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ON THE L^p ESTIMATES FOR ELLIPTIC BOUNDARY PROBLEMS

LEIF ARKERYD

0. Introduction.

Let α be a multi-index, i.e. a sequence $\alpha_1,\ldots,\alpha_\nu$ of indices α_j, an index being an integer between 1 and n. Let $|\alpha|$ be the number ν of indices in α. Write

$$D_\alpha = i^{-|\alpha|}\frac{\partial}{\partial x_{\alpha_1}}\cdots\frac{\partial}{\partial x_{\alpha_\nu}}.$$

We denote by

$$A = A(x,D) = \sum_{|\alpha|\leqq 2m} a_\alpha(x)D_\alpha$$

and

$$B_j = B_j(x,D) = \sum_{|\alpha|\leqq m_j} b_{j\alpha}(x)D_\alpha, \qquad j=1,\ldots,m\,,$$

differential operators in R^n, with coefficients submitted to suitable regularity conditions. Suppose A is properly elliptic and the B_j satisfy certain "complementary" conditions with respect to A. We consider the boundary problem

$$Au = f \text{ in } R_+^n = \{x_1>0\}, \qquad B_ju = g_j \text{ on } R^{n-1} = \{x_1=0\}\,.$$

The following inequality is fundamental in the theory of elliptic boundary problems

$$\|u\|_{s;p}^+ \leqq C\left(\|Au\|_{s-2m;p}^+ + \sum_{j=1}^m \langle B_ju\rangle_{s-m_j-p^{-1};p} + \|u\|_{s-1;p}^+\right) \tag{0.1}$$

for $u\in H_{s;p}^+$. Concerning the definition of the spaces $H_{s;p}^+$ and their norms, cf. Section 1.

For s integer $\geqq 2m$, the most important case, inequality (0.1) was proved by Agmon–Douglis–Nirenberg [1], Browder [2] and Slobodeskij [20]. By complex interpolation this result was extended to non-integer $s>2m$ by Lions–Magenes [9] and Schechter [17]. (Real interpolation leads to similar results in $W_{s;p}^+$, cf. Section 1, instead of $H_{s;p}^+$. We shall not consider that case here.) The case of general s (no lower bound) is

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treated by Schechter [17]. For a survey and more complete references, see e.g. Geymonat–Grisvard [4].

Because of the great importance of (0.1) and the not very good connections between the relevant papers, a new and rather independent proof might be motivated. We follow an idea of Peetre [13], [16] for the deduction of (0.1) in the special case of Dirichlet's problem. It is based on a simple, explicit solution in the case of homogeneous boundary conditions. After this case has been treated, the extension to inhomogeneous boundary conditions is rather easy to do using well-known results on the "trace". We then pass to variable coefficients by the usual Korn's argument. Finally it is possible to treat other problems than Dirichlet's, cf. Theorem 4.1, our main result.

Properly elliptic systems (cf. [1, Part II]), however, create certain difficulties, when approached by this method. We have not treated them here.

Instead of the Calderon–Zygmund theorem on n-dimensional Hilbert transforms in L^p-space, which is most often used in similar cases, we shall use Mihlin's theorem on Fourier multipliers. Then the proof becomes as transparent as in the L^2-case. Moreover our assumptions on the coefficients are rather weak. For instance, it is sufficient that the coefficients of A belong to $\mathrm{Lip}_{s+\varepsilon}$ (cf. Campanato [3]). This cannot be done by interpolation. Our method has a greater range of applicability than the case treated in this article. We hope to return to the subject in a forthcoming paper and then to give similar results for the so-called quasi-elliptic operators, including thus the elliptic (treated here) and the parabolic cases.

I want to take this opportunity to express my gratitude to professor Jaak Peetre, who suggested the subject. He has given much constructive criticism and shown kind interest in my work.

1. Preliminaries concerning $H_{s;p}$-spaces.

We denote by L^p, $1 \leqq p \leqq \infty$, the space of equivalence classes of Lebesgue measurable functions in R^n with integrable pth power, and write

$$\|u\|_p = \left(\int |u(x)|^p \, dx\right)^{1/p}, \qquad u \in L^p .$$

Let S be the space of infinitely differentiable functions u such that

$$\sup_{x \in R^n} |x_p D_\alpha u(x)| < \infty$$

for all α and p, and with the topology defined by these seminorms. The dual space S' is the space of tempered distributions. (See Schwartz [18]).

The Fourier transform of an element f in S or S' is denoted by Ff, the inverse Fourier transform by $\bar{F}f$, $\bar{F}Ff=f$. We take formally

$$Ff = \int \exp(-ix\xi) f(x)\, dx$$

We use the notation

$$K(D)f = \bar{F}KFf$$

where $K(\xi)$ is a function on R^n. We shall often employ

$$K(\xi) = \Lambda(\xi) = \delta\xi_1 + i(1+\delta^2|\xi'|^2)^{\frac{1}{2}}$$

and

$$K(\xi) = \Lambda_1(\xi) = (1+\delta^2|\xi'|^2)^{\frac{1}{2}}, \qquad \delta > 0 .$$

Here $\xi' = (\xi_2, \ldots, \xi_n)$. The dependence on δ will not be explicitely indicated. Constants are denoted by C or K. Different constants sometimes get the same symbol in different expressions, when this does not lead to confusion.

We shall often use the following well-known theorem by Mihlin [10], [11].

THEOREM 1.1. *If*

$$|D_\alpha f(\xi)| \leqq C_\alpha |\xi|^{-|\alpha|} ,$$

for $|\alpha| \leqq n$, *then* f *is a Fourier multiplier on* L^p, *that is,*

$$T \to \bar{F}fFT$$

defines a continuous linear mapping of L^p *into* L^p, $1<p<\infty$.

For a proof, see Hörmander [7, Theorem 2.5, p. 120].

We now introduce the space

$$H_{s;p} = \{f;\ f\in S', \bar{F}\Lambda^s Ff \in L^p\}$$

normed by

$$\|f\|_{s;p} = \|\bar{F}\Lambda^s Ff\|_p .$$

Theorem 1.1. gives that the definition of the space is independent of $\delta>0$, and that the corresponding norms are equivalent. Further

$$\|f\|_{s_1;p} \leqq C\|f\|_{s_2;p}, \quad \text{when } s_1 \leqq s_2 .$$

The δ-dependent norms were used by Hörmander [8]. For $1<p<\infty$, $H_{s;p}$ is a reflexive Banach space with dual space $H_{-s;p'}$, where p' is defined by $p^{-1}+p'^{-1}=1$. We also notice that $H_{0;p}=L^p$. For further properties of the $H_{s;p}$-spaces, cf. e.g. [9, Part III].

