MATH. SCAND. 38 (1976), 265–284

GAUSSIAN RADON MEASURES ON LOCALLY CONVEX SPACES

CHRISTER BORELL[1]

1. Introduction.

Throughout this paper E denotes a *locally convex Hausdorff vector space over the field of real numbers* (l.c.s.). A Radon probability measure μ on E is said to be a (centred) Gaussian Radon measure on E if the image measure $\xi(\mu)$ is a (centred) Gaussian measure on R for every ξ belonging to the topological dual E' of E. The class of all (centred) Gaussian Radon measures on E is denoted by $\mathscr{G}(E)$ ($\mathscr{G}_0(E)$).

In Section 2, it will be proved that every $\mu \in \mathscr{G}(E)$ has a *reproducing kernel Hilbert space* (RKHS) $\mathscr{H}(\mu)$ contained in E. One of the main results of this paper shows that $\mathscr{H}(\mu)$ is separable (Theorem 7.1). This conclusion has many corollaries. For example, it follows that $L_p(\mu)$ $(1 \leqq p < +\infty)$ and $\operatorname{supp}(\mu)$ are separable (Corollaries 8.1 and 8.2).

Theorem 2.1 shows that every $\mu \in \mathscr{G}(E)$ has a barycentre $b \in E$. Setting $\mu_0(\cdot)=\mu(\cdot+b)$, it follows that $\mathscr{H}(\mu)$ and the closure $E_2'(\mu)$ of E' in $L_2(\mu_0)$ are isomorphic (Theorem 2.1). This makes it possible to give a very simple representation of μ_0-measurable additive functions. A real-valued function f on E is said to be a μ-measurable additive (subadditive) function on E, if $f(\pm\,\cdot)$ are μ-measurable, and there exists an additive μ-measurable subgroup G of E with μ-measure one so that

$$f(x+y) = (\leqq)\, f(x)+f(y), \quad x,y \in G\,.$$

Since $E_2'(\mu)$ is separable, there exists an at most denumerable orthonormal basis $\{e_n\}$ for this Hilbert space so that every e_n belongs to E'. If f is a μ_0-measurable additive function on E, it will be proved that there exists an $(a_n) \in l^2(\mathsf{N})$ such that

$$f(x) = \sum a_n e_n(x), \quad \mu_0\text{-almost all } x\,,$$

(Theorem 8.1). The class of all μ-measurable subadditive functions is much more complicated and a simple representation of this class is unknown to us. In any case, we discuss some of its properties in Sections 5 and 9.

---

[1] Partly supported by Statens naturvidenskabelige Forskningsråd, Danmark.
Received July 6, 1975.

As said above, the topological support of a Gaussian Radon measure μ is separable. In particular, μ sits on a separable Borel subspace of probability one. It can be conjectured that it sits on a Souslin subspace of probability one. This would be an extremely important result. It can be said that some results of this paper are trivial if E is a Souslin space. In Section 11, we will point out that a Gaussian Radon measure will not, in general, sit on an ultrabornological Borel subspace of probability one.

Many results of this paper have, clearly, earlier been obtained on special l.c.s.'s or in terms of Gaussian stochastic processes. Again it can be said that some results are trivial on locally convex Souslin spaces. On the other hand, the proper setting for Gauss measures on l.c.s.'s is Gaussian Radon measures on arbitrary l.c.s.'s. This paper is thus devoted to a study of this class.

A large part of this paper was written during my visit at Matematisk Institut, Aarhus Universitet. I am indepted to S. E. Graversen, J. Hoffmann-Jørgensen, and W. Słowikowski for many valuable discussions on the subject of this paper.

2. Every $\mu \in \mathscr{G}(E)$ has an RKHS.

To start with, we introduce some notation and conventions.

If $\mathscr{H}$ is a Hilbert space, the canonical cylinder Gauss measure on $\mathscr{H}$ is denoted by $\gamma_{\mathscr{H}}$, that is, the Fourier transform of $\gamma_{\mathscr{H}}$ equals $\exp(-\|\cdot\|^2/2)$, where $\|\cdot\|$ denotes the Hilbert norm in $\mathscr{H}$.

If μ is a positive measure on E, elements of $L_p(\mu)$ $(1 \leqq p < +\infty)$ are regarded as functions defined everywhere on E, and two such elements are identified if they coincide a.s. $[\mu]$.

We shall prove

THEOREM 2.1. *Suppose $\mu \in \mathscr{G}(E)$. Then*

a) *μ has a barycentre $b \in E$,*
b) *every measure $\xi\mu_0$, $\xi \in E_2'(\mu)$, has a barycentre $\Lambda\xi \in E$.*

The map $\Lambda: E_2'(\mu) \to E$ is linear and injective. We define,

$$\mathscr{H}(\mu) = \text{range}(\Lambda)\,,$$

$$\tilde{h} = \Lambda^{-1}h, \quad h \in \mathscr{H}(\mu)\,,$$

and

$$\|h\|^2 = \mu_0(\tilde{h}^2), \quad h \in \mathscr{H}(\mu)\,,$$

respectively. Then,

c) $(\mathscr{H}(\mu), \|\cdot\|)$ *is a Hilbert space and the canonical injection* θ *of* $(\mathscr{H}(\mu), \|\cdot\|)$ *into* E *is weakly continuous. Furthermore,*

$$\theta(\gamma_{\mathscr{H}(\mu)}) = \mu_0 . \tag{2.1}$$

Here μ_0 and $E_2'(\mu)$ are defined as in Section 1.

The notation introduced in Theorem 2.1 will be fixed throughout this paper. The Hilbert space $\mathscr{H}(\mu)$ is called the RKHS of μ.

Theorem 2.1 is well-known if E is a μ-Lusin space, that is if

$$\sup\{\mu(K) \mid K \text{ compact convex} \subseteq E\} = 1 .$$

[10, Theorem 4]. It can be conjectured that E is a μ-Lusin space when $\mu \in \mathscr{G}(E)$. If that is the case, many proofs of this paper can be a little bit simplified.

Let us at once point out two corollaries of Theorem 2.1.

COROLLARY 2.1. *Suppose* $\mu \in \mathscr{G}(E)$. *Then*

$$\mu_h = [\exp(\tilde{h} - \|h\|^2/2)]\cdot\mu_0, \quad h \in \mathscr{H}(\mu) . \tag{2.2}$$

(Compare [26, Prop. 8.1].)

Here $\mu_x(\cdot) = \mu_0(\cdot - x)$, $x \in E$.

COROLLARY 2.2. *Suppose* $\mu \in \mathscr{G}(E)$, *and let* G *be an additive,* μ*-measurable subgroup of* E *with positive* μ*-measure.*

Then, $2b \in G$ *and* $\mathscr{H}(\mu) \subseteq G$.

Compare [15], [23], and [2, Exposition IX].)

Note also that $\mu(G) = 1$ [5, Theorem 4.1]. (Compare also [15], [21], [23], [2, Exposition IX], and [11, p. 7].)

We have not been able to prove that $b \in G$.

