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Titel: New results on Ramanujan's function

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PURL: https://resolver.sub.uni-goettingen.de/purl?320141322_0011|log35

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Séminaire de Théorie des Nombres
Année 1981-1982 - exposé n° 36

NEW RESULTS ON RAMANUJAN'S FUNCTION

by

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Ramanujan's function $\tau(n)$ is defined by

$$q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n , \quad |q| < 1 . \tag{1}$$

There is no need to explain here why $\tau(n)$ is an interesting and important function of number theory. Let us rather recall some of its basic properties to set the stage for our new results. Roughly speaking one has studied $\tau(n)$ by elementary, arithmetical and analytical means.

In principle every $\tau(n)$ is computable from (1) in a finite number of steps : $\tau(1) = 1$, $\tau(2) = -24$, $\tau(3) = 253, \ldots$. Although this most elementary method is very uneconomical in practice because the required number of steps increases too rapidly it should at least convince everyone quickly that $\tau(n)$ is a rather wild function of n . Therefore it is all the more remarkable that $\tau(n)$ enjoys many nice arithmetical and analytical properties. Indeed these properties often say that for various notions of regularity the behaviour of $\tau(n)$ is extremely regular.

In the arithmetical theory one is mainly concerned with questions of divisibility or congruence properties. The subject got started with S. Ramanujan who established some congruences and stated e. g. that $\tau(n)$ is a multiplicative function of n. This was proved by Mordell soon afterwards while the Ramanujan conjecture survived as conjecture until it was settled by Deligne [2], [3] a few years ago. It amounts

$$\tau(n) \ll n^{11/2+\varepsilon} \tag{2}$$

for every $\varepsilon > 0$ and is the best possible bound of this type. Also some years ago Serre [8] and Swinnerton-Dyer [9] fully classified the congruences satisfied by $\tau(n)$. Thus we have a complete understanding of at least two important problems on the arithmetical side.

Unfortunately we cannot say the same thing about the analytical theory of $\tau(n)$. In this theory the interest is not in the individual $\tau(n)$ but various averages involving a collection of $\tau(n)$ are studied. The problem is to describe the asymptotic of these averages. The degree to which this is possible measures the regularity with which $\tau(n)$ grows. Here we meet the defects one is familiar with in analytic number theory. Namely there always remains a gap between the upper and lower bounds one can prove. So in the analytical theory no major problem has yet been fully settled.

For the classical averages the most important results are as follows. For every $\varepsilon > 0$

$$\sum_{n \le x} \tau(n) = \begin{cases} O(x^{11/2+1/3+\varepsilon}) , \\ \Omega(x^{11/2+1/4}) , \end{cases} \tag{3}$$

and

$$\sum_{n \le x} |\tau(n)| = \begin{cases} O(x^{13/2}) , \\ \Omega(x^{13/2-\varepsilon}) . \end{cases} \tag{4}$$

Furthermore, if

$$T_x(\theta) = \sum_{n \le x} \tau(n)\, e(n\theta), \quad e(\alpha) = e^{2\pi i \alpha} ,$$

then there exists $A > 0$ such that

$$\int_0^1 |T_x(\theta)|^2 d\theta = \sum_{n\le x} \tau^2(n) = Ax^{12} + O(x^{12-2/5}) \qquad (5)$$

and

$$\sup_\theta |T_x(\theta)| \ll x^6 \log x . \qquad (6)$$

The results in (3) follow for instance from the general O- and Ω-theorems in [1]. The problem of determining the true exponent in (3) strongly resembles Dirichlet's divisor problem. The O-result in (4) is an immediate consequence of (5) which is due to Rankin [7]. The Ω-result in (4) was proved by Walfisz [10]. Finally (6) is due to Wilton [11]. Since the O-result in (3) and the Ω-result in (4) rely on (2) they are not quite so innocent as they look.

To (3) - (6) we now add results on some new types of averages.