Let $(H_{s;p})_-$ be the subspace of $H_{s;p}$ of elements in $H_{s;p}$, whose supports are contained in the half-space $x_1 \leqq 0$. Let $H^+_{s;p}$ be the corresponding quotient space

$$H^+_{s;p} = H_{s;p}/(H_{s;p})_-$$

and $\|\cdot\|^+_{s;p}$ the quotient norm

$$\|f\|^+_{s;p} = \inf \|\tilde{f}\|_{s;p} ,$$

where inf is taken over all $\tilde{f}$ in $H_{s;p}$, whose restriction to $x_1 > 0$ is f. We shall sometimes need a somewhat more general space than $H_{s;p}$, viz.

$$H_{s,r;p} = \{f;\ f \in S',\ \Lambda_1{}^r(D) f \in H_{s;p}\}$$

and the corresponding norm

$$\|f\|_{s,r;p} = \|\overline{F}\Lambda_1{}^r \Lambda^s F f\|_p .$$

The quotient space $H^+_{s,r;p}$ is defined analogously to $H^+_{s;p}$ above.

We now give a few theorems on $H_{s;p}$-spaces, which will be needed later on.

THEOREM 1.2. *If $0 < s < 1$, the norm $\|f\|_{s;p}$ is equivalent to*

$$\delta^s \left\| \int_0^\infty \int_{-\infty}^\infty \frac{f(x+h) - f(x)}{|h|^{s+n}} dh_1\, dh' \right\|_p + \|f\|_p ,$$

and the norm $\|f\|^+_{s;p}$ is equivalent to

$$\delta^s \left\| \int_0^\infty \int_{-\infty}^\infty \frac{f(x+h) - f(x)}{|h|^{s+n}} dh_1\, dh' \right\|^+_p + \|f\|^+_p ,$$

where $\int_{-\infty}^\infty$ denotes the $n-1$ times repeated integral. The equivalence constants are independent of $\delta > 0$.

PROOF. The theorem follows essentially from Theorem 1.1. and the observation

$$F \int \frac{f(x+h) - f(x)}{|h|^{s+n}} dx = |\xi|^s \frac{\exp(i|\xi|^{-1} \sum \xi_j h_j) - 1}{|h|^{s+n}} Ff .$$

This is essentially a theorem by Stein [21] and Peetre (unpublished).

For $s \geqq 0$, set

$$M_s = \Big\{ \varphi;\ D_\alpha \varphi \in L_\infty,\ 0 \leqq |\alpha| \leqq [s], \text{ and in addition, if } s \text{ not an integer },$$

$$\int_0^\infty \int_{-\infty}^\infty \frac{\|D_\alpha \varphi_h - D_\alpha \varphi\|_\infty \, dh_1\, dh'}{|h|^{s-[s]+n}} < \infty, \qquad \text{for } |\alpha| \leqq [s] \Big\},$$

where $\varphi_h(x)=\varphi(x+h)$. In particular $M_s \supset \mathrm{Lip}_{s+\varepsilon}$, $\varepsilon>0$. Here $\mathrm{Lip}_{s+\varepsilon}$ denotes the space of bounded Lipschitz continuous functions. Define M_s for $s<0$ by $M_s=M_{-s}$. The elements in M_s take $H_{s;p}$ into $H_{s;p}$. We have

THEOREM 1.3. *If $f \in H_{s;p}$ and $\varphi \in M_s$ then $\varphi f \in H_{s;p}$ and*

$$\|\varphi f\|_{s;p} \leqq (K\|\varphi\|_\infty + o(1))\|f\|_{s;p} \quad \text{as } \delta \to +0 .$$

K is here independent of φ and $\delta \in (0,1]$ and $o(1)$ depends on φ.

Analogously we have,

THEOREM 1.3′. *If $f \in H^+_{s;p}$ and $\varphi \in M_s$, then $\varphi f \in H^+_{s;p}$ and*

$$\|\varphi f\|^+_{s;p} \leqq (K\|\varphi\|_\infty + o(1))\|f\|^+_{s;p} \quad \text{as } \delta \to +0 ,$$

where K is independent of φ and $\delta \in (0,1]$, and $o(1)$ depends on φ.

PROOF OF THEOREM 1.3. We first consider the case $0<s<1$, the case $s=0$ being evident. Then, according to Theorem 1.2.,

$$\begin{aligned}
\|\varphi f\|_{s;p} &\leqq K'\left\{\delta^s \left\| \int_0^\infty\int_{-\infty}^\infty \frac{\varphi(x+h)f(x+h)-\varphi(x)f(x)}{|h|^{s+n}}\,dh_1\,dh' \right\|_p + \|\varphi f\|_p\right\} \\
&\leqq K'\left\{\delta^s \left(\int\left|\varphi(x)\int_0^\infty\int_{-\infty}^\infty \frac{f(x+h)-f(x)}{|h|^{s+n}}\,dh_1\,dh'\right|^p dx\right)^{p^{-1}} + \right. \\
&\quad \left. + \delta^s \left(\int\left|\int_0^\infty\int_{-\infty}^\infty f(x+h)\,\frac{\varphi(x+h)-\varphi(x)}{|h|^{s+n}}\,dh_1\,dh'\right|^p dx\right)^{p^{-1}} + \|f\|_p\|\varphi\|_\infty\right\} \\
&\leqq K\left\{\|\varphi\|_\infty \|f\|_{s;p} + \|f\|_p\,\delta^s \int \frac{\|\varphi_h-\varphi\|_\infty}{|h|^{s+n}}\,dh\right\} \\
&\leqq K\left(\|\varphi\|_\infty + \delta^s \int \frac{\|\varphi_h-\varphi\|_\infty}{|h|^{s+n}}\,dh\right)\|f\|_{s;p} .
\end{aligned}$$

Thus

$$\|\varphi f\|_{s;p} \leqq (K\|\varphi\|_\infty + o(1))\|f\|_{s;p} \quad \text{as } \delta \to +0 .$$