PROOF OF COROLLARY 2.1. (Compare [26, Proposition 8.1].) The measures in the left-hand, and right-hand sides of (2.2), respectively, have the same Fourier transforms. Since they are Radon probability measures, they, clearly, coincide. This proves Corollary 2.1.

PROOF OF COROLLARY 2.2. We already know that $\mu(G) = 1$. Hence

$$\mu_0(-b + G) = 1 \quad \text{and} \quad \mu_0(-(-b + G)) = 1 .$$

In particular $\mu_0((-b+G) \cap (b+G)) = 1$. The set $(-b+G) \cap (b+G)$ is thus nonempty, which proves that $2b \in G$. Furthermore Corollary 2.1 shows that

$$\mu_0(h-b+G) = 1 \quad \text{whenever } h \in \mathscr{H}(\mu) .$$

The set $(h-b+G)\cap(-b+G)$ is thus non-empty and it follows that $\mathscr{H}(\mu) \subseteq G$.

Proof of Theorem 2.1. Denote by F the completion of E, and let $j: E \to F$ be the canonical injection. Set $\nu = j(\mu)$ and observe that $\nu \in \mathscr{G}(F)$. Since F is a ν-Lusin space, it is readily seen that the map

$$\mathrm{Var}(\nu): F' \ni \eta \to \nu(\eta^2) \in \mathsf{R}$$

is $\tau(F',F)$-continuous (Mackey continuous) (Compare [10, p. 403].) In particular, the measure ν has a barycentre $b \in F$. In the same way it follows that the measures $\eta\nu_0, \eta \in F_2'(\nu)$ have barycentres in F. Corollaries 2.1 and 2.2 thus apply to the measure ν. Since E is a ν-measurable subspace of F with ν-measure one, we have that $b \in E$. It is obvious that b is the barycentre of μ and $j(\mu_0) = \nu_0$. Using this and Corollary 2.2 again, it is readily seen that the measures $\xi\mu_0$, $\xi \in E_2'(\mu)$ have barycentres in E. Finally, observe that

$$\xi(h) = \langle \Lambda\xi, h \rangle, \quad h \in \mathscr{H}(\mu),\ \xi \in E' ,$$

where $\langle \cdot , \cdot \rangle$ denotes the scalar product in $\mathscr{H}(\mu)$. The mapping θ is thus weakly continuous and a simple calculation proves (2.1). This concludes the proof of Theorem 2.1.

We define

$$O(\mu) = \{h \in \mathscr{H}(\mu) \mid \|h\| \leqq 1\}, \quad \mu \in \mathscr{G}(E) ,$$

and have

Corollary 2.3. *Suppose $\mu \in \mathscr{G}(E)$. Then $O(\mu)$ is a compact subset of E, and*

(2.3) $$\mu_0(\xi^2) = \max\{\xi(x)^2 \mid x \in O(\mu)\}, \quad \xi \in E' .$$

In particular, the mapping $\mathrm{Var}(\mu)$ *is $\tau(E',E)$-continuous.*

Proof. Using the same notation as above, we have $O(\mu) = O(\nu) \subseteq E$. To prove that $O(\mu)$ is a compact subset of E, it therefore suffices to show that $O(\nu)$ is a compact subset of F. To this end let K be a compact convex subset of F, symmetric about the origin, and so that $\nu(K) > 0$. By Corollary 2.2, we have $O(\nu) \subseteq \bigcup_{n \in \mathsf{N}} nK$. Theorem 2.1 shows that $O(\nu)$ is a weakly compact subset of F, and K is, clearly, a weakly compact subset of F. An application of the closed graph theorem now easily gives

$O(\nu) \subseteq nK$ for a suitable $n \in \mathsf{N}$. Hence $O(\nu)$ is compact, which was to be proved. The identity (2.3) follows at once from the definitions, and the last statement is now also obvious. This proves Corollary 2.3.

3. Inequalities of the Brunn-Minkowski type.

The purpose of this section is merely to point out three known inequalities satisfied by Gaussian Radon measures. We think it can be convenient doing so since almost all subsequent results will depend on these estimates. In fact, we have already touched upon one of these estimates in connection with the zero-one law [5, Theorem 4.1].

If μ is a Radon probability measure on E, we set

$$\mu_*(A) = \sup\{\mu(K) \mid K \text{ compact} \subseteq A\},$$

whenever $A \subseteq E$. Furthermore, we define

$$\Phi(a) = \int_{-\infty}^{a} \exp(-x^2/2)\,dx/(2\pi)^{\frac{1}{2}}, \quad -\infty \leqq a \leqq \infty. \tag{3.1}$$

THEOREM 3.1. *Let $\mu \in \mathscr{G}(E)$ and suppose A is a μ-measurable subset of E. Choose $a \in \bar{\mathsf{R}}$ so that $\mu(A) = \Phi(a)$. Then,*

$$\mu_*(A + tO(\mu)) \geqq \Phi(a+t), \quad t > 0.$$

Equality occurs if A is a half space.

In particular, if $A + \mathscr{H}(\mu) = A$, then $\mu(A) = 0$ or 1.

Since the map $\operatorname{Var}(\mu)$ is Mackey continuous by Corollary 2.3, the result is contained in [6, Theorem 3.1].

THEOREM 3.2. *Suppose $\mu \in \mathscr{G}(E)$. Then, the inequality*

$$\mu_*(\lambda A + (1-\lambda)B) \geqq \mu^{\lambda}(A)\mu^{1-\lambda}(B)$$

is valid for all μ-measurable subsets A and B of E, and every $0 < \lambda < 1$.

(See [5, Corollary 2.1].)

The following simple corollary will play a fundamental rôle later on.

COROLLARY 3.1. *Let $\mu \in \mathscr{G}_0(E)$ and suppose A is a convex Borel measurable subset of E, symmetric about the origin. Then,*

$$\mu(A) \geqq \mu(A+x), \quad x \in E.$$

(See [5, Theorem 6.1].)

4. Some continuity properties of translations.

Let μ be a Radon probability measure on E and A a μ-measurable subset of E. Suppose $\varepsilon > 0$ is a given real number. Then, by definition, there exists a compact subset K of A so that $\mu(K) > \mu(A) - \varepsilon/2$. Furthermore, the Hahn–Banach separation theorem gives us a finite-dimensional cylinder set $C \supseteq K$ so that $\mu(C) < \mu(K) + \varepsilon/2$. To say that C is a finite-dimensional cylinder set means that there exist $\xi_1, \ldots, \xi_n \in E'$, and a Borel set M in R^n so that

$$C = \{(\xi_1, \ldots, \xi_n) \in M\} . \tag{4.1}$$

It is always possible to choose the ξ_k so that the family $\{\xi_1, \ldots, \xi_n\}$ constitutes an orthonormal family in $L_2(\mu)$ whenever $E' \subseteq L_2(\mu)$. In addition, we thus have

$$\mu(|I_A - I_C|) < \varepsilon . \tag{4.2}$$

From this important approximation property, we easily get

THEOREM 4.1. *Let $\mu \in \mathscr{G}(E)$ and suppose f is a bounded μ-measurable function on E. Then, for every fixed p, $1 \leqq p < +\infty$, the mapping*

$$(\mathscr{H}(\mu), \|\cdot\|) \ni h \to f(\cdot + h) \in L_p(\mu)$$

is continuous.

