots s_2 s_1 upu^* s_1 s_2 \ldots s_l = p,$$

and hence

$$u = s_1 s_2 \ldots s_l \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

where u_1 is a unitary of $p\mathscr{A}p$ and u_2 is a unitary of $(1 - p)\tilde{\mathscr{A}}(1 - p)$, as wanted.

4.3. THEOREM. *Assume that $\mathscr{A}$ is any C^*-algebra and $n > m \geqq n - m$. Let $p_m = \operatorname{diag}(\underbrace{1, 1, \ldots, 1}_{m\text{times}}, 0, \ldots, 0)$, an $n \times n$ diagonal matrix in $M_n(\tilde{\mathscr{A}})$. Then the following hold:*

(i) *The orbit $\mathscr{G}_{p_m}$ of p_m under the action of $U_n(\mathscr{A})$ is connected subset of $\mathscr{P}(M_n(\tilde{\mathscr{A}}))$ if and only if $\phi_{m,n}$: $U_m(\mathscr{A}) \to U_n(\mathscr{A})/U_n^0(\mathscr{A})$ is surjective.*

(ii) *If $\phi_{m,n}$ is surjective, then every unitary u_n of $U_n(\mathscr{A})$ can be written as*

$$u_n = s_1 s_2 \ldots s_l \begin{pmatrix} u'_m & 0 \\ 0 & I_{n-m} \end{pmatrix}$$

for some $l \geqq 1$ and symmetries $s_1, s_2, \ldots, s_l$ of $M_n(\tilde{\mathscr{A}})$, where u'_m is a unitary of $M_m(\tilde{\mathscr{A}})$.

REMARK. The condition $n - m \leqq m$ is not essential, because either $n - m \leqq m$ or $m \leqq n - m$ holds and that $\mathscr{G}_{p_m}$ is a connected subset of $\mathscr{P}(M_n(\tilde{\mathscr{A}}))$ iff $\mathscr{G}_{p_{n-m}}$ is.

PROOF. (i) Assume that $\phi_{m,n}$ is surjective. Then every unitary u_n of $U_n(\mathscr{A})$ is homotopic to a unitary v with the following form

$$\begin{pmatrix} u_m & 0 \\ 0 & I_{n-m} \end{pmatrix}.$$

It follows that $u_n p_m u_n^*$ is homotopic to $v p_m v^* = p_m$. Thus, the orbit of p_m is connected as a subset of $\mathscr{P}(M_n(\tilde{\mathscr{A}}))$.

Conversely, if the orbit $\mathscr{G}_{p_m}$ is a connected subset of $\mathscr{P}(M_n(\mathscr{A}))$, and if u_n is any unitary of $M_n(\mathscr{A})$, then $u_n p_m u_n^*$ is homotopic to p_m. It follows from Lemma (4.2) that u_n is homotopic to a unitary w with the form

$$\begin{pmatrix} u_m & 0 \\ 0 & u_{n-m} \end{pmatrix},$$

where u_m is a unitary of $M_m(\mathscr{A})$ and u_{n-m} is a unitary of $M_{n-m}(\mathscr{A})$. If $n - m \leqq m$, there is a partial isometry v of $M_n(\mathscr{A})$ such that

$$vv^* = I_{n-m} \quad \text{and} \quad v^*v = \sum_{i=1}^{n-m} 1 \otimes e_{ii}.$$

Then

$$\begin{pmatrix} u_m & 0 \\ 0 & u_{n-m} \end{pmatrix}$$

can be written as the product w_1w_2, where

$$w_1 := \begin{pmatrix} (v^*u_{n-m}^*v) & 0 & 0 \\ 0 & (\sum_{i=n-m+1}^{m} 1 \otimes e_{ii}) & 0 \\ 0 & 0 & u_{n-m} \end{pmatrix} \quad \text{and}$$

$$w_2 := \begin{pmatrix} (v^*u_{n-m}v \oplus \sum_{i=n-m+1}^{m} 1 \otimes e_{ii})u_m & 0 \\ 0 & I_{n-m} \end{pmatrix}.$$