For a general $s \geqq 0$ put $s=q+\eta$ with $0 \leqq \eta < 1$ and q an integer. By Theorem 1.1 the norms

$$\|f\|_{s;p} \quad \text{and} \quad \sum_{|\alpha|\leqq q} \delta^{|\alpha|}\|D_\alpha f\|_{\eta;p}$$

are equivalent. Hence we obtain

$$\begin{aligned}
\|\varphi f\|_{s;p} &\leqq C \sum_{|\alpha|\leqq q} \delta^{\alpha} \|D_{\alpha}\varphi f\|_{\eta;p} \\
&\leqq C \sum_{|\alpha|\leqq q} \sum_{\beta+\gamma=\alpha} \|\delta^{|\beta|} D_{\beta}\varphi\, \delta^{|\gamma|} D_{\gamma} f\|_{\eta;p} \\
&\leqq K'\left\{ \sum_{|\alpha|\leqq q} \sum_{\beta+\gamma=\alpha} \left(\delta^{|\beta|}\|D_{\beta}\varphi\|_{\infty} + o(1)\right) \|\delta^{|\gamma|} D_{\gamma} f\|_{\eta;p} \right\} \\
&\leqq (K\|\varphi\|_{\infty} + o(1))\|f\|_{s;p} \qquad \text{as } \delta \to +0\,.
\end{aligned}$$

For $s<0$ we get the inequality by duality.

Proof of Theorem 1.3′. $H^{+}_{s;p}$ is a quotient space of $H_{s;p}$.

Theorem 1.4. *If* $f \in H_{s;p}$, *and*

$$g = \begin{cases} f & \text{if } x_1 > 0\,, \\ 0 & \text{if } x_1 < 0\,, \end{cases}$$

then $g \in H_{s;p}$ *for* $-p'^{-1} < s < p^{-1}$. Further

$$\|g\|_{s;p} \leqq K\|f\|_{s;p}$$

with K *independent of* $\delta > 0$.

This theorem is by Shamir [19] to whom we refer for a proof.

To treat inhomogeneous boundary problems we need another type of spaces beside the $H_{s;p}$ ones. For $0<\theta<1$ let $W_{\theta;p}$ be the space of functions $u \in L^p$ such that

$$|u|_{\theta} = \delta^{\theta} \sum_{i=1}^{n} \left(\int_0^{\infty} t^{-\theta p} \left(\int_{R^n} |u(\ldots, x_i + t, \ldots) - u(x)|^p \, dx \right) t^{-1}\, dt \right)^{p^{-1}} < \infty$$

normed by

$$\langle u \rangle_{\theta;p} = \|u\|_p + |u|_{\theta}\,.$$

The space of functions u such that $D_{\beta}u \in L^p$, $0 \leqq |\beta| < m$, $D_{\beta}u \in W_{\theta;p}$, $|\beta| = m$, $0<\theta<1$, normed by

$$\langle u \rangle_{m+\theta;p} = \sum_{|\beta|<m} \delta^{|\beta|} \|D_{\beta}u\|_p + \sum_{|\beta|=m} \delta^m \langle D^{\beta}u\rangle_{\theta;p}$$

is denoted by $W_{m+\theta;p}$, and $W_{s;p} = (W_{-s;p'})'$ for $s<0$. An introduction to these spaces and a bibliography is found in e.g. [9].

We need in $W_{s;p}$ a theorem of the same type as Theorem 1.3. For $s>0$, s not integer, set

$$K_{s;p} = \left\{\varphi;\ D_\alpha\varphi \in L_\infty, |\alpha| \leqq [s],\ \sum_{|\alpha|\leqq]s[}\ \sum_{i=1}^{n} \left(\int_0^\infty t^{-p(s-[s])} \cdot \right.\right.$$
$$\left.\left. \cdot \|D_\alpha\varphi(\ldots,\cdot+t,\ldots)-D_\alpha\varphi\|_\infty^p\, t^{-1}\,dt\right)^{p^{-1}} < \infty\right\}$$

and for $s<0$ set $K_{s;p}=K_{-s;p'}$. Then we have

THEOREM 1.5. *If $f \in W_{s;p}$ and $\varphi \in K_{s;p}$ then $\varphi f \in W_{s;p}$ and there is a constant K independent of φ and $\delta \in (0,1]$ such that*

$$\langle \varphi f\rangle_{s;p} \leqq (K\|\varphi\|_\infty + o(1))\langle f\rangle_{s;p}$$

as $\delta \to +0$, with $o(1)$ depending on φ.

The proof is quite similar to the proof of Theorem 1.3. and will therefor be omitted.

Let $s>k+p^{-1}$, $s \neq p^{-1} \bmod 1$, be given. For $u \in H^+_{s;p}(R^n)$ it is possible to define $D_1{}^k u(0,.')$. The properties of the trace are described by the following

THEOREM 1.6. [5], [21]: *The mapping*

$$u \to \delta^k D_1{}^k u(0,.')$$

from $H^+_{s;p}(R^n)$ to $W_{s-k-p^{-1};p}\ (R^{n-1})$ is continuous, and

$$\delta^{k+p^{-1}}\langle D_1{}^k u(0,.')\rangle_{s-k-p^{-1};p} \leqq C\|u\|^+_{s;p}\,,$$

where C does not depend on $\delta>0$. Conversely there is a linear, continuous mapping

$$(\varphi_0,\ldots,\varphi_k) \to u$$

from $\prod_{j=0}^k W_{s-j-p^{-1};p}$ to a set $V_{s;p} \subset H^+_{s;p}$ such that

$$D_1{}^j u(0,.') = \varphi_j$$

and

$$\|u\|^+_{s;p} \leqq C\sum \delta^{j+p^{-1}}\langle D_1{}^j u(0,.')\rangle_{s-j-p^{-1};p}\,, \qquad u \in V_{s;p}\,.$$

Here C does not depend on $\delta>0$.

2. Dirichlet's problem in the constant coefficient case.

Let

$$A(D) = \sum_{|\alpha|=2m} a_\alpha D_\alpha$$

be a homogeneous properly elliptic differential operator with constant coefficients, i.e.

(i) for every $\xi \in R^n$, $\xi \neq 0$, we have $A(\xi) \neq 0$,
(ii) for every $\xi' \in R^{n-1}$, $\xi' \neq 0$, $A(\xi_1, \xi')$ has exactly m roots with positive imaginary part as a polynomial in ξ_1.

As well known (i) implies (ii) for $n > 2$.