Note here that the function $f(\cdot + h)$ is μ-measurable for every $h \in \mathscr{H}(\mu)$ by Corollary 2.1.

A very special case of Theorem 4.1 is proved in [14, Corollary p. 740].

PROOF. It is enough to prove the continuity at the origin. Furthermore, it can be assumed that $\mu \in \mathscr{G}_0(E)$ and $f \geqq 0$. Let (s_k) be a denumerable sequence of μ-measurable simple functions so that $s_k \nearrow f$ as k tends to $+\infty$. By Corollary 2.1, we have

$$\mu(|f(\cdot + h) - s_k(\cdot + h)|^p) = \mu(|f - s_k|^p \exp(\tilde{h} - \|h\|^2/2)) ,$$

for every $h \in \mathscr{H}(\mu)$. The right-hand side here is less or equal to

$$(\mu(|f - s_k|^{2p}))^{\frac{1}{2}} \cdot (\mu(\exp 2\tilde{h}))^{\frac{1}{2}} .$$

Since $\tilde{h}$ is a centred Gaussian random variable with variance $\|h\|^2$, we get a uniform bound for the second factor if $\|h\|$ is bounded. It can therefore be assumed that $f = I_A$, where A is a μ-measurable set.

Now let $\varepsilon > 0$ be given. There exists a cylinder set C of the form (4.1) so that (4.2) holds. Furthermore, as said above, it can be assumed that the image measure $\gamma=(\xi_1,\ldots,\xi_n)(\mu)$ equals canonical Gauss measure in R^n. As above, we conclude that it suffices to prove

$$\mu(|I_C(\cdot+h)-I_C|) < \varepsilon ,$$

for small $\|h\|$, or, equivalently,

$$\gamma\big(|I_M(\cdot+(\xi_1(h),\ldots,\xi_n(h)))-I_M|\big) < \varepsilon ,$$

for small $\|h\|$.

Since the mapping θ in Theorem 2.1 is weakly continuous this follows at once from a well-known property of Lebesgue measure in n-space. This concludes the proof of Theorem 4.1.

COROLLARY 4.1. *Let $\mu\in\mathscr{G}(E)$ and assume A is a μ-measurable subset of E with positive μ-measure. Then there exists a positive number δ so that*

$$\delta O(\mu) \subseteq A-A .$$

(Compare [23, Proposition 1].)

Note that Corollary 4.1 again shows that $O(\mu)$ is a compact subset of E.

PROOF. The map

$$(\mathscr{H}(\mu),\|\cdot\|)\ni h \to \mu((h+A)\cap A)\in \mathsf{R}$$

is continuous by Theorem 4.1 and positive at the origin. From this the result follows at once.

COROLLARY 4.2. *Suppose $\mu\in\mathscr{G}(E)$. Then*

a) *if f is a μ-measurable additive function on E, the function $f|(\mathscr{H}(\mu),\|\cdot\|)$ is continuous and linear,*

b) *if f is a μ-measurqble subadditive function on E, positively homogeneous of degree one on $\mathscr{H}(\mu)$, the function $f|(\mathscr{H}(\mu),\|\cdot\|)$ is continuous.*

PROOF. Let us first prove Part a).

Suppose G is a μ-measurable additive subgroup of E with μ-measure one and so that the function $f|G$ is additive. Note that $\mathscr{H}(\mu)\subseteq G$ by Corollary 2.2 (or 4.1). Suppose $t\in\mathsf{R}$, and

$$\mu(\{f<t\}\cap\{f(-\cdot)<t\}\cap G) > 0 .$$

By Corollary 4.1, there exists a $\delta>0$ so that $(f|\delta O(\mu))<2t$. This proves Part a). Part b) follows in exactly the same way.

A subset of E (function from E into R) is said to be universally Gauss measurable, if it is μ-measurable with respect to every $\mu\in\mathscr{G}(E)$.

COROLLARY 4.3. *Let E be an ultrabornological* l.c.s. *and Ω an open convex subset of E. Furthermore, assume that f is a universally Gauss measurable convex function on E. Then f is continuous.*

Corollary 4.3 is essentially proved in [6, Lemma 5.1]. (Compare also [9, Theorem 7.3], [25], [28], and [31].) The following proof is probably simpler since it only depends on Theorem 4.1.

PROOF. It can be assumed that E is a Banach space and f a seminorm on E. Let $F = l^\infty(\mathsf{N})$, equipped with the $\sigma(l^\infty(\mathsf{N}), l^1(\mathsf{N}))$-topology, and set $\Pi_n(x) = x_n$, $x = (x_n) \in F$. Choose a $\nu \in \mathscr{G}_0(F)$ so that $\nu(\Pi_m \Pi_n) = 0$, $m \neq n$, $= \sigma_n > 0$, $m = n$. The existence of such a measure ν follows at once from Kolmogorov's consistence theorem. Suppose $(e_n)_{n \in N}$ is an arbitrary sequence in E such that the series $\sum e_n$ is absolutely convergent. It is enough to show that $(\sigma_n f(e_n)) \in F$. To this end set $u(x) = \sum x_n e_n$, $x \in F$, and observe that u is a continuous linear mapping of F into E by the Banach–Steinhaus theorem. Hence $\mu = u(\nu) \in \mathscr{G}_0(E)$. Furthermore, the submartingale convergence theorem shows that

$$\int \xi^2 d\mu = \sum \sigma_n{}^2 \xi^2(e_n), \quad \xi \in E' .$$

In particular, $\sigma_n e_n \in O(\mu)$ and Corollary 4.2(b) proves the result.

COROLLARY 4.4. *Let E be an ultrabornological* l.c.s. *and A a convex universally Gauss measurable subset of E. Furthermore, suppose $\mu \in \mathscr{G}(E)$ and set*

$$f(x) = \mu(x + A), \quad x \in E .$$

Finally, assume that there exists an open convex subset Ω of E so that $f(x) > 0$, $x \in \Omega$. Then, $f|\Omega$ is continuous.

A special case of Corollary 4.4 is proved in [13, Propositions 4 and 5].

PROOF. By Fubini's theorem f is universally Gauss measurable. Furthermore, the function $-\log f|\Omega$ is finite-valued and Theorem 3.2 easily shows that it is convex. Corollary 4.3 thus proves the result.

We observe that Corollary 4.4 applies to generalized Gaussian processes.

5. An inequality of Berwald's type.

Let $\mu \in \mathscr{G}(E)$ and suppose $f \geqq 0$ is a μ-measurable, subadditive function, positively homogeneous of degree one on $\mathscr{H}(\mu)$. By Corollary 4.2b), we have

$$\|f\|_{\mathscr{H}(\mu)} = \sup_{O(\mu)} f < +\infty .$$

Throughout this section, it will be assumed that

$$\|f\|_{\mathscr{H}(\mu)} \leqq 1 .$$

We here prefer to use the $\leqq$ sign instead of the $=$ sign, since it seems convenient to include the case when $\|f\|_{\mathscr{H}(\mu)}=0$.