THEOREM 1. - Let $\varepsilon > 0$ and c_n, $n = 1, 2, \ldots$, be arbitrary complex numbers. Then

$$\sum_{\ell \le x} \Big| \sum_{n=1}^{N} c_n \tau(\ell+n) \Big|^2 = Ax^{12} \sum_{n=1}^{N} |c_n|^2 \left(1 + O\left(x^{\varepsilon} \left(\frac{N^2}{x}\right)^{1/3}\right)\right)$$

for $N \le x^{1/2}$ with the implicit constant depending on ε alone.

Remarks. - (i) We have a truly asymptotic result as long as N is somewhat smaller than $x^{1/2}$.

(ii) By Cauchy's inequality and (5) the trivial upper bound would be

$$\sum_{\ell \le x} \Big| \sum_{n=1}^{N} c_n \tau(\ell+n) \Big|^2 \ll Nx^{12} \sum_{n=1}^{N} |c_n|^2 .$$

This is weaker than what follows from Theorem 1 by a factor N. Note that this trivial bound could not be improved if $\tau(n)$ did not change sign very often. The fact that, in addition, we have a nontrivial remainder term is even stronger evidence for a deep regularity in the growth and the sign changes of $\tau(n)$.

If we take $c_n = e(n\theta)$ in Theorem 1 we obtain

COROLLARY 1. - For every real θ and every Θ with $0<\Theta<1/2$ we have

$$\sum_{|m-\ell|\le \ell^{\Theta}} \tau(m)\, e(m\theta) = \Omega(\ell^{(11+\Theta)/2})$$

and

$$T_x(\theta) = \Omega(x^{(11+\Theta)/2}) .$$

Remarks. - (i) The Ω-result for $T_x(0)$ comes pretty close to the classical Ω-result in (3).

(ii) Classical methods yield for rational θ only and $1/2<\Theta\le 1$

$$\sum_{|m-\ell|\le \ell^{\Theta}} \tau(m)\, e(m\theta) = \Omega(x^{11/2+1/4})$$

while our Ω-result holds for all real and much shorter intervals namely with $<1/2$.

Theorem 1 and duality yield

COROLLARY 2. - Let $\varepsilon>0$ and $N\le x^{1/2-\varepsilon}$. Then

$$\sum_{n=1}^{N} \Big| \sum_{\ell\le x} c_\ell\, \tau(\ell+n) \Big|^2 \ll x^{12} \sum_{\ell\le x} |c_\ell|^2$$

for arbitrary complex numbers c_ℓ where the implicit constant depends on ε alone.

Remark. - Up to the determination of the implicit constant Corollary 2 is best possible. This follows from (5) on taking $c_\ell = \tau(\ell+1)$ and on considering the contribution to the left hand side from $n=1$ only.

Our next result should be compared with (5).

THEOREM 2. - If $\varepsilon>0$ then

$$\Big\{\frac{1}{2N} \sum_{0<|n|\le N} \Big| \int_0^1 |T_x(\theta)|^2\, e(n\theta)\, d\theta \Big|^2 \Big\}^{1/2} \ll \frac{x^{12+\varepsilon}}{(Nx)^{1/3}}$$

for $1\le N\le x^{1/2}$ with the implicit constant depending on ε alone. In particular,

$$\int_0^1 T_x(\theta)\, e(n\theta)\, d\theta \ll x^{12-1/3+\varepsilon}$$

<u>for every non-zero integer</u> n , <u>where the implicit constant now depends on</u> n <u>and</u> ε .

<u>Remarks</u>. - (i) Theorem 2 shows that $\theta \longmapsto T_x(\theta)$ has a tendency to stay close to its mean-value.

(ii) On average over $0 < |n| \le x^{1/2}$ Theorem 2 even yields

$$\int_0^1 T_x(\theta)\, e(n\theta)\, d\theta \ll x^{12-1/2+\varepsilon}$$

for every $\varepsilon > 0$. This bound presumably comes close to the truth because there are rasons to believe that no such bound can hold with an $\varepsilon < 0$.

Using only little of Theorem 2 we can also show

COROLLARY 3

$$\int_\alpha^\beta |T_x(\theta)|^2\, d\theta = A(\beta-\alpha)x^{12} + O(x^{12-1/3+\varepsilon})$$

<u>uniformly for</u> $0 \le \alpha < \beta \le 1$.