REMARK. It follows that there is a $C > 0$, such that for every $\xi \in R^n$

$$C^{-1}|\xi|^{2m} \leqq A(\xi) \leqq C|\xi|^{2m}.$$

Moreover (cf. e.g. [4]), for all roots $\varrho(\xi')$ of $A(\xi_1, \xi') = 0$, it holds

$$|\varrho| \leqq K_1|\xi'| \quad \text{and} \quad |\operatorname{Im}\varrho| \geqq K_2|\xi'|,$$

for some K_1 and $K_2 > 0$.

We let $a_{1,\ldots,1} = 1$, which is no restriction. Let $\varrho_j^{(\pm)} = \varrho_j^{(\pm)}(\xi')$ be the roots of $A(\xi_1, \xi') = 0$ with positive (negative) imaginary part and set

$$A_{(\pm)} = \prod_{j=1}^{m} (\xi_1 - \varrho_j^{(\pm)}).$$

Set

$$\mathscr{M} = \{f;\ f \in S',\ Ff = 0 \text{ for } |\xi'| < \delta^{-1}\}.$$

LEMMA 2.1. *In* $\mathscr{M} \cap H_{m+s;p}$

$$\delta^m \|A_+ \Lambda^s f\|_p$$

is a norm equivalent to $\|f\|_{m+s;p}$.

LEMMA 2.1'. *In* $\mathscr{M} \cap H_{-m+s;p}$,

$$\delta^{-m}\|(A_-)^{-1}\Lambda^s f\|_p$$

is a norm, equivalent to $\|f\|_{-m+s;p}$.

PROOF. The proofs are analogous in the two cases, so we prove only Lemma 2.1. Take a function $\varphi_1(t) \in C^\infty(R^1_+)$, such that $\varphi_1(t) = 0$ for $t < \frac{1}{2}$, $\varphi_1(t) = 1$ for $t \geqq 1$. Let $\varphi(\xi) = \varphi_1(\delta|\xi|')$. Because of Theorem 1.1, we get

$$\delta^m \|A_+ \Lambda^s f\|_p = \left\| \frac{\delta^m \varphi A_+}{\Lambda^m} \Lambda^{m+s} f \right\|_p \leqq C\|f\|_{m+s;p},$$

where C does not depend on $\delta > 0$, and

$$\|f\|_{m+s;p} = \|\Lambda^{m+s} f\|_p = \left\| \frac{\varphi \Lambda^m}{\delta^m A_+} \delta^m A_+ \Lambda^s f \right\|_p \leqq C\delta^m \|A_+ \Lambda^s f\|_p,$$

where C does not depend on $\delta > 0$.

We shall consider the following form of Dirichlet's problem (with homogeneous boundary conditions) for the half-space $x_1 > 0$ when $-p'^{-1} < \varepsilon < p^{-1}$:

given $f \in H_{-m+\varepsilon;p}$ find $u \in H_{m+\varepsilon;p}$ such that

$$u = 0 \quad \text{for } x_1 < 0, \qquad Au = f \quad \text{for } x_1 > 0 . \tag{2.1}$$

Condition (2.1) is clearly equivalent to

$$\operatorname{supp} u \subset \{x_1 \geqq 0\}, \qquad \operatorname{supp}(Au - f) \subset \{x_1 \leqq 0\} . \tag{2.2}$$

Let

$$Ff(\xi) = 0 \quad \text{for } |\xi'| < \delta^{-1} . \tag{2.3}$$

By the Paley–Wiener theorem (as $(A_+(\xi))^{-1}$ is analytic in ξ_1 for $\operatorname{Im} \xi_1 < 0$ and ξ' fixed, and as $(A_-(\xi))^{-1}$ is analytic in ξ_1 for $\operatorname{Im} \xi_1 > 0$ and ξ' fixed, $|\xi'| \geqq \delta^{-1}$), we obtain from (2.2)

$$\begin{cases} A_+ u = 0 & \text{for } x_1 < 0 , \\ (A_+)^{-1} u = 0 & \text{for } x_1 < 0 , \\ A_+ u = (A_-)^{-1} f & \text{for } x_1 > 0 . \end{cases} \tag{2.4}$$

Let Y_+ be the characteristic function of the half-space $x_1 \geqq 0$. Then according to Theorem 1.4, for $-p'^{-1} < s < p^{-1}$ we have: if $g \in H_{s;p}$, then $Y_+ g \in H_{s;p}$. So evidently

$$Y_+ g \in H_{s;p} \cap \mathscr{M} \quad \text{if} \quad g \in H_{s;p} \cap \mathscr{M} .$$

Then

$$(A_+)^{-1} F(Y_+ g)(\xi) = 0 \quad \text{for } |\xi'| < \delta^{-1} .$$

Moreover from (2.4) we get

$$\operatorname{supp} (A_+)^{-1} Y_+ g \subset \{x_1 \geqq 0\}$$

and

$$A_+ u = Y_+ (A_-)^{-1} f .$$

Thus

$$u = (A_+)^{-1} Y_+ (A_-)^{-1} f \tag{2.5}$$

is the unique solution of (2.1) submitted to the restriction (2.3).

The solution (2.5) will now be used to obtain à priori estimates for elliptic boundary problems. Let

$$u \in H_{m+\varepsilon} \cap \mathscr{M}, u(x) = 0 \quad \text{for} \quad x_1 < 0,\ -p'^{-1} < \varepsilon < p^{-1} .$$

Because of Theorem 1.4 we immediately obtain

$$\|A_+ u\|_{\varepsilon;p} \leqq C \inf \|(A_-)^{-1} f\|_{\varepsilon;p} ,$$

where inf is taken over all $f \in H_{-m+\varepsilon;p}$ such that the restriction of f to

$t \geqq 0$ coincides with Au for $x_1 > 0$. According to Lemma 2.1 and 2.1′, we also have

(2.6) $$\|u\|^+_{m+s;p} \leqq C\delta^{2m}\|Au\|^+_{-m+s;p}$$

for these u. Here C is independent of $\delta > 0$. More generally we shall prove

LEMMA 2.2. *If* $u \in H_{s;p} \cap \mathscr{M}$, $D_1^j u = 0$ *for* $x_1 = 0$, $0 \leqq j \leqq m-1$, *then*

$$\|u\|^+_{s;p} \leqq C\delta^{2m}\|Au\|^+_{s-2m;p},$$

where C *does not depend on* $\delta > 0$, $s \neq p^{-1} \bmod 1$, $s > m - p'^{-1}$.