With these assumptions, we have the following estimate of Berwald's type [3]:

THEOREM 5.1. *Let* $\varphi\colon [0,+\infty[\to \mathsf{R}$, $\varphi(0)=0$, *be a strictly increasing continuous function, and* $\alpha\colon \mathrm{range}(\varphi) \to [0,+\infty[$, $\alpha(0)=0$, *a strictly increasing, continuous, and convex function. Set* $\psi=\alpha(\varphi)$. *Furthermore, suppose* $\varphi(f) \in L_1(\mu)\setminus\{0\}$ *and choose* $\tau \in \mathsf{R}$ *such that*

$$\mu(\varphi(f)) = \int \varphi((t-\tau)^+)\,d\Phi(t) .$$

Then

$$\mu(\psi(f)) \leqq \int \psi((t-\tau)^+)\,d\Phi(t) .$$

Equality occurs if $f=(\xi/\|\xi\|_{\mathscr{H}(\mu)})^+$, *where* $\xi \in E'$ *and* $\xi \,|\, \mathscr{H}(\mu) \not\equiv 0$.

Here Φ is as in (3.1), and $t^+=\max(0,t)$.

PROOF. By definition,

$$\mu(\varphi(f)) = \int_0^\infty \mu(f \geqq s)\,d\varphi(s) ,$$

and

$$\mu(\varphi(\cdot-\tau)^+) = \int_0^\infty \Phi((\cdot-\tau)^+ \geqq s)\,d\varphi(s) ,$$

respectively. Here we identify Φ and the canonical Gauss measure on R.

Suppose $s_0>0$ and

$$\mu(f<s_0) \geqq \Phi((\cdot-\tau)^+<s_0) . \tag{5.1}$$

Using Theorem 3.1 and the assumptions on f, we have

$$\mu(f<s_0+t) \geqq \Phi((\cdot-\tau)^+<s_0+t), \qquad t>0 .$$

Let s_1 denote the infimum of all $s_0>0$ such that (5.1) is valid, and choose $s_2 \in]s_1,+\infty[$ arbitrarily but fixed. Writing $d\psi=\bar{\alpha}\,d\varphi$, where $\bar{\alpha}$ increases, we get

$$\begin{aligned}
&\int_0^{s_2} \mu(f\geqq s)\,d\psi(s) - \int_0^{s_2} \Phi((\cdot-\tau)^+\geqq s)\,d\psi(s)\\
&\quad = \int_0^{s_1} \big(\mu(f\geqq s)-\Phi((\cdot-\tau)^+\geqq s)\big)\bar{\alpha}(s)\,d\varphi(s)+\\
&\qquad + \int_{s_1}^{s_2} \big(\mu(f\geqq s)-\Phi((\cdot-\tau)^+\geqq s)\big)\bar{\alpha}(s)\,d\varphi(s)\\
&\quad \leqq \bar{\alpha}(s_1)\Big[\int_0^{s_2} \mu(f\geqq s)\,d\varphi(s) - \int_0^{s_2} \Phi((\cdot-\tau)^+\geqq s)\,d\varphi(s)\Big] .
\end{aligned}$$

The desired estimate now follows at once. This concludes the proof of Theorem 5.1.

Corollary 5.1. *Let* μ *and* f *be as in Theorem* 5.1, *and suppose* $\psi: [0, +\infty[\to [0, +\infty[$ *is a strictly increasing continuous function such that* $\psi((\cdot - \tau)^+) \in L_1(\Phi)$ *for every* $\tau \in \mathsf{R}$.

Then $\psi(f) \in L_1(\mu)$.

(Compare [11, pp. 10–15], [21], [24], and [6, Th. 5.2].)

Proof. Set first $\varphi(t) = \arctan t$, $t \geqq 0$, and $\alpha(t) = \tan t$, $0 \leqq t < \pi/2$. Theorem 5.1 then tells us that $f \in L_1(\mu)$. Then by setting $\varphi(t) = \alpha(t) = t$, $t \geqq 0$, and assuming, as we can, that $\psi(0) = 0$, the result follows at once from Theorem 5.1.

6. Extension of Anderson's inequality.

A function f of E into $[0, +\infty[$ is called a symmetric quasi-convex function, if all the level sets $\{f \leqq t\}$, $t \geqq 0$, are convex and symmetric about the origin. The following result is proved by Anderson in Euclidean n-space [1].

Theorem 6.1. *Let* $\mu, \nu \in \mathscr{G}_0(E)$. *Then the following statements are equivalent*:

(i) $\operatorname{Var}(\mu) \geqq \operatorname{Var}(\nu)$,

(ii) *there exists a* $\sigma \in \mathscr{G}_0(E)$ *such that* $\mu = \nu * \sigma$,

(iii) $\mu(f) \geqq \nu(f)$ *for every Borel measurable, symmetric, and quasi-convex function on* E.

Furthermore, if any of these conditions are fulfilled and

$$\dim \mathscr{H}(\nu) < +\infty\,, \tag{6.1}$$

then

$$\infty > \mu(f^2) \geqq \nu(f^2)\,,$$

for every μ*-measurable additive function on* E.

Later on we shall see that the condition (6.1) can be omitted. Note here that $\mathscr{H}(\nu)$, clearly, is embedded into $\mathscr{H}(\mu)$ so f is, clearly, continuous on $\operatorname{supp}(\nu) = \mathscr{H}(\nu)$ by Corollary 4.2 a).

Proof. (i) $\Rightarrow$ (ii). As in the proof of Theorem 2.1, denote by F the completion of E and let $j: E \to F$ be the canonical injection. Clearly, there exists a centred Gaussian cylinder measure τ on F such that $j(\mu) = j(\nu) * \tau$. Let $\varepsilon > 0$ be given. Since F is a $j(\mu)$-Lusin space, there exists a compact and convex subset K of F, symmetric about the origin, and so that $j(\mu)(K) > 1-\varepsilon$. Let G be a closed subspace of F with finite codimension, and denote by $\Pi_{F/G}$ the canonical surjection of F onto F/G. Using the same notation as in [29, pp. 172], we have

$$1-\varepsilon < [(j(\nu))_{F/G} * \tau_{F/G}](\Pi_{F/G}(K)) .$$

From Fubini's theorem and Corollary 3.1, we get $\tau_{F/G}(\Pi_{F/G}(K)) > 1-\varepsilon$. The cylinder measure τ thus comes from a Radon measure on F, again denoted by τ [29]. Now it only remains to be proved that there exists a Radon probability measure σ on E such that $\tau = j(\sigma)$. To this end let $\varepsilon > 0$ be given, and choose a compact subset K of E such that $\mu(K) > 1-\varepsilon$. From Fubini's theorem we now get an $x \in E$ such that $\tau(j(K+x)) > 1-\varepsilon$. Hence $\tau(\bigcup j(K_n)) = 1$ for an appropriate denumerable family of compact subsets K_n of E. The existence of a σ with the above mentioned properties now follows from [29, Theorem 12, p. 39]. This proves (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Suppose first A is a convex Borel set, symmetric about the origin. Fubini's theorem and Corollary 3.1 yield $\mu(A) \leqq \nu(A)$. From this the result follows at once.