On the other hand, w_1 can be written as a product of two symmetries as follows:

$$w_1 = \begin{pmatrix} 0 & 0 & v^* \\ 0 & \sum_{i=n-m+1}^{m} 1 \otimes e_{ii} & 0 \\ v & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & v^*u_{n-m} \\ 0 & \sum_{i=n-m+1}^{m} 1 \otimes e_{ii} & 0 \\ u_{n-m}^*v & 0 & 0 \end{pmatrix}.$$

Set $u'_m = (v^*u_{n-m}v \oplus \sum_{i=n-m+1}^{m} 1 \otimes e_{ii})u_m$. Then w_1w_2 is homotopic to a unitary with the form

$$\begin{pmatrix} u'_m & 0 \\ 0 & I_{n-m} \end{pmatrix},$$

which is the image of an element in $U_m(\mathscr{A})$ under $\phi_{m,n}$. We have proved the conclusion (i).

(ii) If $\phi_{m,n}$ is surjective, then $\mathscr{G}_{p_m}$ is a connected subset of $\mathscr{P}(M_n(\tilde{\mathscr{A}}))$. It follows from Lemma 4.2 that

$$u_n = s_1s_2 \ldots s_{1-2} \begin{pmatrix} v_m & 0 \\ 0 & u_{n-m} \end{pmatrix}$$

for a finite set of symmetries $\{s_j\}_{j=1}^{l-2}$ in $U_n(\mathscr{A})$. As in the last paragraph, we can rewrite

$$\begin{pmatrix} v_m & 0 \\ 0 & u_{n-m} \end{pmatrix}$$

as a product of two symmetries and a unitary of the form

$$\begin{pmatrix} u'_m & 0 \\ 0 & I_{n-m} \end{pmatrix}.$$

We then have the desired factorization.

The following corollary is immediate from Theorem 4.3 (just let $m = n - 1$):

4.4. COROLLARY. *Let $\mathscr{A}$ be any unital C^*-algebra and $n \geqq 2$. Then*

$$\phi_{n-1,n}: U_{n-1}(\mathscr{A})/U^0_{n-1}(\mathscr{A}) \to U_n(\mathscr{A})/U^0_n(\mathscr{A})$$

is surjective if and only if $\mathscr{G}_n(\mathscr{A})$ is a connected subset of $\mathscr{P}(M_n(\tilde{\mathscr{A}}))$; where $\mathscr{G}_n(\mathscr{A})$ is the orbit, under the action of $U_n(\mathscr{A})$, of the projection

$$e_n = \begin{pmatrix} 0 & \cdots & \cdots \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & 0 \\ 0 & \cdots & 1 \end{pmatrix}_{n \times n}.$$

4.5. COROLLARY. *Let $m \geqq 1$ be a fixed integer. Then the following implications hold:* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv):

(i) *$\mathscr{G}_n(\mathscr{A})$ is connected subset of $\mathscr{P}(M_n(\mathscr{A}))$ for all $n > m$.*

(ii) *For any $n > m$, every unitary $u_n \in U_n(\mathscr{A})$ can be written as a product of the following form*

$$u = s_1 s_2 \dots s_l \begin{pmatrix} u_m & 0 \\ 0 & I_{n-m} \end{pmatrix},$$

where $\{s_j\}_{j=1}^l$ is a set of symmetries in $U_n(\mathscr{A})$ and $u_m \in U_m(\mathscr{A})$.

(iii) *Every unitary u of $U_\infty(\mathscr{A})$ can be factored as the following form:*

$$u = u_\varepsilon s_1 s_2 \dots s_l \begin{pmatrix} u_m & 0 \\ 0 & \lambda(\sum_{i=m+1}^\infty 1 \otimes e_{ii}) \end{pmatrix},$$

where λ is a complex number with $|\lambda| = 1$, $\{s_j\}_{j=1}^l$ is a set of symmetries in $U_\infty(\mathscr{A})$ such that $s_j(1 - \sum_{i=1}^{n_0} 1 \otimes e_{ii}) = 1 - \sum_{i=1}^{n_0} 1 \otimes e_{ii}$ for some fixed n_0, u_ε is a unitary in $U_\infty(\mathscr{A})$ close to the identity in norm within any given positive number ε, and u_m is a unitary in $U_m(\mathscr{A})$.

