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Titel: L-functions at s=0 and Galois-modules

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L-FUNCTIONS AT S=0 AND GALOIS MODULES

by

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This is a very partial report on new connections between Galois Module Structure of units, or **S**-units, on the one hand and integral variants of definitions, problems and conjectures related to the Stark conjecture on **L**-functions at **s=0.** Our emphasis is totally on the latter aspect, with invariants of multiplicative Galois modules brought in only to show that they also give insight into "L-value-regulator-quotients". The centre piece is the introduction of a new type of arithmetic regulator in terms of pairings.

<u>Notations.</u> As usual $\mathbb{Z},\mathbb{Q},\mathbb{C}$, are the ring of integers, the field of rationals and the field of complex numbers. The algebraic closure of a field **K** is K^c, and the Galois Group of K^c/K is Ω_K. If **N** is a number field its ring of integers is $\mathcal{O}_N$.

1. Pairings. Field theoretic version.

Our aim here is to give an alternative definition of the regulator à la Tate and of the Stark conjecture for it (cf. [Ta]), in terms of pairings. This will enable us to compare it to the subsequent integral version (see [F1], Appendix and §1).

Throughout Γ is a finite group, M_1 and M_2 are $\mathbb{Z}\Gamma$-lattices, Γ acting on M_1 from the right, on M_2 from the left. We consider a pairing

$$(1,1) \qquad b: M_1 \times M_2 \to \mathbb{C}$$

which admits Γ, i.e., so that

$$(1.2) \qquad b(x\gamma, y) = b(x, \gamma y), \ \forall\ x \in M_1,\ y \in M_2,\ \gamma \in \Gamma.$$

We assume the extended pairing

$$b \otimes_{\mathbb{Z}} \mathbb{C} : M_1 \otimes_{\mathbb{Z}} \mathbb{C} \times M_2 \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}$$

to be non-degenerate.

Let F be a subfield of $\mathbb{C}$, of finite degree over $\mathbb{Q}$. We shall often drop the tensor product notation, i.e., write MF in place of $M \otimes_{\mathbb{Z}} F$, when there is no danger of misunderstanding, and we shall still use the symbol b for the extended pairing

$$M_1 F \times M_2 F \to \mathbb{C},$$

$$b(x \otimes c_1,\ y \otimes c_2) = c_1 c_2\ b(x,y).$$

Let now V be a right $F\Gamma$-module (always finite dimensional over F). Write

$$\hat{V} = \mathrm{Hom}_F(V,F), \text{ a left } F\Gamma\text{-module}$$

with

$$(1.3) \qquad \langle\ ,\ \rangle : V \times \hat{V} \to F$$

as defining pairing. We also have a homomophism

$$(\ , \ ) : \hat{V} \otimes_F V \to F\Gamma$$

of $F\Gamma$-bimodules, given by

$$(1.4) \qquad (u,v) = \sum_{\gamma} \langle v\gamma^{-1}, u \rangle \gamma, \ v \in V, \ u \in \hat{V}.$$

Moreover we get a further isomorphism

$$(1.5) \qquad [\,,] : M_1 F \otimes_{F\Gamma} \hat{V} \simeq \mathrm{Hom}_{F\Gamma}(V, M_1 F)$$

where

$$[x,u](v) = x(u,v).$$

We end up with a commutative diagram

$$\begin{tikzcd}
M_1 F \otimes_{F\Gamma} \hat{V} \times V \otimes_{F\Gamma} M_2 F \simeq \mathrm{Hom}_{F\Gamma}(V, M_1 F) \times V \otimes_{F\Gamma} M_2 F \arrow[dr, "b_V"] & \\
& \mathbb{C}
\end{tikzcd} \qquad (1.6)$$

identifying the two modules in the top row, and with b_V as the derived pairing, given e.g., by

$$(1.6a) \qquad b_V(f, v \otimes y) = b(f(v), y)$$

for $f \in \mathrm{Hom}_{F\Gamma}(V, M_1 F)$, $v \in V$, $y \in M_2$.

Suppose that $\mathrm{Hom}_{F\Gamma}(V, M_1 F) \neq 0$. Then the pairing b_V is again non-degenerate. With $\{h_i\}$, $\{z_i\}$ F-bases of $\mathrm{Hom}_{F\Gamma}(V, M_1 F)$ and of $V \otimes_{F\Gamma} M_2 F$ respectively we form the discriminant

$$(1.7) \qquad \mathrm{Det}(b_V\ (h_i,\ z_j)) \in \mathbb{C}^* .$$

All these discriminants will lie in a unique coset

$$(1.8) \qquad D_{b_V} = D_\rho \in \mathbb{C}^*/F^*$$

where $\rho : \Gamma \to GL_F(V)$ is the associated representation of Γ, and

$$(1.9) \qquad D_{\rho+\tau} = D_\rho D_\tau .$$

We shall also view the elements of $\mathbb{C}^\times/F^\times$ as the one dimensional F-subspaces of $\mathbb{C}$. If $\mathrm{Hom}_{F\Gamma}(V, M_1\ F) = 0$ we put $D_\rho = F$.