(iii) $\Rightarrow$ (i). Choose $f = \xi^2$, where $\xi \in E'$.

We shall now prove the last part of Theorem 6.1. First note that $\mu(f^2) < +\infty$ by Corollary 5.1. Let σ be as in (ii), and suppose G is a μ-measurable additive subgroup of E with μ-measure one and such that $f|G$ is additive. There is no loss of generality to assume that G is Borel measurable. Furthermore, Corollary 2.2 b) shows that $\mathscr{H}(\mu) \subseteq G$. Hence, $\operatorname{supp}(\nu) \subseteq G$ and $\nu(G) = 1$. By Fubini's theorem, we also get $\sigma(x+G) = 1$ for a suitable $x \in \operatorname{supp}(\nu)$. Hence $\sigma(G) = 1$. By assumption, the function $E \times E \ni (x,y) \to f(x+y) \in \mathsf{R}$ is $\sigma \otimes \nu$-measurable. Fubini's theorem thus yields

$$\begin{aligned} \int f^2 d\mu &= \int_G d\sigma(x)[\int_G f^2(x+y)\,d\nu(y)] \\ &= \int_G d\sigma(x)[f^2(x) + 2f(x)\nu(f) + \nu(f^2)] . \end{aligned}$$

Since $\nu(f) = 0$, the result follows at once.

Corollary 6.1. *Let $\mu \in \mathscr{G}_0(E)$ and suppose $\{h_1, \ldots, h_n\}$ is a finite orthonormal family in $\mathscr{H}(\mu)$. Then*

$$\mu(A) \leqq \mu(\textstyle\sum_1^n h_i \tilde{h}_i \in A)$$

for every convex Borel set A in E, which is symmetric about the origin.

Proof. Note first that the function $g = \sum h_i \tilde{h}_i$ is μ-measurable. Therefore the image measure $\nu = g(\mu) \in \mathscr{G}_0(E)$ since $\dim(\operatorname{supp}(\nu)) < +\infty$. Furthermore, the random variables $\xi(g)$ and $\xi-\xi(g)$ are orthogonal in $L_2(\mu)$ for every $\xi \in E'$. Hence

$$(6.2) \qquad [\mathrm{Var}(\mu)](\xi) = \mu\big(((\xi(g))^2\big) + \mu\big((\xi-\xi(g))^2\big) \geqq \mu\big((\xi(g))^2\big) = [\mathrm{Var}(\nu)](\xi), \quad \xi \in E' .$$

Corollary 6.1 now follows from Theorem 6.1.

7. The RKHS is separable.

We shall thus prove

THEOREM 7.1. *Suppose $\mu \in \mathscr{G}(E)$. Then the* RKHS $(\mathscr{H}(\mu), \|\cdot\|)$ *is separable.*

A wrong proof of this result is given in [19, Theorem 2] when E is complete. Furthermore, the paper [10, p. 403] claims that the RKHS need not be separable when E is a μ-Lusin space and μ is a Gaussian Borel measure. In view of Prohorov's theorem this statement contradicts Theorem 7.1. Finally, the paper [2, footnote 1, p. 376] imagines that $\mathrm{supp}(\mu)$ need not be separable when $\mu \in \mathscr{G}(E)$. As we shall see below this would contradict Theorem 7.1.

To prove Theorem 7.1 we need the following well-known

LEMMA 7.1. *Let $(B, \|\cdot\|_B)$ be a Banach space, and (a_n) a denumerable sequence in B, which converges weakly to zero and such that $\inf \|a_n\|_B > 0$.*

Then there exist an infinite subsequence (a_{n_k}), and a positive number C such that

$$\max_{0 \leq k \leq m} |t_k| \leqq C \|\textstyle\sum_0^m t_k a_{n_k}\|_B$$

for all $t_0, \ldots, t_m \in \mathsf{R}$, and all $m \in \mathsf{N}$.

PROOF. By a familiar theorem, the sequence (a_n) contains an infinite subsequence (a_{n_k}), which is a basic sequence [4, Corollary 1]. The result thus follows by an application of the Banach–Steinhaus theorem.

PROOF OF THEOREM 7.1. It can be assumed that $\mu \in \mathscr{G}_0(E)$. Let $(h_\alpha)_{\alpha \in Q}$ be an orthonormal family in $(\mathscr{H}(\mu), \|\cdot\|)$. It shall be proved that Q is at most denumerable. To this end it is, clearly, no loss of generality to assume that E is a μ-Lusin space. Let us choose a compact convex subset K of E, symmetric about the origin, and with positive μ-measure. Set $B = \bigcup_{n \in \mathsf{N}} nK$, and denote by $\|\cdot\|_B$ the Minkowski functional of K

in B. The zero-one law tells us that $\mu(B)=1$ and Corollary 2.2 b) that $\mathscr{H}(\mu)\subseteq B$. Furthermore, Corollary 6.1 implies the inequality

$$(7.1) \qquad \mu(\|\textstyle\sum_I h_\alpha \tilde{h}_\alpha\|_B \leqq 1) \geqq \mu(K)\,,$$

valid for every nonempty, finite subset I of Q.

Now let $\varepsilon>0$ be a given real number. We claim that the set

$$(7.2) \qquad \{\alpha \in Q \mid \|h_\alpha\|_B \geqq \varepsilon\} \textit{ is finite}\,.$$

Suppose on the contrary that this is wrong. Then there exists a denumerable subset $\{h_{\alpha_n}\}$ of the set $\{h_\alpha\}$ such that $\|h_{\alpha_n}\|_B \geqq \varepsilon$ and $h_{\alpha_m} \neq h_{\alpha_n}$, $m \neq n$. Let f be a continuous linear form on the Banach space B, $\|\cdot\|_B)$ with the norm one. The inequality (7.1) then yields

$$\mu(|\textstyle\sum_0^m f(h_{\alpha_n})\tilde{h}_{\alpha_n}| \leqq 1) \geqq \mu(K), \qquad m \in \mathbf{N}\,.$$

Since the random variable $\sum_0^m f(h_{\alpha_n})\tilde{h}_{\alpha_n}$ is a centred Gaussian random variable with variance $\sum_0^m f^2(h_{\alpha_n})$, we deduce that $\sum_0^\infty f^2(h_{\alpha_n}) < +\infty$. The sequence (h_{α_n}) thus converges weakly to zero in the Banach space $(B,\|\cdot\|_B)$. Lemma 7.1 now applies to the sequence $(a_n)=(h_{\alpha_n})$. The inequality (7.1) together with Lemma 7.1 yield

$$\mu(\max\nolimits_{0\leqq k\leqq m} |\tilde{h}_{\alpha_{n_k}}| \leqq C) \geqq \mu(K), \qquad m \in \mathbf{N}\,.$$

Hence

$$(\Phi(C)-\Phi(-C))^m \geqq \mu(K) > 0, \qquad m \in \mathbf{N}\,.$$

This contradiction proves (7.2). The family $(h_\alpha)_{\alpha\in Q}$ is thus at most de numerable, which proves Theorem 7.1.

