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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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ON WEINGARTEN SURFACES

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Abstract: A global characterization of spheres.

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Following the ideas of [1], I am going to prove the following

Theorem. Let G be a bounded domain in $\mathbb{R}^2$, ∂G its boundary and $M: G \cup \partial G \longrightarrow E^3$ a surface such that: (i) $M(\partial G)$ consists of umbilical points; (ii) there are functions $g(x,y)$, $F(x,y)$ and, on M, orthonormal tangent vector fields v_1, v_2 satisfying

$$(1) \qquad v_1 g(h,K) = 0 , \qquad v_2 f(h,K) = 0 ,$$

H and K being the mean and Gauss curvatures of M resp. Further, let

$$(2) \qquad K>0 , \qquad g_H g_K > 0 , \qquad f_H f_K > 0$$

Then $M(G \cup \partial G)$ is a part of a sphere.

Proof. On M, consider a field of orthonormal moving frames $\{m; v_1, v_2, v_3\}$. Then

(3) $$dm = \omega^1 v_1 + \omega^2 v_2 ,$$
$$dv_1 = \omega_1^2 v_2 + \omega_1^3 v_3 ,$$
$$dv_2 = -\omega_1^2 v_1 + \omega_2^3 v_3 ,$$
$$dv_3 = -\omega_1^3 v_1 - \omega_2^3 v_2$$

with the usual integrability conditions. We have

(4) $$\omega_1^3 = a\omega^1 + b\omega^2 , \quad \omega_2^3 = b\omega^1 + c\omega^2 ,$$

(5) $$da - 2b\omega_1^2 = \alpha\omega^1 + \beta\omega^2$$
$$db + (a - c)\omega_1^2 = \beta\omega^1 + \gamma\omega^2$$
$$dc + 2b\omega_1^2 = \gamma\omega^1 + \delta\omega^2$$

(6) $$dH = \frac{1}{2}(\alpha + \gamma)\omega^1 + \frac{1}{2}(\beta + \delta)\omega^2 ,$$
$$dK = (a\gamma + c\alpha - 2b\beta)\omega^1 + (a\delta + c\beta - 2b\gamma)\omega^2 .$$

On M, choose a curvilinear coordinate system (u,v) such that

(7) $$I = r^2 du^2 + s^2 dv^2 , \quad \omega^1 = r\,du, \quad \omega^2 = s\,dv, \quad rs \neq 0$$

Then

(8) $$\omega_1^2 = -\frac{r_v}{s} du + \frac{s_u}{r} dv .$$

From (5) and (7),

(9) $$d(a - c) = 4b\omega_1^2 + (\alpha - \gamma)\omega^1 + (\beta - \delta)\omega^2 ,$$
$$db = -(a - c)\omega_1^2 + \beta\omega^1 + \gamma\omega^2 ;$$