Suppose now that $F/\mathbb{Q}$ is Galois. Let $\omega \in \Omega_{\mathbb{Q}}$. An ω-semilinear isomorphism $g : U \simeq W$ of F-vector spaces (of $F\Gamma$-modules) is an isomorphism of Abelian groups with

$$g(uc) = g(u)c^\omega \text{ for } u \in U,\ c \in F$$

$$(\text{and } g(u\gamma) = g(u)\gamma \text{ for } u \in U,\ \gamma \in \Gamma).$$

<u>Example</u> : The action of ω on $M_1 \otimes_{\mathbb{Z}} F$ via its action on F defines an ω-semilinear automorphism of this $F\Gamma$-module again denoted by ω : $(x \otimes c)^\omega = x \otimes c^\omega$.

Given V as above there is a pair (g, V^ω), where V^ω is an $F\Gamma$-module,

$$g = g_{V,\omega} : V \simeq V^\omega$$

an ω-semilinear isomorphism of $F\Gamma$-modules, all unique to within (ordinary, i.e.,1-) isomorphism of $F\Gamma$-modules. Given (g, V^ω) there is then a unique ω-semilinear isomorphism

$$\hat{g} : \hat{V} \simeq \widehat{V^\omega}\ (=\hat{V}^\omega)$$

of $F\Gamma$-modules, so that for all $v \in V$, $u \in \hat{V}$,

$$\langle gv, \hat{g}u\rangle = \langle v, u\rangle^\omega \qquad (\text{cf.}(1.3)),$$

hence

$$(\hat{g}u, gv) = (u, v)^\omega \qquad (\text{cf.}(1.4)).$$

We then obtain an ω-semilinear isomorphism of F-vector spaces (see (1.6))

$$(1.10)\quad \begin{cases} \tilde{g}: M_1 F \otimes_{F\Gamma} \hat{V} \otimes_F V \otimes_{F\Gamma} M_2 F \simeq M_1 F \otimes_{F\Gamma} \widehat{V^\omega} \otimes_F V^\omega \otimes_{F\Gamma} M_2 F \\ \tilde{g}(x \otimes u \otimes v \otimes y) = (x^\omega \otimes \hat{g}u \otimes gv \otimes y^\omega) \\ x \in M_1 F,\ y \in M_2 F,\ u \in \hat{V},\ v \in V. \end{cases}$$

If now $g_1 : V \simeq V_1$ is a second ω-semilinear isomorphism of $F\Gamma$-modules then we have

$$(1.11))\qquad b_{V^\omega} . \tilde{g} = b_{V_1} . \tilde{g}_1 .$$

This is proved by first considering the case $\omega = 1$. As a consequence we get the following result :

(1.12) <u>There is a unique</u> ω-<u>semilinear isomorphism</u>

$$t_{\rho,\omega} : D_\rho \simeq D_{\rho^\omega}$$

<u>of</u> F-<u>vector spaces</u>, <u>depending only on</u> ρ <u>and</u> ω, <u>so that for any</u> g <u>as above</u>, <u>any</u> F-<u>basis</u> $\{h_i\}$ <u>of</u> $\mathrm{Hom}_{F\Gamma}(V, M_1 F)$, <u>and any</u> F-<u>basis</u> $\{z_i\}$ <u>of</u> $V \otimes_{F\Gamma} M_2 F$ <u>we have</u>

$$t_{\rho,\omega}(\mathrm{Det}(b_V(h_i, z_j))) = \mathrm{Det}(b_{V^\omega}(\tilde{g}(h_i, z_j))).$$

(Remark : The given ω-semilinear isomorphism $M_1 F \otimes_{F\Gamma} \hat{V} \simeq M_1 F \otimes_{F\Gamma} \widehat{V^\omega}$ is, under the isomorphism [,] of (1.5), translated into a ω-semilinear isomorphism

$$(1.13)\quad \begin{cases} \mathrm{Hom}_{F\Gamma}(V, M_1 F) \simeq \mathrm{Hom}_{F\Gamma}(V^\omega, M_1 F) \\ f \mapsto f^\omega \\ f^\omega(gv) = f(v)^\omega . \end{cases}$$