We shall discuss some applications of Theorem 7.1 in the following section.

8. Applications of Theorem 7.1.

Suppose $\mu \in \mathscr{G}_0(E)$. We shall say that a μ-measurable additive function f on E is equal to zero, if there exists an additive μ-measurable subgroup G of E with μ-measure one so that $f|G=0$.

THEOREM 8.1. *Suppose $\mu \in \mathscr{G}_0(E)$, and let $\{e_n\}_{n\in\mathbf{N}}$ be an orthonormal basis for $E_2'(\mu)$, where each e_n belongs to E'. Furthermore, let f be a μ-measurable additive function on E. Then,*

a) *$f=0$ iff f vanishes on $\mathscr{H}(\mu)$,*
b) *$f \in E_2'(\mu)$,*

c) *there exist a Borel measurable additive subgroup G of E, with μ-measure one, and an $(a_n) \in l^2(\mathsf{N})$ so that the series*

$$\sum a_n e_n(x) \tag{8.1}$$

converges for every $x \in G$, and has the sum $f(x)$.

Note that $\mathscr{H}(\mu)$ and $E_2'(\mu)$ are isomorphic so the existence of the basis $\{e_n\}$ follows from Theorem 7.1.

Theorem 8.1 is well-known on many spaces. (See [7], [12], [18], [17, Theorem 2.3], and [16, Section 7].) These papers essentially prove Theorem 8.1 when E is a μ-Lusin space which also is Souslin. Note that Theorem 7.1 is trivial when E is a Souslin space, since the Borel σ-algebra in such a space is countably generated.

Proof. Corollary 2.2 b) proves the "only if" part of Part a). Let us now prove the "if" part of Part a). To this end let G be a μ-measurable additive subgroup of E with μ-measure one so that $f\,|\,G$ is additive. Set

$$A_- = \{x \in G \mid f(x) \leqq 0\}, \quad A_+ = \{x \in G \mid f(x) \geqq 0\}.$$

Then $A_- = -A_+$ and $A_- \cup A_+ \supseteq G$. Hence $\mu(A_-) = \mu(A_+) \geqq \frac{1}{2}$. Furthermore,

$$A_{\binom{-}{+}} + \mathscr{H}(\mu) = A_{\binom{-}{+}}$$

since $f\,|\,\mathscr{H}(\mu) = 0$. The last part of Theorem 3.1 thus gives $\mu(A_+) = \mu(A_-) = 1$. The μ-measurable additive subgroup $A_- \cap A_+$ of E therefore has μ-measure one and f vanishes on this subgroup. This proves the "if" part of Part a).

Let us now prove Part b). Theorem 6.1 shows that $f \in L_2(\mu)$. Let g denote the orthogonal projection of f onto $E_2'(\mu)$. We can choose g (as a point-function) so that g is a μ-measurable additive function on E. (See Section 2.) Set $a = f - g$, and note that a is a μ-measurable additive function on E. It only remains to be proved that $a = 0$ in $L_2(\mu)$. To this end choose $h \in \mathscr{H}(\mu)$ with $\|h\| = 1$, arbitrarily but fixed. We can choose $\check{h}$ (as a point-function) so that $\check{h}$ is a μ-measurable additive function on E. Set $\nu = (h\check{h})(\mu)$ and observe that $\operatorname{Var}(\mu) \geqq \operatorname{Var}(\nu)$ by (6.2). The last part of Theorem 6.1 thus tells us that

$$\mu((\check{h} + ta)^2) \geqq \nu((\check{h} + ta)^2), \quad t \in \mathsf{R}. \tag{8.2}$$

The zero-one law and Corollary 2.2 b) show that $\check{h}(h) = \mu(\check{h}^2) = 1$. Corollary 4.2 now easily gives that $\nu(\check{h}^2) = 1$. Furthermore, $\mu(\check{h}a) = 0$. From (8.2), we therefore have

$$t^2\mu(a^2) \geqq 2t\nu(\check{h}a) + t^2\nu(a^2), \quad t \in \mathsf{R}.$$

Hence $\nu(\check{h}a)=0$. But $a(h)=\nu(\check{h}a)$. Part a) now shows that $a=0$ as a μ-measurable additive function on E. This also proves that $a=0$ in $L_2(\mu)$. Part b) is thereby completely proved.

Part c) follows at once from Part b) and the zero-one law since the series (8.1) converges in $L_2(\mu)$ if and only if it converges a.s. $[\mu]$.

This concludes the proof of Theorem 8.1.

COROLLARY 8.1. *Let $\mu \in \mathscr{G}(E)$ and suppose $\{e_n\}_{n\in\mathbf{N}}$ is an orthonormal basis for $E_2'(\mu)$, where every $e_n \in E'$.*

Then, for every Borel set A in E, there exists a set B belonging to the σ-algebra generated by the e_n so that

$$\mu(|I_A - I_B|) = 0 .$$

In particular, the space $L_p(\mu)$ $(1 \leqq p < +\infty)$ is separable.[1])

PROOF. Corollary 8.1 follows at once from (4.2) and Theorem 8.1.

It is also possible to pick out an orthonormal basis for $L_2(\mu)$ when $\mu \in \mathscr{G}_0(E)$. In fact, let H_n, $n \geqq 0$, denote the nth Hermite polynomial normalized in a convenient way and set

$$f_{n_1,\ldots,n_k} = H_{n_1}(e_1)\cdot \ldots \cdot H_{n_k}(e_k), \quad n_1,\ldots,n_k \geqq 0,\ k \geqq 1 ,$$

where the e_n are as in Theorem 8.1. The family $\{f_{n_1,\ldots,n_k}\}$ then constitutes an orthonormal basis for $L_2(\mu)$. The proof is exactly the same as in case of classical Wiener measure [8].

COROLLARY 8.2. *Suppose $\mu \in \mathscr{G}(E)$. Then,*

$$\operatorname{supp}(\mu) = b + \overline{\mathscr{H}(\mu)} . \tag{8.3}$$

In particular, supp(μ) *is a closed, separable, affine subspace of E.*[2])

PROOF. The representation (8.3) follows from [2, Theorem (IX, 2.1)] and Corollary 2.3. Furthermore, the mapping θ in Theorem 2.1 is continuous for the Mackey topologies on $(\mathscr{H}(\mu), \|\cdot\|)$, E [27, Theorem 7.4]. The last statement thus follows from Theorem 7.1

9. Estimates of tail probabilities.

Suppose $\mu \in \mathscr{G}(E)$ and let f be a μ-measurable sublinear function on E, that is $f(\pm\,\cdot)$ are μ-measurable, and there exists a μ-measurable linear subspace F with μ-measure one so that $f|F$ is sublinear. Note that $\|f\|_{\mathscr{H}(\mu)} < +\infty$ by Corollary 4.2 b). With these assumptions, we have the following result, which partly generalizes Corollary 5.1.