$$(10)\qquad (a-c)_u + 4b\frac{r_v}{s} = (\alpha-\gamma)\,r\,,$$

$$b_u - (a-c)\frac{r_v}{s} = \beta r\,,$$

$$(a-c)_v - 4b\frac{s_u}{r} = (\beta-\delta)\,s\,,$$

$$b_v + (a-c)\frac{s_u}{r} = \gamma s$$

and

$$(11)\qquad \alpha rs = s(a-c)_u + rb_v + (.)(a-c) + (.)b\,,$$

$$\beta rs = s\,b_u + (.)(a-c) + (.)b\,,$$

$$\gamma rs = r\,b_v + (.)(a-c) + (.)b\,,$$

$$\delta rs = -r(a-c)_v + s\,b_u + (.)(a-c) + (.)b\,.$$

From (1),

$$(12)\qquad g_H\,dH(v_1) + g_K\,dK(v_1) = 0\,,$$

$$f_H\,dH(v_2) + f_K\,dK(v_2) = C\,,$$

i.e.,

$$(13)\qquad g_H(\alpha+\gamma) + 2g_K(a\gamma + c\alpha - 2b\beta) = 0\,,$$

$$f_H(\beta+\delta) + 2f_K(a\delta + c\beta - 2b\gamma) = 0\,.$$

Because of (11), (13) turns out to be

$$(14)\qquad \{sg_H - 2cs\,g_K\}(a-c)_u - 4bs\,g_K\,b_u +$$

$$+\{2rg_H + 2rg_K(a+c)\}b_v = (.)(a-c) + (.)b,$$

$$\{ - rf_H - 2\, ar\, f_K \}(a - c)_v + \{ 2\, sf_H + 2\, sf_K(a +$$

$$+ c) \}\, b_u - 4\, br\, f_K\, b_v = (.)\,(a - c) + (.)b\ .$$

This system can be written in the form

$$(15)\quad a_{11}(a - c)_u + a_{12}(a - c)_v + b_{11}b_u + b_{12}b_v =$$

$$= c_{11}(a - c) + c_{12}b\ ,$$

$$a_{21}(a - c)_u + a_{22}(a - c)_v + b_{21}b_u + b_{22}b_v =$$

$$= c_{21}(a - c) + c_{22}b$$

with

$$(16)\quad a_{11} = s(g_H + 2\, ag_K), \qquad a_{12} = 0\ ,$$

$$b_{11} = - 4\, b\, sg_K\ , \qquad b_{12} = 2\, r(g_H + 2\, H\, g_K)\ ,$$

$$a_{21} = 0\ , \qquad a_{22} = - r(f_H + 2\, a\, f_K)\ ,$$

$$b_{21} = 2\, s(f_H + 2\, H\, f_K)\ , \quad b_{22} = - 4\, br\, f_K$$

The system (15) is called elliptic if the form

$$(17)\quad \Phi = (a_{12}b_{22} - a_{22}b_{12})\mu^2 - (a_{11}b_{22} - a_{21}b_{12} +$$

$$+ a_{12}b_{21} - a_{22}b_{11})\,\mu\nu + (a_{11}b_{21} - a_{21}b_{11})\nu^2$$

is definite. From (16),

$$a_{12}b_{22} - a_{22}b_{12} = 2\, r^2(g_H + 2\, H\, g_K)(f_H + 2\, a\, f_K)\ ,$$

$$a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11} = - 4\, b\, sr\, \{ f_K(g_H +$$

$$+ 2\,c\,g_K) + g_K(f_H + 2\,a\,f_K)\}\ ,$$

$$a_{11}b_{21} - a_{21}b_{11} = 2\,s^2(g_H + 2\,c\,g_K)(f_H + 2\,H\,f_K)\ .$$

The discriminant of Φ being denoted by Δ, we get

$$\begin{aligned}(18)\qquad \Delta &= 16\,r^2s^2\,[2\,H\,f_Hf_Kg_H^2 + 2\,H\,g_Hg_Kf_H^2 + \\ &+ 2(\tfrac{1}{2}\,f_Hg_H + c\,f_Hg_K)^2 + 2(\tfrac{1}{2}\,f_Hg_H + \\ &+ a\,f_Kg_H)^2 + (2\,K + b^2)(f_H^2g_K^2 + f_K^2g_H^2) + \\ &+ (12\,H^2 + 4\,K + 2\,b^2)\,f_H\,f_K\,g_H\,g_K + \\ &+ 8\,H(b^2 + c^2 + 2\,K)\,f_H\,f_K\,g_K^2 + \\ &+ 8\,H(a^2 + b^2 + 2\,K)\,g_H\,g_K\,f_K^2 + \\ &+ 16\,H^2\,K\,f_K^2\,g_K^2\,]\ .\end{aligned}$$

From (2), $\Delta > 0$, and the system (14) is elliptic. On ∂G, $a - c = b = 0$; the ellipticity of (14) implies $a - c = b = 0$ on G. Because of $4(H^2 - K) = (a - c)^2 + 4\,b^2 = 0$, M is a part of a sphere. QED.

The H- and K-theorems are now trivial consequences of our Theorem.

Reference

[1] A. ŠVEC: Several new characterizations of the sphere, Czech. Math. J. (to appear).

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