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# REPRESENTATION OF SEMIGROUPS BY PRODUCTS OF TOPOLOGICAL SPACES WITH PRESCRIBED CARDINAL FUNCTIONS

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Abstract. Given an m-tuple $\Phi_1,\dots,\Phi_m$ of cardinal functions and an m-tuple of cardinal numbers $\alpha_1,\dots,\alpha_m$, we examine when there exists a space X homeomorphic to $X^3$ but not to $X^2$ such that $\Phi_i(X) = \alpha_i$ for all $i = 1,\dots,m$. We show that under some natural assumptions about the cardinal functions, such space X exists provided that there exists at least one space Y with $\Phi_i(Y) = \alpha_i$ for all $i = 1,\dots,m$. A more general setting of representations of commutative semigroups by products of spaces is investigated.

Key words: Representation, semigroup, topological product, cardinal function.

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I. Introduction and the Main Theorems. Let (S,+) be a commutative semigroup, $\mathcal{K}$ a class of topological spaces. A mapping $r: S \longrightarrow \mathcal{K}$ such that for all $s_1, s_2 \in S$

(i) $r(s_1)$ is homeomorphic to $r(s_2)$ iff $s_1 = s_2$ and

(ii) $r(s_1 + s_2)$ is homeomorphic to $r(s_1) \times r(s_2)$

is called a representation of (S,+) by products in $\mathcal{K}$.

Representations of commutative semigroups by products of topological, algebraic or relational structures have been investigated by many authors, for a survey of topological results see [7].

In the present paper, we investigate representations of semigroups by products of topological spaces with prescribed values of some cardinal functions. The notion of a cardinal function is as in [2] or [3], i.e. an arbitrary function from the class of topological spaces into the class of all cardinal numbers such that homeomorphic spaces have the same value.

Given a class T of topological spaces, an m-tuple of cardinal functions $\Phi_1,\dots,\Phi_m$ and an m-tuple of cardinal numbers $\alpha_1,\dots,\alpha_m$, we denote by

$$T(\Phi_1\longrightarrow\alpha_1,\dots,\ \Phi_m\longrightarrow\alpha_m)$$

the class of all spaces $X\in T$ such that $\Phi_i(X)=\alpha_i$ for all $i=1,\dots,m$. The aim of the present paper is to prove the two theorems below.

<u>Theorem 1</u>. Let T be a class of topological spaces containing all metrizable continua and closed with respect to homeomorphic images, finite products and countable coproducts (= disjoint unions as closed-and-open subspaces). Let $\Phi_1,\dots$ $\dots,\Phi_m$ be an m-tuple of cardinal functions such that, for every $i=1,2,\dots,m$,

(a) $\Phi_i(X)\geq\aleph_0$ for every space $X\in T$ and $\Phi_i(K)=\aleph_0$ whenever K is a metrizable continuum;

(b) $\Phi_i(X\times K)=\Phi_i(X)$ whenever $X\in T$ and K is a metrizable continuum;

(c) $\Phi_i(\coprod_{n=1}^{\infty}X_n)=\sup_{n=1,2,\dots}\Phi_i(X_n)$ (where $\coprod$ denotes the coproduct), whenever $X_n\in T$ for all $n=1,2,\dots$ ,

(d) $\Phi_i(X)=\Phi_i(X^n)$ for all $n=1,2,3,\dots$ and all $X\in T$.

Then for every m-tuple $\alpha_1,\dots,\alpha_m$ of infinite cardinal numbers every countable commutative semigroup has a representation by products in the class

$$\mathbb{T} = T(\Phi_1 \longrightarrow \alpha_1, \ldots, \Phi_m \longrightarrow \alpha_m)$$

provided that the class $\mathbb{T}$ is non-empty.

Remark. It is well-known that e.g. the weight, $\pi$-weight, net weight, density, character, pseudocharacter fulfil (a)-(d) above, so the Theorem 1 can be applied on them; the class $T$ can be chosen to be e.g. the class of all topological spaces, all $T_1$-spaces, Hausdorff, regular, completely regular, metrizable and many others. Unfortunately, the class of compact spaces does not fit in it, this class is not closed with respect to countable disjoint unions. We present here another theorem for the compact Hausdorff spaces.