It follows from (1.12) that

$$(1.14)\qquad t_{\rho^\omega,\sigma} \circ t_{\rho,\omega} = t_{\rho,\omega\sigma},$$

and of course

$$(1.14a)\qquad t_{\rho,\omega} = 1 \text{ if } \omega \in \Omega_F$$

Let E be an extension field of F, of finite degree and still Galois over $\mathbb{Q}$. Indicating the underlying field by a subscript we have now

$$t_{E,\rho,\omega} = t_{F,\rho,\omega} \otimes_F E. \tag{1.15}$$

Next, given ρ, V as before, the stabiliser of ρ in $\Omega_{\mathbb{Q}}$ is of form Ω_H, where $H \subset F$ and $H/\mathbb{Q}$ is Abelian. ρ need not be defined over H, but for all $\omega \in \Omega_H$ the map $t_{\rho,\omega} = t_{F,\rho,\omega}$ is an ω-semilinear automorphism of $D_\rho = D_{F,\rho}$ and therefore

$$D_{F,\rho} = D_{H,\rho} \otimes_H F \tag{1.16}$$

with $D_{H,\rho}$ a one-dimensional H-vector space. Moreover

$$(w \otimes c)\ t_{F,\rho,\omega} = w \otimes c^{\omega},\ w \in D_{H,\rho},\ c \in F. \tag{1.16a}$$

We call $H=H_\rho$ the <u>field of definition</u> of D_ρ.

Examples. 1) M_1 a given $\mathbb{Z}\Gamma$-lattice, $M_2 = \mathrm{Hom}_{\mathbb{Z}}(M_1,\mathbb{Z})$ its dual (<u>canonical pairing</u>).

2) X a $\mathbb{Q}$-vector space with a non-degenerate quadratic form. Γ acts on X by orthogonal transformation. To translate into terms of our theory view this as a pairing of X and its opposite module $\overline{X}$, admitting Γ. M_1 any $\mathbb{Z}\Gamma$-lattice spanning X, $M_2 = \overline{M_1}$ its opposite.

In both these examples $D_{F,\rho} = F$. The lattices do at this stage not enter at all. In the next three examples K is an algebraic number field,

$$q:\ \Omega_K \to \Gamma \tag{1.17}$$

a continuous surjection, with kernel Ω_N. Thus

$$\Gamma \simeq \mathrm{Gal}(N/K). \tag{1.17a}$$

3) The ring of integers $\mathcal{O}_N$ of N is a right $\mathbb{Z}\Gamma$-module. We let $S=S(N)$ be the set of embeddings $s: N \to \mathbb{C}$. The free Abelian group $\mathbb{Z}S$ on S is

viewed as a left $\mathbb{Z}\Gamma$-module and the "additive pairing" is

$$b_N : \mathcal{O}_N \times \mathbb{Z}S \to \mathbb{C}$$
$$b_N(x,s) = x^s,\ x \in \mathcal{O}_N,\ s \in \mathbb{Z}S.$$

This leads to the theory of generalized resolvents (cf.[Fl]).

4) This is the example with which we are concerned with here. S is a finite set of places of K including the infinite ones, S(N) the set of places of N above those in S. The group of S(N)-units in N modulo roots of unity is $\mathcal{U}_{S,N}$, viewed as a right $\mathbb{Z}\Gamma$-module. $\mathbb{Z}$ S(N) is the free Abelian group on S(N), viewed as a left $\mathbb{Z}\Gamma$-module, and

$$\mathcal{G}_{S,N} = \mathbb{Z}S(N)/(\Sigma s), \tag{1.18}$$

the quotient of $\mathbb{Z}S(N)$ modulo the group generated by the sum of the $s \in S(N)$. The pairing is

$$\begin{cases} b_N : \mathcal{U}_{S,N} \times \mathcal{G}_{S,N} \to \mathbb{C} \\ b_N(u,s) = \log\|u\|_s, \quad u \in \mathcal{U}_{S,N},\ s \in S(N). \end{cases} \tag{1.19}$$

5) There is a variant of this, where we start with a finite set T of finite places of K and get analogously a pairing

$$\mathcal{I}_{T,N} \times \mathbb{Z}T(N) \to \mathbb{C}$$

where $\mathcal{I}_{T,N}$ is the group of fractional ideals in N whose support lies in T(N).

The basis discriminants with respect to the pairings $b_{N,v} = b_v$ for $b_N = b$ given by (1.19) (see (1.7)) are the <u>element regulators</u>. The regulators à la Tate (cf. [Ta] I §4) are of this form, as we shall now show.