For the proof we need the following

LEMMA 2.3 *If* $u \in H_{s,r;p}$, *then*

$$\|u\|^+_{s,r;p} \leqq C(\delta^{2m}\|Au\|^+_{s-2m,r;p} + \|u\|^+_{s-1,r+1;p}),$$

where C *does not depend on* $\delta > 0$.

PROOF. Noting that

$$\Lambda^{2m}u = \sum_{j=0}^{2m} \binom{2m}{j} (\delta D_1)^j (i\Lambda_1)^{2m-j} u,$$

we obtain

(2.7) $$\|u\|^+_{s,r;p} = \|\Lambda^{2m}u\|^+_{s-2m,r;p} \leqq C\big(\|(\delta D_1)^{2m}u\|^+_{s-2m,r;p} + \|u\|^+_{s-1,r+1;p}\big).$$

But

(2.8) $$\begin{aligned}\delta^{2m}\|D_1^{2m}u\|^+_{s-2m,r;p} &\leqq \delta^{2m}\|Au\|^+_{s-2m,r;p} + C'\|u\|^+_{s-1,r+1;p} \\ &\leqq C(\delta^{2m}\|Au\|^+_{s-2m,r;p} + \|u\|^+_{s-1,r+1;p}).\end{aligned}$$

From (2.7) and (2.8) follows

(2.9) $$\|u\|^+_{s,r;p} \leqq C(\delta^{2m}\|Au\|^+_{s-2m,r;p} + \|u\|^+_{s-1,r+1;p}),$$

where C does not depend on $\delta > 0$.

PROOF OF LEMMA 2.2. If $u \in H_{s,r;p}$, then $\Lambda_1^r u \in H_{s;p}$.
From

$$Au = f \quad \text{for } x_1 > 0, \qquad u = 0 \quad \text{for } x_1 < 0$$

it follows by partial Fourier transformation that

$$A\Lambda_1^r u = \Lambda_1^r f \quad \text{for } x_1 > 0, \qquad \Lambda_1^r u = 0 \quad \text{for } x_1 < 0.$$

We get

$$F\Lambda_1^r u(\xi) = 0 \quad \text{for } |\xi'| < \delta^{-1},$$

if

$$Fu(\xi) = 0 \quad \text{for } |\xi'| < \delta^{-1},$$

and by (2.6) that

(2.10)
$$\|u\|^+_{m+\varepsilon, r; p} = \|\Lambda_1{}^r u\|^+_{m+\varepsilon; p} \leqq C\delta^{2m}\|A\Lambda_1{}^r u\|^+_{-m+\varepsilon, p} = C\delta^{2m}\|Au\|^+_{-m+\varepsilon, r; p},$$

if $-p'^{-1} < \varepsilon < p^{-1}$. Suppose that

$$\|u\|^+_{s-1, r; p} \leqq C\delta^{2m}\|Au\|^+_{s-1-2m, r; p}$$

for every real r. Then by Lemma 2.3

$$\begin{aligned}\|u\|_{s, r; p} &\leqq C\delta^{2m}(\|Au\|^+_{s-2m, r; p} + \|Au\|^+_{s-1-2m, r+1; p}) \\ &\leqq C'\delta^{2m}\|Au\|^+_{s-2m, r; p}.\end{aligned}$$

From this inequality and (2.10) the lemma follows.

We shall now extend the à priori estimates given by Lemma 2.2 by dropping the restriction (2.3) and the homogeneity. Thus we consider the boundary problem

(2.11) $\quad Au = f$ for $x_1 > 0$, $\quad D_1{}^j u = 0$ for $x_1 = 0, 0 \leqq j \leqq m-1$,

where

$$A(D) = \sum_{|\alpha| \leqq 2m} a_\alpha D_\alpha = A_0(D) + \sum_{|\alpha| < 2m} a_\alpha D_\alpha .$$

A_0 is homogeneous and a_α are constants.

THEOREM 2.1. *Let $s \not\equiv p^{-1} \bmod 1, s > m - p'^{-1}$. If u is a solution of* (2.11), *then*

$$\|u\|^+_{s; p} \leqq C(\delta^{2m}\|f\|^+_{s-2m; p} + \|u\|^+_{s-1; p}),$$

where C does not depend on $\delta \in (0, 1]$.

PROOF. Choose a function $\varphi_1 \in C_0^\infty(R^1)$ so that

$$\varphi_1(t) = 1 \text{ for } 0 \leqq t \leqq 1, \qquad \varphi_1(t) = 0 \text{ for } t \geqq 2,$$

and take $\varphi(\xi) = \varphi_1(\delta|\xi'|)$. Set $u = u_0 + u_1$, where

$$Fu_0(\xi) = \varphi(\xi)\, Fu(\xi), \qquad Fu_1(\xi) = Fu(\xi)\,(1 - \varphi(\xi)).$$

Then (e.g. by Theorem 1.1) it follows that $u_0 \in H_{s; p}$, $u_1 \in H_{s; p} \cap \mathcal{M}$ if $u \in H_{s; p}$. By Lemma 2.2 and Theorem 1.1,

$$\|u_1\|^+_{s; p} \leqq C\delta^{2m}\|A_0 u_1\|^+_{s-2m; p} \leqq C'\delta^{2m}\|A_0 u\|^+_{s-2m; p}$$

and

$$\|u_0\|^+_{s; p} = \left\|\frac{\Lambda_1 \varphi \Lambda^s u}{\Lambda_1}\right\|^+_p \leqq C\left\|\frac{\Lambda^s u}{\Lambda_1}\right\|^+_p = C\|u\|^+_{s, -1; p}.$$

But according to Lemma 2.3

$$\|u\|^{+}_{s,-1;p} \leqq C(\delta^{2m}\|A_0 u\|^{+}_{s-2m,-1;p} + \|u\|^{+}_{s-1;p}) .$$

Hence

$$\|u_0\|^{+}_{s;p} \leqq C(\delta^{2m}\|A_0 u\|^{+}_{s-2m,-1;p} + \|u\|^{+}_{s-1;p}) .$$

We have now shown

$$\|u\|^{+}_{s;p} \leqq C(\delta^{2m}\|A_0 u\|^{+}_{s-2m;p} + \|u\|^{+}_{s-1;p}) .$$

For $A(D)$ we then get

$$\begin{aligned}\|u\|^{+}_{s;p} &\leqq C(\delta^{2m}\|A_0 u\|^{+}_{s-2m;p} + \|u\|^{+}_{s-1;p}) \\ &\leqq C\left(\delta^{2m}\|Au\|^{+}_{s-2m;p} + \|u\|^{+}_{s-1;p} + \delta \sum_{|\alpha|<2m}^{2m} |a_\alpha|\, \|D_\alpha u\|^{+}_{s-2m;p}\right) \\ &\leqq C'(\delta^{2m}\|Au\|^{+}_{s-2m;p} + \|u\|^{+}_{s-1;p}) .\end{aligned}$$

The proof is complete.