Given a seventuple $\alpha_1, \ldots, \alpha_7$ of cardinal numbers, let us denote by

$$\mathbb{C}_{\alpha_1,\ldots,\alpha_7} = \text{Comp}\,(w \longrightarrow \alpha_1,\ \pi \longrightarrow \alpha_2, nw \longrightarrow \alpha_3, d \longrightarrow \alpha_4,$$
$$\chi \longrightarrow \alpha_5,\ \psi \longrightarrow \alpha_6,\ t \longrightarrow \alpha_7$$

the class of all compact Hausdorff spaces X such that its weight $w(X) = \alpha_1$, $\pi$-weight $\pi(X) = \alpha_2$, net weight $nw(X) = \alpha_3$, density $d(X) = \alpha_4$, character $\chi(X) = \alpha_5$, pseudocharacter $\psi(X) = \alpha_6$, and tightness $t(X) = \alpha_7$.

Theorem 2. For every seventuple $\alpha_1, \ldots, \alpha_7$ of infinite cardinal numbers, every finite cyclic group has a representation by products in the class $\mathbb{C}_{\alpha_1,\ldots,\alpha_7}$ provided that this class is non-empty.

Remark. If $\{r(0), r(1)\}$ is a representation of the cyclic group $c_2 = \{0,1\}$, then the space $X = r(1)$ is homeomorphic to $X^3 \simeq r(1+1+1) \simeq r(1)$ but not to $r(0) \simeq r(1+1) \simeq X^2$. Hence the result mentioned in the abstract is obtained as the special case of Theorem 1. The Theorem 2 can be also applied on the group $c_2$.

II. Proofs of the Theorems

1. Let N be the additive semigroup of all non-negative integers, $N^M$ the semigroup of all functions from M into N with the pointwise addition (i.e. $(f+g)(m) = f(m) + g(m)$) and by $\exp N^M$ the semigroup of all its subsets with the addition defined by

$$A + B = \{f+g \mid f \in A \text{ and } g \in B\}.$$

By [5], for every commutative semigroup (S,+) there exists a homomorphism $h:(S,+) \longrightarrow \exp N^M$ such that

(i) card $M = \aleph_0$ card S; for each $s \in S$, card $h(s) = \aleph_0$ card S; for every $s \in S$ and every $f \in h(s)$, $f(m) \neq 0$ for infinitely many $m \in M$;

(ii) if $s \neq s'$ then $h(s) \cap h(s') = \emptyset$.

Let C be a Cook continuum, i.e. a metrizable continuum such that for any subcontinuum $K \subset C$ and any continuous map $c:K \longrightarrow C$, c is either the inclusion map or a constant, see [1]. Let $\{K_n \mid n = 0,1,\dots,\infty\}$ be a pairwise disjoint system of non-degenerate subcontinua of C, so if $c:K_n \longrightarrow K_m$ is a continuous map then either $n = m$ and c is the identity or c is a constant map. For every map $f:N \longrightarrow N$ put

$$K(f) = \prod_{m \in N} K_n^{f(n)},$$

where $K_n^{f(n)}$ is the product of f(n) copies of $K_n$ (if $f(n) = 0$, then it is a one-point space) and $\prod$ denotes the (countable) product. By [4], if $f,g: N \longrightarrow N$ and $f \neq g$, then $K(f)$ is not homeomorphic to $K(g)$.

2. Proof of Theorem 1. Let a countable commutative semigroup $(S,+)$ be given. We find a homomorphism $h:(S,+) \longrightarrow \exp N^N$ such that (i) and (ii) from 1 are fulfilled. Since $\mathbb{T} = T(\Phi_1 \longrightarrow \alpha_1, \dots, \Phi_m \longrightarrow \alpha_m)$ is supposed to be non-empty, we choose Y in it. Let $\{K_n \mid n = 0,1,\dots,\infty\}$ and $K(f)$ be as in 1. We denote $Y \times K_\infty$ by Z and for every $s \in S$ define $r(s)$ as a coproduct (we denote coproduct in the class of all topological spaces by $\coprod$ ) of $\aleph_0$ copies of the space

$$\coprod_{n \in N, f \in h(s)} Z^n \times K(f).$$

The properties of T and (a)-(d) in Theorem 1 guarantee that $r(s) \in \mathbb{T}$. It is easy to verify that $r(s+s')$ is homeomorphic to $r(s) \times r(s')$. Hence to prove that $\{r(s) \mid s \in S\}$ is a representation of $(S,+)$ in $\mathbb{T}$, it is sufficient to verity that

if $s \neq s'$, then $r(s)$ is not homeomorphic to $r(s')$.