Let

$$\mathcal{H}_{S,N} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{G}_{S,N},\mathbb{Z}). \tag{1.20}$$

In other words, viewing now $\mathbb{Z}S(N)$ as a right $\mathbb{Z}\Gamma$-module,

$$\mathcal{H}_{S,N} = \mathrm{Ker}[\mathbb{Z}S(N) \to \mathbb{Z}] \tag{1.20a}$$

where the map is $s \mapsto 1$, for all $s \in S(N)$. We then have isomorphisms

$$g: \mathcal{H}_{S,N}\mathbb{Q} \simeq \mathcal{U}_{S,N}\mathbb{Q} \tag{1.21}$$

of $\mathbb{Q}\Gamma$-modules, and the Tate regulator depends on the choice of g. We let

$$c : \mathcal{H}_{S,N} \times \mathcal{G}_{S,N} \to \mathbb{Z} \subset \mathbb{C}$$

denote the canonical pairing. As before we get a derived pairing

$$c_V : \mathrm{Hom}_{F\Gamma}(V,\mathcal{H}_{S,N}F) \times V \otimes_{F\Gamma} \mathcal{G}_{S,N} F \to F \subset \mathbb{C}.$$

Choose dual bases $\{h_i\}$ of $\mathrm{Hom}_{F\Gamma}(V,\mathcal{H}_{S,N}F)$, $\{z_j\}$ of $V \otimes_{F\Gamma}\mathcal{G}_{S,N}F$, i.e., so that

$$\mathrm{Det}\ c_V(h_i,z_j) = 1. \tag{1.22}$$

The map

$$\ell : \mathcal{U}_{S,N} \to \mathcal{H}_{S,N}$$

$$\ell(u) = \sum_s s \log \|u\|_s$$

extends to an isomorphism

$$\ell : \mathcal{U}_{S,N}\mathbb{C} \simeq \mathcal{H}_{S,N}\mathbb{C}.$$

By (1.22)

$$\mathrm{Det}\ c_V(\log.h_i,z_j) = \mathrm{Det}\ \mathrm{Hom}_{F\Gamma}(V,\log),$$

and -disregarding the different notation- this is the Tate regulator. But in terms of our pairing $b = b_N$ of (1.19)

$$c_V(\log.h,z) = b_V(g.h,z)$$

whence

$$(1.23)\quad \mathrm{Det}\ \mathrm{Hom}_{F\Gamma}(V,\log) = \mathrm{Det}\,b_V(g.h_i, z_j)$$

Our discussion also shows that the Tate regulator, for given V, is unique in $\mathbb{C}^\times$ modulo the group of determinants $\mathrm{Det}(\mathrm{Hom}_{F\Gamma}(V,fF))$ for $f \in \mathrm{Aut}_{\mathbb{Q}\Gamma}(\mathscr{U}\mathbb{Q}), \mathscr{U} = \mathscr{U}_{S,N}$.

2. Parameters. The Stark conjecture.

Let **b** be a pairing as in § 1 and let **G** be a set of (equivalence classes of) representations ρ of Γ over $\mathbb{C}$, closed under addition and under the action of $\Omega_{\mathbb{Q}}$. A parameter with respect to **b** for **G** is a map

$$\mathbf{k} : \mathbf{G} \to \mathbb{C}^{\times}$$

so that

(2.1) (i) $\mathbf{k}(\rho) \in D_{F,\rho}$ (so $\mathbf{k}(\rho)\, F = D_{F,\rho}$)

(where of course ρ will be realisable over **F**).

(2.1) (ii) $t_{\rho,\omega}(\mathbf{k}(\rho)) = \mathbf{k}(\rho^{\omega})$

(whence $\mathbf{k}(\rho) \in D_{H,\rho}$, **H** the field of definition of D_{ρ}).

(2.1) (iii) $\mathbf{k}(\rho_1 + \rho_2) = \mathbf{k}(\rho_1)\, \mathbf{k}(\rho_2)$.

We immediately have

(2.2) Let **k** be a parameter with respect to **b** for **G**. Then a map $\mathbf{k}_1 : \mathbf{G} \to \mathbb{C}^{\times}$ is a parameter with respect to **b** for **G** if and only if the map $\mathbf{a}=\mathbf{k}_1.\mathbf{k}^{-1}$ $(\rho \mapsto \mathbf{k}_1(\rho)\mathbf{k}(\rho)^{-1})$ satisfies

(2.2(i)) $\mathbf{a}(\rho) \in (\mathbb{Q}^{c})^{\times}$ for all $\rho \in \mathbf{G}$,

(2.2.(ii)) $\mathbf{a}(\rho^{\omega}) = \mathbf{a}(\rho)^{\omega}$ for all $\rho \in \mathbf{G}$, all $\omega \in \Omega_{\mathbb{Q}}$,

(2.2.(iii)) $\mathbf{a}(\rho_1 + \rho_2) = \mathbf{a}(\rho_1)\mathbf{a}(\rho_2)$.