Next we give an estimate of the corresponding inhomogeneous problem

(2.12) $\quad Au = f \quad \text{for } x_1 > 0, \qquad D_1{}^j u = \varphi_j \quad \text{for } x_1 = 0,\ 0 \leqq j \leqq m-1 .$

If $u \in H_{s;p}$, then by Theorem 1.6 there exists a function v satisfying

$$D_1{}^j u(0,.') = D_1{}^j v(0,.')$$

and such that, if $s > m - p'^{-1}$, $s \neq p^{-1} \bmod 1$

$$\|v\|^{+}_{s;p} \leqq C\left(\sum_{j=0}^{m-1} \delta^{j+p^{-1}} \langle D_1{}^j v(0,.')\rangle_{s-j-p^{-1};p}\right).$$

Then

$$\begin{aligned}\|u\|^{+}_{s;p} &\leqq \|u-v\|^{+}_{s;p} + \|v\|^{+}_{s;p} \\ &\leqq C'(\delta^{2m}\|A(u-v)\|^{+}_{s-2m;p} + \|u-v\|^{+}_{s-1;p} + \|v\|^{+}_{s;p}) \\ &\leqq C''(\delta^{2m}\|Au\|^{+}_{s-2m;p} + \|v\|^{+}_{s;p} + \|u\|^{+}_{s-1;p}) \\ &\leqq C\left(\delta^{2m}\|Au\|^{+}_{s-2m;p} + \|u\|^{+}_{s-1;p} + \sum_{j=0}^{m-1} \delta^{j+p^{-1}} \langle D_1{}^j v(0,.')\rangle_{s-j-p^{-1};p}\right),\end{aligned}$$

and we have proved

THEOREM 2.2. *If* $u \in H^{+}_{s;p}$, $s \neq p^{-1} \bmod 1$, $s > m - p'^{-1}$, *then*

$$\|u\|^{+}_{s;p} \leqq C\left(\delta^{2m}\|Au\|^{+}_{s-2m;p} + \|u\|^{+}_{s-1;p} + \sum_{j=0}^{m-1} \delta^{j+p^{-1}} \langle D_1{}^j u(0,.')\rangle_{s-j-p^{-1};p}\right).$$

3. General boundary problems in the constant coefficient case.

Let

$$A = A(D) = \sum_{|\alpha|=2m} a_\alpha D_\alpha$$

and let

$$B_j = B_j(D) = \sum_{|\alpha|=m_j} b_{j\alpha} D_\alpha ,$$

of order μ_j in D_1, $j=1,\ldots,m$, be homogeneous partial differential operators with constant coefficients. We consider the boundary value problem

$$(3.1) \qquad Au = f, \quad B_j u = g_j .$$

Denote by $F(\xi')=(F_{ij}(\xi'))$ the corresponding characteristic matrix (cf. e.g. [6]). The elements $F_{ij}(\xi')$ are infinitely differentiable for $\xi' \neq 0$ and

$$\sum_{j=0}^{m-1} F_{ij}(\xi') D_1{}^j u(0,\xi') = B_i(D_1,\xi') u(0,\xi')$$

if $Au=0$, $i=1,\ldots,m$. Further

$$|F_{ij}(\xi')|^2 \leqq C(1+|\xi'|^2)^{m_i-j} ,$$

and $F_{ij}(\xi')$ is homogeneous.

We say that (3.1) satisfies (E_0) if

(I) $A(D)$ is properly elliptic (cf. p. 67)

(II) $\det(F_{ij}(\xi')) \neq 0$ for ξ' with $|\xi'| \geqq L > 0$.

From (II) follows

$$\sum_{i=1}^{m} (1+|\xi'|^2)^{-m_i} \left| \sum_{j=1}^{m} F_{ij}(\xi')\lambda_j \right|^2 \geqq C \sum_{j=1}^{m} (1+|\xi'|^2)^{-j} |\lambda_j|^2$$

for every $\lambda \in C^m$. Instead of (II) we can use formally simpler conditions. See e.g. [1]. See also [12] and [13].

We take $L=1$ which is no essential restriction and take in this section

$$s > s_0 = \max(m-p'^{-1}, \mu_1+p^{-1}, \ldots, \mu_m+p^{-1}) .$$

From II follows the existence of F^{-1}. Let ${}^{co}F(\xi')=({}^{co}F_{ij}(\xi'))$ be the matrix of cofactors in $(F_{ij}(\xi'))=F'=$the transpose of F. That gives ${}^{co}F \cdot F = \det F \cdot I$ and with, for the moment, $\delta=1$

$$D_1{}^i u(0,\xi') = \sum \frac{{}^{co}F_{ij}(\xi')\, B_j u(0,\xi')}{\det F(\xi')}$$

if $u \in H^+_{s;p} \cap \mathcal{M}$ and $Au=0$. Thus, noting that

$$\frac{{}^{co}F_{ij}\, \Lambda_1{}^{m_j-i}}{\det F}$$

is a Fourier multiplier (cf. [6, Lemma 2.2]), we get

$$\sum_{i=0}^{m-1} \langle D_1{}^i u(0,.')\rangle_{s-i-p^{-1};p} = \sum_{i=0}^{m-1} \left\langle \sum_{j=1}^{m} \frac{\mathrm{co}F_{ij} B_j u(0,.')}{\det F} \right\rangle_{s-i-p^{-1};p}$$
$$\leqq \sum_{ij} \left\langle \frac{\mathrm{co}F_{ij}}{\det F} B_j u(0,.') \right\rangle_{s-i-p^{-1};p}$$
$$\leqq C \sum_j \langle B_j u(0,.')\rangle_{s-m_j-p^{-1};p} .$$