Denote by $Z(s)$ the subspace of $r(s)$ consisting of all components L of $r(s)$ such that there exists no homeomorphism of $K_\infty$ into L. Since $Z = Y \times K_\infty$, $Z(s)$ consists of all $Z^n \times K(f)$ with $n = 0$, i.e. $Z(s)$ is the coproduct of $\aleph_0$ copies of the space $\coprod_{f \in h(s)} K(f)$. We define analogously $Z(s')$, hence $Z(s')$ is a coproduct of $\aleph_0$ copies of $\coprod_{g \in h(s')} K(g)$. Since $h(s) \cap h(s') = \emptyset$ and $K(f)$ is not homeomorphic to $K(g)$ for $f \neq g$, $Z(s)$ is not homeomorphic to $Z(s')$, hence $r(s)$ is not homeomorphic to $r(s')$.

3. Proof of Theorem 2. The method of [6], simplified by A. Úlehlová is used. Let a seventuple $\alpha_1,\dots,\alpha_7$ be given such that $\mathbb{C}_{\alpha_1,\dots,\alpha_7}$ is non-empty, choose a connected

space Y in it. Let a finite cyclic group $c_n = \{0,1,\dots,n-1\}$ be given (the operation of $c_n$ is the addition modulo n). We construct the spaces $\{r(s) \mid s \in c_n\}$ as in ? (by means of the Y chosen in $\mathbb{C}_{\alpha_1,\dots,\alpha_7}$) i.e. $r(s) = \coprod_{k,m \in N, f \in h(s)} (Z^n \times K(f))_k$. Let $\tilde{X} = X \cup \{\xi\}$ be a one-point compactification of the space $X = r(1)$. Then $\tilde{X}$ is in $\mathbb{C}_{\alpha_1,\dots,\alpha_7}$. Let $f: \tilde{X}^{n+1} \longrightarrow \tilde{X}$ be the continuous map which maps $X^{n+1} = r(n+1) \simeq r(1)$ onto X as a homeomorphism and sends $\tilde{X}^{n+1} \setminus X^{n+1}$ to $\xi$. Let us denote by V the inverse limit of the chain

$$\tilde{X} \xleftarrow{f} \tilde{X}^{n+1} \xleftarrow{f^{n+1}} \tilde{X}^{(n+1)^2} \xleftarrow{f^{(n+1)^2}} \tilde{X}^{(n+1)^3} \longleftarrow \dots$$

Then V is in $\mathbb{C}_{\alpha_1,\dots,\alpha_7}$ and is homeomorphic to $V^{n+1}$. The union of all closed-and-open components of V is homeomorphic to X. Since the spaces $X, X^2, \dots, X^n$ are pairwise non-homeomorphic, the spaces $V, V^2, \dots, V^n$ are also pairwise non-homeomorphic and they form a representation of $c_n$ by $r(0) = V^n$, $r(i) = V^i$ for $i = 1,\dots,n-1$.

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## R e f e r e n c e s

[1] H. COOK: Continua which admit only the identity mapping onto non-degenerate subcontinua, Fund. Math. 60 (1967), 241-249.

[2] R. ENGELKING: General Topology, Warszawa, 1977.

[3] I. JUHÁSZ: Cardinal functions in topology, Math. Centre Tracts 34 (Amsterdam 1971).

[4] V. TRNKOVÁ: Representation of semigroups by products in a category, J. Algebra 34(1975), 191-204.

[5] V. TRNKOVÁ: Isomorphism of products and representation of commutative semigroups, Coll. Math. Soc. J. Bolyai 20. Algebraic theory of semigroups, Szeged (Hungary) 1976, North-Holland 1979, 657-683.

[6] V. TRNKOVÁ: Categorial aspects are useful for topology, General Topology and its Relation to Modern Analysis and Algebra IV, Proc. Fourth Prague Topol. Symp. 1976, Lect. N. in Math. 609(1977), 211-225.

[7] V. TRNKOVÁ: Representations of commutative semigroups by products of topological spaces, General Topology and its Relation to Modern Analysis and Algebra V, Proc. Fifth Prague Topol. Symp. 1981, Helderman Verlag Berlin 1982, 631-541.

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