In our principal example 4, fix the map **g** of (1.21) once and for all. Let **G** be the set of all representations of Γ. Put (cf. (1.23))

$$(2.3) \qquad k(\rho) = \mathrm{Det}\ \mathrm{Hom}_{F\Gamma}(V, \log)$$

where ρ corresponds to the $F\Gamma$-module V. Then indeed k is a parameter with respect to the pairing $b = b_N$ of (1.19).

Let now $L_S(s,\rho.q)$ be the truncated Artin L-function for the representation

$$\rho.q : \Omega_K \to GL(V)$$

with the Euler factors for S omitted. Here the set S is the same as that in the definition (1.19) and q is as in (1.17). With N as in (1.17a) we write

$$L_S(\rho, N)$$

for the leading non zero coefficient in the Taylor expansion of $L_S(s,\rho.q)$ at $s = 0$. Let $\hat{\rho}$ be the contragredient of ρ. The Stark conjecture in the Tate formulation, translated by (2.2), (2.3) then states

(2.4) $\rho \mapsto L_S(\hat{\rho}, N)$ <u>is a parameter with respect to</u> b,

(for the set of all representations).

We know classically :

(2.4(i)) (2.4) <u>holds if</u> $K = \mathbb{Q}$, Γ <u>is abelian and</u> N <u>real</u>.

Also

(2.4.(ii)) (2.4) <u>holds if</u> G <u>is the set of all representations</u> ρ <u>whose character is rational valued</u> (cf.[Ta] II § 7).

In the case of the pairing of example 5 in § 1 we have a parameter, given by the leading coefficient in the Taylor expansion of $\prod_t L_t(s,\hat{\rho}.q)^{-1}$ at $s = 0$, where L_t is the local L-function at the finite place t.

We briefly return to the additive case (Example 3 in § 1). The role of $\mathcal{H}_{S,N}$, in the above, is now taken e.g., by the $\mathbb{Z}\Gamma$-lattice.

$\mathcal{O}_K\Gamma$ leading to "norm resolvents" (cf [F1] §9, [F2] I §4), the additive analogue to the Tate regulator. By a basic theorem of the additive theory (cf. [F2] Theorem 20) the Galois Gauss sums yield a parameter

$$\rho \mapsto \tau(\mathrm{ind}\ \rho.q)$$

(ind=induction from Ω_K to $\Omega_{\mathbb{Q}}$) (compare [F1] § 3).

3. Pairings. (Integral version)

We return to the general theory of § 1. Let now P be a $\mathcal{O}_F\Gamma$-lattice spanning the $F\Gamma$-module V. Then $\mathrm{Hom}_{\mathcal{O}_F\Gamma}(P, M_1\mathcal{O}_F)$ is an $\mathcal{O}_F$-lattice spanning $\mathrm{Hom}_{F\Gamma}(V, M_1F)$ and the "torsion free tensor product"

$$P_{\mathcal{O}_F}\cdot_\Gamma M_2\mathcal{O}_F = \mathrm{Im}[P\otimes_{\mathcal{O}_F\Gamma} M_2\mathcal{O}_F \to V\otimes_{F\Gamma} M_2F]$$

is an $\mathcal{O}_F$-lattice spanning $V \otimes_{F\Gamma} M_2\ F$. If both these $\mathcal{O}_F$-lattices are free then we can define a discriminant in terms of $\mathcal{O}_F$-bases. In the general case we proceed via localisation. We denote this discriminant by

$$\mathcal{R}(P, M_1, M_2).$$

Thus for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$

$$(3.1) \qquad \mathcal{R}(P, M_1, M_2)_\mathfrak{p} = \mathcal{O}_{F,\mathfrak{p}}\ \mathrm{Det}\ b_V(h_{i,\mathfrak{p}}, z_{j,\mathfrak{p}}),$$

where $\{h_{i,\mathfrak{p}}\}$ and $\{z_{j,\mathfrak{p}}\}$ are $\mathcal{O}_{F,\mathfrak{p}}$-bases of $\mathrm{Hom}_{\mathcal{O}_F\Gamma}(P, M_1\ \mathcal{O}_F)_\mathfrak{p}$ and of $(P_{\mathcal{O}_F}\cdot_\Gamma M_2\ \mathcal{O}_F)_\mathfrak{p}$ respectively. If $\mathrm{Hom}_{F\Gamma}(V, M_1\ F) = 0$ we again put $\mathcal{R} = (1)$. Thus $\mathcal{R}(P, M_1, M_2)$ is a rank one $\mathcal{O}_F$-module spanning $D_{F,\rho}$ over F (where $\rho : \Gamma \to GL(V)$ as before). The rank one $\mathcal{O}_F$-lattices embedded in $\mathbb{C}$ form an Abelian group, via multiplication in $\mathbb{C}^\times$. Then

$$(3.2) \qquad \mathcal{R}(P_1 \oplus P_2, M_1, M_2) = \mathcal{R}(P_1, M_1, M_2).\mathcal{R}(P_2, M_1, M_2).$$

This group structure is also used to evaluate module indices, thus of importance in applications.