[1]), [2]) See Note added in proof.

THEOREM 9.1. *There holds,*

a) $\lim_{t\to\infty} t^{-2}\log\mu(f \geqq t) = -\frac{1}{2}\cdot\frac{1}{\|f\|^2_{\mathscr{H}(\mu)}}$ *if* $\|f\|_{\mathscr{H}(\mu)} > 0$,

b) $\|f\|_{\mathscr{H}(\mu)} = 0$ *iff* $f = \text{const.}$ a.s. $[\mu]$.

(Compare [6, Theorem 5.2] and [24].)

PROOF. Let us first prove Part a). Theorem 3.1 easily gives

$$\text{(9.1)} \quad \limsup_{t\to+\infty} t^{-2}\log\mu(f\geqq t) \leqq -\frac{1}{2}\cdot\frac{1}{\|f\|^2_{\mathscr{H}(\mu)}}.$$

(Compare [6, pp. 12].)

Now let F be as above and observe that $b \in F$ by Corollary 2.2 a). Hence

$$-b+\{f\geqq t\}\cap F \supseteq \{f\geqq t+f(-b)\}\cap F .$$

If $\mu_0{}^*$ denotes the outer $\mu_0{}^*$-measure, we thus have

$$\mu(f\geqq t) \geqq \mu_0{}^*(\{f\geqq t+f(-b)\}\cap F) .$$

Setting $g(x)=\max(f(x),f(-x))$, $x\in E$, we easily get

$$\mu(f\geqq t) \geqq \tfrac{1}{2}\mu_0{}^*(\{g\geqq t+f(-b)\}\cap F) .$$

Now let $h\in\mathscr{H}(\mu)$ be chosen such that $\|h\|=1$ and $f(h)>0$, and set $\nu=(h\check{h})(\mu_0)$. Corollary 6.1 now easily gives

$$\mu(f\geqq t) \geqq \tfrac{1}{2}\mu_0(g(h)|\check{h}|\geqq t+f(-b)) .$$

Hence

$$\liminf_{t\to\infty} t^{-2}\log\mu(f\geqq t) \geqq -\frac{1}{2}\cdot\frac{1}{g^2(h)} \geqq -\frac{1}{2}\cdot\frac{1}{f^2(h)}.$$

Combining this estimate with (9.1), Part a) follows at once.

If $\|f\|_{\mathscr{H}(\mu)}=0$, Theorem 3.1 easily shows that $f=\text{const.}$ a.s.$[\mu]$. Conversely, if $\|f\|_{\mathscr{H}(\mu)}>0$, Part a) shows that f cannot be constant a.s.$[\mu]$. This concludes the proof of Theorem 9.1.

It can clearly, happen that $f=1$ a.s.$[\mu]$ and f vanishes on $\mathscr{H}(\mu)$. In fact, this can happen even if $\mu\in\mathscr{G}_0(E)$ and f is a seminorm. We will illustrate this in Section 11. On the other hand, if $\mu\in\mathscr{G}_0(E)$ and f is a μ-measurable additive function on E, which vanishes on $\mathscr{H}(\mu)$, then Theorem 8.1 shows that $f=0$ a.s.$[\mu]$.

10. A log law for Gaussian random variables.

Let E be a metrizable l.c.s. and d a translation-invariant metric on E. Suppose $\mu \in \mathscr{G}(E)$ and let $(X_n)_{n\in\mathrm{N}}$ be an E-valued stochastic process so that $X_n(P)=\mu$ for every n.

Set

$$Y_n = (2\log n)^{-\frac{1}{2}} X_n, \quad n \geqq 2\,,$$

and

$$M = \bigcap_2^\infty \{Y_n, Y_{n+1}, \ldots\}^-\,,$$

where the bar denotes the closure operation in E.

We then have

THEOREM 10.1. *The random set M is contained in $O(\mu)$ a.s. Furthermore, if the X_n are independent, the random set M equals $O(\mu)$ a.s.*

It is readily seen that Theorem 10.1 implies Strassen's law of the iterated logarithm for Brownian motion [30] and also various extensions of this result (see [22], [20], and [19]). We omit the obvious proofs. Although Theorem 10.1 is very close to these results, it can perhaps be worth pointing out since it has a simpler character and an almost trivial proof.

PROOF. Choose $\varepsilon>0$ arbitrarily but fixed. We claim that

$$P[Y_n \in O_\varepsilon, \text{large}\, n] = 1\,, \tag{10.1}$$

where $O_\varepsilon=\{d(\cdot,O(\mu))\leqq\varepsilon\}$. To see this let V be a closed, convex, and symmetric neighbourhood of the origin such that $V\subseteq\{d\leqq\varepsilon\}$. Denote by f the Minkowski functional of the set $O(\mu)+V$. By Corollary 2.3, there is a $\lambda>0$ such that $O(\mu)\subseteq\lambda V$. Hence $\|f\|_{\mathscr{H}(\mu)}\leqq(1+\lambda^{-1})^{-1}$. Since $O(\mu)+V\subseteq O_\varepsilon$, Theorem 9.1 yields

$$\lim_{n\to\infty}(2\log n)^{-1}\log P[Y_n\notin O_\varepsilon] \leqq -\tfrac{1}{2}\cdot(1+\lambda^{-1})^2\,.$$

Hence

$$\sum P[Y_n\notin O_\varepsilon] < +\infty\,,$$

which proves (10.1). From this the first part of Theorem 10.1 follows at once.

Let us now prove the last part.

It can be assumed that $\mu\in\mathscr{G}_0(E)$. Note that Theorem 7.1 implies that $O(\mu)$ is a separable subset of E. It is therefore enough to show that

$$P[d(Y_n,h) < \varepsilon, \text{i.o.}] = 1\,,$$

for every $h \in O(\mu)$ and every $\varepsilon > 0$. Now fix h and ε, and let V be as above. It suffices to prove that

$$P[X_n \in (2 \log n)^{\frac{1}{2}} h + (2 \log n)^{\frac{1}{2}} V. \text{ i.o.}] = 1 . \tag{10.2}$$

Corollary 2.1 easily implies that the left-hand side l_n of (10.2) is greater or equal to

$$\tfrac{1}{2} n^{-\|h\|^2} \cdot \mu((2 \log n)^{\frac{1}{2}} V) .$$

Hence $\sum l_n = +\infty$, and the Borel–Cantelli lemma proves (10.2) and the theorem.

11. Some concluding remarks.

Suppose $\mu \in \mathscr{G}_0(E)$ for simplicity. It is often desirable to have a slightly different representation of μ than that given by Theorem 2.1. Let namely F be another l.c.s. and $u: F \to E$ a continuous linear mapping. If F is a "nice" space, the existence of a $\nu \in \mathscr{G}_0(F)$ such that

$$\mu = u(\nu) \tag{11.1}$$

is often of great value. Corollary 8.2 shows that F can be chosen as a separable l.c.s. Better results would be desirable. Here we will point out that F cannot be chosen too "simple".