By homogeneity this gives for an arbitrary $\delta>0$ if $u \in H^+_{s;p} \cap \mathscr{M}$ and $Au=0$

$$\sum_{j=0}^{m-1} \delta^{j+p^{-1}} \langle D_1{}^j u(0,.')\rangle_{s-j-p^{-1};p} \leqq C \sum_{j=1}^{m} \delta^{m_j+p^{-1}} \langle B_j u(0,.')\rangle_{s-m_j-p^{-1};p} .$$

To treat an arbitrary element in $H^+_{s;p}$ we make the same decomposition $u=u_0+u_1$ as in the proof of Theorem 2.1. Set $f=Au_1$. Then

$$f \in H_{s-2m;p} \cap \mathscr{M} .$$

Define $v=\bar{F}(A^{-1}Ff)$. Then

$$v \in H_{s;p} \cap \mathscr{M} \quad \text{and} \quad Av = Au_1 = f \text{ if } x_1>0 .$$

Setting $w=u_1-v$ we have

$$w \in H^+_{s;p} \cap \mathscr{M} \quad \text{and} \quad Aw = 0 \text{ if } x_1>0 .$$

Then

$$\sum_0^{m-1} \delta^{j+p^{-1}} \langle D_1{}^j u_1(0,.')\rangle_{s-j-p^{-1};p} \leqq \sum_{j=0}^{m-1} \delta^{j+p^{-1}} \langle D_1{}^j w(0,.')\rangle_{s-j-p^{-1};p} + $$
$$+ \sum_{j=0}^{m-1} \delta^{j+p^{-1}} \langle D_1{}^j v(0,.')\rangle_{s-j-p^{-1};p}$$
$$\leqq C' \left(\sum_{j=1}^{m} \delta^{m_j+p^{-1}} \langle B_j w(0,.')\rangle_{s-m_j-p^{-1};p} + \|v\|^+_{s;p} \right)$$
$$\leqq C'' \left(\sum_{j=1}^{m} \delta^{m_j+p^{-1}} \langle B_j u_1(0,.')\rangle_{s-m_j-p^{-1};p} + \right.$$
$$\left. + \sum_{j=1}^{m} \delta^{m_j+p^{-1}} \langle B_j v(0,.')\rangle_{s-m_j-p^{-1};p} + \|v\|^+_{s;p} \right)$$
$$\leqq C''' \left(\sum_{j=1}^{m} \delta^{m_j+p^{-1}} \langle B_j u_1(0,.')\rangle_{s-m_j-p^{-1};p} + \|v\|^+_{s;p} \right)$$
$$\leqq C \left(\delta^{2m} \|Au_1\|^+_{s-2m;p} + \sum_{j=1}^{m} \delta^{m_j+p^{-1}} \langle B_j u_1(0,.')\rangle_{s-m_j-p^{-1};p} \right) .$$

We thus have

$$\sum_{0}^{m-1} \delta^{m_j+p^{-1}} \langle D_1{}^j u_1(0,.')\rangle_{s-j-p^{-1};p} \leqq C\left(\sum_{j=1}^{m} \delta^{m_j+p^{-1}} \langle B_j u_1(0,.')\rangle_{s-m_j-p^{-1};p} + \delta^{2m} \|Au_1\|^+_{s-2m;p}\right). \tag{3.2}$$

Moreover

$$\|u_0\|^+_{s;p} \leqq C(\delta^{2m}\|Au\|^+_{s-2m;p} + \|u\|^+_{s-1;p}) \tag{3.3}$$

according to the proof of Theorem 2.1 and

(3.4)
$$\|u_1\|^+_{s;p} \leqq C\Big(\delta^{2m}\|Au_1\|^+_{s-2m;p} + \|u_1\|^+_{s-1;p} + \sum_j \delta^{j+p^{-1}} \langle D_1{}^j u_1(0,.')\rangle_{s-j-p^{-1};p}\Big)$$

according to Theorem 2.2. By (3.2), (3.3) and (3.4) follows

$$\begin{aligned}\|u\|^+_{s;p} &\leqq \|u_1\|^+_{s;p} + \|u_0\|^+_{s;p} \\ &\leqq C\left(\delta^{2m}\|Au\|^+_{s-2m;p} + \|u\|^+_{s-1;p} + \sum_{j=1}^{m} \langle \delta^{m_j+p^{-1}} B_j u(0,.')\rangle_{s-m_j-p^{-1};p}\right).\end{aligned} \tag{3.5}$$

Next we consider (3.1), when A and B_j are inhomogeneous. Thus

$$A(D) = A_0(D) + \sum_{|\alpha|<2m} a_\alpha(D_\alpha)$$

and

$$B_j(D) = B_{j0}(D) + \sum_{|\alpha|<m_j} b_{j\alpha} D_\alpha$$

are differential operators with constant coefficients.

We say that (3.1) satisfies (E) if

(i) A_0 and B_{j0} satisfy (E_0), and

(ii) the order of D_1 in B_j equals the order of D_1 in B_{j0} (equals μ_j).

We notice that

$$\begin{aligned}\delta^{2m}\|A_0 u\|^+_{s-2m;p} &\leqq \delta^{2m}\|Au\|^+_{s-2m;p} + \sum_{|\alpha|<2m} \delta^{2m}|a_\alpha|\, \|D_\alpha u\|^+_{s-2m;p} \\ &\leqq C(\delta^{2m}\|Au\|^+_{s-2m;p} + \|u\|^+_{s-1;p}).\end{aligned} \tag{3.6}$$

In the same way

$$\begin{aligned}&\delta^{m_j+p^{-1}}\langle B_{j_0} u(0,.')\rangle_{s-m_j-p^{-1};p} \\ &\leqq \delta^{m_j+p^{-1}}\langle B_j u(0,.')\rangle_{s-m_j-p^{-1};p} + \sum_{|\alpha|<m_j} \delta^{m_j+p^{-1}}|b_{j\alpha}|\, \langle D_\alpha u(0,.')\rangle_{s-m_j-p^{-1};p} \\ &\leqq C\big(\delta^{m_j+p^{-1}}\langle B_j u(0,.')\rangle_{s-m_j-p^{-1};p} + \|u\|^+_{s,-1;p}\big) \\ &\leqq C\big(\delta^{2m}\|Au\|^+_{s-2m;p} + \|u\|^+_{s-1;p} + \delta^{m_j+p^{-1}}\langle B_j u(0,.')\rangle_{s-m_j-p^{-1};p}\big),\end{aligned} \tag{3.7}$$

the last inequality by Lemma 2.3. By (3.5), (3.6) and (3.7) we conclude for $A(D)$ and $B_j(D)$ satisfying (E)

THEOREM 3.1. *If* $u \in H^+_{s;p}$, $s \neq p^{-1} \bmod 1$, $s > s_0$ (p. 71) then

$$\|u\|^+_{s;p} \leqq C\left(\delta^{2m}\|Au\|^+_{s-2m;p} + \sum_{j=1}^{m} \delta^{m_j+p^{-1}}\langle B_j u(0,.')\rangle_{s-m_j-p^{-1};p} + \|u\|^+_{s-1;p}\right),$$

where C *does not depend on* $\delta \in (0,1]$.