(iv) *The map $\phi_{m,\infty}: U_m(\mathscr{A})/U^0_m(\mathscr{A}) \to K_1(\mathscr{A})$ is a surjective homomorphism (and hence $K_1(\mathscr{A})$ is a quotient group of $U_m(\mathscr{A})/U^0_m(\mathscr{A})$).*

PROOF. Assume that $\mathscr{G}_n(\mathscr{A})$ is a connected subset of $\mathscr{P}(M_n(\mathscr{A}))$. By Theorem 4.3 there are symmetries $s_1, s_2, \dots, s_{l_1}$ such that

$$u = s_1 s_2 \dots s_{l_1} \begin{pmatrix} u_{n-1} & 0 \\ 0 & 1 - I_{n-1} \end{pmatrix},$$

where u_{n-1} is a unitary of $U_{n-1}(\mathscr{A})$. Repeating this argument $n-m$ times recursively, one sees that (i) implies (ii).

Assume that (ii) holds. Let u_∞ be any unitary of $U_\infty(\mathscr{A})$. For any positive number ε, there exist an integer $n_0 \geqq 1$, a unitary $u_{n_0} \in U_{n_0}(\mathscr{A})$, and a complex number λ with $|\lambda| = 1$ such that

$$\begin{pmatrix} u_{n_0} & 0 \\ 0 & \lambda(\sum_{i=n_0+1}^{\infty} 1 \otimes e_{ii}) \end{pmatrix}$$

is close to u_∞ in norm within ε. Then one can write

$$u_\infty = u_\varepsilon \begin{pmatrix} u_{n_0} & 0 \\ 0 & \lambda(\sum_{i=n_0+1}^{\infty} 1 \otimes e_{ii}) \end{pmatrix},$$

where

$$u_\varepsilon = u_\infty \begin{pmatrix} u_{n_0}^* & 0 \\ 0 & \lambda(\sum_{i=n_0+1}^{\infty} 1 \otimes e_{ii}) \end{pmatrix}$$

such that $\|u_\varepsilon - 1\| < \varepsilon$. Applying (ii) to u_{n_0}, the conclusion (iii) follows.

It is obvious that (iii) implies (iv).

4.6. COROLLARY. *If* $\operatorname{tsr}(\mathscr{A}) < +\infty$ *(in particular, if* $\mathscr{A} = C(X)$ *for some compact Hausdorff space* X *whose covering dimension* $\dim(X) < +\infty$*), then for any* $n \geqq \operatorname{tsr}(\mathscr{A})$

(1) $\mathscr{G}_{n+1}(\mathscr{A})$ *is a connected subset of* $\mathscr{P}(M_n(\tilde{\mathscr{A}}))$;

(2) *every unitary* u_∞ *of* $U_\infty(\mathscr{A})$ *can be written as a product of the following form*

$$u_\infty = u_\varepsilon s_1 s_2 \dots s_l \begin{pmatrix} u_n & 0 \\ 0 & \lambda(\sum_{i=m+1}^{\infty} 1 \otimes e_{ii}) \end{pmatrix},$$

where $s_1, \dots, s_l$, u_n, *and* u_ε *are as explained in Corollary* 4.5 (iii); *and*

(3) $\tilde{\mathscr{A}} \otimes \mathscr{K}$ *is generated (as a* C^**-algebra) by unitaries in* $U_n(\mathscr{A})$ *and projections.*

PROOF. If $\operatorname{tsr}(\mathscr{A}) < +\infty$, then $\phi_{n,n+1}$ is surjective for any $n \leqq \operatorname{tsr}(\mathscr{A})$ [40, 2.10]). Hence, the conclusions (1) and (2) follows immediately from Corollary 4.4 and Corollary 4.5. Since each symmetry is of the form $2r - 1$ for some projections in $\tilde{\mathscr{A}} \otimes \mathscr{K}$, the conclusion (3) follows from the factorization in (2).