With P,V and ρ as before, and with $\omega \in \Omega_\mathbb{Q}$, there is a semilinear isomorphism g: $P \simeq P^\omega$ of $\mathcal{O}_F\Gamma$-lattices extending to $V \simeq V^\omega$. One now verifies easily that

$$(3.3) \qquad \mathcal{R}(P^\omega, M_1, M_2) = t_{\rho,\omega}(\mathcal{R}(P, M_1, M_2)).$$

Now let $\mathbf{k}$ be a parameter with respect to our given pairing $\mathbf{b}$ for a set G (as in §2). Then we have (assuming $F/\mathbb{Q}$ to be Galois)

(3.4) <u>For all</u> $\omega \in \Omega_{\mathbb{Q}}$,

$$k(\rho^{\omega}) \cdot \mathcal{R}(P^{\omega},M_1,M_2)^{-1}$$

<u>is a fractional ideal of</u> $\mathcal{O}_F$, <u>and</u>

$$k(\rho^{\omega}) \cdot \mathcal{R}(P^{\omega},M_1,M_2)^{-1} = [k(\rho)\ \mathcal{R}(P,M_1,M_2)^{-1}]^{\omega}.$$

<u>Remark 1</u>.- We had chosen F so that P was defined over F. The fractional ideals in (3.4) may in fact "belong" to a smaller field. As fractional ideals embed on extension of number fields, this makes sense.

<u>Remark 2</u>.- There are in fact two basic constructions, the one we have used here ; we then used the symbol $\mathcal{R}^{\times}$ in [F1]. A second construction uses $M_1 \otimes_{\mathcal{O}_F} \hat{P}$ ($\hat{P}=\mathrm{Hom}_{\mathcal{O}_F}(P,\mathcal{O}_F)$ in place of $\mathrm{Hom}_{\mathcal{O}_F\Gamma}(P,M_1 \otimes \mathcal{O}_F)$ (see [F1] Appendix).

<u>Remark 3</u>.- If, say, Γ is Abelian we can associate an invariant $\mathcal{R}(\rho,M_1,M_2)$ with every representation ρ of Γ over F. If first ρ is irreducible over F, let $\mathcal{M}_{\rho}$ be the component of the maximal order of $F\Gamma$ corresponding to ρ, and define

$$(3.5) \qquad \mathcal{R}(\rho,M_1,M_2) = \mathcal{R}(\mathcal{M}_{\rho},M_1,M_2)$$

and extend the definition by

$$(3.5a) \quad \mathcal{R}(\Sigma n_i \rho_i,M_1,M_2) = \prod_i \mathcal{R}(\rho_i,M_1,M_2)^{n_i}.$$

The dependence on F is only superficial : if ρ is defined over F and so over an extension E of F then in the obvious notation

$$(3.6) \qquad \mathcal{R}(\rho,M_1,M_2)_F \otimes \mathcal{O}_E = \mathcal{R}(\rho,M_1,M_2)_E.$$

(This is not quite as obvious as it looks!).

Still with Γ Abelian let k be a parameter with respect to b for the set of all representations. Write Γ^{+} for the character group $\mathrm{Hom}(\Gamma,\mathbb{C}^{\times})$. We then have

(3.7) <u>For all</u> $\Theta \in \Gamma^+$ <u>the quotient</u>

$$k(\Theta) \cdot \mathfrak{A}(\Theta, M_1, M_2)^{-1}$$

<u>is a fractional ideal in the field</u> $\mathbb{Q}(\Theta)$ <u>of values of</u> Θ, <u>and for all</u> $\omega \in \mathrm{Gal}(\mathbb{Q}(\Theta)/\mathbb{Q})$ <u>we have</u>

$$k(\Theta^\omega) \cdot \mathfrak{A}(\Theta^\omega, M_1, M_2)^{-1}$$

$$= [k(\Theta) \cdot \mathfrak{A}(\Theta, M_1, M_2]^{-1}]^\omega .$$

4.- Integral properties.