REMARK. The restriction $s \neq p^{-1} \bmod 1$ is not essential and can be removed by complex interpolation (cf. e.g. [9, Part III]).

4. General boundary problems in the variable coefficient case.

Let

$$A(x,D) = \sum_{|\alpha| \leqq 2m} a_\alpha D_\alpha$$

be a differential operator with M_{s-2m}-coefficients and

$$B_j(x,D) = \sum_{|\alpha| \leqq m_j} b_{j\alpha} D_\alpha$$

be differential operators with $K_{s-m_j-p^{-1};p}$-coefficients, such that at every point x° with $x_1{}^\circ = 0$, condition (E) of Section 3 is satisfied for $A(x^\circ, D)$, $B_j(x^\circ, D)$ and such that at every point x° with $x_1{}^\circ > 0$, $A(x^\circ, D)$ is properly elliptic.

We can extend the inequality of Theorem 3.1 to such operators for functions u with compact support (cf. [15]). To do so, it is sufficient to treat operators $A(x,D)$ such that

$$a_\alpha(x) = a_\alpha(0) + c_\alpha(x), \qquad c_\alpha(x) \in M_{s-2m},$$

and operators $B_j(x,D)$ such that

$$b_{j\alpha}(x) = b_{j\alpha}(0) + d_{j\alpha}(x), \qquad d_{j\alpha}(x) \in K_{s-m_j-p^{-1};p}$$

and with

$$\varepsilon = \sum_{|\alpha| \leqq 2m} \sup_x |c_\alpha(x)| + \sum_{j=1}^{m} \sum_{|\alpha| \leqq m_j} \sup_x |d_{j\alpha}(x)| < \varepsilon_0,$$

ε_0 being a number specified below. For those we show

THEOREM 4.1. *If* $s > s_0$, $s \neq p^{-1} \bmod 1$, *then there is a number* $\varepsilon_0 > 0$ *and a number* $\delta_0 = \delta_0(\varepsilon) > 0$ *such that*

$$\|u\|^+_{s;p} \leqq C\left(\delta^{2m}\|Au\|^+_{s-2m;p} + \|u\|^+_{s-1;p} + \sum_{j=1}^{m} \delta^{m_j+p^{-1}}\langle B_j u(0,.')\rangle_{s-m_j-p^{-1};p}\right)$$

for every $\varepsilon < \varepsilon_0$, $\delta < \delta_0$ *and* $u \in H^+_{s;p}$.

REMARK. The δ-dependent $\|.\|^+_{s;p}$-norms are essential for this proof and can be found in Hörmander [8]. See also [14]. For constant coefficients we only need $\delta = 1$. From such estimates, the δ-dependent norms and inequalities can be obtained alternatively by homogeneity considerations.

PROOF OF THEOREM 4.1. According to Theorem 3.1

$$\|u\|^+_{s;p} \leqq C\Big(\delta^{2m}\|A(0,D)u\|^+_{s-2m;p} + \|u\|^+_{s-1;p} + + \sum_{j=1}^{m} \delta^{m_j+p^{-1}}\langle B_j(0,D)\,u(0,.')\rangle_{s-m_j-p^{-1};p}\Big) \tag{4.1}$$

and by Theorem 1.3′

$$\begin{aligned}(4.2)\qquad & \delta^{2m}\|A(0,D)u\|^+_{s-2m;p} \\ & \leqq \delta^{2m}\|A(x,D)\,u\|^+_{s-2m;p} + \delta^{2m}\sum_{|\alpha|\leqq 2m}\|c_\alpha(x)\,D_\alpha u\|^+_{s-2m;p} \\ & \leqq \delta^{2m}\|A(x,D)\,u\|^+_{s-2m;p} + \delta^{2m}K\Big(\sum_{|\alpha|\leqq 2m}\sup|c_\alpha| + o(1)\Big)\,\|D_\alpha u\|^+_{s-2m;p} \\ & \leqq \delta^{2m}\|A(x,D)u\|^+_{s-2m;p} + K\Big(\sum_{|\alpha|\leqq 2m}\sup|c_\alpha| + o(1)\Big)\,\|u\|^+_{s;p}\end{aligned}$$

for $\delta \in (0,1]$. By Theorem 1.5

$$\begin{aligned}(4.3)\qquad & \delta^{m_j+p^{-1}}\langle B_j u(0,D)u\rangle_{s-m_j-p^{-1};p} \\ & \leqq \delta^{m_j+p^{-1}}\langle B_j(x,D)u\rangle_{s-m_j-p^{-1};p} + \delta^{m_j+p^{-1}}\sum_{|\alpha|\leqq m_j}\langle d_{j\alpha}D_\alpha u\rangle_{s-m_j-p^{-1};p} \\ & \leqq \delta^{m_j+p^{-1}}\langle B_j(x,D)u\rangle_{s-m_j-p^{-1};p} + \\ & \qquad + \delta^{m_j+p^{-1}}\sum_{|\alpha|\leqq m_j} K(\sup|d_{j\alpha}| + o(1))\langle D_\alpha u\rangle_{s-m_j-p^{-1};p} \\ & \leqq \delta^{m_j+p^{-1}}\langle B_j(x,D)u\rangle_{s-m_j-p^{-1};p} + \sum_{|\alpha|\leqq m_j} K(\sup|d_{j\alpha}| + o(1))\,\|u\|^+_{s;p}\,.\end{aligned}$$

The last inequality followed from Theorem 1.6 and the following observation: If $D_\alpha u \in W_{s;p}$, then $u \in W_{s+|\alpha|;p}$, $2 \leqq \alpha_\nu \leqq n$, $1 \leqq \nu \leqq |\alpha|$ (cf. [10]). By (4.1), (4.2) and (4.3), choosing first ε and then δ small enough, the desired inequality follows.

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INSTITUTE OF MATHEMATICS
LUND, SWEDEN