All these new results on $\tau(n)$ are based on the fact that

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)\, e(nz) \qquad (7)$$

is a cusp form of weight 12 with respect to $\Gamma = SL_2(Z)$. This means that, in addition to (7),

$$\Delta\left(\frac{az+b}{cz+d}\right)(cz+d)^{-12} = \Delta(z)$$

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ . Thus global properties of Δ yield information about its Fourier coefficients at infinity which are of a local nature.

The basic ideas in the proof of our results are as follows. To begin with everything is reduced to the study of

$$\sum_{n=1}^{N} c_n \sum_{\ell \le x} \tau(\ell)\, \tau(\ell+n) \ .$$

For this double sum we want to obtain the desired estimates by proceeding similarly as in Voronoi's summation formula. This requires a thorough study of the Dirichlet series

$$D(s,n) = \sum_{\ell=1}^{\infty} \frac{\tau(\ell)\,\tau(\ell+n)}{(\ell+n/2)^{11+s}} , \quad \mathrm{Re}\, s > 1 , \; n = 1, 2, \ldots .$$

First it is shown that

$$D(s,n) = \int_{\mathfrak{F}} f(z)\, P(z,s,n)\, d\omega(z) , \quad \mathrm{Re}\, s > 1 ,$$

where $\mathfrak{F}$ denotes a fundamental domain for Γ,

$$f(z) = y^{12}\, |\Delta(z)|^2 , \; d\omega(z) = \frac{dx\,dy}{y^2} , \; z = x+iy ,$$

and $P(z,s,n)$ is a suitably defined Poincaré series. This series depends analytically on the second variable in $\mathrm{Re}\, s > 1$.

In a second step $P(z,s,n)$ is continued to a meromorphic function in s. This requires the spectral theory of the Laplacian

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

in $L_2(\Gamma \backslash H)$, the Hilbert space of Γ-invariant functions g with $\langle g, g\rangle < \infty$, where the inner product $\langle\ ,\ \rangle$ is given by

$$\langle g, h\rangle = \int_{\mathfrak{F}} g(z)\, \overline{h}(z)\, d\omega(z) .$$

A fundamental result of this theory says that every g in $L_2(\Gamma \backslash H)$ is representable in the L_2-sense as

$$g(z) = \sum_{j \geq 0} \langle g, e_j\rangle\, e_j(z) + \frac{1}{4\pi i} \int_{(1/2)} \langle g, E(\cdot, s)\rangle\, E(z,s)\, ds . \qquad (8)$$

Here $(e_j)_{j=0}^{\infty}$ is a complete orthonormal system of eigenfunctions of Δ in $L_2(\Gamma \backslash H)$, $E(z,s)$ denotes the so-called Eisenstein series and $\int_{(1/2)}$ stands for integration along $\mathrm{Re}\, s = 1/2$ in the direction of increasing imaginary parts

If

$$\Delta e_j(z) = s_j(s_j-1)\, e_j(z)$$

we assume, as we may, that $s_o = 1$ and $s_j = 1/2 + it_j$ with $t_j > 0$ for $j > 0$. Moreover these functions have Fourier expansions of the form

$$e_j(z) = \sum_{n \neq 0} \alpha_{jn}\, y^{1/2}\, K_{s_j - 1/2}(2\pi\, |n|\, y)\, e(nx), \quad j > 0,$$

and

$$E(z, s) = y^s + \alpha_o(s)\, y^{1-s} + \sum_{n \neq 0} \alpha_n(s)\, y^{1/2}\, K_{s-1/2}(2\pi\, |n|\, y)\, e(nx)$$

where $K_\nu(z)$ is the modified K-Bessel function.