Based on the preceding §3, we now have regulators which give rise to natural problems and theorems of integrality and divisibility. We restrict ourselves to Abelian groups Γ. We assume we are given a surjection **(1.17)**, with **N** as in **(1.17a)** and we are in the situation of example 4 in §1 (cf. **(1.18)**, **(1.19)**). We thus consider the quotient

$$(4.1) \qquad L_S(\hat{\rho},N) \cdot \mathcal{R}(\rho,\mathcal{U}_{S,N}, \mathcal{G}_{S,N})^{-1}$$

(see **(3.5)** for the definition). If **(2.4)** holds then in particular we have here fractional ideals in the appropriate number-field. We shall consider then the two cases **(2.4(i))** and **(2.4(ii))** separately.

First **(2.4.(i))**, i.e., $K = \mathbb{Q}$ and **N** is a real Abelian field. Then **(3.7)** holds with $M_1=\mathcal{U}_{S,N}$, $M_2=\mathcal{G}_{S,N}$. We shall get two theorems on the arithmetic nature of this L-value-regulator quotient.

THEOREM A.- *Let* **S** *consist of the infinite prime only. Then for all* $\Theta \in \Gamma^+$, *different from the identity character* ε,

$$L_S(\Theta^{-1},N) \cdot \mathcal{R}(\Theta,\mathcal{U}_{S,N},\mathcal{G}_{S,N})^{-1}$$

is an integral ideal.

(cf [F1] Theorem 2).

THEOREM B.- *Suppose* **S** *is sufficiently large so that the* **S**-*class number* $h_{S,N} = 1$. *Then for all* $\Theta \in \Gamma^+$, $\Theta \neq \varepsilon$ *any prime ideal ocurring in the decomposition of*

$$L_S(\Theta^{-1},N)\ \mathcal{R}(\Theta,\mathcal{U}_{S,N},\mathcal{G}_{S,N})^{-1}$$

divides $\mathrm{ord}(\Gamma)$.

(cf [F1] Theorem 3).

Theorem B may seem unsatisfactory, depending as it does on a good choice of the set **S**. There are however stability results which tell us that we have here a property attached naturally to the field **N**.

Next we come to **(2.4(ii))**, i.e., with Γ still Abelian, we consider a relative extension **N/K**. In the character group Γ^+ we introduce an equivalence relation : Θ,Θ' are equivalent if $\langle\Theta\rangle=\langle\Theta'\rangle$ ($\langle\Theta\rangle$ the subgroup generated by Θ) or, which is the

same, if $\Theta' = \Theta''$ for some $\omega \in \Omega_{\mathbb{Q}}$. The equivalence classes are the <u>divisions</u> of Γ^+. The divisions D then correspond biuniquely to the irreducible rational representations

$$\rho_D = \sum_{\Theta \in D} \Theta$$

of Γ.

We consider maps from the set of subgroups of Γ^+ into an Abelian group A which we write multiplicatively. Such a map f can be extended to the divisions D of Γ^+ by the Möbius formula

$$(4.2) \qquad f(D) = \prod_{C < \overline{D}} f(C)^{\mu(\overline{D}:C)}$$

(product of the subgroups C of the cyclic group $\overline{D}$ generated by any $\Theta \in D$.) μ is the Möbius function.

We have to use two such functions associated with our extension N/K:
Let $\Delta < \Gamma$ be given by

$$\Delta = \bigcap_{\Theta \in G} \operatorname{Ker} \Theta,$$

and let

$$(4.3) \qquad \begin{cases} w(G) = w_{N^\Delta} = \text{no. of roots of unity in } N^\Delta \\ h_S(G) = \text{S-class number of } N^\Delta = h_{S,N^\Delta}. \end{cases}$$

Then $w(D))$, $h_S(D)$ are defined as in (4.2).

By (2.4(ii))

$$L_S(\rho_D, N) \cdot \mathcal{R}(\rho_D, \mathcal{U}_{S,N}, \mathcal{G}_{S,N})^{-1}$$

is a rational fractional ideal.

THEOREM A' : *Suppose that* S *splits in* N, *i.e., is a union of orbits each isomorphic to* Γ. *Then*

$$L_S(\rho_D, N) \cdot \mathcal{R}(\rho_D, \mathcal{U}_{S,N}, \mathcal{G}_{S,N})^{-1} \cdot w(D) \cdot h_S(D)^{-1}$$

is integral at all primes not dividing $(w_N, \operatorname{ord}(\Gamma))$.
(cf [F1] supplement 2 to Theorem 2.)

REMARK 1.- If N is real and Γ of odd order then $(w_N, \text{ord}(\Gamma)) = w(D) = 1$.

REMARK 2.- If S is large enough so that $h_{S,N} = 1$ and if N/K is totally ramified (i.e. contains no non-ramified intermediary extension) then $h_S(D) = 1$.