4.7. COROLLARY. *If* $\mathscr{G}_m(\mathscr{A})$ *is connected for all* $m = 2, 3, \dots, n$, *then every unitary* u *of* $M_n(\mathscr{A})$ *can be written as*

$$u = s_1 s_2 \dots s_l \begin{pmatrix} u_1 & 0 & \dots & 0 \\ 0 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & u_n \end{pmatrix} = s_1' s_2' \dots s_k' \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \dots & 0 & v \end{pmatrix},$$

where $\{s_j, s_i'\}$ *is a finite set of symmetries of* $M_n(\tilde{\mathscr{A}})$ *and* $\{v, u_i\} \subset U(\mathscr{A})$. *Consequently,* $U_1(\mathscr{A})/U^0(\mathscr{A}) \xrightarrow{\phi_{1,\infty}} K_1(\mathscr{A})$ *is surjective, and* $M_n(\mathscr{A})$ *is generated (algebraically) by all diagonal unitaries and projections in* $M_n(\tilde{\mathscr{A}})$.

4.8. REMARK. For any $n \geqq 2$, by Corollary 4.5, one sees that there is no new path component created passing from $U_n(\mathscr{A})$ to a larger group $U_{n+1}(\mathscr{A})$ (under the natural embedding $i_{n,n+1}$) iff $\mathscr{G}_{n+1}(\mathscr{A})$ is connected.

4.9 EXAMPLES. The following two examples indicate that $\mathscr{G}_n(\mathscr{A})$ may not be connected in general as a subset of $\mathscr{P}(M_n(\tilde{\mathscr{A}}))$.

(1) Take $\mathscr{A} = C(S^3)$ and $n = 2$. Let u be the unitary in $M_2(C(S^3))$ given by the homeomorphism $S^3 \cong \mathrm{SU}(2)$. As pointed out in the second section, u is not homotopic to a diagonal unitary. This implies that ue_2u^* is not homotopic to e_2 in the Grassmann space of $M_2(C(S^3))$. Thus, the unitary orbit $\mathscr{G}_2(\mathscr{A})$ of e_2 in $M_2(C(S^3))$ is not connected. On the other hand, if $n \geqq 3$, then $\mathscr{G}_n(\mathscr{A})$ is connected; this follows from Corollary 4.6(1) and the fact that $\mathrm{tsr}(C(S^3)) = 2$ which is equal to the integer part of $\dfrac{\dim(X)}{2} + 1$ ([39]). In view of Remark 4.8, for any $n \geqq 3$ there is no new path component appearing in $U_n(C(S^3))$ different from the path components in $U_2(C(S^3))$, while, as a well known fact, new path component does arise when $U(C(S^3))$ passes to $U_2(C(S^2))$.

(2) Take $\mathscr{A} = C(S^2)$ and again $n = 2$. Let $p(.)$ be a projection in $M_2(C(S^2))$ defined by the homeomorphism $S^2 \cong G_{1,2}$, where $G_{1,2}$ is the Grassmann space of all one-dimensional projections in $M_2(\mathbf{C})$. Explicitly, $p(.)$ is the line bundle

$$p(z,\theta) = \begin{pmatrix} 1+z & \theta(1-z^2)^{\frac{1}{2}} \\ \bar{\theta}(1-z^2)^{\frac{1}{2}} & 1-z \end{pmatrix},$$

where (z, θ) is the cylindrical coordinates with $-1 \leqq z \leqq 1$ and $\theta \in S^1$. It is well known thatthe Stiefel manifold $V_{1,2}$ of all unit vectors in $\mathbf{C}^2$ is homeomorphic to S^3, and the Hopf map $S^3 \to S^2$ does not admit a cross-section. It follows that there is no unitary in $M_2(C(S^2))$ such that $up(.)u^*$ is a diagonal projection ([26]). It follows that $p(.)$ is not homotopic to e_2. Therefore, the unitary orbit of e_2 in $M_2(C(S^2))$ is not connected. Due to the same reason as in Example 1, if $n \geqq 3$, then $\mathscr{G}_n(\mathscr{A})$ is connected (because $\mathrm{tsr}(C(S^2)) = 2$, again); and hence there is no new path component appearing in $U_n(C(S^2))$ for $n \geqq 3$.