The analytic continuation of $P(z, s, n)$ results from applying (8) with $g(z) = P(z, s, n)$. In addition we have to determine residues and, to some extent, the rate of growth of the Poincaré series in $\operatorname{Re} s \leq 1$. We find that $D(s, n)$ has at most simple poles in $\operatorname{Re} s \geq 1/2$ namely at $s = s_j$ or $s = 1 - s_j$, $j > 0$. There e_j contributes

$$\frac{1}{2}(4\pi)^{23/2}\, \left(\frac{4}{n}\right)^{s-1/2}\, \frac{\Gamma(s-1/2)}{\Gamma(s+11)}\, \bar{\alpha}_{jn} \langle e_j, f \rangle$$

to the residue where $\Gamma(s)$ is the Γ-function. Moreover we prove a functional equation for $P(z, s, n)$ and a fortiori for $D(s, n)$, namely

$$D(s, n) = \left(\frac{4}{n}\right)^{s-1}\, \frac{\Gamma(s+1/2)\, \Gamma(12-s)}{\Gamma(3/2-s)\, \Gamma(s+11)}\, D(1-s, n) + \frac{1}{2}(4\pi)^{23/2}\, \left(\frac{4}{n}\right)^{s-1/2} \cdot$$

$$\cdot\, \frac{\Gamma(s-1/2)}{\Gamma(s+11)}\, \alpha_{-n}(1-s) \langle E(\cdot, s), f \rangle + R(s, n).$$

Here the second term on the right hand side is due to the presence of the integral term in (8). It has lots of poles in $0 < \operatorname{Re} s < 1/2$. On the other hand, $R(s, n)$ is shown to be relatively small for $-\frac{1}{2} < \operatorname{Re} s \leq \frac{1}{2}$, although it is given by a fairly complicated expression. This has the consequence that all our results depend in a crucial way on good estimates for

$$\Sigma(U) = \sum_{t_j \le U} e^{\pi t_j} |< e_j , f>|^2 + \frac{1}{4\pi} \int_{-U}^{U} e^{\pi|t|} |< E(\cdot , \tfrac{1}{2}+it) , f>|^2 \, dt$$

and

$$\Sigma_1(U,N) = \sum_{t_j \le U} e^{-\pi t_j} \Big| \sum_{n=N}^{2N} c_n \alpha_{jn} n^{it_j} \Big|^2 + \frac{1}{4\pi} \int_{-U}^{U} e^{-\pi|t|} \Big| \sum_{n=N}^{2N} c_n \alpha_n (\tfrac{1}{2}+it) n^{it} \Big|^2 dt \, .$$

We prove

THEOREM 3. - (i) $\Sigma(U) \ll U^{24}$.

(ii) For arbitrary complex numbers c_n and $\varepsilon > 0$

$$\Sigma_1(U,N) \ll U(U+N^{1+\varepsilon}) \sum_{n=N}^{2N} |c_n|^2 ,$$

where the implicit constant depends on ε alone.

The first part of Theorem 3 is a special case of the main result in [4], the second is contained in [5] Lemma 3. The proof of Theorem 3 is the most difficult part in our argument. At an early stage we apply a sum formula à la Bruggeman-Kusznietsov. Then we perform a series of purely analytical estimates. Finally an arithmetical device is needed. Fort part (i) this is the functional equation for

$$L(s , \frac{a}{c}) = \sum_{n=1}^{\infty} \tau(n) \, e(n \frac{a}{c}) \, n^{-s} ,$$

where a , c are coprime integers, while for part (ii) this is the large sieve inequality for Farey points.

Apart from the determination of the implicit constant our bound for $\Sigma(U)$ is best possible. On the other hand, Iwaniec [6] showed for the bilinear form $\Sigma_2(U,N)$ which results from $\Sigma_1(U,N)$ by deleting the factor n^{it_j} or n^{it} that

$$\Sigma_2(U,N) \ll (U^2+N^{1+\varepsilon}) \sum_{n=N}^{2N} |c_n|^2 .$$

The same bound probably holds for $\Sigma_1(U,N)$. We must point out, however, that it would not lead to any improvement of the results we stated earlier.

Finally we should mention that our method does not only work for $\tau(n)$. Indeed similar results hold for all Fourier coefficients of cusp forms of weight k ($k > 2$, $k \equiv 0 \pmod 2$) with respect to any finitely generated Fuchsian group of the first kind. In general the estimates are somewhat weaker since the Ramanujan conjecture (2) is used for Theorem 1 and 2. For detailed proofs the reader sould consult [4], [5].

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(texte reçu le 8 novembre 1982)

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