THEOREM B'.- *Suppose that* S *is sufficiently large, so that* $h_{S,L} = 1$ *for all fields* L *with* $K \subset L \subset N$. *Then any prime which occurs in*

$$L_S(\rho_{D,N}) \cdot \mathcal{R}(\rho_D, \mathcal{U}_{S,N}, \mathcal{G}_{S,N})^{-1}$$

will divide ord (Γ) or w_N/w_K.
(w_N = number of roots of unity)
(cf. [F1] supplement to Theorem 3).

There are analogous, but even stronger results in the additive case.

All the theorems are based on the fact that we have a regulator attached to the $\mathbb{Z}\Gamma$-lattice $\mathcal{U}_{S,N}$, which itself is a rank one module over a suitable coefficient ring of algebraic integers.

For Theorems A' and B' we also use an invariant attached to $\mathbb{Z}\Gamma$-lattices, the "module defect" first defined in [F3]. We briefly describe its properties in the relevant special case.

Write $\mathcal{H} = \mathcal{H}_{S,N}$ and let M be a $\mathbb{Z}\Gamma$-lattice so that

$$\mathcal{H}\mathbb{Q} \simeq M\mathbb{Q} \text{ (isomorphism of } \mathbb{Q}\Gamma\text{-modules)}.$$

Let $M^{\mathcal{M}}$ be the maximal $\mathcal{M}_\Gamma$-sublattice of M, $\mathcal{M}_\Gamma = \mathcal{M}$ the maximal order of $\mathbb{Q}\Gamma$. This is of finite index $[M : M^{\mathcal{M}}]$ in M. Define the module defect by

$$(4.4) \qquad \mathcal{j}(\mathbb{Z}\Gamma, \mathcal{H}, M) = [\mathcal{H} : \mathcal{H}^{\mathcal{M}}]/[M : M^{\mathcal{M}}] = \mathcal{j}(\mathcal{H}, M).$$

This has the following properties :

(i) $\mathcal{j}(\mathbb{Z}\Gamma, \mathcal{H}, M)$ is integral, and its prime factors all divide ord(Γ),

(ii) $\mathcal{j}(\mathbb{Z}\Gamma, \mathcal{H}, M) = [\mathcal{H} : \mathcal{H}^{\mathcal{M}}]$
if and only if M is an $\mathcal{M}$-lattice,

(iii) $\mathcal{j}(\mathbb{Z}\Gamma, \mathcal{H}, M) = 1$ if and only if $\mathcal{H} \sim M$, i.e., $\mathcal{H}$ and M lie in the same genus of $\mathbb{Z}\Gamma$-lattices.

Now let Δ be a subgroup of Γ and replace, in (4.4), $\mathcal{X}$ and M by $\mathcal{X}^{\Delta}$ and M^{Δ} (fixed submodules). Define a function on the subgroups G of Γ^{+}, again denoted by j, by the rule

$$(4.5)\qquad \begin{cases} j_{\mathcal{X},M}(G) = [\mathcal{X}^{\Delta} : \mathcal{X}^{\Delta\#}]/[M^{\Delta} : M^{\Delta\#}], \\ \text{if } \Delta = \bigcap_{\Theta\in G} \operatorname{Ker}\Theta \end{cases}$$

Now via the formula (4.2) we get a function $j_{\mathcal{X},M}(D)$ on the divisions of Γ^{+}, which we view as having values fractional rational ideals. The crucial property is :

$$(4.6)\qquad j_{\mathcal{X},M}(D) \text{ is integral}$$

Trivially also

(4.7) The prime divisors of $j_{\mathcal{X},M}(D)$ divide ord (Γ).
The derivation of Theorems A' and B' then hinges on (4.6), (4.7) and the formula

$$(4.8)\qquad \begin{cases} L_S(\rho_D,N) \cdot \mathcal{R}(\rho_D, \mathcal{U}_{S,N}, \mathcal{G}_{S,N})^{-1} \\ \simeq j_{\mathcal{X},\mathcal{U}}(D) \cdot h_S(D)\ w(D)^{-1} \end{cases}$$

(notations as in (4.3), (4.5)). Here the $\simeq$ indicates equality outside certain specified prime factors. For instance (4.8) will hold as a proper equation if S splits in N and $(w_N, \operatorname{ord}(\Gamma)) = 1$.

Example : N totally real, Γ of odd order. In this case also $w(D) = 1$ except for $D = \langle\varepsilon\rangle$.

Final remark. Without going too deeply into the conjecture game it is reasonable to expect integrality theorems of the type of Theorems A,A' and divisibility theorems of the type of Theorems B,B' whenever the basic, field theoretic form of the Stark conjecture can be established.

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