To this end let Π be the canonical product Gauss measure on the Fréchet space R^{N}. Set

$$E = \{x \in \mathsf{R}^{\mathsf{N}} \mid \limsup_{n\to\infty} |x_n|/(2 \log n)^{\frac{1}{2}} < +\infty\} ,$$

and observe that Theorem 10.1 shows the well-known fact that $\Pi(E) = 1$. Let μ be the restriction of Π to E and define

$$f(x) = \limsup_{n\to\infty} |x_n|/(2 \log n)^{\frac{1}{2}}, \quad x \in E .$$

Theorem 10.1 also shows the well-known fact that $f = 1$ a.s. $[\mu]$. Note that $\mu \in \mathscr{G}_0(E)$ and $f | \mathscr{H}(\mu) = 0$ since $\mathscr{H}(\mu) = l^2(\mathsf{N})$. We claim that the space F above cannot be chosen to be an ultrabornological l.c.s. Suppose to the contrary that this is possible. Corollary 4.3 then tells us that the seminorm $f(u)$ is continuous. Hence

$$\nu(f(u) < \tfrac{1}{2}) > 0 , \tag{11.2}$$

since $0 \in \operatorname{supp}(\nu)$. But (11.1) implies that

$$\nu(f(u) < \tfrac{1}{2}) = \mu(f < \tfrac{1}{2}) = 0 .$$

This contradiction proves the assertion. This discussion extends [19, Sec. 3].

It is also easy to see that the l.c.s. F cannot be chosen so that the seminormed linear space $(F, f(u))$ is separable. In fact, if that is the case, there holds

$$\nu(a + \{f(u) < \tfrac{1}{2}\}) > 0$$

for an appropriate $a \in F$. Corollary 3.1 now implies (11.2) and we have again a contradiction.

NOTE ADDED IN PROOF. Independently, H. Sato and Y. Okazaki (Separabilities of a Gaussian Radon measure, Ann. Inst. H. Poincaré Sect. B, 9 (1975), 287–298) have proved that $L_2(\mu)$ ís separable for every Gaussian Radon measure. The same work also shows that supp(μ) is separable, if μ is centred or E is a μ-Lusin space.

REFERENCES

1. T. W. Anderson, *The integral of a symmetric convex set and some probability inequalities*, Proc. Amer. Math. Soc. 6 (1955), 170–176.
2. A. Badrikian, and S. Chevet, *Mesures cylindriques, espaces de Wiener et fonctions aléatoires gaussieus*, Lecture Notes in Mathematics 379, Springer-Verlag, Berlin · Heidelberg · New York, 1974.
3. L. Berwald, *Verallgemeinerung eines Mittelwertsatzes von J. Favard für positive konkave Funktionen*, Acta Math. 79 (1947), 17–37.
4. C. Bessaga, and A. Pelczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), 151–164.
5. C. Borell, *Convex measures on locally convex spaces*, Ark. Mat. 12 (1974), 239–252.
6. C. Borell, *The Brunn–Minkowski inequality in Gauss space*, Invent. Math. 30 (1975), 207–216.
7. R. H. Cameron, and R. E. Graves, *Additive functionals on a space of continuous functions*, I, Trans. Amer. Math. Soc. 70 (1951), 160–176.
8. R. H. Cameron, and W. T. Martin, *The orthogonal development of non-linear functionals in series of Fourier–Hermite functionals*, Ann. of Math. 48 (1947), 385–392.
9. J. P. R. Christensen, *Topology and Borel structure*, Mathematics Studies 10, North-Holland Publ. Co., Amsterdam, 1974.
10. R. M. Dudley, J. Feldman, and L. LeCam, *On seminorms and probabilities, and abstract Wiener spaces*, Ann. of Math. 93 (1971), 390–408.
11. X. Fernique, *Regularité des trajectoires des functions aleatoires gaussiennes*, École d'ête calcul des probabilitites, St. Flour, 1974.
12. R. E. Graves, *Additive functionals on a space of continuous functions*, II, Ann. of Math. 54 (1951), 275–285.
13. L. Gross, *Potential theory on Hilbert space*, J. Functional Analysis 1 (1967), 123–181.
14. B. Jamison and S. Orey, *Subgroups of sequences and paths*, Proc. Amer. Math. Soc. 24 (1970), 739–744.
15. G. Kallianpur, *Zero-one laws for Gaussian processes*, Trans. Amer. Math. Soc. 149 (1970), 199–211.

16. G. Kallianpur, *The role of reproducing kernel Hilbert spaces in the study of Gaussian processes* in *Advances in probability and related topics*, Vol. 2, edited by P. Ney, Marcel Dekker, Inc., New York, 1970.
17. M. Kanter, *Linear sample spaces and stable processes*, J. Functional Analysis 9 (1972), 441–459.
18. J. Kuelbs, *Abstract Wiener spaces and applications to analysis*, Pacific J. Math. 31 (1969), 433–450.
19. J. Kuelbs, *Some results for probability measures on linear topological vector spaces with an application to Strassen's* loglog *law*, J. Functional Analysis 14 (1973), 28–43.
20. J. Kuelbs and R. LePage, *The law of the iterated logarithm for Brownian motion in Banach space*, Trans. Amer. Math. Soc. 185 (1973), 253–264.
21. H. J. Landau and L. A. Shepp, *On the supremum of a Gaussian process*, Sankhyā Ser. A. 32 (171), 369–378.
22. R. LePage, Loglog *law for Gaussian processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 25 (1973), 103–108.
23. R. LePage, *Subgroups of paths and reproducing kernels*, Ann. of Prob. 1 (1973), 345–347.
24. M. B. Marcus and L. A. Shepp, *Sample behaviour of Gaussian processes*, Proc. of the Sixth Berkeley Symposium on Math. Stat. and Probability, Vol. 2, 423–441, Berkeley 1972.
25. A. Martineau, *Sur le théorème du graph fermé*, C. R. Acad. Sci. Paris Sér. A. 263 (1966), 870–871.
26. J. Neveu, *Processes aléatoires gaussiens*, Les presses de l'Univ. de Montréal, Montréal, 1968.
27. H. H. Schaefer, *Topological vector spaces*, The MacMillan Comp., New York, 1967.
28. L. Schwartz, *Sur le théorème du graph fermé*, C. R. Acad. Sci. Paris Sér. A. 263 (1966), 602–605.
29. L. Schwartz, *Radon measures*, Oxford Univ. Press, London 1973.
30. V. Strassen, *An invariance principle for the law of the iterated logarithm*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 3 (1964), 211–226.
31. J. D. M. Wright, *On the continuity of mid-point convex functions*, Bull. London Math. Soc. 17 (1975), 89–92.

DEPARTMENT OF MATHEMATICS
SYSSLOMANSGATAN 8
752 23 UPPSALA
SWEDEN