It follows from Corollary 4.6 that every unitary in the unitizations of both $C(S^2) \otimes \mathscr{K}$ and $C(S^3) \otimes \mathscr{K}$ and $C(S^3) \otimes \mathscr{K}$ can be approximated by a product of some symmetries and a unitary of the form $u_2 \oplus (1 - I_2)$, where u_2 is a unitary of $M_2(C(S^2))$ or $M_2(C(S^3))$, respectively.

4.10. PROPOSITION. *Assume that* $\mathscr{A}$ *is any* C^**-algebra. Under the standard*

embedding of $M_n(\mathscr{A})$ into $M_{n+1}(\tilde{\mathscr{A}})$, $\mathscr{G}_n(\mathscr{A})$ is identified with a subset of $\mathscr{G}_{n+1}(\mathscr{A})$. Then $\mathscr{G}_n(\mathscr{A})$ is contained in a connected component of $\mathscr{G}_{n+1}(\mathscr{A})$.

PROOF. Obviously, the projection e_n, regarded as an element in $M_{n+1}(\tilde{\mathscr{A}})$ under the embedding, is homotopic to e_{n+1}. In turn, in $\mathscr{G}_{n+1}(\mathscr{A})$, e_{n+1} is clearly equivalent to, and hence homotopic to, any projection in $\mathscr{G}_n(\mathscr{A})$.

Before concluding this section, we would like to point out the following fact.

4.11. PROPOSITION. *Suppose that $\mathscr{A}$ is a C*-algebra with property that unitarily equivalent projections are homotopic, and p is a nontrivial projection of $\mathscr{A}$. Then there are symmetries $s_1, s_2, \ldots, s_l$ of $\tilde{\mathscr{A}}$ such that*

$$u = s_1 s_2 \ldots s_l \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

where $u_1 \in U(p\mathscr{A}p)$ and $u_2 \in U((1-p)\tilde{\mathscr{A}}(1-p))$.

PROOF. Use exactly the same proofs in this section and the section 2.

4.12. REMARKS. The following three facts (1)–(3) are well known, where all subsets of $\mathscr{A}$ involved are equipped with the relative norm topology.

(1) The group of invertible elements of a unital *C*-algebra* $\mathscr{A}$, denoted by $\mathrm{GL}(\mathscr{A})$, is homotopy equivalent to $U(\mathscr{A})$, since $U(\mathscr{A})$ is a homotopy retract of $\mathrm{GL}(\mathscr{A})$.

(2) The space of all nontrivial ($\neq 0, 1$) indempotents of $\mathscr{A}$ is homotopy equivalent to $\mathscr{P}(\mathscr{A})$.

(3) The space of symmetries of $\mathscr{A}$ is homotopy equivalent to the subspace

$$\{x \in \mathrm{GL}(\mathscr{A}) : x^2 = 1\}.$$

Clearly, under the actions of $\mathrm{GL}(\mathscr{A})$ and $U(\mathscr{A})$ on $\mathscr{P}(\mathscr{A})$, the orbit of $\mathrm{GL}(\mathscr{A})$ is the space of all nontrivial idempotents, while the orbit of $U(\mathscr{A})$ is $\mathscr{P}(\mathscr{A})$. *Thus, one can rephrase all results in the first section and the second section in terms of* $\mathrm{GL}(\mathscr{A})$ *and the space of non-trivial idempotents.* This is left to the interested reader.

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